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Slope number of graphs with bounded degree

BSc thesis

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Contents

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1 Introduction

Drawing of graphs is a popular topic in mathematics. It has practical applications in engineering, network research, traffic control, computer networks and it also raises interesting theoretical questions.

A *straight-line* drawing of a graph is a layout of its vertices in the plane such that the vertices are distinct points, and the line segments connecting the corresponding point pairs are not passing through any other point that represents a vertex. In this thesis we will only consider straight-line drawings. These drawings can be measured in different ways.

The *crossing number* of a drawing is the number of pairs of edges that cross each other. If a drawing with no crossings exists, we call the graph planar. Kuratowski gave a nice characterization of planar graphs: A finite graph is planar if and only if it does not contain a subgraph that is a subdivision of K_5 or $K_{3,3}$ [6].

Angular resolution is a measure of the sharpest angle in a graph drawing. If a graph has vertices with high degree, then it necessarily will have small angular resolution. Malitz and Papakostas showed that every planar graph with maximum degree d has a planar drawing whose angular resolution is at worst an exponential function of d, independent of the number of vertices in the graph [7].

The *slope number* of a graph G is the smallest number k such that G has a straight-line drawing using k distinct edge slopes. It was introduced by Wade and Chu [10], who showed that the slope number of K_n is n if $n > 2$. It is hard to tell the slope number of a graph in general. It has been shown that it is NP-complete to decide whether a graph has slope number two [3].

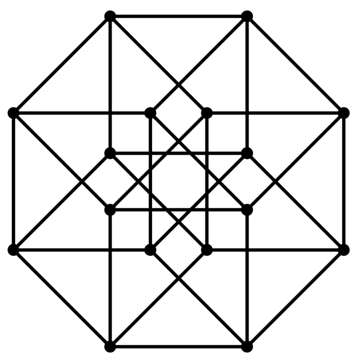

Figure 1: A drawing of the hypercube using 4 slopes.

We might be able to determine the slope number if we restrict our graph in some way. Obviously, if the maximal degree is d, then the slope number is at least $\frac{d}{2}$, so it is natural to consider graphs with bounded degree. We can also ask the slope number for trees, planar graphs, Hamiltonian graphs, etc.

The main topic of my thesis is the slope number of graphs. I will prove some new result concerning 3-regular Hamiltonian graphs.

1.1 Know results

The slope number of graphs with bounded degree is an active topic in mathematics. Let us say that the degrees are bounded by d. In recent years the following results have been established.

Theorem 1.1 (Barát et al. [1], Pach and Pálvölgyi [9]). *If $d > 4$, then the slope number can be arbitrary large.*

Theorem 1.2 (Keszegh et al. [5]). *If $d = 3$, then the slope number is at most 5.*

Theorem 1.3 (Mukkamala and Pálvögyi [8]). *If $d = 3$, then the slope number is at most 4, and we can always use the horizontal, vertical and two diagonal slopes.*

Unfortunately this last theorem had some mistakes in its proof. The idea of the proof is to draw parts of the graphs first and then build up the drawing from the smaller parts. This idea only worked for graphs with at least 18 vertices. For most of the graphs with less then 18 vertices they used the following theorem.

Theorem 1.4 (Engelstein [2]). *For all 3-regular 3-connected Hamiltonian graphs the slope number is at most 4.*

This theorem only states that 3-regular, 3-connected Hamiltonian graphs can be drawn with *some* set of four distinct slopes. It does not prove that it can be done with the horizontal, vertical and two diagonal slopes.

To fix this we will improve these results on 3-regular Hamiltonian graphs with the following theorem.

Theorem 1.5. *Let G be a simple 3-regular Hamiltonian graph. If G is not K_4 or the prism graph, then the slope number is at most 3, where the prism graph is the graph on Figure 2. Moreover, we can choose integer lengths for the edges.*

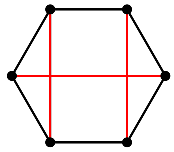

Figure 2: The prism graph.

In this theorem we did not pick a set of slopes to be used in the drawing. The reason for this is the following.

Lemma 1.6. *If we can draw a graph with 3 different slopes, then we can draw it with any set of 3 different slopes.*

Proof. There is an affine transformation that takes the original slopes into the new ones. This transformation can be used on the drawing. Since a set of parallel lines remain parallel after an affine transformation, the new drawing is a straight-line drawing with the required set of slopes. □

1.2 Definitions

From Lemma 1.6 we see that we can choose 3 arbitrary slopes for our proofs. Although Theorem 1.3 requires the horizontal, vertical and diagonal slopes, we will always use the slopes that are defined by 3th roots of unity. So the slopes are 0, $\sqrt{3}$ and $-\sqrt{3}$, and we call these the *basic slopes*.

If it leads to no confusion, we will make no distinction between the vertices and the corresponding points in the drawing, and similarly between the edges and segments.

There are drawings of Hamiltonian graphs in the thesis. In most of them the black edges are representing the Hamiltonian cycle and the rest of the edges are red.

Let the following names define the 6th roots of unity:

- V_{East}: $(1, 0)$
- $V_{NorthEast}$: $(\frac{1}{2}, \frac{\sqrt{3}}{2})$
- $V_{NorthWest}$: $(\frac{-1}{2}, \frac{\sqrt{3}}{2})$
- V_{West}: $(-1, 0)$
- $V_{SouthWest}$: $(\frac{-1}{2}, \frac{-\sqrt{3}}{2})$
- $V_{SouthEast}$: $(\frac{1}{2}, \frac{-\sqrt{3}}{2})$

We say that point A lies east to B if A-B is a positive constant times V_{East}. Similarly we use the other defined directions.

2 Lemmas and ideas

In this section we will prove some useful lemmas and explain some ideas. We use them later to prove Theorem 1.5.

Definition 2.1. *Let G be a 3-regular Hamiltonian graph. We can divide the edges of G into two groups. The first group is the edges of the Hamiltonian cycle. We will call these* cycle edges*. Since G is 3-regular, the rest of the edges form a perfect matching. We will call these* matching edges*. Define the* pair *of a vertex as the other vertex of the matching edge it is on.*

The main idea behind our proof is the following lemma.

Lemma 2.2. *Let G be a 3-regular Hamiltonian graph. If we delete an edge from the Hamiltonian cycle, the remaining graph can be drawn with the 3 basic slopes. Furthermore, we can draw the matching edges to be horizontal, and we can prescribe the y-coordinates of these edges to be any set of different numbers.*

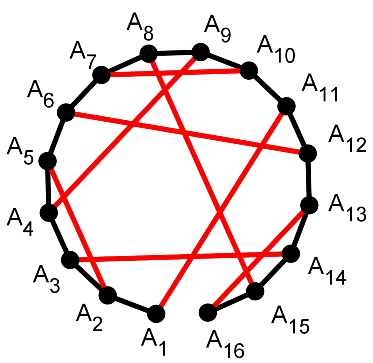

Figure 3: A 3-regular hamiltonian graph with a deleted cycle edge.

Proof. Since we deleted an edge from the cycle, a Hamiltonian path remains. Let us name the vertices following the path, $A_1, ..., A_k$ (see Figure 3).

The y-coordinates are given for the matching edges. Since each vertex is incident to exactly one matching edge, we know the y-coordinate for each vertex. Let us call the y-coordinates $y_1, ..., y_k$. If we place the vertices with the prescribed y-coordinates, then automatically the matching edges will be horizontal, and since we assigned different coordinates to the edges, they will not go through other vertices.

We will calculate $x_1, ..., x_k$ in a recursive way. x_1 is arbitrary. If $i > 1$, then

$$x_i = x_{i-1} + |(y_i - y_{i-1})| \cdot \frac{1}{\sqrt{3}} \tag{2.1}$$

From the equation we see that the slope of the edge A_iA_{i+1} is either $\sqrt{3}$ or $-\sqrt{3}$. This means that all edges have correct slopes. We need to check that the edges do not go through vertices. We have seen that the matching edges do not cause problems. The x-coordinates of the vertices are monotone increasing along the Hamiltonian path, so the cycle edges are also good. □

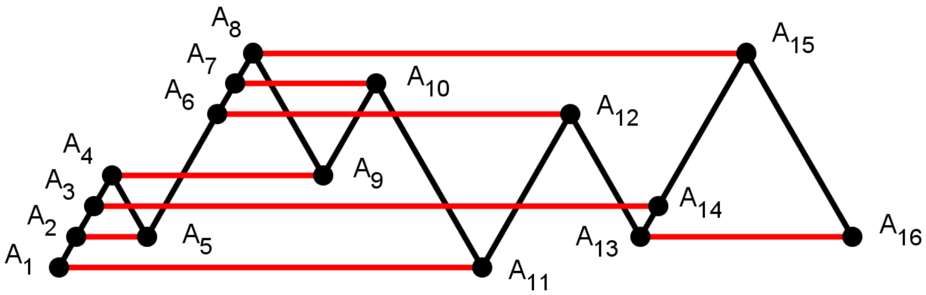

Figure 4: A solution provided by Lemma 2.2.

Remark 2.3. *It is important to note that if the y-coordinates we choose are multiples of $\sqrt{3}$, then the lengths of the edges are integers.*

Proof. From Equation (2.1) we can see that if the y-coordinates are multiples of $\sqrt{3}$, then $x_i - x_{i-1}$ is integer for every i. From this it follows that $x_i - x_j$ is integer for every i and j. So the horizontal edges have integer length.

The length of A_iA_{i+1} is

$$\sqrt{(x_i - x_{i+1})^2 + (y_i - y_{i+1})^2} = \sqrt{((y_i - y_{i+1}) \cdot \frac{1}{\sqrt{3}})^2 + (y_i - y_{i+1})^2} = 2 \cdot \frac{|y_i - y_{i+1}|}{\sqrt{3}}$$

Since the y-coordinates are multiples of $\sqrt{3}$, this is an integer. Hence all of the edges have integer length. It is easy to see that we draw the graph onto a triangular grid. □

Remark 2.4. *Another important observation is that if we follow the Hamiltionian path in the drawing, we always go in either southeast or northeast direction. In other words, for every $i < j$ pair there are $b_{i,j}$ and $c_{i,j}$ non-negative numbers such that $A_j = A_i + b_{i,j} \cdot V_{NorthEast} + c_{i,j} \cdot V_{SouthEast}$.*

Proof. If $j = i + 1$, this is trivial. A_{i+1} has strictly bigger x-coordinate than A_i therefore A_i has to be either to the southeast or to the northeast of A_i.

If $j > i + 1$, then $A_j - A_i = (A_j - A_{j-1}) + (A_{j-1} - A_{j-2}) + ... + (A_{i+1} - A_i)$. We have seen that the parts of the sum are good, so the sum is also good.

We can also see that $b_{i,j} = b_{i,i+1} + b_{i+1,i+2} + ... + b_{j-1,j}$ and similarly $c_{i,j} = c_{i,i+1} + c_{i+1,i+2} + ... + c_{j-1,j}$. □

As we can see, Lemma 2.2 almost proves Theorem 1.5. The only problem is the deleted edge. It might have a wrong slope and it can also go through other vertices. Unfortunately it is not too easy to fix this problem. We will divide the graphs into groups and solve them in different ways. The following lemmas and ideas will help us with some of the cases.

Definition 2.5. *Let G be a 3-regular graph which contains a triangle. We can delete the triangle from the graph and replace it with a vertex, and then connect the vertex to the vertices the triangle was connected to. We will call this process the* contraction *of the triangle (see Figure 5).*

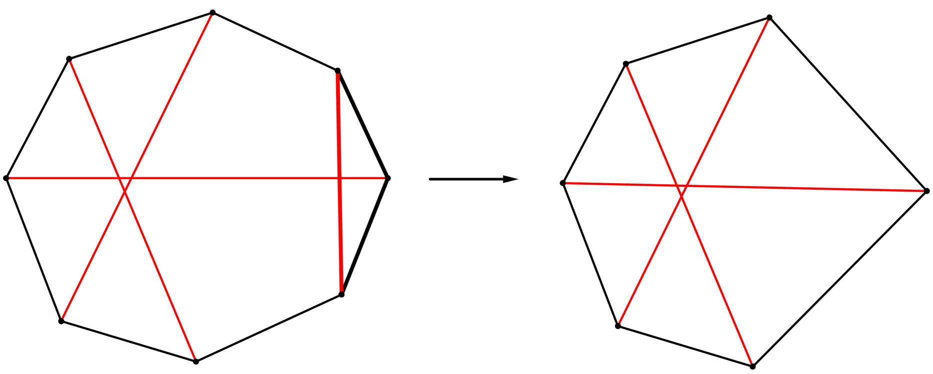

Figure 5: Contraction of a triangle.

Definition 2.6. *Let G be a 3-regular graph which contains a triangle. Each vertex of the triangle is incident to an edge which is not in the triangle. If these 3 edges are independent, then we call the triangle* contractible. *If they are not, we will call it* non-contractible.

The meaning behind the definition is the following. It will be useful to us to contract the triangle into one point. If the three edges are independent, then the new graph we get is simple, but if they are not independent, it will not be simple, since there will be a multiple edge.

Lemma 2.7. *Contraction of a* contractible triangle *preserves 3-regularity and Hamiltonicity.*

Proof. The degrees of the unchanged vertices remain 3. The new vertex has degree 3 since the triangle has 3 vertices and each vertex adds an edge to the new vertex. So the new graph is 3-regular.

The three independent edges connected to the triangle form a cut, so the Hamiltonian cycle contains exactly two of them. If we contract the triangle, these two edges will meet in the new vertex. Hence we lose two edges from the Hamiltonian cycle, but the rest of the edges of the cycle will from a Hamiltonian cycle in the new graph. □

Definition 2.8. *If we have a drawing, the vertices can be divided into groups. Two vertices are in the same group if the edges incident to them are in the same direction. We will call this property the* type *of the vertex.*

Definition 2.9. *Let G be a graph. We say that that a drawing of G using the 3 basic slopes is* good *if none of the vertices has any of the types shown in Figure 6. We will call these types* bad *types and the rest of the types will be called* good.

With the help of contraction, we will use induction in some part of our proof. If we have a triangle we will contract it, draw the new graph and then insert the triangle back in the graph. The reason we defined good and bad vertex types is that if we have a vertex drawn with a good type, then there is an easy way to insert the triangle back in its palce.

Lemma 2.10. *Let G be a Hamiltonian, 3-regular graph that has a contractible triangle. Let G' be the graph that we get, when we contract the triangle. Suppose that we have a* good *drawing of G'. Then we can create a* good *drawing of G.*

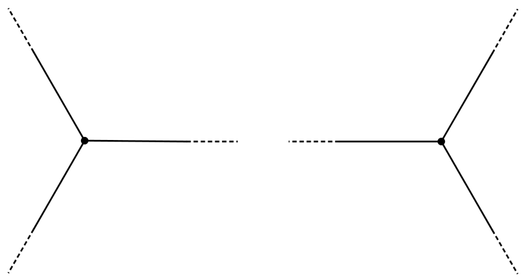

Figure 6: Bad vertex types.

Proof. We will modify the drawing of G' to get a drawing for G. Let A be the vertex that we made from the triangle. Since it is a *good* drawing, there are two edges that are connected to A and the angle of their slopes is $60°$. We make a new vertex on both of these edges at a fixed distance from A. We connect these new vertices. Let B and C denote the two new vertex. Since $BAC\angle = 60°$ and $AB = AC$, the ABC triangle is equilateral. This means that the new BC edge has a good slope. We have to check that BC does not go through other vertices. Since the graph has finitely many vertices, A has a small neighbourhood that does not contain other vertices. We can choose the length of AB to be so small, that the ABC triangle will be in the neighbourhood. So the BC edge cannot go through other vertices.

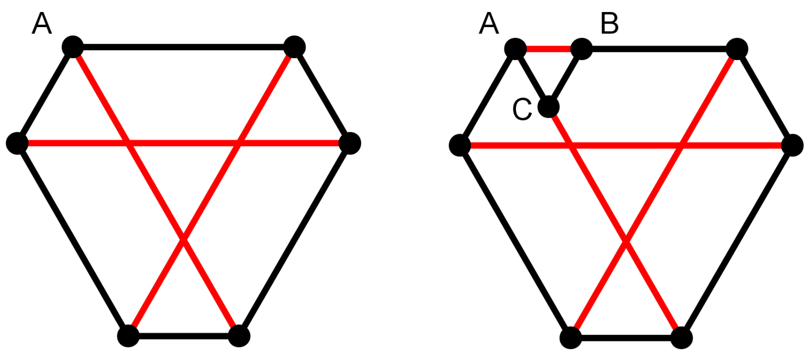

Figure 7: Inflation of a triangle.

It is easy to see that the new drawing is *good*. The type of the old vertices did not change. The two new vertices are part of an equilateral triangle, so they cannot have a bad type.

If the edges in the original drawing had integer length they can be integer in the new drawing too. If we chose the length of AB to be rational, all of the edges will have rational length. Therefore, we can simply enlarge the drawing by an integer factor to receive integer lengths. □

Since Lemma 2.10 requires a *good* drawing, we will modify Theorem 1.5 to be slightly stronger:

Theorem 2.11. *Let G be a simple 3-regular Hamiltonian graph. If G is not K_4 or the prism graph, then the slope number is at most 3. Moreover, we can choose integer lengths for the edges, and we can also make it a* good *drawing.*

Remark 2.12. *It is worth to note that Lemma 2.2 gives us a* good *drawing.*

Proof. The $A_{i-1}A_i$ edge meets the A_iA_{i+1} edge either in 60° or 180°. In either case A_i cannot have a bad type. We only have two edges at A_0 and A_k, but that two meet in 60° so even if we extend the drawing, they will have a good type. □

Lemma 2.13. *K_4 and the prism graph cannot be drawn with 3 slopes.*

Proof. Let us name the vertices of K_4 by A, B, C and D. A triangle can only be drawn as an equilateral triangle. If A and C are already drawn in the plane, there are only two points that form an equilateral triangle with them. B and D must be that two point, but then the slope of the BD edge will be wrong.

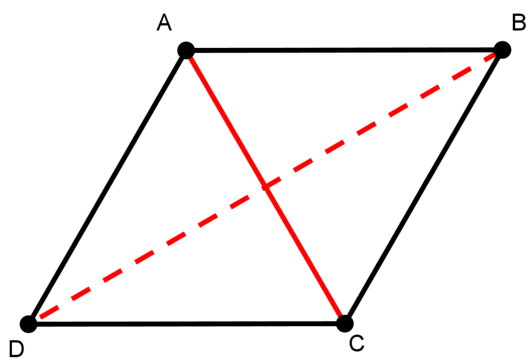

Figure 8: K_4 cannot be drawn using 3 slopes.

Let us name the vertices of the prism graph A, B, C, D, E, and F, such that ABC and DEF are a triangles. Suppose we have a solution. We can assume that AB is horizontal and C is to the northeast of A without loss of generality. Since the graph is symmetric we can also assume that D lies to the northwest or to the west of A. Figure 9 shows the two cases. If D lies to the northwest, E must lie to the northeast of B. If it lies to the west, then E must lie to the southwest of B. In either case we cannot draw F properly. DEF has to be an equilateral triangle, which leaves two possible position for F. It can be seen from the Figure 9 that neither is good. Either FC will have a wrong slope or it will go trough vertices. □

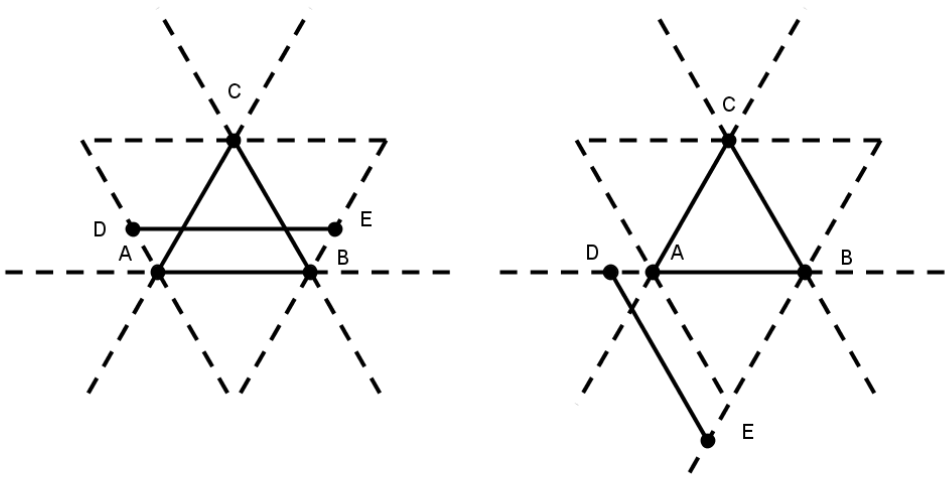

Figure 9: The prism graph cannot be drawn using 3 slopes.

3 Proof of main theorem

As we have seen in Lemma 2.2, we have an algorithm which is almost good. To prove Theorem 2.11 we will consider different cases. In each case we will use Lemma 2.2 to draw most of the graph, and draw some vertices in a different way.

3.1 Triangle-free graphs

First we will prove that Theorem 2.11 holds for triangle-free graphs. If we have a triangle, we must draw it as an equilateral triangle. It is a big constraint so we will deal with them in a different way.

The following lemma proves that we can find a special part of the graph if it is triangle-free. We will draw this part differently.

Lemma 3.1. *Let G be a 3-regular, Hamiltonian, triangle-free graph, with at least 10 vertices. We can find a Hamiltonian cycle and A, B, C, D, E, F, X, Y vertices with the following properties:*

1. *A, B, C, D, E, F, X and Y are distinct.*
2. *A, B, C, D, E, F are consecutive vertices on the Hamiltonian cycle in this order.*
3. *CY and DX are matching edges.*
4. *The order of C, D, X, Y along the cycle is $CDXY$.*

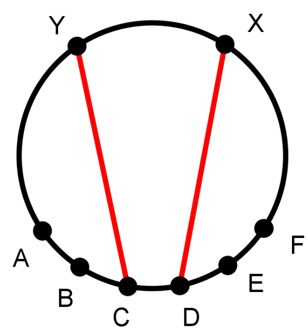

Figure 10: The required configuration.

Proof. If we have already chosen C and D, and CD is a cycle edge, we can trivially choose A, B, E, F, X, Y to satisfy condition 1, 2 and 3. We choose A, B, E and F to be the vertices that precede and follow C and D on the cycle. We choose X and Y to be the pairs of C and D. Since the graph is triangle-free, these will be different vertices. Therefore, the main goal is to satisfy condition 4.

Let us pick a Hamiltonian cycle. There are two cases to consider.

First case: Each vertex is adjacent to the opposite vertex in the cycle. In this case we will use another Hamiltonian cycle instead. We can simply switch two opposite cycle edges with two matching edges as Figure 11 shows. We can use the second case on the new Hamiltonian cycle.

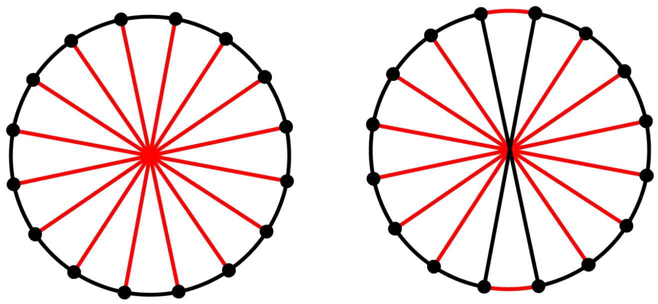

Figure 11: Redefining the Hamiltonian cycle.

Second case: Not every vertex is adjacent to the opposite vertex. We can pick a matching edge that divides the Hamiltonian cycle into two parts that are not equal in size. Denote by $P_2, P_3, ...P_r$ the vertices of the bigger part, starting from P_1 and going along the cycle. Similarly denote by $Q_2, Q_3, .., Q_s$ the vertices in the smaller (see Figure 12). We pick the smallest l such that P_l is connected to P_j by a matching edge. There is an l like that because there are more P_i's than Q_i's. Denote by Q_k the pair of P_{l-1}. We will choose the vertices the following way: $C = P_l$, $D = P_{l-1}$ and as we have seen, we can chose A, B, E, F, X, Y to satisfy condition 1, 2 and 3.

Let us check condition 4. P_l and P_j divides the cycle into two part. Q_k is in the same part as P_{l-1}, since in the other part there are only P_i's. So the order of P_l, P_{l-1}, P_j and Q_k is appropriate. □

We will even need a slightly stronger lemma.

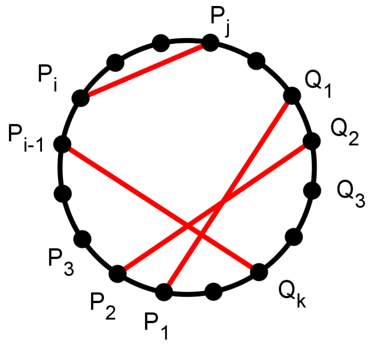

Figure 12: The labeling of the graph.

Lemma 3.2. *Let G be a 3-regular, Hamiltonian, triangle-free graph, with at least 10 vertices. We can find a Hamiltonian cycle and A, B, C, D, E, F, X, Y vertices with the following properties:*

1. *A, B, C, D, E, F, X and Y are distinct.*
2. *A, B, C, D, E, F are consecutive vertices on the Hamiltonian cycle in this order.*
3. *CY and DX are matching edges.*
4. *The order of C, D, X, Y along the cycle is $CDXY$.*
5. *At least one of AE or BF is not a matching edge.*

Proof. As in the previous lemma, we will choose C and D, which will give us A, B, E, F, X, Y, and we will have to check condition 4 and 5.

First we choose the vertices according to the previous lemma. We know that the first four condition is met, so we only have consider the case when the fifth is not (Figure 13). In this case continue the naming of the vertices with G, H and I along the cycle. G, H and I can be the same as X or Y, but they are not A, B, C, D, E or F, since we have at least 10 vertices. We will choose new vertices to satisfy this lemma, and to avoid confusion we will mark them with an apostrophe.

First case: If $G \neq X$ then we choose $C' = F$ and $D' = G$, and the rest to satisfy condition 1, 2 and 3.

B and F divides the cycle into two parts. One part only consists C, D and E, so if condition 4 is not true, then G is adj to one of them. G is not X, Y or A which means its pair cannot be C, D or E.

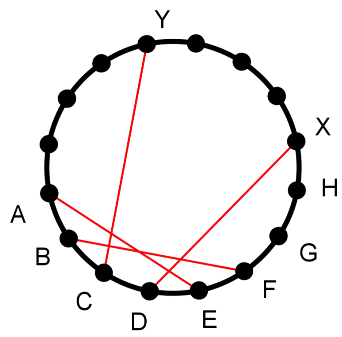

Figure 13: Condition 5 of Lemma 3.2 is not met in this graph.

$B' = E$ and it is adjacent to A. A is not I, so $B'F'$ is not a matching edge. So condition 5 is met.

Second case: If $G = X$, then we choose $C' = G$ and $D' = H$, and the rest to satisfy condition 1, 2 and 3.

H cannot be adjacent to E or F since they are adjacent to A and B. This means that condition 4 is met.

If condition 5 would not be true, A and B would come after H on the cycle. It would mean that we only have 8 vertices. Since we have at least 10, it is not possible. □

Now we can prove Theorem 2.11 for triangle-free graphs. We will use the graph in Figure 14 to illustrate the process.

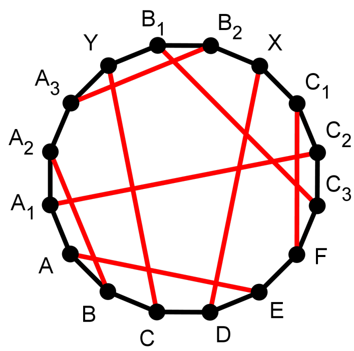

Figure 14: A triangle-free graph with A,B,C,D,E,F,X,Y chosen.

We use the Lemma 3.2 to choose A, B, C, D, E, F, X, Y vertices. Let $A_1, A_2, ...A_l$ be the vertices between A and Y. Let $B_1, B_2, ...B_m$ be the vertices between Y and

X. Let $C_1, C_2, ...C_n$ be the vertices between X and F (see Figure 14). This is a point where we use condition 4 from Lemma 3.2. If the order of $CDXY$ would not be right, the A-s, B-s and C-s would not be distinct vertices.

First we will draw the graph using Lemma 2.2. The deleted edge is CD. We need to assign y-coordinates to the matching edges.

Lemma 3.3. *We can choose the y-coordinates to satisfy the following properties:*

1. *CY and DX have smaller y-coordinate than the rest.*
2. *B has smaller y-coordinate than A.*
3. *E has smaller y-coordinate than F.*
4. *All y-coordinates are multiples of $\sqrt{3}$.*
5. *The y-coordinate of CY and DX are consecutive multiples of $\sqrt{3}$*

Proof. We will assign the $0, \sqrt{3}, 2 \cdot \sqrt{3}, ...$ numbers to the edges. CY will have 0 and DX will have $\sqrt{3}$. From condition 5 in Lemma 3.2 we know that either AE or BF is not a matching edge. The reason to require condition 5 in Lemma 3.2 was to be able to satisfy condition 2 and 3 at the same time in this lemma.

If AE is not a matching edge, then the edge coming from E receives $2 \cdot \sqrt{3}$ and the edge coming from B receives $3 \cdot \sqrt{3}$. The rest of the coordinates can be assigned arbitrarily. Since AE is not an edge, this means that A will have a bigger y-coordinate than B. Condition 3 is also true since the only two smaller y-coordinates than E has, were assigned to the CY and DX edges.

If BF is not a matching edge, then the edge coming from B receives $2 \cdot \sqrt{3}$ and the edge coming from E receives $3 \cdot \sqrt{3}$. The rest of the coordinates can be assigned arbitrarily. Since BF is not an edge, this means that F will have a bigger y-coordinate than E. Condition 3 is also true since the only two smaller y-coordinates than B has, were assigned to the CY and DX edges.

□

With the combination of Lemma 2.2 and Lemma 3.3 we get a drawing of the graph using 3 slopes (see Figure 15). We only have to fix the missing CD edge. We delete the C and D vertices and redraw them differently. We draw a ray from B in southeast direction. We have seen in Remark 2.4 that all of the vertices can be written as $B + c_1 \cdot V_{NorthEast} + c_2 \cdot V_{SouthEast}$ where c_1 and c_2 are non-negative. Since

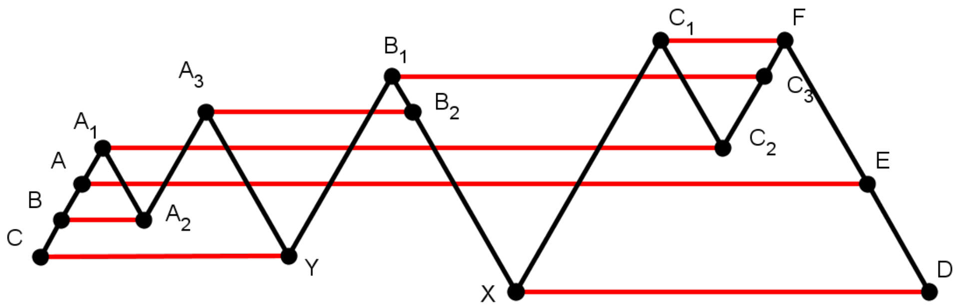

Figure 15: The drawing after the first stage.

A has a bigger y-coordinate than B, c_1 is never zero. It means that none of the vertices can lie southeast to B. We draw a ray from Y in the direction of southwest. Since only X can have smaller y-coordinate than Y, none of the vertices but X can be on this ray. From condition 4 in Lemma 3.2 we know that X was drawn later then Y, so it has a bigger x-coordinate. Therefore, not even X can lie on this ray. We place C in the intersection of the two rays. In a symmetric way we place D southeast of X and southwest of E.

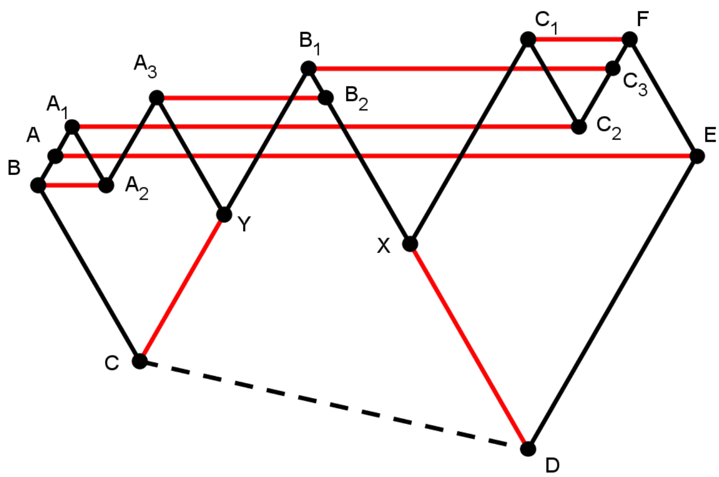

Figure 16: The drawing after redrawing C and D.

As we can see in Figure 16 the CD edge still can have a wrong slope but since C and D are lower the rest of the vertices, it will not go trough any vertices. We will make the CD edge horizontal. Without loss of generality we can assume that

C has a bigger y-coordinate than D. Let d denote the difference of the y-coordinate of C and D. Let m denote the number of vertices between X and Y. We have to consider three cases.

First case: $m > 0$. In this case B_1 exists and there is an edge between Y and B_1. Since Y has smaller y-coordinate than B_1, B_1 lies to the northeast of Y.

We will fix the drawing by translating some of its parts. Since translation preserves directions, the edges inside the translated parts will remain good.

Let us translate C and Y by $d \cdot \frac{2}{\sqrt{3}} \cdot V_{SouthWest}$. This moves C downwards with d, so this will make CD horizontal. Also C remains southwest of Y and Y remains southwest of B_1. We translate B, A, A_1, ... A_l by $d \cdot \frac{2}{\sqrt{3}} \cdot V_{West}$. Since it is a horizontal translation the matching edges will remain good. C will still be southeast of B since $B - C$ changed with $d \cdot (V_{SouthWest} - V_{West}) \cdot \frac{2}{\sqrt{3}} = d \cdot V_{SouthEast} \cdot \frac{2}{\sqrt{3}}$. Similarly Y remains southeast of A_l.

We also have to check that the drawings are *good*. By remark 2.12 we know that the vertices are correct except for C, D, X and Y. $BCY\angle = 60°$, $XDE\angle = 60°$, $A_lYB\angle = 60°$, $B_mXC_1\angle = 60°$, so they are good too.

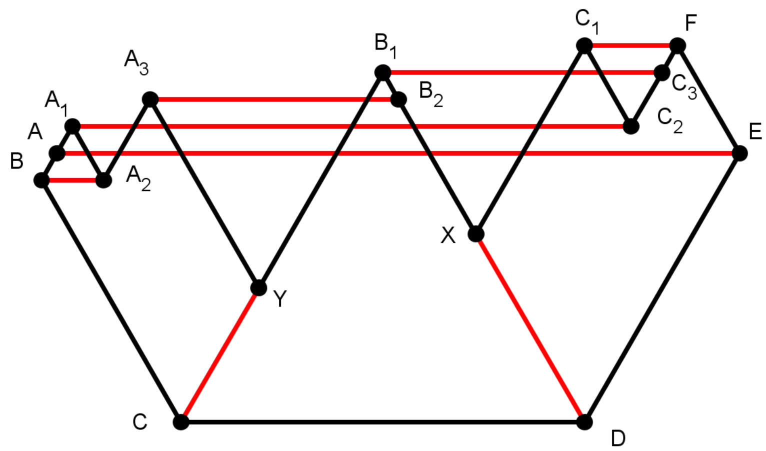

Figure 17: The completed drawing.

Second case: $m = 0$ and $d = \sqrt{3}$, so X and Y are neighbours. If X has a bigger y-coordinate than Y, everything works like in the first case. The only difference is that Y is adjacent to X instead of B_1.

If X has smaller y-coordinate we translate Y and C by $2 \cdot V_{SouthWest}$ and we translate B, A, A_1, ... A_l by $2 \cdot V_{West}$. Similarly to the previous case everything remains good, we just have to check YX and CD. From condition 5 in Lemma 3.3 we know that the y-coordinate of Y and X are consecutive multiples of $\sqrt{3}$. Since Y has a bigger y-coordinate, translating it by $2 \cdot V_{SouthWest}$ will turn YX into horizontal. Since $d = \sqrt{3}$, we know that the y coordinate of C is bigger than the y-coordinate of D by $\sqrt{3}$. We translated C by $2 \cdot V_{SouthWest}$, therefore CD will be horizontal too.

We now have a drawing with 3 slopes but it is not a *good* drawing. The type of X and Y is bad. We can easily fix this:

We translate C, Y, B, A, A_1, ... A_l by $3 \cdot V_{East}$. Notice that there are only horizontal edges between C, Y, B, A, A_1, ... A_l and the rest of the vertices. Since it is a horizontal translation, the slopes of the edges will not change. We only have to check that none of the edges go trough any of the vertices. The structure of the two parts does not change. The only possible problem is that a vertex form one part of the graph moves onto an edge in the other part of the graph. It is easy to see from the x-coordinates of the vertices that this cannot happen.

The new drawing is a *good* drawing. The type of the vertices did not change except for X and Y. Now $XYA_l\angle = 60°$ and $YXC_1\angle = 60°$, therefore X and Y have a good type now.

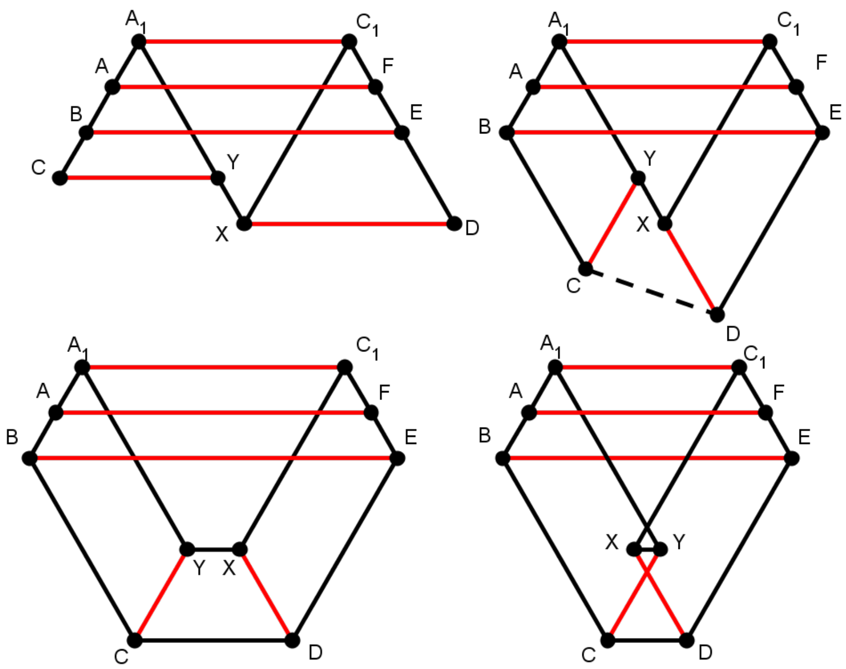

Figure 18: An example to the second case.

Third case: $m = 0$ and $d > \sqrt{3}$. If X has a bigger y-coordinate than Y, everything works like in the first case. If X has smaller y-coordinate then we translate C and Y by $2 \cdot V_{East} + d \cdot \frac{2}{\sqrt{3}} \cdot V_{SouthWest}$. This moves C downwards with d, so this will make CD horizontal. C remains southwest of Y and Y will be southwest of X. We translate B, A, A_1, ... A_l by $d \cdot \frac{2}{\sqrt{3}} \cdot V_{West} + 2 \cdot V_{East}$. Since it is a horizontal translation the matching edges will remain good. C will still be southeast of B since $B - C$ changed with $d \cdot (V_{SouthWest} - V_{West}) \cdot \frac{2}{\sqrt{3}} = d \cdot V_{SouthEast} \cdot \frac{2}{\sqrt{3}}$. Similarly Y remains southeast of A_l. We cannot use this method in the second case since translating Y by $2 \cdot V_{East} + 2 \cdot V_{SouthWest}$ would take Y onto X.

We also have to check that the drawings are *good*. By the remark 2.12 we know that the vertices are correct except for C, D, X and Y. $BCY\angle = 60°$, $XDE\angle = 60°$, $A_lYB\angle = 60°$, $B_mXC_1\angle = 60°$, so they are good too.

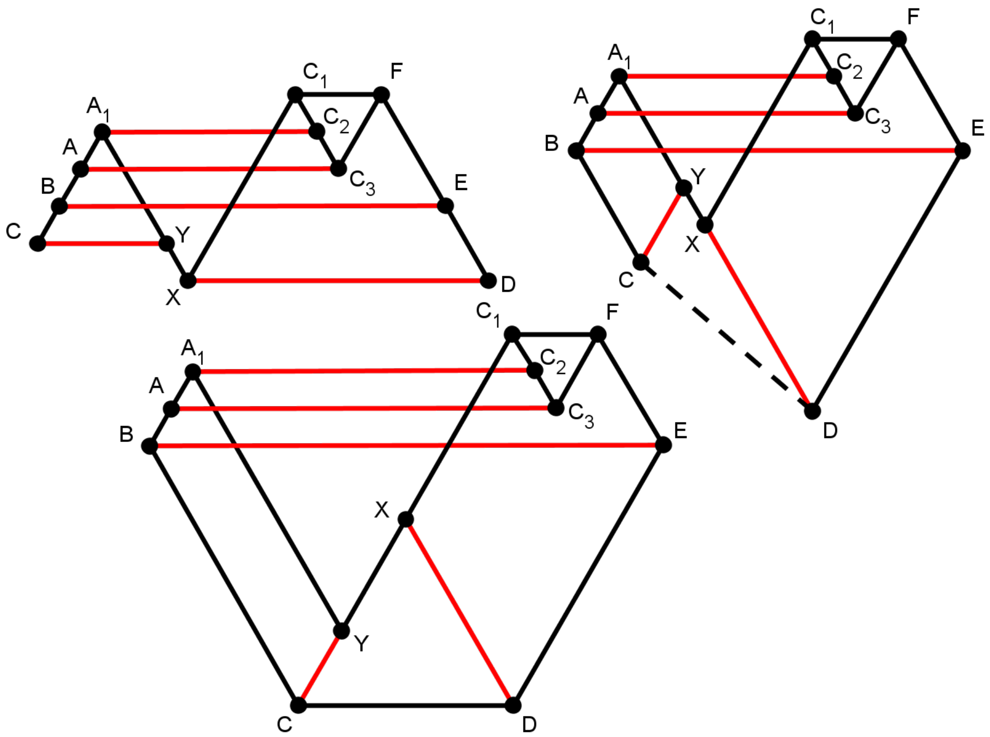

Figure 19: An example to the third case.

In each case the vertices remained on a triangular grid, hence the lengths of the edges are integer.

3.2 Graphs with triangles

3.2.1 Non-contractible triangle

Suppose that we have a triangle but it is non-contractible. By definition the three edges that are connected to the triangle are not independent. There are two cases, first if these three edges have a common vertex. It can only happen if the graph is K_4. As we have seen K_4 cannot be drawn with 3 slopes.

The other case is when only two of the three edges has a common vertex. It means the graph contains the graph we can see in Figure 20.

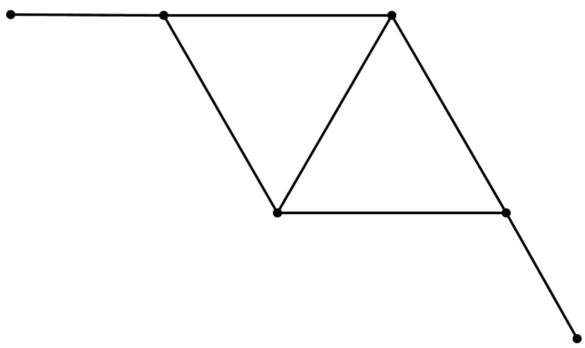

Figure 20: A non-contractible triangle.

Let A, B and C denote the vertices of the triangle. Let D denote the vertex that is connected to two vertices of the triangle. Without loss of generality we can assume that the order of these vertices is $ABCD$ along the cycle. Let us call X the vertex before A on the cycle. Let us call Y the vertex after D on the cycle.

First we delete A, B, D, C from the graph. If we would connect X and Y, we would get a 3-regular Hamiltonian graph. This means, that we can use Lemma 2.2 on the remaining graph. We choose the y-coordinates to be multiples of $\sqrt{3}$ and we choose X to have the smallest y-coordinate. Using Lemma 2.2 we get a drawing. We only have to draw A, B, C and D. We start a ray from X and Y in the southeast direction. Since X has the smallest y-coordinate and Y have the biggest x-coordinate these rays will not go trough any vertices. We choose an y-coordinate for D that is smaller than the y-coordinate of X and draw it on the ray coming from Y. We draw B and C to be at an equal distance from the two rays. We also place C to the west of D, and B to the southwest of D. Since now DBC form an equilateral

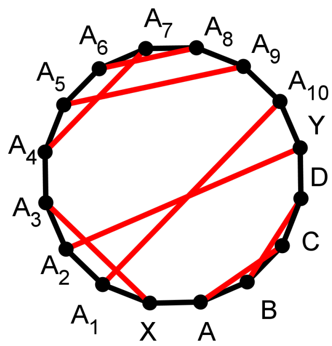

Figure 21: A graph with a non-contractible triangle.

triangle whose altitude is half of the distance between the rays, we can place A on the ray coming from X to form an equilateral triangle with B and C. This gives us an appropriate drawing.

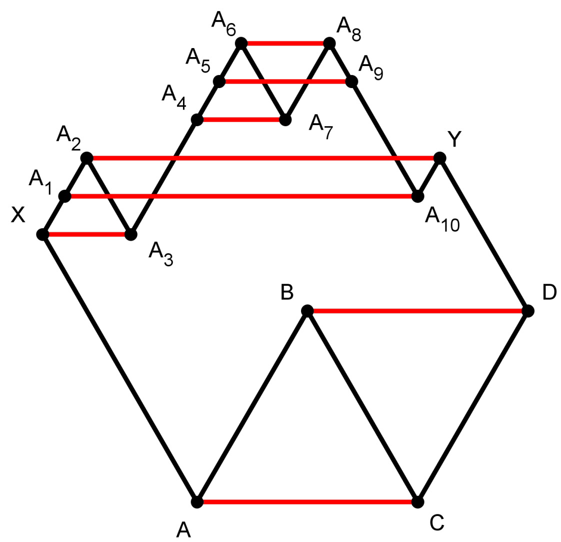

Figure 22: The drawing of Figure 21.

We have to check that the drawing is *good*. By remark 2.12 we only have to check A, B, C and D. Since each part of a triangle, they cannot have a bad type.

If choose the y-coordinate of D to be a multiple of $\sqrt{3}$ then the length of the edges will be integer.

3.2.2 Contractible triangle

We are going to use induction on the number of the vertices. Since the graph is 3-regular we only have to consider graphs with an even number of vertices.

Lemma 3.4. *Theorem 2.11 holds if the number of the vertices is 6 or 8.*

Proof. There is only one 3-regular connected simple graph with 6 vertices that is not the prism graph. There are only five 3-regular connected simple graphs with 8 vertices [11]. All of them are Hamiltonian. Their drawings can be seen in the following figure.

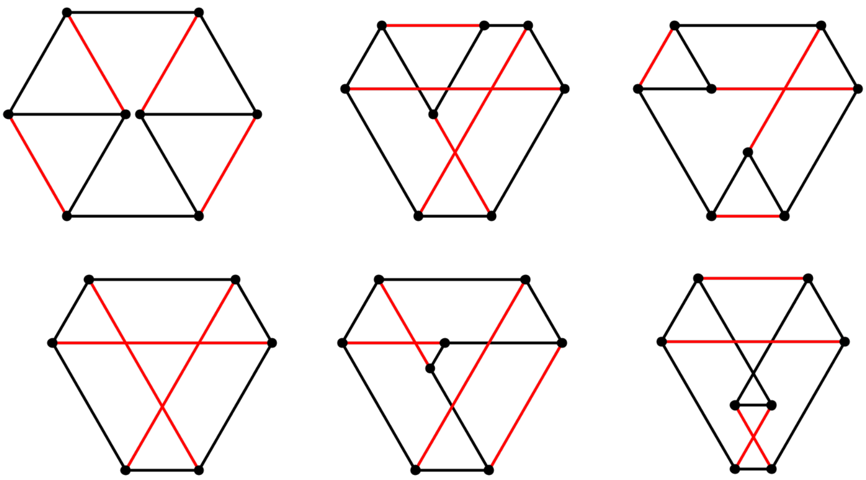

Figure 23: 3-regular connected simple graphs with 6 or 8 vertices.

Notice that all of the drawings are *good*, and the edges can have integer lengths. □

Suppose the graph has more than 8 vertices. Since we have a contractible triangle, we can contract it into one point. By Lemma 2.10 we know that it is enough to draw the smaller graph. By induction we know that if there is a non-contractible triangle, then we have a good drawing for it. If there is no contractible triangle, we can use the other parts of the proof to get a good drawing.

A graph is either triangle-free or it contains a triangle. We proved both cases, hence Theorem 2.11 is true. Theorem 2.11 is stronger than Theorem 1.5, therefore we proved that one too.

4 Open questions and remarks

We have shown that we can have integer lengths for the edges. It is easy to see that we can have integer coordinates instead. If we have a drawing that uses the basic slopes and the edges have integer lengths, we can just multiply the y-coordinates by $\frac{1}{\sqrt{3}}$. Since the y-coordinates were multiples of $\sqrt{3}$, this will make them integer. We have seen previously that the difference of the x-coordinates are integers, thus we can translate the drawing horizontally to get integer coordinates.

We proved that 3-regular Hamiltonian graphs can be drawn with 3 slopes. An interesting question is to determine which graphs can be drawn with 3 slopes? How important is Hamiltonicity?

Lemma 4.1. *The following graph cannot be drawn using 3 slopes.*

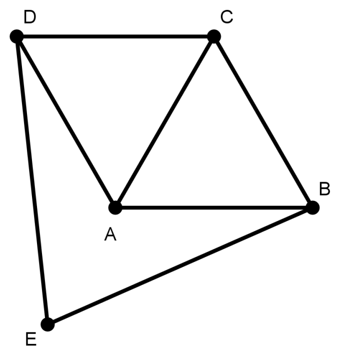

Figure 24: This graph cannot be drawn using 3 slopes.

Proof. Suppose we have a drawing. ABC is a triangle in the graph so they will form an equilateral triangle in the drawing. ACD is also a triangle, so they will also form an equilateral triangle. Since there are only two points in the plane that forms an equilateral triangle with A and C, one on each side of the AC line, the points D and B will lie on different side of the AC line. E is connected to B, but it cannot lie on BA or BC. There is 4 other possible half-line where we can place E. Since none of those intersect the AC line, B and E will be on the same side of the AC line. Similarly E and D will be on the same side of the AC line. It is not possible since D and B must lie on different side of the AC line. □

From this it follows that every graph that contains this graph as a subgraph cannot be drawn using 3 slopes.

Conjecture 4.2. *A 3-regular graph can be drawn using 3 slopes if it is not the prism graph or K_4, and it does not contain the graph in Figure 24 as a subgraph.*

Let d denote the maximal degree of the graph. One of the most interesting open question in this topic is the case of $d = 4$. Theorem 1.1 solves the $d > 4$ case and 1.3 solves the $d = 3$ case. Nothing is know in the $d = 4$ case.

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