BSc thesis

Rigidity of modified polyhedral graphs

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1. Introduction

One of the first conjectures considered to be in the realm of mathematical rigidity theory was posed by Euler back in 1766, when he proposed that "closed polyhedral surfaces made up of rigid polygon plates that hinge along the edges where plates meet allow no changes in their structure as long as they are not ripped apart". [21] Essentially, this question can be reformulated to asking whether the graphs of spherical polyhedra can be realised as 3-dimensional bar and joint frameworks - joints assigned to the vertices and fixed length bars assigned to the edges - such that triangulating their faces with additional bars results in a space-rigid framework. In 1813, Cauchy made a great contribution by providing a proof for the case of strictly convex polyhedra, and in 1975, Gluck showed that the conjecture holds for polyhedra realised in generic configurations - bar and joint frameworks in which the coordinates of the joints form an algebraically independent set over $\mathbb{Q}$. Finally, Connelly constructed a non-generic triangulated polyhedral framework that is nonrigid in $\mathbb{R}^3$, disproving Euler's original conjecture.

One of the main benefits of investigating generic configurations is that the rigidity of a generic framework depends only on the underlying graph. Hence, the question arises whether we can combinatorially characterize the class of graphs that are rigid when realised as a generic bar and joint framework in $\mathbb{R}^d$. While in one and two dimensions the characterization turned out to be relatively easy, from $d = 3$ onward it still remains a major unsolved problem of the field. Instead of tackling the seemingly complicated general problem of characterising 3-rigid graphs, special classes of graphs related to the well-understood spherical polyhedral frameworks have been studied more extensively over the recent years. These graphs arise from the modification of plane triangulations involving the excision of the disjoint interiors of certain discs and the insertion of minimally rigid blocks into some of the resulting holes, possibly with a few additional bracing edges inserted between nonadjacent vertices. The main focus of my thesis is to present the most important results known about the rigidity of such classes of graphs along with the most frequently used proof methods, and to provide an overview of the major open problems related to the investigated topics.

My thesis is structured as follows. After introducing the fundamental definitions and concepts in Section 2, the most relevant existing results involving the rigidity properties of certain classes of block and hole graphs and braced plane triangulations will be listed in Section 3, without proof. Since the proof of almost all the presented results relies heavily on utilizing rigidity preserving inductive techniques, the main purpose of Section 4 is to introduce some of these inductive methods along with a few proof sketches where they are applied. After that, Section 5 will provide a short summary of the fundamental results and applicable inductive moves known about a stronger version of rigidity, global rigidity, with respect to braced plane triangulations. Additionally, a probabilistic algorithm for determining the d-rigidity and global d-rigidity of graphs will also be presented in Section 5. Section 6 is devoted to summarizing the main open questions and providing a few ideas about how these problems might be solved in the (hopefully close) future. In order to verify the major conjectures for certain graphs with small number of vertices, I also conducted some computer-aided experimentations whose results will be presented in Section 7.

2. Preliminaries

The first part of the Preliminaries deals with introducing the main vocabulary and the basic concepts used in rigidity theory. To gain a wider perspective on the basics of rigidity theory, see [16, 18, 19, 24] for more information, and [10] for an introduction in Hungarian. After that, the second part of the section includes the definitions related to block and hole graphs, following the notations of [7].

2.1. Rigidity of bar and joint frameworks

A d-dimensional **bar and joint framework** is a pair (G, p) where $G = (V, E)$ is a simple undirected graph with no isolated vertices, and $p : V \to \mathbb{R}^d$ is a point configuration of the vertices of G in $\mathbb{R}^d$. The framework (G, p) is also said to be a d-dimensional *realisation* of G as a bar and joint framework.

The term bar and joint framework comes from the application of rigidity theory in statics: given a bar and joint framework (G, p), one can view the vertices of G as joints and the edges of G as rigid, fixed length bars connecting the joints. The joints allow the connecting bars to rotate relative to each other in a free way, facilitating the framework to change its shape when the positions of the joints are locally perturbed. If no such perturbation alters the shape of a given framework, the framework is considered to be statically rigid. This motivates the mathematical definition of *rigid* frameworks.

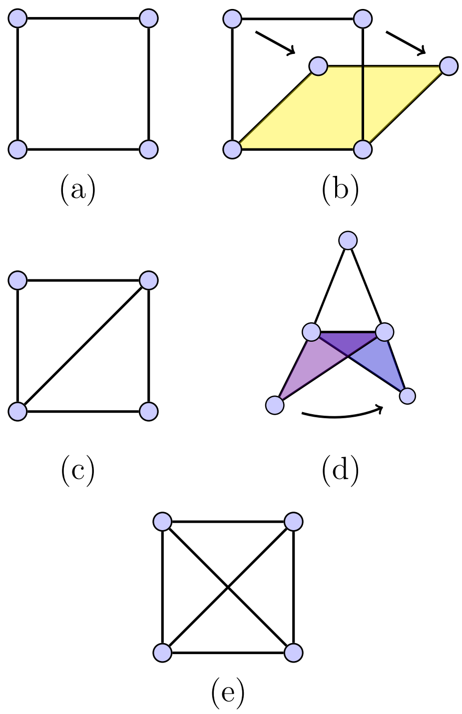
Let $||.||$ denote the Euclidean distance in $\mathbb{R}^d$. Two d-dimensional bar and joint frameworks (G, p) and (G, q) are said to be

- **equivalent** if $||p(u) - p(v)|| = ||q(u) - q(v)||$ for every $uv \in E$,
- **congruent** if $||p(u) - p(v)|| = ||q(u) - q(v)||$ for every $u, v \in V$.

2.1. Definition. *A d-dimensional framework (G, p) is said to be* ***rigid*** *if there exists an $\varepsilon > 0$ such that every d-dimensional framework (G, q) that is equivalent to (G, p) and satisfies $||p(v) - q(v)|| < \varepsilon$ for every $v \in V$, is congruent to (G, p).*

A stronger version of rigidity, *global rigidity* encompasses the case when not only the local perturbations keep the overall shape of a framework, but any global displacement of the joints allows no changes in the pairwise distances of the joints.

2.2. Definition. *A d-dimensional framework (G, p) is said to be* ***globally rigid*** *if every d-dimensional framework (G, q) equivalent to (G, p) is congruent to (G, p).*



2.1. Figure. The 2-dimensional framework in (a) is nonrigid since the smooth motion illustrated in (b) alters the shape of the framework without changing any of the bar lengths. The framework in (c) is rigid as a 2-dimensional framework, but there is a 3-dimensional realisation of the same underlying graph, depicted in (d), that is nonrigid. The 2-dimensional framework illustrated in (e) is an example of a globally rigid framework.

In the limiting case, local perturbations of the joints can also be thought of as giving starting velocities to the joints. This motivates the definition of *infinitesimal motions.*

2.3. Definition. *An infinitesimal motion of a d-dimensional bar and joint framework (G, p) is a map $u : V \to \mathbb{R}^d$ such that for every $v_iv_j \in E$ we have*

$$(u(v_i) - u(v_j))(p(v_i) - p(v_j)) = \mathbf{0}.$$

The coefficients of the above equations can be arranged into an $|E| \times d|V|$ matrix which is called the *rigidity matrix* of the framework (G,p). The rigidity matrix $R_d(G,p)$ can be constructed as follows: for every edge $v_iv_j \in E$ let the entries in the row corresponding to v_iv_j and the d columns corresponding to v_i be the coordinates of $p(v_i) - p(v_j)$, while in the same row let the entries in the columns of v_j be the coordinates of $p(v_j) - p(v_i)$; finally, define the other entries of the row to be zeros. Then the infinitesimal motions of (G,p) are exactly the $d|V|$-dimensional vectors u satisfying the equation $R_d(G,p)u = \mathbf{0}$.

Every d-dimensional bar and joint framework has a space of trivial infinitesimal motions induced by the isometries of $\mathbb{R}^d$ that preserve orientation. If, namely, a d-dimensional framework (G,p) is transformed to (G,q) with a smooth motion that realizes such an isometry and preserves all distances between the vertices of G along the way, then the collection of the starting velocities of each vertex always gives an infinitesimal motion. From this, it is easy to see that the dimension of the space generated by the trivial infinitesimal motions equals the dimension of the orientation preserving isometries of $\mathbb{R}^d$, which is equal to $\binom{d+1}{2}$.

For integers $n \geq 2$ and $d \geq 1$, let

$$S(n,d) = \begin{cases} nd - \binom{d+1}{2} & \text{if } n \geq d+2 \\ \binom{n}{2} & \text{if } n \leq d+1. \end{cases}$$

Note that for $n = d$ and $n = d+1$ the two values above are equal. Now, since the infinitesimal motions lie in the kernel of the rigidity matrix, we obtain the following.

2.4. Lemma. *Suppose (G,p) is a d-dimensional bar and joint framework on $n \geq 2$ vertices. Then the rank of the rigidity matrix $R_d(G,p)$ is at most $S(n,d)$.*

Furthermore, Asimow and Roth [1] showed that in case of equality, the framework (G,p) has to be rigid.

2.5. Theorem. *[1] Let (G,p) be a d-dimensional bar and joint framework on $n \geq 2$ vertices. If the rank of the rigidity matrix $R_d(G,p)$ equals to $S(n,d)$, then (G,p) is rigid.*

A bar and joint framework (G,p) is called **infinitesimally rigid** if $\text{rank}(R_d(G,p)) = S(n,d)$. If $n \geq d+1$, this condition is satisfied exactly when (G,p) has no nontrivial infinitesimal motions. Theorem 2.5 implies that every infinitesimally rigid framework is necessarily rigid. The converse is not true in general. However, in case of *generic* frameworks, infinitesimal rigidity is equivalent to rigidity [1].

2.6. Definition. *A bar and joint framework* (G, p) *is called* ***generic*** *if the collection of the* $d|V|$ *coordinates determined by the vectors* $p(v), v \in V$ *forms an algebraically independent set over* $\mathbb{Q}$.

The generic configurations p of a graph G form an open dense subset of $\mathbb{R}^{d|V|}$, implying that almost every configuration of G is generic. Moreover, frameworks in generic configurations possess many other convenient properties on top of the equivalence of infinitesimal rigidity and rigidity.

The genericity of a framework (G, p) implies that a square submatrix M in $R_d(G, p)$ can only have zero determinant if the polynomial determined by the expansion of $\det M$ with variables $p(v_i)$ is the zero polynomial. Therefore the rank of $R_d(G, p)$, and hence the infinitesimal rigidity of (G, p) does not depend on which generic configuration p was chosen. In this sense, infinitesimal rigidity is a generic property: for generic configurations, the infinitesimal rigidity of a framework (G, p) only depends on the graph G. Since infinitesimal rigidity is equivalent to rigidity for generic frameworks, rigidity is also a generic property. Finally, global rigidity also turns out to be a generic property, proven in [12]. This motivates the following definitions.

2.7. Definition. *A graph* G *is called*

1. ***d-rigid*** *if some (or equivalently, every) generic realisation of* G *as a* d*-dimensional bar and joint framework is rigid.*

2. ***globally d-rigid*** *if some (or equivalently, every) generic realisation of* G *as a* d*-dimensional bar and joint framework is globally rigid.*

Furthermore, a graph G is said to be *minimally d-rigid* if it is d-rigid, but the omission of any edge leaves a graph that is not d-rigid. A graph G is *k-edge d-rigid* if the omission of any $k-1$ edges leaves a d-rigid graph. 2-edge d-rigid graphs are also referred to as *redundantly d-rigid* graphs. Similarly, a graph is said to be *redundantly globally d-rigid* if omitting any edge leaves a globally d-rigid graph.

As an easy consequence of Theorem 2.5 and the equivalence of infinitesimal rigidity and rigidity in generic configurations, every d-rigid graph on at least $d+1$ vertices must have at least $d|V| - \binom{d+1}{2}$ edges. In particular, for $d = 1, 2, 3$ this lower bound on the edges is $|V| - 1, 2|V| - 3$ and $3|V| - 6$ respectively.

2.8. Definition. *A graph $G = (V, E)$ is said to satisfy the d-dimensional Maxwell count if*

- $|E| = d|V| - \binom{d+1}{2}$,
- $|E(V')| \leq d|V'| - \binom{d+1}{2}$ *for every* $V' \subset V$ *with* $|V'| \geq d$.

The graphs satisfying the d-dimensional Maxwell count are also referred to as *d-dimensional Maxwell graphs.* Moreover, the conditions above are also called *d-dimensional Maxwell conditions.*

An easy argument applied to the row matroid of the rigidity matrix using basic linear algebra yields the following.

2.9. Lemma. *Let G be a minimally d-rigid graph. Then G is a d-dimensional Maxwell graph.*

Throughout the thesis, most of the theorems involving rigidity will contain certain connectivity assumptions. Therefore it is worth mentioning that the d-rigid graphs are automatically d-connected.

2.10. Lemma. *Every d-rigid graph is d-connected.*

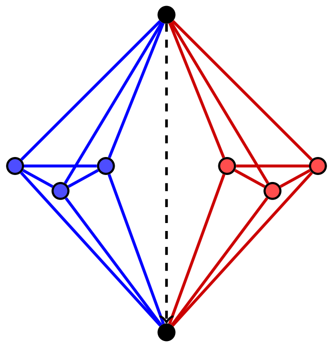
In the $d = 1$ and $d = 2$ case, a combinatorial characterisation of d-rigidity and minimal d-rigidity is already known.

In the simplest case $d = 1$, the vertices and edges of a bar and joint framework (G, p) all lie on a line. If G is not connected, then each connected component can move separately on the line without changing any of the bar lengths, implying that (G, p) is nonrigid. Conversely, if G is connected, then it is easy to see that no nontrivial motions of the framework exist on the line, hence (G, p) is rigid. Therefore a graph is 1-rigid if and only if it is connected. Moreover, a graph is minimally 1-rigid if and only if it is a tree. Since the 1-dimensional Maxwell graphs are exactly the trees, it follows that the conditions of Lemma 2.9 are necessary and sufficient in the $d = 1$ case.

The situation is a bit more complicated in two dimensions, but it still remains true that the minimally 2-rigid graphs are exactly the 2-dimensional Maxwell graphs. This result is attributed to Laman (1970).

2.11. Theorem. *[17] (Laman) A graph $G = (V, E)$ is minimally 2-rigid if and only if it is a 2-dimensional Maxwell graph. Furthermore, a graph is 2-rigid if and only if it has a spanning subgraph which satisfies the 2-dimensional Maxwell count.*

However, it still remains a major open problem to characterise the class of d-rigid graphs for $d \geq 3$. In particular, when $d = 3$, the double banana graph provides an example for a graph that satisfies the 3-dimensional Maxwell conditions, but fails to be 3-rigid.



2.2. Figure. The double banana graph can be constructed by gluing together two copies of K_5 along a pair of vertices, and removing the two resulting parallel edges. The blue and red "banana" subgraphs can be rotated in $\mathbb{R}^3$ independently along the dashed axis, therefore the graph is not 3-rigid.

2.2. Block and hole graphs

Although the general problem of characterising 3-rigid graphs as well as minimally 3-rigid graphs still remains a great challenge, many interesting results are known about the 3-rigidity properties of certain special classes of graphs obtained by modifying plane triangulations: block and hole graphs and braced plane triangulations. In the next lines, we aim to introduce the necessary related definitions and concepts, mainly following the notations and vocabulary of [7].

2.12. Definition. *A simple, 3-connected planar graph is called a polyhedral graph. A simple triangulated planar graph is called a plane triangulation.*

Note that the plane triangulations are exactly the maximal planar graphs, and that a plane triangulation is always 3-connected. Moreover, it is well known that a plane triangulation G has exactly $3|V|-6$ edges. The term polyhedral graph derives from Steinitz theorem, which states that the graphs formed by the vertices and edges of convex spherical polyhedra are exactly the polyhedral graphs.

2.13. Theorem. *[20] (Steinitz) A graph can be realized as the edge skeleton of a convex spherical polyhedron if and only if it is a simple, 3-connected planar graph.*

Let P be a plane triangulation with a given planar realisation. Every cycle c in P of length at least four divides the graph into an inner and outer region bounded by the edges of c. Each of the subgraphs determined by the edges of such a region and the edges of c are referred to as *simplical disks* with boundary cycle c. The boundary cycle of a simplical disk D is denoted by ∂D, and the set of edges not belonging to ∂D is called the *edge interior* of D. Two simplical disks D_1 and D_2 are said to be *internally disjoint* if every edge contained in both D_1 and D_2 lies in $\partial D_1 \cap \partial D_2$.

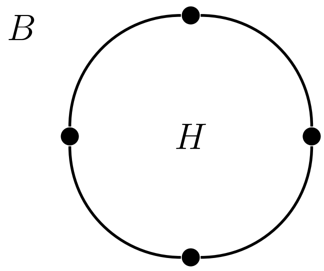
2.14. Definition. *Let P be a plane triangulation with a given planar realisation, and $\mathcal{B} \cup \mathcal{H}$ a collection of pairwise internally disjoint simplical disks in P.*

The graph $G = (G, \mathcal{B}, \mathcal{H})$ is called a face graph if G is obtained from P by

1. *removing the edge interiors of each simplical disk $D \in \mathcal{B} \cup \mathcal{H}$,*
2. *labelling the new faces bounded by the boundary cycles ∂D by either the letter B or H depending on whether $D \in \mathcal{B}$ or $D \in \mathcal{H}$.*

Thus, a face graph is essentially a special planar graph in which every face with boundary length at least four is labelled by either B or H. The B-labelled and H-labelled faces of G are called *blocks* and *holes*, respectively. A face graph G is said to be of type (m, n) if the number of B-labelled faces is m and the number of H-labelled faces is n. Plane triangulations are also considered to be face graphs as they constitute the class of type $(0, 0)$ face graphs.

If the triangular faces are furthermore labelled by the letter T, every face of G has a corresponding label B, H or T, and the edges of G can be classified into 6 different types according to the labelling of their two adjacent faces: BB, BH, HH, BT, HT or TT.



2.3. Figure. The smallest face graph of type (1,1).

2.15. Example. The complete graph K_4 is a plane triangulation which may be expressed as the union of two internally disjoint simplical disks with a common 4-cycle boundary. The edge interiors of the simplical disks each contain a single edge, with the two edges being nonadjacent. Removing these edge interiors and labelling the two resulting faces by B and H gives us a face graph of type $(1, 1)$, which is the smallest face graph of this type.

The reason why we have defined face graphs is that we would like to determine the rigidity properties of graphs that arise from plane triangulations by removing the edge interiors of certain disks to create holes, and in order to compensate for the loss of edges, bracing certain disks to create blocks. The base structure of such graphs can be described by a face graph, while the resulting braced graphs will be called *block and hole graphs.*

2.16. Definition. *Let $G = (G, \mathcal{B}, \mathcal{H})$ be a face graph with B-labelled faces $\mathcal{B} = \{B_1, B_2, \ldots, B_m\}$. A block and hole graph on G is a graph $\hat{G} = (\hat{G}, \hat{\mathcal{B}})$ of the form $\hat{G} = G \cup \hat{B}_1 \cup \hat{B}_2 \cup \ldots \cup \hat{B}_m$ where*

1. *$\hat{B}_1, \hat{B}_2, \ldots, \hat{B}_m$ are minimally 3-rigid graphs which might only intersect at vertices and edges of G,*
2. *$G \cap \hat{B}_i = \partial B_i$ for every $1 \leq i \leq m$.*

The subgraphs $\hat{B}_i$ are referred to as the blocks or isostatic blocks of $\hat{G}$. Given a block and hole graph $\hat{G}$, the letter G without the bar will denote the unique underlying face graph. Note that $\hat{G}$ need not be simple; this is going to be apparent when we define the *spherical block and hole graphs* on the next page.

The following useful lemma called the *isostatic block substitution principle* asserts that one may substitute isostatic blocks without altering the rigidity properties of $\hat{G}$, hence the minimal 3-rigidity of a block and hole graph only depends on the underlying face graph.

2.17. Lemma. *[7, 8] (Isostatic block substitution principle) Let $(G, \mathcal{B}, \mathcal{H})$ be a face graph and suppose there exists a block and hole graph on G which is simple and minimally 3-rigid. Then every simple block and hole graph on G is minimally 3-rigid.*

As a consequence, for determining whether the block and hole graphs on a face graph G are minimally 3-rigid, it suffices to investigate the most natural block and hole graph constructions on G. Throughout this thesis, two special classes of block and hole graph constructions will be used: the *discus and hole graph*, and the *spherical block and hole graphs.*

2.18. Definition. *(Discus and hole graph) Let $(G, \mathcal{B}, \mathcal{H})$ be a face graph with block set $\mathcal{B} = \{B_1, B_2, \ldots, B_m\}$. The discus and hole graph $G^\dagger$ is constructed from G by*

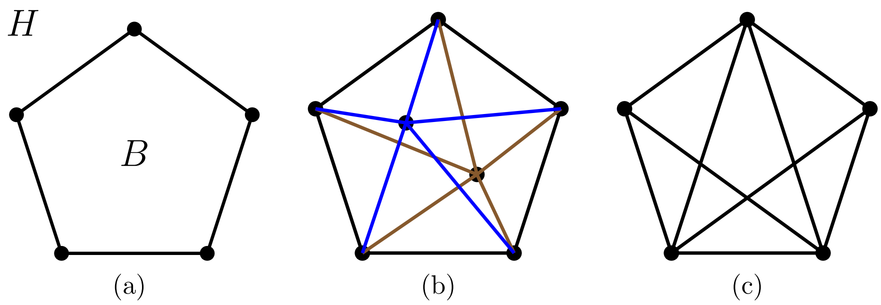
1. *inserting $2m$ different vertices $x_1, x_2, \ldots, x_m$ and $y_1, y_2, \ldots, y_m$ into G,*
2. *connecting x_i and y_i to the vertices of ∂B_i for every $1 \leq i \leq m$.*

Note that the graph $G^\dagger$ is simple and uniquely determined by G.

2.19. Definition. *(Spherical block and hole graph) Let $(G, \mathcal{B}, \mathcal{H})$ be a face graph with block set $\mathcal{B} = \{B_1, B_2, \ldots, B_m\}$. The graph G° is said to be a spherical block and hole graph on G if G° is constructed from G by adjoining $2|\partial B_i| - 6$ edges to each block B_i such that the vertices of ∂B_i induce the graph of a plane triangulation.*

Note that since every cycle is a planar graph, every boundary cycle ∂B_i can be extended to a plane triangulation by an appropriately chosen $2|\partial B_i| - 6$ edges, and the included $2|\partial B_i| - 6$ edges together with the $|\partial B_i|$ edges lying in the boundary indeed give the $3|\partial B_i| - 6$ edges required to build a plane triangulation.

Observe moreover that G° is not uniquely determined and may not be simple. The minimal 3-rigidity of the isostatic blocks in $G^\dagger$ and G° is guaranteed by Gluck's Theorem which will be presented in the next section (Theorem 3.1).



2.4. Figure. The graph in (b) depicts the discus and hole graph of the face graph illustrated in (a). The graph in (c) is a spherical block and hole graph on the same face graph.

Throughout the thesis, we will also frequently investigate *braced plane triangulations.*

2.20. Definition. *A braced plane triangulation is a graph* $G = (V, E \cup B)$ *where* $H = (V, E)$ *is a plane triangulation, and* B *is a set of additional edges which we refer to as the bracing edges of* G.

A braced plane triangulation is called *uni-braced* if $|B| = 1$ and *doubly-braced* if $|B| = 2$.

3. Rigidity of modified plane triangulations

3.1. Fundamental results

First of all, we aim to list the most fundamental existing results known about the 3-rigidity of block and hole graphs and braced plane triangulations. As we have already mentioned in the Introduction, the oldest and most profound result of generic rigidity theory, attributed to Gluck (1975), states that plane triangulations are minimally 3-rigid.

3.1. Theorem. *[11] (Gluck's Theorem) Every plane triangulation is minimally 3-rigid.*

We will present two short proofs of this theorem in Section 4, one using an inductive move called *vertex splitting* (Theorem 4.3), and another using *proper subnets* (Theorem 4.22).

A very convenient and widely exploited property of face graphs is that the block and hole graphs on a face graph G are minimally 3-rigid if and only if the block and hole graphs on the face graph G_t obtained from G by replacing the B labels by H labels and the H labels with B labels are minimally 3-rigid. This is referred to as the *block and hole transposition principle.*

3.2. Theorem. *[8] (Block and hole transposition principle) The block and hole graphs on the face graph G are minimally 3-rigid if and only if the block and hole graphs on G_t are minimally 3-rigid.*

Now, we will move on to the results known about the 3-rigidity of block and hole graphs in the simplest cases. To begin, we will investigate block and hole graphs with one hole and one block concentrating on the rigidity properties that can be concluded from certain connectivity assumptions. In the following results, two sets of vertices A and B in a graph are said to be *k-connected* if no vertex of A can be separated from a vertex of B by a set of fewer than k vertices.

3.3. Theorem. *[22] Let G be a face graph with one block B and one hole H such that $|\partial B| = |\partial H| = k$. Then the block and hole graphs on G are minimally 3-rigid if and only if the vertices of ∂B and ∂H are k-connected.*

If we are satisfied with 3-rigidity instead of minimal 3-rigidity, we can increase the size of the block to still ensure the 3-rigidity of the block and hole graphs, provided that the same connectivity assumptions hold.

3.4. Corollary. *[22] Let G be a face graph with one block B and one hole H such that $|\partial B| \geq |\partial H| = k$. Then the block and hole graphs on G are 3-rigid if and only if the vertices of ∂B and ∂H are k-connected.*

If the face graph G is itself k-connected, the connectivity condition of Theorem 3.3 is automatically satisfied. Thus in the special cases of $k = 4$ and $k = 5$ we obtain the following results.

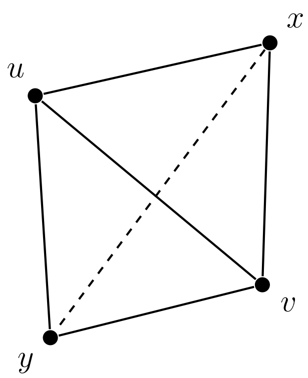
3.5. Corollary. *[22]*

1. *Every block and hole graph on a 4-connected face graph with one quadrilateral block and one quadrilateral hole is minimally 3-rigid.*
2. *Every block and hole graph on a 5-connected face graph with one pentagonal block and one pentagonal hole is minimally 3-rigid.*

Now we would like to investigate graphs which are obtained from a 4-connected plane triangulation by inserting an additional bracing edge. Since the case when the bracing edge connects distant parts of the graph is a bit trickier to handle, firstly we will consider only *dihedral* bracing edges.

3.6. Definition. *Let G be a plane triangulation, and construct the uni-braced plane triangulation $H = G + xy$ by adjoining the brace xy. The edge xy is said to be a dihedral bracing edge in H if there is an edge $uv \in E$ such that $F_1 = uvx$ and $F_2 = uvy$ are two triangular faces in G.*

Suppose now that G is obtained from a 4-connected plane triangulation by omitting an edge. Since G has $3|V|-7$ edges, G cannot be 3-rigid, hence it has a nontrivial infinitesimal motion in every generic realisation in $\mathbb{R}^3$. However, if H is a uni-braced plane triangulation obtained from G by inserting a dihedral bracing edge xy disjoint from the four vertices of the hole, the 4-connectivity together with Corollary 3.5 implies that H is 3-rigid. Thus, every nontrivial infinitesimal motion of G has been eliminated by inserting the dihedral edge xy into G. This in turn means that given a



3.1. Figure. A dihedral bracing edge.

generic realisation (G, p) as a bar and joint framework, every nontrivial infinitesimal motion u must *flex* the distance between x and y, that is, for any two vertices x, y in a dihedral position we have

$$(u(x) - u(y))(p(x) - p(y)) \neq \mathbf{0}.$$

Considering a generic realisation of G as a spherical polyhedron in $\mathbb{R}^3$, the argument above shows that every nontrivial infinitesimal motion flexes each of the dihedral angles formed by adjacent triangular faces. This means that omitting an edge from a 4-connected triangulated spherical polyhedron not only destroys the 3-rigidity of the global framework, but allows every local part of the framework (sufficiently far away from the omitted edge) to change its shape through an appropriate infinitesimal flex.

Now we present a stronger version of the first statement of Corollary 3.5 which was proven by Whiteley [22].

3.7. Theorem. *[22] Let G be a uni-braced plane triangulation with a dihedral bracing edge. If G is* 4*-connected, then G is redundantly* 3*-rigid.*

Whiteley [22] also showed that this result holds in the more general case when the bracing edge connects two arbitrary vertices of the graph.

3.8. Theorem. *[22] Let G be a uni-braced plane triangulation. If G is* 4*-connected, then G is redundantly* 3*-rigid.*

The natural generalization would be to ask whether 5-connected doubly-braced plane triangulations are 3-edge 3-rigid, that is, they remain 3-rigid if any two edges are omitted. This particular question is conjectured in [22], and is also one of the main open questions of the field.

3.9. Conjecture. *[22] Let G be a doubly-braced braced plane triangulation. If G is 5-connected, then G is 3-edge 3-rigid.*

A major partial result for the case when the bracing edges are two different diagonals of a pentagon, and the vertices of the pentagon have all degree 5 was proven in [14].

3.10. Theorem. *[14] Let G be a 5-connected plane triangulation on $n \geq 12$ vertices, and fix a planar realisation of G. Suppose the graph H is obtained from G by*

1. *choosing a five-cycle C in G whose interior contains three triangular faces and two diagonal edges, and whose vertices have all degree 5,*
2. *inserting two diagonals of C as bracing edges such that the subgraph induced by the vertices of C forms the graph of a plane triangulation.*

Then H is 3-edge 3-rigid.

In [7], another partial result for the case when the two bracing edges and the two omitted edges are dihedral and the bracing edges form a pentagonal block was established. With regards to the block and hole transposition principle, the corresponding extended theorem is the following.

3.11. Theorem. *[7] If a 5-connected face graph G has*

- *one pentagonal block and one pentagonal hole or,*
- *one pentagonal block and two quadrilateral holes or,*
- *two quadrilateral blocks and one pentagonal hole,*

then every block and hole graph on G is minimally 3-rigid.

A main milestone in getting closer to proving Conjecture 3.9 would be to address the missing case of two quadrilateral blocks and two quadrilateral holes.

3.12. Conjecture. *[22] Every block and hole graph on a 5-connected face graph with two quadrilateral blocks and two quadrilateral holes is minimally 3-rigid.*

3.2. Maxwell conditions

In the following two subsections we describe two characterisations of minimally 3-rigid block and hole graphs in the special cases of a single block or a single hole, based on the results established in [7]. The main theorem of this subsection will state that the 3-dimensional Maxwell count provides a necessary and sufficient condition for the minimal 3-rigidity of such block and hole graphs, whereas in the following subsection the girth inequalities will be introduced, which provide another equivalent condition for minimal 3-rigidity in terms of lower bounds on the lengths of cycles around collections of blocks and holes.

For simplicity, in the rest of the thesis will refer to the 3-dimensional Maxwell conditions as Maxwell conditions, and to the 3-dimensional Maxwell graphs as Maxwell graphs.

3.13. Theorem. *[7] Let $\hat{G}$ be a block and hole graph with a single block or a single hole. Then $\hat{G}$ is minimally 3-rigid if and only if $\hat{G}$ satisfies the Maxwell conditions.*

A proof sketch for the one block case of this theorem following the argument used in [7] will be presented in Subsection 4.2. The single hole case will then follow from the block and hole transposition principle described in Theorem 3.2.

Now we will list some other notable results about the relationship of minimal 3-rigidity and the Maxwell conditions in the case of block and hole graphs of type $(1, n)$ and $(m, 1)$. The following lemma notes that a block and hole graph of type $(1, n)$ which satisfies the Maxwell conditions is always 3-connected. Note that in general a block and hole graph is not necessarily 3-connected.

3.14. Lemma. *[7] Let $\hat{G}$ be a block and hole graph with a single block. If $\hat{G}$ satisfies the Maxwell count, then $\hat{G}$ is 3-connected.*

Although in [7] the above lemma was proven directly, it is apparent that it also follows from Theorem 3.13 and Lemma 2.10.

Similarly to the isostatic block substitution principle of Lemma 2.17, one may exchange an isostatic block of a block and hole graph satisfying the Maxwell count to another minimally 3-rigid block such that the resulting block and hole graph is also a Maxwell graph.

3.15. Lemma. *[7] Let G be a face graph. If there exists a block and hole graph on G which satisfies the Maxwell count, then every block and hole graph on G satisfies the Maxwell count.*

Therefore the underlying face graph G determines completely whether the block and hole graphs on G satisfy the Maxwell count.

3.3. Girth inequalities

The girth inequalities defined in [7] provide the second characterization of minimally 3-rigid block and hole graphs having one block or one hole.

3.16. Definition. *Let $(G, \mathcal{B}, \mathcal{H})$ be a face graph and let $\mathcal{B}' \subset \mathcal{B}$ and $\mathcal{H}' \subset \mathcal{H}$ respectively denote collections of B-labelled and H-labelled faces of G. Define the index of the collection $\mathcal{B}' \cup \mathcal{H}'$ by*

$$ind(\mathcal{B}' \cup \mathcal{H}') = \sum_{B \in \mathcal{B}'} (|\partial B| - 3) - \sum_{H \in \mathcal{H}'} (|\partial H| - 3).$$

3.17. Definition. *(Girth inequalities) A face graph G is said to satisfy the girth inequalities if for every cycle c in G and planar realisation of G we have*

$$|c| \geq |ind(\mathcal{C})| + 3,$$

where $\mathcal{C}$ is the collection of B-labelled and H-labelled faces interior to c.

Now let us see what the girth inequalities actually mean in the case of face graphs of type $(1, 1)$.

3.18. Example. Let G be a face graph with a single block B and a single hole H, with boundary lengths r_1 and r_2 respectively, and suppose G satisfies the girth inequalities. Choosing a planar realisation of G with a triangular face as the unbounded region, the girth inequality on the boundary 3-cycle gives us $3 \geq |r_1 - r_2| + 3$, therefore $r_1 = r_2$. Moreover, every cycle in G which winds around either B or H must have length at least r_1, which is satisfied exactly when every cycle around H

has length at least r_1. From here it is easy to show that G satisfies the girth inequalities if and only if $r_1 = r_2$ and every cycle winding around H has length at least r_1.

The main characterisation theorem is then the following.

3.19. Theorem. *[7] Let G be a face graph with one block or one hole. Then the block and hole graphs on G are minimally* 3*-rigid if and only if G satisfies the girth inequalities.*

In the more general setting of type (m, n) face graphs the following result holds.

3.20. Lemma. *[7] Let G be a face graph, and suppose the block and hole graphs on G satisfy the Maxwell conditions. Then G satisfies the girth inequalities.*

In particular, if the block and hole graphs on G are minimally 3-rigid, then G must satisfy the girth inequalities.

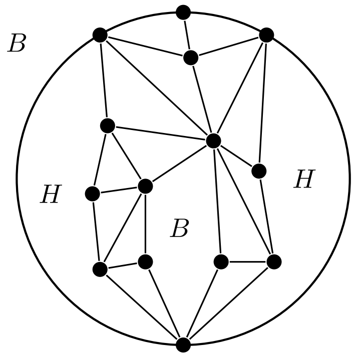
The following necessary conditions for minimally 3-rigid block and hole graphs known as *separation conditions* arise directly from the application of Lemma 3.20 and Lemma 2.10.

3.21. Corollary. *[7] Let G be a face graph, and suppose that the block and hole graphs on G are minimally* 3*-rigid. Then the following statements hold.*

1. *There are no edges in G between nonadjacent vertices in the boundary of a labelled face of G.*

2. *Each pair of labelled faces in G with the same label share at most two vertices and these vertices must be adjacent.*

To conclude this section, we give an example from [7] of a face graph G of type $(2, 2)$ which satisfies the girth inequalities, and the block and hole graphs on G satisfy the Maxwell count, but the block and hole graphs on G fail to be minimally 3-rigid. This in turn means that Theorem 3.13 and Theorem 3.19 do not generalize to block and hole graphs with at least two blocks and two holes.



3.2. Figure. A face graph of type $(2,2)$ which satisfies the girth inequalities, has block and hole graphs satisfying the Maxwell conditions, but has no minimally 3-rigid block and hole graphs.

4. Inductive techniques

In this section we are going to present the most frequently used inductive methods that are applied to generate various families of 3-rigid graphs and to prove the results of Section 3. In particular, in order to see the inductive moves in application, we include two proof sketches of Gluck's Theorem (see Theorem 4.3 and Theorem 4.22), detail the argument used to prove Theorem 3.13 (see Subsection 4.2), and illustrate the proof for a special case of Theorem 3.3 (see Theorem 4.21).

4.1. Vertex splitting

The first and perhaps the most widely applied inductive move is the *d-dimensional vertex splitting operation.*

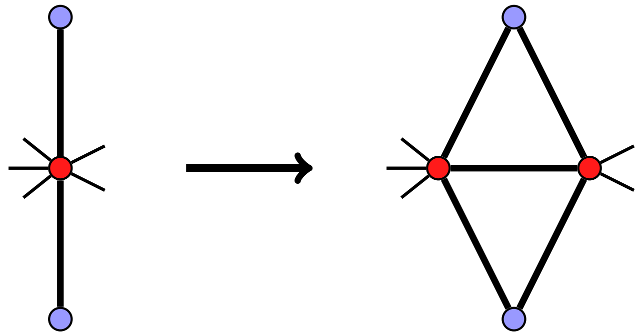
4.1. Definition. *Let $G = (V, E)$ be a graph and fix a vertex $v \in V$ with degree at least $d - 1$ and neighbor set $N = N(v)$. The graph operation $G \to G'$ is said to be a d-dimensional* ***vertex splitting operation*** *on the vertex v if G' is obtained from G by*

1. *partitioning N into three pairwise disjoint sets $N = N_0 \cup N_1 \cup N_2$ such that $|N_0| = d - 1$,*
2. *removing v from G, and inserting two different vertices v_1 and v_2 connected by an edge into G,*
3. *connecting v_1 to all vertices of $N_0 \cup N_1$ and connecting v_2 to all vertices of $N_0 \cup N_2$.*

Since we are mostly interested in results involving 3-rigidity, the term 3*-dimensional vertex splitting operation* will be referred to as simply *vertex splitting operation* in the rest of the thesis.

A d-dimensional vertex splitting operation on v is called *nontrivial* if N_1 and N_2 are both nonempty. This is equivalent to saying that both v_1 and v_2 have degree at least $d + 1$ in G'.

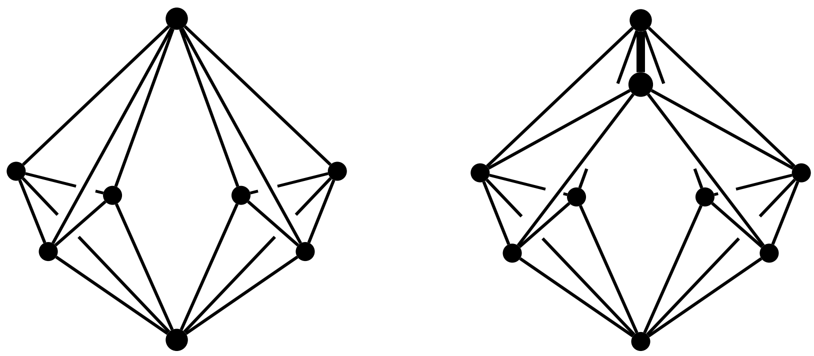
Whiteley showed in [23] that d-dimensional vertex splitting is a d-rigidity preserving operation.



4.1. Figure. A 3-dimensional vertex splitting operation

4.2. Theorem. *[23] Let G be a d-rigid graph, and suppose G' is obtained from G by a d-dimensional vertex splitting operation. Then G' is also d-rigid.*

The d-dimensional vertex splitting operation can be reversed by shrinking an edge v_1v_2 with $|N(v_1) \cap N(v_2)| = d-1$ into one vertex v such that v gets connected exactly to the vertices v_1 and v_2 were connected to (with exception to v_1 and v_2 itself). Note that this so called (d-dimensional) *inverse vertex splitting operation* does not necessarily preserve d-rigidity; a counterexample in the $d = 3$ case is illustrated below.



4.2. Figure. The graph depicted in (b) derives from the double banana graph (a) by a vertex splitting operation on the top vertex. It can be easily verified that this graph is 3-rigid, but the double banana graph which is obtained from the extended graph by an inverse vertex splitting operation is not 3-rigid.

It can be seen that in a face graph G', shrinking an edge e by a (3-dimensional) inverse vertex splitting operation results in a face graph G whose labelled faces have identical boundary cycles as the respective labelled faces of G' if and only if e is contained in no 3-cycle in G' that separates two nonempty sets of vertices from each other. Such a 3-cycle is called a *separating* 3-*cycle*, and if an edge is not contained in any separating 3-cycle, it is said to be a *contractible* edge. Moreover, the graph move $G' \to G$ involving the contraction of a contractible edge e is referred to as the *edge contraction* move on e.

If e is a TT or a BH type edge, the edge contraction move on e is called TT *edge contraction* or BH *edge contraction*, respectively. These two operations will receive special attention in the next subsection.

Finally, to show one of the most straightforward applications of the Vertex Splitting Theorem 4.2, we include a short proof of Gluck's Theorem based on the argument described in [7].

4.3. Theorem. *[11] (Gluck) Every plane triangulation is minimally* 3*-rigid.*

Proof. [7] We first prove by induction on n that for every plane triangulation G on $n \geq 4$ vertices and every planar realisation of G, there is a contractible *interior* edge, that is, an edge which is not contained in the three boundary edges of the unbounded face of G. If $n = 4$, the only plane triangulation is K_4, and since every edge of K_4 is contractible, the statement follows. Now suppose that the claim is proven for plane triangulations on at most n vertices, and let G be a plane triangulation on $n+1$ vertices.

Fix a plane realisation of G and an interior edge uv in G. If uv is contractible, the inductive step is complete. If not, uv is contained in a separating 3-cycle uvz. Suppose the two triangular faces bounded by uv are uvx and uvy. Then it is clear that either x or y is contained in the interior of uvz, implying that the plane triangulation G' determined by the 3-cycle uvz and its interior edges has at least 4, but at most n vertices. Therefore, by the inductive hypothesis, G' contains a contractible interior edge, and this edge remains a contractible interior edge in the bigger graph G.

Now we are ready to prove that every plane triangulation is 3-rigid. The smallest plane triangulation is the complete graph K_3 which is 3-rigid. Suppose we have already shown that every plane triangulation on at most n vertices is 3-rigid, and take a plane triangulation G with $n+1$ vertices. We have shown that G contains a contractible edge; shrinking this edge creates a plane triangulation with n vertices which is 3-rigid by the induction hypothesis. Since G can be constructed from this graph by a vertex splitting operation that is known to preserve 3-rigidity, we obtain that G is also 3-rigid.

Minimal 3-rigidity then follows from 3-rigidity and the fact that a plane triangulation satisfies $|E| = 3|V| - 6$. □

From the argument used in the proof we also obtain the following corollary, which is an independent result on its own, proven by Steinitz and Rademacher [20].

4.4. Corollary. *[20] Every plane triangulation can be derived from K_3 by a sequence of vertex splitting operations.*

4.2. Proof sketch of Theorem 3.13

Theorem 3.13 about the equivalence of minimal 3-rigidity and the Maxwell conditions in the case of block and hole graphs with one hole or one block is proven in [7] using TT and BH edge contraction moves and the separating cycle division operation defined below. In the following lines we will sketch the proof of the one block case of this theorem by focusing on the key definitions and lemmas, following the argument presented in [7].

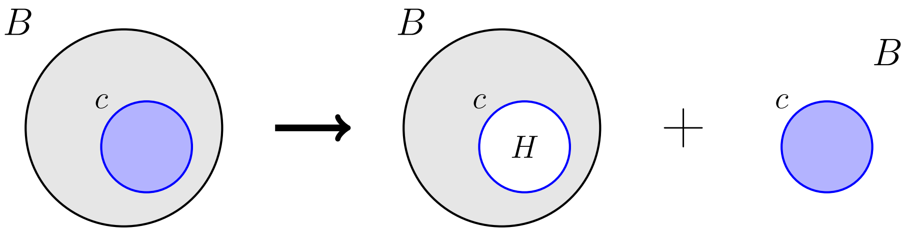
For simplicity, face graphs whose block and hole graphs satisfy the Maxwell conditions will be referred to as *Maxwell face graphs.*

4.5. Definition. *(Separating cycle division) Let G be a face graph with one block, and fix a planar realisation of G with the unbounded face labelled by B. Let c be a cycle in G.*

Define G_1 as the face graph obtained from G by deleting the edges and vertices interior to c, and if $|c| \geq 4$, assigning the label H to the new face bounded by c. Moreover, let G_2 be the face graph obtained from G by deleting the edges and vertices outside of c, and if $|c| \geq 4$, assigning the label B to the newly formed outer region bounded by c.

The graph operation $G \to \{G_1, G_2\}$ is referred to as a ***separating cycle division*** *on the cycle c.*

Let $G \to \{G_1, G_2\}$ be a separating cycle division on the cycle c. Define $Ext(c)$ as the discus and hole graph of G_1 and $Int(c)$ as the discus and hole graph of the face graph derived from G_2 by relabelling the outside region by H. Note that these



4.3. Figure. Separating cycle division

graphs satisfy $Int(c) \cap Ext(c) = c$ and $Int(c) \cup Ext(c) = G^{\dagger}$, where $G^{\dagger}$ is the discus and hole graph of G.

4.6. Definition. *Let G be a face graph and c a cycle in G. The cycle c is called a critical separating cycle if either $Ext(c)$ or $Int(c)$ satisfies the Maxwell conditions.*

Observe that if c is a boundary cycle of a face in the Maxwell face graph G, then $Int(c)$ is a Maxwell graph. Hence the boundary cycle of any face in G gives us a trivial critical separating cycle. A critical separating cycle c is called *nontrivial* if it is not a boundary cycle of any face.

4.7. Definition. *A critical separating cycle division on a face graph G is a separating cycle division on a critical separating cycle.*

A critical separating cycle division on c is called nontrivial if c is a nontrivial critical separating cycle. A nontrivial critical separating cycle division has the convenient property that both of the generated graphs G_1 and G_2 have fewer vertices than G.

The following lemma asserts that every Maxwell face graph with one block and at least one hole admits a TT edge contraction, a BH edge contraction, or a nontrivial critical separating cycle division.

4.8. Lemma. *[7] Let G be a Maxwell face graph with one block and at least one hole. Then one of the following statements must hold.*

1. *G contains a contractible TT edge.*
2. *G contains a contractible BH edge.*
3. *G contains a nontrivial critical separating cycle.*

It turns out that in most cases, all of the three inductive moves above preserve the Maxwell property of face graphs.

4.9. Lemma. *[7] Let G be a Maxwell face graph with one block.*

1. *If $G \to G'$ is a TT edge contraction move on a contracible TT edge which is not contained in any nontrivial critical separating cycle of G, then G' is also a Maxwell face graph.*

2. *If $G \to G'$ is a BH edge contraction move on a contracible BH edge, then G' is also a Maxwell face graph.*

3. *If $G \to \{G_1, G_2\}$ is a critical separating cycle division, then both G_1 and G_2 are Maxwell face graphs.*

On the other side, we have already seen that the inverse TT and BH edge contraction moves are special vertex splitting operations, hence they preserve minimal 3-rigidity. Moreover, given a critical separating cycle division $G \to \{G_1, G_2\}$ such that the discus and hole graphs $G_1^\dagger$ and $G_2^\dagger$ are both minimally 3-rigid, observe that $G^\dagger$ can be obtained from $G_2^\dagger$ by substituting the minimally 3-rigid graph $G_1^\dagger$ into the B-labelled outside region of G_2. Therefore, the isostatic block substitution principle implies that $G^\dagger$ is also minimally 3-rigid.

We are now ready to prove the one block case of Theorem 3.13.

4.10. Theorem. *[7] Let $\hat{G}$ be a block and hole graph on the face graph G with a single block. Then $\hat{G}$ is minimally 3-rigid if and only if $\hat{G}$ satisfies the Maxwell conditions.*

Proof. [7] It is already known that minimally 3-rigid graphs are necessarily Maxwell graphs, therefore we only need to prove the opposite direction of the implication.

We will proceed by induction on the number of vertices of $\hat{G}$. If $V(\hat{G}) = 4$, then $\hat{G}$ is a Maxwell graph if and only if $\hat{G} = K_4$. We have already seen that K_4 is minimally 3-rigid, hence the base case of the induction is complete.

Suppose now that we have already proven the claim for block and hole graphs on at most $n-1$ vertices. Let $\hat{G}$ be a block and hole graph with one block on n vertices, and suppose $\hat{G}$ satisfies the Maxwell conditions. According to Lemma 4.8, the underlying face graph G admits a TT edge contraction, a BH edge contraction, or a nontrivial critical separating cycle division.

In the former two cases, the contracted face graph G' is also a Maxwell graph, and G' has fewer than n vertices. Therefore, according to the induction hypothesis, the block and hole graphs on G' are minimally 3-rigid. Since G can be constructed from G' by applying a vertex splitting operation, the block and hole graphs on G must also be minimally 3-rigid.

In the case of a nontrivial critical separating cycle division $G \to \{G_1, G_2\}$, the face graphs G_1 and G_2 both inherit the Maxwell property and have fewer than n vertices. This means that the block and hole graphs on G_1 and G_2 are minimally 3-rigid, and we have already seen that in this case the block and hole graphs on G are also minimally 3-rigid.

This concludes the proof. □

The induction step in the proof essentially relied upon the fact that every Maxwell face graph with one block can be constructed from smaller Maxwell face graphs by applying either a vertex splitting operation or an isostatic block substitution move. Since both of these inductive operations preserve minimal 3-rigidity, we obtain another equivalent condition on the minimal 3-rigidity of face graphs which have one block.

4.11. Theorem. *[7] Let $\hat{G}$ be a block and hole graph with a single block. Then $\hat{G}$ is minimally 3-rigid if and only if $\hat{G}$ is constructible from K_3 by the moves of vertex splitting and isostatic block substitution.*

Summa summarum, combining Theorem 3.19, Theorem 3.13, Theorem 4.11, and the block and hole transposition principle, we obtain the following complete characterisation theorem for minimally 3-rigid block and hole graphs which have one block or one hole.

4.12. Theorem. *[7] Let $\hat{G}$ be a block and hole graph with a single block or a single hole. Then the following statements are equivalent.*

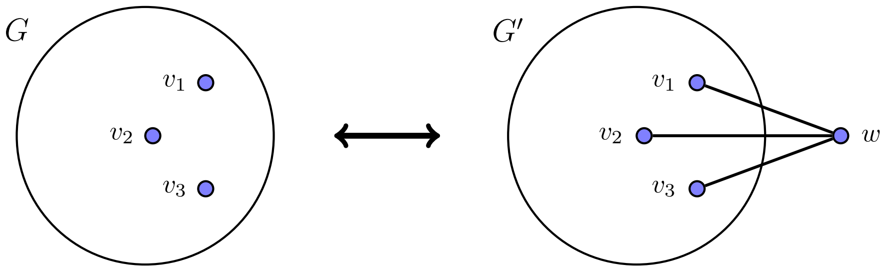
1. *$\hat{G}$ is minimally 3-rigid.*
2. *$\hat{G}$ satisfies the Maxwell-conditions.*
3. *$\hat{G}$ can be constructed from K_3 by the moves of vertex splitting and isostatic block substitution.*
4. *G satisfies the girth inequalities.*

4.3. Henneberg extension operations

Similarly to the vertex splitting operation, the *Henneberg extension operations* are also widely applied simple graph moves that are known to preserve 3-rigidity. Although the later results of this thesis will not make use of the Henneberg operations, we are still going to introduce them for the sake of completeness.

4.13. Definition. *Let $G = (V, E)$ be graph and $v_1, v_2, v_3 \in V$ distinct vertices of G. Suppose G' is obtained from G by inserting an additional vertex w and the edges wv_1, wv_2, wv_3 into G. The operation $G \to G'$ is said to be a 3-dimensional Henneberg 0-extension operation.*

4.14. Lemma. *[24] Suppose G' is obtained by a 3-dimensional Henneberg 0-extension operation from G. Then G' is 3-rigid if and only if G is 3-rigid.*

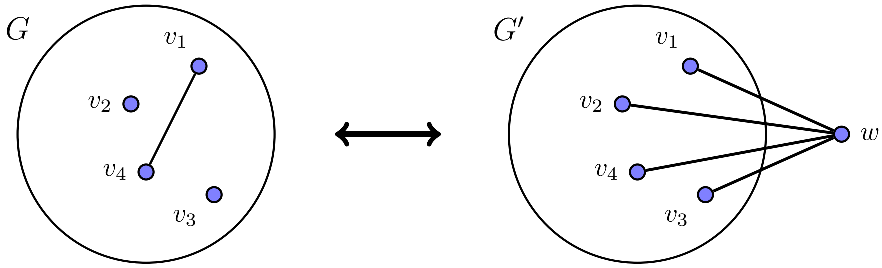


4.4. Figure. A 3-dimensional Henneberg 0-extension operation

4.15. Definition. *Let $G = (V, E)$ be a graph and $v_1, v_2, v_3, v_4 \in V$ distinct vertices of G with $v_1v_4 \in E$. Suppose G' is obtained from G by inserting an additional vertex w and the edges wv_1, wv_2, wv_3, wv_4 into G, and removing the edge v_1v_4 from G. The operation $G \to G'$ is said to be a 3-dimensional Henneberg 1-extension operation.*

4.16. Lemma. *[24] Suppose G' is obtained from G by a 3-dimensional Henneberg 1-extension operation. If G is 3-rigid, then G' is also 3-rigid.*

Conversely, if G' is a 3-rigid graph and wv_1, wv_2, wv_3, wv_4 are edges of G', then there is a pair of vertices v_i and v_j $(1 \le i < j \le 4)$ such that removing the four edges wv_k $(1 \le k \le 4)$ and connecting v_i to v_j results in a 3-rigid graph G.



4.5. Figure. A 3-dimensional Henneberg 1-extension operation

4.4. Proper subnets

The concept of *subnets* and *proper subnets* was introduced by Whiteley [22] as he tried to come up with an effective tool that can be utilized to determine the rigidity properties of block and hole graphs with certain connectivity properties. In particular, this method is applied in [22] to prove Theorem 3.3 and Theorem 3.8.

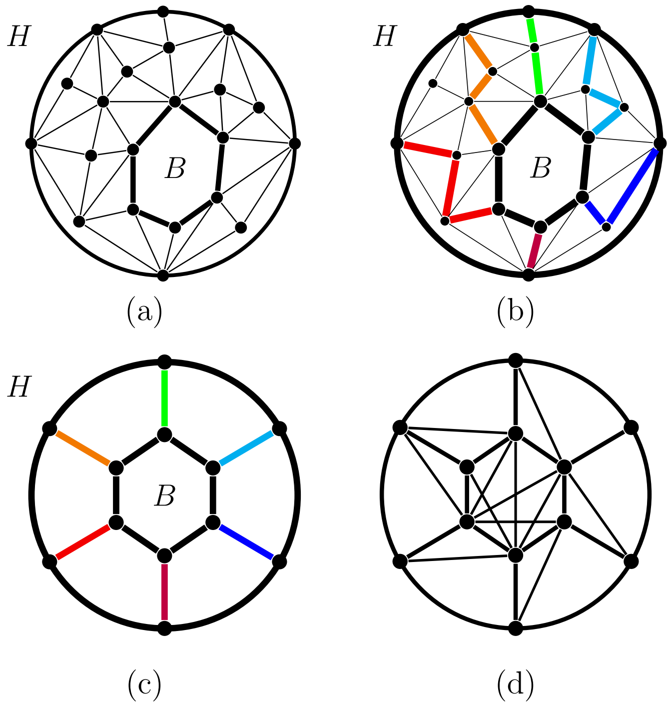
From now on we say that two paths in a graph are *disjoint* if they are internally disjoint.

4.17. Definition. *The graph $G' = (V', E', \mathcal{H})$ is said to be a subnet of the block and hole graph $\hat{G} = (V, E)$ if*

1. *$V' \subset V$,*
2. *the elements of E' are pairwise disjoint paths in $\hat{G}$ with endpoints contained in V',*
3. *V' contains the boundary vertices of the blocks and holes of $\hat{G}$,*
4. *E' contains the edges of the isostatic blocks of $\hat{G}$, and the boundary edges of the holes of $\hat{G}$,*
5. *$\mathcal{H}$ contains the simplical disks of the holes labelled by the letter H,*
6. *the resulting graph G' is 2-connected with no multiple edges.*

A subnet of a block and hole graph may have vertices of degree 2, and may contain faces with boundary lengths at least 4 that are different from the faces determined by the blocks and holes. This is the main reason why it is necessary to mark the position of the holes by the label H.

It is clear that if the isostatic blocks and the holes are replaced with B-labelled faces, the above definition can be generalized to face graphs, i.e. we can define a *subnet face graph* $G' = (V', E', \mathcal{B}, \mathcal{H})$ of a given face graph G with the property that substituting the B labelled faces of both G and G' by the same set of isostatic blocks results in a block and hole graph $\hat{G}$ and one of its subnets $\hat{G}'$.



4.6. Figure. The boundary vertices and edges of the labelled faces of Figure (b) together with the colored paths determine a subnet of the face graph illustrated in (a). The graph of the subnet is depicted in (c), while (d) illustrates a subnet triangulation of a spherical block and hole graph on the subnet (see Definition 2.19 for the definition of spherical block and hole graphs).

Let G' be a subnet of the block and hole graph $\hat{G}$. The vertices of G' with degree at least 3 are called *natural vertices* of G', and paths of degree 2 vertices in G' joining two natural vertices are called *natural edges* of G'. The natural vertices with the corresponding natural edges determine a planar graph, which is referred to as the *skeleton graph* of G'.

If a subnet G' in a block and hole graph $\hat{G}$ has a 3-connected skeleton graph, then G' is said to be a *proper subnet* of $\hat{G}$. Note that if $\mathcal{H}$ denotes the collection of the holes of $\hat{G}$, and each hole in $\hat{G}$ is labelled by H, then $(\hat{G}, \mathcal{H})$ can be considered to be a proper subnet of $\hat{G}$.

Finally, observe that every planar embedding of the underlying face graph of $\hat{G}$ also determines a planar embedding of G' with a corresponding face structure. We can now proceed with the definition of *subnet triangulations*.

4.18. Definition. *A graph T is called a subnet triangulation if T is obtained from a subnet G' of a block and hole graph by fixing a planar realisation of G' and triangulating the unlabelled faces of G'.*

A convenient property of subnet triangulations is that the 3-rigidity of a particular subnet triangulation does not depend on the chosen planar realisation of G' and the applied face triangulation. [22]

Given a block and hole graph $\hat{G}$ with a proper subnet G', Whiteley's original idea was to deduce the rigidity properties of $\hat{G}$ by only looking at the structure of G'. In order to achieve this, he defined a set of inductive moves that can be used to construct $\hat{G}$ from any of its proper subnets with the additional nice property that all these moves preserve the 3-rigidity of the generated subnet triangulations along the way. The corresponding theorem describing these construction moves is as follows.

4.19. Theorem. *[22] Suppose $\hat{G}$ is a block and hole graph and G' is a proper subnet of $\hat{G}$. Then there is a sequence of proper subnets $G' = G_1, G_2, \ldots, G_k = \hat{G}$ such that G_{i+1} can be constructed from $G_i = (V_i, E_i)$ using one of the following operations:*

1. ***Edge stretching:*** *replacing a path $P \in E_i$ by a longer path joining the same end vertices, pairwise disjoint from the other paths in E_i,*
2. ***Edge splitting:*** *dividing a path $P = (v_1, v_2, \ldots, v_n) \in E_i$ into two disjoint paths $P_1 = (v_1, \ldots v_l)$ and $P_2 = (v_l, \ldots, v_n)$ and adding v_l to V_i,*
3. ***Face splitting:*** *inserting a new path P into E_i across a face of G_i that is neither a block nor a hole, pairwise disjoint from the paths in E_i, such that the endpoints of P can be separated by the remaining natural vertices of G_i,*
4. ***Vertex inserting:*** *inserting a vertex u of degree 3 into V_i such that the edges of the triangle determined by the three neighbors of u are boundary edges of some block or hole in G_i.*

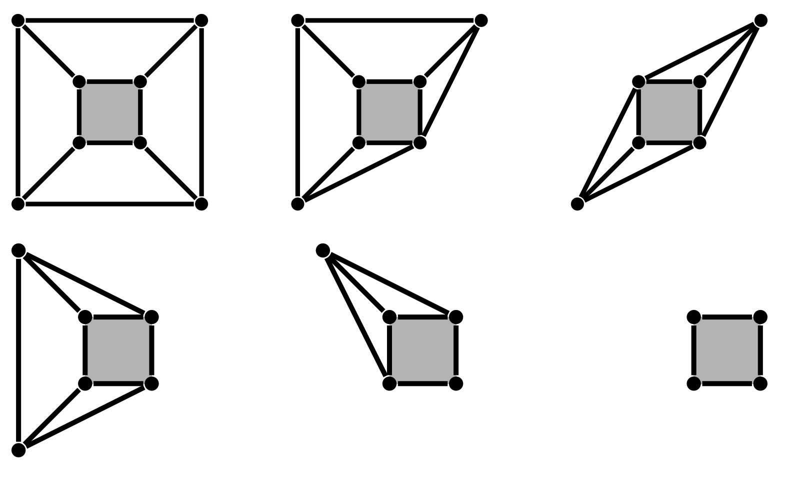
Furthermore, if the subnet triangulations of G_i are 3-rigid, then the subnet triangulations of G_{i+1} are also 3-rigid.

4.20. Corollary. *[22] If a block and hole graph $\hat{G}$ has a proper subnet with a* 3*-rigid subnet triangulation, then $\hat{G}$ is* 3*-rigid.*

Now we illustrate the use of Corollary 4.20 by giving a short proof of a weaker version of Theorem 3.3 in the special case $k = 4$, and a second proof of Gluck's Theorem (Theorem 3.1).

4.21. Theorem. *[22] (A special case of Theorem 3.3) Let G be a face graph with one quadrilateral block B and one quadrilateral hole H. If the vertices of ∂B and ∂H are* 4*-connected, then the block and hole graphs on G are minimally* 3*-rigid.*

Proof. Let $\hat{G}$ be a block and hole graph on G. The 4-connectivity condition together with Menger's theorem implies that there are 4 disjoint paths connecting all boundary vertices of the block of $\hat{G}$ with the boundary vertices of the hole. Depending on whether these paths have length 0 or at least 1, we obtain that $\hat{G}$ contains one of the subnets illustrated below. It can be easily verified that all these subnets are proper and have 3-rigid subnet triangulations (e.g. by calculating the rank of the rigidity matrix of a subnet triangulation in an appropriately chosen generic configuration, or by constructing the subnet triangulations from smaller 3-rigid graphs using rigidity-preserving graph operations). Therefore, Corollary 4.20 concludes the proof. □



4.7. Figure. The subnets used in the proof of Theorem 4.21

4.22. Theorem. *[11] (Gluck) Every plane triangulation is minimally* 3*-rigid.*

Proof. [22] We will prove that every plane triangulation (as a face graph) contains the complete graph K_4 as a proper subnet. Let G be a plane triangulation and $v_1v_2v_3$ a face of G. It is evident that G is 3-connected. For any vertex v_4 not on the chosen face, Menger's theorem asserts that there exist three pairwise disjoint paths p_1, p_2, p_3 such that p_i connects v_i and v_4 $(i = 1, 2, 3)$. Then the subnet on the vertices v_1, v_2, v_3, v_4 determined by the paths and the edges of the triangular face gives us a proper subnet isomorphic to K_4.

Since K_4 is already triangulated and its 3-rigidity can be easily verified, Corollary 4.20 concludes the proof. □

5. Global rigidity

This section will focus on listing a few relevant results about the global rigidity of braced plane triangulations along with some inductive techniques preserving global rigidity. Recall that a graph G is called globally d-rigid if for every generic configuration in $\mathbb{R}^d$ as a bar and joint framework, the distances between any two vertices are invariant under all bar-length preserving transformations.

5.1. Global rigidity of braced plane triangulations

The following theorem describes two useful properties of globally d-rigid graphs on at least $d+2$ vertices.

5.1. Theorem. *[13] Let G be a globally d-rigid graph. Then either G is a complete graph on at most $d+1$ vertices, or G is $(d+1)$-connected and redundantly d-rigid.*

The necessary conditions of Theorem 5.1 are known to be sufficient to characterize global d-rigidity for $d = 1, 2$. However, for $d \geq 3$, there are counterexamples showing that the necessary conditions are not sufficient.

A recent result of Jordán and Tanigawa [15] characterises global 3-rigidity for braced plane triangulations on $n \geq 5$ vertices.

5.2. Theorem. *[15] Let $G = (V, E \cup B)$ be a braced plane triangulation on at least 5 vertices. Then G is globally 3-rigid if and only if G is 4-connected and $|B| \geq 1$.*

Note that the necessity follows from Theorem 5.1 and the fact that plane triangulations are not redundantly 3-rigid (since they are minimally 3-rigid by Gluck's theorem).

Another remark worth mentioning is that Theorem 5.2 and Theorem 5.1 together imply Theorem 3.8.

Following Conjecture 3.9, a similar but stronger conjecture is proposed in [15].

5.3. Conjecture. *[15] Suppose $G = (V, E)$ is a 5-connected doubly-braced plane triangulation. Then G is redundantly globally 3-rigid.*

5.2. Inductive moves

Since the most utilized inductive move in the case of rigidity is the vertex splitting operation, the question arises whether it can be made use in arguments involving global rigidity. Note that a d-dimensional vertex splitting move may create a vertex with degree at most d, which would violate the $(d+1)$-connectivity condition of Theorem 5.1 for the resulting graph. Therefore, only nontrivial vertex splitting operations have a chance to preserve global rigidity. Unfortunately, however, it is still not known for $d \geq 3$ whether the nontrivial vertex splitting move preserves global rigidity.

5.4. Conjecture. *[3] Let G be a globally d-rigid graph with at least $d+2$ vertices, and suppose G' is obtained from G by a nontrivial d-dimensional vertex splitting operation. Then G' is globally d-rigid.*

A weaker result is proven in [6] in the case when G' admits a $v'v''$- coincident infinitesimally rigid realisation in $\mathbb{R}^d$.

5.5. Theorem. *[6] Let G be a globally d-rigid graph and $v \in V$. Suppose that G' is obtained from G by a nontrivial d-dimensional vertex splitting operation splitting v into v' and v''. If G' has an infinitesimally rigid realisation (G,p) in $\mathbb{R}^d$ with $p(v') = p(v'')$, then G' is globally d-rigid.*

This theorem is the main tool used in [6] for providing a combinatorial proof of Theorem 5.2.

Another weaker result for graphs constructible from smaller graphs with maximum degree at most $d+2$ was established in [15].

5.6. Theorem. *[15] Let G be a globally d-rigid graph with maximum degree at most $d+2$, and suppose G' is obtained from G by a sequence of nontrivial d-dimensional vertex splitting operations. Then G' is globally d-rigid.*

It is easy to verify that K_{d+2} is globally d-rigid for $d \geq 1$, therefore from Theorem 5.6 we obtain the following corollary.

5.7. Corollary. *[15] Suppose G is constructible from K_{d+2} by a sequence of nontrivial d-dimensional vertex splitting operations. Then G is globally d-rigid.*

This confirms the global d-rigidity of an interesting infinite family of graphs.

Now, Corollary 5.7 together with the following theorem establishes a proof of Theorem 5.2.

5.8. Theorem. *[15] Let G be a 4-connected uni-braced plane triangulation. Then G is constructible from K_5 by a sequence of 3-dimensional nontrivial vertex splitting operations.*

5.3. Testing global rigidity

As we have already seen, checking the d-rigidity of a graph is relatively easy: it suffices to find a configuration of the vertices such that the rank of the rigidity matrix at the given configuration is $S(n, d)$. This can be done by e.g. calculating the rank of the rigidity matrix at randomly chosen configurations until we obtain a matrix with the required rank. If no such configuration was found, we can safely assume the graph is not d-rigid.

However, in the case of global rigidity the situation is a bit more complicated. In the next few paragraphs we introduce some new definitions such as the stress equations and the stress matrices [15], and give a randomized algorithm for testing the global d-rigidity of a graph [5, 12].

5.9. Definition. *Let $G = (V, E)$ be a graph and $p : V \to \mathbb{R}^d$ a configuration of the vertices. An equilibrium stress (or stress, for short) on the framework (G, p) is an assignment $\omega : E \to \mathbb{R}$ such that for each vertex $v_i \in V$ we have*

$$\sum_{j: v_i v_j \in E} \omega_{i,j}(p(v_i) - p(v_j)) = \mathbf{0},$$

where $\omega_{i,j}$ is used to denote $\omega(v_i, v_j)$.

The equations above are referred to as the *equilibrium equations.*

Notice that the equilibrium stresses for a framework (G, p) are exactly the $|E|$ dimensional vectors ω satisfying $\omega^T \cdot R_d(G, p) = \mathbf{0}$. Therefore, there is a direct relationship between the space of stress vectors and the space of infinitesimal motions of a framework: they are the orthogonal complements of the column space and the row space of $R_d(G, p)$, respectively.

5.10. Definition. *Let ω be an equilibrium stress on the framework (G, p). The stress matrix Ω associated to ω is the $|V| \times |V|$ symmetric matrix in which the entries are defined such that $\Omega_{i,j} = \Omega_{j,i} = -\omega_{i,j}$ if $v_i v_j$ is an edge, $\Omega_{i,j} = 0$ for all nonadjacent vertex pairs $v_i, v_j \in V$, and $\Omega_{i,i} = \sum_{j \neq i} \omega_{i,j}$ for all $1 \leq i \leq |V|$.*

It is easy to show that the rank of Ω is at most $|V| - d - 1$.

Using these new definitions, the following characterisation theorems provide computable conditions for testing the global d-rigidity of a graph.

5.11. Theorem. *[4, 12] Let (G, p) be a d-dimensional generic framework on at least $d+2$ vertices. Then (G, p) is globally d-rigid if and only if (G, p) has an equilibrium stress ω for which the rank of the associated stress matrix Ω is $|V| - d - 1$.*

5.12. Theorem. *[5] Let $G = (V, E)$ be a graph. Suppose there is a configuration $p : V \to \mathbb{R}^d$ such that (G, p) is infinitesimally rigid, and there is an equilibrium stress where the rank of the associated stress matrix Ω is $|V| - d - 1$. Then G is globally d-rigid.*

Theorem 5.12 already provides a randomized algorithm for testing the global d-rigidity of a graph G (on at least $d+2$ vertices): pick a random configuration p, and check if (G, p) is infinitesimally rigid; after that, choose a random stress vector ω, and calculate the rank of the stress matrix associated to ω; if it equals $|V| - d - 1$, conclude that G is globally d-rigid; else, try choosing another configuration, or if number of configurations that were already tried out exceeds a critical number, declare that G is probably not globally d-rigid. However, the main drawback of this algorithm is that when implementing it as a program, it requires special attention to handle the rounding issues that may emerge during the rank calculations.

Moreover, a deterministic algorithm for testing global d-rigidity can also be given based on the statement of Theorem 5.12: treat the coordinates p as formal symbols, and calculate the formal rank of $R_d(G, p)$; if it equals $S(n, d)$, calculate a stress vector ω by formally solving the equilibrium equations; then, check if the formal rank of the associated stress matrix equals $|V| - d - 1$. For a more detailed description of the algorithm, see Algorithm 5.1 in [12]. While this algorithm is guaranteed to give the correct answer, the running time can be very slow as the size of the polynomials we need to manipulate grows rapidly.

A more efficient probabilistic algorithm was developed by Connelly and Whiteley [5] which is based on a more general form of Theorem 5.12 and only requires calculations with integers.

5.13. Theorem. *[5] Let $G = (V, E)$ be a graph and q be a prime number. Suppose there is a configuration $p : V \to \mathbb{Z}_q^d$ such that (G, p) is infinitesimally rigid (i.e. the modulo q rank of the rigidity matrix equals $S(n, d)$), and there is an equilibrium stress where the modulo q rank of the associated stress matrix Ω is $|V| - d - 1$. Then G is globally d-rigid.*

The corresponding algorithm for determining the global d-rigidity of a graph is then the following.

5.14. Algorithm. *[5] Let $G = (V, E)$ be a graph, $d \geq 1$ the dimension, and $N \geq 1$ an integer parameter.*

1. *If $|E| < d|V| - \binom{d+1}{2} + 1$, then G cannot even be redundantly d-rigid; hence, declare that G is not globally d-rigid.*
2. *Choose a prime number q. (This should be such that q^d is sufficiently larger than $|V|$.)*
3. *Pick a random configuration $p : V \to \mathbb{Z}_q^d$.*
4. *Compute the modulo q rank of $R_d(G, p)$; if it does not equal $d|V| - \binom{d+1}{2}$, return to step 2, or if the number of primes that were already considered exceeds N, conclude that G is probably not globally d-rigid.*
5. *Calculate a random equilibrium stress $\omega \in \mathbb{Z}_q^{|E|}$ by solving the equilibrium equations; since $|E| \geq d|V| - \binom{d+1}{2} + 1$ and G is d-rigid, the solution space of the equilibrium equations is at least one dimensional.*
6. *Compute the modulo q rank of the stress matrix Ω associated to ω; if it does not equal $|V| - d - 1$, return to step 2, or if the number of primes that were already considered exceeds N, declare that G is probably not globally d-rigid; else, stop and conclude that G is globally d-rigid.*

While the above algorithm gives no false positives, false negatives are possible with low probability. This probability can be reduced by either considering greater prime numbers q, or increasing the number of tries N. Experimentation suggests that if $d = 3$ and $|V| \leq 200$, the values $q \approx 1000$ and $N \approx 20$ already give a negligible probability of false negatives.

6. Open questions

In this section we include a summary of the most important conjectures and open questions related to the topics we have investigated throughout the thesis. Special attention will be devoted to Conjecture 3.9 and Conjecture 5.3 as we are going to list some relaxed versions of these conjectures along with possible directions on how these questions might be approached and hopefully solved in the future.

6.1. Rigidity

The main unsolved question presented in this thesis involving rigidity is Whiteley's Conjecture 3.9, stated again down below.

6.1. Conjecture. *[22] Let G be a doubly-braced braced plane triangulation. If G is 5-connected, then G is 3-edge 3-rigid.*

A slightly relaxed variation of this conjecture is the following.

6.2. Conjecture. *Suppose G is constructed from a 5-connected plane triangulation by inserting two additional bracing edges. Then G is 3-edge 3-rigid.*

An even weaker version concerning the case when the two bracing edges as well as the two omitted edges form dihedral edges and the final block and hole graph has two blocks and two holes is described below.

6.3. Conjecture. *[22] Every block and hole graph on a 5-connected face graph with two quadrilateral blocks and two quadrilateral holes is minimally 3-rigid.*

6.2. Global rigidity

It is still a major open question whether nontrivial vertex splitting operations preserve global rigidity. If proven, this particular statement would immediately imply the global rigidity of a wide range of graphs and might contribute to the solution of most questions involving global rigidity.

6.4. Conjecture. *[3] Let G be a globally d-rigid graph with at least $d+2$ vertices, and suppose G' is obtained from G by a nontrivial d-dimensional vertex splitting operation. Then G' is globally d-rigid.*

A similar and equally important conjecture as Conjecture 6.1 is Conjecture 5.3 which was proposed by Jordán and Tanigawa [15].

6.5. Conjecture. *[15] Suppose G is a 5-connected braced triangulation with at least two braces. Then G is redundantly globally 3-rigid.*

As in the case of Conjecture 6.1, we can weaken the connectivity assumption to obtain a slightly weaker version of the question.

6.6. Conjecture. *Suppose G is constructed from a 5-connected plane triangulation by inserting two additional bracing edges. Then G is redundantly globally 3-rigid.*

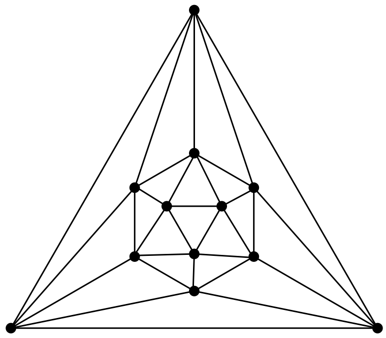
It would already be an interesting result if for every $n \geq 12$, a graph with exactly n vertices could be found that satisfies the statement of Conjecture 6.6, or at least if the statement of the conjecture was proven for an infinite family of graphs. Note that in the case of the similar Conjecture 6.1, Theorem 3.10 gives a graph for every $n \geq 12$ satisfying the conjecture.

6.7. Conjecture. *For every $n \geq 12$, there exists a 5-connected doubly-braced plane triangulation that is redundantly globally 3-rigid and has exactly n vertices.*

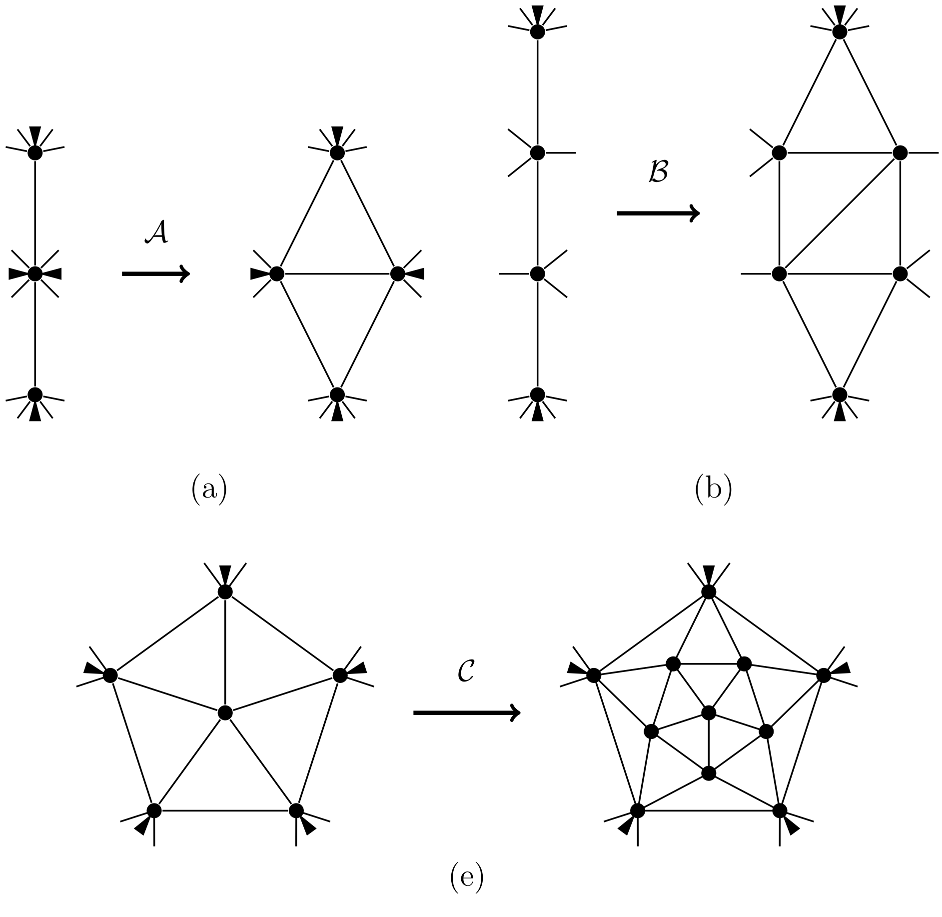
6.8. Conjecture. *There exists an infinite family of 5-connected doubly-braced plane triangulations that are redundantly globally 3-rigid.*

In order to find a proof for Conjecture 5.3, the first idea that comes to mind is that we should consider a construction sequence of 5-connected plane triangulations and then apply an inductive argument. Luckily, Brinkmann and McKay [2] solved the former problem by showing that 5-connected plane triangulations can be generated from the graph of the icosahedron using operations $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$, all three moves illustrated in the next page.

6.9. Theorem. *[2] Let G be a plane triangulation. Then G is 5-connected if and only if G can be constructed from the graph of the icosahedron by the operations $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$.*



6.1. Figure. The graph of the icosahedron.



6.2. Figure. The construction moves $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$, generating every 5-connected plane triangulation from the graph of the icosahedron. The black wedges at certain vertices indicate that other graph edges on top of the illustrated edges may connect to the marked vertex.

Note that the operation $\mathcal{A}$ is a (nontrivial) vertex splitting operation. More interestingly, the following can be shown.

6.10. Observation. *Suppose G' is constructed from G by one of construction operations $\mathcal{A}, \mathcal{B}$, or $\mathcal{C}$. Then G' can be constructed from G by a sequence of nontrivial vertex splitting operations.*

The reason why we are still concerned with the three more complicated operations is that they all preserve 5-connectivity, while the sequence of nontrivial vertex splitting moves generating them may create 4-connected plane triangulations along the way. Therefore, in order to make progress in verifying Conjecture 6.5, it would be sufficient to prove the following weaker version of the $d = 3$ case of Conjecture 6.4.

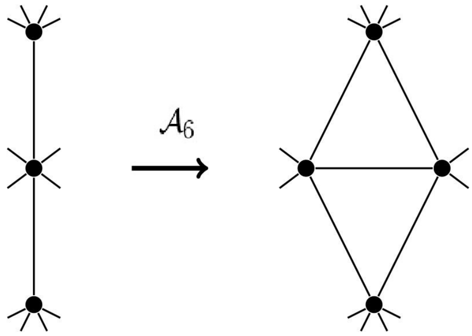
6.11. Conjecture. *The operations $\mathcal{A}, \mathcal{B}, \mathcal{C}$ preserve global 3-rigidity.*

It would be a great start if Conjecture 6.5 could be verified at least for the class of 5-connected plane triangulations with maximum degree 6. It is easy to see that these graphs can be constructed from the graph of the icosahedron using the special versions of the three illustrated inductive moves where zero additional edges are entering the vertices through the black wedges. Let us call these three special moves $\mathcal{A}_6, \mathcal{B}_6$, and $\mathcal{C}_6$.

6.12. Conjecture. *The operations $\mathcal{A}_6, \mathcal{B}_6, \mathcal{C}_6$ preserve global 3-rigidity.*

Observe that the operations $\mathcal{A}_6, \mathcal{B}_6$ and $\mathcal{C}_6$ each need 2, 4 and 6 vertices of degree 5 to be applied, respectively. Since a 5-connected plane triangulation with maximum degree 6 has exactly 12 vertices of degree 5, it seems to be unlikely that such graphs with a great number of vertices possess degree 5 vertices in configurations required to apply either operation $\mathcal{B}_6$ or $\mathcal{C}_6$. Therefore, if we restrict our attention to prove Conjecture 6.7 by generating an infinite family of 5-connected, doubly-braced, maximum degree 6 plane triangulations satisfying Conjecture 6.5, it might also be sufficient to show that operation $\mathcal{A}_6$ preserves global 3-rigidity.

6.13. Conjecture. *The operation $\mathcal{A}_6$ preserves global 3-rigidity.*



6.3. Figure. The operation $\mathcal{A}_6$

7. Computer-aided investigations

In order to give hope that the conjectures investigated in the last section might indeed be true, I implemented the randomized global rigidity testing algorithm described in Algorithm 5.14 as well as the randomized rigidity testing algorithm decribed in the beginning of Subsection 5.3 in Python, and verified some of the conjectures for a range of graphs with a small number of vertices. This section is dedicated to presenting the results of these computer-aided tests.

7.1. Conjecture 6.2

The main conjecture involving 3-rigidity, Conjecture 6.2 was tested in certain special cases. First of all, the following special case of the conjecture was checked and verified.

7.1. Lemma. *If the doubly-braced plane triangulation G is obtained from the graph of the icosahedron by inserting two bracing edges, then G is* 3*-edge* 3*-rigid.*

Since greedily checking the 3-edge 3-rigidity of each doubly-braced triangulation obtained from the icosahedron turned out to be very slow even with exploiting the symmetry properties of the icosahedron, I did not go any further with investigating the corresponding statement for base plane triangulations bigger than the graph of the icosahedron.

After that, I implemented the following randomized procedure, and verified the 3-edge 3-rigidity of the graphs encountered along the way.

1. Insert two chosen bracing edges into the graph of the icosahedron.
2. Successively generate graphs up to a given number of vertices, starting with the doubly-braced icosahedron graph, via randomly choosing an applicable $\mathcal{A}_6, \mathcal{B}_6$ or $\mathcal{C}_6$ operation and then performing it.

This procedure, together with checking the 3-edge 3-rigidity of the generated graphs, was then repeated for multiple other configurations of the initial two bracing edges. As expected, the program for this simplified task run considerably faster compared to greedily checking the 3-edge 3-rigidity of each possible doubly-braced plane triangulation on a given base plane triangulation, and I was able to show the 3-edge 3-rigidity of graphs with up to 60 vertices via following the procedure above. Moreover, the computational speed was further accelerated by exploiting the following observation.

7.2. Observation. *Let G be a 3-edge 3-rigid graph. Suppose G' is constructed from G by one of the operations $\mathcal{A}, \mathcal{B}$ or $\mathcal{C}$, and denote the set of vertices in G' that were modified during the construction move by V_0. Then G' is 3-edge 3-rigid if and only if $G' - e$ is redundantly 3-rigid for every edge e that has at least one endpoint in V_0.*

The reason why this observation holds is that if two edges e_1 and e_2 are omitted from G' whose endpoints are not contained in V_0, then the resulting graph can be obtained from the 3-rigid graph $G - e_1 - e_2$ by applying one of the operations $\mathcal{A}, \mathcal{B}$ or $\mathcal{C}$, and these operations are just successive applications of vertex splitting moves which are known to preserve 3-rigidity.

Matching our expectations, the program could not find practically any applicable inductive operation other than $\mathcal{A}_6$ after a critical number of vertices (around 20) has been reached.

7.2. Conjecture 6.6

Now, we are turning our focus to investigating Conjecture 6.6. Analogously to Conjecture 6.2, Conjecture 6.6 was verified for all doubly-braced plane triangulations obtained from the graph of the icosahedron by adjoining two bracing edges.

7.3. Lemma. *If the doubly-braced plane triangulation G is obtained from the graph of the icosahedron by inserting two bracing edges, then G is redundantly globally 3-rigid.*

Since checking the redundant global rigidity of a braced plane triangulation only involves the deletion of each edge as opposed to every pair of edges as in the case of the investigation of Conjecture 6.2, the program checking the statement of Lemma 7.3 was considerably faster than the program verifying Lemma 7.1. This enabled the verification of the analogous version of Lemma 7.3 for a few bigger base plane triangulations on up to 20 vertices, each obtained from the icosahedron graph by applying a short sequence of the operations $\mathcal{A}_6, \mathcal{B}_6$ and $\mathcal{C}_6$.

After that, again similarly to the investigation of Conjecture 6.2, the following (randomized) procedure was implemented, and the redundant global 3-rigidity of the encountered graphs was verified.

1. Insert two chosen bracing edges into the graph of the icosahedron.
2. Successively generate graphs up to a given number of vertices, starting with the doubly-braced icosahedron graph, via randomly applying an applicable $\mathcal{A}_6, \mathcal{B}_6$ or $\mathcal{C}_6$ operation.

This procedure, together with verifying the redundant global 3-rigidity of the generated graphs, was then repeated for multiple other initial configurations of the two bracing edges.

Note that the corresponding version of Observation 7.2 cannot be made use of in this case, since the operations $\mathcal{A}_6, \mathcal{B}_6$ and $\mathcal{C}_6$ are not yet known to preserve global rigidity. Accordingly, even though verifying redundant global rigidity is faster than testing 3-edge 3-rigidity, the program was only able to investigate graphs up to 60 vertices.

However, since all evidence to this moment suggests that Conjecture 6.12 might indeed be true, I also looked at what happens if we assume Conjecture 6.12 holds. Then, the previous procedure can be accelerated by applying the following observation similar to Observation 7.2.

7.4. Observation. *Suppose Conjecture 6.12 holds, and let G be a redundantly globally 3-rigid graph. Suppose G' is constructed from G by one of the operations $\mathcal{A}_6, \mathcal{B}_6$ or $\mathcal{C}_6$, and denote the set of vertices in G' that were modified during the construction move by V_0. Then G' is redundantly globally 3-rigid if and only if $G' - e$ is globally 3-rigid for every edge e that has at least one endpoint in V_0.*

Implementing the previous procedure with this simplification resulted in a much faster program, enabling the size of the investigated doubly-braced plane triangulations to grow up to around 100 vertices. However, this boost in the computational speed comes with the trade-off that the obtained results can only be approved if Conjecture 6.12 holds.

Note that if Conjecture 6.12 was proven, we would obtain a strong tool that could possibly be utilized to prove Conjectures 6.7 and 6.8 based on Lemma 7.3. Moreover, the experimentation suggests that starting from a critical number of vertices n_0 (where n_0 was found to be around 20), applying only the operation $\mathcal{A}_6$ is sufficient to generate an infinite sequence of 5-connected doubly-braced plane triangulations with maximum degree 6, covering every number of vertices $n \geq n_0$. Therefore, Conjecture 6.7 might already follow from Conjecture 6.13 alone.

With this in mind, I also checked Conjecture 6.13 separately in some special cases by performing the following procedure, and verifying the global 3-rigidity of the graphs encountered along the way.

1. Insert two chosen bracing edges into the graph of the icosahedron, and remove an edge from the obtained graph that is different from the two bracing edges.

2. Successively generate graphs up to a given number of vertices, starting with this modified icosahedron graph, via randomly performing an applicable $\mathcal{A}_6$ move.

This procedure was then repeated for multiple other initial configurations of the two inserted bracing edges and the omitted third edge. During the above process, graphs on up to 200 vertices were generated, and every encountered graph was found to be globally 3-rigid. This gives us a further indication that Conjecture 6.13 might indeed be correct.

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NYILATKOZAT

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ELTE Természettudományi Kar, szak: Matematika BSc

NEPTUN azonosító: TPNPN3

Szakdolgozat címe:
Rigidity of modified polyhedral graphs

A **szakdolgozat** szerzőjeként fegyelmi felelősségem tudatában kijelentem, hogy a dolgozatom önálló szellemi alkotásom, abban a hivatkozások és idézések standard szabályait következetesen alkalmaztam, mások által írt részeket a megfelelő idézés nélkül nem használtam fel.

Budapest, 2021.05.24.

Daróczi Sándor

a hallgató aláírása