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SPECTRA OF GRAPHS IN A CONVERGENT GRAPH SEQUENCE

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Contents

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Introduction

We consider a graph sequence (G_n) and define various notions of convergence in Chapter 2. In the sparse graphs, the convergence notion has been introduced by Benjamini and Schramm, defined in terms of neighbourhood sampling statistics. We also define it in terms of the homomorphism frequencies from connected graphs into G_n. By [1], we show that these two definitions are equivalent. As to the dense graph sequence, we define the convergence by subgraph density (the number of subgraph in G_n over a suitable normalization). G_n is convergent if subgraph density from any graph to G_n is convergent. The convergence of subgraph density leads to the definition of graphon. It can be comprehended as a symmetric measurable function. It was introduced in [2] (Fulkerson Prize 2012). In [3], P. Frenkel unifies two definitions in sparse and dense graph sequence in one. The homomorphism density now relies on the degree bound parameter.

The spectral (eigenvalue) measure of a graph is the empirical measure of the graph's eigenvalue random variables. For example, the spectral measure of the cycle sequence and path sequence tends to an arcsine distribution. In general, the spectral measure of bounded degree graph sequence converges weakly and usually goes to non-trivial distribution. In the dense graph sequence, we inspect the convergence of non-negative $i-$th largest eigenvalue and non-positive $i-$th smallest eigenvalue of the uniformly convergent graphon sequence (Theorem 2.2). In theorem 2.3, we take the empirical measure of r largest (smallest) eigenvalue. It also converges to probability measures supported on $[0, 1]$ ($[-1, 0]$).

In chapter 3, we rescale the eigenvalue measure by taking eigenvalues divided by the square root of the degree bounded and study it in some special graph sequences. The rescaled spectral measure of $n-$ hypercube sequence converges to standard normal distribution. Or else the rescaled spectral measure of $\alpha-$ regular, large girth graph sequence tends to Wigner distribution. Because of large girth property, we can consider the graph sequence as a tree sequence when n is big enough. In a tree, the eigenvalue measure equals to the matching measure. By using some results in [3], [4] it follows.

In Chapters 2 and 3, we only consider finite graphs while in Chapter 4, we extend for infinite graphs of bounded degree. In particular, we define the expected spectral measure for a random rooted graph and prove that it equals the usual spectral measure in a uniform rooted finite graph. Then we examine the spectral measure of some infinite graph sequences considered as Cayley graphs. From these conceptions, we show the pointwise convergence of the expected spectral measure in the special case of Lück Approximation theorem, the detailed proof we can see in [5].

1 Some basic notions and preliminaries

In this section, we introduce our preparations. Related monographs include [6], [7], [8].

1.1 Graph notions

Graphs mostly considered in the literature are simple graphs. A **simple graph** $G = (V, E)$ consists of the vertex set $V(G)$ with $v(G) = |V(G)|$ vertices and the edge set $E(G)$ with $e(G) = |E(G)|$ edges.

The **adjancency matrix** $A(G)$ of the simple graph G is the $v(G) \times v(G)$ matrix, defined by

$$(A_G)_{ij} = \begin{cases} 1, & \text{if } \ ij \in E(G) \\ 0, & \text{if } \ ij \notin E(G) \end{cases}$$

A_G is real and symmetric, so its eigenvectors can be chosen to provide an orthonormal basis for $\mathbb{R}^{v(G)}$. We denote the eigenvalues of A_G by $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_{v(G)}$ and the corresponding orthonormal eigenvectors by $v_1, v_2, ..., v_{v(G)}$.

The **diameter** of a graph is the maximum eccentricity of any vertex in the graph. That is, it is the greatest distance between any pair of vertices.

X, Y be graphs. A mapping $f : V(X) \to V(Y)$ is a **homomorphism** if $f(X)$ and $f(Y)$ are adjacent in Y whenever x and y are adjacent in X. When X and Y have no loops, which is our usual case, this definition implies that if $x \sim y$ then $f(x) \neq f(y)$.

X, Y are **isomorphic** if there is a bijection, ϕ say, $V(X) \to V(Y); x \longmapsto \phi(X)$ such that $x \sim y$ in $X \Longleftrightarrow \phi(x) \sim \phi(y)$ in Y. We say that ϕ is an **isomorphism** from X to Y.

Automorphism is an isomorphism from a graph X to itself. The set of all automorphisms of X forms a group, which is called the **automorphism group** of X and denoted by $\mathrm{Aut}(X)$.

A subgraph Y of X is an **induced subgraph** if two vertices of $V(Y)$ are adjacent in Y if and only if they are adjacent in X.

For two finite simple graphs F and G, let $\mathbf{hom}(\boldsymbol{F}, \boldsymbol{G})$ denote the number of homomorphisms of F into G, $\mathbf{inj}(\boldsymbol{F}, \boldsymbol{G})$ be the number of injective homomorphisms of F into G, and $\mathbf{ind}(\boldsymbol{F}, \boldsymbol{G})$ be the number of embeddings of F into G as an induced subgraph.

The **girth** of a graph is the length of the shortest cycle contained in it. If a graph contains no cycles, its girth is defined to be ∞.

A **k-matching** in a graph G is a set of k edges, no two of which have a vertex in common (i.e., an independent edge set of size k).

S_n	a star graph with n vertices	$\lambda_1 = \sqrt{n-1}$; $\lambda_n = -\sqrt{n-1}$; $\lambda_i = 0 \ \forall i = 2, 3, ..., n-1$.
K_n	a complete graph with n vertices	$\lambda_1 = n-1$; $\lambda_2 = -1$ with multiplicity $n-1$
C_n	a cycle graph with n vertices	$\lambda_i = 2\cos\frac{2\pi}{n}i$
P_n	a path with n vertices	$\lambda_i = 2\cos\frac{\pi}{n+1}i$
Q_n	$n-$ dimensional hypercube	$\lambda_i = n - 2i$ with multiplicity $\binom{n}{i}$ where i ranges from 0 to n.

Table 1: Some typical graph's spectra

1.2 Spectra of graphs

We will determine the spectra of some graphs. It is given in the table (1).

Proof. (see [7] 11.1,11.4,11.9 problems)

- The adjancency matrix of S_n

$$A_{S_n} = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

  If $v = (v_1, v_2, \cdots, v_n)^T$ is an eigenvector with eigenvalue λ, then

$$\lambda v_2 = \lambda v_3 = \cdots = \lambda v_n = v_1 \text{ and } v_2 + \cdots + v_n = \lambda v_1.$$

  If $\lambda \neq 0$, that implies $(n-1)/\lambda = \lambda$, $\lambda = \pm\sqrt{n-1}$
  If $\lambda = 0$ then $v_1 = 0$, $v_2 + \cdots + v_n = 0$. We can choose $n-2$ linearly independent eigenvectors of S_n because $\dim\{(v_1, \cdots, v_n) | v_1 = 0, v_2 + \cdots + v_n = 0\} = n-2$. Hence the multiplicity of $\lambda = 0$ is $n-2$.

- For the complete graph, we have $(1, 1, \cdots, 1)$ is an eigenvector with the eigenvalue $n-1$ and we also have $v_i = (1, 0, \cdots 0, -1, 0, \cdots, 0)$ (-1 is the $(i+1)$-th component and $i = 1, 2, \cdots, n-1$) are $(n-1)$ linearly independent eigenvectors corresponding to the eigenvalue -1.

- Let

$$W = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

  Notice that W^k is then the permutation matrix whose first row has a single 1, in position $k+1$, and whose subsequent rows are shifted to the right as for W. This makes sense for any k by just taking it modulo k. And

$$A_{C_n} = W + W^{-1}$$

  So if $v = (v_1, v_2, \cdots, v_n)^T$ is an eigenvector with eigenvalue λ , then we have $v_1 = \lambda v_n = \lambda^2 v_{n-1} = \cdots = \lambda^{n-1} v_2 = \lambda^n v_1$. That implies the eigenvalues are among the $n-$th roots of unity and $\lambda_k = e^{2\pi ik/n} = \epsilon^k$ with multiplicity 1. Because if we choose $v_1 = 1$, the equations above give the eigenvector $u_k = (1, \epsilon^k, \epsilon^{2k}, \cdots, \epsilon^{(n-1)k})$. Because each W^k has the same eigenvector for corresponding eigenvalues, we get the eigenvalues of C_n

$$A_{C_n} u_k = W u_k + W^{-1} u_k = e^{2\pi ik/n} u_k + e^{-2\pi ik/n} u_k = 2 \cos \frac{2\pi k}{n} u_k.$$

- By taking an eigenvalue of multiplicity 2 of the $(2n+2)$-cycle, we get the eigenvalue of P_n. Precisely, let $\zeta = e^{\frac{2\pi ij}{2n+2}}$ be a $(2n+2)$-th root of unity for a fixed $j \in \{0, 1, \ldots, 2n+1\}$. Then, the vectors $u(\zeta)$ and $u(\zeta^{-1})$ are eigenvectors of C_{2n+2} having the common eigenvalue $2\cos(\pi j/(n+1))$. All eigenvectors have the property that opposite vertices have opposite values. So if one vertex has value 0 for an eigenvector, then so has the opposite one, and we find an eigenvector$(u(\zeta) - u(\zeta^{-1}))$ for the disjoint union of two paths P_n.

- As to the hypercube, we prove by induction. Denote $A_n = A_{Q_{n-1}}$ the adjancency matrix of the $n-1$-cubes then

$$A_{n+1} = \begin{pmatrix} A_n & I_{2^{n-1}} \\ I_{2^{n-1}} & A_n \end{pmatrix}$$

  Let $\chi_n(\lambda) = \det(\lambda I_{2^n} - A_{n+1})$ be the characteristic polynomial of the n-dim hypercube.

  Notice for any $2m \times 2m$ matrix $X(A)$ of the form $\begin{pmatrix} A & I_m \\ I_m & A \end{pmatrix}$. When A is invertible, we have:

$$\begin{pmatrix} A & I_m \\ I_m & A \end{pmatrix} \begin{pmatrix} I_m & -A^{-1} \\ 0 & I_m \end{pmatrix} = \begin{pmatrix} A & 0 \\ I_m & A - A^{-1} \end{pmatrix}$$

This implies

$$\det X(A) = \det(A)\det(A - A^{-1}) = \det(A^2 - I_{2m}) = \det(A - I_{2m})\det(A + I_{2m})$$

Since both side of this identity are polynomials in entries of A, this identity is true even when A is not invertible.
Apply this to $A_{n+1} - \lambda I_{2^n}$, we immediately obtain:

$$\chi_n(\lambda) = \chi_{n-1}(\lambda + 1)\chi_{n-1}(\lambda - 1)$$

So if the roots for the $(n-1)$-dim hypercube are $n-1, n-3, ..., -(n-1)$ with multiplicities

$$\binom{n-1}{0}, \binom{n-1}{1}, \binom{n-1}{2}, \ldots$$

then the roots for the n-dim hypercube are $n = (n-1)+1, n-2 = (n-3)+1 = (n-1)-1, ..., -n$ with multiplicities

$$1 = \binom{n}{0}, \binom{n}{1} = \binom{n-1}{0} + \binom{n-1}{1}, \binom{n}{2} = \binom{n-1}{1} + \binom{n-1}{2}, \ldots$$

■

Lemma 1.1. *Let λ_1 be the largest eigenvalue of the graph G and let d_{min}, D be the minimum and maximum degree of G. Then*

$$\max\{d_{min}, \sqrt{D}\} \leq \lambda_1 \leq D.$$

Proof. Using the classical characterization of the largest eigenvalue λ_1 in terms of the Rayleigh quotient of A where A be the adjancency matrix of G, we have:

$$\lambda_1 = \max_{y \neq 0} \frac{y^T A y}{y^T y}$$

Let $e = (1, 1, ..., 1)^T$ all of whose entries are 1. Then $\lambda_1 \geq \dfrac{e^T A e}{e^T e} \geq \dfrac{e^T d_{min} e}{e^T e} = d_{min}$.
Let v_1 be eigenvector corresponding to λ_1 and we can suppose that all elements of v is not exceed 1 expect the first element is 1, ie. $v_1 = (1, v_{12}, ..., v_{1n})$ where $v_{12}, ..., v_{1n} \leq 1$ then

$$\lambda_1 v_1 = A v_1 \leq Ae \leq De.$$

Compare the first coordinate $(\lambda_1 v_1)_1 = \lambda_1 \leq D = (De)_1$
Let G' be a subgraph of G which is a star graph with D point. Let λ_1' be the largest eigenvalue of G', then $\lambda_1' \leq \lambda_1$. The eigenvalues of G' are $\pm\sqrt{n-1}$ and 0 ($n-2$ times), so G' has the largest eigenvalue is $\sqrt{D}$ implies $\lambda_1 \geq \lambda_1' = \sqrt{D}$. ■

Lemma 1.2. *All eigenvalues of a graph belong to the interval $[-D; D]$.*

Proof. Recall the Gershgorin circle theorem, it claims that every eigenvalue of A lies within at least one of the Gershgorin discs $\left(a_{ii}, \sum_{j\neq i}|a_{ij}|\right)$. Here $a_{ii}=0, \sum_{j\neq i}|a_{ij}|$ be the degree of the $i-$th vertex which is bounded by D for all i. So all the eigenvalues will lie within $(0,D)$ disk. ∎

Lemma 1.3. *Let T be a tree in which every vertex has degree at most D, $D\geq 2$. Then, all eigenvalues of T have absolute value at most $2\sqrt{D-1}$.*

Proof. Choose aimlessly a vertex to be the root of the tree, and define its height to be 0. For every other vertex v_i, define its height, $h(v_i)$, to be its distance to the root. Let H be the diagonal matrix as below

$$H=\begin{pmatrix} \left(\sqrt{D-1}\right)^{h(v_1)} & 0 & \cdots & 0 \\ 0 & \left(\sqrt{D-1}\right)^{h(v_2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \left(\sqrt{D-1}\right)^{h(v_n)} \end{pmatrix}$$

We know two similar matrices have the same eigenvalues, so the eigenvalues of A_T are the same as the eigenvalues of HA_TH^{-1}. Using the Gershgorin circle theorem, we need to prove that all row sums of HA_TH^{-1} are at most $2\sqrt{D-1}$. The sum of $i-$th row is

$$\sum_{v_iv_j\in E(G)}\left(\sqrt{D-1}\right)^{h(v_i)-h(v_j)}$$

If v_i is the root, the row of the root has up to D entries that are all $1/\sqrt{D-1}$ then the row's sum is $\leq \dfrac{D}{\sqrt{D-1}}\leq 2\sqrt{D-1}$.
If v_i is the intermediate vertex, one entry in their row that equals $\sqrt{D-1}$ and up to $D-1$ entries that are equal to $1/\sqrt{D-1}$, for a total of $2\sqrt{D-1}$.
If v_i is a leaf (the degree of v_i is 1), then its row has only one nonzero entry and this entry equals $\sqrt{D-1}$. ∎

1.3 Convergence of probability measures

Let X be a set with $\sigma-$algebra F. We say that $\mu: F\to[0,\infty]$ is a **measure** if $\mu(\emptyset)=0$ and $\mu(\bigcup_{j=1}^{\infty}A_j)=\sum_{j=1}^{\infty}\mu(A_j)$ if $A_j\bigcap A_k=\emptyset$ for all $j\neq k$ (σ-additivity). If μ returns results in the unit interval $[0,1]$, returning 0 for the empty set and 1 for the entire space, then μ is called a **probability measure**.
If X is a metric space and $f: X\to\mathbb{R}$ is continuous, the support of f is defined by $\text{supp}f=\{x\in X: f(x)\neq 0\}$. We define the **support** of a measure μ on X by

$$\begin{aligned}\text{supp}(\mu)&=\{x\in X:\mu(B(x))>0 \ \text{ for any open neighborhood } B(x) \text{ of } x\}\\ &=X\setminus\{x\in X: \text{there exists an open neighborhood } \ B(x) \text{ of } x \text{ with } \mu(B(x))=0\}\end{aligned}$$

Convergence of probability measures

Let (S,d) be a metric space with Borel $\sigma-$field $\mathcal{S} = \mathbb{B}(S)$ (the smallest σ-algebra in S that contains all open subsets of S). Let μ and μ_n, $n \in \mathbb{N}$ be probability measures on $(S, \mathcal{S})$.

A sequence μ_n is said to *converge strongly* to a limit μ if $\lim_{n\to\infty} \mu_n(A) = \mu(A)$ for all Borel sets A.

We say that $(\mu_n)_{n\in\mathbb{N}}$ **converges weakly** to μ if

$$\int f\, d\mu_n \to \int f\, d\mu$$

for all bounded continuous functions $h : S \to \mathbb{R}$.

$(\mu_n)_{n\in\mathbb{N}}$ *converges in variation* to μ if

$$\|\mu_n - \mu\| = \sup_{A\in\mathcal{S}} |\mu_n(A) - \mu(A)| \longrightarrow 0 \text{ for } n \to \infty.$$

If one thinks of μ_n, μ as the distributions of $S-$valued random variables X_n, X, one often uses instead of weak convergence of μ_n to μ the terminology that the X_n converge to X *in distribution.*

Portmanteau's Theorem : The following statements are equivalent:

(i) $\mu_n \to \mu$ weakly.

(ii) $\int f\ d\mu_n \to \int f\ d\mu$ for all uniformly continuous and bounded $f : S \to \mathbb{R}$.

(iii) $\limsup_{n\to\infty} \mu_n(F) \le \mu(F)$ for all measurable closed subsets F.

(iv) $\liminf_{n\to\infty} \mu_n(U) \ge \mu(U)$ for all measurable open subsets U.

(v) $\lim_{n\to\infty} \mu_n(A) = \mu(A)$ for all measurable A with $\mu(\partial A) = 0$.

Wigner semicircle distribution

The Wigner semicircle distribution is the probability distribution supported on the interval $[-R, R]$ the graph of whose probability density function f is a semicircle of radius R centered at $(0, 0)$ and then suitably normalized:

$$f(x) = \frac{2}{\pi R^2}\sqrt{R^2 - x^2}$$

for $-R \le x \le R$, and $f(x) = 0$ if $|x| > R$.

Arcsine distribution

The arcsine distribution is the probability distribution supported on the interval $[a, b]$ whose cumulative distribution function is

$$F(x) = \frac{2}{\pi} \arcsin\left(\sqrt{\frac{x-a}{b-a}}\right)$$

for $a \le x \le b$, and whose probability density function is

$$f(x) = \frac{1}{\pi\sqrt{(x-a)(b-x)}} \mathbb{1}_{x\in(a,b)}$$

1.4 Spectra of operators

Let $(X, \|\cdot\|_1)$ and $(Y, \|\cdot\|_2)$ be normed spaces over the field $\mathbb{K}$ (the scalar field here is either $\mathbb{R}$ or $\mathbb{C}$). We say that $T : X \longrightarrow Y$ is a **bounded linear operator** if T satisfies:
(i) linearity: $T(ax + by) = aT(x) + bT(y)$ for all $x, y \in X$ and $a, b \in \mathbb{K}$.
(ii) bounded property:

$$\|T\| := \sup_{x \neq 0} \frac{\|Tx\|_2}{\|x\|_1} < \infty$$

$\|T\|$ was defined above is the operator norm of T . Let $B(X, Y)$ denote the set of all bounded linear operators from X to Y.
A **compact operator** is a linear operator L from a Banach space X to another Banach space Y, such that the image under L of any bounded subset of X is a relatively compact subset (has compact closure) of Y.
Let $(H_1, \langle \cdot, \cdot \rangle)_1$ and $(H_2, \langle \cdot, \cdot \rangle)_2$ be Hilbert spaces and $T \in B(H_1, H_2)$. We define the **adjoint** T^* of T to be the unique operator $T^* \in B(H_2, H_1)$ satisfying

$$\langle Tx, y \rangle_2 = \langle x, T^* y \rangle_1 \qquad \forall\, x \in H_1, y \in H_2$$

The existence and uniqueness can by proved by the Riesz representation theorem. Moreover, $\|T^*\| = \|T\|$.
We say that $T : H \to H$ is **self-adjoint** if $T^* = T$.
A **Hilbert–Schmidt operator** is a bounded operator T on a Hilbert space H with finite Hilbert–Schmidt norm

$$\|T\|_{\mathrm{HS}}^2 = \mathrm{Tr}(T^* T) := \sum_{i \in I} \|Te_i\|^2$$

where $\|\cdot\|$ is the norm of H, $\{e_i : i \in I\}$ an orthonormal basis of H and Tr is the trace of a nonnegative self-adjoint operator. Note that the index set need not be countable; however, at most countably many terms will be non-zero.

Definition 1.1. Let $\ell^2(G) = \{f : V(G) \to \mathbb{C} : \sum_{v \in V(G)} |f(v)|^2 < \infty\}$ is the set of square summable functions on the vertex set of G. Note that if the graph is finite then $\ell^2(G) = \mathbb{C}^{V(G)}$.
Assume that G has a degree bound D. For $f \in \ell^2(G)$ and $x \in V(G)$ we define the **adjacency operator** $A : \ell^2(G) \to \ell^2(G)$ as follows.

$$(Af)(x) = \sum_{(x,y) \in E(G)} f(y).$$

Lemma 1.4. *The adjacency operator A of a graph G with degrees bounded by D is a self-adjoint, bounded operator with $\|A\| \leq D$.*

Proof.

$$\|Af\|^2 = \sum_{x\in V(G)} |(Af)(x)|^2 = \sum_{x\in V(G)} \left| \sum_{(x,y)\in E(G)} f(y) \right|^2 .$$

By Cauchy-Schwarz inequality,

$$\left| \sum_{(x,y)\in E(G)} f(y) \right| \le \sum_{(x,y)\in E(G)} |f(y)| \le \left(\sum_{(x,y)\in E(G)} 1^2 \right)^{1/2} \left(\sum_{(x,y)\in E(G)} |f(y)|^2 \right)^{1/2}$$
$$= \sqrt{\deg(x)} \left(\sum_{(x,y)\in E(G)} |f(y)|^2 \right)^{1/2} .$$

Hence,

$$\|Af\|^2 \le D \sum_{x\in V(G)} \sum_{(x,y)\in E(G)} |f(y)|^2 = D \sum_{y\in V(G)} \sum_{(x,y)\in E(G)} |f(y)|^2$$
$$\le D^2 \sum_{y\in V(G)} |f(y)|^2 = D^2 \|f\|^2 .$$

This gives $\|A\| \le D$. As to self-adjointness,

$$\langle Af, g\rangle = \sum_{x\in V(G)} (Af)(x)\overline{g(x)} = \sum_{x\in V(G)} \sum_{(x,y)\in E(G)} f(y)\overline{g(x)}$$
$$= \sum_{y\in V(G)} \sum_{(x,y)\in E(G)} f(y)\overline{g(x)} = \sum_{y\in V(G)} f(y)\overline{(Ag)(y)} = \langle f, Ag\rangle .$$

∎

λ is an **eigenvalue** of A if there exists $v \neq 0, v \in H$ such that $Av = \lambda v$. Equivalently, λ is an eigenvalue if and only if $(A - \lambda I)$ is not injective.

The eigenvalues of a self-adjoint operator, A, are real. Indeed, $\lambda \langle v, v\rangle = \langle Av, v\rangle = \langle v, Av\rangle = \overline{\lambda} \langle v, v\rangle$.

The **resolvent set** of T, $\rho(T)$ is the set of all complex numbers λ such that $R_\lambda(T) := (\lambda I - T)^{-1}$ if $(\lambda I - T)$ is a bijection with a bounded inverse. The **spectrum** of T, $\sigma(T)$ is then given by $\mathbb{C} \setminus \rho(T)$.

Lemma. The spectrum of a bounded linear operator is a closed and bounded subset of $\mathbb{C}$. In fact, $\sigma(T) \subseteq \{z \in \mathbb{C} : |z| \le \|T\|\}$

The spectral theorem : For any self-adjoint operator A on a separable Hilbert space H, there is a unique projection-valued measure P_A on $\mathbb{R}$ such that

$$A = \int_{\mathbb{R}} \lambda dP_A(\lambda)$$

More generally, for any Borel function $f : \sigma(A) \to \mathbb{C}$, we can define an operator

$$f(A) = \int_{\mathbb{R}} f(\lambda) dP_A(\lambda)$$

Let see an example on Hilbert space $H = \mathbb{C}^n$ and a square matrix $A : H \to H$. $u \in H$ is an eigenvector of A with corresponding eigenvalue $\lambda \in \mathbb{C}$ if $Au = \lambda u$. The matrix $A = (a_{ij})_{i,j=1}^n$ is said to be Hermitian if $a_{ij} = \overline{a_{ji}}$ for all $i, j = 1, ..., n$, is self-adjoint. Let $\lambda_1 < \lambda_2 < ... < \lambda_m$ $(m \leq n)$ be the eigenvalues of A. Let $\varphi_1(\lambda_k), ..., \varphi_{r_k}(\lambda_k)$ be the eigenvectors corresponding to λ_k ($r_k = 1$ if the eigenvalue is simple). Let $H_k = \text{span}\{\varphi_j(\lambda_k)\}_j$. The spectral theorem then tells us that the eigenvectors $\{\varphi_j(\lambda_k)\}_{j,k}$ form an orthonormal basis of H. So any $f \in H$ has an expansion

$$f = \sum_{k=1}^{m} \sum_{j=1}^{r_k} \langle f, \varphi_j(\lambda_k) \rangle \, \varphi_j(\lambda_k)$$

$$Af = \sum_{k=1}^{m} \sum_{j=1}^{r_k} \lambda_k \, \langle f, \varphi_j(\lambda_k) \rangle \, \varphi_j(\lambda_k)$$

Let $P(\lambda_k)$ be the orthogonal projection onto H_k

$$P(\lambda_k)f = \sum_{j=1}^{r_k} \langle f, \varphi_j(\lambda_k) \rangle \, \varphi_j(\lambda_k)$$

Then

$$\mathbb{1} = \sum_{k=1}^{m} P(\lambda_k) \quad \text{and} \quad A = \sum_{k=1}^{m} \lambda_k P(\lambda_k)$$

Let P_A be the projection-valued measure such that $P_A(J) = \sum_{\lambda_k \in J} P(\lambda_k)$ for any Borel set J. Then

$$\mathbb{1} = \int_{\mathbb{R}} dP_A(\lambda) \quad \text{and} \quad A = \int_{\mathbb{R}} \lambda dP_A(\lambda)$$

2 Local convergence in a graph sequence

2.1 Benjamini-Schramm convergence

Local convergence via neighbourhood sampling

A graph $G = (V, E)$ has degree bound d if degree of any vertex is at most d.
We denote $B_G(x, r)$ is the subgraph spanned by $\{y \in V(G), d(x, y) \leq r\}$.
A rooted graph (G, o) is a connected graph $G = (V(G), E(G))$ with a distinguished vertex $o \in V(G)$, the root. Two rooted graphs (G_1, o_1) and (G_2, o_2) are isomorphic if there exists a an isomorphism $\phi : V(G_1) \to V(G_2)$ such that $\phi(o_1) = o_2$. We will denote this equivalence relation by $(G_1, o_1) \simeq (G_2, o_2)$.

Definition 2.1. Let (G_n) be a graph sequence and all graphs G_n have degree bound d. The graph sequence (G_n) is **Benjamini-Schramm convergent** or **locally convergent** if and only if for all r and for all rooted graphs (Γ, v) with radius smaller than r, degree bound d then

$$\mathbb{P}(B_{G_n}(x, r) \simeq (\Gamma, v))$$

converges where x is an uniformly random node of G_n.

In other words, the graph sequence (G_n) be convergent if for all $r \in \mathbb{N}$ the distribution on rooted r -balls around a randomly selected root of G_n converges when n tends to ∞.

Example 2.1. **(Cycles)** The obvious example is graph sequence of cycles with length tending to infinity. $(\Gamma; v)$ always is the path graph P_{2r+1} and $\mathbb{P}(B_{G_n}(x, r) \simeq (\Gamma, v)) = 1$ for n large enough.

Example 2.2. **(Grids)** Let G_n be the $n \times n$ grid in the plane. If we choose x not far away from the center of grid and r small enough then $B(x, r)$ actually is $(2r+1)\times(2r+1)$ grid. Let denote (Γ, v) be $(2r + 1) \times (2r + 1)$ grid the we calculate

$$\mathbb{P}(B_{G_n}(x, r) \simeq (\Gamma, v)) = \frac{(n - 2r)^2}{n^2} \overset{n\to\infty}{\longrightarrow} 1.$$

for all other (Γ', v') cases we got $\mathbb{P}(B_{G_n}(x, r) \simeq (\Gamma', v')) \overset{n\to\infty}{\longrightarrow} 0$. By definition, we say (G_n) grids local convergent.

Example 2.3. **($d-$ regular trees)** Consider a rooted $d-$regular tree where the root denoted by o has d children and every child has $d - 1$ children (see the figure 2). If this

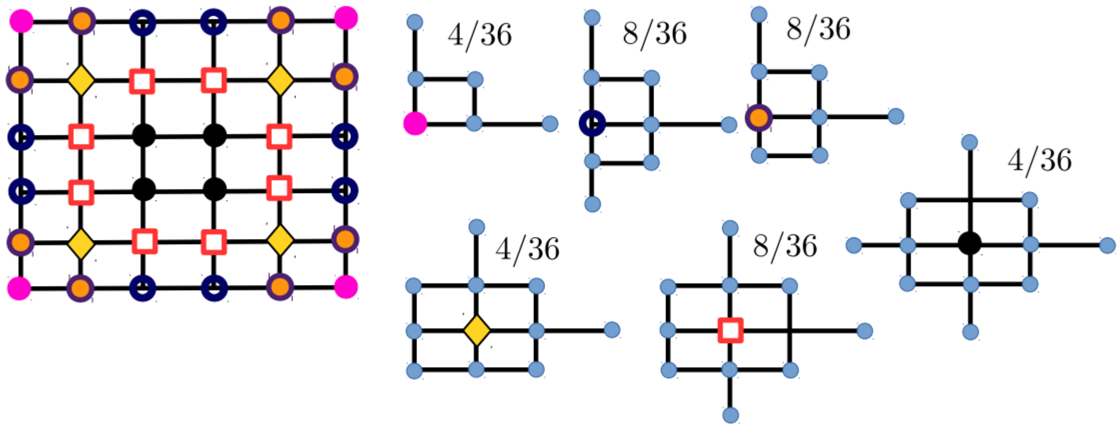


Figure 1: Distribution on rooted 2-balls around a uniformly random root of the 6×6 grid. [9]

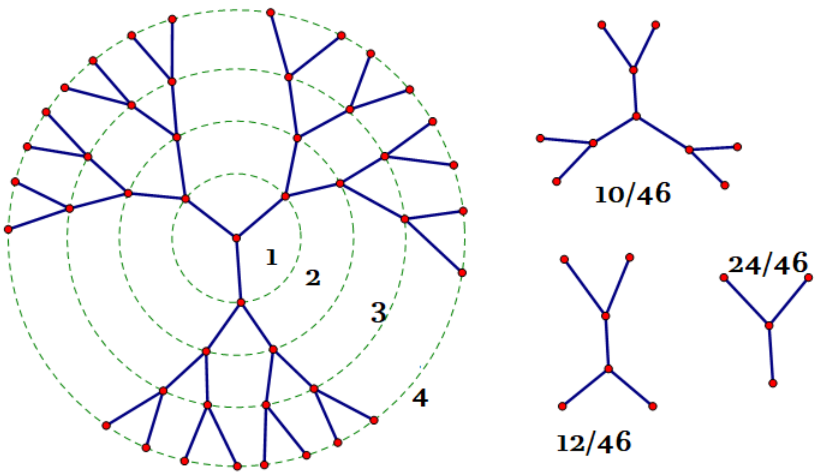


Figure 2: 3-regular tree with depth 4 and all rooted 2-balls

tree has depth n it is uniquely defined and we refer to it as G_n.

$$
\mathbb{P}(B_{G_n}(x,r) \simeq (\Gamma, v))
= \begin{cases}
\dfrac{d(d-1)^{n-1}}{d(d-1)^n - 2} \to \dfrac{1}{d-1} & \text{if } \Gamma \text{ is r-ball at the root } v \text{ where } dis(o,v) = n \\
\dfrac{d(d-1)^{n-2}}{d(d-1)^n - 2} \to \dfrac{1}{(d-1)^2} & \text{if } \Gamma \text{ is r-ball at the root } v \text{ where } dis(o,v) = n-1 \\
\dfrac{d(d-1)^{n-3}}{d(d-1)^n - 2} \to \dfrac{1}{(d-1)^3} & \text{if } \Gamma \text{ is r-ball at the root } v \text{ where } dis(o,v) = n-2 \\
 & \dots \\
\dfrac{d(d-1)^r - 2}{d(d-1)^n - 2} \to 0 & \text{if } \Gamma \text{ is d-regular tree with depth r and the root } v
\end{cases}
$$

Hence, the $d-$ regular tree converges locally.

If (G_n) is a sequence of d–regular graphs and has large girth (see definition 3.4), then

(G_n) Benjamini–Schramm converges, since every $r-$ball of G_n will be isomorphic to the $r-$ball of the $d-$regular tree for large enough n.

Local convergence via homomorphism density

Homomorphisms of some simple graphs into G are related to sampling. We can express a number of graph parameters as homomorphism numbers into and from the given graph.

Example 2.4. (Stars) Homomorphisms from stars into G give the $k-$th power sum of the degree sequence:

$$\hom(S_k, G) = \sum_{v_i \in V(G)} \deg(v_i)^{k-1}$$

Example 2.5. (**Cycles**) $\hom(C_k, G)$ is the trace of the of $k-$th power the adjacency matrix of the graph G.

$$\hom(C_k, G) = \mathrm{Tr}(A_G^k) = \sum \lambda_i^k$$

where λ_i are the eigenvalues of the adjacency matrix of G.

Definition 2.2. We define the (usual) **homomorphism density** from F to G

$$t(F, G) = \frac{\hom(F, G)}{v(G)^{v(F)}}$$

where F be connected graph and G be bounded degree graph. We define similarly for the injective and induced homomorphism densities

$$t_{\mathrm{inj}}(F, G) = \frac{\mathrm{inj}(F, G)}{(v(G))_{v(F)}} \quad , \quad t_{\mathrm{ind}}(F, G) = \frac{\mathrm{ind}(F, G)}{(v(G))_{v(F)}}$$

where $(v(G))_{v(F)} = v(G)(v(G) - 1) \cdots (v(G) - v(F) + 1)$. When $v(G) >> v(F)$ then$(v(G))_{v(F)} \approx v(G)^{v(F)}$

Definition 2.3. $t^*(F, G)$ is defined as the **homomorphism frequency** as below:

$$t^*(F, G) = \frac{\hom(F, G)}{v(G)}$$

which we consider for connected graph F.
We can interpret the injective and induced homomorphism densities similarly

$$t^*_{\mathrm{inj}}(F, G) = \frac{\mathrm{inj}(F, G)}{v(G)} \quad , \quad t^*_{\mathrm{ind}}(F, G) = \frac{\mathrm{ind}(F, G)}{v(G)}$$

Definition 2.4. The bounded degree graph sequence (G_n) be **local convergent** if and only if the homomorphism frequency is convergent for any connected graph F.

The equivalence between 2 definitions

Proposition 2.1. *([1],5.6) Homomorphism densities and neighborhood sampling are equivalent. Specifically, let G be a bounded degree graph and F be a connected graph.*
(a) Each density $t^*(F,G)$ can be expressed as a linear combination (with coefficients independent of G) of the neighborhood sample densities $\mathbb{P}(B_G(x,r) \simeq (\Gamma, v))$ with $r = v(F) - 1$.
(b) For every $r \neq 0$ there are a finite number of connected simple graphs $F_1, \cdots, F_m$ such that $\mathbb{P}(B_G(x,r) \simeq (\Gamma, v))$ can be expressed as a linear combination (with coefficients independent of G) of the densities $t^*(F_i, G)$.

Proof. (a) Instead of counting the number of homomorphism from F to G, we can count the number of homomorphisms from F to a ball with fixed root and radius $v(F) - 1$, then sum up with probability. From this we got:

$$t^*(F,G) = \sum_{(\Gamma,v)} \mathbb{P}(B_G(x,r) \simeq (\Gamma,v)) \cdot \hom((F,u),(\Gamma,v))$$

where x is a node of G selected uniform randomly, $r = v(F) - 1$ and u is a fixed node of F.
So homomorphsim density can be represented by the linear combination of neighborhood sampling distribution with coefficients that do not depend on G.
(b) In reverse, we expect that neighborhood sampling density also can be expressed as a linear combination of homomorphsim density from some graphs to G. It is not easy to see directly, we will use induced densities in alternative form.
Recall $\mathrm{ind}(F,G)$ be the number of embeddings of F into G as an induced graph. Let $\delta : V(F) \to \mathbb{R}$ be the degree of vertices of F in G under a homomorphism then $\mathrm{ind}(F,\delta,G)$ is the number of injective homomorphisms $\varphi : V(F) \longrightarrow V(G)$ that also preserve non-adjancency and the degree of $\varphi(v)$ is $\delta(v)$. We determine:

$$t^*_{\mathrm{ind}}(F,\delta,G) = \frac{\mathrm{ind}(F,\delta,G)}{v(G)}$$

where $t^*(F,\delta,G)$ is called as induced density with δ map.
For a ball Γ of radius r we have:

$$\mathbb{P}(B_G(x,r) \simeq (\Gamma,v)) = \sum_{\delta} \frac{t^*_{\mathrm{ind}}(\Gamma,\delta,G)}{\mathrm{aut}(\Gamma)}$$

We can express $t^*_{\mathrm{inj}}(F_i,G)$ as a linear combination of $t^*(F,G)$. Indeed, by inclusion-exclusion principle, induced homomorphism can be obtained via injective homomorphism.

$$\mathrm{ind}(F,G) = \sum_{F' \supseteq F; V(F')=V(F)} (-1)^{e(F')-e(F)} \,\mathrm{inj}(F',G)$$

Again by inclusion-exclusion principle, injective homomorphism can be obtained via homomorphism.

$$\mathrm{inj}(F, G) = \sum_P \mu_P \hom(F/P, G)$$

where P ranges over all partitions of $V(F)$, and μ_p is Möbius functions (see [7] , A.1) ■

Convergence of eigenvalue measures

Let G be a graph with degree bound d, eigenvalues of A_G are $\lambda_1, \lambda_2, ...\lambda_{v(G)}$. We know all spectrum of A_G belong to $[-d, d]$ interval .For all i we put weight $\frac{1}{v(G)}$ on λ_i to get a probability distribution μ_G.

Example 2.6. (Complete graph) The eigenvalues of the adjacency matrix of the complete graph K_n on n vertices, are $n-1$ with multiplicity 1 and -1 with multiplicity $n-1$. It follows that

$$\mu_{K_n} = \frac{1}{n}\delta_{n-1} + \frac{n-1}{n}\delta_{-1}$$

Notice that μ_{K_n} converges weakly to δ_{-1} as $n \to \infty$.

Example 2.7. (Cycle) The eigenvalue measure of C_n is

$$\mu_{C_n} = \frac{1}{n}\sum_{k=1}^{n} \delta_{2\cos 2\pi k/n}$$

Notice that μ_{C_n} is supported on $[-2, 2]$ interval. Let x be a number such that $-2 \leq x \leq 2$ then

$$\begin{aligned}
\mu_{C_n}([-2, x)) &= \frac{1}{n}\sum_{k=1}^{n} \delta_{2\cos 2\pi k/n}([-2, x)) = \frac{1}{n}\#\left\{k \in \{1, \cdots, n\} | 2\cos\frac{2\pi k}{n} < x\right\} \\
&= \frac{1}{n}\#\left\{k \in \{1, \cdots, n\} | \frac{2\pi k}{n} \in \left(\arccos\frac{x}{2}, 2\pi - \arccos\frac{x}{2}\right)\right\} \\
&\to 1 - \frac{1}{\pi}\arccos\frac{x}{2} = \frac{2}{\pi}\arcsin\sqrt{\frac{x+2}{4}}
\end{aligned}$$

which is the cumulative distribution function of arcsine distribution on $[-2, 2]$. Hence, μ_{C_n} converges weakly to an arcsine distribution on bounded support $[-2, 2]$

Example 2.8. (Path) Similarly, as n goes to infinity, μ_{P_n} converges weakly to a arcsine distribution on $[-2, 2]$ where

$$\mu_{P_n} = \frac{1}{n}\sum_{k=1}^{n} \delta_{2\cos \pi k/n+1}$$

Proposition 2.2. *If the graph sequence (G_n) with bounded degree is local convergent then μ_{G_n} is weakly convergent.*

Proof. By example 2.5,

$$\int_{\mathbb{R}} x^k d\mu_{G_n} = \frac{1}{v(G_n)} \sum_{i=1}^{n} \lambda_i^k = \frac{\mathrm{Tr}(A_{G_n}^k)}{v(G_n)} = \frac{\hom(C_k, G_n)}{v(G_n)}$$

which is the homomorphism frequency from cycle graph to G_n. By definition of graph convergence, the homomorphism frequency converges when $n \to \infty$. By the Weierstrass approximation theorem, the convergence of μ_{G_n} follows. ■

2.2 Convergence in dense graphs

A dense graph is a graph in which the number of edges is close to the maximal number of edges. The opposite, a graph with only a few edges, is a sparse graph. For undirected simple graphs G, the graph density is defined as:

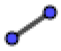
$$Den(G) = \frac{2|E|}{|V|\left(|V|-1\right)}$$

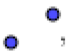
Convergence via subgraph sampling

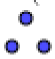
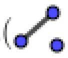
Definition 2.5. Let F be a fixed graph and G be a dense graph. We define subgraph density as:

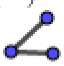
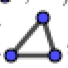
$$t_{sub}(F,G) = \frac{\#\text{copies of } F \text{ in } G}{\binom{v(G)}{v(F)}}$$

Example 2.9. Let G be a complete 3-partite graph of order $3k$. (see Figure 3). The graph density of G is $Den(G) = 3k^2/\binom{3k}{2} \to 2/3$.

If $v(F) = 2$, then $t_{sub}($$, G) = e(G)/\binom{3k}{2} = Den(G) \to 2/3$

$t_{sub}($$, G) = (\binom{3k}{2} - e(G))/\binom{3k}{2} = 1 - Den(G) \to 1/3$

If $v(F) = 3$, then $t_{sub}($$, G) \to 1/9$, $t_{sub}($$, G) \to 0$

$t_{sub}($$, G) \to 2/3$, $t_{sub}($$, G) \to 2/9$

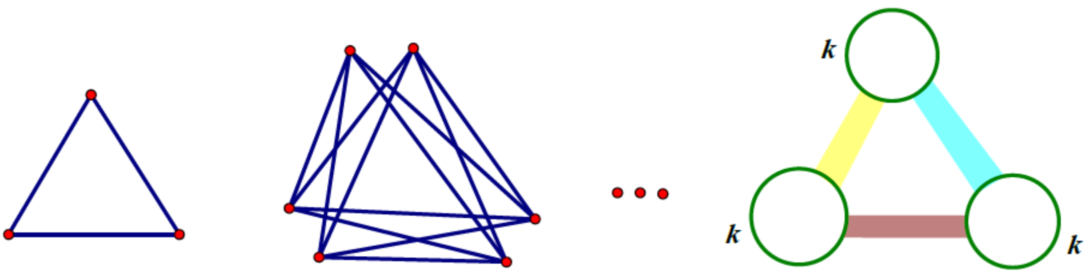


Figure 3: A complete 3-partite graph sequence

Definition 2.6. A sequence of graphs G_n converges if for each F, the sequence $t_{sub}(F, G_n)$ converges as $n \to \infty$.

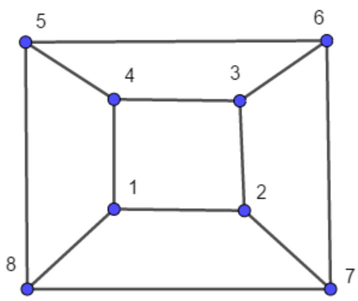
Convergence via graphons

Definition 2.7. A graphon is a bounded measurable function $W : [0,1]^2 \to \mathbb{R}$ such that $W(x,y) = W(y,x)$ for all $(x,y) \in [0,1]^2$

Example 2.10. Every finite simple graph G can be represented by a graphon $W : [0,1]^2 \to [0,1]$. We will split the interval $[0,1]$ into $v(G)$ equal intervals $J_1, J_2, \cdots, J_{v(G)}$ where $J_i = \left[\frac{i}{v(G)}, \frac{i+1}{v(G)}\right]$ for $i = 1, 2, \cdots, v(G)$. We define a graphon as a function below:

$$W_G(x,y) = \begin{cases} 1 & \text{if } ij \in E(G) \\ 0 & \text{if } ij \notin E(G) \end{cases}$$

for all $x \in J_i$ and $y \in J_j$. We image if we replace the i-th column and $j-$th row entry in the adjacency matrix of G by a square of size $(1/v(G) \times 1/v(G))$, and define the value of the function W_G on this square equals the entry of the adjacency matrix.



$$\begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

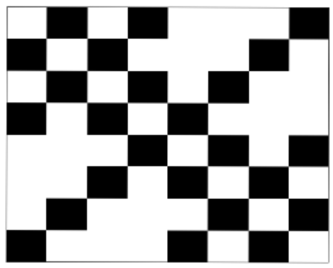


Figure 4: The 3-cube graph, its adjacency matrix, and its pixel picture

Every graphon $W : [0,1]^2 \to [0,1]$ can be represented by a grayscale picture on the unit square: the point (x,y) is black if $W(x,y) = 1$, it is white if $W(x,y) = 0$, and it is grey if $0 < W(x,y) < 1$. For a graph, this picture gives a black-and-white picture consisting of a finite number of "pixels". The origin is in the upper left corner (as for a matrix). ([10])

Example 2.11. (Half graph) A half graph G is a type of bipartite graph, say $G = (A, B, E(G))$ where $A = \{v_1, \cdots, v_n\}$; $B = \{v_{n+1}, ..., v_{2n}\}$ and $v_i v_{n+j} \in E(G)$ whenever $i \geq j$. In figure [5], We represent the pixel picture of the half graph when $n = 7$ and $n = 1000$ and we can guess when n goes to infinity, the "stairs part" will become a "smooth slope".

Example 2.12. (Complete bipartite graph) Note that the function associated with a graph depends on the ordering of the nodes. Look at an example when we enumerate the vertices in different ways leads to the limit of each graphon does not look the same.

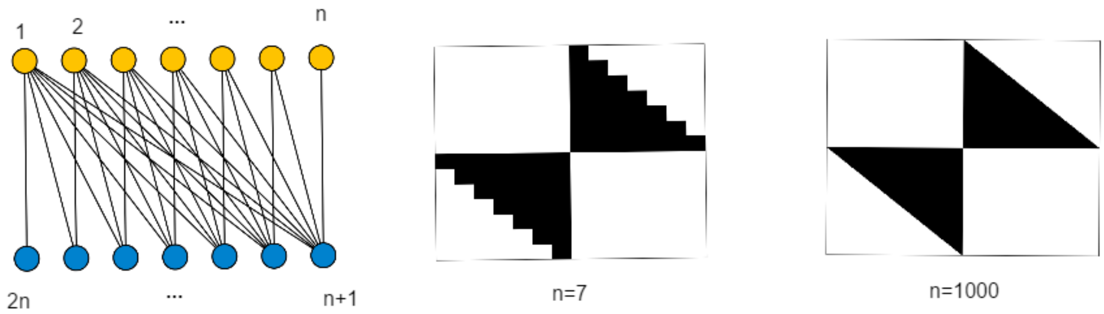


Figure 5: Half graph and its pixel picture(n=7, n=1000)

Let $G = (A, B, E(G))$be a complete bipartite graph of the order $2n$. If we embed the vertice as $A = \{v_1, v_2, \cdots, v_n\}$; $B = \{v_{n+1}, ..., v_{2n}\}$ then all complete bipartite graphs with equal color classes have the same pixel picture see figure ([6] (*)). But if we list $A = \{v_1, v_3, \cdots, v_{2n-1}\}$; $B = \{v_2, v_4, \cdots, v_{2n}\}$ then its pixel picture look like black-and-white square stripes and the limit function is a uniformly grey square.

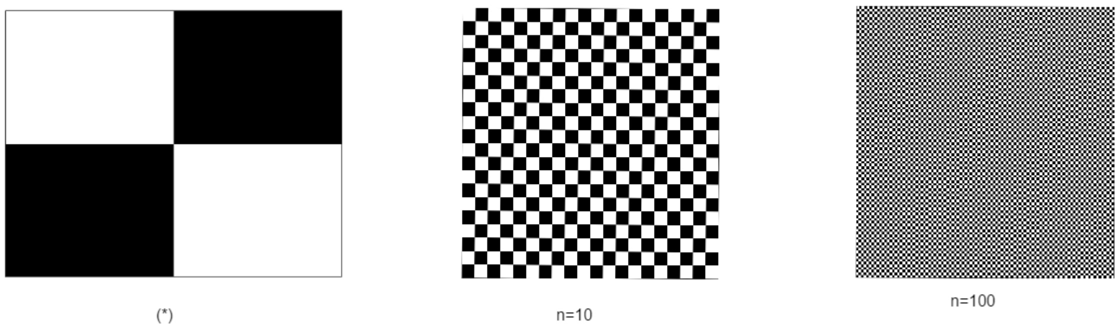


Figure 6: The pixel pictures of a complete bipartite graph

Definition 2.8. Let W be a graphon, F be a simple graph. We define the homomorphism density of W

$$t(F, W) = \int_{[0,1]^{|V(F)|}} \prod_{ij \in E(F)} W(x_i, x_j) \prod_{i \in V(F)} dx_i$$

By definition, for every finite simple graph G, $t(F, G) = t(F, W_G)$.
In [2], L.Lovász and B. Szegedy proved that if a sequence of dense graphs G_n has the property that for every fixed graph F, the density of copies of F in G_n tends to a limit, then there is a natural "limit object", namely a symmetric measurable function $W : [0, 1]^2 \to [0, 1]$. This limit object determines all the limits of subgraph densities. Conversely, every such function arises as a limit object.

Theorem 2.1. *([2])For every convergent graph sequence (G_n) there is a W graphon such that $t(F, G_n) \to t(F, W)$ for every simple graph F. And every graphon W : $[0, 1]^2 \to [0, 1]$ arises as the limit of a convergent graph sequence.*

Convergence of eigenvalues of graphon sequence

Definition 2.9. The operator

$$T_W : L_2[0,1] \to L_2[0,1]$$

$$T_W(f(x)) = \int_0^1 W(x,y)f(y)dy$$

is called kernel operator.

T_W is a Hilbert-Schmidt operator which is self-adjoint, so T_W has a countable multiset Spec(W) of non zero real eigenvalues $\{\lambda_1, \lambda_2, ...\}$ and $\lambda_n \to 0$. More details, it can be written in the form

$$W(x,y) \sim \sum_k \lambda_k f_k(x) f_k(y)$$

where f_k is the eigenfunction corresponding to eigenvalue λ_k. The $\sim$ indicates that the series is convergent in L_2 only.
We denote $\lambda_i(W)$ be the non negative $i-$th largest eigenvalue (included multiplicities), $\lambda'_i(W)$ be the non positive $i-$th smallest eigenvalue (included multiplicities), otherwise $\lambda_i = 0 (\lambda'_i = 0)$.We have

$$t(C_2, W) = \int_{[0,1]^n} (W(x,y))^2 \, dxdy = \|W\|_2^2 = \sum_{\lambda \in \text{Spec}(W)} \lambda^2$$

In general, homomorphism densities can be expressed in terms of this spectrum as

$$t(C_n, W) = \int_{[0,1]^n} W(x_1,x_2)....W(x_{n-1},x_n)W(x_n,x_1)dx_1....dx_n = \sum_{\lambda \in \text{Spec}(W)} \lambda^n$$

And from this, we obtain

$$\sum_{i=1}^n \lambda_i^3 \leq t(C_3, W) \Rightarrow \lambda_n \leq \left(\frac{t(C_3, W)}{n} \right)^{1/3}$$

Theorem 2.2. *([1]) If $W_1, W_2, \cdots, W_n, \cdots$ are bounded graphons which the homomorphism densities converge, namely, there exists a graphon W such that $t(F, W_n) \to t(F, W)$ for all finite simple graphs F, then*

$$\lambda_i(W_n) \to \lambda_i(W) \qquad \textit{and} \qquad \lambda'_i(W_n) \to \lambda'_i(W)$$

Proof. Suppose that the statement is not true $\lambda_i(W_n) \nrightarrow \lambda_i(W)$ or $\lambda'_i(W_n) \nrightarrow \lambda'_i(W)$. W_n are bounded graphons so eigenvalues are bounded also. By Bolzano–Weierstrass theorem, there exists a convergent subsequence $(n_j) \in \mathbb{N}$ such that $\lambda_i(W_{n_j})$ converges for all i. Let

$$\lim_{n_j \to \infty} \lambda_i(W_{n_j}) := \zeta_i \qquad \text{and} \qquad \lim_{n_j \to \infty} \lambda'_i(W_{n_j}) := \zeta'_i$$

For $k \geq 4$, we have

$$\lim_{n_j \to \infty} \sum_{i=1}^{\infty} \lambda_i^k(W_{n_j}) \to \sum_{i=1}^{\infty} \zeta_i^k \qquad \text{and} \qquad \lim_{n_j \to \infty} \sum_{i=1}^{\infty} \lambda_i'^k(W_{n_j}) \to \sum_{i=1}^{\infty} \zeta_i'^k$$

It is not always true for all k because we need $\sum_{i=1}^{\infty} \lambda_i^k(W_{n_j})$ converges. Because (W_n) be bounded graphons so let say c be a number such that $t(C_3, W_n) \leq c \quad \forall n$. Using the inequality between eigenvalues and the homomorphism density from C_3 to a graphon above, we obtain

$$\sum_{i=1}^{\infty} \lambda_i^k(W_{n_j}) \leq \sum_m \left(\frac{c}{m}\right)^{k/3} \leq \sum_m \left(\frac{c}{m}\right)^{4/3} = c^{4/3}\zeta(4/3) \quad \text{is finite}$$

$(W_1, W_2, ...)$ converges to W so $t(C_n, W_m) \to t(C_n, W)$ as $n \to \infty$

$$\lim_{n_j \to \infty} \left(\sum_{i=1}^{\infty} \lambda_i^k(W_{n_j}) + \sum_{i=1}^{\infty} \lambda_i'^k(W_{n_j}) \right) \to \sum_{i=1}^{\infty} \lambda_i^k(W) + \sum_{i=1}^{\infty} \lambda_i'^k(W)$$

Hence

$$\sum_{i=1}^{\infty} \lambda_i^k(W) + \sum_{i=1}^{\infty} \lambda_i'^k(W) = \sum_{i=1}^{\infty} \zeta_i^k + \sum_{i=1}^{\infty} \zeta_i'^k \tag{*}$$

We expect to prove $\lambda_i(W) = \zeta_i$ and $\lambda_i'(W) = \zeta_i'$ by induction on i. If $i = 1$, WLOG $\lambda_1(W) = \max\{\lambda_1(W), \zeta_1, -\lambda_1'(W), -\zeta_1'\}$ from $(*)$ we get:

$$\lambda_1^k(W) \left(a + \sum_{i=a}^{\infty} \left(\frac{\lambda_i(W)}{\lambda_1(W)} \right) \right)^k + (-\lambda_1(W))^k \left(a' + \sum_{i=a'}^{\infty} \left(\frac{\lambda_i'(W)}{-\lambda_1(W)} \right) \right)^k$$

$$= \lambda_1^k(W) \left(b + \sum_{i=b}^{\infty} \left(\frac{\zeta_i^k}{\lambda_1(W)} \right) \right)^k + (-\lambda_1(W))^k \left(b' + \sum_{i=b'}^{\infty} \left(\frac{\zeta_i'^k}{-\lambda_1(W)} \right) \right)^k$$

where λ_1 was counted a times with multiplicities in $(\lambda_1, \lambda_2, ...)$ and b times appears in $(\zeta_1, \zeta_2, ...)$; $-\lambda_1$ was counted a' times with multiplicities in $(\lambda_1', \lambda_2', ...)$ and b' times occurs in $(\zeta_1', \zeta_2', ...)$.

We take a limit as $k \to \infty$, k is a even numbers sequence then $a + a' = b + b'$. We also take a limit as $k \to \infty$, k is a odd numbers sequence then we get $a - a' = b - b'$. So $a = b$; $a' = b'$, hence $\lambda_1 = \zeta_1$ implies $\lambda_1' = \zeta_1'$.

We suppose that $\lambda_j = \zeta_j$; $\lambda_j' = \zeta_j'$ for all $j < n$ and $\lambda_n(W) = \max\{\lambda_n(W), \zeta_n, -\lambda_n'(W), -\zeta_n'\}$. With the same definition for a, b, a', b' is the number of appearances of λ_n , we obtain

$$\lambda_n^k(W) \left(a + \sum_{i=n+a}^{\infty} \left(\frac{\lambda_i(W)}{\lambda_n(W)} \right) \right)^k + (-\lambda_n(W))^k \left(a' + \sum_{i=n+a'}^{\infty} \left(\frac{\lambda_i'(W)}{-\lambda_n(W)} \right) \right)^k$$

$$= \lambda_n^k(W) \left(b + \sum_{i=n+b}^{\infty} \left(\frac{\zeta_i^k}{\lambda_n(W)} \right) \right)^k + (-\lambda_n(W))^k \left(b' + \sum_{i=n+b'}^{\infty} \left(\frac{\zeta_i'^k}{-\lambda_n(W)} \right) \right)^k$$

Take the limit on both sides when k is even and odd sequence, we get $a = b$ and $a' = b'$ so $\lambda_n = \zeta_n$. ■

2.3 Convergence in general

Definition 2.10. Let F be a connected graph and G be a graph having degree bound d , $d \geq 1$. The homomorphism density from F to G with *degree bound* d are defined by

$$t(F, G, d) = \frac{\hom(F, G)}{v(G)d^{v(F)-1}}$$

And similarly, we define the injective homomorphism density

$$t_{\text{inj}}(F, G, d) = \frac{\text{inj}(F, G)}{v(G)d(d-1)^{v(F)-2}}$$

for a connected graph F

Definition 2.11. A bounded degree graph sequence (G_n) by d_n is **convergent** if $t(F, G_n, d_n)$ converges for every connected graph F.

Definition 2.12. We define ϱ_G as the **normalized eigenvalue measure** of G if

$$\varrho_G = \frac{1}{v(G)} \sum_{i=1}^{v(G)} \delta_{\lambda_i}$$

where δ_x is the Dirac measure at x.

Definition 2.13. $\varrho_{G,d}$ be the uniform probability measure on the $v(G)$ numbers $\dfrac{\lambda}{d}$.

$$\varrho_{G,d} = \frac{1}{v(G)} \sum_{i=1}^{v(G)} \delta_{\lambda_i/d}$$

Using lemma 1.2, $\varrho_{G,d}$ is supported on $[-1, 1]$ interval.

Lemma 2.1. ***(The moments of eigenvalue distribution)***
The zeroth moment of $\varrho_{G,d}$ is 1, the first moment is 0 and

$$\int_{-1}^{1} x^k d\varrho_{G,d}(x) = \frac{\hom(C_k, G)}{v(G)d^k} = \frac{t(C_k, G, d)}{d}$$

for all $k \geq 2$. In case $k = 2$, C_2 is comprehended as K_2.

Proof. We will prove that, $\hom(C_k, G)$ is the trace of the $k-th$ power of the adjacency matrix of the graph. That is $\hom(C_k, G) = \text{Tr}(A_G^k)$ where A_G be the adjacency matrix of G.

$$\hom(C_k, G) = \sum_{x \in V(G)} w_G(x, k)$$

where $w_G(x, k)$ denotes the number of walks in G of length k starting and ending at x.

$$w_G(v_{i_1}, k) = \sum a_{i_1 i_2} a_{i_2 i_3} ... a_{i_k i_1} = (A_G^k)_{i_1, i_1}$$

where $(A_G^k)_{i_1,i_1}$ is the (i_1, i_1) entry of A_G^k. So $\hom(C_k, G) = \sum_i (A_G^k)_{i,i} = \mathrm{Tr}(A_G^k)$.
If $\lambda_1, \lambda_2, ..., \lambda_{v(G)}$ be eigenvalues of matrix A_G then $\mathrm{Tr}(A_G^k) = \sum_{i=1}^{v(G)} \lambda_i^k$

$$\int_{-1}^{1} x^k d\varrho_{G,d}(x) = \frac{1}{v(G)} \sum_{i=1}^{v(G)} \left(\frac{\lambda_i}{d}\right)^k = \frac{1}{v(G)d^k} \sum_{i=1}^{v(G)} (\lambda_i)^k = \frac{\hom(C_k, G)}{v(G)d^k} = \frac{t(C_k, G, d)}{d}$$

■

Subsequent. Let (G_n) be convergent graph sequence bounded by d_n and $f : [-1, 1] \to \mathbb{R}$ be continuous, α is a integer number, $\alpha \geq 2$. Then

$$d_n \int_{-1}^{1} x^{\alpha} f(x) d\varrho_{G_n,d_n}(x)$$

converges when n tends to infinity.

Proof. If $f(x) = x^k$ then

$$d_n \int_{-1}^{1} x^{\alpha} f(x) d\varrho_{G_n,d_n}(x) = d_n \int_{-1}^{1} x^{k+\alpha} d\varrho_{G_n,d_n}(x) = t(C_{k+\alpha}, G_n, d_n) \text{ converges}.$$

By Weierstrass approximation, f is a continuous real-valued function on $[-1, 1]$ and if any $\epsilon > 0$ is given, then there exists a polynomial $P(x)$ on $[-1, 1]$ such that: $|f(x) - P(x)| < \epsilon \ \forall x \in [-1, 1]$. Hence, $d_n \int_{-1}^{1} x^{\alpha} f(x) d_{\varrho_{G_n,d_n}}(x)$ converges. ■

Proposition 2.3. *Let (G_n) be a graph sequence with all degrees $\leq d_n$, $d_n \to \infty$ then ϱ_{G_n,d_n} converges weakly to the Dirac measure concentrated on* 0.

$$\varrho_{G_n,d_n} \xrightarrow{w} \delta_0$$

Proof. For any $\epsilon > 0$, we have

$$\epsilon^2 \varrho_{G,d}(\{|x| \geq \epsilon\}) \leq \int_{-1}^{1} x^2 d\varrho_{G,d}(x) = \frac{t(K_2, G, d)}{d} \leq \frac{1}{d}$$

Apply this for (G_n) graph sequence and take d_n to infinity, we get

$$\varrho_{G_n,d_n}((-\epsilon, \epsilon)) \geq 1 - \frac{1}{\epsilon^2 d_n} \to 1$$

That implies $\varrho_{G_n,d_n}([-1; 1] \setminus (-\epsilon, \epsilon)) \to 0$ then $\varrho_{G_n,d_n} \xrightarrow{w} \delta_0$. ■

Definition 2.14. $\varrho_{G,d,r}$ and $\varrho'_{G,d,r}$ be the uniform probability measure on the $v(G)$ numbers $\dfrac{\lambda}{d}$, where λ runs over the r largest and r smallest eigenvalues, respectively.

$$\varrho_{G,d,r} = \frac{1}{r} \sum_{i=1}^{r} \delta_{\lambda_i/d} \qquad ; \qquad \varrho'_{G,d,r} = \frac{1}{r} \sum_{i=v(G)-r+1}^{v(G)} \delta_{\lambda_i/d}$$

$\varrho_{G,d,r}$ is supported on $\left[\dfrac{\lambda_r}{d},1\right]$ and $\varrho'_{G,d,r}$ is supported on $\left[-1,\dfrac{\lambda_{v(G)-r+1}}{d}\right]$.

Theorem 2.3. *Let (G_n) be a graph sequence, all degrees of G_n vertice small than d_n, $d_n \to \infty$. For each d_n there is given an integer number r_n such that $\dfrac{r_n d_n}{v(G_n)} \to \alpha \;\; ; \;\; \alpha > 0$. Then $\varrho_n = \varrho_{G_n,d_n,r_n}$ ($\varrho'_n = \varrho'_{G_n,d_n,r_n}$) weakly convergent to probability measures ϱ(ϱ') supported on $[0,1]$ ($[-1;0]$), respectively.*

Proof. To prove ϱ_n converges, by Portmanteau theorem, we need to show that

$$\liminf \varrho_n([b,1]) \leq \limsup \varrho_n([b,1]) \leq \liminf \varrho_n([a,1]) \leq \limsup \varrho_n([a,1])$$

for any $0 < a < b < 1$.

Let $g : [-1,1] \to [0,1]$ be continuous, non decreasing function and $g(a) = 0, g(b) = 1$.

If $\liminf \varrho_n([a,1]) = 1$ then $\limsup \varrho_n([b,1]) \leq 1$

If $\liminf \varrho_n([a,1]) < 1$ then

$$\varrho_n([a,1]) = \frac{\#\{\lambda_i \mid \; \lambda_i/d_n \in [a,1] \; ; \; i \leq r_n\}}{r_n} = \frac{\#\{\lambda_i \mid \lambda_i/d_n \in [a,1] \;\}}{r_n} = \frac{v(G_n)}{r_n}\varrho_{G_n,d_n}([a,1])$$

That implies

$$\begin{aligned}\liminf \varrho_n([a,1]) \geq \liminf \varrho_n([a,b]) &\geq \liminf \left(\frac{d_n}{\alpha}\int g d\varrho_{G_n,d_n}\right)\\ &= \liminf \frac{1}{\alpha}\left(d_n \int g d\varrho_{G_n,d_n}\right) = \frac{1}{\alpha}\lim\left(d_n\int g d\varrho_{G_n,d_n}\right)\end{aligned}$$

Or then

$$\varrho_n([b,1]) \leq \int g d\varrho_n \leq \frac{v(G_n)}{r_n}\int g d\varrho_{G_n,d_n}$$

So

$$\begin{aligned}\limsup \varrho_n([b,1]) &\leq \limsup \left(\frac{d_n}{\alpha}\int g d\varrho_{G_n,d_n}\right)\\ &= \limsup \frac{1}{\alpha}\left(d_n \int g d\varrho_{G_n,d_n}\right) = \frac{1}{\alpha}\lim\left(d_n\int g d\varrho_{G_n,d_n}\right)\end{aligned}$$

Hence, ϱ_n converges weakly to ϱ and we will prove ϱ is supported on $[0,1]$. Since all eigenvalues of G are smaller than d, we have

$$0 = \operatorname{Tr}(A_G) = \sum_{i=1}^{v(G)} \lambda_i = \sum_{i=1}^{k}\lambda_i + \sum_{i=k+1}^{v(G)}\lambda_i \leq k.d + (v(G)-k)\lambda_k$$

$$\frac{\lambda_k}{d} \leq \frac{k}{k - v(G)}$$

Let $k = r_n$ then

$$\frac{\lambda_{r_n}}{d} \leq \frac{r_n}{r_n - v(G_n)} \sim \frac{\alpha}{\alpha - d_n} \to 0$$

Similar proof with ϱ' measure. ∎

3 The rescaled spectral measure

In the *section 2.3*, when we define the general eigenvalue distribution by put the weight $\dfrac{\lambda}{d}$ on each vertex. In this section, we will rescale with the weight $\dfrac{\lambda}{\sqrt{d}}$ and will examine the convergence of this spectral measure in some special cases.

Definition 3.1. Let $\xi_{G,d}$ be the uniform probability measure on the $v(G)$ numbers $\dfrac{\lambda}{\sqrt{d}}$.

$$\xi_{G,d} = \frac{1}{v(G)} \sum_{i=1}^{v(G)} \delta_{\lambda_i/\sqrt{d}}$$

It is clear that $\xi_{G,d}$ is a probability measure and is supported on $[-\sqrt{d}, \sqrt{d}]$.

3.1 The hypercube

In case of the hypercube graph sequence Q_n, choose $d_n = n$. By table 1, we know the eigenvalues of Q_n are $\lambda_i = n - 2i$ with multiplicity $\binom{n}{i}$.

$$\xi_{Q_n,n}\left(\frac{n-2i}{\sqrt{n}}\right) = \xi_{Q_n,n}\left(\frac{\lambda_i}{\sqrt{n}}\right) = \frac{1}{2^n}\binom{n}{i}$$

So the probability measure $\xi_{Q_n,n}$ is the distribution of $\dfrac{X}{\sqrt{n}} = \dfrac{\sum_{i=1}^{n} Y_i}{\sqrt{n}}$ where X be the binomial distribution and Y_i are modified Bernoulli distributions with $\mathbb{P}(Y_i = 1) = \mathbb{P}(Y_i = -1) = 1/2$. By the central limit theorem, $\dfrac{\sum_{i=1}^{n} Y_i}{\sqrt{n}} \to N(0,1)$ as to $t \to \infty$.

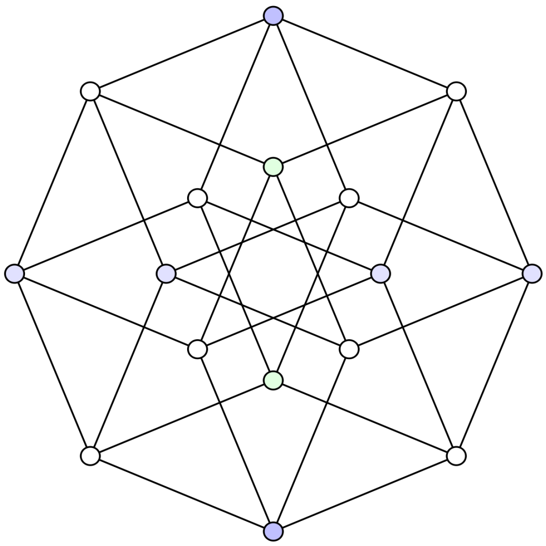


Figure 7: 4- dimensional hypercube graph Q_4

3.2 The $\alpha-$ regular, large girth graph sequence

Definition 3.2. Let a graph sequence G_n is bounded by d_n. G_n is α**- regular** if the degree of v_i divided by d_n tends stochastically to α where $0 \leq \alpha \leq 1$, v_i is a uniform random vertex of G_n.

Proposition 3.1. *(see [3], proposition 1.19) If the graph sequence G_n is bounded by d_n is $\alpha-$regular then for all forest F, we have $t(F, G_n, d_n) \to \alpha^{e(F)}$ as $n \to \infty$.*

Definition 3.3. The graph sequence (G_n) has **large girth** if for any $k \geq 3$ there exists $n_o(k)$ such that $\text{inj}(C_k, G_n) = 0$ for all $n \geq n_o(k)$. In other words, the length of the shortest cycle (girth) tends to infinity.

Definition 3.4. Let G be a graph with $v(G)$ vertices and let m_k be the number of $k-$edge matchings. $m_0 = 1, m_1 = e(G)$. The **matching polynomial** is

$$M_G(x) := \sum_{k \geq 0} (-1)^k m_k x^{v(G)-2k}.$$

Our convention that $m_o = 1$ ensures that this is a polynomial of degree n.
Some properties of the matching polynomial:

(i) $M_{G \cup H}(x) = M_G(x) M_H(x)$

(ii) $M_G(x) = x M_{G \backslash v_i}(x) - \sum_{v_i v_j \in E(G)} M_{G \backslash v_i \backslash v_j}(x)$

Example 3.1. If G is a forest, then its matching polynomial is equal to the characteristic polynomial of its adjacency matrix (see [7],11.4). If G is a path or a cycle, then $M_G(x)$ is a Chebyshev polynomial.

Theorem 3.1. *([11])The roots of the matching polynomial are real and in case of the bounded graph by $d \geq 2$ then all roots in $[-2\sqrt{d-1}, 2\sqrt{d-1}]$.*

Proof. Look at the recursion $M_\emptyset(x) = 1$

$$M_G(x) = x M_{G \backslash v_i}(x) - \sum_{v_i v_j \in E(G)} M_{G \backslash v_i \backslash v_j}(x)$$

We prove by induction on $v(G)$ that $M_G(x)$ is real-rooted, with distinct simple roots and $M_{G \backslash v_i}(x)$ strictly interlaces $M_G(x)$ .That means if $y_1, y_2, ..., y_{v(G)-1}$ and $x_1, x_2, ..., x_{v(G)}$ be real and distinct roots of $M_{G \backslash v_i}(x)$ and $M_G(x)$ corresponding then $x_1 < y_1 < x_2 < ... < x_{v(G)-1} < y_{v(G)-1} < y_{v(G)}$.
If $v(G) = 1$ then $m_G(x) = x$ and $M_{G \backslash V(G)}(x) = 1$.
If $v(G) = n$, define (y_i) as above. By the inductive hypothesis,

$$M_G(y_i) = x M_{G \backslash v_i}(y_i) - \sum_{v_i v_j \in E(G)} M_{G \backslash v_i \backslash v_j}(y_i) = - \sum_{v_i v_j \in E(G)} M_{G \backslash v_i \backslash v_j}(y_i)$$

To prove $M_G(x)$ has n real and distinct roots, we will show $M_G(y_i)$ alternates signs for $i = 1, 2, 3, ...$ By the inductive hypotesis, $M_{G\backslash v_i \backslash v_j}(x)$ strictly interlaces $M_{G\backslash v_i}(x)$, that means $M_{G\backslash v_i \backslash v_j}(y_i)$ alternates signs for $i = 1, 2, 3, ..., n-1$. All roots of $M_{G\backslash v_i \backslash v_j}(x)$ is in the interval $[y_1, y_{n-1}]$ and the highest coefficient of $M_{G\backslash v_i \backslash v_j}(x)$ is 1 then $M_{G\backslash v_i \backslash v_j}(y_1) > 0$ and $(-1)^{i+1} M_{G\backslash v_i \backslash v_j}(y_i) > 0$. This implies that $(-1)^i M_G(x) > 0$
The highest coefficient of $M_G(x)$ is 1, then $\lim_{x\to\infty} M_G(x) = +\infty$ and $\lim_{x\to-\infty} M_G(x) = \operatorname{sign}((-1)^n)\infty$. Hence, as $M_G(x)$ has exactly n roots interlacing $\{y_1, y_2, ..., y_n\}$.

We will summarize what Godsil proved in [12].

- For v a vertex of G, the path tree of G starting at v, written $T_v(G)$ is a tree whose vertices correspond to paths in G that start at a and do not contain any vertex twice. One path is connected to another if one extends the other by one vertex.
- $\dfrac{M_{G\backslash v}(x)}{M_G(x)} = \dfrac{M_{T_v(G)\backslash v}(x)}{M_{T_v(G)(x)}}$
- For every vertex v of G, the polynomial $M_G(v)$ divides the polynomial $M_{T_v(G)}(x)$.
- By *lemma 1.3*, all eigenvalues of a tree T have absolute value at most $2\sqrt{d-1}$ if T is a bounded graph by d. This implies that that matching polynomial of a graph with all degrees at most d has all of its roots bounded in absolute value by $2\sqrt{d-1}$.

■

Definition 3.5. Let x_i be all roots of the matching polynomial then $\kappa_{G,d}$ can be defined as the **matching measure** of G

$$\kappa_{G,d} = \frac{1}{v(G)} \sum_{i=1}^{v(G)} \delta_{x_i/\sqrt{d}}$$

Definition 3.6. Let G be a graph with $v(G)$ vertices. The **modified matching polynomial** is

$$M'_G(x) := \sum_{k\geq 0} (-1)^k m_k x^{v(G)-k}.$$

Definition 3.7. Let y_i be all roots of the modified matching polynomial then $\kappa_{G,d}$ can be defined as the **modified matching measure** of G

$$\nu_{G,d} = \frac{1}{v(G)} \sum_{i=1}^{v(G)} \delta_{y_i/d}$$

If λ is a root of $M'_G(x)$ then λ^2 is a root of $M_G(x)$.

$$M_G(x^2) = x^{v(G)} M'_G(x)$$

By *theorem 3.1*, the matching measure κ_{G_n,d_n} is supported on the interval $[-2;2]$ and that implies the modified matching measure ν_{G_n,d_n} is supported on $[0,4]$. And

$$\int_{-2}^{2} x^{2k} d\kappa_{G_n,d_n}(x) = 2\int_{0}^{4} x^k d\nu_{G_n,d_n}(x)$$

Theorem 3.2. *If the bounded convergent graph sequence (G_n) has degree bound d_n is $\alpha-$regular and $d_n \to \infty$ then the modified matching measure ν_{G_n,d_n} converges weakly.*

Proof. The graph sequence (G_n) is convergent so $t(F, G_n, d_n)$ converges for any connected garph F. Denote $t(F) = \lim_{n\to\infty} t(F, G_n, d_n)$

$$\int_0^4 x^k d\nu_{G_n,d_n}(x) = \frac{1}{v(G)} \sum_{M'_{G_n}(y_i)=0} \left(\frac{y_i}{d_n}\right)^k = \frac{\sum y_i^k}{v(G)d_n^k}$$

By ([4], 5.6), $k-$th power sum of the roots of the modified matching polynomial can be represented by the linear combination of the injective homomorphism density. It also counts the number of closed tree-like walks of length k in the graph G (see chapter 6 of [12]).

$$\begin{aligned}\sum_{i=1}^{v(G)} y_i^k &= \sum_{2\le v(F)\le k+1} c_k(F)\,\mathrm{inj}(F, G_n) \\ &= \sum_{2\le v(F)\le k+1} c_k(F)v(G)d_n(d_n-1)^{v(F)-2} t_{\mathrm{inj}}(F, G_n, d_n)\end{aligned}$$

where F runs over the isomorphism classes of connected graphs, $C_k(F)$ is constant.
By ([1] ,5.22), the homomorphism and injective homomorphism density are almost the same in the dense case

$$t(F, G_n, d_n) - t_{\mathrm{inj}}(F, G_n, d_n) = o\left(\frac{1}{d_n}\right) \to 0$$

So

$$\lim_{n\to\infty} t_{\mathrm{inj}}(F, G_n, d_n) = \lim_{n\to\infty} t(F, G_n, d_n) = t(F)$$

Hence

$$\begin{aligned}\int_0^4 x^k d\nu_{G_n,d_n}(x) &= \sum_{2\le v(F)\le k+1} \frac{c_k(F)(d_n-1)^{v(F)-2}}{d_n^{k-1}} t_{\mathrm{inj}}(F, G_n, d_n) \\ &\longrightarrow \sum_{v(F)=k+1} c_k(F)t(F)\end{aligned}$$

The modified matching polynomial has only real roots,this implies ν_{G_n,d_n} converges weakly. ■

Theorem 3.3. *If the bounded graph sequence (G_n) has degree bound d_n is $\alpha-$regular then the matching measure κ_{G_n,d_n} converges weakly Wigner semicircle distribution supported on $[-2\sqrt{\alpha}; 2\sqrt{\alpha}]$.*

Proof. The matching measure κ_{G_n,d_n} converges weakly because of the convergence of the modified matching measure ν_{G_n,d_n}.

$$\int_{-2}^{2} x^{2k} d\kappa_{G_n,d_n}(x) = 2\int_{0}^{4} x^k d\nu_{G_n,d_n}(x) \to \sum_{v(F)=k+1} 2c_k(F)t(F)$$

where $t(F) = \lim_{n\to\infty} t(F, G_n, d_n) = \alpha^k$.
and

$$c_k(F) = \frac{N(F,k)}{\mathrm{aut}(F)} = \frac{1}{k+1}\binom{2k}{k}$$

where $N(F,k)$ is the number of tree-like, closed walk of length $2k$ which is include all edges of F. In this case F has $k+1$ vertices . By induction, we can prove $N(F,0) = \mathrm{aut}(F)$ and $N(F,n+1)\,\mathrm{aut}(F) = \sum_{i=0}^{n} N(F,i)\,N(F,n-i)$ for $n \geq 0$. That imples $c_k(F)$ equals to Catalan number.
On the other hands, the $2n$-th moment of Wigner distribution is

$$\left(\frac{R}{2}\right)^{2n} \frac{1}{n+1}\binom{2n}{n}$$

and the odd-order moments are zero. Let X have a semicircle distribution with radius $R = 2$ then

$$\lim_{n\to\infty} \int_{-2}^{2} x^{2k} d\kappa_{G_n,d_n}(x) = \alpha^k \mathbb{E}X^{2k} = \mathbb{E}(\sqrt{\alpha}X)^{2k}.$$

■

Theorem 3.4. *Let (G_n, d_n) be an $\alpha-$ regular sequence of large girth, $d_n \to \infty$. Then ξ_{G_n,d_n} converges weakly to the Wigner semicircle distribution supported on the interval $[-2\sqrt{\alpha}; 2\sqrt{\alpha}]$.*

Proof.

$$\int_{-\sqrt{d}}^{\sqrt{d}} x^k d\xi_{G_n,d_n} = \frac{1}{v(G_n)} \sum \left(\frac{\lambda_i}{d_n}\right)^k$$

Note that G_n has large girth so there is $n_o(k)$ such that for all $n \geq n_o(k)$, $\mathrm{inj}(C_k, G_n) = 0$. That means G_n does not contain any cycle when n is big enough, we can consider G_n as a tree because all walks of length k in G_n are not a cycles. So,

$$\frac{1}{v(G_n)} \sum \left(\frac{\lambda_i}{d_n}\right)^k = \frac{1}{v(G_n)} \sum \left(\frac{x_i}{d_n}\right)^k = \int_{-2}^{2} x^k d\kappa_{G_n,d_n}$$

By theorem 3.3,

$$\int_{-2}^{2} x^k d\kappa_{G_n,d_n} \to \int_{-2}^{2} (\sqrt{\alpha}x)^k w(x)dx$$

■

4 Pointwise convergence of the expected spectral measure

4.1 The expected spectral measure

Let A be the adjacency operator of a bounded degree graph, say G (G can be infinite). We already proved that A is a self-adjoint, bounded operator and A is independent of the choice of the root. The spectral theorem for bounded self-adjoint operators gives a projection valued measure

$$\{P_X : \ell^2(G) \to \ell^2(G) | X \subseteq \mathbb{R} \text{ is a Borel set } \}$$

For any Borel function $f : \sigma(A) \to \mathbb{R}$, it satisfies

$$f(A) = \int_{\mathbb{R}} f(x) dP(x)$$

The projection P_X can be comprehended as the orthogonal projection to the span of eigenvectors corresponding to the eigenvalues in X. We can define a measure $\mu_{G,f}$ for any $f \in \ell^2(G)$

$$\mu_{G.f}(X) = \langle P_X f, f \rangle$$

In case G is finite, if $e_1, e_2, \cdots, e_{v(G)}$ is an orthonormal basis of eigenvectors associated to eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_{v(G)}$, we can rewrite

$$\mu_{G.f} = \sum_{i=1}^{v(G)} \langle f, e_i \rangle^2 \delta_{\lambda_i}$$

Definition 4.1. We define the **spectral measure** of G at the root $o \in V(G)$ as

$$\mu_{G,o} = \mu_{G,\chi_o}$$

where χ_o is the characteristic function of o.

Note that for any $o_1, o_2 \in V(G)$, $\langle \chi_{o_1}, A^k \chi_{o_2} \rangle$ is the number of paths of length k from o_1 to o_2 in G. Hence,

$$\int x^k d\mu_{G.o} = \langle \chi_o, A^k \chi_o \rangle = \#\{\text{closed path of length } k \text{ starting from } o\}$$

Definition 4.2. Let G be a fixed graph and choose a root o uniformly at random from the vertex set $V(G)$. This defines a random rooted graph. The **expected spectral measure** of a random rooted graph G is

$$\mu_G = \mathbb{E}(\mu_{G,o})$$

where o is the root of the graph G.

It is well-defined for an arbitrary random rooted graph. By the following lemma, the expected spectral measure recovers the usual spectral measure for finite graphs.

Lemma 4.1. *When G is finite and the root is chosen uniformly then*

$$\mu_G = \frac{1}{v(G)} \sum_{i=1}^{v(G)} \delta_{\lambda_i}$$

Proof. Let (e_i) be the orthonormal basis of A. Since G is finite, by the definition of the spectral measure we get

$$\mu_{G,o} = \sum_{i=1}^{v(G)} \langle \chi_o, e_i \rangle^2 \delta_{\lambda_i} = \sum_{i=1}^{v(G)} e_i(o)^2 \delta_{\lambda_i}$$

By definition of the expected value,

$$\begin{aligned}\mu_G = \mathbb{E}(\mu_{G,o}) &= \frac{1}{v(G)} \sum_{o \in V(G)} \mu_{G,o} = \frac{1}{v(G)} \sum_{o \in V(G)} \sum_{i=1}^{v(G)} e_i(o)^2 \delta_{\lambda_i} \\ &= \frac{1}{v(G)} \sum_{i=1}^{v(G)} \left(\sum_{o \in V(G)} e_i(o)^2 \right) \delta_{\lambda_i} = \frac{1}{v(G)} \sum_{i=1}^{v(G)} \delta_{\lambda_i}\end{aligned}$$

■

It is shown above that the expected spectral measure equals to the eigenvalue distribution of G when G is finite.

4.2 The spectral measure of Cayley graphs

Definition 4.3. Let G be a group. Then a subset $S \subseteq G$ is called a generating set for the group G if every element of G can be expressed as a product of the elements of S or the inverses of the elements of S. If a group has a finite set of generators, it is called a finitely generated group.

Definition 4.4. Let G be a group and S be a subset of G not containing the identity element. Assume $S^{-1} = S$. The **Cayley graph** of a group G with respect to a subset S, denoted $\Gamma(G, S)$, is defined as follows:

(i) $V(\Gamma) = G$

(ii) $E(\Gamma) = \{(g, sg) | g \in G,\ s \in S\}$.

Example 4.1.
- If $G = \mathbb{Z}$ is the infinite cyclic group and the set S consists of the standard generator 1 and its inverse (-1 in the additive notation) then the Cayley graph is an infinite path.

- Similarly, if $G = \mathbb{Z}_n$ is the finite cyclic group of order n and the set S consists of two elements, the standard generator of G and its inverse, then the Cayley graph is the cycle C_n.

- A $d-$regular tree is the Cayley graph of a group freely generated by d involutions.

- The d-dimensional hypercube can be constructed as the Cayley graph

$$\mathrm{Cay}\{((\mathbb{Z}/2\mathbb{Z})^d, (1,0,...,0), (0,1,0,...,0), ..., (0,...,0,1)\}$$

  where the group is the set $\{0,1\}^d$ with the operation of bit-wise xor, and the set S is the set of bit-vectors with exactly one 1.

A vertex-transitive graph is a graph G in which, given any two vertices v_1 and v_2 of G, there is some automorphism $f\colon V(G) \to V(G)$ such that $f(v_1) = v_2$. The Cayley graph is also vertex transitive. Hence, the spectral measure of a vertex-transitive graph at the root o does not depend on the choice of o. It is then natural to define the spectral measure of G as $\mu_G = \mu_{G,o}$ (same notion with the expected spectral measure, but it only makes sense for vertex-transitive graphs in general).

Example 4.2. (Two-way infinite path) The spectral measure of the bi-infinite path

$$d_{\mu_{\mathbb{Z}}}(x) = \frac{1}{\pi\sqrt{4-x^2}} \mathbb{1}_{|x|<2} dx$$

Example 4.3. (Lattice) A lattice is a group which is isomorphic to the additive group $\mathbb{Z}^n$. Let $S = \{(\pm 1, 0, \cdots, 0), (0, \pm 1, 0, \cdots, 0), \cdots, (0, \cdots, 0, \pm 1)\}$ .The Cayley graph of a lattice with respect to a subset S can be considered as the Cartesian product n times of the two-way infinite path. By [13], the spectral measure of the Cartesian product of two graphs equals to convolution of two spectral measures on each graph. So,

$$\mu_{\mathbb{Z}^n}(x) = \varpi * \varpi * \cdots * \varpi$$

where ϖ is the arcsine distribution on $[-2, 2]$.

Example 4.4. (d- regular tree) Let T_d be the infinite d-regular tree. T_d is isomorphic to the Cayley graph of the free group with d generators. In [14], Kesten has proved that

$$d_{\mu_{T_d}}(x) = \frac{d\sqrt{4(d-1)-x^2}}{2\pi(d^2-x^2)} \mathbb{1}_{|x|\le 2\sqrt{d-1}} dx$$

4.3 The pointwise convergence of the expected spectral measure

Theorem 4.1. *Let G_n be a sequence of random rooted finite graphs, and G_n has a common degree bound. Suppose G_n locally converges to G. Then μ_{G_n} weakly converges* *to* μ_G *and*

$$\lim_{n\to\infty} \mu_{G_n}(\{x\}) = \mu_G(\{x\})$$

for any $x \in \mathbb{R}$.

Proof. The first part of the statement can be proved similarly to *Proposition 2.2.* The $k-$th moment of μ_{G_n} equals to the expected value of the number of walks in G_n of length k and starting and ending at o. Convergence in neighbourhood sampling statistics implies convergence of k-th moment of μ_{G_n}.

Since we proved the weak convergence of μ_{G_n}, by Portmanteau theorem we get

$$\limsup_{n\to\infty} \mu_{G_n}(\{x\}) \le \mu_G(\{x\})$$
$$\text{and } \liminf_{n\to\infty} \mu_{G_n}((x-\epsilon, x+\epsilon)) \le \mu_G((x-\epsilon, x+\epsilon))$$

From this we get

$$\begin{aligned}\limsup_{n\to\infty} \mu_{G_n}(\{x\}) \le \mu_G(\{x\}) &\le \mu_G((x-\epsilon, x+\epsilon)) \\ &\le \liminf_{n\to\infty} \mu_{G_n}((x-\epsilon, x+\epsilon)) \\ &\le \liminf_{n\to\infty} \mu_{G_n}(\{x\}) + \liminf_{n\to\infty} \mu_{G_n}((x-\epsilon, x+\epsilon) \setminus \{x\})\end{aligned}$$

Choose ϵ such that $\mu_{G_n}((x-\epsilon, x+\epsilon) \setminus \{x\}) \to 0$

Lemma: Let $D > 0$ be an integer. Then for any $x \in [-D^2, D^2]$ there exists a sequence ϵ_k of positive real numbers converging to 0, such that

$$\mu_G((x-\epsilon_k, x+\epsilon_k) \setminus \{x\}) \le \sqrt{\frac{\log(2D^2)}{\log(1/2\epsilon_k)} + \frac{1}{k}}$$

Apply this lemma for all G_n and choose $\epsilon = \epsilon_k$ then

$$\begin{aligned}\limsup_{n\to\infty} \mu_{G_n}(\{x\}) \le \mu_G(\{x\}) \le \liminf_{n\to\infty} \mu_{G_n}(\{x\}) + \sqrt{\frac{\log(2D^2)}{\log(1/2\epsilon_k)} + \frac{1}{k}} \\ \longrightarrow \liminf_{n\to\infty} \mu_{G_n}(\{x\}) \;\text{ if }\; k \to \infty\end{aligned}$$

We omit the lemma's proof for brevity (see [5],p.9). ■

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Szakdolgozat címe:

Spectra of graphs in a convergent graph sequence

A **szakdolgozat** szerzőjeként fegyelmi felelősségem tudatában kijelentem, hogy a dolgozatom önálló szellemi alkotásom, abban a hivatkozások és idézések standard szabályait következetesen alkalmaztam, mások által írt részeket a megfelelő idézés nélkül nem használtam fel.

Budapest, 2020.

Hieu

a hallgató aláírása