Eötvös Loránd Tudományegyetem
Természettudományi Kar

Abstract homotopy theory in categories

Szűcs András
egyetemi tanár

Kanalas Kristóf
Matematikus BSc

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Contents

0 Introduction

The aim of the present thesis is to introduce the basic notions of model categories, which provide the standard framework for doing homotopy theory in a categorical manner. It allows one to generalise the concepts of topology to certain categories (e.g. the notion of a cylinder, homotopy groups, and suspension), while the most important theorems remain true.

In the first section some of the basic concepts of category theory are discussed in a short and incomplete way. Then a few classical theorems are reconstructed, such as Whitehead's theorem, fibre and cofibre sequences, etc. In the last section a possible construction of model structures is given, which is applied to prove the category of chain complexes to be a model category.

Thanks for the help of Szűcs András, Szabó Endre and Makkai Mihály.

1 Categorical constructions

The main reference for this section is [4].

1.1 (co)limits

Definition 1.1.1. *A diagram in the category $\mathscr{C}$ is a functor $j : \mathscr{J} \to \mathscr{C}$, where $\mathscr{J}$ is any category. The diagram is (locally) small, if $\mathscr{J}$ is (locally) small. Generally j is identified with its image.*

Definition 1.1.2. *A cone over the diagram j consists of an object A of $\mathscr{C}$, and of arrows $jX \to A$ for all $X \in \mathscr{J}$, such that all triangles which contain A commute. A cocone is the same with arrows $A \to jX$. A morphism of (co)cones is an arrow $A \to A'$, such that all triangles having A and A' as vertices commute.*

Definition 1.1.3. *The limit of the diagram j is the universal cone over it, that means, from any other such cone there is a unique morphism of cones to it. The colimit is the universal cocone (there is a unique morphism to every other from it). Universality implies that if they exist they are unique up to (unique) isomorphism (of (co)cones).*

Example 1.1.1. Observe the following diagram in Ab:

$$\mathbb{Z}_p \hookrightarrow \mathbb{Z}_{p^2} \hookrightarrow \mathbb{Z}_{p^3} \hookrightarrow \dots$$

Its colimit is the group $\mathbb{Z}_{p^\infty} = \bigcup_1^\infty \mathbb{Z}_{p^k}$, together with the evident inclusions. To see this, take any sequence of homomorphisms $\mathbb{Z}_{p^k} \to G$, such that all of them are extensions of the previous ones. Then it induces a unique map $\mathbb{Z}_{p^\infty} \to G$, whose restrictions are the given morphisms.

Example 1.1.2. In Table 1 the most important limits and colimits are listed (without their proofs). Their names are: terminal object, initial object, product, coproduct, equaliser, coequaliser, pullback, pushout.

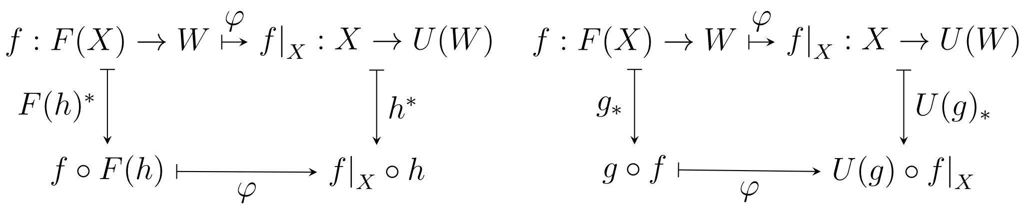
1.2 adjunctions

Example 1.2.1. Let $F : \mathbf{Sets} \to \mathbf{Vect_K}$ be the functor that assignes to each set X the K-vector space generated by its elements and let $U : \mathbf{Vect_K} \to \mathbf{Sets}$ be the usual forgetful functor. Any function $f : X \to U(W)$ has a unique K-linear extension $\hat{f} : F(X) \to W$ given by $\hat{f}(\sum c_i e_i) = \sum c_i f(e_i)$. This has an inverse, namely the restriction of a linear map $g : F(X) \to W$ to the base set X, and hence it gives a bijection $\varphi : \mathbf{Vect_K}(F(X), W) \to \mathbf{Sets}(X, U(W))$.

diagram	$\emptyset$		$\cdot \quad \cdot$		$\cdot \rightrightarrows \cdot$		$\cdot \to \cdot \leftarrow \cdot$		$\cdot \leftarrow \cdot \to \cdot$	
	lim	colim	lim	colim	lim	colim	lim	colim		
Sets	$\{*\}$	$\emptyset$	$\prod$	$\coprod$	$\{x \mid f(x) = g(x)\}$	$Y/\sim$ $f(x) \sim g(x)$	$\{\langle a, b\rangle \mid f(a) = g(b)\}$	$(X \sqcup Y)/\sim$ $i_1 f(z) \sim i_2 g(z)$		
Top	$\{*\}$	$\emptyset$	$\prod$	$\coprod$	$\{x \mid f(x) = g(x)\}$ + subspace topology	$Y/\sim$ $f(x) \sim g(x)$ + quotient topology	$\{\langle a, b\rangle \mid f(a) = g(b)\}$ + subspace topology	$(X \sqcup Y)/\sim$ $i_1 f(z) \sim i_2 g(z)$ + quotient topology		
Top$_*$	$\{x_0\}$	$\{x_0\}$	$\prod$	$\bigvee$	$\{x \mid f(x) = g(x)\}$ + subspace topology	$Y/\sim$ $f(x) \sim g(x)$ + quotient topology	$\{\langle a, b\rangle \mid f(a) = g(b)\}$ + subspace topology	$(X \sqcup Y)/\sim$ $i_1 f(z) \sim i_2 g(z)$ + quotient topology		
Grp	1	1	$\prod$	$*$	$\ker\,(f \cdot (1/g))$	$Y/\sim$ $\sim$ gen. by $f(x) \sim g(x)$	$\{\langle a, b\rangle \mid f(a) = g(b)\}$	$(X \sqcup Y)/\sim$ $\sim$ gen. by $i_1 f(z) \sim i_2 g(z)$		
Cat	$\{*\}$	$\emptyset$	$\prod$	$\coprod$	$\{x \mid f(x) = g(x)\}$	$Y/\sim$ $\sim$ gen. by $f(x) \sim g(x)$	$\{\langle a, b\rangle \mid f(a) = g(b)\}$	$(X \sqcup Y)/\sim$ $\sim$ gen. by $i_1 f(z) \sim i_2 g(z)$		

Table 1: main examples for (co)limits

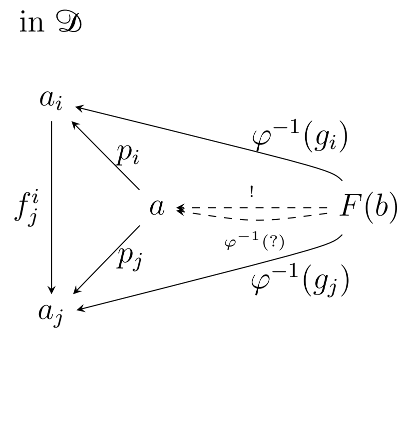
This assignment is defined in a homogenous way, this (so called naturality) can be expressed by the following commutative diagrams. Here $h : X' \to X$ is any set-set function, while $g : W \to W'$ is any K-linear map.

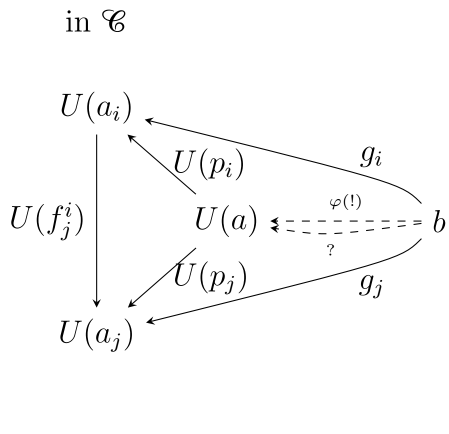


Definition 1.2.1. *An adjoint pair of functors is the triple* $\langle F, U, \varphi\rangle$*, where* $F : \mathscr{C} \to \mathscr{D}$ *and* $U : \mathscr{D} \to \mathscr{C}$ *are functors, while* $\varphi = \{\varphi_{X,Y}\}_{X\in\mathscr{C}, Y\in\mathscr{D}}$ *is a set of functions, such that* $\varphi_{X,-} : \mathscr{D}(F(X), -) \to \mathscr{C}(X, U(-))$ *and* $\varphi_{-,Y} : \mathscr{D}(F(-), Y) \to \mathscr{C}(-, U(Y))$ *are natural isomorphisms. Then* F *is said to be the left adjoint of* U *(and* U *is called the right adjoint of* F*).*

Proposition 1.2.1. *Right adjoints preserve limits.*

Proof. Take a diagram in $\mathscr{D}$ and assume it has the limiting cone $\{p_i : a \to a_i\}$. Its U-image gives a cone on the U-image of the diagram, for proving its universality take an arbitrary cone $\{g_i : b \to U(a_i)\}$. By the naturality of φ^{-1} in a_- we have $\varphi^{-1}(g_j) = \varphi^{-1}(U(f^i_j) \circ g_i) = f^i_j \circ \varphi^{-1}(g_i)$, so $\{\varphi^{-1}(g_-)\}$ is a cone in $\mathscr{D}$, therefore there is a unique $! : F(b) \to a$ making each triangle commute. φ is natural in $U(a_-)$, so $\varphi(!)$ is a morphism of cones in $\mathscr{C}$. If there was another such arrow ?, then by the naturality of φ^{-1} and by the uniqueness of ! we get $\varphi^{-1}(?) =!$, applying φ gives $? = \varphi(!)$.





□

Corollary 1.2.1.1. *By dualising the proof we get that left adjoints preserve colimits.*

Definition 1.2.2. *The maps $\eta_X = \varphi(1_{FX}) : X \to UFX$ form a natural transformation $\eta : Id_{\mathscr{C}} \to UF$ by the naturality of φ. Analogously the natural transformation $\varepsilon : FU \to Id_{\mathscr{D}}$ is defined on an object Y as $\varphi^{-1}(1_{UY}) : FUY \to Y$. η is called the **unit** and ε is called the **counit** of the adjunction.*

1.3 (co)monads and (co)algebras

Example 1.3.1. Let $\mathscr{G}$ be a fixed group and T : **Sets** $\to$ **Sets** be the endofunctor with $TX = G \times X$, where G denotes the underlying set of $\mathscr{G}$. Then two natural transformations: $\mu : T^2 \to T$ and $\eta : Id \to T$ are defined, given by the components $\mu_X : G\times(G\times X) \ni \langle g_1, \langle g_2, x\rangle\rangle \mapsto \langle g_1g_2, x\rangle \in G\times X$ and $\eta_X : X \ni x \mapsto \langle u, x\rangle \in G \times X$, where u is the unit element of $\mathscr{G}$. By the associativity of group multiplication and by the properties of the unit, the following diagrams commute:

$$\begin{array}{ccc} \langle g_1,\langle g_2,\langle g_3,x\rangle\rangle\rangle & \xmapsto{T\mu_X} & \langle g_1,\langle g_2g_3,x\rangle\rangle \\ {\scriptstyle \mu_{TX}}\downarrow & & \downarrow{\scriptstyle \mu_X} \\ \langle g_1g_2,\langle g_3,x\rangle\rangle & \xmapsto[\mu_X]{} & \langle g_1g_2g_3,x\rangle \end{array}$$

$$\begin{array}{ccc} \langle g,x\rangle & \xmapsto{\eta_{TX}} & \langle u,\langle g,x\rangle\rangle \\ & \diagdown\!\!\diagdown & \downarrow{\scriptstyle \mu_X} \\ & & \langle ug,x\rangle \end{array} \qquad \begin{array}{ccc} \langle g,\langle u,x\rangle\rangle & \xleftarrow{T\eta_X} & \langle g,x\rangle \\ {\scriptstyle \mu_X}\downarrow & \diagup\!\!\diagup & \\ \langle gu,x\rangle & & \end{array}$$

Definition 1.3.1. *A monad on the category $\mathscr{C}$ is a triple $\mathbf{T} = \langle T, \eta, \mu\rangle$, where $T : \mathscr{C} \to \mathscr{C}$ is an endofunctor, $\eta : Id \to T$ and $\mu : T^2 \to T$ are natural transformations, making the following diagrams commute:*

$$\begin{array}{ccc} T^3 & \xrightarrow{T\mu} & T^2 \\ {\scriptstyle \mu T}\downarrow & & \downarrow{\scriptstyle \mu} \\ T^2 & \xrightarrow[\mu]{} & T \end{array} \qquad \begin{array}{ccccc} IdT & \xrightarrow{\eta T} & T^2 & \xleftarrow{T\eta} & TId \\ & \diagdown\!\!\diagdown & \downarrow{\scriptstyle \mu} & \diagup\!\!\diagup & \\ & & T & & \end{array}$$

Example 1.3.2. Let X be a fixed set and $h : G \times X \to X$ an action of the group $\mathscr{G}$ on it, that is $h(g_1, h(g_2, x)) = h(g_1 g_2, x)$ and $h(u, x) = x$. Using the notations of Example 1.3.1, this means the commutativity of the following diagrams:

$$
\begin{array}{ccc}
\langle g_1, \langle g_2, x\rangle\rangle & \xmapsto{Th} & \langle g_1, h(g_2, x)\rangle \\
{\scriptstyle \mu_X}\downarrow & & \downarrow{\scriptstyle h} \\
\langle g_1 g_2, x\rangle & \xmapsto[h]{} & h(g_1 g_2, x) = h(g_1, h(g_2, x))
\end{array}
\qquad
\begin{array}{ccc}
x & \xmapsto{\eta_X} & \langle u, x\rangle \\
& \searrow\!\!= & \downarrow{\scriptstyle h} \\
& & h(u, x)
\end{array}
$$

A $\mathscr{G}$-equivariant map between the $\mathscr{G}$-actions $\langle X, h\rangle$ and $\langle X', h'\rangle$ is a function $f : X \to X'$, making

$$
\begin{array}{ccc}
h(g, x) & \xleftarrow{h} & \langle g, x\rangle \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Tf} \\
f(h(g, x)) = h'(g, f(x)) & \xleftarrow[h']{} & \langle g, f(x)\rangle
\end{array}
$$

commutative.

Definition 1.3.2. *A* $\mathbf{T}$*-algebra for the monad* $\mathbf{T} = \langle T, \eta, \mu\rangle$ *consists of an object* X *of* $\mathscr{C}$ *(called the underlying object) and a morphism* $h : TX \to X$ *(called the structure map), such that*

$$
\begin{array}{ccc}
T^2X & \xrightarrow{Th} & TX \\
{\scriptstyle \mu_X}\downarrow & & \downarrow{\scriptstyle h} \\
TX & \xrightarrow[h]{} & X
\end{array}
\qquad
\begin{array}{ccc}
X & \xrightarrow{\eta_X} & TX \\
& \searrow\!\!= & \downarrow{\scriptstyle h} \\
& & X
\end{array}
$$

commute. A morphism of $\mathbf{T}$*-algebras is an arrow* $f : X \to X'$*, such that the following square commutes:*

$$
\begin{array}{ccc}
X & \xleftarrow{h} & TX \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Tf} \\
X' & \xleftarrow[h']{} & TX'
\end{array}
$$

The category of **T**-*algebras is denoted by* $Alg(\mathbf{T})$.

Example 1.3.3. $\mathbf{T} = \langle T_\tau : \mathbf{Sets} \to \mathbf{Sets}, \eta, \mu \rangle$, τ is an algebra type.
$T_\tau X = \{\tau\text{-terms on } X\}$
$\eta_X : X \hookrightarrow T_\tau X$ is inclusion.
$\mu_X : T_\tau^2 X \to T_\tau X$ is identity.

Then $Alg(\mathbf{T})$ is the category of algebras of type τ, with objects $\langle A, \alpha \rangle$, where A is the underlying set and $\alpha : T_\tau A \to A$ is the evaluation function of the algebra, having the properties $\alpha(g(\overrightarrow{f(\vec{a})})) = \alpha(g(\overrightarrow{\alpha(f(\vec{a}))}))$ and $\alpha(a) = a$ for all $a \in A$.

A morphism $h : A \to A'$ of $Alg(\mathbf{T})$ should satisfy $h(\alpha(f(\vec{a}))) = \alpha'(f(\overrightarrow{h(a)}))$, which means precisely that h is a homomorphism.

In [1] a stronger version of the following is proved.

Proposition 1.3.1. *If* $\mathscr{C}$ *is complete and cocomplete (i.e. has all small limits and colimits), and* $T : \mathscr{C} \to \mathscr{C}$ *preserves colimits, then* $Alg(\mathbf{T})$ *is complete and cocomplete.*

2 Model categories

The material of this section is taken from [2].

2.1 homotopies

Definition 2.1.1. *An object* a *is the retract of the object* b *in the category* C*, if there are maps* $i : a \to b$ *and* $r : b \to a$*, such that* $ri = Id_a$*. An arrow* f *is the retract of the arrow* g*, if it is a retract in the category* $Mor\ C$*, whose objects are the maps of* C *and whose morphisms are commutative squares.*

Definition 2.1.2. *Let* $i : a \to b$ *and* $p : x \to y$ *be maps of the category* C*. Then* i *has the left lifting property wrt.* p *(and* p *has the right lifting property wrt.* i*) if for every commutative square*

$$\begin{array}{ccc} a & \xrightarrow{f} & x \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ b & \xrightarrow{g} & y \end{array}$$

there is a lift $h : b \to x$ *that makes both triangle commute.*

Definition 2.1.3. *A model structure on a category C consists of three subcategories of it, called weak equivalences, fibrations and cofibrations, together with functors α, β, γ, $\delta : Mor\ C \to Mor\ C$, such that*

- *if two out of f, g and gf is a weak equivalence then so is the third,*
- *if f is a retract of g and g belongs to one of the above subcategories, then so does f,*
- *trivial cofibrations (maps, that are both cofibrations and weak equivalences) have the left lifting property wrt. fibrations, while trivial fibrations (that are both fibrations and weak equivalences) have the right lifting property wrt. cofibrations and*
- *every morphism f splits as $\beta(f) \circ \alpha(f)$, where $\alpha(f)$ is a cofibration and $\beta(f)$ is a trivial fibration and as $\delta(f) \circ \gamma(f)$, where $\gamma(f)$ is a trivial cofibration and $\delta(f)$ is a fibration.*

Definition 2.1.4. *A model category is a category with all small limits and colimits and with a model structure on it.*

Proposition 2.1.1. *If $\mathscr{C}$ is a model category, with the subcategories $\mathscr{Fib}$ (of fibrations), $\mathscr{Cof}$ (of cofibrations), and $\mathscr{W}$ (of weak equivalences), then there is a model structure on $\mathscr{C}^{op}$, whose fibrations form the category $\mathscr{Cof}^{op}$, cofibrations form $\mathscr{Fib}^{op}$, and its weak equivalences form $\mathscr{W}^{op}$. This follows from the self-duality of the axioms.* □

Lemma 2.1.2 (Retract argument)**.** *Assume $f = pi$ and f has l.l.p. wrt. p. Then f is the retract of i. Dually, if f has r.l.p. wrt. i, then f is the retract of p.*

Proof. Complete the factorisation of f with 1_B to get a commutative square. Then there is a lift r, that can be drawn as

$$\begin{array}{ccccc} A & = & A & = & A \\ \downarrow f & & \downarrow i & & \downarrow f \\ B & \xrightarrow{r} & C & \xrightarrow{p} & B \end{array}$$

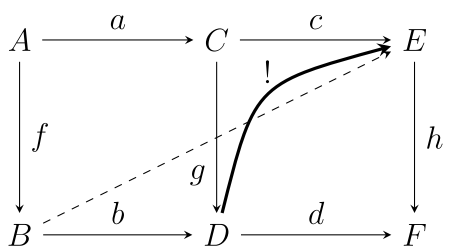
. □

Lemma 2.1.3. *f is a (trivial) fibration/ (trivial) cofibration iff the related lifting property holds.*

Proof. For the nontrivial direction factor f to a cofibration followed by a trivial fibration; $f = pi$. If f has l.l.p. wrt. p, then by the retract argument f is a retract of i, and hence it is a cofibration. The other cases are similar. □

Proposition 2.1.4. *Cofibrations and trivial cofibrations are closed under pushouts. Dually, fibrations and trivial fibrations are closed under pullbacks.*

Proof. Assume f has left lifting property wrt. h, and the left square in the following diagram is a pushout.



! is induced by the universal property of the pushout, therefore commutativity of the CDE-triangle is immediate. Then take the composites hc, $hl = db$. Using universality again, commutativity of the lower triangle is deduced, hence g has the same l.l.p.

□

Proposition 2.1.5 (Ken Brown's lemma)**.** *Let $\mathscr{C}$ be a model category and $\mathscr{D}$ be a category with a distinguished subcategory $\mathscr{W}$ that has the 2-out-of-3 property. If $F : \mathscr{C} \to \mathscr{D}$ is a functor that maps triv. cof. between cofibrant objects to $\mathscr{W}$, then it does the same with all weak eqivalences between cofibrant objects. The analogous result (concerning trivial fibrations between fibrant objects) also holds.*

Proof. Let $f : A \to B$ be a weak equivalence of cofibrant objects. Take the factorisation $(f, 1_B) : A \vee B \xrightarrow[cof]{q} C \xrightarrow[triv.fib.]{p} B$. Both components $p \circ q \circ i_1$, $p \circ q \circ i_2$ and p are weak equivalences, so by the 2-out-of-3 axiom $q \circ i_1$ and $q \circ i_2$ are also. The coproduct is equivalently the pushout

$$\begin{array}{ccc} 0 & \longrightarrow & A \\ \big\downarrow & & \big\downarrow{\scriptstyle i_1} \\ B & \xrightarrow[i_2]{} & A \vee B \end{array}$$

hence i_1, i_2 are cofibrations and $q \circ i_1$, $q \circ i_2$ are trivial cofibrations. It also follows that C is cofibrant, so by assumption $F(q \circ i_1)$ and $F(q \circ i_2)$ are in $\mathscr{W}$. Its 2-out-of-3 property implies that all identity maps of domains and codomains of maps from $\mathscr{W}$ are included, therefore $F(p \circ q \circ i_2) = F(1_B) \in \mathscr{W}$. Then $F(p)$ and finally $F(f) = F(p \circ q \circ i_1)$ are also in it.

□

The homotopy category is at first given by the inversion of weak equalences.

Definition 2.1.5. *The homotopy category of a model category $\mathscr{C}$ is the free category generated by $\mathscr{C} \cup \mathscr{W}^{-1}$ (where $\mathscr{W}^{-1}$ is the dual category of the subcategory of weak equalences), factored by the relations $ww^{-1} = 1_{dom\ w^{-1}}$, $w^{-1}w = 1_{dom\ w}$, $fg = f \circ g$ (for all $w \in \mathscr{W}$, $f, g \in \mathscr{C}$).*

It is not trivial, that $Ho\ \mathscr{C}$ is locally small (as the objects might form a proper class), but it will be proven later. Therefore it is not needed to move to a higher set theoretical universe.

Proposition 2.1.6. *There is $\gamma : \mathscr{C} \to Ho\ \mathscr{C}$, which is identity on objects, and takes the arrows of $\mathscr{W}$ to isomorphisms. If $F : \mathscr{C} \to \mathscr{D}$ also takes $\mathscr{W}$ to isos, then there is a unique $Ho\ F : Ho\ \mathscr{C} \to \mathscr{D}$ with $(Ho\ F) \circ \gamma = F$. If $\delta : \mathscr{C} \to \mathscr{E}$ has the same universal property, then there is a unique isomorphism $\varphi : Ho\ \mathscr{C} \to \mathscr{E}$, such that $\varphi\gamma = \delta$ holds.*

Proof. γ is identity on $\mathscr{C}$ and $\gamma(w^{-1}) = (\gamma w)^{-1}$. $Ho\ F|_{\mathscr{C}} = F|_{\mathscr{C}}$ and $Ho\ F(w^{-1}) = (Fw)^{-1}$. Therefore γ is the initial object in the category of functors mapping from $\mathscr{C}$, taking $\mathscr{W}$ to isos, so it is unique up to unique isomorphism. □

Definition 2.1.6. *The categories $\mathscr{C}$ and $\mathscr{D}$ are equivalent if there are functors $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$, such that the compositions are naturally isomorphic to the identity functors of the domains.*

Example 2.1.1. Let $\mathscr{A}$ be the skeleton of $\mathscr{C}$ (i.e. a full subcategory, where any object from $\mathscr{C}$ is isomorphic to exactly one from $\mathscr{A}$) and let K denote the inclusion. The isomorphisms $\theta_c : c \cong Tc \in \mathscr{A}$ uniquely determine a functor $T : \mathscr{C} \to \mathscr{A}$ for which θ is a natural isomorphism from $Id_{\mathscr{C}}$ to KT. $TK = Id_{\mathscr{A}}$, therefore a category is equivalent to its scaleton.

The proof of the following can be found in [4].

Proposition 2.1.7. *A functor $F : \mathscr{C} \to \mathscr{D}$ is part of a weak equivalence, iff it is bijective on Hom-sets and for all $x \in \mathscr{D}$ there is an isomorphism $x \cong y$ in $\mathscr{D}$, such that $y = F(b)$ for an object $b \in \mathscr{C}$.*

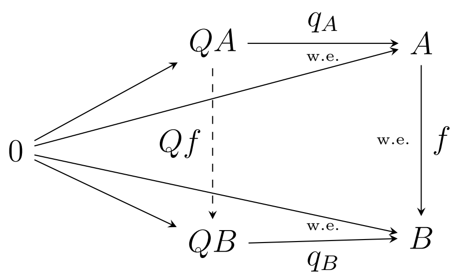
Let $\mathscr{C}_c$, $\mathscr{C}_f$, $\mathscr{C}_{cf}$ denote the full subcategories of $\mathscr{C}$ consisting of the cofibrant, fibrant and both fibrant and cofibrant objects and all morphisms between them.

For any object A the functorial factorisation of $A \to *$ to a trivial cofibration followed by a fibration gives a fibrant object RA, which is weakly equivalent to it. By functoriality this extends to the fibrant replacement functor R. The dual construction (factorising $0 \to A$ as a cofibration followed by a trivial fibration) gives the cofibrant replacement functor Q. These induce equivalence between the related homotopy categories.

Proposition 2.1.8. *The categories $Ho\ \mathscr{C}_c$, $Ho\ \mathscr{C}_f$,$Ho\ \mathscr{C}_{cf}$ and $Ho\ \mathscr{C}$ are equivalent.*

Proof. The inclusion $i : \mathscr{C}_c \to \mathscr{C}$ preserves weak equivalences, so the functor $Ho\ i : Ho\ \mathscr{C}_c \to Ho\ \mathscr{C}$ can be defined.

Q also preserves them, as it is illustrated here.



Therefore the functor $Ho\ Q : Ho\ \mathscr{C} \to Ho\ \mathscr{C}_c$ exists.

This picture also shows, that $q|_{\mathscr{C}_c} : Q \circ i \to 1_{\mathscr{C}_c}$ and $q : i \circ Q \to 1_{\mathscr{C}}$ are natural transformations, whose components are weak equivalences. Therefore $Ho\ q|_{\mathscr{C}_c} : Ho\ (Q \circ i) = Ho\ Q \circ Ho\ i \to Ho\ 1_{\mathscr{C}} = 1_{Ho\ \mathscr{C}_c}$ and $Ho\ q : Ho\ i \circ Ho\ Q \to 1_{Ho\ \mathscr{C}}$ are natural isomorphisms. The rest is similar.

□

The homotopy category can also be defined through the notion of homotopy.

Definition 2.1.7. *Let $f, g : B \to X$ be morphisms of the model category $\mathscr{C}$.*

- *A cylinder object for B is an object B', for which there is a factorisation of $1 \vee 1 : B \vee B \to B$ as $B \vee B \xrightarrow{i_0 \vee i_1} B' \xrightarrow{s} B$, where $i_0 \vee i_1$ is a cofibration and s is a weak equivalence.*

- *Dually, a path object for X is X' if there is a factorisation of $1 \times 1 : X \to X \times X$ as $X \xrightarrow{r} X' \xrightarrow{p_0 \times p_1} X \times X$, where r is a weak equivalence and $p_0 \times p_1$ is a fibration.*

- *A left homotopy from f to g is a map $H : B' \to X$, such that $Hi_0 = f$ and $Hi_1 = g$. The notation will be $f \overset{l}{\sim} g$.*

- *The dual notion is right homotopy from f to g, which is a map $K : B \to X'$, such that $p_0 K = f$ and $p_1 K = g$. For right homotopy we will write $f \overset{r}{\sim} g$.*

- *f is homotopic to g (in symbols: $f \sim g$), if $f \overset{l}{\sim} g$ and $f \overset{r}{\sim} g$. f is a homotopy equivalence if there is $h : X \to B$, for which $hf \sim 1_B$ and $fh \sim 1_X$.*

Proposition 2.1.9. *The cylinder objects $B \times I_0$ that are obtained from the (α, β) functorial factorisation of $1 \vee 1$ form the object function of a functor $- \times I_0$. For any other cylinder object B' of B, $B \times I_0$ and B' are weakly equivalent.*

Dually, there is a functorial path object X^{I_0}, coming from the (γ, δ) factorisation. It is weakly equivalent to any other path object of X.

Lemma 2.1.10. *$\mathscr{C}$ is a model category, $f, g : B \to X$ are arrows.*

1. *$f \overset{l}{\sim} g$, $h : X \to Y \Rightarrow hf \overset{l}{\sim} hg$.*
   $f \overset{r}{\sim} g$, $h : A \to B \Rightarrow fh \overset{r}{\sim} gh$.

2. *X is fibrant, $f \overset{l}{\sim} g$, $h : A \to B \Rightarrow fh \overset{l}{\sim} gh$.*
   B is cofibrant, $f \overset{r}{\sim} g$, $h : X \to Y \Rightarrow hf \overset{r}{\sim} hg$.

3. *B is cofibrant $\Rightarrow \overset{l}{\sim}$ is an equivalence relation on $\mathscr{C}(B, X)$.*
   X is fibrant $\Rightarrow \overset{r}{\sim}$ is an equivalence relation on $\mathscr{C}(B, X)$.

4. *B is cofibrant, $h : X \to Y$ is a trivial fibration/ weak equivalence of fibrant objects $\Rightarrow$ h induces a $\mathscr{C}(B, X)/\overset{l}{\sim} \xrightarrow{\simeq} \mathscr{C}(B, Y)/\overset{l}{\sim}$ isomorphism (of sets).*
   X is fibrant, $h : A \to B$ is a trivial cofibration/ weak equivalence of cofibrant objects $\Rightarrow$ h induces a $\mathscr{C}(B, X)/\overset{r}{\sim} \xrightarrow{\simeq} \mathscr{C}(A, X)/\overset{r}{\sim}$ isomorphism.

5. *B is cofibrant $\Rightarrow$ if $f \overset{l}{\sim} g$, then $f \overset{r}{\sim} g$, and for all X' path object there is a $K : B \to X'$ right homotopy from f to g.*
   X is fibrant $\Rightarrow$ if $f \overset{r}{\sim} g$, then $f \overset{l}{\sim} g$, and for all B' cylinder object there is an $H : B' \to X$ left homotopy from f to g.

Proof. By duality it is enough to prove statements about left homotopies.

1. $h \circ H : B' \to Y$ is a left homotopy from hf to hg if H was an $f \overset{l}{\sim} g$ homotopy.

2. The cylinder object of B is given by the factorisation $B \vee B \xrightarrow[cof]{(i_0,i_1)} B' \xrightarrow[w.e.]{s} B$. s is assumed to be a trivial fibration; otherwise factor it as $s : B' \xrightarrow[tr.cof]{s_1} B'' \xrightarrow[fib]{s_2} B$, where s_2 is trivial by the 2-out-of-3 property and therefore B'' is a cylinder object. Then there would be a lift H' in the diagram

$$\begin{array}{ccccc} B \vee B & \xrightarrow{(i_0,i_1)} & B' & \xrightarrow{H} & X \\ & & {\scriptstyle \text{tr.cof}}\downarrow & \overset{H'}{\nearrow} & \downarrow{\scriptstyle \text{fib}} \\ & & B'' & \longrightarrow & * \end{array}$$

which is also a left homotopy from f to g by commutativity.

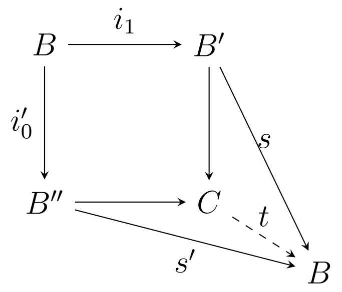
Assume $A \vee A \xrightarrow{j} A' \xrightarrow{t} A$ gives the cylinder object A'. Now form the square

$$\begin{array}{ccc} A \vee A & \xrightarrow{(i_0,i_1)\circ(h,h)} & B' \\ {\scriptstyle \text{cof}}\downarrow{\scriptstyle j} & \overset{k}{\nearrow} & {\scriptstyle s}\downarrow{\scriptstyle \text{tr.fib}} \\ A' & \xrightarrow{h \circ t} & B \end{array}$$

The composite $H \circ k$ is a left homotopy from fh to gh.

3. Using the above notation, fs is a left homotopy from f to f, which gives reflexivity. For symmetry use the same homotopy $H : B' \to X$, but with the factorisation $B \vee B \xrightarrow{(i_1,i_0)} B' \xrightarrow{s} B$. It remains to prove transitivity.

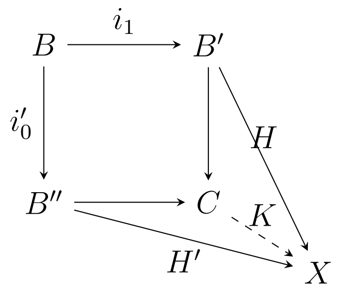
   Let $H : B' \to X$ be a left homotopy from f to g, and $H' : B'' \to X$ from g to h. Form the pushout of $B'' \xleftarrow{i_0'} B \xrightarrow{i_1} B'$. Then the map $t : C \to B$ is induced, as in



Define the maps $j_0 : B \xrightarrow{i_0} B' \to C$ and $j_1 : B \xrightarrow{i'_1} B'' \to C$. A factorisation of $1_B \vee 1_B$ is given by $B \vee B \xrightarrow{(j_0,j_1)} C \xrightarrow{t} B$.

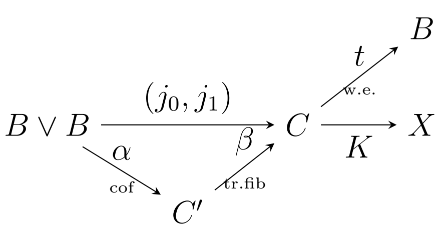
The coproduct is equivalently a pushout from 0, using that B is cofibrant we get, that the inclusions $B \to B \vee B \leftarrow B$ are cofibrations, hence i_1 (the restriction of (i_0, i_1) to the second component) is also. $s \circ i_1 = 1_B$, so by the 2-out-of-3 property i_1 is a trivial cofibration, then so is its pushout $B'' \to C$. s' is a weak equvalence, so t is a weak equivalence too.

The pushout



gives a map $K : C \to X$, such that $Kj_0 = Hi_0 = f$ and $Kj_1 = H'i'_1 = h$. The problam is, that C is not a cylinder object.

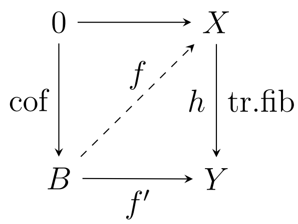
The suitable cylinder object C' is given by



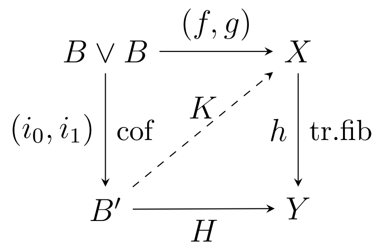
$K\beta$ is a left homotopy from f to h.

4. The case when h is a weak equivalence of fibrant object can be deduced from the second case (when h is a trivial fibration) and the Ken Brown lemma, applied to the functor $\mathscr{C}(B,-)/{\sim_l} : \mathscr{C} \to \mathbf{Sets}$ which is well defined by the first and third statement of this lemma. The subcategory $\mathscr{W} \leq \mathbf{Sets}$ containes the bijections.

   Now assume h is a trivial fibration. The map $\mathscr{C}(B,h)/{\sim_l} : \mathscr{C}(B,X)/{\sim_l} \to \mathscr{C}(B,Y)/{\sim_l}$ is surjective by the existence of the lift f in

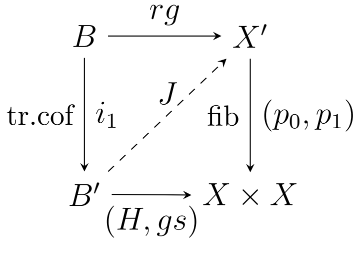


   For injectivity assume $H : B' \to Y$ is a left homotopy from hf to hg. The lift K in



   is a left homotopy from f to g.

5. Let $H : B' \to X$ be a left homotopy from f to g. Using the same arguments as in 3., we get that $i_0 : B \to B'$ is a trivial cofibration. Let X' be a path object via $X \xrightarrow{r} X' \xrightarrow{(p_0,p_1)} X \times X$. Form the square



Define $K = Ji_0$. Then $p_0K = p_0Ji_0 = Hi_0 = f$ and $p_1K = p_1Ji_0 = gsi_0 = g$, so K is a right homotopy from f to g.

□

Corollary 2.1.10.1. *$\mathscr{C}$ is a model category, B is cofibrant, X is fibrant ⇒ $\overset{l}{\sim}=\overset{r}{\sim}$ is an equivalence relation on $\mathscr{C}(B,X)$. If $f \sim g$, then any cylinder/ path object realises the homotopy.* □

Corollary 2.1.10.2. *$\sim$ is an equivalence relation on Arr $\mathscr{C}_{cf}$, which is compatible with composition, hence the category $\mathscr{C}_{cf}/\sim$ exists.* □

It is worth to discuss the relation between left and right homotopies explicitely. $A \times I$, $A \times I'$, etc. will denote arbitrary cylinder objects, while $A \times I_0$ will stand for the functorial one, and the analogous notation will be used for path objects.

Definition 2.1.8. *A correspondence between the left homotopy $H : B \times I \to X$ and the right homotopy $K : B \to X^I$ (both going from f to g) is a map $\varphi : B \times I \to X^I$, such that $\varphi i_0 = K$, $\varphi i_1 = rg$, $p_0\varphi = H$, and $p_1\varphi = gs$.*

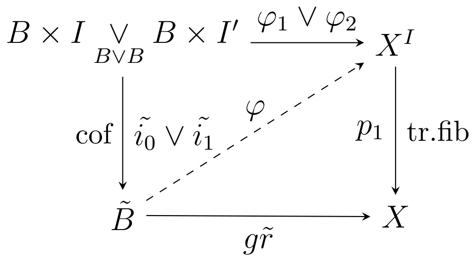
Corollary 2.1.10.3. *Starting from any left homotopy H, there is a corresponding right homotopy K.* □

Our goal now is to define (generalised) homotopy groups. As a first step, we discuss (left) homotopies between (left) homotopies. The reason why a new notion is introduced, is that the cylinder objects are not assumed to coincide, and also because these homotopies should fix the original maps.

Definition 2.1.9. *Let $H : B\times I \to X$ and $H' : B\times I' \to X$ be left homotopies from f to g, with the usual maps i_0, i_1, r and i'_0, i'_1, r' of the cylinder objects. Denote the pushout of $B \times I \xleftarrow{(i_0,i_1)} B \vee B \xrightarrow{(i'_0,i'_1)} B \times I'$ by $B \times I \bigvee_{B\vee B} B \times I'$. Factor the map $r \vee r'$ into a cofibration followed by a weak equivalence, as $B \times I \bigvee_{B\vee B} B \times I' \xrightarrow{\tilde{i_0}\vee\tilde{i_1}} \tilde{B} \xrightarrow{\tilde{r}} B$. A left homotopy from H to H' is a map $\tilde{H} : \tilde{B} \to B$, such that $\tilde{H}\tilde{i_0} = H$ and $\tilde{H}\tilde{i_1} = H'$.*

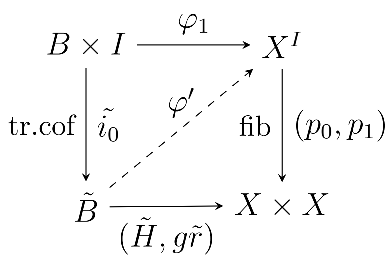
Proposition 2.1.11. *As before, H and H' are left homotopies, K is a right homotopy from f to g ($f, g : B \to X$, B is cofibrant, X is fibrant). Assume K corresponds to H. Then it corresponds to H', iff H and H' are left homotopic.*

Proof. First assume φ_1 and φ_2 are correspondences between the above homotopies. Form the square



Then $p_0\varphi$ is a left homotopy from H to H', as $p_0\varphi\tilde{i_0} = p_0\varphi_1 = H$ and $p_0\varphi\tilde{i_1} = p_0\varphi_2 = H'$

Now assume $\tilde{H}$ is a left homotopy from H to H'. $\tilde{i_0}$ is a cofibration (as the inclusion $B \times I \to B \times I \underset{B \vee B}{\vee} B \times I'$ is the pushout of a cofibration) and it is a weak equivalence (by the 2-out-of-3 property, used for $\tilde{r}\tilde{i_0} = r$). Therefore there is a lift in

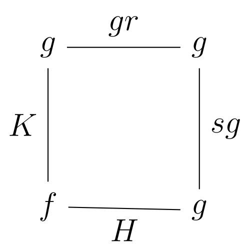


and $\varphi'\tilde{i_1} : B \times I' \to X^I$ is a correspondence from H' to K, as $p_0\varphi'\tilde{i_1} = \tilde{H}\tilde{i_1} = H$, $p_1\varphi'\tilde{i_1} = g\tilde{r}\tilde{i_1} = gr'$, $\varphi'\tilde{i_1}i_0 = \varphi'\tilde{i_0}i_0 = \varphi_1 i_0 = K$, and $\varphi'\tilde{i_1}i_1 = \varphi'\tilde{i_0}i_1 = \varphi_1 i_1 = sg$

□

Corollary 2.1.11.1. *Left homotopy is an equivalence relation on left homotopies from $f : B \to X$ to $g : B \to X$ (dually: right homotopy on right homotopies is an equivalence relation), the equivalence classes form a set in both cases (as any left homotopy corresponds to a right homotopy with fixed X^I), and correspondence gives a bijection between these two sets (denoted by $\pi_1^l(B, X; f, g)$ and $\pi_1^r(B, X; f, g)$).* □

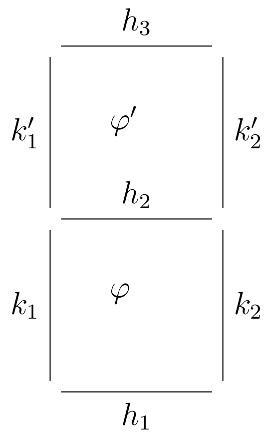
It will be convenient to illustrate correspondences as



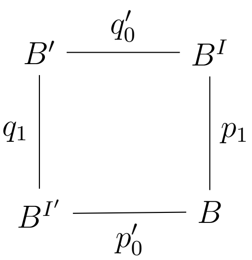
and in general, any map from a cylinder object to a path object, with lower (resp. upper) side standing for composition with p_0 (resp. p_1), and with left (resp. right) side standing for precomposition with i_0 (resp. i_1).

Lemma 2.1.12. *These squares can be glued together at edges which refer to the same homotopy (and therefore for vertical gluing the cylinder objects, for horisontal, the path objects are assumed to coincide); i.e there is a map $B \times I' \to X^{I'}$, whose restrictions and projections are the compositions of homotopies, written on the sides of the rectangle. This composition is defined in the proof of Proposition 2.1.10, and it is written in the order of application (unlike the composition of arrows).*

Proof. Take the squares

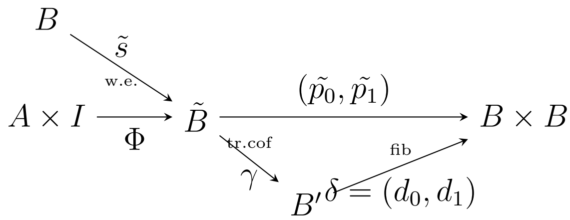


Form the pullback:



Then s and s' induce $\tilde{s}$, which is a weak equivalence by the pullback stability of trivial fibrations, and by the 2-out-of-3 property. Take $\tilde{p_0} = p_0 q'_0$ and $\tilde{p_1} = p_1 q'_1$, then $(\tilde{p_0}, \tilde{p_1}) \circ \tilde{s} : B \to B \times B$ is identity in both coordinates.

The maps φ and φ' induce $\Phi : A \times I \to \tilde{B}$, with $\tilde{p_0}\Phi = p_0\varphi = h_1$ and $\tilde{p_1}\Phi = p_1\varphi' = h_3$. The only problam is that $\tilde{B}$ may not be a path object, as (p_0, p_1) may not be a fibration. Therefore factor it into a trivial cofibration, followed by a fibration, as



Then $\Phi' = \gamma\Phi : A \times I \to B'$ has a path object as codomain, and $d_0\Phi' = h_1$, $d_1\Phi' = h_3$, $\Phi' j_0 = k_1 k_1'$, and $\Phi' j_1 = k_2 k_2'$. The last two equivality is by definition; precomposing j_0 in the above diagram results the definition of composition of right homotopies (whose dual was described in Proposition 2.1.10). □

Proposition 2.1.13.

- *Composition of left homotopies respects the equivalence classes, i.e. there is an induced map* $\pi_1^l(A, B; f_1, f_2) \times \pi_1^l(A, B; f_2, f_3) \xrightarrow{\circ} \pi_1^l(A, B; f_1, f_3)$. *The dual statement also holds.*

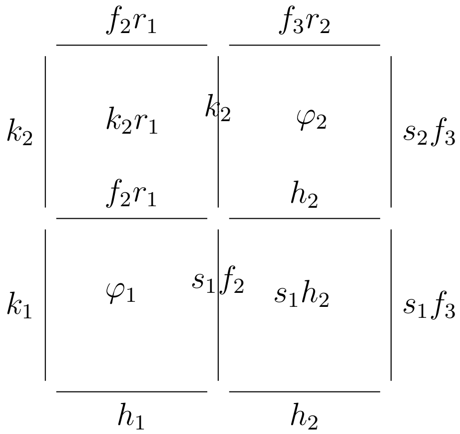
- *If* H_1 *corresponds to* K_1 *(via* φ_1*), and* H_2 *corresponds to* K_2 *(via* φ_2*), then* $H_1 \circ H_2$ *corresponds to* $K_1 \circ K_2$ *(where* $H_1 : f_1 \overset{l}{\sim} f_2$, $H_2 : f_2 \overset{l}{\sim} f_3$, $K_1 : f_1 \overset{r}{\sim} f_2$, $K_2 : f_2 \overset{r}{\sim} f_3$*).*

- $(Hom(A, B), \circ)$ *is a grupoid.*

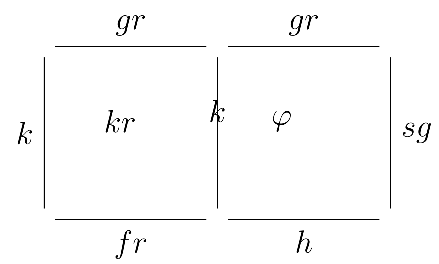
Proof.

- The second statement together with Proposition 2.1.11 implies the first one.

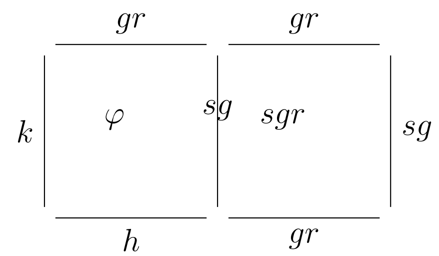
- This is proved by forming the squares



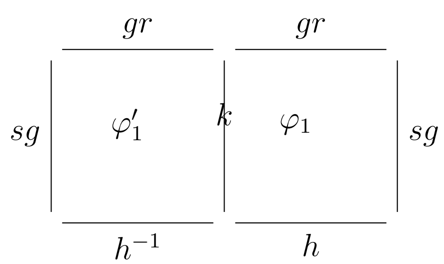
- Associativity of composition follows from the fact, that it is defined through a universal arrow from a pushout. The identity on $[f]$ is $[fr]$. It can be seen by

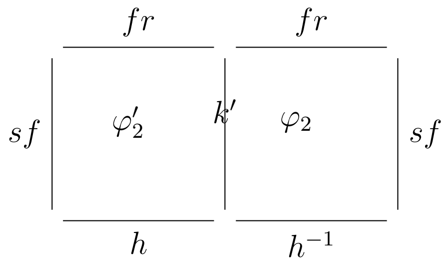


and



The inverse of h (as it was constructed in Proposition 2.1.10) is the same map $h^{-1} : A \times I \to B$, but j_0 and j_1 is reversed in the factorisation through the cylinder object. Let φ_1 be a correspondence between h and k, and φ_2 be a correspondence between h^{-1} and k', and let φ_1' and φ_2' be the same maps, but with the reversed cylinders as domains. Then the following squares prove that the composition of $[h]$ and $[h^{-1}]$ is identity in both ways:





□

Definition 2.1.10. $\pi_1^A(B) = \pi_1(A, B) = \pi_1(A, B; 0, 0)$ *is the first A-homotopy group of B, where A is cofibrant and B is fibrant. It is a group by the previous proposition.*

The suspension of an object A is the pushout of $* \leftarrow A \vee A \xrightarrow{(i_0,i_1)} A \times I_0$ (this and related concepts will be examined in the next section). This pushout in the category of pointed topological spaces ($\mathbf{Top}_*$) results the reduced suspension, therefore it is meaningful to define $\pi_n(A, B)$ as $\pi_1(\Sigma^{n-1}A, B)$. In the remaining part the notion of homotopy is exposed to further discussion.

It can be proved, that $\mathbf{Top}_*$ admits a model structure, where f is a weak equivalence iff $\pi_n(f)$ is isomorphism for all $n = 0, 1 \ldots$, all spaces are fibrant and the cofibrant ones are CW-complexes. Therefore the following is a generalisation of Whitehead's theorem.

Theorem 2.1.14. *Let $\mathscr{C}$ be a model category. A map of $\mathscr{C}_{cf}$ is a weak equivalence iff it is a homotopy equivalence.*

Proof. First assume $f : A \to B$ is a weak equivalence in $\mathscr{C}_{cf}$. By the previous lemma $f_* : {}^{\mathscr{C}_{cf}}\!/\!_{\sim}(X, A) \to {}^{\mathscr{C}_{cf}}\!/\!_{\sim}(X, B)$ is bijective. Taking $X = B$, there is a map $g : B \to A$, such that $fg \sim 1_B$, and this g is unique up to homotopy. Then $fgf \sim f$, taking $X = A$ this means $f_*(gf) = f_*(1_A) = f$, therefore $gf \sim 1_A$.

Now take $f : A \to B$ to be a homotopy equivalence. Factor it as $A \underset{tr.cof}{\xrightarrow{g}} C \underset{fib}{\xrightarrow{p}} B$. Then C is both fibrant and cofibrant, so g is a homotopy equivalence by the first part of the theorem. Assume the homotopy inverse of f is $f' : B \to A$, and $H : B' \to B$ is a left homotopy from ff' to 1_B. Observe the square

$$\begin{array}{ccc} B & \xrightarrow{gf'} & C \\ {\scriptstyle \text{tr.cof}}\; \Big\downarrow {\scriptstyle i_0} & \overset{H'}{\nearrow} & {\scriptstyle p} \Big\downarrow \;{\scriptstyle \text{fib}} \\ B' & \xrightarrow[H]{} & B \end{array}$$

The lift H' is a left homotopy from gf' to $q = H'i_1$. We also get $pq = Hi_1 = 1_B$.

Let $g' : C \to A$ be a homotopy inverse for g. Then $p \sim pgg' \sim fg'$, therefore $qp \sim (gf')(fg') \sim 1_C$. Let $K : C' \to C$ be a left homotopy from 1_C to qp. $Ki_0 = 1_C$, i_0 and 1_C are weak equivalences, so K, and by that $Ki_1 = qp$ are weak equivalences too. The following commutes (with horisontal composites being identical)

$$\begin{array}{ccccc} C & = & C & = & C \\ \Big\downarrow {\scriptstyle p} & & \Big\downarrow {\scriptstyle qp} & & \Big\downarrow {\scriptstyle p} \\ B & \xrightarrow{q} & C & \xrightarrow{p} & B \end{array}$$

therefore p is the retract of qp, hence p and $f = gp$ are weak equivalences. □

Note, that this generalisation of Whitehead's theorem has nothing to do with homotopy groups, although they were introduced for arbitrary cofibrant A, and natural number n. It would be nice to see, how these notions are related. The following is an important consequence of the previous theorem.

Proposition 2.1.15. *If $\gamma : \mathscr{C}_{cf} \to Ho\ \mathscr{C}_{cf}$, $\delta : \mathscr{C}_{cf} \to \mathscr{C}_{cf}/\sim$ are the evident canonical functors, then $\exists!\ j : \mathscr{C}_{cf}/\sim\ \to Ho\ \mathscr{C}_{cf}$ isomorphism of categories, such that $j\delta = \gamma$. j is identity on objects.*

Proof. δ takes homotopy equivalences, therefore weak equivalences to isomorphisms. It would be enough to see, that it has the same universal property as γ, then Proposition 2.1.6 is applied.

Let $F : \mathscr{C}_{cf} \to D$ be a functor, that takes weak equivalences to isomorphisms. Assume $A \vee A \xrightarrow{(i_0,i_1)} \xrightarrow{s} A$ is a cylinder object. Then $s_{i_0} = s_{i_1} = 1_A$, s is a weak equivalence, so $Fi_0 = Fi_1$ is the inverse of Fs. For a left homotopy $H : A' \to B$ from f to g, $Ff = (FH)(Fi_0) = (FH)(Fi_1) = Fg$, so F identifies left (and dually: right) homotopic maps.

It implies, that F factors through ${}^{\mathscr{C}_{cf}}\!/\!_{\sim}$ as $F = G\delta$ by $G : {}^{\mathscr{C}_{cf}}\!/\!_{\sim} \to D$, where $G([f]) = Ff$. This G is unique and it is idenetity on objects, so δ has the required universal property. $\square$

The main goal of this section was the following theorem.

Theorem 2.1.16. *$\mathscr{C}$ is a model category.*

1. *$\mathscr{C}_{cf} \hookrightarrow \mathscr{C}$ induces ${}^{\mathscr{C}_{cf}}\!/\!_{\sim} \cong Ho\, \mathscr{C}_{cf} \to Ho\, \mathscr{C}$ equivalence of categories.*

2. *There are isomorphisms $Ho\, \mathscr{C}(\gamma X, \gamma Y) \cong {}^{\mathscr{C}(QRX, QRY)}\!/\!_{\sim} \cong {}^{\mathscr{C}(RQX, RQY)}\!/\!_{\sim} \cong {}^{\mathscr{C}(QX, RY)}\!/\!_{\sim}$ natural in all components. Hence $Ho\, \mathscr{C}$ is locally small.*

3. *$\gamma : \mathscr{C} \to Ho\, \mathscr{C}$ identifies left or right homotopic maps.*

4. *If γf is an isomorphism in $Ho\, \mathscr{C}$, then f is a weak equivalence.*

Proof. The first statement is already seen, the third one is in the proof of the previous proposition. For the second statement, recall that there is a natural transformation $qr : i \circ R \circ Q \to 1_{\mathscr{C}}$, whose components are weak equivalences. So we get a natural isomorphism from $Ho\ i \circ Ho\ R \circ Ho\ Q$ to $1_{\mathscr{C}}$, that gives $Ho\, \mathscr{C}_{cf}(\gamma QRX, \gamma QRY) \cong Ho\, \mathscr{C}(\gamma X, \gamma Y)$. Then use the isomorphism ${}^{\mathscr{C}_{cf}}\!/\!_{\sim} \cong Ho\, \mathscr{C}_{cf}$.

As $Ho\, i \circ Ho\, R \circ Ho\, Q \cong Ho\, i \circ Ho\, Q \circ Ho\, R (\cong 1_{\mathscr{C}})$ we have that $\gamma RQX \cong \gamma QRX$ in $Ho\, \mathscr{C}_{cf}$. Hence $Ho\, \mathscr{C}_{cf}(\gamma QRX, \gamma QRY) \cong Ho\, \mathscr{C}_{cf}(\gamma RQX, \gamma QRY)$

In the functorial factorisations $QX \to X$ is a trivial fibration and $X \to RX$ is a trivial cofibration. Using the forth statement of Proposition 2.1.10, we have ${}^{\mathscr{C}(\gamma QRX, \gamma QRY)}\!/\!_{\sim} \cong {}^{\mathscr{C}(\gamma RQX, \gamma QRY)}\!/\!_{\sim} \cong {}^{\mathscr{C}(\gamma QX, \gamma QRY)}\!/\!_{\sim} \cong {}^{\mathscr{C}(\gamma QX, \gamma RY)}\!/\!_{\sim}$ isomorphisms, natural in both components.

Finally, if γf is an isomorphism in $Ho\, \mathscr{C}$, then by the commutativity of

$$\begin{array}{ccc} X & \xrightarrow{\gamma f} & Y \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\ QRX & \xrightarrow{QRf} & QRY \end{array}$$

QRf is an isomorphism in $Ho\, \mathscr{C}_{cf}$, therefore in ${}^{\mathscr{C}}\!/\!_{\sim}$. So QRf is a homotopy equivalence in $\mathscr{C}_{cf}$, hence a weak equivalence. The following shows Rf and f to be weak equivalences:

$$
\begin{array}{ccc}
QRX & \xrightarrow{QRf} & QRY \\
\downarrow & & \downarrow \\
RX & \xrightarrow{Rf} & RY \\
\uparrow & & \uparrow \\
X & \xrightarrow{f} & Y
\end{array}
$$

□

2.2 exact sequences

Definition 2.2.1. *The category $\mathscr{C}$ is pointed, if it has an initial and a terminal object, and the unique map between them is an isomorphism.*

Some examples are **Top**$_*$ the category of pointed topological spaces, **Cat**$_*$ the category of small categories with a distinguished object and base-point preserving functors, and its full subcategory **Grp**. In general, one can always create a pointed category from a category with terminal object $(*)$ by concerning maps $* \to A$ as objects and commutative triangles as arrows.

In this section all categories are assumed to be pointed.

Definition 2.2.2. *The kernel of a map $f : A \to B$ is the equalizer of f and the zero-map $0 : A \to * \to B$. Dually, the cokernel of f is the coequalizer of f and 0.*

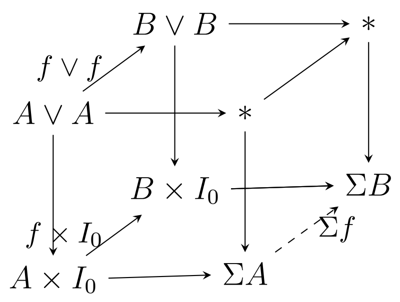
These (co)limits have equivalent characterisation as the pullback of f through $* \to B$ and dually, as the pushout of f through $A \to *$. Motivated by their realisation in **Top**, the (co)kernel is also referred as the (co)fibre of the map, hence the below terminology.

Definition 2.2.3. *In a pointed model category, the suspension of an object A is the cokernel of the cofibration map $A \vee A \to A \times I_0$ described in Definition 2.1.7.*

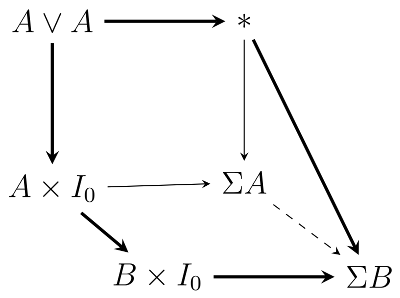
The loop space of A is the kernel of the (dual) fibration $A^{I_0} \to A \times A$.

Proposition 2.2.1. *This extends to the suspension and loop functors; $\Sigma, \Omega : Ho\,\mathscr{C} \to Ho\,\mathscr{C}$.*

Proof. By the commutativity of the left and back face of the cube



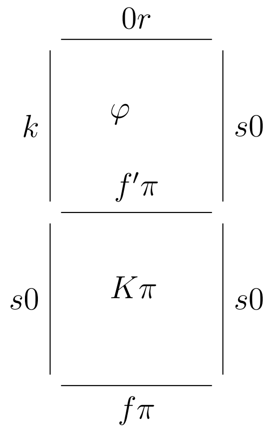
The outer square in the pushout diagram



commutes, this induces Σf. If f is a weak equivalence, then $f \times I_0$ is also, then by the 2-out-of-3 property so is Σf. Now it follows, that Σ and Ω are endofunctors of $Ho\ \mathscr{C}$. □

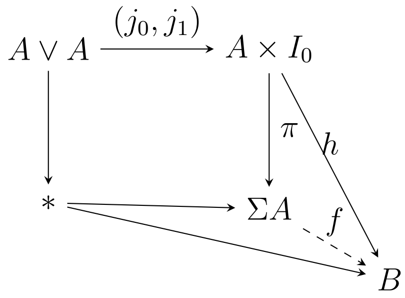
Proposition 2.2.2. *Σ is left adjoint to Ω (in $Ho\ \mathscr{C}$). Moreover, if $[A, B]_1$ is defined as $\pi_1(QA, RB)$, then there are isomorphisms $[\Sigma A, B] = [A, B]_1 = [A, \Omega B]$, natural in all components ($[X, Y]$ stands for $Ho\ \mathscr{C}(X, Y)$).*

Proof. First assume, that A is cofibrant and B is fibrant. Then the map $\rho : (f : \Sigma A \to B) \mapsto (f\pi : A \times I_0 \to B)$ (where $\pi : A \times I_0 \to \Sigma A$ is the cofibre of $A \vee A \to A \times I_0$) is a bijection. It is well-defined; i.e. let $K : \Sigma A \to B^I$ be a right homotopy from f to f', and let φ be a correspondence between $f'\pi$ and some right homotopy k. Form the squares



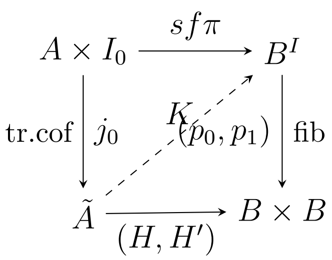
This shows, that $f\pi$ and $f'\pi$ correspond to the same right homotopy $k = (s0)k$, hence belong to the same homotopy class.

To see surjectivity take a left homotopy $h : A \times I_0 \to B$, with $h(j_0, j_1) = 0$ (Proposition 2.1.10 and 2.1.11 together imply, that any left homotopy is homotopic to one which maps from the functorial cylinder object). Then there is an induced map $f : \Sigma A \to B$ in the pushout



hence $h = f\pi$.

To prove injectivity assume $[f\pi] = [f'\pi]$, and let $H : \tilde{A} \to B$ be a left homotopy between them. Define $H' : \tilde{A} \to B$ by $H'j_0 = H'j_1 = f\pi$. Then there is a lift in



and $Kj_1 : A \times I_0 \to B^I$ is a right homotopy from $f\pi$ to $f'\pi$, with $Kj_1(i_0, i_1) = 0$. Therefore it factors through ΣA as $K'\pi$, where $p_0K'\pi = f\pi$ and $p_1K'\pi =$

$f'\pi$. It is enough to see, that π is an epimorphism (i.e. left-cancelable) which holds, as it is the pushout of an epi (its codomain is the terminal object).

This argument shows, that $\pi_1(A, B)$ is invariant under weak equivalences, in both variables. To see that $\pi_1 : Ho\,\mathscr{C}_c^{op} \times Ho\,\mathscr{C}_f \to \mathbf{Grp}$ is a functor, one has to check that for maps $a : A' \to A$, $b : B \to B'$ the following commutes

$$\begin{array}{ccc} \pi_1(A,B) \ni \alpha & \xmapsto{a^*} & \alpha(a \times I_0) \in \pi_1(A',B) \\ {\scriptstyle b_*}\downarrow & & \downarrow{\scriptstyle a^*} \\ \pi_1(A,B) \ni b\alpha & \xmapsto[b_*]{} & b\alpha(a \times I_0) \in \pi_1(A',B') \end{array}$$

associativity of composition in a fixed variable is immediate.

It remains to prove naturality. In the second variable it is automatic, in the first it follows from the equality $A \times I_0 \xrightarrow{a \times I_0} A' \times I_0 \xrightarrow{\pi'} \Sigma A' = A \times I \xrightarrow{\pi} \Sigma A \xrightarrow{\Sigma a} \Sigma A'$, which holds by definition.

Finally, we have the natural isomorphisms of functors

$$[\Sigma A, B] \cong [\Sigma QA, RB] \cong [A, B]_1 \cong [QA, \Omega RB] \cong [A, \Omega B]$$

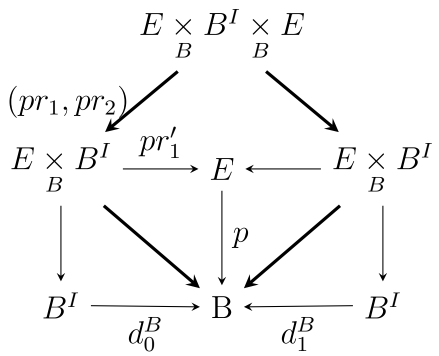
which complete the proof

□

In the case of topological spaces, there is a map $m : F \times \Omega B \to F$, that lifts the given loop with the given endpoint, and results the element of F, in which the lift ends. The analogous construction is given for any model category.

Let $p : E \to B$ be a fibration with fibre F. A path object of B is given by $B \xrightarrow[w.e.]{s^B} B^I \xrightarrow[fib]{(d_0^B, d_1^B)} B \times B$.

Factor $1_E \underset{B}{\times} s^B \circ p \underset{B}{\times} 1_E$ as $E \xrightarrow[w.e.]{s^E} E^I \xrightarrow[fib]{(d_0^E, p^I, d_1^E)} E \underset{B}{\times} B^I \underset{B}{\times} E$. Observe the following pullbacks:



By pullback stability (pr_1, pr_2) is a fibration, hence $(pr_1.pr_2)\circ(d_0^E, p^I, d_1^E) = (d_0^E, p^I)$ is also. pr'_1 is a trivial fibration. In the equation $1_E = pr'_1\circ(d_0^E, p^I)\circ s^E$ all arrows except (d_0^E, p^I) were shown to be weak equivalences, so by the '2-out-of-3' axiom it is a trivial fibration.

Our next goal is to show, that $\exists!\alpha$, such that

$$\begin{array}{ccc} \Omega B & \xrightarrow{\ j\ } & B^I \\ \alpha \Big\uparrow & \nearrow{\scriptstyle p^I \circ pr''_2} & \\ F \underset{E}{\times} E^I \underset{E}{\times} F & & \end{array}$$

commutes, where j is the inclusion of the fibre.

Using the universal property of the defining pullback of ΩB, it is enough to see, that $(d_0^B, d_1^B)\circ p^I \circ pr''_2 : F \underset{E}{\times} E^I \underset{E}{\times} F \to B \times B$ is equal to the zero-map. From the construction of the functorial path object it follows, that the square

$$\begin{array}{ccc} E^I & \xrightarrow{\ p^I\ } & B^I \\ d_0^E \Big\downarrow & & \Big\downarrow d_0^B \\ E & \xrightarrow{\ p\ } & B \end{array}$$

commutes. Therefore $(d_0^B, d_1^B)\circ p^I = p \times p \circ (d_0^E, d_1^E)$, and $(d_0^B, d_1^B)\circ p^I \circ pr''_2 = p \times p \circ (d_0^E, d_1^E) \circ pr''_2 = p \times p \circ i \times i \circ (pr_1, pr_3) = 0$

It is enough to prove, that the square

$$\begin{array}{ccc} F \underset{E}{\times} E^I \underset{E}{\times} F & \xrightarrow{\ pr''_2\ } & E^I \\ \pi = (pr''_1, \alpha) \Big\downarrow & & \Big\downarrow (d_0^E, p^I) \\ F \times \Omega B & \xrightarrow{\ i \underset{B}{\times} j\ } & E \underset{B}{\times} B^I \end{array}$$

is a pullback. Then π is a trivial fibration, so there is a map $m : F \times \Omega B \xrightarrow{\gamma(\pi)^{-1}} F \underset{E}{\times} E^I \underset{E}{\times} F \xrightarrow{\gamma(pr''_3)} F$ in $Ho\mathscr{C}$.

Definition 2.2.4. *A fibre sequence is the functorial image of the defining pullback of the fibre of a fibration p, and of* $m : \Omega B \times F \to F$ *at the functor* $\gamma : \mathscr{C} \to Ho\,\mathscr{C}$.

A cofibre sequence is the induced diagram in $Ho\,\mathscr{C}$ *by the defining pushout of the cofibre, and by the map* $n : C \vee \Sigma A \to C$.

In [5] Quillen proves the following result.

Theorem 2.2.3. *For a fibre sequence* $F \xrightarrow{i} E \xrightarrow{p} B$ *define the boundary map as* $\partial : \Omega B \xrightarrow{1_B \times 0} \Omega B \times F \xrightarrow{m} F$. *Then*

$$\cdots \to [A, \Omega E] \xrightarrow{(\Omega p)_*} [A, \Omega B] \xrightarrow{\partial_*} [A, F] \xrightarrow{i_*} [A, E] \xrightarrow{p_*} [A, B]$$

is an exact sequence of groups from $[A, \Omega E]$ *to the left, half exact at all stages (as a sequence of pointed sets), and*

- $\delta_*(\lambda_1) = \delta_*(\lambda_2) \Leftrightarrow \exists \mu \in [A, \Omega E] : \lambda_1 = \lambda_2 \cdot (\Omega p)_*(\mu)$
- $i_*(\alpha_1) = i_*(\alpha_2) \Leftrightarrow \exists \lambda \in [A, \Omega B] : \alpha_1 = \alpha_2 \cdot \lambda$
- $p_*^{-1}(0) = Im(i_*)$

Here central dot abbreviates $m_* : [A, \Omega B] \times [A, F] \to [A, F]$.

For a cofibre sequence $A \xrightarrow{i} X \xrightarrow{\psi} C$ *with* $\partial : C \xrightarrow{1_C + 0} C + \Sigma A \xrightarrow{n} \Sigma A$

$$\dots [\Sigma X, B] \xrightarrow{(\Sigma i)^*} [\Sigma A, B] \xrightarrow{\partial^*} [C, B] \xrightarrow{\psi^*} [X, B] \xrightarrow{i^*} [A, B]$$

is exact in the same sense, except that $\cdot$ *stands for the right action* $n^* : [C, B] \times [\Sigma A, B] \to [C, B]$.

For a straightforward application observe the following model structure on $\mathbf{Cat}_*$, the category of small pointed categories. The proof is given in [3].

Theorem 2.2.4. *There is a model structure on* $\mathbf{Cat}_*$, *where fibrations are isofibrations (i.e. whenever there is an isomorphism* $\varphi : a \cong b$ *in* $\mathscr{D}$, *such that* $a = F(x)$, *for the isofibration* $F : \mathscr{C} \to \mathscr{D}$, *then there is an isomorphism* ψ *in* $\mathscr{C}$ *for which* $\varphi = F(\psi)$*) cofibrations are functors, that are injective on objects and weak equivalences are those, which are part of an equivalence of categories.*

Proposition 2.2.5. *In the above model structure homotopy (between functors* $F, G : \mathscr{C} \to \mathscr{D}$*) is natural isomorphism (*η*), for which* $\eta_{x_0} = 1_{x_0}$ *(*x_0 *is the base point of* $\mathscr{C}$*).*

Proof. In the proof of the next corollary the functorial cylinder object is characterised, then it follows immediately. □

Corollary 2.2.5.1. *If $\varphi : \mathscr{C} \to \mathscr{D}$ is a functor that is injective on objects, $\psi : \mathscr{D} \to \mathscr{D}' = {}^{D}\!/_{< Im\ \varphi >}$ is the factor map, and $[\mathscr{C}, \mathscr{D}]$ denotes the homotopy classes of $\mathscr{C} \to \mathscr{D}$ functors, then*

$$0 \to Hom(F_{d'-1}, \mathscr{A}) \xrightarrow{(\Sigma\psi)^*} Hom(F_{d-1}, \mathscr{A}) \xrightarrow{(\Sigma\varphi)^*} Hom(F_{c-1}, \mathscr{A}) \xrightarrow{\partial^*}$$
$$\xrightarrow{\partial^*} [\mathscr{D}', \mathscr{A}] \xrightarrow{\psi^*} [D, \mathscr{A}] \xrightarrow{\varphi^*} [C, \mathscr{A}]$$

is an exact sequence of pointed sets. Here $\mathscr{A}$ is any (pointed, small) category, and a, c, d, d' is the number of connected components of the related categories.

If $p : \mathscr{E} \to \mathscr{B}$ is an isofibration, and $i : Ker\ p \hookrightarrow \mathscr{E}$ is the inclusion of the kernel, then

$$0 \to [\mathscr{A}, (ker\ p)^{\mathbb{Z}}] \xrightarrow{(\Omega i)_*} [\mathscr{A}, \mathscr{E}^{\mathbb{Z}}] \xrightarrow{(\Omega p)_*} [\mathscr{A}, \mathscr{B}^{\mathbb{Z}}] \xrightarrow{\delta_*}$$
$$\xrightarrow{\delta_*} [\mathscr{A}, ker\ p] \xrightarrow{i_*} [\mathscr{A}, \mathscr{E}] \xrightarrow{p_*} [\mathscr{A}, \mathscr{B}]$$

and

$$0 \to Hom(F_{a-1}, ker\ p) \to Hom(F_{a-1}, \mathscr{E}) \to Hom(F_{a-1}, \mathscr{B}) \to$$
$$\to [\mathscr{A}, ker\ p] \xrightarrow{i_*} [\mathscr{A}, \mathscr{E}] \xrightarrow{p_*} [\mathscr{A}, \mathscr{B}]$$

is exact.

Proof. The following is a pushout square:

$$\begin{array}{ccc} \mathscr{C} & \longrightarrow & * \\ {\scriptstyle\varphi}\downarrow & & \downarrow \\ \mathscr{D} & \xrightarrow{\psi} & \mathscr{D}' \end{array}$$

Therefore there is a cofibre sequence $\mathscr{C} \xrightarrow{\varphi} \mathscr{D} \xrightarrow{\psi} \mathscr{D}'$ with some suitable ∂.

Define $\mathscr{C} \times I$ as $\mathscr{C} \vee \mathscr{C}$ together with isomorphisms between the two copies of an object (except x_0), such that any square containing two of them commutes. Then the composition

$$\mathscr{C} \vee \mathscr{C} \xrightarrow{i} \mathscr{C} \times I \xrightarrow{pr} \mathscr{C}$$

is a factorisation of $1_{\mathscr{C}} \vee 1_{\mathscr{C}}$ to a cofibration followed by a trivial fibration. Here pr identifies the two copies of $\mathscr{C}$ and maps the vertical isomorphisms to identities. This is surjective, and by commutativity it is bijective on Hom-sets, so it is a trivial fibration. It follows, that $\mathscr{C} \times I$ is a cylinder object. Then the pushout

$$\begin{array}{ccc} \mathscr{C} \vee \mathscr{C} & \longrightarrow & * \\ {\scriptstyle i}\downarrow & & \downarrow \\ \mathscr{C} \times I & \longrightarrow & F_{c-1} \end{array}$$

shows, that the suspension of a category is the free group generated by $c-1$ elements, where c is the number of connected components in $\mathscr{C}$.

For any monoid $\mathscr{M}$ and category $\mathscr{C}$, $Hom(\mathscr{M}, \mathscr{C}) = [\mathscr{M}, \mathscr{C}]$, as in this case, a natural isomorphism would have only one component, which is the identity of the base point.

Now we construct the loop object. Let $\mathscr{C}^I$ be the functor category $\mathscr{C}^{\leftrightarrow}$. Note, that $\cdot \leftrightarrow \cdot$ is itself not pointed, the base point of $\mathscr{C}^{\leftrightarrow}$ is the functor, whose image is the identity of $x_0 \in \mathscr{C}$. Then

$$\mathscr{C} \xrightarrow{i} \mathscr{C}^I \xrightarrow{pr} \mathscr{C} \times \mathscr{C}$$

is a decomposition of $1_{\mathscr{C}} \times 1_{\mathscr{C}}$ to a trivial cofibration followed by a fibration. Here i maps every object to its identity, and pr takes an isomorphism to its endpoints.

The following is a pullback:

$$\begin{array}{ccc} \mathscr{C}^{\mathbb{Z}} & \longrightarrow & \mathscr{C}^I \\ \downarrow & & \downarrow {\scriptstyle pr} \\ * & \longrightarrow & \mathscr{C} \times \mathscr{C} \end{array}$$

Finally, exactness of the last sequence follows by adjointness. □

Corollary 2.2.5.2. $Hom(-,-) : \mathbf{Mon} \times \mathbf{Mon} \to \mathbf{Mon}$ *is left-exact in both variables.*

Proof. In the observed model structure every monoid homomorphism is a cofibration, and the fibrations are those, which are surjective on invertable elements (arrows). Therefore a short exact sequence is both a fibre and cofibre sequence, then the above result is applied. □

3 Constructions for model categories

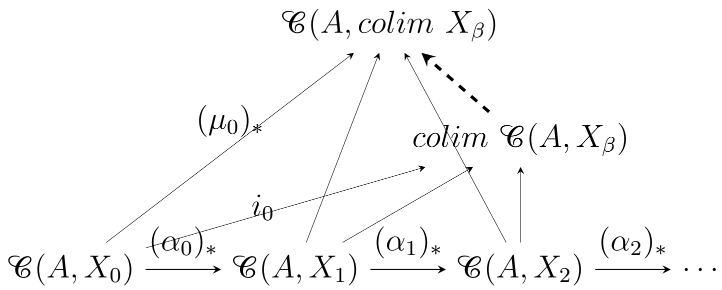
3.1 cofibrantly generated model categories

In this section a possible construction of model structures is considered. The hardest part is to create functorial factorisation; this motivates the following technicalities.

Definition 3.1.1. *Assume, the category* $\mathscr{C}$ *has all small colimits, and* λ *is an ordinal (represented as a category). Then a* λ*-sequence in* $\mathscr{C}$ *is a colimit-preserving functor* $X : \lambda \to \mathscr{C}$ *(i.e. for each limit ordinal* $\gamma < \lambda$ *the induced map* $\underset{\beta<\gamma}{colim}\ X_\beta \to X_\gamma$ *is an isomorphism). The map* $X_0 \to \underset{\beta<\lambda}{colim}\ X_\beta$ *of the universal cone is said to be the transfinite composition of the* λ*-sequence.*

Definition 3.1.2. *For a cardinal* γ*, the limit ordinal* α *is* γ*-filtered, if* $A \subseteq \alpha$*,* $|A| \leq \gamma \Rightarrow sup\ A < \alpha$*.*

Definition 3.1.3. *Assume* $\mathscr{C}$ *has all small colimits,* D *is a collection of arrows,* A *is an object,* κ *is a cardinal. Then* A *is* κ*-small rel.* D*, if for all* κ*-filtered ordinal* λ *and for all* λ*-sequence* $X : X_0 \xrightarrow{\alpha_0} X_1 \xrightarrow{\alpha_1} \dots$*, with* $\alpha_i \in D$*, the induced map* $! : colim\ \mathscr{C}(A, X_\beta) \to \mathscr{C}(A, colim\ X_\beta)$ *is bijective.*



Example 3.1.1. Every set A is $|A|$-small. To see this, take $f : A \to colim\ X_\beta$. For all $a \in A$ there is β_a, such that $f(a) \in im\ \mu_{\beta_a}$. $sup\ \beta_a = \gamma < \lambda$, as λ was $|A|$-filtered.

Now $f(a) \in im\ \mu_\gamma$, for all $a \in A$, as $\mu_{\beta_a} = \mu_\gamma \circ \alpha_{\beta_a\gamma}$, therefore f can be factored as $A \xrightarrow{g} X_\gamma \xrightarrow{\mu_\gamma} colim\ X_\beta = (\mu_\gamma)_*(g) = !i_\gamma(g)$, so ! is surjective.

To see injectivity, assume $!f_1 = !f_2$, then for all $a \in A$ there is β_a, such that $f_{1,\beta_a}(a) = f_{2,\beta_a}(a)$, where $f_i = \mu_{\beta_a} f_{i,\beta_a}$. Then $f_{1,\beta} = f_{2,\beta}$, where $\beta = \underset{a\in A}{sup}\ \beta_a$. Therefore $f_1 = f_2$.

Example 3.1.2. Every R-module A is $|A|(|A|+|R|)$-small. It can be proved similarly, and the proof is given in [2].

Definition 3.1.4. *$I \subseteq Arr\ \mathscr{C}$ is a subclass.*

- *I-inj is the class of morphisms, that have r.l.p. wrt. I.*
- *I-proj is the class of morphisms, that have l.l.p.wrt. I.*
- *I-cof=(I-inj)-proj*
- *I-fib=(I-proj)-inj*

Proposition 3.1.1. *The above classes are subcategories.* □

Lemma 3.1.2. *$I \subseteq I$-cof, $I \subseteq I$-fib, (I-cof)-inj=I-inj, (I-fib)-proj=I-proj, $I \subseteq J \Rightarrow I$-inj $\supseteq J$-inj, I-proj $\supseteq J$-proj, I-cof $\subseteq J$-cof, I-fib $\subseteq J$-fib.*

If $\mathscr{C}$ is a model category and I is the class of cofibrations, then I-inj is the class of trivial fibrations and I-cof=I.

If I is the class of fibrations, then elements of I-proj are trivial cofibrations and I-fib=I. □

Example 3.1.3. Take $\mathscr{C}$ = **Top** and let I consist of the boundary inclusions $S^{n-1} \hookrightarrow D^n$. Then a pushout from I is the attachment of an n-cell;

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & X \\ \downarrow & & \downarrow \\ D^n & \longrightarrow & X \coprod_{S^{n-1}} D^n \end{array}$$

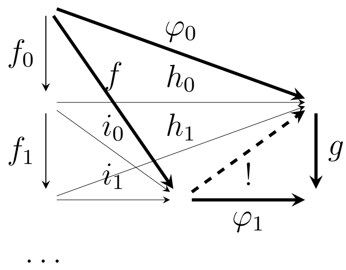
Take a transfinite composition from these maps, whose domain is a point. Codomains of such arrows are exactly the CW-complexes.

Definition 3.1.5. *I is a subset of $Arr\ \mathscr{C}$, $\mathscr{C}$ has all small colimits. A relative I-cell complex is a map, that is the transfinite composition of arrows, that are pushouts from I.*

Lemma 3.1.3. *All isomorphisms are in I-cell. I-cell $\subseteq I$-cof.*

Proof. An isomorphism $f : A \xrightarrow{\sim} B$ is the transfinite composition of the sequence whith only one object, A. For the second statement, it is enough to prove, that I-cof is closed under pushouts and transfinite composition. It was already seen, that left lifting properties are preserved by pushouts.

Assume $g \in I$-inj, $f_\alpha \in I$-cof, $(\alpha < \lambda)$. Take φ_0, φ_1 such that $g\varphi_0 = \varphi_1 f$. Then the lifts h_α $(\alpha < \lambda)$ exist, and the relations $gh_\alpha = \varphi_1 i_\alpha$, $h_\alpha f_\alpha = \varphi_0$ hold.



□

Lemma 3.1.4. *Take a λ-sequence, where each arrow is from I, or is an isomorphism. Then its transfinite composition is in I-cell.*

Proof. The transfinite composition of isomorphisms is itself an isomorphism. I.e. it can be choosen to be the first map in the sequence, while the rest of the arrows of the cone are defined via transfinite recursion. Composing an isomorphism after a pushout from I is again a pushout. □

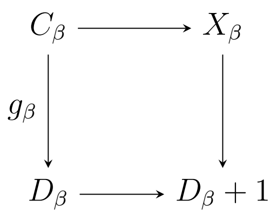
Lemma 3.1.5. *$\mathscr{C}$ has all small colimits, $I \subseteq Arr\,\mathscr{C}$ is a subset. Then I-cell is closed under transfinite composition.*

Proof. Such a sequence consists of isomorphisms and of transfinite compositions of pushouts from I. Replacing these with the sequence of composable arrows, the previous lemma can be applied. □

Lemma 3.1.6. *Pushout of coproduct of maps from I is in I-cell.*

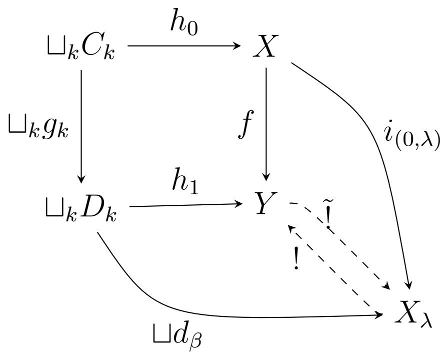
Proof. A λ-sequence is created by transfinit recursion, whose composition is isomorphic to the pushout $f : X \to Y$ of $\sqcup_{k \in K}\{g_k : C_k \to D_k\}$ through $h_0 : \sqcup_k C_k \to X$, and where the cardinality of λ is $|K|$.

For the recursion define $X_0 = X$, and for the successor ordinal $\beta + 1$ take the pushout



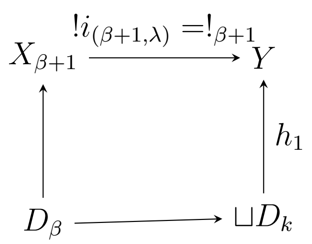
where the above vertical map is given by the composition $C_\beta \to \sqcup_k C_k \xrightarrow{h_0} X \xrightarrow{i_{(0,\beta)}} X_\beta$. When β is a limit ordinal, take $X_\beta = \underset{\alpha<\beta}{colim}\; X_\alpha$.

Using the universal properties of the defining colimits, we get for each β a universal map $!_\beta : X_\beta \to Y$, such that $!_\beta i_{(0,\beta)} = f$. f was also a pushout, therefore there is a map $\tilde{!} : Y \to X_\lambda = \underset{\alpha<\lambda}{colim}\; X_\alpha$, making



commute (where d_β is the map $D_\beta \to X_{\beta+1}$, appearing in the defining pushout of $X_{\beta+1}$, composed with $i_{(\beta+1,\lambda)}$). It is enough to prove that the triangles including $! =!_\lambda$ commute, then the outer square can also be proved to be a pushout, and finally ! turns out to be an isomorphism (hence $! \in I$-cell).

Commutativity of the upper triangle is immediate, for the lower note the commutativity of



(which follows from the construction of $!_{\beta+1}$, using universality of the defining pushout of $X_{\beta+1}$). Then composing the edges of the tringle after the coproduct cocone over the set $\{D_k\}_{k<\lambda}$ leads to the desired commutativity, by the universal property if the coproduct.

□

Theorem 3.1.7 (Small object argument)**.** *$\mathscr{C}$ has all small colimits, $I \subseteq$Arr $\mathscr{C}$ is a subset, domains of I are small rel. I-cell. Then there is (γ, δ) functorial factorisation, such that $\forall f \in$Arr $\mathscr{C}$: $\gamma(f) \in I$-cell, $\delta(f) \in I$-inj.*

Proof. Assume κ is a cardinal, such that domains of I are κ-small. Let λ be a κ-filtered ordinal and take an arrow $f : X \to Y$ from $\mathscr{C}$.

First the λ-sequence $Z^f : \lambda \to \mathscr{C}$ is defined, together with a natural transformation $\rho^f : Z^f \to !_Y$, where $!_Y$ is the constant Y functor (i.e. maps every object to Y and each arrow to 1_Y). It is defined through transfinite recursion, as

- $Z^f_0 = X$, $\rho^f_0 = f$

- for the successor ordinal $\beta + 1$:

  Take all pairs of arrows $(g_s, h_s) : g_s \in I$ $(s \in S)$, such that there is a commutative square

$$\begin{array}{ccc} A_s & \xrightarrow{h_s} & Z^f_\beta \\ {\scriptstyle g_s}\downarrow & & \downarrow{\scriptstyle \rho^f_\beta} \\ B_s & \longrightarrow & Y \end{array}$$

  Then the pushout of $\sqcup g_s : \sqcup_s A_s \to \sqcup_s B_s$ through $\sqcup h_s : \sqcup_s A_s \to Z^f_\beta$ is defined to be $i_{\beta,\beta+1} : Z^f_\beta \to Z^f_{\beta+1}$, and $\rho^f_{\beta+1} : Z^f_{\beta+1} \to Y$ is induced by the universal property of the pushout. This makes sense even when S is the empty set; in this case $i_{\beta,\beta+1} = 1_{Z^f_\beta}$.

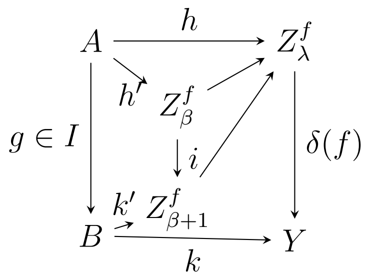
- When β is a limit ordinal take $Z^f_\beta = \underset{\alpha<\beta}{colim}\ Z^f_\alpha$. It is proved by transfinite induction, that the maps $\rho^f_\alpha : Z^f_\alpha \to Y$ form a cocone, then $\rho^f_\beta : Z^f_\beta \to Y$ is the induced morphism of cocones.

Define $\gamma(f) : X \to Z^f_\lambda = \underset{\alpha<\lambda}{colim}\ Z^f_\alpha$ as the transfinite composition of the previous λ-sequence, and $\delta(f) : Z^f_\lambda \to Y$ as the induced morphism of cocones. Then $f = \delta(f)\gamma(f)$, and from the previous lemmas it follows that $\gamma(f) \in I$-cell.

Now take a commutative square

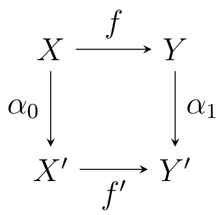
$$\begin{array}{ccc} A & \xrightarrow{h} & Z^f_\lambda \\ {\scriptstyle g \in I}\downarrow & & \downarrow{\scriptstyle \delta(f)} \\ B & \xrightarrow[k]{} & Y \end{array}$$

Domains of I are κ-small rel. I-cell, therefore h factores through $h' : A \to Z^f_\beta$ for some $\beta < \lambda$. The pair (g, h') is part of a commutative square ending in Y, so from the definition of $Z^f_{\beta+1}$ it follows, that there is a $k' : B \to Z^f_{\beta+1}$, such that everything commutes in

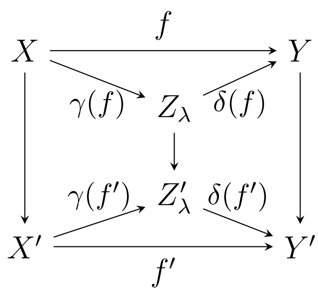


Therefore $\delta(f) \in I$-inj.

It remains to prove, that this factorisation is functorial. It is enough to see, that $Z^{(-)} : \ Arr\ \mathscr{C} \to \mathbf{Cat}$ is a functor. In this case a commutative square

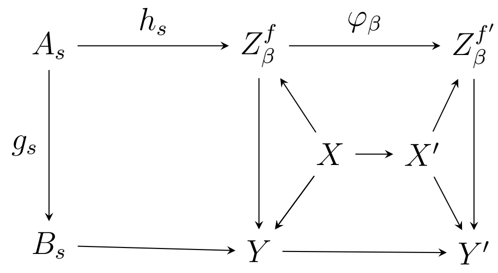


induces a natural transformation from Z^f to $Z^{f'}$, using its components, we get a cocone over the image of Z^f, ending in Z'_λ. Therefore we get a map $Z_\lambda \to Z'_\lambda$, that makes the left square commute in

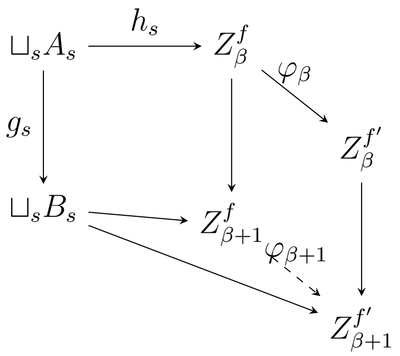


To see the commutativity of the right square, whose vertexes are all endings of cocones over Z^f (constructed by the composition of the square's edges after the colimiting object Z_λ), take the lift $Z_\lambda \to Y'$, that exists by universality, and what makes both of the resulting triangles commute.

The required natural transformation is constructed via transfinite recursion, with $\varphi_0 = \alpha_0$. First note, that for all $s \in S_\beta$ there is a commutative diagram



therefore there is a commutative outer square, and the universal property of the pushout induces $\varphi_{\beta+1}$:



When β is a limit ordinal, the component φ_β is induced by the universal property of the colimiting cocone.

□

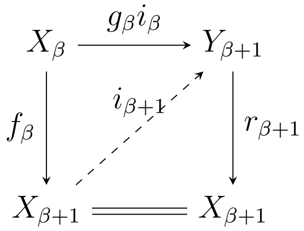
Corollary 3.1.7.1. *$\mathscr{C}$ has all small colimits, $I \subseteq$ Arr $\mathscr{C}$ is a subset, domains of I are small rel. I-cell. Then for all $f : A \to B \in I$-cof, there is a $g : A \to C \in I$-cell, such that f is the retract of g by a map, that fixes A.*

Proof. Factor f as $h \circ g$, where g is in I-cell and h is in I-inj. $f \in I$-cof, therefore f has l.l.p. wrt. h, so by the retract argument f is the retract of g, with 1_A in the appearing commutative squares. □

Theorem 3.1.8 (Hirschhorn)**.** *$\mathscr{C}$ has all small colimits, $I \subseteq$ Arr $\mathscr{C}$ is a subset, domains of I are small rel. I-cell. Assume A is small rel. I-cell. Then A is small rel. I-cof.*

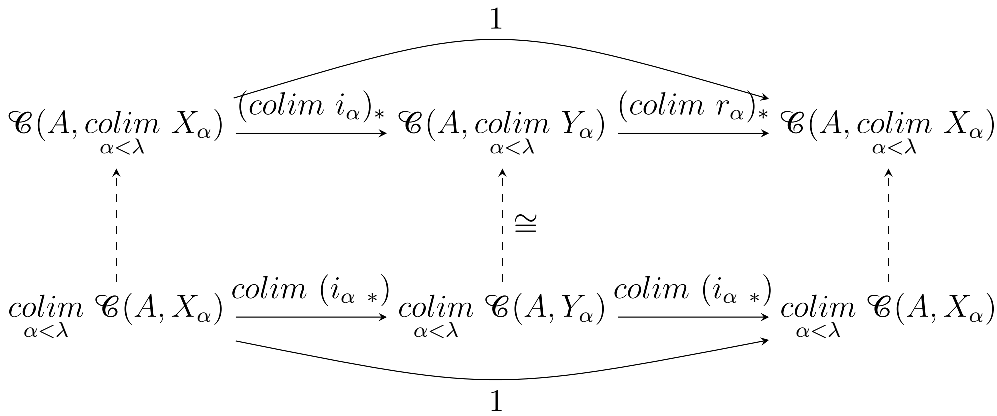
Proof. Assume A is κ-small rel. I-cell, λ is a κ-filtered ordinal, and $X : \lambda \to \mathscr{C}$ is a λ-sequence from I-cof arrows (f_β). The idea of the proof is to construct a λ-sequence $Y : \lambda \to \mathscr{C}$ from I-cell arrows (g_β), together with morphisms $i : X \to Y$ and $r : Y \to X$, such that $ri = 1_X$. The construction goes by transfinite recursion.

$Y_0 = X_0$, $i_0 = r_0 = 1_{X_0}$. By the retract argument there is a factorisation of $f_\beta r_\beta : Y_\beta \to X_{\beta+1}$ into an I-cell arrow $g_\beta : Y_\beta \to Y_{\beta+1}$, followed by $r_{\beta+1} : Y_{\beta+1} \to X_{\beta+1}$, which is in I-inj. As $f_\beta \in I$-cof, there is a lift $i_{\beta+1}$ in the square



When β is a limit ordinal, define Y_β to be $\underset{\alpha<\beta}{colim}\ Y_\alpha$, then (using the previously constructed i_α-s) we have a cocone over $X|_\beta$, ending in Y_β, now the induced map $X_\beta \to Y_\beta$ defines i_β. The analogous construction using r_α-s ($\alpha < \beta$) gives $r_\beta : Y_\beta \to X_\beta$. It follows from the universal property of the colimiting cocone over $X|_\beta$, that the equality $r_\beta i_\beta = 1_{X_\beta}$ holds.

The diagram



commutes. To see this, first remove the colimit operations from the lower line, then remove the hom-functors. Now put them back. The retract of an isomorphism is also an isomorphism.

□

Definition 3.1.6. *The model category $\mathscr{C}$ is cofibrantly generated if there are subsets I, J of Arr C, such that*

- *domains of I are small rel. I-cell,*
- *domains of J are small rel. J-cell,*

- *J-inj is the class of fibrations,*

- *I-inj is the class of trivial fibrations.*

I is referred as the set of generating cofibrations, J is the set of generating trivial cofibrations.

The followings are immediate from the previous results.

Proposition 3.1.9. *$\mathscr{C}, I, J$ are as before.*

- *I-cof is the class of cofibrations.*

- *Every cofibration is the retract of a map from I-cell.*

- *Domains of I are small relative to cofibrations.*

- *J-cof is the class of trivial cofibrations.*

- *Every trivial cofibration is the retract of a map from J-cell.*

- *Domains of J are small relative to trivial cofibrations.*

□

Finally, the main theorem of this section can be stated.

Theorem 3.1.10. *$\mathscr{C}$ has all small limits and colimits, $\mathscr{W}$ is a subcategory of $\mathscr{C}$, I and J are subsets of $Arr\,\mathscr{C}$. Then $\mathscr{C}$ is a cofibrantly generated model category with generating cofibrations I, trivial cofibrations J, and with $\mathscr{W}$ being the category of weak equivalences, iff*

- *$\mathscr{W}$ has the 2-out-of-3 property, and is closed under retracts,*

- *domains of I are small rel. I-cell,*

- *domains of J are small rel. J-cell,*

- *J-cell $\subseteq \mathscr{W} \cap I$-cof,*

- *I-inj $\subseteq \mathscr{W} \cap J$-inj,*

- *$\mathscr{W} \cap I$-cof $\subseteq J$-cof or $\mathscr{W} \cap J$-inj $\subseteq I$-inj.*

Proof. The implication from the left to the right is already proved. Now, take J-inj to be the class of fibrations, I-cof to be the class of cofibrations. These were shown to be subcategories.

Any lifting property is preserved by retracts, and both functorial factorisations are given by the small object argument. For the lifting axiom first assume $\mathscr{W} \cap I\text{-cof} \subseteq J\text{-cof}$. Then trivial cofibrations have l.l.p. wrt. J-inj, which are the fibrations. Take a trivial fibration $p : X \to Y$ and factor it as $\beta(p) \circ \alpha(p)$, where $\alpha(p) \in I$-cof and $\beta(p) \in I$-inj. It follows from the 2-out-of-3 property, that $\alpha(p)$ is a trivial cofibration, so p has right lifting property wrt. it. By the retract argument p is the retract of $\beta(p)$, so it is in I-inj. But I-inj $=(I$-cof$)$-inj, so p has r.l.p. wrt. cofibrations.

Now, assume $\mathscr{W} \cap J\text{-inj} \subseteq I\text{-inj}$. This means, trivial fibrations have r.l.p. wrt. I, which implies wrt. I-cof. Let $i : X \to Y$ be a trivial cofibration, and factor it as $\delta(i) \circ \gamma(i)$, where $\gamma(i) \in J$-cell and $\delta(i) \in J$-inj, which is a trivial fibration. So i has l.l.p. wrt. $\delta(i)$ and is a retract of $\gamma(i)$. Therefore $i \in J$-cof (as it is closed under retracts), then i has l.l.p. wrt. J-inj, which are the fibrations. □

3.2 chain complexes

Let R be a ring with unit element 1. The category $Ch(R)$ consists of chain complexes as objects (i.e. a $\mathbb{Z}$-indexed family of R-modules and R-module homomorphisms: $\dots X_k \xleftarrow{d_{k+1}} X_{k+1} \dots$, such that $d_{n-1} \circ d_n = 0$ for all $n \in \mathbb{Z}$), and of chain maps as arrows (i.e. commutative ladders). The n-th homology group of the chain complex X is defined as $H_n = {}^{ker\ d_n}\!/_{im\ d_{n+1}}$. This category admits a cofibrantly generated model structure:

Definition 3.2.1. *$S^n(M)$ is the chain complex, whose n-th module is M, the rest is zero. The $n-1$-th and n-th module of $D^n(M)$ is M, the rest is zero, d_{n-1} is identity. The abbreviations $S^n(R) = S^n$ and $D^n(R) = D^n$ are used.*

I consists of the inclusions $S^{n-1} \to D^n$, elements of J are the maps $0 \to D^n$. The class of fibrations is J-inj, the class of cofibrations is I-cof. An arrow φ is a weak equivalence, iff $H_n(\varphi)$ is an isomorphism for all $n \in \mathbb{Z}$. The weak equivalences trivially form a subcategory.

To prove, that this is a model structure, the requirements of Theorem 3.1.10 must be checked.

Proposition 3.2.1. *$\mathscr{W}$ has the 2-out-of-3 property, and is closed under retracts.*

Proof. The functorial image of a composition (resp. retract) is again a composition (or retract), and this property holds for the class of isomorphisms in any category. □

Proposition 3.2.2. *Every object X in $Ch(R)$ is small. Therefore domains of I are small rel. I-cell, and domains of J are small rel. J-cell.*

Proof. Let $\gamma > |R \times \bigcup_n X_n|$ be an infinite cardinal, λ is a γ-filtered ordinal, $Y : \lambda \to Ch(R)$ is a λ-sequence. The induced map $! : \underset{\alpha<\lambda}{colim}\ Hom(X, Y_\alpha) \to Hom(X, \underset{\alpha<\lambda}{colim}\ Y_\alpha)$ should be proved to be an isomorphism.

For surjectivity take a chain map $f : X \to \underset{\alpha<\lambda}{colim}\ Y_\alpha$. As X_α is γ-small $f_\beta : X_\beta \to \underset{\alpha<\lambda}{colim}\ (Y_\alpha)_\beta$ factors through Y_{α_β}. Then f factors through $Y_{\alpha = \underset{\beta<\lambda}{sup}\ \alpha_\beta}$, by a map $g : X \to Y$, which is a R-module homomorphism in each coordinate, although may not be a chain map. As for all $x \in X_k$ $f_{k-1}(d_k^X(x)) = d_k^{colim\ Y}(f_k(x))$ holds, there is $\beta_x > \alpha$, such that $g'_{k-1}(d_k^X(x)) = d_k^{Y_{\beta_x}}(g'_k(x))$, where $g' = i_{\alpha,\beta_x} g$ and $i_{\alpha,\beta_x} : Y_\alpha \to Y_{\beta_x}$ is in the λ-sequence Y. Then f factors through $Y_{\beta = \underset{x \in \bigcup_n X_n}{sup}\ \beta_x}$ as $f = \mu_\beta g''_\beta = !i_\beta(g''_\beta)$.

To see injectivity, assume $!f_1 = !f_2$, then (using previous notations) for all x there is β_x, such that $g''_{1,\beta_x}(x) = g''_{2,\beta_x}(x)$. Then $g''_{1,\beta}(x) = g''_{2,\beta}(x)$, where $\beta = \underset{x \in \bigcup_n X_n}{sup}\ \beta_x$. Therefore $f_1 = f_2$. □

Proposition 3.2.3. *p is a fibration, iff for each n, p_n is surjective.*

Proof. A chain map $D^n \to Y$ is completely determined by the image of 1 in Y_n (denoted by y_n). Then a lift in the square

$$\begin{array}{ccc} 0 & \longrightarrow & X \\ \downarrow & & \downarrow p \\ D^n & \longrightarrow & Y \end{array}$$

means precisely that $p_n^{-1}(y_n)$ is nonempty.

□

Proposition 3.2.4. *$p : X \to Y$ is a trivial fibration, iff $p \in I$-inj.*

Proof. A diagram of the form

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & X \\ \big\downarrow & & \big\downarrow p \\ D^n & \longrightarrow & Y \end{array}$$

is in one-to-one correspondence with a pair $(y, x) \in Y_n \oplus Z_{n-1}X$, for which $p_{n-1}x = dy$, while a lift is given by $z \in X_n$, such that $d_n z = x$ and $pz = y$.

First assume $p \in I$-inj (i.e. in any of the above squares there is a lift). For $n \in \mathbb{N}$ and $y \in Z_nY$, the pair $(y, 0)$ has the property $p_{n-1}0 = dy = 0$, therefore there is a $z \in X_n$, with $dz = 0$ and $pz = y$, which means, that $Z_np : Z_nX \to Z_nY$ is surjective. An element of $H_nY = {}^{\ker d_n}\!\diagup_{\text{im } d_{n+1}}$ is a set $\{y_0 + \tilde{y} : \tilde{y} \in \text{im } d_{n+1}\}$, where $y_0 \in Z_nY$. As $0 \in \text{im } d_{n+1}$, if y_0 is hit by Z_np, then $[y_0]$ is hit by H_np. Hence H_np is surjective.

p is also surjective (and therefore it is a fibration). I.e. if $y \in Y_n$, then $dy \in Z_{n-1}Y$, and by the surjectivity of $Z_{n-1}X$, there is $x \in Z_{n-1}X$, such that $p_{n-1}x = d_ny$. Using the lifting property again, an element z of X_n exists, for which $pz = y$.

Let $H_nX \ni [x] = [x_0 + 0]$, where $x_0 \in Z_nX$. $H_np([x_0]) = 0$ means $\exists y \in Y_{n+1} : p_nx = d_{n+1}y$. Therefore there is a $z \in X_{n+1}$ with $dz = x$, so $[x] = 0$.

Now assume p is a trivial fibration. We have to prove, that whenever (y, x) is given, and $y \in Y_n$, $x \in Z_{n-1}X$, $p_{n-1}x = d_ny$ holds, one can construct a $z \in X_n$ with $p_nz = y$ and $d_nz = x$. As p is a fibration, there is a short exact sequence

$$0 \to K \to X \xrightarrow{p} Y \to 0$$

p is a weak equivalence, therefore $H_*K = 0$, as H_* is an exact functor and $H_*(p)$ is an isomorphism. Choose $w \in X_n$, for which $p_nw = y$.

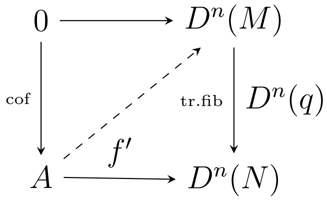
$p(dw) = d(pw) = dy = px$, so $p(dw - x) = 0$, i.e. $dw - x \in K$. $d(dw - x) = -dx = 0$ (as $x \in Z_{n-1}X$) and $H_*K = 0$ implies, that there is a $v \in K_n$, such that $dv = dw - x$. Now let z be $w - v$. Then $pz = y - pv = y$ (as $v \in K_n$) and $dz = dw - dv = x$ □

It only remains to prove J-cell $\subseteq \mathscr{W} \cap I$-cof. This is proved through several lemmas.

Lemma 3.2.5. *If the chain complex A is cofibrant, then for all n A_n is a projective R-module.*

Proof. Starting from any surjection $q : M \twoheadrightarrow N$ between the R-modules M and N, there is a fibration $D^n(M) \xrightarrow{D^n(q)} D^n(N)$, where $(D^n(q))_k = q$ for

$k = n,\ n-1$, and all other maps are zero. It is also a weak equivalence, as all homology groups are trivial. Given $f : A_{n-1} \to N$ there is an induced chain map $f' : A \to D^n(N)$, with $(f')_{n-1} = f$ and $(f')_n = f d_n$ being the nonzero maps. The lift in

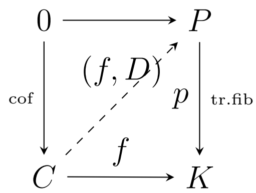


proves that A_{n-1} is projective. □

Definition 3.2.2. *The chain maps $f, g : X \to Y$ are chain homotopic, iff there are maps $D_n : X_n \to Y_{n+1}$, such that $d^Y_{n+1} D_n + D_{n-1} d^x_n = f_n - g_n$.*

Lemma 3.2.6. *If C is a cofibrant chain complex, and all homology groups of K are trivial, then any chain map $f : C \to K$ is chain homotopic to 0.*

Proof. Define the chain complex P by $P_n = K_n \oplus K_{n+1}$ and $d(x, y) = (dx, x - dy)$ (then $d^2(x, y) = (d^2x, dx - (dx - d^2y)) = 0$ as required). The projection $p : P \to K$, given by $(x, y) \mapsto x$ is surjective, hence is a fibration. $H_*(\ker p) = H_{*+1}(K) = 0$, which shows that p is also a weak equivalence. So there is a lift in



(f, D) is a chain map, therefore $(d^P_n f_n, f_n - d^K_{n+1} D_n) = (f_{n-1} d^C_n, D_{n-1} d^C_n)$, equivality of the second coordinates completes the proof. □

Proposition 3.2.7. *$i : A \to B$ is a cofibration, iff for all n i_n is a split injection, with cofibrant cokernel.*

Proof. First assume i is cofibrant. Then it has l.l.p. wrt. the trivial fibration $D^{n+1}(A_n) \to 0$ $(H_* D^{n+1}(A_n) = 0)$. Use it for a square, where the upper map $A \to D^{n+1}(A_n)$ is d in the $n+1$-th dimension and identity in the n-th, the rest are zero). Then $h_n i_n = 1_{A_n}$, so i_n is a split injection. L.l.p-s are preserved by pushouts, and $0 \to$coker i is the pushout of i through $A \to 0$.

Assume i_n is split injection, and C, the cokernel of i is cofibrant. Then by Lemma 3.2.5 C_n is projective, therefore B_n can be written as $A_n \oplus C_n$.

We have to prove that i has l.l.p. wrt. the trivial fibration $p : X \to Y$. Take a commutative square with horisontal maps $f : A \to X$ and $g : B \to Y$, and let $j : K = \ker p \hookrightarrow X$ denote the inclusion of the kernel.

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p}\ {\scriptstyle \text{tr.fib}} \\ B = A \oplus C & \xrightarrow{g} & Y \end{array}$$

Using that i is a chain map we get, that the differential of B can be written as $d(a, c) = (da + \tau c, dc)$, where $d\tau + \tau d = 0$. Because of the commutativity of the given square we can write $g(a, c) = pf(a) + \sigma(c)$. The fact that g is also a chain map implies, that $pf(da + \tau c) + \sigma(dc) = dpf(a) + d\sigma(c)$, hence $d\sigma = pf\tau + \sigma d$.

A lift in the above square is given by a pair (f, ν), where $p\nu = \sigma$ and $fda + f\tau c + \nu dc = dfa + d\nu c$, hence $d\nu = \nu d + f\tau$. The goal now is to construct such ν.

C_n is projective, so there is $G_n : C_n \to X_n$ with $p_n G_n = \sigma_n$. Define $r = dG - Gd - f\tau : C_n \to X_{n-1}$. As $pr = pdG - pGd - pf\tau = d\sigma - d\sigma = 0$, there is $s : C_n \to K_{n-1}$ with $js = r$. $dr = -dGd - df\tau = -dGd - fd\tau = -dGd + f\tau d = -rd$ shows that $s : C \to \Sigma K$ is a chain map (where $(\Sigma K)_n = K_{n-1}$ and $d^{\Sigma K} = -d^K$). By Lemma 3.2.6 s is chain homotopic to 0, i.e. there are arrows $D_n : C_n \to K_n$ with $-dD + Dd = s$. Take $\nu = G + jD$. Then $p\nu = pG = \sigma$ and $d\nu = \nu d + f\tau$, so $h = (f, \nu)$ is a lift.

□

Proposition 3.2.8. *$i : A \to B \in J$-cof, iff coker i is a projective chain complex, and i_n is injective for all n.*

Proof. First go from right to left. Take a square

$$\begin{array}{ccc} A & \xrightarrow{f} & M \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p}\ {\scriptstyle \text{fib}} \\ B & \xrightarrow{g} & N \end{array}$$

p is a surjection as it is in J-inj. As C is cofibrant, there is $r : B \to A$, such that $ri = 1_A$. $(pfr - g)i = 0$, so $pfr - g$ factors through $s : C \to N$. C is cofibrant, the lift $t : C \to M$ exists, then $fr - tj$ is a lift in the original diagram.

By Proposition 3.2.3 and 3.2.4 I-inj $\subseteq$ J-inj, therefore J-cof $\subseteq$ I-cof. Take $i \in J$-cof. Then i is injective, let $q : B \to C$ the factor map onto the cokernel. There is a lift in

$$\begin{array}{ccc} A & \xrightarrow{0} & M \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p}\ {\scriptstyle \in J\text{-inj}} \\ B & \xrightarrow{fq} & N \end{array}$$

for any surjective p, and $hi = 0$, so h factors through C as gq. q is an epimorphism, so the equality $pgq = fq$ implies $pg = f$, hence C is projective. □

Corollary 3.2.8.1. *J-cof $\subseteq \mathscr{W} \cap I$-cof.*

Proof. Again, take $i : A \to B \in J$-cof. Its cokernel C is projective, so it has l.l.p. wrt. fibrations, so it is cofibrant (i.e. $0 \to C \in I$-cof). Therefore $i : A \to B = A \oplus C$ is also a cofibration.

It is enough to prove, that $H_*C = 0$. Define P as in the proof of Lemma 3.2.6. Let $p : P \to C$ be projection to the first component; there is a retraction $(1, D) : C \to P$, where $dDx + Ddx = x$. If $dx = 0$ then $x = dDx$, so $H_*C = 0$ □

This proves, that $Ch(R)$ is a cofibrantly generated pointed model category, therefore the previous results can be applied. Now it is possible to define suspension and loop space for chain complexes, write out the exact sequences, or put Whitehead's theorem and cofibrant replacement in context. The details are not worked out here.

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