Credit Risk Valuation

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Chapter 1

Introduction

An investor that enters into a financial transactions is faced to various risks, like market risk, credit risk, liquidity risk or operational risk. Market risk is the risk of value change in a financial asset due to changes in market variables, e.g. interest rates, exchange rates, equity prices, and commodity prices. Credit risk can be defined as the risk of a financial loss due to changes in the credit quality of market participants. This includes both losses due to defaults and losses caused by changes in credit quality. Liquidity risk is the risk that a given security or asset cannot be bought or sold quickly enough to prevent or minimize a loss. The boundaries of these three risk categories are not always clearly defined, nor do they form an exhaustive list of the full range of possible risks affecting a financial institution.

1.1 Types of Credit Risk

The most common type of credit risk is default risk, which occurs when a company or an individual is unable or unwilling to make the required payments on their debt obligations. This also includes that the borrower or debtor can not repay on schedule or in full. In a banking context, by far the oldest risk type to be regulated is default risk. For most banks, loans are the largest and most obvious source of default risk, however, banks face credit risk in other various financial instruments, including interbank transactions, foreign exchange transactions, futures, swaps, options.

For firms, default is normally triggered by a failure of the firm to meet its debt servicing obligations, which usually leads to the firm's reorganisation or liquidation. Thus default is considered a rare event, which can cause large financial losses to security holders. With some flexible thinking, this type of credit risk can be extended to sovereign bonds issued by countries with a non-negligible risk of default. Many countries have faced this sovereign risk in the late-2000s global recession.

Counterparty risk is also a sub-class of credit risk and usually is a result of trading activities. Counterparty risk is the risk of default by the counterparty in many forms of contracts, often over-the-counter (OTC) contracts. OTC contracts, such as interest rate swaps, are unfunded bilateral contract, and if one party defaults, it may expose the other party to significant financial losses.

With loans or bonds, the exposure is easy to determine, it is the outstanding amount that the counterparty has yet to repay. The counterparty risk is measured as the replacement cost of the position if the counterparty defaults prior to transaction, plus the estimated future loss if the counterparty defaults on its obligations.

Counterparty credit risk has become one of the biggest issues and challenges in the global financial crisis. A famous example is the case of AIG, which sold credit default swaps to counterparties who wanted default protection, usaly on CDO tranches. In September 2008 AIG required a federal bailout because it did not have the financial strength to support its many CDS commitments as the crisis progressed.

Another important issue in credit risk management is the estimation of credit risk correlation. Understanding how corporate defaults are correlated is particularly important for the risk management of portfolios of corporate debt. For example, banks determine their capital requirements on the basis of default correlation models. If defaults are more heavily clustered in time than envisioned in these default risk models, then significantly greater capital might be required in order to survive default losses, especially in times of distress.

An understanding of the sources and degree of default clustering is also crucial for the rating and risk analysis of derivatives that are heavily exposed to correlations in credit risk, such as collateralized debt obligations (CDOs) and options on portfolios of default swaps. This is especially true given the rapid growth of structured credit products in the financial markets.

Several explanations have been explored, why corporate defaults cluster. First, firms may be exposed to common risk factors whose movements cause correlated changes in default probabilities. Second, default may be "contagious," in other words a default event may induce other corporate failures. Third, learning from default may generate default correlation. For example, the default of one firm may reveal some systematic irregularities that could be present in other firms, what increases default probabilities for other firms.

1.2 Measurement and Management

The central issue of financial risk management has been market risk, since a firm's business is constantly affected by this risk. Therefore a lot of methods have been developed to measure and reduce market risk. In 1974 Robert Merton has noted that the Black Scholes option pricing formula can be used to evaluate the credit risk of a corporation's debt. Since then the the field of credit risk research has seen many theoretical developments. Most of this research has concentrated on the pricing of corporate and sovereign defaultable bonds as the basis of credit risk pricing.

Credit risk can be modeled with different approaches. The literature distinguishes between methods that use (historical) accounting information to asses or forecast the credit risk of a firm, and methods that use market prices of assets to model credit risk.

Accounting based models like credit rating, rely purely on accounting data in estimating the default probability. The advantages of these models are that they can take several factors into account at the same time and that they are very easy to use. However, this models do not provide an up-to-date indication of credit risk, since they are not frequently updated. In the recent credit crisis we saw the consequences of this: the sudden changes in financial conditions of firms were not yet reflected in the firm's credit rating. Firms that used credit ratings as the only source of credit risk information were thereby unable to make a correct assessment of their counterparty's credit risk and incurred high losses.

Market price methods are developed to take the sudden nature of default events into account. There are two primary types of market price methods in the literature: reduced form models and structural models.

Structural models are based on the option pricing theories developed by Merton. In these models when the asset value of the firm falls below a certain threshold, the firm fails to meet its obligations to the debt holders, thus triggering a default event.

Unlike structural models, intensity models specify neither firm value processes nor default boundaries explicitly. The credit risk is specified by the occurrence of default and the recovery rate. The first is modeled as a stochastic event whose arrival rate is governed by a given intensity. The intensities are typically obtained from market prices of defaultable instruments, such as bonds and credit default swaps. These models are flexible and computationally fast but they are less grounded in economic theory than the structural models. Recovery rates are usually assumed to be constant.

Chapter 2

Structural Models

In this chapter we review some models for measuring credit risk based on the structural approach and also discuss the empirical evidence in the literature on the performance of structural models of credit risk.

A model of default is known as a structural or firm-value model when it attempts to explain the mechanism by which default takes place. Structural models have their origins in the framework of Merton [59], which has been the key foundation of corporate debt pricing. Under these structural models, all the relevant credit risk elements, including default, are a function of the structural characteristics of the firm, like asset volatility and leverage. The main advantage of structural models is that they provide an intuitive picture, and an endogenous explanation for default.

Structural models can be divided depending on, how they explain the mechanism of default. In the Merton model default can occur only at the maturity of the dept. First passage model are based on the assumptions, that the default occurs when the asset value drops below a barrier, allowing default to occur at any time. The boundary can be either endogenous or exogenous. The former can be considered to be a safety covenant while the latter can result from the equity holder's option to default, as in Geske [42]. The recent theoretical models on structural credit risk allow the equityholder's to default strategically or examine optimal leverage using a first-passage model.

2.1 The Merton model

The literature on structural models for credit risk starts with the paper of Merton [59] and it is based on the option pricing theory developed by Black & Scholes [9]. Many extensions of this model have been developed over the years, but Merton's original model is still popular with practitioners in credit risk analysis.

Merton considers a firm whose asset value follows some stochastic process V_t. The firm finances itself by equity (i.e. by issuing shares) and by debt. In Merton's model debt has a very simple structure: it consists of one single debt obligation or zero-coupon bond with face value D and maturity T. In practice a debt maturity T is chosen such that all debts are mapped into a zero-coupon bond. Let E_t and D_t denote the time t value of equity and debt. According the capital structure given by the balance sheet relationship the value of the firm's assets is simply the sum of these, i.e. $V_t = E_t + D_t, \ 0 \le t \le T$. Default occurs if the firm misses a payment to its debt holders, which in the Merton model can occur only at the maturity of the bond. At maturity T we have to distinguish between two cases.

- $V_T \ge D$: the firm's asset value exceeds the face value of the dept. In this case the debt holders receives D, the share holders receive the residual value $E_T = V_T - D$, and there is no default.
- $V_T < D$: the firm's asset value is less than the face value of the debt. In this case the firm defaults and the share holders hand over the firm's control to the bondholders, who distribute the firm's capital among themselves. Shareholders pay and receive nothing, so we have $D_T = V_T, \ E_T = 0$.

Summarizing, we have the relations:

$$\begin{aligned} E_T &= \max(V_T - D, 0) = (V_T - D)^+, \\ D_T &= \min(V_T, D) = D - (D - V_T)^+, \end{aligned} \tag{2.1}$$

and thus the firm's equity can be seen as a a call option on the firm's assets, with the strike price D and the maturity T.
Merton made the following assumptions to develop his model:

- There are no transaction costs, bankruptcy costs or taxes.
- The risk-free interest rate r is constant and known. Therefore, the price at time t of the unit default-free zero-coupon bond with maturity T is easily seen to be $B(t,T) = e^{-r(T-t)}$.
- Assets are traded and trading takes place continuously in time with no restrictions on short selling.
- The firm's asset-value process V_t is independent of the debt level D and under the real-world probability measure $\mathbf{P}$ the process V_t follows a diffusion model of the form

  $$\mathrm{d}V_t = \mu_V V_t \mathrm{d}t + \sigma_V V_t \mathrm{d}W_t \tag{2.2}$$

  where $\mu_V \in \mathbb{R}$ is the expected return on the firm's assets and $\sigma_V > 0$ is the asset's return volatility and W_t is a standard Brownian motion under the real-world measure.

According to (2.1), the firm's equity corresponds to a European call on V_t with exercise price D and maturity T. Thus the value of equity is simply given by the Black and Scholes option pricing model:

$$E_t = C^{BS}(V_t, r, \sigma_V, D, T-t) = V_t \Phi(d_+) - De^{-r(T-t)}\Phi(d_-) \tag{2.3}$$

where $\Phi(\cdot)$ denotes the $N(0,1)$ cumulative distribution function, with the quantities d_+ and d_- given by:

$$d_+ = \frac{\ln(V_t/D) + (r + \frac{1}{2}\sigma_V^2)(T-t)}{\sigma_V \sqrt{T-t}}, \tag{2.4}$$

$$d_- = d_+ - \sigma_V \sqrt{T-t}. \tag{2.5}$$

Under this framework, a credit default at time T is triggered by the event that shareholder's call option matures out-of-money. Equation (2.2) implies that $V_T = V_0 \exp((\mu_V - \frac{1}{2}\sigma_V^2)T + \sigma_V W_T)$, and thus the firm's default probability under the real-world probability measure is:

$$\mathbf{P}(V_T \le D) = \mathbf{P}(\ln V_T \le \ln D) = \Phi\left(\frac{\ln(D/V_0) - (\mu_v - \frac{1}{2}\sigma_V^2)T}{\sigma_V^2 \sqrt{T}}\right) = \Phi(-d_-).$$

It is easy to see that the default probability is increasing in D, decreasing in V_0 and μ_V and for $V_0 > D$ increasing in σ_V, which is in line with economic intuition.

Next we turn to the valuation of the risky debt issued by the firm. According to (2.1) we have

$$D_t = De^{-r(T-t)} - P^{BS}(V_t, r, \sigma_V, D, T-t),$$

where $P^{BS}(V_t, r, \sigma_V, D, T-t)$ denotes the Black-Scholes price of a European put option on V_t. It is known that

$$P^{BS}(V_t, r, \sigma_V, D, T-t) = De^{-r(T-t)}\Phi(-d_-) - V_t\Phi(-d_+).$$

Combining this with the last equation, we get that

$$D_t = V_t\Phi(-d_+) + De^{-r(T-t)}\Phi(d_-).$$

The Term Structure of Credit Spreads

It is natural to consider yields instead of prices when trying to compare the the values of risky and riskless debt. Credit spread compensates for exposure to credit risk, and it is defined as the difference between the (promised) yield on the risky bond and the yield on riskless bonds.

Let r_t^D denote the continuously compounded yield on the firm's dept at time $t < T$, so that

$$D_t = De^{-r_t^D(T-t)}.$$

From this equality follows, that

$$r_t^D = -\frac{\ln D_t - \ln D}{T-t}.$$

For $t < T$ the credit spread s_t is defined as the excess return over the risk-free rate, $s_t = r_t^D - r_t$. In Merton's model, we have

$$s_t = -\frac{\ln\left(\Phi(d_-) + V_t/De^{-r(T-t)}\Phi(-d_+)\right)}{T-t} > 0 \tag{2.6}$$

The term structure of credit spreads refers to a plot of spreads against the time to maturity. By varying $T-t$ in (2.6), we obtain the term structure of credit spreads implied by the Merton model. From the definition of d_+ and d_-, this term structure depends only on four variables: (i) the volatility of firm value σ_V, (ii) the risk-free rate r , (iii) the current leverage of the firm $L = D/V_t$, and, of course, (iv) the time to maturity $T-t$. To better understand implications of this model, we examine the term structure of credit spreads:

- A medium-leverage company has a humped-shape credit spread term structure. The very short-term spreads are low as the company currently has just enough assets to cover debts. Spread then rises quickly since asset value fluctuations could easily result in insufficient assets. At very long maturities the credit spread starts to decrease, since the spread is determined by the value of an European put option. The maximum payoff from the put at maturity is D, which has a present value of PV(D), so $PV(D)$ is an upper bound for the value of the put at any point, and as $T \to \infty$, $PV(D) \to 0$.

- A low-leverage company has a flat credit spread term structure with initial spreads close to zero since default is an unlikely event. Spread slowly increases with debt maturity, reflecting that there is sufficient time for the firm value to drop below the value of debt. Then it starts to decrease at the long end.

- A high-leverage company has a downward-sloping credit spread term structure which starts very high and decreases for longer maturities as more time is allowed for the company's assets to grow higher and cover liabilities.

- For fixed T the spreads decline as risk-free rates increase. The reason for this was noted earlier: the spread is determined by the value of a put option on the firm's assets, and put option values are negatively related to interest rates.

- For any fixed maturity, an increase in firm volatility increases spreads. Since increasing volatility means future firm values become more spread out. Debt holders, whose payoffs are maximised, cannot get additional benefit from the higher firm values but stand to lose from the lower firm values. Thus, the higher volatility lowers debt value, and transfers value from debt holders to equity holders.

Implementation

The description in the previous section makes the Merton model appear easy to implement in practice. However, there are two problems to be addressed before the model can be applied to real-world firms:

- The current value of the company's assets V_t and its volatility σ_V are unobservable.
- The model assumes that the firm has a single issue of zero-coupon debt outstanding. Real-world debt structures are more complex.

We discuss each of these problems below. The procedures we highlight here are those commonly used in practice to resolve these problems. One popular way of estimating V_t and σ_V is using the approach suggested by Jones et al. [52], which allows us to solve for credit spread when $T-t$, D and r are given. If the firm is publicly traded, it has observable equity prices, and the volatility of equity prices σ_E may also be estimated from the data.

The Black-Scholes-Merton equation (2.3) is expressing the value of a firm's equity as a function of V_t and σ_V. Since equity prices are observable, we have one equation in the two unknowns V_t and σ_V. To be able to solve for these quantities, we need a second equation. For this Jones assumes that the equity price E_t also follows a geometric Brownian motion, ie.

$$\mathrm{d}E_t = \mu_E E_t \mathrm{d}t + \sigma_E E_t \mathrm{d}Z_t \tag{2.7}$$

where μ_E is the expected continuously compounded return on E_t, σ_E is the volatility of equity value and Z_t is a standard Wiener process.

By using Ito's Lemma, we can also represent the process for equity as:

$$\mathrm{d}E_t = \left(\frac{\partial E_t}{\partial V_t}\mu_V V_t + \frac{\partial E_t}{\partial t} + \frac{1}{2}\frac{\partial^2 E_t}{\partial V_t^2}\sigma_V^2 V^2\right)\mathrm{d}t + \frac{\partial E_t}{\partial V_t}\sigma_V V_t \mathrm{d}W_t \tag{2.8}$$

Since the diffusion terms in the equity process in (2.7) and (2.8) are equal, we can write,

$$E_t\sigma_E = \frac{\partial E_t}{\partial Vt}V_t\sigma_V.$$

Under the Merton model's assumptions, it can be shown that $\frac{\partial E_t}{\partial V_t} = \Phi(d_+)$ using Black-Scholes results on call option delta. So, the volatilities of the firm and its equity are related by

$$\sigma_E = \frac{V_t}{E_t}\sigma_V\Phi(d_+), \tag{2.9}$$

where d_+ is defined in equation (2.4).

With (2.3) and (2.9) we have the required two equations in the two variables V_t and σ_V, and the two unobservable quantities may be identified.

The second problem with the Merton model is that it assumes a too simplistic capital structure. Capital structures in practice are far more complex than assumed by the model. There are usually many issues of debt outstanding, with varied coupons, maturities, and subordination structures. From an implementation standpoint, this presents us with two alternatives:

- Simplify reality to make it fit within the existing model.
- Extend the theoretical structure of the model to enable it to handle more complex debt structures.

There are many ways one could simplify reality. One, for example, is to take a zero-coupon bond that has the same duration as the given debt structure. An alternative, that is used in the popular KMV model is to take a zero-coupon having a maturity of one year and a face value that is the sum of (i) the face value of all short-term (less than one year) dept, and (ii) half the face value of all longer-term liabilities. This approach is based on the observation that, in practice, default tends to occur when the market value of the firm's assets drops below a critical point that typically lies below the book value of all liabilities but above the book value of short-term liabilities. Of course, mapping all debts into a single zero-coupon bond is not always feasible, but both empirical evidences as well as the popularity of the KMV approaches suggests, that this simplification of the dept structure works well in practice.

It has been shown that the second solution can also be applied using a structural approach. Robert Geske [42] was the first to relax the simplistic capital structure assumption made in Merton. He models company debt as a risky coupon bond, this model allows for the simultaneous existence of multiple debt issues that can differ in the size of coupons, maturity, and other dimensions such as seniority. Geske assumes that the company only faces default on payment dates, what matches the realistic assumption that without the pressure of payment, companies wont default.

The first assumption is that the equity owners are not allowed to sell the firm's assets to pay the debt. In other words they have to finance the debt by issuing new equity or by paying "out of their own pockets". In this simple model, there is no difference between the two options.

Lets assume that there are n different debt payment dates, and at each of these payment dates $t_1, t_2, \ldots, t_n = T$ a respective payment of $D_1, D_2, \ldots D_n$ is due. Just like in Merton's model, equity receives no dividend payment, only a terminal payment at maturity, $\max(V_T - D_n, 0)$. The value of equity at t can be calculated using a recursive procedure starting at time T, and working backwards to t.

If t is between t_{n-1} and t_n and the firm is still alive, we can value the only remaining dept payment D_n simply using the Merton model, so $E_t = C^{BS}(V_t, D_n, t_n - t)$

If t is between t_{n-2} and t_{n-1} we think of equity owners as they decide an instant before t_{n-1} whether to make the payment at date t_{n-1} for the right to continue ownership of the firm, or allow the firm to default.

If the equity holders choose to pay the dept it will leave them with equity worth $C^{BS}(V_{t_{n-1}}, D_n, t_n - t_{n-1})$ and hence it is optimal to pay D_{n-1} if $D_{n-1} < C^{BS}(V_{t_{n-1}}, D_n, t_n - t_{n-1})$. If this is not true, they will choose to default and debt holders take over the firm. Applying this line of reasoning leads to the following recursion when pricing equity.

The value of equity immediately before the ith debt payment is

$$E_{t_i^-} = \begin{cases} E_{t_i^+} - D_i & \text{if } V_{t_i} > \hat{V}_{t_i} \\ 0, & \text{if } V_{t_i} \leq \hat{V}_{t_i} \end{cases} \tag{2.10}$$

where $E_{t_i^+}$ denotes the value of equity immediately after the ith coupon payment and $\hat{V}_{t_i}$ is the value for which the equity is worth D_i right after the coupon has been paid at date t_i.

Since the firm has no payout between t_{i-1} and t_i the equity value immediately after dept payment D_{i-1} is computed as the risk neutral expectation of the values in the next period, that is $E_{t_{i-1}^+} = \mathbf{E}(e^{-r(t_i - t_{i-1})} E_{t_i^-} | \mathcal{F}_{t_i - 1})$.

With each additional payment, the resulting solution for equity value has a a closed-form representation but an increasingly complex one. If there are n payments, the closed-form expression involves n-dimensional multivariate normal integrals, what cannot be computed analytically.

Pricing risky debt in the extended model is conceptually not difficult but is technically more complex than in the Merton model.The model requires precise and complete information on the actual debt structure and the process to identify the unobservable variables using equity data gets significantly more complicated than in the Merton model.

One solution to this added complication was described in Delianedis and Geske [24]. Delianedis and Geske collapse the firm's capital structure into two debt buckets, a short-term debt bucket and a long-term debt bucket. Since there are two possible default time, useful information can be obtained by comparing the short-term and long-term default probabilities.

2.2 The KMV model

The KMV model, which probably is one of the most famous industry model for valuing default probability, is a development of the Merton model. It was developed by KMV (a private company named after its founders Kealhofer, Mc Quown and Vasicek) in the 1990s and which is now maintained by Moody's KMV. The model uses a three-step procedure to calculate an Expected Default Frequency (EDF) credit measure for publicly-traded firms as described in Crosbie and Bohn ([21]). Just like the Merton model this method also takes the information contained in the firm's stock price and balance sheet as primitive.

The first step in KMV's model is to identify the market value of the firm's asset V_t, and its volatility σ_V, by implementing an iterative procedure.

Recall, that the market value and the volatility of the firm's assets can be calculated by solving (2.3) and (2.9) as showed before. However most of the empirical studies argue that the relationship between σ_E and σ_V from (2.9) holds only instantaneously and the solution is quite sensitive to the change in assets value, also there is no simple way to measure precisely σ_E from market data. To solve the problem, an iterative procedure was introduced.

The value of equity, E_t is directly observable and (2.3) forms a one-to-one relationship between the asset value and the equity price, so we can back out V_t from the equity pricing equation if the asset volatility is known. Let $g(\cdot;\sigma_V)$ denote the equity pricing equation, that is $E_t = g(V_t;\sigma_V)$. Since this function is invertible at any given asset volatility, we can express $V_t = g^{-1}(E_t,\sigma_V)$, and this inversion can be easily performed numerically.

Assume that we have observed a series of equity prices $E_0, E_1, \ldots, E_n$ at regular time points. The KMV two-step iterative algorithm begins with an arbitrary value of the asset volatility and repeats the two steps until achieving convergence. The two steps going from the m-th to $(m+1)$-th iteration are:

- Compute the implied asset value time series $V_0(\hat{\sigma}^m), V_1(\hat{\sigma}^m), \ldots, V_n(\hat{\sigma}^m)$ corresponding to the observed equity value data set $E_0, E_1, \ldots, E_n$. where $V_i(\hat{\sigma}^m) = g^{-1}(E_i, \hat{\sigma}^m)$.
- Estimate $\hat{\sigma}^{m+1}$ by assuming that $V_0(\hat{\sigma}^m), V_1(\hat{\sigma}^m), \ldots, V_n(\hat{\sigma}^m)$ follows a Geometric Brownian Motion. First we compute the implied asset returns $R_0^m, \ldots, R_n^m$ where $R_i^m = \ln(V_i(\hat{\sigma}^{(m)})/V_{i-1}(\hat{\sigma}^{(m)}))$. Then we update the asset volatility as follows:

$$\overline{R}^m = \frac{1}{n}\sum_{k=1}^{n} R_k^m$$

$$\left(\hat{\sigma}^{m+1}\right)^2 = \frac{1}{nh}\sum_{k=1}^{n}\left(R_k^m - \overline{R}^m\right)^2$$

where h is the length of time, measured in years, between two observations, the division by h is to state the parameter values on a per annum basis.

Crosbie & Bohn argue that "if the future distribution of the asset value were known, the default probability would simply be the likelihood that the final asset value was below the default point." However, in practice, distribution of the asset value is difficult to measure, the usual assumptions of normal or lognormal distribution of asset return and the simplifying assumptions about the capital structure don't hold. For all these reasons, KMV implements an intermediate phase before computing the probabilities of default.

So, as the second step, KMV computes the so-called distance to default (DD) as the number of standard deviations between the asset value, and a critical threshold, the "default point". It can be calculated as

$$DD_t = \frac{V_t - DP}{\sigma_V V_t}.$$

The idea of putting in standard deviation terms is to enable inter-firm comparisons. This latter task wold be more difficult in percentage. For example, suppose firm A has to make a 75% drop in value before it is in default, while firm B had to make only a 50% drop. Does this mean firm A is more likely to default that firm B? Not necessarily, since firm A may be more volatile, making a 75% drop in its value more likely than a 50% drop in B's value. Thus, conversion into volatility terms is necessary for a meaningful comparison.

KMV has observed that firms default when the asset value reaches a level somewhere between the value of total liabilities and the value of short-term debt, so they set the default point to equal the short term debt plus 50% of the long term debt.

The last step consists of mapping the DD to the actual probabilities of default over a specified period of time, usually 1 year. These probabilities are called EDF, for Expected Default Frequencies.

In the KMV model it is assumed that firms with equal DD have equal default probabilities, ie. $EDF = f(DD)$, where $f(\cdot)$ is determined empirically, using a database of historical default events. KMV estimates for every horizon the proportion of firms with DD in a given small range that defaulted within the given horizon. This proportion is the empirically estimated EDF. As one would expect, the empirically estimated EDF is a decreasing function; its precise form is proprietary to Moody's KMV.

EDFs have proved to be a useful leading indicator of default, or at least of the degradation of the creditworthiness of issuers. When the financial situation of a company starts to deteriorate, EDFs tend to shoot up quickly. EDFs are not biased by periods of high or low defaults, but distant-to-default can be observed to decrease during recession, and increase during periods of prosperity, so it appears that these differences are captured by the distance-to-default measure.

Detailed information about actual implementation and calibration procedures is hard to obtain and such procedures are likely to change as the model is developed further.

2.3 First Passage Models

First Passage Models extend the Merton model to the case when the firm may default at any time. In these first-passage models a firm defaults when the firm's value process V_t drops for the first time below either a deterministic or a random barrier. If we are at time t and default has not been triggered yet, then the time of default τ is given by

$$\tau = \inf\big\{s \geq t : V_s \leq K_s\big\}.$$

In most models the value of the firm's assets follow a diffusion process, while different authors make different assumptions about the threshold level. The literature mainly distinguish between exogenous barriers that are defined outside the model, and endogenous barriers defined within the model.

If one is looking for closed-form valuation formula for corporate bonds, the challenge is to appropriately specify the lower threshold. In general, the type of boundary that we are interested in, are those that give closed-form solutions of the distribution of the first passage times. These models usually bring us back to the case of Brownian motion hitting a linear boundary.

Suppose now that a Y_t process follows a Brownian motion with the standard deviation $\sigma > 0$ and drift $\nu \in \mathbb{R}$ specifically:

$$Y_t = y_0 + \nu t + \sigma W_t, \quad \forall t \in \mathbb{R}, \tag{2.11}$$

where (W_t) is a Wiener process under the measure $\mathbf{Q}$. Let τ stand for the first passage time to zero by the process Y_t, that is, $\tau = \inf\{t \geq 0 : Y_t \leq 0\}$. Then for any $t < s$, we have on the set $\{\tau > t\}$

$$\mathbf{Q}\{\tau \leq s | \mathcal{F}_t\} = \Phi\left(\frac{-Y_t - \nu(s-t)}{\sigma\sqrt{s-t}}\right) + e^{-2\nu\sigma^{-2}Y_t}\Phi\left(\frac{-Y_t + \nu(s-t)}{\sigma\sqrt{s-t}}\right), \tag{2.12}$$

where Φ is the standard normal cumulative distribution function, for proof see Bielecki and Rutkowski [12].

Black and Cox Model

The first paper witch extends Merton's research in this directions was written by Black and Cox [10]. They assume that the firm's stockholders receive continuous dividend payment, which is proportional to the current asset value. In their model, they assume that under the risk-neutral measure the value of asset follows the process

$$dV_t = V_t(r - \kappa)\mathrm{d}t + V_t\sigma_V\mathrm{d}W_t.$$

where the constant $\kappa \geq 0$ represents the payout ratio.

Similarly as in Merton's model, the debt consists of one zero-coupon bond with face value D at maturity T, and the short-term interest rate r is assumed to be constant.

In the original approach of Black and Cox, the default occurs at the first time that the firm's asset value drops below a certain time dependent barrier K_t, what represents the point at which bond safety covenants cause default. Safety covenants provide the bondholders with the right to liquidate the firm if it is doing poorly according to a set standard. For the choice of the barrier, observe that if $K_t > D$ then bondholders are always completely covered, which is certainly unrealistic. On the other hand, one should clearly have $K_T \leq D$ for a consistent definition of default.

The default barrier chosen by Black and Cox is an exponentially rising function over time $K(t) = Ke^{-\gamma(T-t)}$, where $0 < K \leq D$ is an exogenously given constant. A particular case of the threshold specification is setting $\gamma = r$ and $K = D$, to ensures that the payoff to the bondholder at the default time is the face value of debt, discounted at a risk-free rate.

Otherwise, default can take place at debt's maturity, depending on whether $V_T < D$ or not. This means that even though the default may not have occurred premature it may occur when the bond is due. Formally, the default time equals $\tau = \hat{\tau} \wedge \overline{\tau}$, where the early default time $\hat{\tau}$ equals

$$\hat{\tau} = \inf\left\{0 \leq t < T : V_t \leq K_t\right\}$$

and $\overline{\tau}$ stands for Merton's default time, that is

$$\overline{\tau} = T\mathbb{1}_{\{V_T<D\}} + \infty\mathbb{1}_{\{V_T\geq D\}}.$$

Note that since $V_t = V_0 \exp((r-\kappa)t - \frac{1}{2}\sigma^2 t + \sigma W_t)$, the default time $\hat{\tau}$ is given as

$$\hat{\tau} = \inf\left\{0 \leq t \leq T : \log V_0 + ((r-\kappa) - \frac{1}{2}\sigma^2)t + \sigma W_t \leq \log K - \gamma(T-t)\right\}$$

$$= \inf\left\{0 \leq t \leq T : \sigma W_t + (r - \kappa - \frac{1}{2}\sigma^2 - \gamma)t \leq \log K - \log V_0 - \gamma T\right\}, \quad (2.13)$$

i.e. the first passage time to the default barrier can now be reduced to the first passage time of a Brownian motion with drift.

The value of the defaultable bond at time t prior to default is a sum of the recovery value of the bond if the boundary is hit before maturity ,and the value of the payment at maturity if the bond didn't default before it, $D_t = D_t^b + D_t^m$.

In the event of default, the pay-off for debt holders is $V(\tau) = K(\tau)$ at the time of default, and the fair recovery value is

$$D_t^b = \mathbf{E_Q}\big(Ke^{-\gamma(T-\hat{\tau})}e^{-r(\hat{\tau}-t)}\mathbb{1}_{\{t<\hat{\tau}<T\}}\big|\mathcal{F}_t\big) = Ke^{rt-\gamma T}\int_t^T e^{(\gamma-r)s}\mathrm{d}\mathbf{Q}\{\hat{\tau} < s|\mathcal{F}_t\}$$

what can be computed by using the conditional probability law of the first passage time (2.12) and (2.13).

The remaining term can be written as

$$D_t^m = \mathbf{E_Q}\big(e^{-r(T-t)}(V_T - (V_T - D)^+)\mathbb{1}_{\{\hat{\tau}>T\}}|\mathcal{F}_t\big).$$

Hence we need to be able to value the difference of barrier call options, one with zero strike, and the technique is available from the results on down-and-out barrier options. Down-and-out options are a type of knock-out barrier option that ceases to exist when the price of the underlying security hits a specific barrier price level. If the price of the underlying does not reach the barrier level, the investor has the right to exercise their European call or put option at the exercise price specified in the contract.

The credit spread resulting from the Black and Cox model will always be higher than that coming out of the Merton model. Since in the Black and Cox model the firm can default at any point in the life of the dept, not only at the dept's expiration date. So the probability of default is always greater then in the Merton model, meaning investors should ask for a higher credit spread as compensation of this extra risk.

Stochastic Interest Rate Models

In addition to the Black-Cox exponential barrier model, Kim, Ramaswamy, and Sundaresan [53] and Longstaff and Schwartz [58] propose flat default barrier models with random interest rates. These models are an improvement over the Black-Cox model, since it is possible to include interest rate risk in them.

Kim, Ramaswamy, and Sundaresan assume the firm's value follows

$$\mathrm{d}V_t = (\mu - \kappa)V_t\mathrm{d}t + V_t\sigma_V\mathrm{d}W_t,$$

where κV_t is the net cash outflow. For the interest rate process they use the CIR term structure model

$$\mathrm{d}r_t = (a - br_t)\mathrm{d}t + \sigma_r\sqrt{r_t}\mathrm{d}\hat{W}_t$$

where W_t and $\hat{W}_t$ are correlated Brownian motions with the instantaneous correlation $\rho\mathrm{d}t$.

Furthermore, the firm's dept consists of a single bond with continuous coupon payment of C and a promised final payment of D. Also the stockholders are prohibited from selling the assets of the firm to pay dividends.

In this model the bondholders have priority and if the firm's cash flows are unable to cover its coupon payment the firm is forced into bankruptcy. Thus the default barrier is an endogenously given constant C/κ. At this level, the total net cash flow per unit time will be just sufficient to pay the contractual coupon.

If the boundary is not reached during the lifetime of the bond, the final payout is $\min(V_T, D)$, so default may occur also at maturity. If, on the other hand, the boundary is reached, then the bond holders receive the recovery payment immediately, what is $\min(V_t, (1-\omega)\hat{B}(t,T))$ where ω is an exogenous writedown rate and $\hat{B}(t,T)$ is the price of an equivalent treasury bond.

Given all the assumptions in their paper, Kim, Ramaswamy, and Sundaresan solve the resulting PDE numerically for noncallable and callable bonds, finding that the introduction of stochastic interest rates leads to higher, more realistic credit spreads than predicted by models with deterministic interest rates. However, they also report that in their examples, spreads are fairly insensitive to the level of interest rate risk or to it's correlation with firm value.

The Longstaff and Schwartz [58] model is similar to the Kim, Ramaswamy and Sundaresan model in the sense that it considers both the default risk and the interest rate risk to price the corporate debt. A major difference however, is that the short-term interest rate is assumed to follow a Vasicek model, that is:

$$\mathrm{d}r_t = (\alpha - \beta r_t)\mathrm{d}t + \sigma_r\mathrm{d}\hat{W}_t, \tag{2.14}$$

and the asset value of the firm follows

$$\mathrm{d}V_t = V_t(r_t\mathrm{d}t + \sigma_V\mathrm{d}W_t),$$

under the martingale measure $\mathbf{Q}$, where the instantaneous correlated between W_t and $\hat{W}_t$ is $\rho\mathrm{d}t$.

The default event is triggered if the value process hits an exogenously given constant barrier K before maturity. Longstaff and Schwartz argue that "the ratio of V_t to K, rather than the actual value of K, that plays the major role in our analysis, allowing a more general specification for K simply makes the model more complex without providing additional insight into the valuation of risky debt." Some researcher criticize this default definition and argue that when the corporate bond reaches maturity the firm can be in a solvent position but with no sufficient assets to pay the face value of the bond at maturity.

In this model default occurs for all debt contracts simultaneously and recovery payment is paid out to the dept holders at maturity. Every bond holder receives $(1-\omega)$ times the face value of the bond, where the writedown rate ω is a fixed constant. This means that the bond pays out $(1-\omega)D\mathbb{1}_{\{\tau\le T\}} + D\mathbb{1}_{\{\tau>T\}}$ at its maturity time T.

The model allows ω to differ across the various bond issues and classes of securities, which is equivalent to allowing the violation of the absolute priority rule.

As a result,the price of a defaultable zero-coupon bond at time t is

$$D_t = \mathbf{E}_{\mathbf{Q}_T}\big(De^{-\int_t^T r_s \mathrm{d}s}(1-\omega\mathbb{1}_{\{\tau\le T\}})|\mathcal{F}_t\big) = DB(t,T)\big(1-\omega\mathbf{Q}_T\big(\tau\le T|\mathcal{F}_t\big)\big).$$

where $B(t,T)$ represents the price of the unite default-free zero-coupon bond whit maturity T and $\mathbf{Q}_T$ is the forward martingale measure associated with $B(t,T)$.

In order to evaluate this expected value, the probability distribution of the first passage time to a constant barrier is needed. To the best of my knowledge, a closed form solution for the probability distribution of the first passage time of this process is not known yet. Longstaff and Schwartz draw a quasi-explicit valuation formula trough the approximation of the default time's distribution,that allows them to analyse the qualitative behavior of the corporate bond.

For example, they find that the correlation between the firm's assets and the interest rate has a significant effect on the properties of the credit spread.

The main drawback of the Longshaff-Swartz model is the complex parameter calibration of the numerous parameters for the bond equation and that the interest rates is not arbitrage-free in the Vasicek model.

Main implication of these valuation frameworks is that the correlation between credit spread and interest rate is negative. The reason for this is that an increase in the interest rate increases the drift of the risk-neutral process for firm value. As a consequence, firm value drifts away at a faster rate from the default boundary K and makes risk-neutral default probability lower. This implication suggests that interest-rate sensitivity of credit spread depends on how strong is the correlation between assets return and changes in the interest rate.

These results provide strong evidence that both default risk and interest rate risk are necessary components for a valuation model for corporate debt.

Stochastic Default Barrier Models

In order to overcome the limitation of an exogenous flat barrier models, like the absence of closed-form solution, many author relax their default barriers to be stochastic. In most of these models, the stochastic default boundaries result from the assumption of stochastic interest rate.

Nielsen at al. [61] examine a model with endogenously specified stochastic barrier and random interest rate. They assume that the asset value process follows the same generalised Wiener process as in the Black and Cox model, and the interest rate process follows the Itô process

$$\mathrm{d}r_t = \mu_r \mathrm{d}t + \sigma_r \mathrm{d}W_t.$$

The recovery rate is an exogenously set fraction of an equivalent default-free bond.

They interpret the exogenously given continuous stochastic barrier v_t used in their model as the representation of the market value of the firm's all liabilities. Notice that the default time τ can be given as

$$\tau = \inf\{0 < t < T : V_t < v_t\} = \inf\{0 < t < T : L_t < 0\}$$

where $L_t = \ln(V_t/v_t)$ is the solvency ratio, representing the firm's credit quality. Nielsen at al. show that under some specific assumptions on the dynamics of the stochastic barrier the solvency ratio follows a generalized Brownian motion under the forward measure. So the first passage time law (2.11) can be directly applied to determinate the distribution of the default times. Under the same assumptions they also give an explicit expression for the value of the defaultable zero-coupon bond.

Briys and de Varenne [13] propose yet another default triggering and pay out rule. They note, that some model at default allows the payment to bondholders to be greater then the firms value, like in the Black-Cox , Kim-Ramaswamy-Sundaresan model and the model of Nielsen at al. In this models the recovery upon default is independent of the value of the asset. On the other hand, in the model of Longstaff and Schwartz it is possible that the firm is "threshold solvent" at maturity but it isn't able to repay the face value of the bond. Briys and de Varenne try to correct these problems by specifying the default threshold and recovery rate such that the firm can not pay more than its assets are worth. Their model is a special case of the Black and Cox model with stochastic interest rate.

The interest rate follows the generalized Vasicek model

$$\mathrm{d}r_t = a(t)(b(t) - r_t)\mathrm{d}t + \sigma(t)\mathrm{d}\hat{W}_t,$$

where $a, b, \sigma : [0, T] \to \mathbb{R}$ are deterministic functions. The firm value evolves according to the following stochastic processes under the martingal measure

$$\mathrm{d}V_t = r_t V_t \mathrm{d}t + \sigma_V V_t\big(\rho \mathrm{d}\tilde{W}_t + \sqrt{1-\rho^2}\mathrm{d}\hat{W}_t\big)$$

where $\hat{W}_t$ and $\tilde{W}_t$ are independent Brownian motions.

The default boundary is given by a safety covenant, that allows bondholders to trigger early bankruptcy, $K(t) = \alpha DB(t,T)$ where D is the face value of the corporate bond and $0 \leq \alpha \leq 1$ is some fixed value. In other words, the default barrier is the discounted face value of the bond scaled down by α, therefor the bondholders will never receive a payoff higher than that of a risk-free bond. Of course the closer α is to 0, the less protective is the safety covenant. Default is triggered at maturity if the asset value at time T is less then the dept value D.

If default occurs at maturity, then an exogenously specified fraction of the asset value is paid out, $\beta_1 V_T$ where $0 \leq \beta_1 \leq 1$. If on the other hand the default threshold is reached before maturity then a fraction of the threshold value is paid out $\beta_2 K(\tau)$, $0 \leq \beta_2 \leq 1$ is also an exogenous fraction. Consequently, the bond's pay out at maturity can bee represented as:

$$D(T,T) = V_T \mathbb{1}_{\tau > T} + \beta_1 V_T \mathbb{1}_{\tau = T} + \beta_2 \alpha D \mathbb{1}_{\tau < T}.$$

If $\beta_1 = \beta_2 = 1$, the strict priority rule applies, since only the bondholders receive a a default payment, i.e. there is no bargaining process between the equity holders and the asset holders. In reality, though, the strict priority rule often does not apply.

By applying change of numeraire and change of time techniques they derive a closed-form solution for a corporate discount bond. Since this model accounts for interest rate risk, default risk, and deviations from the absolute priority rule, this model is capable of producing quite diverse shapes for the term structure of yield spreads. They also present a numerical analysis of the credit spreads, and report generally larger spreads than those obtained using Merton's approach.

Term structures modeled with first passage models are often downward sloping for long maturities, while these are increasing in the marketplace.

Collin-Dufresne, Goldstein **??** introduce a dynamic capital structural for firms in their structural model to overcome this problem. They incorporated the market observation that firms issue more debt when their leverage ratio D_t/V_t drops below a target level and tend to wait with replacing this debt when the leverage ratio is above target. In this model the interest rate is constant and the asset value under the risk-neutra follows a geometric Brownian motion just like in the previous models

$$dV_t/V_t = (r_t - \delta)dt + \sigma_v dW_v$$

with δ payout rate and σ_v asset volatility

The default threshold changes dynamically over time, ie. the dynamics of the log-default threshold k_t is modeled as

$$k_t = \kappa_l(\ln v_t - \nu(\cdot) - k_t)dt.$$

where κ_l is the mean-reversion rate, ν is a buffer parameter for the distance of log-asset value to log-debt value $v_t = \ln V_t$.

This dynamic captures firms tendency to issue debt when their leverage ratio falls below some target, and are more hesitant to replace maturing debt when their leverage ratio is above that target.

The log-leverage is defined as $l_t = k_t - v_t$, and from Ito's lemma we get that

$$dl_t = \kappa_l(\theta_l(\cdot) - l_t)dt + \sigma_v dW_v$$

where $\theta_l = \frac{\delta + \sigma_v^2/2 - r}{\kappa_d} - \nu$

Default is triggered when θ_t reaches zero for the first time.

The model has no closed for solution or bond pricing but CDG implemented an efficient numerical technic to computing the relevant default probabilities and thus bond prices.

2.4 Optimal Capital Structure

Leland [54] and Leland & Toft [55] introduced the concept of endogenous bankruptcy in their studies to explore how a firm best capitalizes itself. These two models have been served as benchmark models in the optimal capital structure literature. Their stationary debt structures and time-independent settings have been widely adopted in the literature.

In particular, firm assets are assumed to follow a geometric Brownian motion under the risk-neutral measure

$$\mathrm{d}V_t = (r - \delta)V_t \mathrm{d}t + \sigma V_t \mathrm{d}W_t,$$

where $\delta > 0$ represents a constant fraction of the asset value available for distribution to debt and equity holders, and r is the riskless interest rate, which is assumed to be constant in this model.

Leland considers a firm that has a perpetual bond outstanding. Perpetual bonds have no maturity date but pay a constant coupon rate C until termination. The reason for using a perpetual coupon debt setup is to construct a time independent stationary debt structure and for ease of solving the optimal leverage problem.

The owners willingness to issue debt comes from a tax advantage, which is represented as an additional cash flow of θC dollars per unit of time , where $\theta \geq 0$ is the tax rate. However, to prevent the firm from using only debt financing, there must be a disadvantage the model. This disadvantage is the bankruptcy costs $\alpha \in [0, 1)$, given as a fraction of firm value, that is lost to a third party if default occurs.

The equity owners are responsible for paying the coupon flow to the bond holders. However, the equity owners have the right to determinate a level of asset, denoted as K, below which they stop making dividend payments and liquidate the firm. In liquidation, the equity owners get nothing and the debt holders get $(1 - \alpha)K$. Thus the default time τ equals $\tau = \inf\{t > 0 : V_t \leq K\}$.

Leland furthermore assumes, that any amount not covered by dividend payout and the tax advantages of debt, firm finances by issuing new equity. And it is known from Black and Cox [10] that any such asset's value must satisfy the partial differential equation

$$\frac{1}{2}\sigma^2 V^2 F_{VV}(V,t) + rVF_V(V,t) - rF(V,t) + vF_t(V,t) + C = 0,$$

where $F(V,t)$ denotes the value of a claim, that continuously pays a nonnegative coupon, C when the firm is solvent.

Leland's model only uses securities with no time-dependency, i.e. they payments depend only on the default timing, witch only depends on the asset value. In these case $F(V)$ has to satisfy the ODE:

$$\frac{1}{2}\sigma^2 V^2 F_{VV}(V) + rVF_V(V) - rF(V) + C = 0, \tag{2.15}$$

for witch The general solution is known to be

$$F(V) = A_0 + A_1 V + A_2 V^{-\gamma}$$

where $\gamma = 2r/\sigma^2$ and the constants A_0, A_1 and A_2 are determined by boundary conditions.

First, consider the case where in addition to the coupon rate C the default boundary K is also known. Let $D(V,C)$ denote the value of debt for $V > K$ and since it is independent of time it must solve the ODE (2.15) and hence has a general solution, with boundary conditions:

$$D(V,C) = (1-\alpha)K \qquad \text{at} \quad V = K$$

$$D(V,C) \to C/r \qquad \text{as} \quad V \to \infty$$

We immediately find that

$$A_1 = 0, \qquad A_0 = \frac{C}{r}, \qquad A_2 = \left((1-\alpha)K - \frac{C}{r}\right)K^{\gamma}$$

Hence

$$D(V,C) = \frac{C}{r}\left(1 - \left(\frac{V}{K}\right)^{-\gamma}\right) + (1-\alpha)\left(\frac{V}{K}\right)^{-\gamma}$$

Here the term $(\frac{V}{K})^{-\gamma}$ has the interpretation of being the present value of a claim, paying 1 dollar at the stochastic time τ when default occurs. So, the value of risky debt equals a weighted average of the value of a risk-free perpetual bond (C/r) and the fraction of recovery bond holders can obtain in case of bankruptcy.

To prove this, let $p(V)$ a denote the value of a security that pays no coupon but at bankruptcy pay 1 dollar As this security satisfies the ODE (2.15) with $C = 0$. The boundary conditions are

$$p(V) = (1-\alpha)K \qquad \text{at} \quad V = K$$

$$p(V) \to 0 \qquad \text{as} \quad V \to \infty$$

and so we obtained that $p(V) = (\frac{V}{K})^{-\gamma}$.

Similarly it can be proven that $q(V) = \frac{1}{r}(1 - p(V))$, where $q(V)$ denotes the value of a security that pays a constant unite coupon as long as the firm is solvent. Notice that $p(V)$ is a strictly convex, decreasing function of V, while $q(V)$ is increasing and strictly concave in V.

The total value of the firm after recapitalization is then given as

$$v(V_0, K, C) = V_0 + \theta C q(V_0) - \alpha K p(V_0) = V_0 + \theta C \frac{1}{r}(1 - (\frac{V_0}{K})^{-\gamma}) - \alpha K (\frac{V}{K})^{-\gamma} \quad (2.16)$$

And since there is only debt and equity, we have

$$\begin{aligned} E(V_0, K, C) &= v(V_0, K, C) - D(V_0, K, C) \\ &= V_0 - (1 - \theta) C q(V_0) - K p(V_0) \end{aligned} \quad (2.17)$$

for $V_0 > K$ and $E(V_0, K, C) = 0$ otherwise.

One can now address the big question: What values of C and K maximize the total firm value at time $t = 0$?

A good way to think about the model is to imagine that at time $t = 0$ the owners of the debt-free firm with value V_0 decide to issue debt. By doing this, the owners seek to maximizes the total value $v(V_0, K, C)$. The total firm value changes with the change of leverage due to the presence of tax benefits and bankruptcy costs. Notice that the total firm value $v(V_0, K, C)$ raises by lowering the default bound K.

After the firm has issued its debt, equity owners optimize equity value by determining the default level K, below which they stop making dividend payments. Rational equity holders choose the default point K high enough to make sure that $E(V_t, K, C)$ isn't negative if $V_t > K$, and low enough that the equity value is equal to zero at the default boundary.

If $K < (1 - \theta)C/r$ then $E(V, C)$ is strictly convex in V and so the solution for the optimization problem is at the point where:

$$\left. \frac{\partial E(V)}{\partial V} \right|_{V=K} = 0$$

This is called the smooth pasting condition, and is solved for $K = K^*(C)$, where

$$K^*(C) = \frac{\gamma(1 - \beta)C}{r(1 + \gamma)}$$

At the initial debt issue, when the owners set the coupon level they take into account that the equity holders will choose the default point $K^*(C)$ optimally. Hence, plugging $K^*(C)$ and C into (2.16) gives the total firm value for a given level of coupons. Finally, to find the optimal coupon level, the total firm value has to be maximized as a function of C. This can be done by differentiating equation

$v(V_0, K*(C), C)$ with respect to C, and solving for the optimal coupon by setting the derivative equal to zero. By solving this equation one finds

$$C* = V_0 \frac{r(1+\gamma)}{\gamma(1-\tau)} \left(\frac{(1+\gamma)\tau + \alpha(1-\tau)\gamma}{\tau} \right)^{-1/\gamma}.$$

The optimal leverage and the default barrier are explicitly linked to taxes, asset volatility, bankruptcy costs, risk-free interest rate, what leads to important differences from previous models. In addition, the default barrier K^* is independent of time and the current asset value, what confirms the assumption of the constant bankruptcy level. The default falls with increases in asset volatility and tax rate, and decreases as the risk-free interest rate, r rises.

The lower the volatility, the higher is the optimal leverage ratio , and also the maximal possible firm value. However the model predicts unreasonably high leverage ratio (e.g., 80%) compared to historical averages. Furthermore, unreasonably large bankruptcy costs as high as 50% is required to lower the optimal leverage to more realistic levels.

The model of Leland and Toft [55] relaxes the extreme assumption of a perpetual debt, while keeping the stationary debt structure. To do this, they assume that the firm continuously sells a constant amount of new debt, always replacing the maturing bonds. This setting allows them to analyze the influence of debt maturity on the optimal leverage.

In this model each individual bond is issued at par value, with constant principal p and with time T to maturity. Bonds pay a constant coupon rate c, implying the total coupon to be paid is $C = cT$. So,the total amount of principal outstanding is always $P = pT$ and the total debt service is equal to $C+p$. Moreover, in case of default a fraction of asset value α is lost and the bondholders with different maturities receive the same fraction of assets $\frac{(1-\alpha)}{T}$. Finally, as in the Leland model, coupon payment gets a tax advantage, and default is triggered by shareholders, when the firm value falls to an endogenously determined value.

They find that the optimal leverage and default barrier is independent of time, but depends on the debt maturity T, for any given values of total bond principal P and coupon rate C. This is an important difference between Leland and Toft model and flow-based bankruptcy models, in which they imply that the default barrier is independent of debt maturity.

The Leland and Toft model implicates that the default barrier and the optimal leverage increases with debt maturity. Moreover, for any maturity, the optimal leverage ratio falls, when firm's risk and bankruptcy costs increase.

They also investigate why firms issue short term debt, when longer-term debt generates higher firm value. They find that short-term debt reduces the asset substitution agency problem. The asset substitution problem refers to the effect that equity holders will try to transfer value from debt to equity by increasing the riskiness of the firm's activities. This potential behavior of equity holders is incorporated into the cost of issuing debt. Leland and Toft find that while short-term debt does not lower tax benefits of debt it can balance against agency costs. This is a consequence of short-term debt holders needing less protection, and so demand lower coupon rates and thus equity holders will also benefit.

In most models considered so far, if there is not enough asset to pay bond holders their contractual amount, they take over the firm and equity holders get nothing. However, in practice, the violation of the absolute priority rule is common, i.e. we see cases where debt contracts are renegotiated in times of financial distress. Structural models can demonstrate the violation of absolute priority rule by incorporating different default boundaries, but can not explain why debt holders are willing to reduce their claims. Therefore, subsequent studies attempt to model dept renegotiation and analyse the cause of the absolute priority rule deviation and its impact on yield spread and capital structure.

In these models a negotiation process between debt and equity holders is assumed. The negotiation can result in a lower debt payment than promised to prevent the firm from default.

Anderson and Sundaresan [6] propose a discrete time model where the firm can be liquidated only at a cost. Liquidations are typically very costly when one takes into account direct and indirect costs associated with it. This supplies the motivation for renegotiation and strategic dept service. They assume that at each time point the firm generates a cash flow proportional to its asset value and equity holders have to make a coupon payment. Even if this cash flow is big enough to cover the coupon payment, equity holders may only pay a reduced strategic debt service.

They use the threat of bankruptcy to its fullest and never offer more than what is necessary to keep bond holders indifferent between liquidation and continuation, in this case dept holders are assumed to choose continuation. If the generated cash flow is not sufficient to make this payment, the bond holders liquidate the firm and receive the promised coupon and principal if there is enough asset after liquidation, otherwise they receive the asset value less the liquidation costs. Hence even in liquidation there could be something left for equity holders.

Anderson and Sundaresan show that accounting for liquidation costs leads much higher credit spreads, even with moderate liquidation costs. In addition, it generates deviations from the absolute priority rule.

Fan and Sundaresan [37] develop a continuous-time model that extends Anderson and Sundaresan approach along several dimensions. The main extension is the inclusion of a tax advantage of debt, what gives bargaining power to debt holders too.

They consider two cases when the firm's asset value falls below an endogenously determined boundary: Debt-equity swap and strategic debt service. Both case can be seen as a distressed exchange where the absolute priority rule is violated to avoid costly liquidation.

In the case of debt-equity swap, debt holders exchange their original debt contract for equity. Here, the firm after the exchange becomes an all-equity firm and the future tax benefits are lost. The parties will bargain over the firm value, and the sharing rule is determined as an equilibrium of a Nash bargaining game, and its optimal value depends on the relative bargaining power assigned to the equity holders. In this framework, stronger equity holders bargaining power and higher liquidation costs implies higher default triggering barrier.

In strategic debt service, when the reorganization boundary is reached, borrowers stop making the contractual coupon payment and offer a strategic debt service until the firm's asset value goes back above the boundary again. During this period the tax benefits are lost, but the firm will not lose its potential future tax benefits and the present value of these tax benefits is included in the bargaining process. This results in debt holders getting less proportion of the firm, but both parties will be better. Since costly liquidation can be avoided, which can be shared by the two parties.

Both the reorganization boundary and the optimal strategic debt servicing upon default are determined endogenously. Fan and Sundaresan show that this strategic debt servicing is decreasing with higher equity holders bargaining power and larger liquidation costs. The role of the liquidation costs derives from the fact that higher liquidation costs generate a stronger incentive for debt-holders to participate in the bargaining game.

Furthermore, by introducing the possibility of renegotiating the debt contract, the default can occur at positive equity value. This is in contrast to the Leland model in that the default occurs when the equity value reaches zero.

In summary, the Fan and Sundaresan model shows that debt renegotiation encourages early default and increases credit spreads on corporate debt, given that shareholders can renegotiate in distress. The basic difference between the two bargaining formulations is that, in the case of debt-equity swap the parties bargain over the value of the firm's assets, while in the second case, the parties bargain over the whole firm value, that is asset value plus future tax benefits.

2.5 Comments on Structural Models

The empirical literature on structural models investigate the explanatory power of credit risk models in predicting default risk or credit spreads. Comparison of results from different studies is quite difficult for a number of reasons. First of all, these studies are performed on different types of data, either CDS or bond data is used, and also the frequency of observations varies across studies. Results are also affected by choices about parameter estimation, or type of regression used in analysis. In addition, when comparing results, a difference should be made between whether analysis is performed using levels or changes in variables.

Jones, Mason, and Rosenfeld [52] provide the first empirical test of a structural models, comparing bond prices predicted by the Merton model with the observed prices. Results were not encouraging, predicted prices were 4.50% too low on average, and errors were largest for speculative-grade firms. Ogden [62] conducts a similar study, finding that the Merton model underpredicts by 104 basis points on average.

Two decades later Eom et al. [33] test the ability of five structural models to predict the yield spread, using a sample of bond prices from firms with simple capital structures. Their main finding is that on average, the Merton and Geske models far underestimate the corporate bond spread, in accordance with the previous literature. However, the Longstaff and Schwartz, Leland and Toft and Collin-Dufresne and Goldstein models overestimate spreads for risky bonds (high volatility and leverage) while they underestimate the spreads for less risky bonds.

Structural models are also examined indirectly using regression analysis that links individual bond yield spreads with certain structural model variables.

For instance, Collin-Dufresne et al. [18] investigate the determinants of credit spread changes. They analyze the impact of traditional structural model inputs on bond yield spreads. They find that these factors are statistically significant, but explain only about 25 percent of the observed credit spread changes. In addition, the results show that the residuals are highly correlated and that they are mostly driven by a single common factor.

Campbell and Taksler [15] studied the sensitivity of credit spread to equity volatility. They conclude that firm specific equity volatility is an important determinant of the bond spread, and that the economic effects of volatility is large. Cremers, Driessen, Maenhout, and Weinbaum [17] give support to this result and argue that option implied volatility contains useful information that is different from historical volatility.

Huang and Huang [47] use a variety of structural models to examine how much of the historically observed yield spreads can be explained by implied default probabilities. The structural models studied include Longstaff and Schwartz with stochastic interest rate, Leland and Toft for endogenous default boundary, Anderson and Sundaresan, for strategic default, and Collin-Dufresne and Goldstein for mean reverting leverage ratio. The main finding of Huang and Huang is that existing structural models generate lower spreads than the corresponding market spreads when calibrated to match historical default and recovery rates. This gap between observed and model-implied credit spreads is known in the literature as the credit spread puzzle. In other words the credit spread puzzle refers to the failure of structural models in explaining yield spreads and default rates simultaneously.

One possible implication of the credit spread puzzle is that perhaps the unexplained portion of bond yield spreads is due to some non-credit factors. Growing evidence shows that multiple firm characteristics and economic conditions are important determinants of corporate credit spread. Most studies agree that liquidity risk could help explain a certain amount of this component. However, the magnitude of the impact differs amongst studies due to differences in data sets and liquidity measures.

Many studies used data on CDS spreads instate of bond spreads to test credit risk models, since CDS spreads are generally considered to be a purer measure of credit risk.

Ericsson et al. [34] investigate the relationship between theoretical determinants of credit risk and actual CDS spreads. Finding that the explanatory power of firm leverage, volatility and the riskless interest rate for levels and differences in CDS spreads is 60% and 23%, respectively. The authors perform principal components analysis on the residuals, finding only weak evidence for a residual common factor.

Longstaff, Mithal, and Neis [57] use CDS data to estimate direct measures of default and nondefault components in corporate bond yields. Their results show that the nondefault component in the credit spreads is time varying and strongly linked to measures of bond-specific liquidity.

Das, Hanouna [22] also observe that CDS spreads are less affected by liquidity and other non-credit risk related factors than bond spreads. However, they show that CDS spreads are negatively related to the equity liquidity of the reference entity.

Summarizing the results of these studies, there is broad agreement that structural models can explain only fraction of actual credit spreads through variables such as leverage, firm value and interest rates. Equity-specific factors such as changes in the stock price, the stock return and implied volatility have additional explanatory power for both bond spreads and CDS premia.

The literature suggests that a large proportion of credit spreads could not be attributed to default risk factors. Most studies agree on that liquidity and tax differential could help explain a certain amount of this component, however, the magnitude of the impact differs amongst studies.

Structural models seem to fit to CDS spreads better than corporate bond spreads but they still cannot fully explain CDS spreads and capture the time series behavior of the CDS term structure.

Structural models have many advantages. First, they offer a intuitive approach to model default, assuming that default is the result of the firm's assets value falling too low. Furthermore most of the structural models links the valuation and hedging of defaultable claims to more traditional corporate finance models as the valuation and hedging of exotic options in the standard default-free setup.

They usually provide closed-from expressions of corporate debt, and shows how the outputs of the model is a function of leverage, asset volatility, taxes, bankruptcy costs, risk-free interest rate, payout rates, and other important variables. Thus structural models can be used to investigate how debt values (and therefore yield spreads) changes with these variables.

This strength, the dependence on fundamental variables, is also one of the models' biggest weaknesses. Often it is hard to define a meaningful asset value process, let alone observe it continuously. It can be very hard to calibrate a firm's value process to market prices, and for some issuers, like sovereign debt, it can't be done at all. Also the assumption that the total value of the firm's assets is a tradeable security is unrealistic.

Structural models are also computationally burdensome. For instance, the pricing of a defaultable zero-coupon bond is as difficult as pricing an option. Just adding coupons transforms the problem into the equivalent of pricing a compound option. If one were to price any subordinated debt one may have to price all of the more senior debts of the firm. This obviously becomes quickly unfeasible.

Another major weakness of structural models is the so-called "predictability of defaults." Generally, structural models consider continuous diffusion processes for the firm's asset value and complete information about asset value and default threshold. The knowledge of the distance of default and the fact that the asset value follows a continuous diffusion process makes default a predictable event, i.e. default does not come as a surprise. This makes the models generate short-term credit spreads close to zero. In contrast, it is observed in the market that even short-term credit spreads are bounded from below incorporating the possibility of an unexpected default. This characteristics of the structural models also imply predictability of recovery, because in case of default the bondholders get the remaining value of the firm, which is precisely the value of the default threshold at default.

Essentially, three solutions have been proposed in the literature to this problem. The first method, developed by Duffie and Lando, assumes that bond investors cannot observe the asset process directly, instead, they receive imperfect information at selected times. The second method assumes the default barrier is stochastic. This is to account for the incomplete knowledge of all the firm's payment obligations. The third method incorporates randomly occurring jumps into the firm's asset value process. Under this jump-diffusion process default could occur expectedly from steady declines in the firm's value, or unexpectedly from a sudden drop in the firm's value. These jumps are due to new information like release of unexpected financial results, the detection of fraud, or a market crash. The usual assumption for the recovery rate is that it is a proportion of the remaining assets after default. Thus using a jump-diffusion process for the asset value one can naturally incorporate the randomness of recovery rates.

Consequently, structural models are not used where rapid and accurate pricing of defaultable securities is needed. Instead, this type of models are well-suited for the analysis of questions of optimal investment and financing decisions, or the relative powers of shareholders and creditors. It is therefore a useful tool in the analysis of counterparty risk for banks and a useful tool in the risk analysis of portfolios of securities. Corporate analysts might also use structural models as a tool for analyzing the best way to structure the debt and equity of a company.

Finally, some researchers argue that the past poor performance of structural models may come from the estimation approaches traditionally used in the empirical studies and we have seen some innovative methods aiming for solving this estimation problem.

Chapter 3

Reduced Form Models

Reduced form models, also known as hazard rate models or intensity-based models form an approach to default complementary to the structural models. In structural models, default was directly linked to the value of the firm, and in the simplest versions, default times are predictable in the filtration available to traders. This makes the models generate short-term credit spreads close to zero. In contrast, it is observed in the market that even short-term credit spreads are bounded from below incorporating the possibility of an unexpected default. Reduced form models make the assumption that default is always a surprise, that is, a totally inaccessible stopping time. The firm value is not modeled, but rather attention is focussed on the instantaneous conditional probability of default.

Reduced form modelling makes a key simplifying assumption on the relation between the timing of defaults, encoded in the filtration $\mathcal{H}$ and the market filtration $\mathcal{F}$. Essentially, one assumes that the fact of a default or not at a given time t has no impact on the evolution of the market filtration beyond t.

3.1 Hazard Rate Approach

Let's begin with the simple case where the market filtration is generated only by a riskless asset, with deterministic interest rate $r(s)$. Default occurs at time τ, where τ is assumed to be a positive random variable with density f, constructed on a probability space $(\Omega, \mathcal{F}, \mathcal{G}, \mathbf{P})$. In this case the information flow available to an agent reduces to the observations of the random time which models the default event. Let H_t denote the right-continuous increasing process $H_t = \mathbb{1}_{\{t \geq \tau\}}$ and by $(\mathcal{H}_t)_{t \geq 0}$ its natural filtration, what is generated by the sets $\tau \leq s$ for $s \leq t$. This is the smallest filtration which makes τ a stopping time.

Let F denote the cumulative distribution function of τ, defined as $F(t) = \mathbf{P}(\tau \leq t) = \int_0^t f(s)\mathrm{d}s$. It is assumed that $F(t) < 1$ for any $t < T$, where T is the maturity date. Otherwise there exists $t_0 < T$ such that $F(t_0) = 1$, and default occurs almost surely before t_0.

Let us introduce the hazard function of default defined by

$$\Gamma(t) = -\ln(1 - F(t))$$

and its derivative $\gamma(t) = \frac{f(t)}{1-F(t)}$, called the hazard rate. Note that Γ is increasing with time, and that

$$\mathbf{P}(\tau > t) = 1 - F(t) = e^{-\Gamma(t)} = \exp(-\int_0^t \gamma(s)\mathrm{d}s)$$

In case τ is defined as the first jump of an inhomogeneous Poisson process with deterministic intensity $\lambda(t)_{t\geq 0}$, then the the density function of τ is

$$f(t) = \mathbf{P}(\tau \in \mathrm{d}t)/\mathrm{d}t = \lambda(t)\exp(-\int_0^t \lambda(s)\mathrm{d}s) = \lambda(t)e^{-\Lambda(t)}$$

where $\Lambda(t) = \int_0^t \lambda(s)\,\mathrm{d}s$ and $\mathbf{P}(\tau \leq t) = F(t) = 1 - e^{\Lambda(t)}$ hence the hazard function is equal to the compensator of the Poisson process, i.e. $\Lambda(t) = \Gamma(t)$.

Conversely, if τ is a random time with density f, setting $\Lambda(t) = -\ln(1 - F(t))$ allows us to think at τ as the first jump time of an inhomogeneous Poisson process with the derivative of Λ as intensity.

The hazard rate can be interpreted as the instantaneous probability of default, i.e. it is the probability of that the default occurs in a small interval dt given that the default did not occur before time t

$$\lim_{h\to 0}\frac{1}{h}\mathbf{P}\big(\tau \in (t, t+h]\,\big|\tau > t\big) = \frac{f(t)}{1-F(t)} = \gamma(t) \tag{3.1}$$

Let $D(t,T)$ denote the value of a defaultable zero-coupon bond with maturity T, which pays one unit at time T if default has not yet occurred, and pays of $R < 1$ units at time τ if $\tau \leq T$.

The price of this defaultable zero-coupon bond at time t before default is

$$\begin{aligned} D(t,T) &= \mathbf{E}\big(B(t,T)\mathbb{1}_{\{T<\tau\}} + RB(t,\tau)\mathbb{1}_{\{\tau\leq T\}}\,\big|\mathcal{H}_t\big) \\ &= \frac{\mathbf{P}(T<\tau)}{\mathbf{P}(t<\tau)}B(t,T) + \frac{\mathbf{P}(T<\tau)}{\mathbf{P}(t<\tau)}R\int_t^T B(s,T)\mathrm{d}F(s) \end{aligned}$$

where $B(t,T)$ denotes the value of the default-free bond zero-coupon at time t.

If F is differentiable, the function $\gamma = \Gamma'$ satisfies $f(t) = \gamma(t)e^{-\Gamma(t)}$, and so

$$\begin{aligned} D(t,T) &= \frac{e^{-\Gamma(T)}}{e^{-\Gamma(t)}}B(t,T) + \frac{1}{e^{-\Gamma(t)}}R\int_t^T B(t,s)\gamma(s)e^{-\Gamma(s)}\mathrm{d}s \\ &= s(t,T)B(t,T) + R\int_t^T \gamma(s)s(s,T)B(s,T)\mathrm{d}s \end{aligned}$$

where the functions $s(t,T) = \frac{e^{-\Gamma(T)}}{e^{-\Gamma(t)}} = \mathbf{P}(\tau > T|\tau > t)$ is the survival probabilities from time t to time T, where $0 \leq t \leq T$.

The first term represents the price of the defaultable zero-coupon bond with no recovery at default, what can be formulated as

$$D^0(t,T) = \exp\Big(- \int_t^T \gamma(s) + r(s)\mathrm{d}s \Big).$$

This result is important because it tells us that the default intensity $\gamma(t)$ plays the same role as interest rates. This property will allow us to taking into account the possibility of default through the inclusion of the hazard rate in the discount rate and view default intensities as credit spreads. However, $\gamma(t)$ is independent of all default free market quantities and represents an external source of randomness that makes these models unrealistic.

The integral in the second term makes reference to the fact that default can happen at any time between t and T. The pre-default value of the defaultable zero-coupon is

$$D(t,T) = D^0(t,T) + R\int_t^T \gamma(s)D^0(s,T)\mathrm{d}s$$

Equivalent Martingale Measure

In order to study the completeness of the financial market, we first need to define the tradeable assets. If the market consists only of the risk-free zero-coupon bond, there exists infinitely many equivalent martingale measure's. The discounted asset prices are constant, hence the set $\mathcal{Q}$ of equivalent martingale measures is the set of probabilities equivalent to the historical one.

The range of prices is defined as the set of prices which do not induce arbitrage opportunities. For a defaultable zero-coupon bond with a constant rebate R paid at maturity if default happens, the range of prices is equal to the set

$$\{\ \mathbf{E_Q}\big(\, B(0,T)\mathbb{1}_{\{T<\tau\}} + RB(0,T)\mathbb{1}_{\{\tau\leq T\}}\,\big), \quad \mathbf{Q}\in\mathcal{Q}\ \}$$

This set is exactly the interval $]\,RB(0,T), B(0,T)[$. Since, in the set $\mathcal{Q}$, one can select a sequence of probabilities $\mathbf{Q}_n$ which converge weakly to the Dirac measure of that the default appears at time 0 or never. Obviously, this range is too large to be efficient.

If defaultable zero-coupon bonds with zero recovery are traded in the market at price $D^*(t,T)$, which belongs to the interval $]\,0,\, B(t,T)[$, then under any risk-neutral probability $\mathbf{Q}$, the process $B(0,t)D^*(t,T)$ is a martingale. So the equivalent martingale measure $\mathbf{Q}^*$ chosen by the market, is such that

$$B(0,t)D^*(t,T) = \mathbf{E}_{\mathbf{Q}^*}(B(0,T)\mathbb{1}_{\{\tau<T\}}|\,\mathcal{H}_t) = B(0,T)\exp\big(- \int_t^T \gamma^*(s)\,\mathrm{d}s\big)$$

Therefore, we can characterize the cumulative distribution function of τ under $\mathbf{Q}^*$ from the market prices of the defaultable zero-coupon bonds as $\mathbf{Q}^*(\tau > T|\tau > t) = D^*(t,T)/B(t,T)$. Of course, this probability may differ from the historical probability.

It is obvious that if $D^*(t,T)$ belongs to the range of viable prices $]0, B(0,T)[$, the process γ^* is strictly positive, and the converse holds true too. The value of $\int_t^T \gamma^*(s)\,\mathrm{d}s$ is known for any $T \geq t$ as soon as there are defaultable zero-coupon bonds for each maturity, and the unique risk-neutral intensity can be obtained from the prices of defaultable zero-coupon bond as

$$\gamma^*(T) = \frac{\partial}{\partial T} \ln \frac{D^*(t,T)}{B(t,T)}.$$

There are two reasons why the calculations for extracting default probabilities from bond prices are, in practice, usually more complicated than this. First, the recovery rates are usually non-zero. Second, most corporate bonds are not zero-coupon bonds. When the recovery rate is non-zero, it is necessary to make an assumption about the claim made by bondholders in the event of default. Jarrow and Turnbull (1995) and Hull and White (1995) assume that the claim equals the no-default value of the bond. Duffie and Singleton (1997) assume that the claim is equal to the value of the bond immediately prior to default.

3.2 The Hull and White Model

Hull and White (2000) provides one of the most famous hazard function models to value credit default swaps. It starts out by estimating the risk-neutral probability of the reference entity defaulting at different times. The prices of bonds issued by the reference entity provide the main source of data for this estimation. By using the estimated risk-neutral default probabilities and making an assumption about the claim amount, they provide an approach for valuation of a credit default swap.

Hull and White formulated their results in terms of f(t) the default probability density, rather than the hazard rate. The hazard rate, $\gamma(t)$, is defined so that $\gamma(t)\Delta t$ is the probability of default between times t and $t + \Delta t$ as seen at time t assuming no default between time zero and time t, while $f(t)\Delta t$ is the probability of default between times t and $t + \Delta t$ as seen at time zero. The two measures provide the same information about the default probability environment and they are related by

$$f(t) = \gamma(t)e^{-\int_0^t \gamma(s)\mathrm{d}s}.$$

It is assumed that all the bonds have the same seniority in the event of default by the reference entity and that the expected recovery rate is independent of time. So the expected value of recovery rate is independent of both j and t, let R denote this expected value.

The model uses a set of N bonds of the reference entity with the maturity of the ith bond being t_i, with $t_1 < t_2 < t_3 < \ldots < t_N$. It then estimates the risk-neutral default probability density function f(t) of the company assuming that $f(t)$ is constant and equal to $f(t_i)$ for $t_{i-1} <\leq t < t_i$.

The model assumes that the only reason for a corporate bond sells for less than a similar treasury bond is the possibility of default. Accordingly, the difference between bond prices is equivalent to the present value of the cost of default of the reference entity. It follows that:

$$D_j - B_j = \sum_{i=1}^{j} f(t_i)\beta_{ij}$$

where B_j is the price of the jth bond today, D_j is the price of a Treasury bond promising the same cash flow as the jth bond. Furthermore, β_{ij} represents the present value of the loss from a default on the jth bond between t_{i-1} and t_i relative to the value of a corresponding risk-free bond and is set to

$$\beta_{ij} = \int_{t_i - 1}^{t_i} v(t)[F_j(t) - \hat{R}C_j(t)]\mathrm{d}t$$

where v(t) is the present value of \$1 certainly received at time t, $F_j(t)$ is the forward price of the jth risk-free bond at time t and if there is the jth bond defaults at time t the bondholder makes a recovery at rate $\hat{R}$ on a claim $C_j(t)$.

Because interest rates are deterministic, $F_j(t)$ is equal to the price of the jth non-defaultable bond at time t.

This equation allows the $f(t_j)$'s to be determined inductively:

$$f(t_j) = \frac{D_j - B_j - \sum_{i=1}^{j-1} f(t_i)\beta_{ij}}{\beta_{jj}}$$

The two sets of zero-coupon bond prices can be bootstrapped from corporate coupon bond prices and treasury coupon bond prices.

To value a CDS with a \$1 notional principal, it is assumed that default events, Treasury interest rates, and recovery rates are mutually independent. It is also assumed that in the event of default the claim is the face value plus accrued interest.

This means that the payoff from a typical CDS is

$$1 - \hat{R}[1 + A(t)] = 1 - \hat{R} - A(t)\hat{R}$$

where $\hat{R}$ is the expected recovery rate, and $A(t)$ is the accrued interest on the reference obligation at time t as a percent of its face value. The present value of the expected payoff from the CDS is

$$\int_0^T [1 - \hat{R} - A(t)\hat{R}]f(t)v(t)\mathrm{d}t$$

Total payments per year made by the protection buyer w last until a credit event or the end of the contract at time T , whichever is sooner. In the case of no default during the lifetime of the CDS, the present value of the payments is $wu(T)$, where $u(T)$ is the present value of payments at the rate of \$1 per year on payment dates between time zero and time t. In the case of default at time $t (t < T)$, the present value of the payments is w[u(t)+e(t)], where $e(t)$ is the present value of an accrual payment at time t The expected present value of the payments is, therefore:

$$w \int_0^T f(t)[u(t) + e(t)]\mathrm{d}t + w\pi u(T)$$

where π denotes the risk-neutral probability of no credit event during the life of the swap. It can be calculated from $f(t)$ as follows

$$\pi = 1 - \int_0^t f(t)\mathrm{d}t$$

The value of the credit default swap to the buyer is the present value of the expected payoff minus the present value of the payments made by the buyer:

$$\int_0^T [1 - \hat{R} - A(t)\hat{R}]q(t)v(t)\mathrm{d}t - w \int_0^T q(t)[u(t) + e(t)]\mathrm{d}t + w\pi u(T)$$

if when entering the contract neither of both parties makes a cash payment.

The one parameter necessary for valuing a credit default swap that cannot be observed directly in the market is the expected recovery rate. It is assume that the same recovery rate can be used for estimating the default probability densities and for calculating the payoff. As the expected recovery rate increases, estimates of the probability of default increase and payoffs decrease. Hull and White show that the overall impact of the recovery rate assumption on the value of a credit default swap is fairly small when the expected recovery rate is in the 0% to 50% range.

Since the value of the CDS to both parties has to be zero at inception, the value of w that makes the last expression zero is

$$s = \frac{\int_0^T [1 - \hat{R} - A(t)\hat{R}]q(t)v(t)\mathrm{d}t}{\int_0^T q(t)[u(t) + e(t)]\mathrm{d}t + w\pi u(T)} \tag{3.2}$$

where the variable s is referred to as the credit default swap spread and represents the value of total payments by year as a share of the notional principal of the CDS.

The presented valuation approaches is based on the assumption that interest rates, default probabilities, and recovery rates are independent. These assumptions are unlikely to be perfectly true in practice. However, Hull and White hypothesize that the effect is small. The model's attractiveness as a reduced form approach is that it can be implemented based on observable market data.

Hull and White (2000) in their paper also present approximate no-arbitrage pricing approach. Skipping most of the derivation, we give a brief review of this valuation approaches.

Put very simple, a portfolio consisting of a CDS and an underlying bond of the same obligor both having maturity T should be risk-free. It should, therefore, have the same payoff as a treasury bond with maturity T. Assuming for simplicity that the treasury curve is flat and that interest rates are constant, a simple no-arbitrage argument then results in the spread

$$s* = \text{maturity-T corporate bond yield - maturity-T treasury bond yield},$$

which will typically overestimate the true spread s. However, there is a way to close most of the gap between $s*$ and s. Let $A^*(t)$ represent the time-t accrued interest as a percentage of the face value on a T-year par yield bond that is issued at time zero by the reference entity with the same payment dates as the swap. Hull and White refer to this bond as the underlying par yield corporate bond. As an approximation, use $a(t)$ as the average value for $A(t)$ under the integral in Equation 3.2 and analogously define $a^*(t)$ as the average value for $A^*(t)$, $0 < t < T$. This yields an approximate formula for s, where

$$s = \frac{s * (1 - R - aR}{(1 - R)(1 + a^*)}$$

In their paper, Hull and White come to the conclusion that this approximate valuation ends up with results only differing marginally from the intensity model spread (Equation 3.2). This would constitute a quick and simple method for the applied valuation of CDSs. Hence, we address both, respective model performance as compared to the market as well as relative pricing performance of the two Hull-White approaches.

Arora Bohn and Zhu (2005) empirically compare three credit-risk model. These models are the Merton model, the Vasicek-Kealhofer (VK) model, and the Hull-White (HW) model. The used data set consists of bond data for each models implementation combined with data on actual CDS spreads used to test each models predicted prices.

They test test the ability of each model to predict spreads in the credit default swap (CDS) market as an indication of each models strength. The VK model tends to do the best across the full sample and relative sub-samples and the HW model approach is not too far behind in terms of accuracy ratio. In cases They find that the quality and quantity of data make a difference for the HW model, many traded issuers will not be well modeled in this way unless they issue more traded debt. This results demonstrate that with proper calibration of default models, both equities and bonds can be effective sources for information about impending defaults.

3.3 Intensity Rate Approach

In the previous section introduced hazard function only changes with time t. In real modeling situations there are other factors that affect the default probability of a firm. Hence we want to be able to build models in which we can condition on a more general information set $\mathcal{F}_t$, where $\mathcal{F}_t$ contains information on the survival up to time t and the hazard process λ_t is predictable with respect to the filtration $\mathcal{F}_t$.

This description of intensity models will focus only on the construction using Cox processes, while this is not the most general setting, it is by far the most convenient to work with in practical pricing problems.

A Cox process is a generalization of the Poisson process in which the intensity is allowed to be time-varying and stochastic, but in such a way that if we condition on a particular realization $\lambda(\,\cdot\,, \omega)$ of the intensity, the jump process becomes an inhomogeneous Poisson process with intensity $\lambda(t, \omega)$. The process is sometimes called doubly stochastic due to the stochastic nature of the jump component and the stochasticity in the probability of jumping, i.e. in the intensity.

Although this setting is very natural, it excludes several plausible situations. For example, the process H_t cannot be adapted to the market information, so default can't be triggered directly by any of the driving processes, as is the case with structural models.

In the following part a formal description is given on how the Cox process can be modeled, it is a very useful tool when performing calculations and simulations of reduced form models.

First, let us recall that one way of simulating the first jump of an inhomogeneous Poisson process is to use the connection between a standard homogeneous Poisson process and a inhomogeneous Poisson process. The first jump time of a standard Poisson process, denoted θ, has an exponential distribution of rate one, i.e. $\mathbf{P}(\theta \geq t) = \exp(-t)$. One can define τ the first jumping time of the inhomogeneous Poisson process N_t as

$$\tau = \inf\{\, t \geq 0 : \int_0^t \lambda(s)\,\mathrm{d}s \geq \theta \,\} = \inf\{\, t \geq 0 : \Lambda(s) \geq \theta \,\} \tag{3.3}$$

Hence the simulation of the first jump of an inhomogeneous Poisson process can be done trough generating exponential variables and taking Λ^{-1} of the generated time instants.

From now on we assume that a probability space $(\Omega, \mathcal{G}, \mathbf{P})$ is given, where Ω is the set of possible states of the economic world, and $\mathbf{P}$ is the physical or real probability measure. A process $X_t = (X_t^1, \ldots, X_t^d)$ with values in $\mathbb{R}^d$ is defined on the probability space, that represents d economy-wide variables. These state variables may include interest rates on riskless debt, stock prices, credit ratings and other variables deemed relevant for determining the likelihood of default. Let $(\mathcal{F}_t)_{t\geq 0}$ denote the "market filtration" generated by X_t, i.e. $\mathcal{F}_t = \sigma\{X_s : 0 \leq s \leq t\}$. And let the intensity rate λ_t be a positive and right continuous $\mathcal{F}_t$-adapted stochastic process.

We also assume that on the probability space $(\Omega, \mathcal{G}, \mathbf{P})$ there exists a random variable θ independent of $\mathcal{F}_\infty$, following the exponential law $\mathbf{P}(\theta > t) = e^{-t}$. The first jump time of a Cox process τ is defined as the first time when the increasing process $\Lambda_t = \int_0^t \lambda_s\,\mathrm{d}s$ is above the random level θ, i.e.

$$\tau = \inf\{\, t \geq 0 : \int_0^t \lambda_s\,\mathrm{d}s \geq \theta \,\} = \inf\{t \geq 0 : \Lambda_s \geq \theta\}.$$

Note that this is an exact analogue to equation (3.3) with a random intensity replacing the deterministic intensity function.

In analogy with Poisson processes, the distribution function of τ can be given as

$$\begin{aligned}\mathbf{P}(\tau > t) &= \mathbf{P}(\Lambda_t < \theta) = \mathbf{P}\Big(\int_0^t \lambda_s \, \mathrm{d}s < \theta\Big) \\ &= \mathbf{E}\Big(\mathbf{P}\Big(\int_0^t \lambda_s \, \mathrm{d}s < \theta \,|\, \mathcal{F}_t\Big)\Big) = \mathbf{E}\big(e^{-\int_0^t \lambda_s \, \mathrm{d}s}\big), \qquad (3.4)\end{aligned}$$

where we used the the independence assumption and the $\mathcal{F}_t$-measurability of λ_s for $s \leq t$.

Pricing Formulas

This section provides a useful tool for showing the general expressions for the price of the derivatives.

As before let $H_t = \mathbb{1}_{\{\tau \leq t\}}$ denote the right continuous default processes and $\mathcal{H}_t = \sigma\{H_s : s \leq t\}$ the filtration generated by it. We introduce the filtration $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$, that is, the enlarged filtration generated by the market filtration and the default time. The following proposition gives the basic analytical tool to "reduce" default related computations from the full filtration $\mathcal{G}_t$ to the default free computations in the market filtration $(\mathcal{F}_t)_{t \geq 0}$.

Lemma 3.3.1. *Let X be an integrable random variable, then*

$$\mathbf{E}(\mathbb{1}_{\{\tau > t\}} X | \mathcal{G}_t) = \mathbb{1}_{\{\tau > t\}} \frac{\mathbf{E}(X \mathbb{1}_{\{\tau > t\}} | \mathcal{F}_t)}{\mathbf{E}(\mathbb{1}_{\{\tau > t\}} | \mathcal{F}_t)}$$

Proof. It is easy to describe the events which belong to the $\mathcal{G}_t$ σ-field on the set $\{\tau > t\}$. Indeed, if $G \in \mathcal{G}_t$, then $G \cap \{\tau > t\} = B \cap \{\tau > t\}$ for some event $B \in \mathcal{F}_t$. Therefore any $\mathcal{G}_t$-measurable random variable X satisfies $\mathbb{1}_{\{\tau > t\}} X = \mathbb{1}_{\{\tau > t\}} x$, where x is an $\mathcal{F}_t$-measurable random variable. Taking conditional expectation with respect to $\mathcal{F}_t$ of both members, we deduce $x = \frac{\mathbf{E}(X \mathbb{1}_{\{\tau > t\}} | \mathcal{G}_t)}{\mathbf{E}(\mathbb{1}_{\{\tau > t\}} | \mathcal{G}_t)}$. □

We now compute the expected value of a a predictable process at time τ.

Lemma 3.3.2. *If h_t is an $(\mathcal{F}_t)_{t \geq 0}$-predictable and bounded stochastic process then*

$$\mathbf{E}(h_\tau | \mathcal{F}_t) = \mathbf{E}\bigg(\int_0^\infty h_s \lambda_s e^{-\Lambda_s} \mathrm{d}s \bigg| \mathcal{F}_t\bigg)$$

and

$$\mathbf{E}(h_\tau | \mathcal{G}_t) = \mathbf{E}\bigg(\int_t^\infty h_s \lambda_s e^{\Lambda_t - \Lambda_s} \mathrm{d}s \bigg| \mathcal{F}_t\bigg) \mathbb{1}_{\{t < \tau\}} + h_\tau \mathbb{1}_{\{\tau \leq t\}}$$

Proof. Let $h_t = \mathbb{1}_{]v,w]}(t)B_v$ where $B_v \in F_v$ be an elementary predictable process. Then

$$\begin{aligned}\mathbf{E}\big(h_\tau|\mathcal{F}_t\big) &= \mathbf{E}\Big(\mathbb{1}_{]v,w]}(\tau)B_v\Big|\mathcal{F}_t\Big) = \mathbf{E}\Big(\mathbf{E}\big(\mathbb{1}_{]v,w]}(\tau)B_v|\mathcal{F}_\infty\big)\Big|\mathcal{F}_t\Big) \\ &= \mathbf{E}\Big(B_v\mathbf{P}(v < \tau \le w)|\mathcal{F}_\infty)\Big|\mathcal{F}_t\Big) = \mathbf{E}\Big(B_v(e^{-\Lambda_v} - e^{-\Lambda_w})\Big|\mathcal{F}_t\Big)\end{aligned}$$

It follows that

$$\mathbf{E}\big(h_\tau|\mathcal{F}_t\big) = \mathbf{E}\Big(B_v\int_v^w \lambda_s e^{-\Lambda_s}\mathrm{d}u\Big|\mathcal{F}_t\Big) = \mathbf{E}\Big(\int_0^\infty h_s\lambda_s e^{-\Lambda_s}\mathrm{d}s\Big|\mathcal{F}_t\Big)$$

and the second result is derived from the monotone class theorem. □

Lemma 3.3.3. *Let X be an $\mathcal{F}_\infty$-measurable random variable. Then*

$$\mathbf{E}(X|\mathcal{G}_t) = \mathbf{E}(X|\mathcal{F}_t)$$

Proof. To prove that $\mathbf{E}(X|\mathcal{G}_t) = \mathbf{E}(X|\mathcal{F}_t)$, it suffices to check that

$$\mathbf{E}(B_h(t\wedge\tau)X) = \mathbf{E}(B_t h(t\wedge\tau)\mathbf{E}(X|\mathcal{F}_t))$$

for any $B_t \in \mathcal{F}_t$ and $h = \mathbb{1}_{[0,a]}$. For $t \le a$, the equality is obvious. For $t > a$, we have

$$\mathbf{E}(B_t\mathbb{1}_{\{\tau\le a\}}\mathbf{E}(X|\mathcal{F}_t)) = \mathbf{E}(B_t\mathbf{E}(X|\mathcal{F}_t)\mathbf{E}(\mathbb{1}_{\{\tau\le a\}}|\mathcal{F}_\infty)) =$$

$$\mathbf{E}(B_t\mathbf{E}(X|\mathcal{F}_t)\mathbf{E}(\mathbb{1}_{\{\tau\le a\}}|\mathcal{F}_t)) = \mathbf{E}(XB_t\mathbf{E}(\mathbb{1}_{\{\tau\le a\}}|\mathcal{F}_t)) = \mathbf{E}(XB_t\mathbb{1}_{\{\tau\le a\}})$$

□

With these technical results in place, we are ready to extend the scope of the model setup. In default modeling we will be concerned with pricing cash flows which in one way or another are tied to the random variable τ.

If X is an integrable $\mathcal{F}_T$ -measurable random variable we note that

$$\mathbf{P}(X\mathbb{1}_{\{\tau>T\}}|\mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}}\mathbf{E}(X\mathbb{1}_{\{\tau>T\}}|\mathcal{G}_t)$$

so using Lemma 3.3.1 gives us

$$\mathbf{E}(X\mathbb{1}_{\{\tau>T\}}|\mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}}\frac{\mathbf{E}(X\mathbb{1}_{\{\tau>T\}}|\mathcal{F}_t)}{\mathbf{E}(\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t)}$$

Now,

$$\mathbf{E}(X\mathbb{1}_{\{\tau>T\}}|\mathcal{F}_t) = \mathbf{E}(X\mathbf{E}(\mathbb{1}_{\{\tau>T\}}|\mathcal{F}_T)|\mathcal{F}_t) = \mathbf{E}\big(Xe^{-\int_0^T\lambda_s\mathrm{d}s}|\mathcal{F}_t\big)$$

$$= e^{-\int_0^t\lambda_s\mathrm{d}s}\mathbf{E}\big(Xe^{-\int_t^T\lambda_s\mathrm{d}s}|\mathcal{F}_t\big) = \mathbf{E}(\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t)\mathbf{E}\big(Xe^{-\int_t^T\lambda_s\mathrm{d}s}|\mathcal{F}_t\big)$$

and therefore,

$$\mathbf{E}(X\mathbb{1}_{\{\tau>T\}}|\mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}}\mathbf{E}(Xe^{-\int_t^T \lambda_s \mathrm{d}s}|\mathcal{F}_t).$$

This corollary of Lemma 3.3.1 admits an interesting interpretation. If we consider a a defaultable claim with a promised payment of X_T at time T and an actual payment of $X_T\mathbb{1}_{\{\tau>T\}}$ at time T, then its value at time t is

$$\mathbf{E}\big(e^{-\int_t^T r_s \mathrm{d}s}X_T\mathbb{1}_{\{\tau>T\}}\big|\mathcal{G}_t\big) = \mathbb{1}_{\{\tau>t\}}\mathbf{E}\big(X_T e^{-\int_0^T r_s+\lambda_s \mathrm{d}s}\big|\mathcal{F}_t\big), \tag{3.5}$$

i.e. the default intensity λ_t can be interpreted as a spread. This simple example shows that the framework obviously holds promise for getting analytically tractable prices of defaultable claims. However, we are not dealing with a risk neutral probability. In the case where the market is assumed to be complete, that means in particular that a defaultable zero-coupon is traded (or duplicable). Then, for pricing purpose, the intensity has to be evaluated under the risk-neutral probability given by the market.

Two more "building blocks" can be priced using this framework, and with these blocks we have a very flexible collection of tools.

Credit insurance products can be thought of as products that pay a stochastic amount at the time of default. Let the amount paid at time τ be Y_τ where Y_t is a process adapted to the market filtration $(\mathcal{F}_t)_{t\geq 0}$. The expected payoff for insurance over the period $[t, T]$ computed at time t is found by using Lemma 3.3.2:

$$\mathbf{E}\big(e^{-\int_t^\tau r_s \mathrm{d}s}Y_\tau\mathbb{1}_{\{t<\tau\leq T\}}\big|\mathcal{G}_t\big) = \mathbb{1}_{\{t<\tau\}}\mathbf{E}\big(\int_t^T Y_s\lambda_s e^{-\int_t^s r_u+\lambda_u \mathrm{d}u}\mathrm{d}s\big|\mathcal{F}_t\big) \tag{3.6}$$

Second, consider a claim paying Z_t continuously until default or until the maturity date T in the case of no default. Then this claim's value with maturity T at time t is

$$\mathbf{E}\big(\int_t^T Z_s\mathbb{1}_{\{\tau>s\}}e^{-\int_t^s r_u \mathrm{d}u}\mathrm{d}s\big|\mathcal{G}_t\big) = \mathbb{1}_{\{\tau>t\}}\mathbf{E}\big(\int_t^T Z_s e^{-\int_t^s r_u+\lambda_u \mathrm{d}u}\mathrm{d}s\big|\mathcal{F}_t\big) \tag{3.7}$$

Observe that in all three cases the claim's value is a $\mathcal{G}_t$-expectation, the reduction from $\mathcal{G}_t$ to $\mathcal{F}_t$ we see here is the essential property why the whole modelling approach is called also reduced form modelling.

Credit spread and Recovery Assumptions

In credit risk valuation methodology, two quantities are crucial. The first is the probability of default and the second is the recovery rate in the event of default. Recovery rates refer to how we model the value that a debt instrument has left, after default. Three main specifications have been used in the literature for the recovery rate parametrization:

Recovery of Market Value fixes a recovery rate equal to an exogenous fraction of the market value of the bond just before default. A contingent claim is said to have a recovery of market value at a default time τ if the amount recovered in the event of default is equal to

$$R_\tau = (1 - L_\tau)V(\tau_-, T)$$

where $V(\tau_-, T)$ is the market price of the claim just before default and L_τ is the expected loss rate conditional on the information available up to time τ. This measures the change in market value at the time of default, and has economic meaning since L_τ is the loss in value associated with default.

Duffie and Singleton [30] show that with this recovery assumption, a defaultable bond can be priced as if it was a default-free zero-coupon bond, by replacing the usual short-term interest rate process r_t with a default-adjusted short rate process $\pi_t = r_t + \lambda_t L_t$.

The advantage of this approach is that that, if $\lambda_t L_t$ does not depend on the value of the defaultable bond, we can apply well known term structure processes to model π_t instead of r_t to price defaultable debt.

One of the main drawbacks of this approach is that since $\lambda_t L_t$ appears multiplied in π_t, in order to be able to estimate λ_t and L_t separately, we would need to have available a collection of bonds that share some, but not all default characteristics, or derivative securities whose payoffs depend, in different ways, on λ_t and L_t.

This identification problem is the reason why most of the empirical work which tries to estimate the default intensity process from defaultable bond data uses an exogenously given constant $L_t = L$ for all t.

Recovery of Face Value considers that, at default, the bond holders of a given seniority receive a fixed fraction of face value, irrespective of the coupon level or maturity. This is the closest we come to legal practice, and it is also the measure typically used in rating-agency studies. One only needs a post-default market price to estimate the quantity and this allows to estimate recovery value based on statistics provided by rating agencies such as Moody's. In mathematical terms, the formula for a bond price is not quite as pretty as in the case of recovery of market value, since we have to compute an integral of the form (3.6).

Recovery of Treasury assumes that, at default, a bond would have a market value equal to an exogenously specified fraction of an otherwise equivalent default-free bond. This could be seen as a more sophisticated approach than recovery of face value, since it at least tries to correct for the fact that amounts of principal with long maturity should be discounted more than principal payments with short maturity. An advantage of the recovery of treasury approach is that it permits (at least with an assumption of independence between the short rate r_t and the default intensityλ_t) an immediate expression for implied survival rates.

With doubly stochastic default times the risk neutral hazard-rate process λ_t and the credit spread

$$c(t,T) = -\frac{1}{T-t}(\ln B(t,T) - \ln D(t,T))$$

of defaultable bonds are closely related. Analytic results for the instantaneous credit spread are most easily derived given by

$$c(t,t) = \lim_{T\to t} c(t,T) = \frac{\partial}{\partial T}\bigg|_{T=t}(\ln B(t,T) - \ln D(t,T)).$$

Assuming that $\tau > t$, so that $B(t,t) = D(t,t) = 1$, one gets that

$$\frac{\partial}{\partial T}\bigg|_{T=t} \ln B(t,T) = \frac{\partial}{\partial T}\bigg|_{T=t} B(t,T),$$

and similarly for $D(t,T)$. In the case of the default-free bound

$$-\frac{\partial}{\partial T}\bigg|_{T=t} B(t,T) = r_t$$

and to compute the derivative for the defaultable bound we need to distinguish between the different recovery models.

Under recovery of market value, we know from Duffie and Singleton [30] that a defaultable bond can be priced as a default-free bond, by replacing the short-term rate with $\pi_t = r_t + \lambda_t L_t$. Exchanging expectation and differentiation,

$$-\frac{\partial}{\partial T}\bigg|_{T=t} D(t,T) = \mathbf{E}\bigg(\frac{\partial}{\partial T}\bigg|_{T=t} e^{-\int_t^T r_s + L_s\lambda_s \mathrm{d}s}\bigg|\mathcal{F}_t\bigg) = r_t + L_t\lambda_t$$

So the instantaneous credit spread equals the product of hazard rate and percentage loss given default, i.e. $c(t,t) = {}_t\lambda_t$, what is quite intuitive from an economic point of view.

Under recovery of face value , $D(t,T)$ is given by the sum of the price of the vulnerable claim (3.7) with $Z_t = 1$ and the recovery payment given by (3.6) where $Y_\tau = (1-L)_\tau$. The derivative of the recovery payment at $T = t$ is equal to

$$\frac{\partial}{\partial T}\bigg|_{T=t} \mathbf{E}\bigg(\int_t^T \lambda_s(1-L_s)e^{-\int_t^s r_u + \lambda_u \mathrm{d}u}\mathrm{d}s\bigg|\mathcal{F}_t\bigg) = \lambda_t(1-L_t)$$

Hence,

$$-\frac{\partial}{\partial T}\bigg|_{T=t} D(t,T) = r_t + \lambda_t - \lambda_t(1-L_t) = r_t + L_t\lambda_t,$$

so that $c(t,t)$ is again equal to $L_t\lambda_t$.

An analogous computation shows that we also have $c(t,t) = \delta_t\lambda_t$ under recovery of treasury. However, for $T-t > 0$ the credit spread corresponding to the different recovery models differs.

One could say that one recovery assumption can always be expressed in terms of another, and therefore they are all equivalent. But the most used credit pricing models treat recovery rate either as a constant parameter or as a stochastic variable independent from the default intensity. And since reduced form models are not able to separately estimate (using only bond data) the parts of the spread coming from the default risk λ_t and from the loss given default L_t, it matters for model fitting which recovery rate we keep constant.

Risk-neutral and Physical Measures

We now turn our attention to reduced form models under a riskneutral measure. Given a reduced form model under the physical measure $\mathbf{P}$, it does not necessarily follow that the model will be of reduced form under the riskneutral measure $\mathbf{Q}$. The doubly stochastic assumption needs to be independently stated for $\mathbf{P}$ and $\mathbf{Q}$. Moreover, the intensities λ_t^Q and λ_t^P themselves can depend differently on the state variables of the model, and may also have different likelihoods for each path. Even in the situation where $\lambda_t^Q = \lambda_t^P$ we can still have that

$$\mathbf{P}(\tau > T|\mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}}\mathbf{P}\big(e^{\int_0^T \lambda_s \mathrm{d}s}|\mathcal{F}_t\big)$$

is different from

$$\mathbf{Q}(\tau > T|\mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}}\mathbf{Q}\big(e^{\int_0^T \lambda_s \mathrm{d}s}|\mathcal{F}_t\big).$$

If $\mathbf{Q}$ is equivalent to $\mathbf{P}$ on $(\Omega, \mathcal{F})$, then according the Radon-Nikodym theorem there is the density process

$$\rho_t = \frac{\mathrm{d}Q}{\mathrm{d}P}\bigg|_{\mathcal{F}_t} \qquad t \in [0, T],$$

and we assume that $\log \rho_t$ is locally bounded.

Then there are predictable processes $\theta : [0, T] \times \Omega \to \mathbb{R}$, and $\eta : [0, T] \times \Omega \to \mathbb{R}$, such that under $\mathbf{Q}$:

$$W_t^Q = W_t - \int_0^t \theta_s \mathrm{d}s$$

is a $\mathcal{G}_t$-Brownian motion and

$$M_t = H_t - \lambda_t \eta_t = H_t - \lambda_t^Q \mathrm{d}s$$

is a $\mathcal{G}_t$-martingale, where W_t is a $\mathcal{G}_t$-Brownian motion and H_t is the counting process determining the default time.

Hence we still define the default event as the first jump of the counting process H_t, but the the intensities under $\mathbf{Q}$ have changed by a (stochastic) factor η_t. All the previous definitions and examples, including the doubly stochastic assumption have risk-neutral analogues in terms of the risk-neutral intensity λ_t^Q. In particular,

$$\mathbf{Q}(\tau > t|\mathcal{G}_s) = \mathbb{1}_{\{\tau>s\}}\mathbf{Q}\big(e^{\int_0^t \lambda_u^Q \mathrm{d}u}\big|\mathcal{F}_s\big).$$

so the survival probability expressions we just derived for intensity models apply for risk-neutral probabilities as well.

The credit risk premium, defined to be the ratio $\eta_t = \lambda_t^Q/\lambda_t^P$ is an object of interest when comparing risk neutral and statistical estimates of the default intensity.

Understanding this relationship is important, as it enables market participants to use information on historical default probabilities in pricing credit-risky securities, and conversely to use prices of defaultable bonds or market quotes for credit default swaps as additional input in determining historical default probabilities.

3.4 Pricing with Doubly Stochastic Default Times

The aim of this section is to develop some tools in the modelling of intensity processes, in order to apply our pricing building blocks obtained in the last section. We need effective ways to evaluate the conditional expectations on the right-hand side of equations (3.5), (3.6) and (3.7).

In most reduced-form models default is modelled by a doubly stochastic random time, r_t and λ_t are modelled as functions of some d-dimensional process $X_t = (X_{1,t}, \dots, X_{d,t})$. Here X_t is some d-dimensional Markovian process representing economic factors, and thus the natural background filtration is given by $\mathcal{F}_t = \sigma\{X_s : s \leq t\}$

Hence, $r_t + \lambda_t$ is of the form $\varphi(X_t)$ for some function $\varphi : \mathbb{R}^d \to \mathbb{R}$ and we have to compute conditional expectations of the form

$$\mathbf{E}\big(\, e^{-\int_t^T \varphi(X_s)\,\mathrm{d}s}\, g(X_T)\,\big|\, \mathcal{G}_t\big) \tag{3.8}$$

for some function $g : \mathbb{R}^d \to \mathbb{R}^+$, to get the time t value of some claim depending on the firm's default. Since X_t is a Markovian process, this conditional expectation can be expressed as a function of time and current value of the state variable $H(t, X_t)$.

Many ways exist to parameterize the intensity model but a useful way can be to specify $\varphi(x) = a_0 + a_1x_1 + \ldots + a_dx_d$ and $g(x) = b_0 + b_1x_1 + \ldots + b_dx_d$ as affine functions of the state variable. One then say that the model is affine if for $t < T$ the the function $H(t, X_t)$ can be expressed in closed form by

$$H(t, X_t) = \exp(\,\alpha(t, T) + \theta(t, T) \cdot X_t), \tag{3.9}$$

for some coefficient functions $\alpha(t, T)$, $\theta_1(t, T), \dots,$ $\theta_d(t, T)$.

It was shown by Duffie and Singleton [30] that in the case where X_t is an affine jump-diffusion process then the function $H(t, X_t)$ can be expressed in closed form by for some coefficients $\alpha(t,T)$, $\theta_1(t,T), \ldots,$ $\theta_d(t,T)$ also given in closed form functions of the model parameters.

The X_t process is called an affine jump diffusion, if it is given by

$$\mathrm{d}X_{j,\,t} = \kappa_j\,(\,\theta_j - X_{j,\,t}\,)\,\mathrm{d}t + \sigma_j\,\sqrt{X_{j,\,t}}\,\mathrm{d}W_{j,\,t} + \mathrm{d}q_{j,\,t}$$

for $j = 1,\ldots,d$, where $W_{j,t}$ is an $\mathcal{F}_t$-Brownian motion. κ_j and θ_j represent the mean reversion rate and reversion level of the process, and σ_j is a constant affecting the volatility of the process. $q_{j,\,t}$ is a pure jump process, independent of $W_{j,\,t}$ process, whose jump times are independent Poisson processes with constant jump intensity γ_j , and jump sizes are exponentially distributed with constant mean μ_j. Jump times and jump sizes are also independent.

Examples of affine processes

One of the simple examples of affine processes is the Ornstein-Uhlenbeck process or the Vasicek model. this is a one dimensional model where $\lambda_t = \kappa(\theta-\lambda_t)\mathrm{d}t+\sigma\mathrm{d}W_t$. As usual, the parameters κ and θ represent the long-term average and the rate of mean reversion for λ_t, while σ is a volatility coefficient. One problem of this approach is that the Ornstein-Uhlenbeck process takes also negative values. A simple solution is to take the modulus of the process, i.e., to reflect the above SDE on the origin.

As our next example of an affine model, let us consider the one factor model where the intensity process λ_t following a CIR dynamics

$$\mathrm{d}\lambda_t = \kappa(\theta-\lambda_t)\mathrm{d}t + \sigma\sqrt{\lambda_t}\mathrm{d}W_t$$

for positive constants κ, θ and σ satisfying the condition $4\kappa\theta > \sigma^2$.

Borrowing from the interest rate models, we have that survival probabilities in the CIR intensity model have the form

$$\mathbf{P}(\tau > t|\mathcal{G}_s) = \mathbb{1}_{\{\tau>s\}}\exp(A(s,t) + B(s,t)\dot{X}_s)$$

where

$$\begin{aligned} A(t,s) &= \frac{\kappa\theta}{\sigma^2}\log\frac{2\gamma\exp\big((\kappa+\gamma)(t-s)/2\big)}{2\gamma+(\kappa+\gamma)\exp\big((t-s)\gamma\big)-1} \\ B(t,s) &= \frac{2(1-\exp\big((t-s)\gamma\big))}{2\gamma+(\kappa+\gamma)\exp\big((t-s)\gamma\big)-1} \end{aligned}$$

and $\gamma^2 = \kappa^2 + \gamma^2$.

It is interesting to notice from this formula that survival probabilities increase if we increase the volatility parameter, while keeping all other parameters fixed. In other words, forward default rates decrease as the volatility in the intensity process increases.

The effects of volatility on survival probabilities and forward rates are compensated, on the other hand, by the rate of mean reversion. Higher values of k mean that λ_t stays close to its long-term average θ. This has the effect of bringing the forward rate close to a long-term level as well. Conversely, smaller values of κ accentuate the impact of the volatility in λ_t, leading to higher survival probabilities and smaller forward rates.

A simple class of multivariate affine processes is obtained by letting $X_t = (X_{1t}, \ldots, X_{dt})$, for independent affine coordinate processes $X_{1t}, \ldots, X_{dt})$. The independence assumption implies that we can break the calculation (3.9) down as a product of terms of the same form as (3.9), but for the one-dimensional coordinate processes.

Making r_t and λ_t dependent on a set of common stochastic factors X_t one can introduce randomness and correlation in the processes of r_t and λ_t. Moreover, our pricing building blocks examined in the previous section are special cases of expressions (3.8), so if we use affine processes for the common factors X_t we get closed form solutions for the building blocks.

Several versions of modelling r_t and λ_t in this framework can be found in the literature, differing from each other in their choices of the state variables and the processes they follow.

When modelling the joint behavior of interest rates and default intensities, the two obvious strategies for incorporating dependence are to use correlated Brownian motions as drivers of the processes or to have an intensity function depend on the interest rate. To illustrate the two approaches in their simplest form, the choice is between correlation through the noise term,

$$\begin{aligned}
\mathrm{d}r_t &= \kappa_r(\theta_r - r_t)\mathrm{d}t + \sigma_r\sqrt{r_t}\mathrm{d}W_{r,t} \\
\mathrm{d}\lambda_t &= \kappa_\lambda(\theta_\lambda - \lambda_t)\mathrm{d}t + \sigma_\lambda\sqrt{\lambda_t}\mathrm{d}W_{\lambda,t} \\
\mathrm{d}W_{r,t}\,\mathrm{d}W_{\lambda,t} &= \rho\mathrm{d}t
\end{aligned}$$

or the intensity is a function of the spot rate of interest like in Jarrow and Yildirim [50],

$$\begin{aligned}
\mathrm{d}r_t &= \kappa_r(\theta_r - r_t)\mathrm{d}t + \sigma_r\mathrm{d}W_{r,t} \\
\lambda_t &= \alpha(t) + \beta r_t
\end{aligned}$$

where $\alpha(t) > 0$ is a deterministic function of time, and β is a constant.

Brigo and Mercurio [14] show that ρ has a negligible impact on CDS prices (within the bid-ask spread), thus allowing to calibrate separately interest rates and credit models, even if $\rho \neq 0$.

3.5 Comments on Reduced Form Models

Reduced-form models are characterized by flexibility in their functional form, what is both a strength and a weakness.

This flexibility, is an advantage that a model can be well fitted to some collection of credit spreads. And the analogy with default-free term structure models makes the reduced-form models with doubly stochastic default times relatively easy to apply.

Also, Jarrow and Protter [?] argue that reduced-form models are more appropriate in an information theoretic context. They point out that while structural models assume complete information, they in fact suffer a lack of information concerning default points and expected recovery. Furthermore, modeler only has as much information as the market, making the reduced form models seem more realistic.

Unfortunately, this flexibility in functional form may result in a model with strong in-sample fitting properties, but poor out-of-sample predictive ability. Since this type of models are less grounded in the economics the default process, in interpreting the results some care is required. In particular, one must bear in mind that in these models the default intensity does not explicitly take into account the structure of a firm's outstanding risky debt. This can lead to nonsensical results.

Empirical evidence concerning reduced-form models is rather limited. Using the Duffie and Singleton framework, Duffee [28] finds that these models have difficulty in explaining the observed term structure of credit spreads across firms of different credit risk qualities. In particular, such models have difficulty generating both relatively flat yield spreads when firms have low credit risk and steeper yield spreads when firms have higher credit risk.

To combine the advantages of structural-form models - a clear economic mechanism behind the default process - and the ones of reduced-form models - unpredictability of default - can be done by modeling the evolution of firm value as a jump-diffusion process.

Chapter 4

Numerical Experiments and Discussions

A valuation model for defaultable claims consists of three components, the model for the default time, the model for the magnitude of default, and the interest rate model that characterizes the dynamics of the risk-free term structure. The fundamental difference between the structural and the reduced-form approach lies in how the models specify the timing risk of default.

As we have seen the two approaches imply a significantly different default timing. The default is predictable in the structural case, but it becomes a purely random event in reduced-form models. This is most obvious for short-term default probabilities. They are predicted to decline to zero as the maturity goes to zero in the first case, while they remain positive also for very short maturities for the second case.

In order to focus on this discriminative modeling of the default time we use an intensity model with structural interpretation. In particular, the state variable is chosen to be the log-leverage process of the firm, and its dynamic is modeled as in Collin-Dufresne,Goldstein. We are simply applying these models, assuming that each of them should match actual historical defaults. Although the models are flexible in all parameters, we assume that firms of different ratings differ only in terms of asset volatility and leverage for which data by ratings is available.

To explore these questions, we need to calibrate our models to a reasonable set of parameters. For each credit rating we use historical default rates from the Moody's 1983-2008 global default rate report to calibrate our models to.

All our models use constant interest rate, so we set $r = 6\%$, this is close to the historical average treasury rates during 1983-2008. Following Huang and Huang, we assume a constant payout rate for the asset, using $\delta_v = 3\%$ for all rating categories.

Leland Model

As a starting point we use these base case estimates of parameters to estimate the optimal asset volatility and leverage in the Leland model for different credit ratings.

The structural models ar known to fit only for longer longer maturities, so we calibrate only for long time cumulative defaults. So, the model was calibrated to match the target initial leverage and the cumulative default probability of 5-10 years. The tax redemption and the bankruptcy cost are set to be $\tau = 15\%, \alpha = 30\%$ just like it is recommended in Leland's original work. The results can be seen in table 4. In the first two rows are the leverage an volatility estimates of Huang and Huang , and the last two columns show our results.

Table 4.1: Parameter estimates for Leland's model

Rating	The base case from HH		Implied	
	Leverage ratio	Asset Vol.	Leverage	Asset Vol.
Aaa	13.08	34.06	37.57	23.63
Aa	21.18	29.23	30.07	27.41
A	31.98	25.25	31.45	27.37
Baa	43.28	25.05	38.14	31.65
Ba	53.53	36.00	45.52	44.49
B	65.70	52.33	63.58	57.83

Our estimates aren't that far away from the HH estimates. The most relevant differences can be found in higher rating classes, where we overestimate the leverage and under estimate the volatility compared to the results of HH. The main cause for this might be the fact that HH also used equity premium data in their estimation process. But even using the volatility estimates proposed by HH the implied leverage for investment grade bonds is higher then in the base case, suggesting that the model with our setting overestimates the optimal leverage for these firms.

A few other conclusions can be reached based on our results. First, the modelled default probabilities are under-predicted for short maturities for all rating categories, as it is expected from a structural mode. This can be seen in Figure **??** For investment grade bonds the long term default probabilities are highly over predicted by the fitted model and the difference between predicted and historical default probabilities increase with time. Second, this over-prediction cumulative defaults for long time periods significantly decreases for junk bonds. This is line with the empirical results of the literature, ie. that the Leland model works well for credit pricing only for longer maturities and for lower bond rating.

Log-leverage Models

As the next step, we we will ask whether our results will be significantly different when we use different approaches to model default, and how good are the different models in capturing the default probabilities across rating categories.

For this we going to use the Collin-Dufresne and Goldstein (CDG) structural model and a closely related intensity model. The CDG model assumes a dynamic capital structure, where the firms log-leverage ratio follows a mean reverting process. This way the firm adjusts its outstanding debt in response to firm value change, to obtain a target leverage ratio. Thus, when the firm value increases the firm will issue more debt, and when the firm value decreases the owners of the firm will wait with issuing new debt, to keep its leverage ratio on target. In the intensity model the economical state variable is the same as the mean reverting log-leverage process in the CDG model.

First we calibrate the CDG model to the cumulative defaults using our parameters and leverage rate estimates form the last section. . But before that, we need to set the value of two more parameters, namely the mean-reversion coefficient $kappal_l = 0.16$, and the buffer parameter $\nu = 0.06$, these numbers are similar to those chosen in CDG.

Table 4.2: Parameter estimates for the CDG model

Rating	The base case from HH		Implied Asset Volatility	
	Leverage ratio	Asset Vol.	for base case	for 60% of leverage
Aaa	13.08	34.06	24.24	26.38
Aa	21.18	29.23	22.28	24.85
A	31.98	25.25	21.16	24.32
Baa	43.28	25.05	21.24	25.29
Ba	53.53	36.00	24.93	31.10
B	65.70	52.33	26.68	36.17

The results in Table 4 show that, our asset volatility estimate in the CDG model are similar to the results in the base case. The U shape is present in our results too but not so significantly and we under predict the base case in all categories. If we decrease the leverage with 40% our result for volatility are more in line with the base case. This might be due to the higher default probability driven by the assumption that firms may increase their debt outstanding.

The two structural models of credit risk have generated very similar default probabilities, despite the fact that these models made very different assumptions on default mechanism. Their predicted credit risk premia do not differ much once each of them is calibrated to match the same historical default loss experience. The result can be seen in figure .

To analyze the impact of the different input parameters on the two models default prediction we take our initial initial input parameters and and set them to their 50% and 150%, to asses their impact on the modelling results.

In the CDG model, we note that if the mean-reverting rate κ is getting large, then the firm returns to its log-leverag faster, decreasing the probability of default. Eventually the firm becomes risk free as κ gets close to 1. Raising the buffer parameter ν has the same effect on the default probabilities but with a much smaller effect. These changes affect lower credit ratings more, since these firm have higher leverage meaning that according the model are much closer to the default boundary. The effect of the initial leverage rate change is in line whit this logic, the changing of the initial leverage has the same effect across rating categories, since with raising the laverage all firs get closer to the default with the same proportion.

Finally, the non-negative parameter $qsig_v$ controls the effect of the stochastic part of a Brownian motion which can be positive or negative and it represents the external market risk. We expect that when σ_v increases, the better capital-structured companies have larger distance to the default boundary while worse capital-structured companies have shorter gap. Thus the impact of increasing σ_v affects firm's default probability in lower categories better.

Now that we have estimated the input parameters for the CDG model we can calibrate the intensity model using the asset volatility and leverage estimates from the previous part. Calibrating the two models to the same data allows as to investigate how much of these behaviors have the intensity modell inherited, and asses the sensitivity of the models for the common parameters across different rating categories.

Using our estimates for asset volatility and initial leverage estimates from the previous section we can estimate the two hazard rate parameters a and c. The results can be found in figure **??**.

It's easy to see that the fitted value of c and a is decreasing as the credit rating increases. This means that the default probabilities generated by the intensity model depend more and more on the constant term a and less on the economical factors. In other word this setting is able to fit default probabilities for higher ratings but the fitted intensities are almost constants. This can be seen from the sensitivity tests, where changing the CDG model parameters has almost no effect. This problem prevails in lower rating if c is chosen to low compared a. An example case for this can be find in figure =**??**. This restriction could make fitting this model cumbersome, and an over fitted model could lead to falls conclusions if a and c is fitted together with other parameters of the model. One solution for this could be to set the value of a for a reasonable constant for different rating categories.

In our brief study we found that our simple parameter estimation methodology did not provide appropriate parameter values for estimating default rates. A more advanced calibration methodology should be used that also includes bond, stock, and balance sheet information data. On the other hand, after a more throughout calibration, it would be interesting to examine the credit spreads generated by our last models.

In this study we made various assumptions that limit the opportunities and performance of the selected credit risk models. Most of our model inputs were constant , but they might differ across credit ratings. Therefore we should determine actual values with our model and make adjustments to modeled CDS spreads if modeled and quoted spreads differ.

Appendix A

Figures

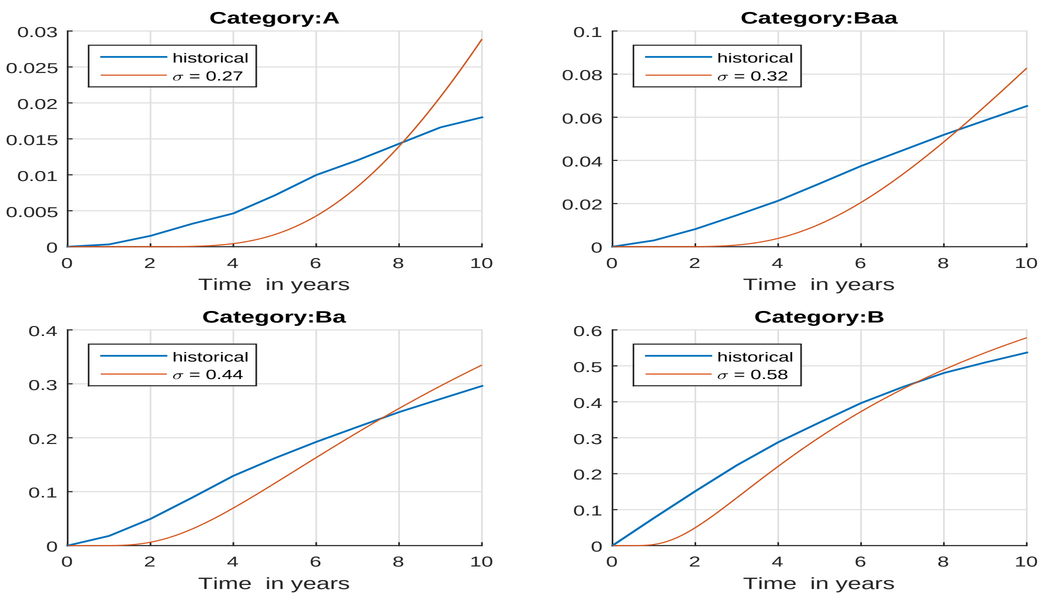

Figure A.1: Predicted cumulative default probabilities for the fitted Lelan model, compared to historical data.

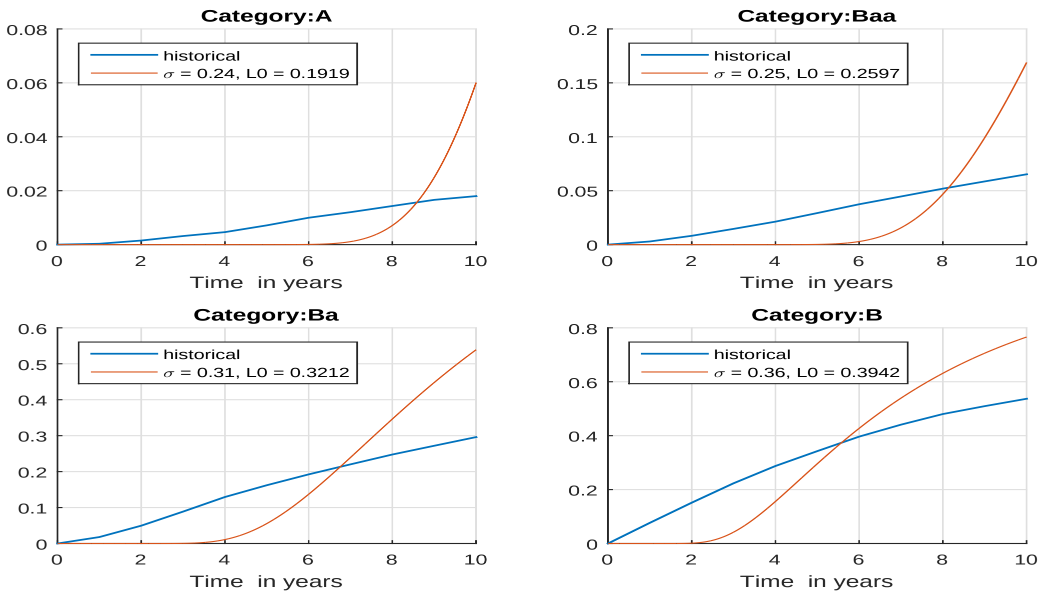

Figure A.2: Predicted cumulative default probabilities for the fitted CDG model, compared to historical data.

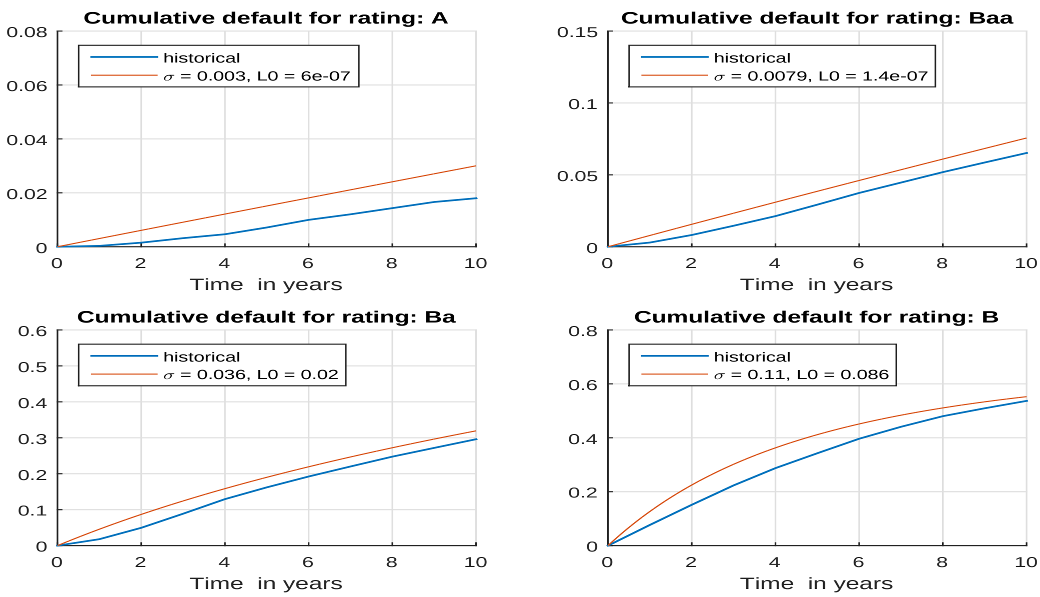

Figure A.3: Predicted cumulative default probabilities for the fitted intensity model, compared to historical data.

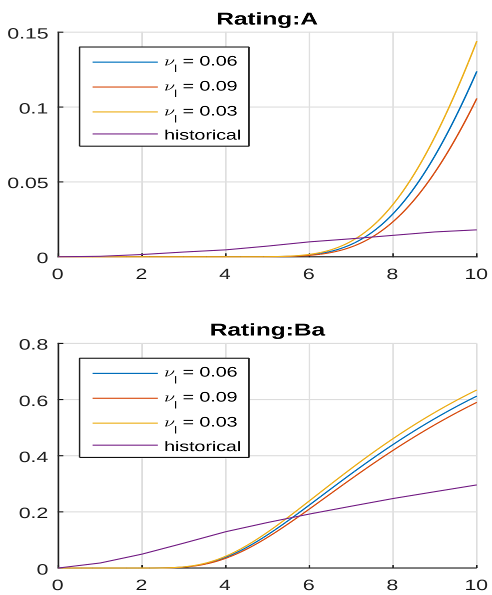

Figure A.4: Sensitivity of the CDG model for parameter σ_v

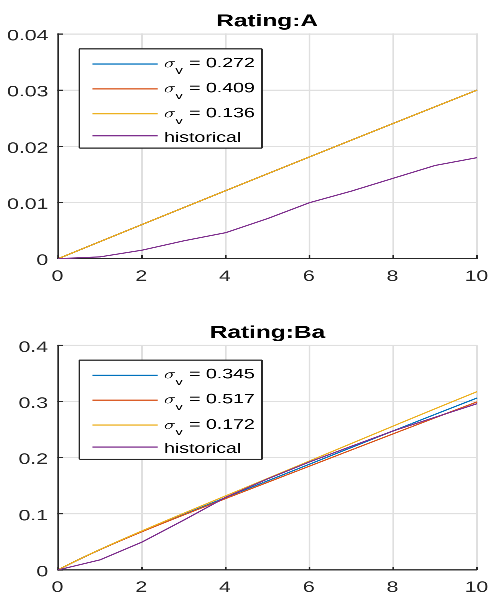

Figure A.5: Sensitivity of the intensity model, for parameter σ_v

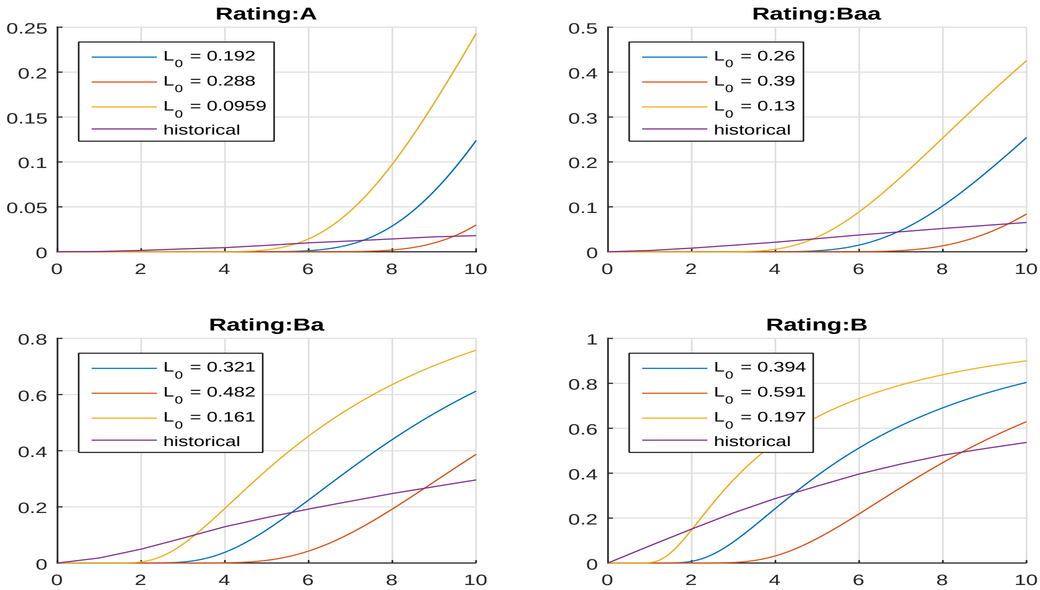

Figure A.6: Sensitivity of the CDG model for the initial leverage

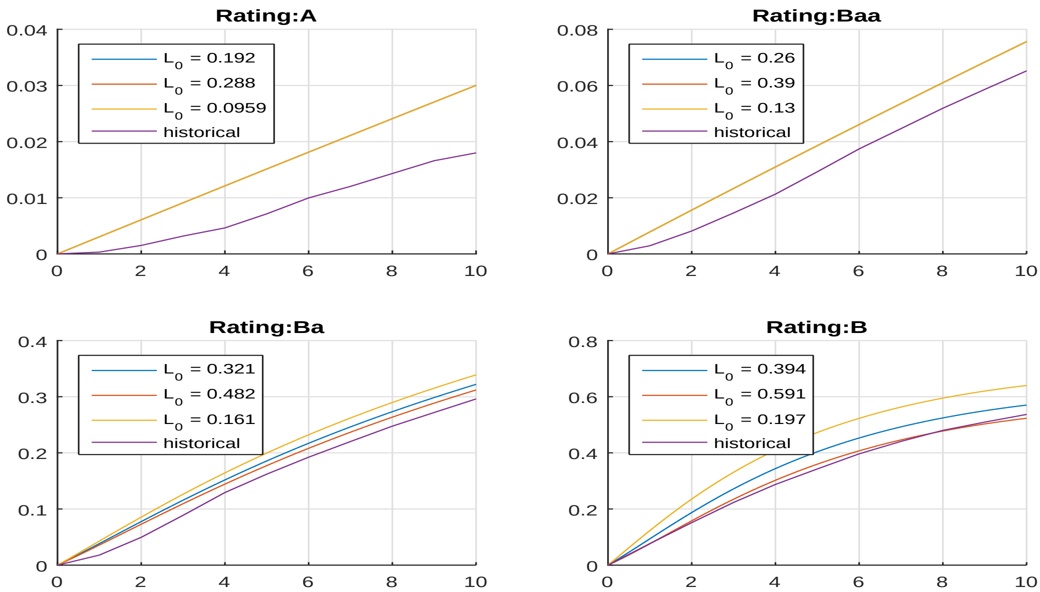

Figure A.7: Sensitivity of the intensity model for the initial leverage

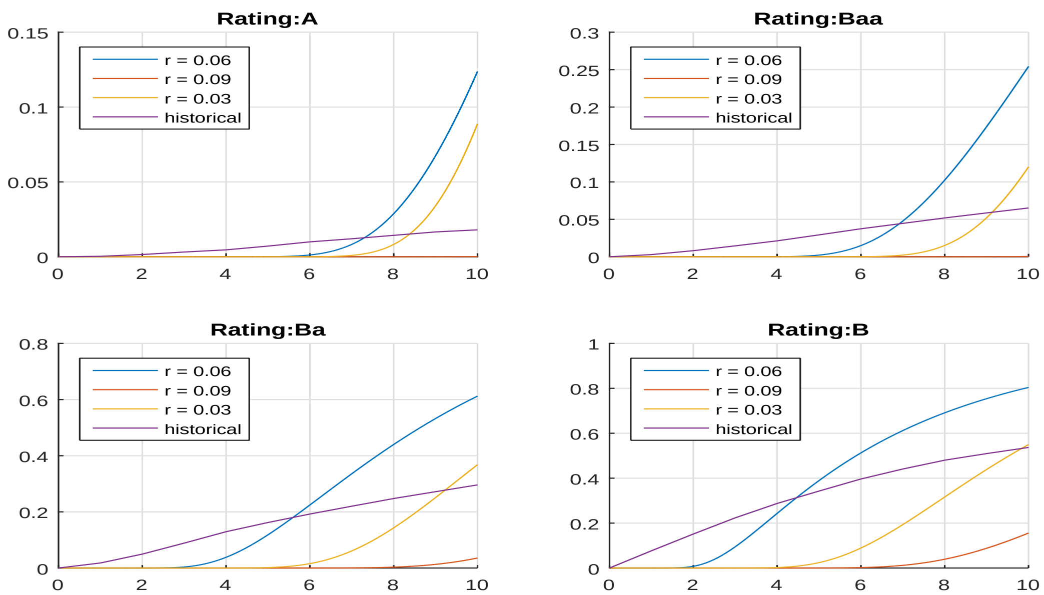

Figure A.8: Sensitivity of the CDG model for the interest rate

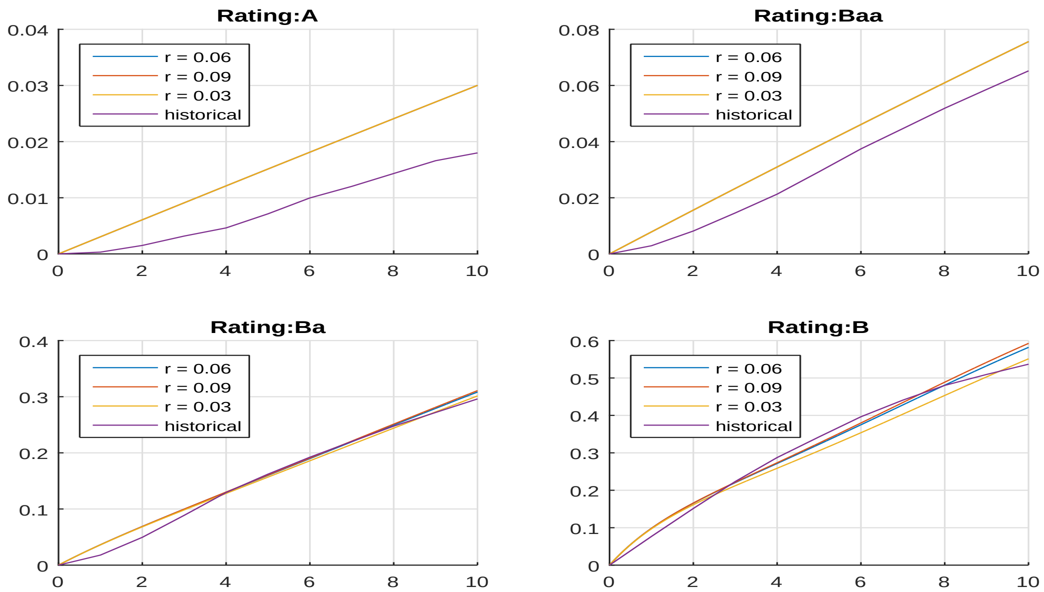

Figure A.9: Sensitivity of the intensity model for the interest rate

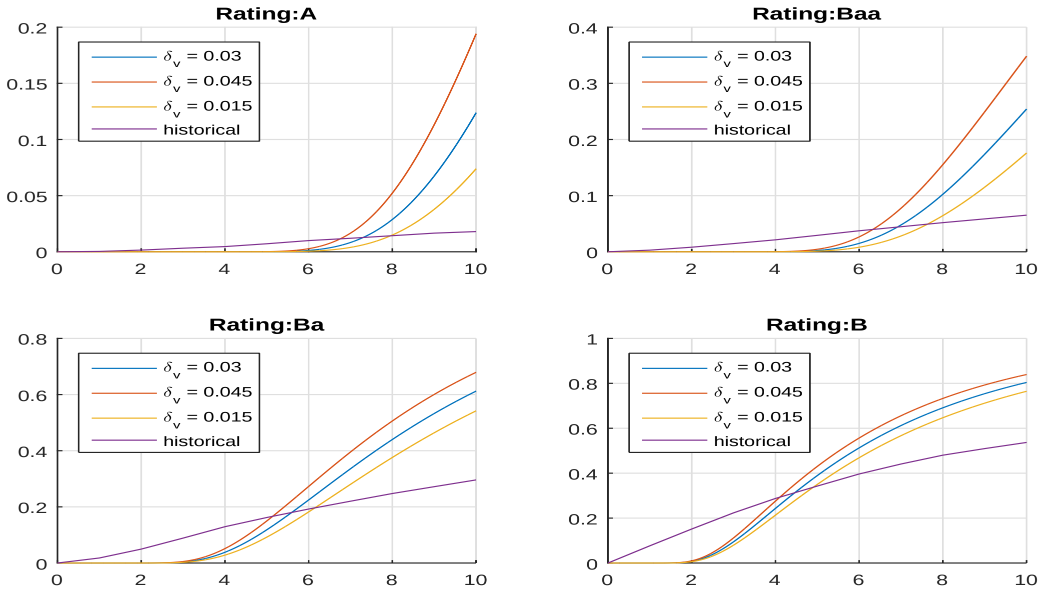

Figure A.10: Sensitivity of the CDG model for the pay out rate

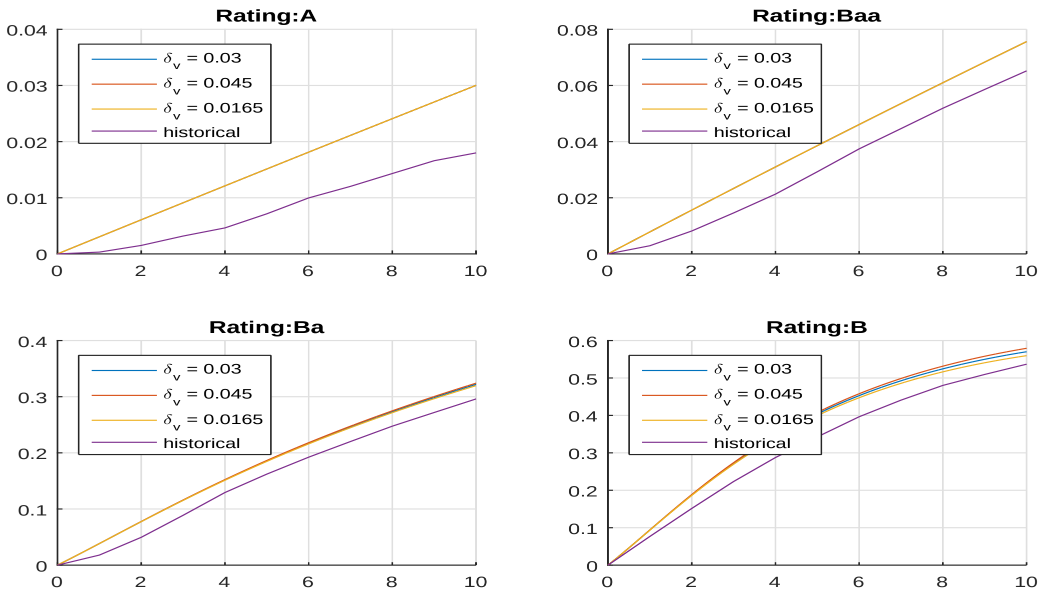

Figure A.11: Sensitivity of the intensity model for the pay out rate

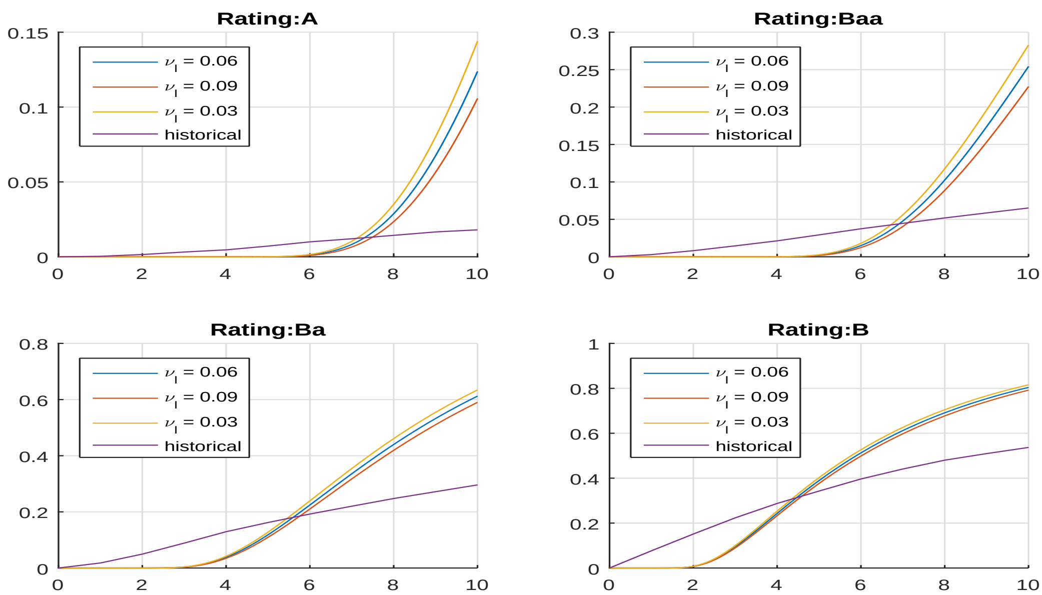

Figure A.12: Sensitivity of the CDG model for parameter ν_l

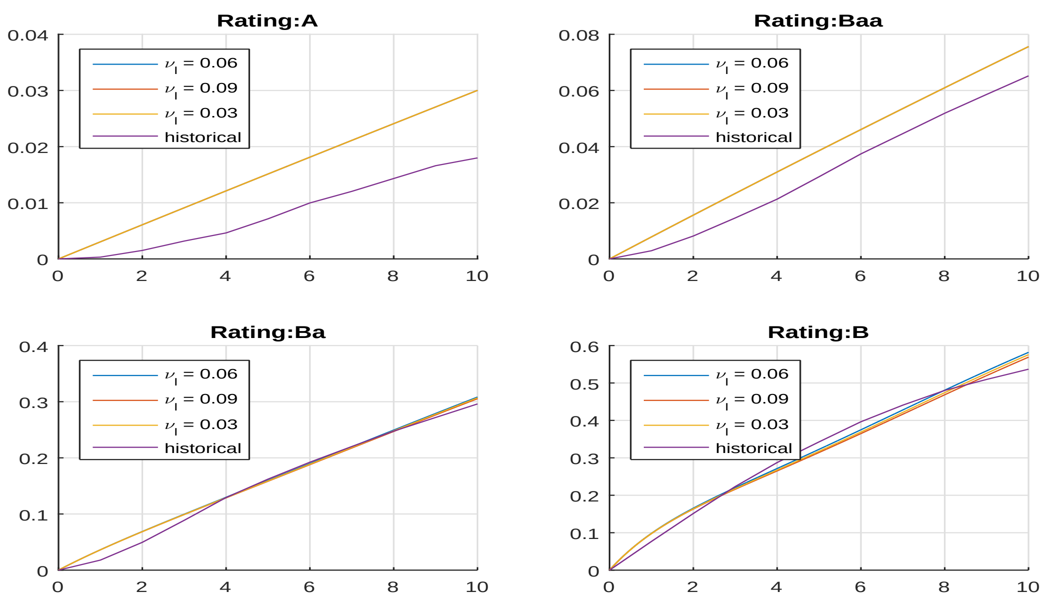

Figure A.13: Sensitivity of the intensity model for parameter ν_l

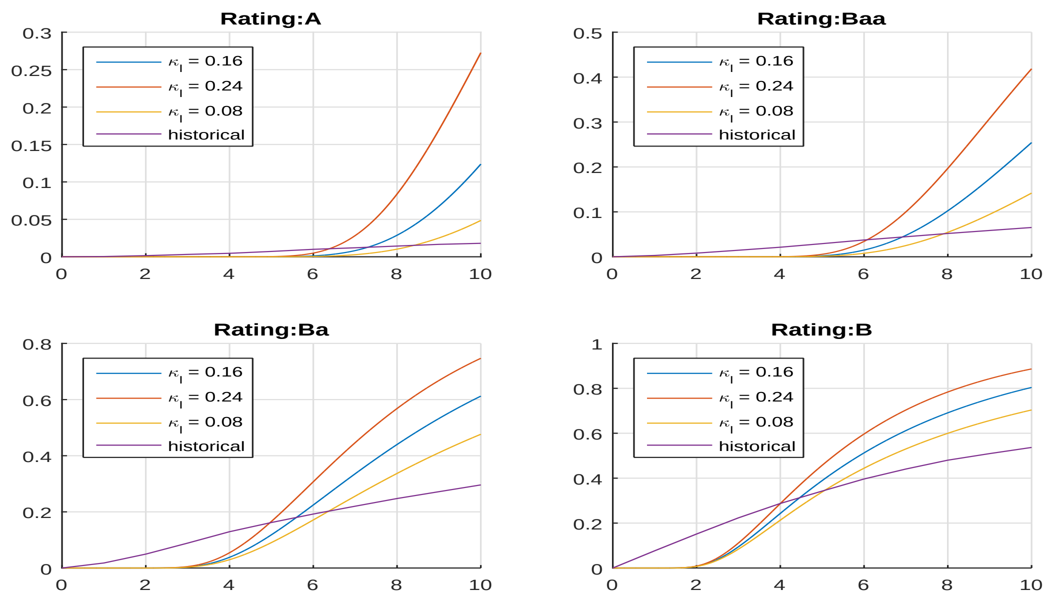

Figure A.14: Sensitivity of the CDG model for parameter κ_l

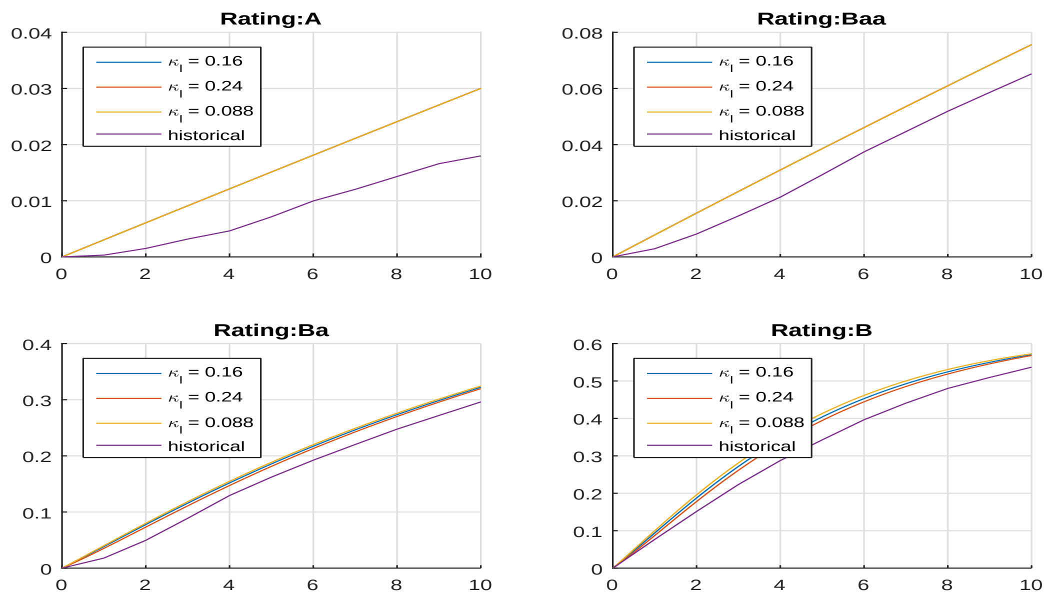

Figure A.15: Sensitivity of the intensity model for parameter κ_l

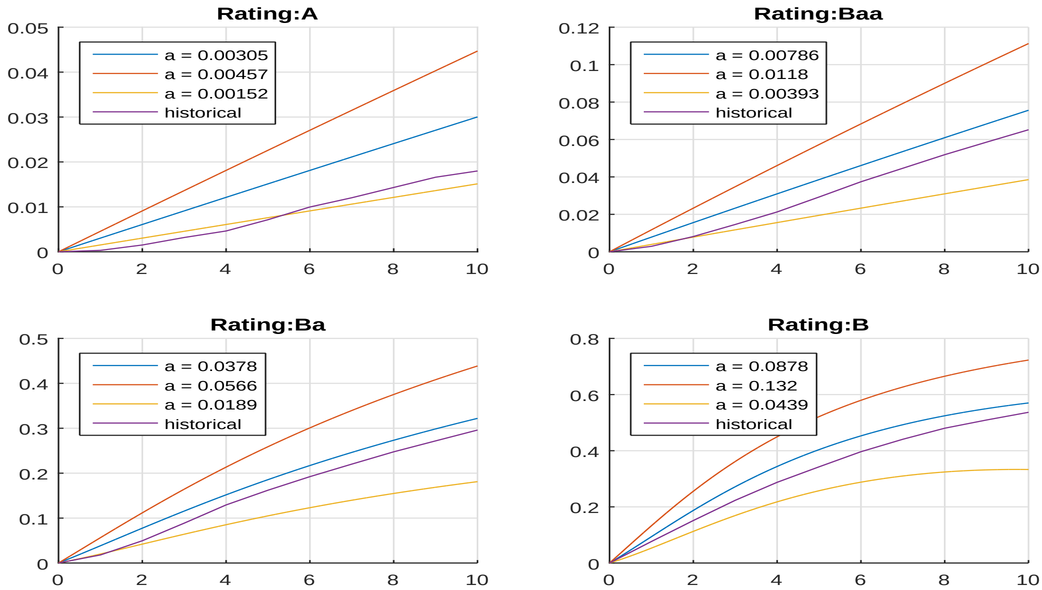

Figure A.16: Sensitivity of the intensity model for parameter a

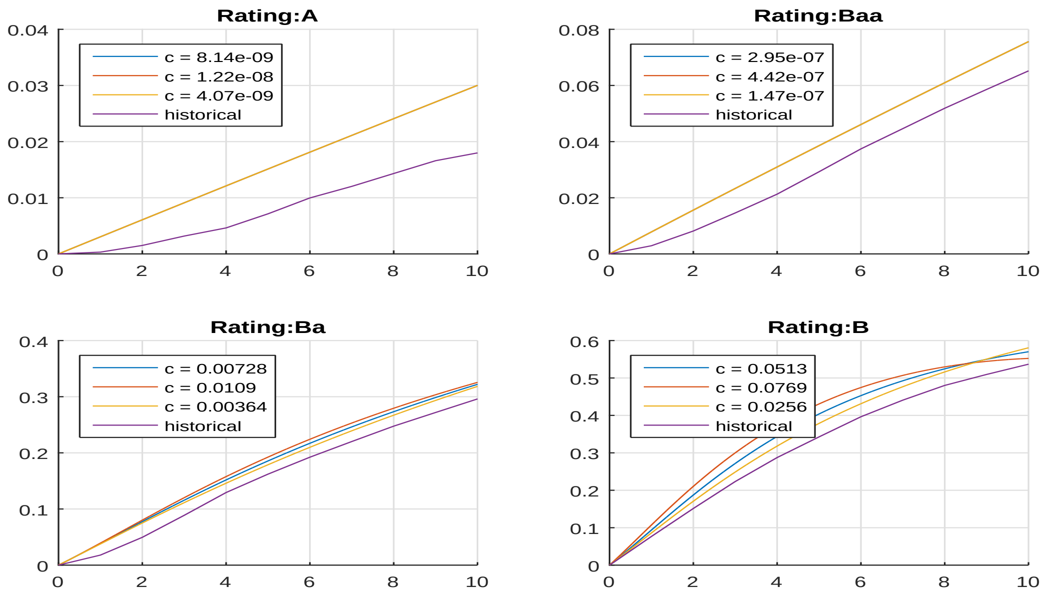

Figure A.17: Sensitivity of the intensity model for parameter c

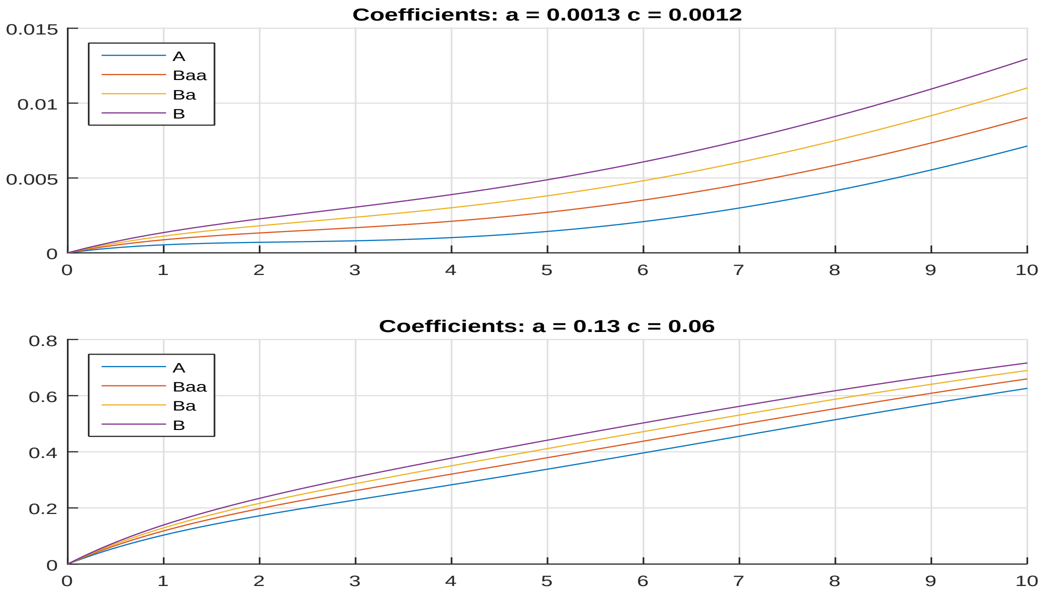

Figure A.18: Sensitivity of the intensity model for parameter c

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