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Groups Acting on Rooted Binary Trees

Thesis

Koppány István Encz

BSc in Mathematics

Supervisor:

Miklós Abért, senior research fellow
Alfréd Rényi Institute of Mathematics

Eötvös Loránd University, Budapest, Hungary
Department of Algebra and Number Theory
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Contents

1 Introduction

In this paper, we aim to examine the automorphism group of the d-ary rooted tree. We mostly confine our observations to the rooted binary tree, bearing in mind that most of the results, when not highlighted otherwise, can be extended to an arbitrary natural number d with insignificant modifications.

We begin our thesis by giving some basic definitions and properties regarding the d-ary tree in Chapter 2. We introduce a useful definition: the profile of an automorphism, which will significantly reduce the complexity for some proofs in the case when $d = 2$.

After listing a few properties of $\Gamma(n)$ as an abstract group, such as determining the order of $\Gamma(n)$; we move to observe it from another perspective, and reveal its nature as the group acting on the binary tree. We establish the connection between orbits at adjacent levels; and using this result we point out every element in $\Gamma(n)$ with maximal order.

In the final part of the chapter, the topic of orbit trees will be discussed, and we establish a method for extracting the order of an automorphism from its orbit tree.

In Chapter 3., the properties of Γ as a topological and measure space (given by the infinite product topology and infinite product measure) will be addressed.

Combining the results from the previous chapters, the thesis concludes in the exploration of Γ from a probabilistic point of view. We notice that the orbit tree of a random element is a Galton-Watson tree, a well-known structure in probability theory. From this observation, we deduce the most significant theorem in this paper, regarding the order of random elements in $\Gamma(n)$.

The closing part of the paper mentions the effect of finitely generated free groups on ∂T, and with the aim of setting path for possible future research, lists a collection of other groups that this result might be extended to.

Throughout the entire paper, we follow in the footsteps of Miklós Abért and Bálint Virág. Most of the results (including the majority of Section 4) can explicitly be found in [1].

2 The automorphism group of the rooted d-ary tree

2.1 Definitions

The exact definition of the rooted d-ary tree is the following:

Definition 2.1. The **rooted d-ary tree** is the unique infinite rooted tree, where every vertex has d descendants.

It is easy to check that up to isomorphism, there exists only one graph with such properties, so it makes sense to talk about the unique rooted d-ary tree. We will often use the following notations:

Notation. T_d denotes the d-ary tree for some $d \in \mathbb{N}$. We call the n-th level of T_d the set of vertices with distance n from the root. The vertices of the first n levels of the d-ary tree, and the edges between nodes from the first $n-1$ levels of T_d together form the **rooted d-ary tree with n levels**, which will be denoted as $T_d(n)$. The root will often be referred to as the 0-th level of T_d.

As we mentioned before, the majority of the theorems will pertain to the case when d is 2, i.e. the so-called infinite rooted binary tree. The reason behind this is that the proofs for the binary tree are much easier to demonstrate, and they can be rephrased without difficulty to the general case as well. To simplify notations, $T(n)$ and T will denote the rooted binary tree with n levels; and the infinite rooted binary tree, respectively. As we will mostly examine the rooted binary tree, we slightly modify the definition of T in order to have a better understanding of the geometry of the tree.

Definition 2.2. The **rooted binary tree** is the unique rooted tree where every vertex has exactly two descendants. To each node v, we can assign 0 or 1, depending on whether it is the left or right descendant of its ancestor. There is a unique path $r = v_0, v_1, \ldots, v_k = v$ from the root to v, so v will be identified with the unique $0-1$ series of length k, whose i-th digit is 0 if and only if v_i is the left descendant of v_{i-1} $(i = 1, \ldots, k)$.

Definition 2.3. ∂T is the set of the infinite rays (infinite paths starting at level 0) of T.

Definition 2.4. The automorphism group of $T_d(n)$ is $\Gamma_d(n)$. The automorphism group of T_d is Γ_d.

In accordance with our previous remark, $\Gamma(n)$ will refer to $\Gamma_2(n)$; and similarly, Γ is the automorphism group of T.

It is trivial that an automorphism of $T(n)$ acts on $T(k)$ as well for every $k < n$, since the distance from the root of an arbitrary vertex is invariant for every automorphism of $T(n)$. This permits the following:

Notation. For some $g \in \Gamma(n)$ and $k \leq n$, g_k denotes the automorphism we get by constraining g to the first k level of $T(n)$.

2.2 Basic properties of $\Gamma(n)$

In the following section, we will take a closer look at the properties of $\Gamma(n)$. First, we introduce an extremely useful tool for our investigations. We say that an automorphism $g \in \Gamma(n)$ **has a switch/flip** at node v, if v_1 and v_2 are the descendants of v, and the last digit of the unique $0-1$ series that encodes the vertices is different for v_i and v_i^g $(i = 1, 2)$.

Definition 2.5. The **profile** of $g \in \Gamma(n)$ is constructed as follows: to each vertex of $T(n)$, we somehow assign a value or a color, depending on whether g switches at v or not (the easiest ways of doing this are writing 1 on v if g switches at v, and 0 otherwise; or coloring the switched nodes with red and the others blue). The coloring we get from this method is the profile of g.

Naturally, every automorphism determines a profile: we simply check for every vertex whether the automorphism switches under it; and assign the color accordingly. Similarly, every profile derives from at least one automorphism. However, it is not trivial that every profile defines a unique automorphism. For this, we first have to calculate the order of $\Gamma(n)$.

Proposition 2.6. *The order of* $\Gamma(n)$ *is* $|\Gamma(n)| = 2^{2^n-1}$.

Proof. We use induction on n. For cases $n = 0$ and 1, the statement is trivial. Consider an automorphism $g \in \Gamma(n)$. Then, as discussed before, g_{n-1} is an automorphism of T_{n-1}, so the effect of g on the first $n-1$ levels is determined by g_{n-1}. There are 2^n vertices at level n, and they can be distributed into 2^{n-1} pairs of vertices, based on whether they have a common ancestor at level $n-1$. Since an automorphism maps neighbouring vertices into neighbouring vertices, a node at level n can only be transferred by an automorphism either to itself, or the other node with which they share a common ancestor. This entails that g can be determined from g_{n-1} by the following method: for every pair $\{v_{2k+1}, v_{2k+2}\}$ at level n, we find the the vertex at level $n-1$ where the common ancestor v of the pair is mapped to; then we know that the pair is mapped to the descendent pair of $w = v^{g_n}$, so we only have to decide if $v_{2k+1}^g = v_{2m+1}$ and $v_{2k+2}^g = v_{2m+2}$; or $v_{2k+1}^g = v_{2m+2}$ and $v_{2k+2}^g = v_{2m+1}$, where v_{2m+1} and v_{2m+2} are the descendent pair of w. Then g can be found by repeating the latter algorithm for every pair at level n. Moreover, for every automorphism of T_{n-1}, the extension we get by assigning the verteces in the related pairs is an automorphism of T_n. Since there are 2^{n-1} such pairs at level n, and the vertices in pairs are assigned independently from each other; there are $2^{2^{n-1}}$ ways to extend an automorphism from level $n-1$ to level n. Additionally, every such extension defines a unique $g \in \Gamma(n)$: if $g = h$, then $g_{n-1} = h_{n-1}$, and it also holds that the pairs at level n are assigned identically in g and h, so g and h were the result of the same distribution, and deriving from the same g_{n-1}.

By the induction hypothesis, $|\Gamma(n-1)| = 2^{2^{n-1}-1}$. As we saw before, each $g \in \Gamma(n-1)$ expands to $2^{2^{n-1}}$ different automorphism of $T(n)$, and each $h \in \Gamma(n)$ can be constructed in that way. As a result, we may conclude that

$$|\Gamma(n)| = |\Gamma(n-1)| \cdot 2^{2^{n-1}} = 2^{2^{n-1}-1} \cdot 2^{2^{n-1}} = 2^{2 \cdot 2^{n-1}-1} = 2^{2^n-1}.$$

□

Proposition 2.7. $\Gamma(n)$ *is isomorphic with the* 2*-Slyow subgroup of* S_{2^n}.

Proof. First, we determine the order of the 2-Sylow subgroup of S_{2^n}. By Sylow's theorem, the order we intend to calculate equals to the exponent of 2 in $2^n!$. By applying Legendre's theorem, we get that the exponent is $\sum_{k=1}^{n} \lfloor \frac{2^n}{2^k} \rfloor = \sum_{k=0}^{n-1} 2^k = 2^n - 1$, so the order of the 2-Sylow subgroup of S_{2^n} is 2^{2^n-1}. By 2.6, $\Gamma(n)$ and the 2-Sylow subgroups have the same order; now it only remains to give an example of a set X of size 2^n, whose symmetry group $\Gamma(n)$ can be embedded to. If we choose X to be the n-th level of $T(n)$, we get a natural representation of $\Gamma(n)$: each $g \in \Gamma$ permutates the nodes at level n; and each automorphism gives a unique permutation. Indeed, note that a vertex v at level n can be identified with the unique ray connecting v and the root; so g and $h \in \Gamma(n)$ have the same effect on the n-th level if and only if they have the same effect on every ray starting from the root and terminating at level n; but the latter is equivalent with g and h being the same automorphism. □

Now we are ready to prove that each automorphism can be identified with its profile.

Proposition 2.8. *The function that assigns a profile to the automorphism it represents is a bijection from $\Gamma(n)$ to the set of profiles on $T(n)$.*

Proof. This is a simple result of 2.6. As there are $\sum_{k=0}^{n-1} 2^k = 2^n - 1$ vertices in $T(n)$ that can be switched, $T(n)$ has 2^{2^n-1} distinctive profiles. We emphasised earlier that the above mapping is surjective; and from 2.6 it follows that it is a surjective function between two finite sets of the same size; ensuring that it is in fact a bijective function. □

2.3 Order of some elements of $\Gamma(n)$

Since $\Gamma(n)$ is the 2-Sylow subgroup of S_{2^n}, and it consists of cycles of 2-power length, the maximal order an element can have is 2^n. In this subsection, we endeavour to classify the elements of $\Gamma(n)$ with order 2^n.

Remark. $g \in \Gamma(n)$ has order 2^n if and only if it acts as a full cycle on the n-th level of $T(n)$.

Proposition 2.9. *Consider the automorphism $g \in \Gamma(n)$ whose profile is comprised of switches placed along a single ray. Then $o(g) = 2^n$.*

Proof. We use induction on n. In case $n = 1$, $\Gamma(1) \simeq Z_2$, and the element that swaps the two nodes at level 1 corresponds to the generator of Z_2. Suppose the statement holds for every $k < n$. Let v be the vertex where the switch at level $n-1$ is located. Then, by the induction hypothesis, g_{n-1} has order 2^{n-1}, and according to the previous remark, it acts as a full cycle on level $n-1$; meaning that first power of g_{n-1} fixing v is $g_{n-1}^{2^{n-1}}$. But $g^{2^{n-1}}$ does not fix every vertex: since g has a switch under node v, the two descendants of v are swapped by $g^{2^{n-1}}$. And because $g^{2^{n-1}}$ is the first power of g fixing v, the two descendants of v have not been changed by the previous powers of g, since at level n two nodes can only be changed if they are located under v at a given step, as level $n-1$ contains only one switch (at node v). As a result, $g^{2^{n-1}} \neq 1$. Reiterating the explication, we get that the smallest power of g fixing every vertex cannot be less than g^{2^n}, since after $g^{2^{n-1}}$, it is the first such power that fixes v. But in $\Gamma(n)$, the order of an element cannot exceed 2^n, proving that $o(g) = 2^n$, as claimed. □

Now we turn to examine the connection between the effect of $g \in \Gamma(n)$ at different levels. Let O_k be the orbit of the effect g has at some level $k \leq n$; i.e. for some vertex v, O_k consists of all the vertices that v is mapped to by the powers of g. Then, as g permutes the vertices of O_k, it also permutes the descendants of these vertices; meaning that these descendants also form the union of some orbits of g at level $k+1$. The following statement addresses the question how these orbits are related to O_k.

Proposition 2.10. *Suppose that O_k is the orbit of the effect g has at some level $k \leq n$. Then the descendants of O_k form two distant orbits of g if there is an even number of switches placed at the vertices of O_k; and they form only one orbit otherwise.*

Proof. Let v be an arbitrary descendant vertex of a node w in O_k. Then, since g is transitive on O_k, the nodes w^{g^m} cover every vertex of O_k, where $m = 0, 1, \ldots, o(g)$.

As a result, the vertices v^{g^m} cover at least $|O_k|$ vertices of the descendants of O_k, since they cover at least one descendant of every vertex from O_k. But the descendants of O_k form a set of size $2 \cdot |O_k|$, so bearing in mind that the orbit of v was arbitrary, we can conclude that the descendants of O_k either form two separate orbits with size $|O_k|$, or one single orbit with size $2 \cdot |O_k|$.

Notice that the latter case holds if and only if the first power of g that fixes w does not fix v. Let us make the assumption that v is a left-vertex, meaning it has an odd index: $v = v_{2l+1}$ for some $l \in \mathbb{N}$. Then v is not fixed by the above power of g only if this power changes the parity of the index of v at an odd number of times (an even number of changes would return v to the descendant of w with the same index parity as v_{2l+1}; which is v_{2l+1}); or equivalently, the parent nodes belonging to the orbit of v contain an odd number of switches. But these parent nodes cover the entire O_k, so the case when O_k does not split up to two different orbits equals to the case when O_k contains an odd number of switches. □

This proposition enables us to classify the elements of $\Gamma(n)$ with order 2^n. An element g has order 2^n if and only if it acts as a full cycle on the n-th level, which can be rephrased as g has only one orbit at level n. But this is equivalent with g having one orbit at every level of $T(n)$, and according to the previous proposition, this holds only if g has an odd number of switches for every level at the first $n-1$ level of $T(n)$. This leads to the following proposition:

Proposition 2.11. *An automorphism $g \in \Gamma(n)$ has order 2^n if and only if the profile of g contains an odd number of switches at every level.*

Remark. This proposition also proves **Proposition 2.9**, since a profile that has switches placed at a single ray clearly has the property that in every level there is an odd number of switches.

It gives an easy way to calculate the number of automorphisms with order 2^n. Because every automorphism can be identified with its profile, we only need to count the profiles that have an odd number of switches in every level. At level $k \leq n-1$, the number of nodes is 2^k, and we have to distribute them into two groups in a way that the group containing the switches has an odd order. It is well known that such a distribution can be arranged 2^{2^k-1} different ways. Since we build up an automorphism by placing the switches at different levels independently, the number of profiles pertaining to automorphisms of order 2^n equals to

$$\prod_{i=0}^{n-1} 2^{2^i-1} = 2^{\sum_{i=0}^{n-1}(2^i-1)} = 2^{2^n-(n+1)}$$

2.4 The center and the solvable length of $\Gamma(n)$

Some proofs may benefit from the fact that Γ_n can either be defined as the automorphism group of $T_2(n)$, or considered as the iterated wreath product of the cyclic group Z_2.

Definition 2.12. The **wreath product** $G \wr Z_2$ of the groups G and Z_2 is the semidirect product $(G \times G) \underset{\varphi}{\rtimes} Z_2$, where φ acts on $G \times G$ via the transposition of the coordinates: if $Z_2 = \langle a \rangle$, then $a^\varphi : (g_1, g_2) \mapsto (g_2, g_1)$.

The n-th **iterated wreath product** of Z_2 is $Z_2 \wr Z_2 \wr \ldots \wr Z_2$, where the $\wr$ operation is repeated n times.

Proposition 2.13. *$\Gamma(n)$ is isomorphic to the n-th iterated wreath product of Z_2.*

Proof. We use proof by induction on the number of levels. First we show that $\Gamma(n) \simeq (\Gamma(n-1) \times \Gamma(n-1)) \underset{\varphi}{\rtimes} Z_2$. The statement will follow from the fact that $\Gamma(1) \simeq Z_2$.

Consider an arbitrary automorphism in $\Gamma(n)$, and its profile. If the profile does not contain the switch in the 0-th level, then the profile can be separated to two independent automorphisms acting on the two T_{n-1} subtrees of the first vertex of T_n. If the profile contains the switch at the 0-th level, then the two subtrees of the first vertex will be interchanged, resulting in that profile pertaining to the left subtree will act on the right subtree, and the profile of the right subtree will act on the left subtree of the first vertex. If we encode the two automorphism of the two T_{n-1} subtrees of the first vertex by the pair (g_1, g_2), then the automorphism acting on the whole tree obtained from g_1 and g_2 will be the pair (g_1, g_2) if the first switch is not inserted, and (g_2, g_1) if it is included. It is easy to check that every automorphism of T_n can be encoded in that way, and every pair (g_1, g_2) (or its transpose (g_2, g_1)) represents an automorphism of T_n. Moreover, if we denote by φ the map from Z_2 that interchanges the pair (g_1, g_2) if and only if the original automorphism g contains the first switch; we get a homomorphism: the product of two automorphism of $\Gamma(n)$ contains the first switch if and only if exactly one of them does. It is also obvious that the transposition of the coordinates of the pair (g_1, g_2) is an automorphism of $\Gamma(n-1) \times \Gamma(n-1)$.

We have that $\Gamma(n) \simeq (\Gamma(n-1)\times\Gamma(n-1)) \rtimes_{\varphi} Z_2$, where φ is a homomorphism from Z_2 to the automorphism group of $\Gamma(n-1) \times \Gamma(n-1)$ acting via the transposition of the coordinates. Thus, by the above definition and the induction hypothesis, our proof is complete. □

The abstract construction of $\Gamma(n)$ in 2.13 enables us to prove the upcoming theorems:

Theorem 2.14. *The solvable length of* $\Gamma(n)$ *is* $d(\Gamma(n)) = n$.

Proof. By 2.13, it is sufficient to show that for every group G, $d(G \wr Z_2) = d(G)+1$. Indeed, $d(\Gamma(1)) = d(Z_2) = 1$, and by induction we have that

$$d(\Gamma(n)) = d(\Gamma(n-1) \wr Z_2) = d(\Gamma(n-1)) + 1 = (n-1) + 1 = n.$$

□

Proposition 2.15. *For any group* G, $d((G \times G) \rtimes_{\varphi} Z_2) = d(G) + 1$.

Proof. Let $H = (G \times G) \rtimes_{\varphi} Z_2$, and $Z_2 = \langle a \rangle$. We will prove that $d(H') = d(G)$. This will be sufficient, as $d(H) = d(H')+1$. $H/G\times G \simeq Z_2$, which is Abelian, and H' is the inclusionwise smallest subgroup with this property; hence $H' \leq G \times G$.

It is easy to check that $H' \ni \big[(a,(1,1)),(1,(1,g))\big] = (1,(g,g^{-1}))\ \forall g \in G$, and by the bijection $(1,(g,g^{-1})) \leftrightarrow (g,g^{-1})$, we can embed G into H': if we denote by π_1 the projection from $G\times G$ to its first coordinate, then the we get the surjection $\pi_1|_{H'} : H' \twoheadrightarrow G$ by the upper bijection. This means that $d(G) \leq d(H')$, since the solvable length is monotonic. Combined with the previous relation $H' \leq G \times G$, we get that $d(G) \leq d(H') \leq d(G \times G) = d(G)$, resulting in $d(H) = d(H') + 1 = d(G) + 1$, as claimed. □

In the following theorem, we determine the center of $\Gamma(n)$ by presenting two different approaches, one of which will rely on 2.13, whereas the other one uses only the definition of $\Gamma(n)$. The first proof uses only the properties of the wreath product, and completely ignores the representation on the binary tree. In spite of requiring more calculations, the second proof has the advantage of showing the profile of the center, as opposed to the previous one.

Theorem 2.16. $Z(\Gamma(n)) \simeq Z_2$.

Proof. For the first proof, we will show a more general statement: $Z(G \wr Z_2) \simeq Z(G)$, where G is an arbitrary group. From this, the theorem follows by induction, since $Z(\Gamma(n)) \simeq Z(\Gamma(n-1) \wr Z_2) \simeq Z(\Gamma(n-1)) \simeq \ldots \simeq Z(\Gamma(1)) \simeq Z(Z_2) \simeq Z_2$.

Let $Z_2 = \langle a \rangle$, $g_1, g_2 \in G$, and make the assumption that $(a,(g_1,g_2)) \in Z(G \wr Z_2)$. Then $(a,(1,1)) \cdot (a,(g_1,g_2)) = (1,(g_1,g_2))$, and

$$(a,(g_1,g_2)) \cdot (a,(1,1)) = = (1,(g_1,g_2)^{\varphi(a)}) = (1,(g_2,g_1)),$$

so $g_1 = g_2$ must hold.

If we suppose that $(1,(g_1,g_2)) \in Z(G \wr Z_2)$, then similarly

$$(a,(g_2,g_1)) = (a,(g_1,g_2)^{\varphi(a)}) = (1,(g_1,g_2)) \cdot (a,(1,1)) =$$

$$= (a,(1,1)) \cdot (1,(g_1,g_2)) = (a,(g_1,g_2));$$

entailing that $g_1 = g_2$.

Now we know that the elements of the center either have the form $(a,(g,g))$ or $(1,(g,g))$. Let

$$h \neq g, (a,(g,g)) \cdot (a,(h,h)) = (1,(gh,gh)) \text{ and } (a,(h,h)) \cdot (a,(g,g)) = (1,(hg,hg)),$$

meaning that g and h must commute for every element h in G; equivalently, $g \in Z(G)$.

Similarly,

$$(1,(gh,gh)) = (1,(g,g)) \cdot (1,(h,h)) = (1,(h,h)) \cdot (1,(g,g)) = (1,(hg,hg)),$$

so $g \in Z(G)$ is true in this case as well.

Now make the assumption that $(a,(g,g)) \in Z(G \wr Z_2)$, where $g \in Z(G)$, and choose $g_1, g_2 \in G$ such that $g_1 \neq g_2$. Now $(a,(g,g)) \cdot (a,(g_1,g_2)) = (1,(gg_1,gg_2))$, whereas

$$(a(g_1,g_2)) \cdot (a,(g,g)) = (1,(g_2g,g_1g)) = (1,(gg_2,gg_1)).$$

But by the choice of g_1 and g_2, $gg_2 \neq gg_1$, so the two elements are not equal. Consequently, $(a,(g,g))$ cannot be in the center of $G \wr Z_2$.

It only remains to check whether $(1,(g,g)) \in Z(G \wr Z_2)$ is true, where $g \in Z(G)$. But this is in fact true, since for any $b \in Z_2$ and $g_1, g_2 \in G$,

$$(b,(g_1,g_2)) \cdot (1,(g,g)) = (b,(g_1g,g_2g)) = (b,(gg_1,gg_2)) = (1,(g,g)) \cdot (b,(g_1,g_2)).$$

To sum up, the map $\psi : Z(G) \to Z(G \wr Z_2)$, $g \mapsto (1,(g,g))$ is a bijection. Furthermore, it is an isomorphism, as $(1,(g,g)) \cdot (1,(h,h)) = (1,(gh,gh))$. □

Theorem 2.17. *$Z(\Gamma(n)) \simeq Z_2$, where the generator of Z_2 has the profile composed of all the switches on the $n-1$-th level.*

Proof. For the case $n = 1$, the statement is trivial. Suppose that the claim is proven for every $k < n$. Our key observation is that if $g \in Z(\Gamma(n))$, then $g_{n-1} \in Z(\Gamma(n-1))$. Indeed, this means that gh has the same effect on T_n as hg has, for every $h \in \Gamma(n)$; but this implies that $g_{n-1}h_{n-1}$ and $h_{n-1}g_{n-1}$ has the same effect on T_{n-1} embedded in T_n.

Suppose that $g \in \Gamma(n)$ is in the center; then by the above observation and the induction hypothesis, the profile of g_{n-1} consists of the switches at level $n-2$, or $g_{n-1} = id_{T_{n-1}}$. We will examine the following cases:

1. $g_{n-1} = id_{T_{n-1}}$ and g does not contain switches from the $n-1$-th level $\Longrightarrow$ $g = id_{T_n}$, and so $g \in Z(\Gamma(n))$.

2. $g_{n-1} = id_{T_{n-1}}$ and g contains a switch from level $n-1$, but not every switch. Suppose that g contains the switch located above the pair of verteces (v_{2k-1}, v_{2k}), but does not contain the switch on top of (v_{2l-1}, v_{2l}). Then let h be an automorphism that takes the parent of (v_{2k-1}, v_{2k}) to the parent of (v_{2l-1}, v_{2l}), and does not include a switch at level $n-1$. It is easy to construct

such h, i.g. if h consists of the only switch located at the intersection of the rays pointing to (v_{2k-1}, v_2) and (v_{2l-1}, v_{2l}), then h has the desired property. In this case, $v_{2k-1}^{hg} = v_{2l-1}^{g} = v_{2l-1}$, and $v_{2k-1}^{gh} = v_{2o}$ for some o, as g changes the parity of the index of v_{2k-1}, but h does not change it back since it does not have a switch at level $n-1$. Since gh and hg takes v_{2k-1} to vertices of index with different parity, they cannot commute in $\Gamma(n)$. As a result, g cannot be in $Z(\Gamma(n))$.

3. $g_{n-1} = id_{T_{n-1}}$ and g does not contain switches from level $n-1$. Consider the permutation of the n-th level induced by g. This permutation only changes the pairs (v_{2k-1}, v_{2k}), and so $g = (12)(34)\ldots(2^n - 1\ 2^n)$. Then every conjugate of g in S_{2^n} has the form $(i_1 i_2)(i_3 i_4)\ldots(i_{2^n-1} i_{2^n})$, and because every conjugate of g in $\Gamma(n)$ is a conjugate in S_{2^n} as well, the conjugate class of g in $\Gamma(n)$ consists of some permutations that can be written as the product of 2^{n-1} disjoint transpositions. But every automorphism of T_n can only contain a transposition with the form $(v_{2k-1}\ v_{2k})$ for some k; so the disjoint product of 2^{n-1} such transpositions must be another permutation form of g. The conjugate class of g in $\Gamma(n)$ comprises of solely g; thus g is in the center of $\Gamma(n)$.

4. g_{n-1} includes the switches from level $n-2$, but g does not include any switch from level $n-1$. Let h be an automorphism of $\Gamma(n)$ that only includes the first switch of level $n-1$. Then $v_1^{gh} = v_3^h = v_3$, but $v_1^{hg} = v_2^g = v_4$.

5. g_{n-1} includes the switches from level $n-2$, g includes switches from level $n-1$, but not all of them. Let us consider the case when g contains the switch above (v_{2k-1}, v_{2k}), and does not contain the one above (v_{2l-1}, v_{2l}). Then let $h \in \Gamma(n)$ such that h maps the parent of (v_{2k-1}, v_{2k}) to the parent of (v_{2l-1}, v_{2l}), and does not include a switch from level $n-1$. Now $v_{2k-1}^{gh} = v_{2m}$ and $v_{2k-1}^{hg} = v_{2o-1}$ for some m, o. As a result, $gh \neq hg$ and $g \notin Z(\Gamma(n))$.

6. g includes every switch from levels $n-2$ and $n-1$. There is only one g satisfying these conditions. However, we have only found two elements of

$Z(\Gamma(n))$ so far, in case 1. and case 3. But since $\Gamma(n)$ is a 2-group, so is $Z(\Gamma(n))$; thus it cannot contain g in addition to the two aforementioned ones.

This case separation completes the proof of the theorem. □

2.5 The orbit tree of an automorphism

In this section, we will examine the orbit tree of an automorphism, that can be defined in a more general case as well:

Definition 2.18. (The orbit tree of an automorphism) Let S be an arbitrary tree, and $g \in \mathrm{Aut}(S)$ an automorphism of S. Consider the graph that has a vertex for every orbit of g, and the vertices u and v are connected if and only if there are two vertices u' and v' in the original tree such that u' is in the orbit belonging to u, v' is in the orbit belonging to v, and u and v are connected.

Notation. S_g will denote the orbit tree of S with respect to g. $O_g(v)$ will denote the orbit of the vertex v

Remark. If there are such vertices u' and v', then every u'' in the orbit of u has a neighbour v'' in the orbit of v: since u'' is in the orbit of u, there exists an $n \in \mathbb{N}$ that has $(u')^{g^n} = u''$. But then, as g^n is an automorphism, u'' and $(v')^{g^n}$ are neighbours in S, so v'' can be chosen as $(v')^{g^n}$.

This enables us to prove that the orbit tree of S with respect to g is in fact a tree, on condition that S has finitely many vertices.

Proposition 2.19. *Let S be a tree with finitely many vertices, and $g \in Aut(S)$ an automorphism. Then S_g is a tree.*

Proof. First we show that S_g is connected. If $O_g(v)$ and $O_g(u)$ are two separate verices in S_g, then so are v and u in S. Since S is connected, there exists a path $v_0 v_1 \dots v_n$ in S where $v_0 = v$ and $v_n = u$. Then $O_g(v) O_g(v_1) \dots O_g(u)$ is a walk in S_g connecting $O_g(v)$ and $O_g(u)$.

Now suppose that S_g has a cycle $O_g(v_1)O_g(v_2)\dots O_g(v_n)O_g(v_1)$. Let us choose u_1 from $O_g(v_1)$. By the above remark, we can choose a neighbour of u_1 in $O_g(v_2)$; let that neighbour be u_2. Then we may choose a vertex u_3 from $O_g(v_3)$ that is a neighbour of u_2. We can repeat this step for every vertex u_i chosen beforehand: we can take a vertex u_{i+1} from $O_g(v_{i+1(\bmod n)})$, such that it is the neighbour of the previous one. By the upper remark, this algorithm can never stop. But since there are only finitely many vertices, there will be an index i such that u_i has already been chosen; but this means that S contained a cycle, namely $u_k u_{k+1} \dots u_i$ where $u_i = u_k$.

As S_g is a graph that is both connected and cycle-free, we may conclude that S_g is a tree. □

In the case when $S = T_n$, we can additionally claim that not only $(T_n)_g$ is a tree, but it also has the property that every vertex $O_g(v)$ has either one or two descendant in $(T_n)_g$. Indeed, by 2.10, every orbit of g at level k can split into two orbits at level $k+1$, or the descendants of the orbit form an entire orbit at level $k+1$.

We may further perceive that there is a strong connection between the order of an automorphism and its orbit tree, that we will elaborate in the followings.

Our first observation is that when we wish to determine the order of an automorphism in $\Gamma(n)$, it is enough to point out the largest cycle in its permutation form: the order of a permutation is the smallest common multiple of the lengths of its cycles; but now $\Gamma(n)$ is a 2-group, meaning that the smallest common multiple of the lengths (along with the length of every cycle) must be a power of 2. Hence, the order of an element equals to the the length of the longest cycle in the permutation form.

Note that the cycles of the permutation form of $g \in \Gamma(n)$ pertain to the orbits of g at level n; so in order to determine the order of g, it suffices to find the longest orbit of g at level n. We have previously established the connection of two orbits at neighbouring levels in 2.10. This result said that an orbit at level k can split into two orbits at level $k+1$ (and the sizes of the two new orbits remain the same), or stays intact (in which case the size of the orbit doubles).

Consider an orbit O of g at level n, and make the assumption that the size of the orbit is 2^l. According to the previous line of thoughts, this can only occur if the following property holds: there is a unique series of orbits containing an orbit from each level of T_n such that each orbit is the descendant of the previous one, and the last orbit is O. If we sum up the orbits in this series that split up in the following level, we must get $n - l$ as the result (or equivalently, the number of orbits that stay intact must be l): the size of the original orbit has to double l times in order to end up as an orbit of size 2^l; and this only occurs if the number of orbits staying intact equals to l (equivalently, the orbits split up $n - l$ times altogether).

This condition can be rephrased to the orbit tree of g: if we take the node representing O in $(T_n)_g$, and the unique ray in $(T_n)_g$ from the first level to the n-th level where the last orbit is O, the ray must contain $n-l$ branches (or equivalently, it must contain l nodes where the ray does not diverge). This leads to the following theorem:

Theorem 2.20. *Let $g \in \Gamma(n)$. The order of g can be determined by the following method: for every vertex v_i of $(T_n)_g$ at level n, consider the unique ray from level 0 ending at the vertex; and let n_i denote the number of nodes in the ray where the ray does not diverge. Then the order of g is 2^m, where $m = \max_i \{n_i\}$.*

Remark. As opposed to the profiles of the automorphisms, the orbit trees do not determine the automorphism they derive from. For instance, an automorphism has order 2^n only if its orbit tree is path of length n; so the path with length n, as an orbit tree, represents every automorphism with order 2^n.

In general, it is hard to give an exact number for the orbit trees that represent automorphisms of an arbitrary order; however, there are some special cases where we can give a recursive formula.

Proposition 2.21. *Let a_n denote the number of orbit trees with depth n belonging to automorphisms with order 2 (up to isomorphism). Then $a_n = \binom{a_{n-1}+2}{2}$.*

Proof. For $n = 1$ the statement is trivial, as there is only one tree of depth n meeting the requirements. Suppose that the statement holds for every $k < n$. The orbit trees of order 2 can be separated into two classes, depending on whether the vertex at the first level has one or two descendants. If the first vertex has only one descendant, then every other vertex must have two descendants, or else the condition for having order 2 would not be satisfied. In this case, we have only one option for an orbit tree of order 2.

If the first vertex has two descendants, then the subtrees of these descendants form orbit trees that both represent automorphisms of order 2, or one of them represents automorphisms of order 2 and the other represents the identity: in each ray there is at most one vertex where the ray does not diverge, and this vertex cannot be the first vertex, as it has two descendants; so each subtree includes at most 1 vertex where the ray splits, but one of them must contain such vertex, or else the entire orbit tree would correspond to the identity. This only leaves two options: both subtrees represent the same orbit tree of order 2 (a_{n-1} cases); or the first subtree represents an orbit tree of order 2 or the identity, and the second represents a different orbit tree with the same property ($a_{n-1}+1$ possibility for the first tree, and a_{n-1} for the second tree).

To summerize,

$$a_n = 1 + a_{n-1} + \frac{(a_{n-1}+1)\cdot a_{n-1}}{2} = \frac{a_{n-1}^2 + 3a_{n-1} + 2}{2} =$$

$$= \frac{(a_{n-1}+1)(a_{n-1}+2)}{2} = \binom{a_{n-1}+2}{2}.$$

□

Giving a recursive formula for the orbit trees with order $n-1$ is much more complicated, we even have to take in consideration the number of orbit trees (c_n) as well.

Proposition 2.22. *Let b_n denote the number of orbit trees with depth n belonging to automorphisms of order $n-1$ (up to isomorphism). Then $b_n = b_{n-1} + c_{n-1}$, where c_k denotes the total number of orbit trees with k levels.*

Proof. We again use the case separation based on the first vertex's descendants.

If the vertex at level 0 has one descendant, then the lower levels of the orbit tree form a rooted tree, where every ray has at most one vertex where the ray diverges, and there is at least one ray with such a vertex. In conclusion, the lower $n-1$ levels form an orbit tree of depth $n-1$ with order $n-2$, so it can be chosen b_{n-1} different ways.

If there are two descendants of the first vertex, then one of the subtrees (the one guaranteeing a ray with no point of divergence) must consist of a single ray; the other one can be an arbitrary orbit tree with $n-1$ levels, leaving us c_{n-1} options to choose the orbit tree from. By adding the two value, we get that $b_n = b_{n-1} + c_{n-1}$, as claimed. □

With a very similar line of thinking, one can give a recursive formula for the total number of orbit trees with n levels: $c_n = \frac{c_{n-1}\cdot(c_{n-1}+3)}{2}$.

Remark. Normally, one of the first things to do while exploring a group would be examining the conjugacy. However, in our case this is a rather hard question, as both of the viable options that come to one's mind fail to give a satisfying answer: neither the profiles, nor the orbit trees determine whether two authomorphisms are conjugate in $\Gamma(n)$.

To be more specific, we state that there are automorphisms g and h such that the profiles of g and h are isomorphic (there exists an automorphism of T_n that maps one of the profiles to the another), but g and h are not conjugate in $\Gamma(n)$. The smallest counterexamples can be found in $\Gamma(3)$: let g be the automorphism with the permutation form $(1625)(3748)$, and h with the permutation form $(16)(25)(37)(48)$. It is easy to check that the profiles of g and h are isomorphic, but they are not conjugate in $\Gamma(3)$, as they are not conjugate in S_8 either.

This example also shows that the conjugacy in $\Gamma(n)$ is not related to the orbit trees of the automorphisms: in spite of having isomorphic orbit trees, g and h are not conjugate.

The upcoming section will demonstrate a few properties of Γ as a topological and measure space.

3 Topology and measure on Γ

In this section, we will mostly rely on the fact that every automorphism of T can be constructed the following way: for $g \in \Gamma$, we know that g_n is the effect g has on the n-th level of T. In other words, every $g \in \Gamma$ is determined by the series $(g_n)_{n\in\mathbb{N}}$: to check the map of a vertex v on level k with respect to g, we only need to find v^{g_k} (or v^{g_m} for some $m > k$, as g_m is extends g_k to the m-th level).

Therefore, to identify g, we only have to give g_n for every $n \in \mathbb{N}$. Consequently, there is a well-defined profile of g (namely, the limit of the profiles of g_n. It is well-defined, since for $m > k$, $(g_m)_k = g_k$); and similarly to the finite case, the profile of g uniquely determines g.

As a result, we may assign a topological structure to Γ: the countable direct product of the discrete topology on $\{0, 1\}$: we somehow enumerate the vertices of T, and $g \in \Gamma$ is represented by the 0/1 series, where the i-th coordinate is 1 if g has a switch under the i-th vertex; and 0 otherwise. As an immediate result of Tychonoff's theorem, we get

Proposition 3.1. Γ *assigned with the product topology is a compact topological space.*

Moreover, we can additionally claim that it is a group topology on Γ.

Notation. Γ provided with the product topology of the discrete topology on $\{0, 1\}$ will be denoted by (Γ, π)

Proposition 3.2. (Γ, π) *is a group topology.*

Proof. It is sufficient to show that the function $h(x, y) = xy^{-1}$ is continuous on $\Gamma \times \Gamma$. Let H be an open set in Γ, and $g \in H$. Then

$$h^{-1}(g) = \{(x, y) : xy^{-1} = g\} = \{(gy, y) : y \in \Gamma\},$$

so

$$h^{-1}(H) = \bigcup_{y\in\Gamma} \bigcup_{g\in H} (gy, y) = \bigcup_{y\in\Gamma} (Hy, y).$$

We know that (Hy, y) is homeomorphic to Hy, as the function

$$f : \Gamma \times \{y\} \mapsto \Gamma : f((x, y)) = x$$

is a homeomorphism; so it is enough to prove that Hy is open in Γ.

As usual, we can make the assumption that H is in the base of Γ; otherwise it can be written in the form $H = \bigcup_{i \in I} H_i$, where all H_i are in the base, and we could turn to proving that $H_i y$ is open for every $i \in I$, since in that case $Hy = \bigcup_{i \in I} H_i y$ would also be open.

An open base in our product topology is composed of all the sets that are restricted to an open subset of $\{0, 1\}$ in finitely many coordinates, and all the other coordinates can be chosen arbitrarily. To rephrase it to our former terminology, an element of the open base $H_{i_1, i_2, \ldots, i_k}(N_1, N_2, \ldots, N_k)$, where $N_1, \ldots, N_k$ are open subsets of $\{0, 1\}$, involves every automorphism subject to the constraints N_j at the vertex v_{i_j}: it involves an automorphism g if and only if the profile of g has the property that $g(i_j) \in N_j$ for $j = 1, 2, \ldots, k$. We may confine to the case where every N_j is a proper subset of $\{0, 1\}$: if $N_j = \{0, 1\}$, we can simply leave out N_j from the conditions, as it does not carry further information about the elements of $H_{i_1, i_2, \ldots, i_k}(N_1, N_2, \ldots, N_k)$. Bearing it in mind, it makes sense to define the profile of $H_{i_1, i_2, \ldots, i_k}(N_1, N_2, \ldots, N_k)$ the following way: for $j = 1, 2, \ldots, k$, the profile of $H_{i_1, i_2, \ldots, i_k}(N_1, N_2, \ldots, N_k)$ has a switch at v_{i_j} if $N_j = \{1\}$, and does not have a switch if $N_j = \{0\}$. According to this definition, g is in $H_{i_1, i_2, \ldots, i_k}(N_1, N_2, \ldots, N_k)$ if and only if the profile of g matches the profile of $H_{i_1, i_2, \ldots, i_k}(N_1, N_2, \ldots, N_k)$ in the coordinates $i_1, i_2, \ldots, i_k$.

One may further perceive that instead of $H_{i_1, i_2, \ldots, i_k}(N_1, N_2, \ldots, N_k)$, it suffices to examine $H_{1, 2, \ldots, 2^m - 1}(A_1, A_2, \ldots, A_{2^m - 1})$, where $2^m - 1 > k$ is arbitrary, since $H_{i_1, i_2, \ldots, i_k}(N_1, N_2, \ldots, N_k)$ can be written as the union of some sets with the upper form.

Now the profile of $H_{1, 2, \ldots, 2^m - 1}(A_1, A_2, \ldots, A_{2^m - 1})$ can be considered as the profile of an automorphism $\alpha_{m-1} \in \Gamma(m - 1)$, and $H_{1, 2, \ldots, 2^m - 1}(A_1, A_2, \ldots, A_{2^m - 1})$ is comprised of every automorphism in Γ that extends α_{m-1} from T_{m-1} to T. It is easy to see that $H_{1, 2, \ldots, 2^m - 1}(A_1, A_2, \ldots, A_{2^m - 1}) \cdot y$ is comprised of the automorphisms in Γ that extends $\alpha_{m-1} \cdot y_{m-1}$ from T_{m-1} to T; thus,

$$H_{1,2,\ldots,2^m-1}(A_1, A_2, \ldots, A_{2^m-1}) \cdot \ y = H_{1,2,\ldots,2^m-1}(M_1, M_2, \ldots, M_{2^m-1}),$$

where $M_1, M_2, \ldots, M_{2^m-1}$ is determined by $\alpha_{m-1} \cdot y_{m-1}$.
As $H_{1,2,\ldots,2^m-1}(M_1, M_2, \ldots, M_{2^m-1})$ is also an open set, our proof is complete. $\square$

Similarly to the product topology, we can provide Γ with the product measure μ, where the σ-algebra of the measurable sets is generated by the open base, and $\mu(H_{i_1,i_2,\ldots,i_k}(N_1, N_2, \ldots, N_k)) = \frac{1}{2^k}$ (on condition that every N_j is a nontrivial subset of $\{0,1\}$).

Proposition 3.3. *The measure μ is Γ-invariant: for every measurable set H and for every $g \in \Gamma$, $\mu(Hg) = \mu(H)$.*

Proof. The proof will be identical with the previous reasoning. First, notice that by definition

$$\mu(H) = \inf \left\{ \sum_{i=1}^{\infty} \mu(H_i) : H \subset \bigcup_{i=1}^{\infty} H_i, \ \forall i \ H_i \text{ is in the open base} \right\};$$

so we only need to prove the case when H is in the open base. Again, the case when $H = H_{i_1,i_2,\ldots,i_k}(N_1, N_2, \ldots, N_k)$ can be traced back to the case when H has the form $H = H_{1,2,\ldots,2^m-1}(A_1, A_2, \ldots, A_{2^m-1})$, since $H_{i_1,i_2,\ldots,i_k}(N_1, N_2, \ldots, N_k)$ can be expressed as the finite disjoint union of sets with the latter form.

Suppose the profile of $H_{1,2,\ldots,2^m-1}(A_1, A_2, \ldots, A_{2^m-1})$ determines the automorphism α_{m-1}, then $H_{1,2,\ldots,2^m-1}(A_1, A_2, \ldots, A_{2^m-1}) \cdot g$ is made up from every automorphism in Γ that extends $\alpha_{m-1} \cdot g_{m-1}$ from level $m-1$ to T, so it can be written in the form $H_{1,2,\ldots,2^m-1}(M_1, M_2, \ldots, M_{2^m-1})$ for some $M_1, M_2, \ldots M_{2^m-1} \subset \{0,1\}$; thus

$$\mu(H_{1,2,\ldots,2^m-1}(A_1, A_2, \ldots, A_{2^m-1}) \cdot g) = \mu(H_{1,2,\ldots,2^m-1}(M_1, M_2, \ldots, M_{2^m-1})) =$$

$$= \frac{1}{2^{2^m-1}} = \mu(H_{1,2,\ldots,2^m-1}(A_1, A_2, \ldots, A_{2^m-1})).$$

$\square$

Remark. As $\mu(\Gamma) = 1$, μ is a probability measure on Γ.

Definition 3.4. μ is the Haar-measure on Γ.

As a matter of fact, the only Γ-invariant probability measure on Γ is the Haar-measure.

Proposition 3.5. *If μ' is a Γ-invariant probability measure on Γ, then $\mu = \mu'$.*

Proof. Consider the set $H_{1,2,\ldots,2^m-1}(A_1, A_2, \ldots, A_{2^m-1})$. It is easy to see that for an arbitrary sequence $B_1, B_2, \ldots B_{2^m-1} \subset \{0, 1\}$, there exists $g \in \Gamma$ such that

$$H_{1,2,\ldots,2^m-1}(A_1, A_2, \ldots, A_{2^m-1}) \cdot g_{m-1} = H_{1,2,\ldots,2^m-1}(B_1, B_2, \ldots, B_{2^m-1}).$$

Furthermore,

$$H_{1,2,\ldots,2^m-1}(A_1, A_2, \ldots, A_{2^m-1}) \cap H_{1,2,\ldots,2^m-1}(B_1, B_2, \ldots, B_{2^m-1}) = \emptyset$$

if $(A_1, \ldots, A_{2^m-1}) \neq (B_1, \ldots B_{2^m-1})$, and the union of $H_{1,2,\ldots,2^m-1}(B_1, B_2, \ldots, B_{2^m-1})$ for every possible sequence $B_1, B_2, \ldots, B_{2^m-1}$ is Γ. From the disjoint additivity of μ', it follows that

$$\mu'(H_{1,2,\ldots,2^m-1}(A_1, A_2, \ldots, A_{2^m-1})) = \frac{1}{2^{2^m-1}} = \mu(H_{1,2,\ldots,2^m-1}(A_1, A_2, \ldots, A_{2^m-1})).$$

This entails that $\mu = \mu'$ constrained to the open base of Γ; and by Carathéodory's unique measure extension theorem, $\mu = \mu'$. □

In the followings, we will consider the so-called 2-ary topology on $\mathbb{Z}$. For this, we first define the **level** of an element $a \in \mathbb{Z}$ by $l(a) = \max\{k \in \mathbb{N} : a \in 2^k\mathbb{Z}\}$. The level of 0 is defined as $l(0) = \infty$.

With the definition of levels, we can further define the **2-ary metric** on $\mathbb{Z}$: for $a, b \in \mathbb{Z}$, let $d(a, b) = \frac{1}{l(a-b)+1}$. The first observation is that d is indeed a metric on $\mathbb{Z}$, and it is shift-wise invariant, i.e. $d(a, b) = d(a + x, b + x)$ for every $x \in \mathbb{Z}$.

Proposition 3.6. *The function d with the above definition is a shift-wise invariant metric on $\mathbb{Z}$.*

Proof. The shift-wise invariant property of d is trivial, as $l(b-a) = l((b+x)-(a+x))$. So is the symmetry of d, since $l(b-a) = l(a-b)$, as the sets $2^k\mathbb{Z}$ are symmetric to 0.

For the proof of the triangle-inequality, let $k, n, m \in \mathbb{Z}$, and make the assumption that $l(n-m) \leq l(m-k)$. Now $n-m$ and $m-k$ are both in $2^{l(n-m)}\mathbb{Z}$, and this infers that $n-k = (n-m)-(m-k)$ is also in $2^{l(n-m)}\mathbb{Z}$, and because of this, $l(n-k) \geq l(n-m)$. But this means that

$$d(n,k) = \frac{1}{l(n-k)+1} \leq \frac{1}{l(n-m)+1} \leq$$

$$\leq \frac{1}{l(n-m)+1} + \frac{1}{l(m-k)+1} = d(n,m) + d(m,k).$$

And finally, if $d(n,k) = 0$, then $l(n-k) = \infty$ must hold, thus $n = k$. □

The topology generated by d on $\mathbb{Z}$ has a strong connection with the product topology on Γ we discussed previously. To explore that relation, our first step will be identifying the natural open base in $(\mathbb{Z}, d)$. Let $a \in \mathbb{Z}$ and $0 < r \in \mathbb{R}$ and $B_a(r) = \{x \in \mathbb{Z} : d(a,x) < r\} = \{x \in \mathbb{Z} : \frac{1}{l(a-x)+1} < r\}$. If $r \geq 1$, then this set is $\mathbb{Z}$, so assume that $r < 1$, and let $n = \lfloor \frac{1}{r} - 1 \rfloor$. Then

$$B_a(r) = \{x \in \mathbb{Z} : \frac{1}{r} - 1 < l(a-x)\} = \{x \in \mathbb{Z} : 2^n | x-a\} = a + 2^n\mathbb{Z}.$$

Proposition 3.7. *$(\mathbb{Z}, d)$ is a topological group.*

Proof. As usual, we will only check the desired properties for elements of the open base. With the above notation, $B_a(r)^{-1} = -B_a(r) = B_a(r)$ and $g + B_a(r) = g + (a + 2^n\mathbb{Z}) = (g+a) + 2^n\mathbb{Z} = B_{a+g}(r)$ are both open, so $(\mathbb{Z}, d)$ is indeed a group topology. □

Now we are ready to examine the resemblance between $\mathbb{Z}$ and Γ regarding their topological properties. Let $\alpha \in \Gamma$ be an odometer, i.e. an automorphism whose profile includes switches along an infinite ray of T. Then for every $n \in \mathbb{N}$,
$o(\alpha_n) = 2^n$, so $o(\alpha) = \infty$, $\langle\alpha\rangle \simeq \mathbb{Z}$.

Let $(\mathbb{Z}, \pi)$ be the topological space, where π is the topology on $\mathbb{Z} = \langle\alpha\rangle$ inherited from (Γ, π) as a subspace topology.

Proposition 3.8. $(\mathbb{Z}, \pi) = (\mathbb{Z}, d)$.

Proof. To simplify notations, the operation on the group $\langle \alpha \rangle$ will be multiplication, instead of addition. Let $H = H_{1,2,\dots,2^{n+1}-1}(A_1, A_2, \dots, A_{2^{n+1}-1})$ be an open subset of Γ according to π. Suppose that there exists $k \in \mathbb{N} : \alpha^k \in H$, or else $H \cap \langle \alpha \rangle = \emptyset$ is trivially open according to d. By the choice of H, $\alpha^l \in H$ if and only if $(\alpha^l)_n = (\alpha^k)_n$. Now

$$(\alpha^{m \cdot 2^n + k})_n = (\alpha^{m \cdot 2^n})_n \cdot (\alpha^k)_n = (\alpha^k)_n;$$

and if $(\alpha^l)_n = (\alpha^k)_n$, then $(\alpha^{l-k})_n = 1$, implying that $2^n | \, l - k$ (as $o(\alpha_n) = 2^n$) and $l = m \cdot 2^n + k$. Consequently, $H \cap \langle \alpha \rangle = \{\alpha^{m \cdot 2^n + k} : m \in \mathbb{Z}\} = \alpha^{k + 2^n \mathbb{Z}} = B_k(r)$ for some $r \in \mathbb{R}$, so $H \cap \langle \alpha \rangle$ is an open set according to d as well. Since every set in the open base of $(\mathbb{Z}, \pi)$ can be written as a finite disjoint union of sets with the form $H_{1,2,\dots,2^m-1}(A_1, A_2, \dots, A_{2^m-1})$, every open set according to π is open according to d as well.

For the other direction, consider the set $\alpha^{k+2^n \cdot \mathbb{Z}}$. Let H be the set

$$H_{1,2,\dots,2^{n+1}-1}(A_1, A_2, \dots, A_{2^{n+1}-1}),$$

whose profile is determined by $(\alpha^k)_n$ (i.e., $A_i = 1$ if and only if α^k contains a switch at v_i). By repeating the upper reasoning, we get that $\alpha^{k+2^n \cdot \mathbb{Z}} = H \cap \langle \alpha \rangle$, proving that every open set in $\langle \alpha \rangle$ according to d is also open according to π. □

4 Random elements of Γ

In this section, we will examine the properties of random elements of Γ according to the Haar-measure defined in 3.4.

4.1 Order of a random element

The following fact is of utter importance, and we will rely on it throughout this entire section:

Proposition 4.1. *Choosing a random element of* Γ *according to the Haar measure equals to independently placing a switch at every vertex of* T *with* $\frac{1}{2}$ *probability.*

Proof. Due to the remark in the beginning of the previous section, a Haar-random $\alpha \in \Gamma$ is uniquely determined by the series $(\alpha_n)_{n\in\mathbb{N}}$, where $(\alpha_{n+1})_n = \alpha_n$, and each α_n is chosen simultaneously from $\Gamma(n)$ according to μ_n, i.e. the constraint of the Haar-measure to T_n. But the above measure is the uniform measure on $\Gamma(n)$: for an arbitrary $g_n \in \Gamma(n)$, let $H_{1,2,\ldots,2^{n-1}-1}(A_1, A_2, \ldots, A_{2^{n-1}-1})$ be the element of the open base whose profile is identical to the profile of g_n. Then $\mu_n(g_n) = \mu(H_{1,2,\ldots,2^{n-1}-11}(A_1, A_2, \ldots, A_{2^{n-1}-1})) = \frac{1}{2^{2^{n-1}-1}}$. Since g_n was arbitrary, μ_n is the uniform measure on $\Gamma(n)$. However, by definition μ_n is also the product measure of $\{0, 1\}$ on the first $n-1$ levels, implying that a μ_n-random automorphism of T_n (which is uniquely given by its profile, as formerly discussed) equals to a profile where every switch is placed independently at the vertices of T_n with probability $\frac{1}{2}$.

Finally; notice that as a result of $(\alpha_{n+1})_n = \alpha_n$, instead of simultaneously choosing α_{n+1} and α_n, we can pick the profile of α_n by independently placing the switches at the first n level, and thereafter move to selecting the switches at level $n + 1$ independently from each other, and also from the first n level. After combining the previous remarks, using induction on n concludes the proof. □

Proposition 4.2. *The orbit tree of a Haar-random element in* Γ *is a Galton-Watson tree, where the specimen of every generation have one or two offspring with probability* $\frac{1}{2}$.

Proof. Let α be a Haar-random element of Γ, and $O_\alpha(v)$ an arbitrary vertex at the n-th level of T_α. The number of descendants v has can be calculated in the following way: we take every vertex of T_n that form the orbit of v, and examine the parity of the number of switches these vertices have. By the previous proposition, every switch is placed (independently from every other vertex) with $\frac{1}{2}$ probability. As a result, the probability that the vertices forming the orbit $O_\alpha(v)$ have an even number of switches altogether equals to $\frac{1}{2}$; and so does the probability of having an odd number of switches. Consequently, by 2.10, $O_\alpha(v)$ has 1 offspring with probability $\frac{1}{2}$, and 2 offspring with probability $\frac{1}{2}$. Since the offspring number of $O_\alpha(v)$ is independent from every other generation, and from members of the same generation as well, T_α is indeed a Galton-Watson tree with the given offspring distribution. □

Corollary 4.3. *Let α_n be a Haar-random element of $\Gamma(n)$. Then $P(\alpha_n$ fixes a vertex on the n-th level of $T_n) \sim \frac{2}{n}$.*

Proof. A vertex v on level n is a fixed point of α_n if and only if $O_{\alpha_n}(v)$ represents an orbit of size 1 at T_{α_n}. The latter condition is equivalent with $O_{\alpha_n}(v)$ being in the subtree T'_{α_n} of T_{α_n} constructed by leaving out every vertex (and its subtree) with only one descendant. But such a subtree T'_{α_n} also corresponds to a branching process as the first n levels of a Galton-Watson tree, where a specimen has either 2 offspring with probability $\frac{1}{2}$, or does not have an offspring at all. The value we are looking for now equals to the probability that this branching process survives until the n-th generation. The expected value of the offspring of a specimen is $\mathbb{E}(N) = \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 0 = 1$ in this branching process; and by the well-known result in the theory of branching processes, it is a critical case, implying that $P(\alpha_n$ fixes a vertex on the n-th level of $T_n)$ is asymptotic to $\frac{2}{n \cdot D^2(N)}$. In our case, $D^2(N) = \mathbb{E}(N^2) - \mathbb{E}^2(N) = 2 - 1 = 1$. □

Let $\alpha \in \Gamma$ be Haar-random. Then

$$\{\alpha \text{ fixes infinitely many vertices}\} = \{\alpha \text{ fixes every vertex of an infinite ray}\} =$$

$$= \bigcap_{n=1}^{\infty} \{\alpha_n \text{ fixes a vertex at level n}\}$$

(the first equation holds because of Kőnig's lemma), and the latter sets are inclusionwise decreasing, so

$$P(\alpha \text{ fixes infinitely many vertices}) =$$

$$= \lim_{n\to\infty} P(\alpha_n \text{ has a fixpoint at the n-th level}) = \lim_{n\to\infty} \frac{2}{n} = 0.$$

By 4.1, α_n is a Haar-random element of $\Gamma(n)$, and we can apply the above proposition. Therefore,

Proposition 4.4. *Let $\alpha \in \Gamma$ be a Haar-random element. Then α fixes a finite number of vertices in T with probability* 1*.*

The following theorem addresses the question of the order of a random element in Γ. Here we sketch a short version of the proof, that is thoroughly described in [1]. For more details and related problems, we refer to [2].

For the sake of this theorem, we return to the general case of $T_p(n)$, the rooted p-ary tree with n levels, where p is a prime number. First we introduce some new definitions.

Definition 4.5. For a prime p, let $p(n)$ be the exponent of p in n, i.e. the highest power of p dividing n.

Definition 4.6. Let H be a group transitive on a set X, and $h \in H$ a uniform random element. $\mu_p(k) = \mu_{H,p}(k)$ denotes the expected number of the orbits $O_h(v)$ which have the property that $p(|O_h(v)|) = k$. $\hat{\mu}_p(z)$ is the generating function for μ_p:

$$\hat{\mu}_p(z) = \sum_{k=0}^{\infty} \mu_p(k) z^k$$

Proposition 4.7. *Let $g_n \in \Gamma_p(n)$ a random element, and let p^{K_n} be its order. Moreover, let*

$$\alpha_p = \min_{\lambda > 0} \frac{\log \hat{\mu}_p(e^{\lambda})}{\lambda}. \tag{1}$$

Then $\frac{\mathbb{E}(K_n)}{n} \to \alpha_p$

The proof for a stronger version of this statement can be found in [1] explained in detail. Guided by this proposition, we are ready to prove the theorem regarding the order of a random element in $\Gamma_p(n)$.

Theorem 4.8. *Let p^{K_n} be the order of a random element in $\Gamma_p(n)$, and let α_p be the solution of the following equation:*

$$x(1-x)^{\frac{1}{x}-1} = 1 - \frac{1}{p}. \tag{2}$$

Then, $\frac{K_n}{n} \to \alpha_p$ *in probability.*

Proof. If we substitute Z_p in the definition of $\hat{\mu}_p(z)$ in 4.6, we get that

$$\hat{\mu}_p(z) = 1 + \frac{p-1}{p} \cdot z.$$

Indeed, $\mu_p(0) = \frac{1}{p} \cdot p + \frac{p-1}{p} \cdot 0 = 1$, since if $g \neq id_X$, then g has X as one orbit with size p, and if $g = id_X$, then g has p orbits; all of which have size $1 = p^0$. Similarly, $\mu_p(1) = \frac{1}{p} \cdot 0 + \frac{p-1}{p} \cdot 1 = \frac{p-1}{p}$.

If we denote by $f(\lambda)$ the function $\log \hat{\mu}_p(e^\lambda)$, then the value given in (1) is $\min_\lambda \frac{f(\lambda)}{\lambda}$. After derivation, we get

$$\frac{f(\lambda)}{\lambda} = f'(\lambda).$$

Provided that this equality holds (so when $\frac{f(\lambda)}{\lambda}$ is minimal), both sides are α_p. By the definition of f and $\hat{\mu}_p$,

$$\alpha_p = f'(\lambda) = (\log \hat{\mu}_p(e^\lambda))' = \frac{\hat{\mu}_p'(\lambda) \cdot e^\lambda}{\hat{\mu}_p(e^\lambda)} = \frac{\frac{p-1}{p} \cdot e^\lambda}{1 + \frac{p-1}{p} \cdot e^\lambda}.$$

From this, we can first express e^λ, then λ:

$$e^\lambda = \frac{\alpha_p}{1-\alpha_p} \cdot \frac{p}{p-1},$$

$$\lambda = \log\left(\frac{\alpha_p}{1-\alpha_p} \cdot \frac{p}{p-1}\right).$$

As a consequence,

$$f(\lambda) = \log \hat{\mu}_p(e^\lambda) = \log\left(\frac{1}{1-\alpha_p}\right),$$

and

$$\alpha_p = \frac{f(\lambda)}{\lambda} = \frac{\log\left(\frac{1}{1-\alpha_p}\right)}{\log\left(\frac{\alpha_p}{1-\alpha_p} \cdot \frac{p}{p-1}\right)}.$$

Using the logarithmic identities, we get

$$\frac{1}{\alpha_p} = \log_{\frac{1}{1-\alpha_p}} \left(\frac{1}{1-\alpha_p}\right)^{\frac{1}{\alpha_p}} = \log_{\frac{1}{1-\alpha_p}} \left(\frac{\alpha_p}{1-\alpha_p} \cdot \frac{p}{p-1}\right),$$

and as the logarithmic function is monotonous,

$$\left(\frac{1}{1-\alpha_p}\right)^{\frac{1}{\alpha_p}} = \frac{\alpha_p}{1-\alpha_p} \cdot \frac{p}{p-1},$$

and this equation can be rearranged to the form

$$\alpha_p(1-\alpha_p)^{\frac{1}{\alpha_p}-1} = \frac{p-1}{p} = 1 - \frac{1}{p}.$$

The latter formula shows that α_p in fact satisfies (2). The convergence in probability follows from 4.7. $\square$

As a generalisation of 4.4, we present the following proposition:

Proposition 4.9. *Let $\alpha \in \Gamma$ be a Haar-random element. Let X_n be the random variable of the number of fixed points with respect to α in the n-th level of T. Then $X_n \xrightarrow{d} 0$.*

Proof. Let $\mathcal{Q}_n$ and $\mathcal{Q}$ be the distribution pertaining to X_n and 0. By the definition of convergence in distribution, we have to show that

$$\int h \; d\mathcal{Q}_n \to \int h \; d\mathcal{Q}$$

for every continuous and bounded function h. Let $p_n(i)$ denote the probability that α fixes i vertices at level n (notice that $p_n(2k+1) = 0$ for every $k \in \mathbb{N}$). Then the upper limit can be expressed in the following form:

$$\sum_{i=0}^{2^n} p_n(i) \cdot h(i) \to h(0), \quad n \to \infty.$$

Let $|h| \leq K$. The result showed in 4.3 implies that

$$|h(0) - \sum_{i=0}^{2^n} p_n(i) \cdot h(i)| \leq (1 - p_n(0)) \cdot h(0) + (1 - p_n(0)) \cdot K \to 0,$$

as K is a constant, and $1 - p_n(0) = P(\alpha_n \text{ has a fixed point at level } n) \sim \frac{2}{n}$.

A particularly interesting phenomenon appears if we compare the joint distribution of the fixed points in $\Gamma(n)$ to that of S_{2^n}. Let $X_{n,i}$ $(i = 1, 2, \dots, 2^n)$ be the random variable which has value 1 if the i-th vertex on the n-th level of $T(n)$ is fixed by a Haar-random element of $\Gamma(n)$; and 0 otherwise. $P(X_{n,i} = 1) = \frac{1}{2^n}$, since the i-th vertex is fixed by α if and only if in $(T_\alpha)_n$ the i-th vertex determines an orbit of size 1; and the latter is equivalent with the fact that all the n ancestors of the orbit have 2 descendants in the orbit tree. As this occurs independently for every ancestor with probability $\frac{1}{2}$, the probability that the i-th vertex is fixed is in fact $\frac{1}{2^n}$.

It is easy to check that this holds for every transitive permutation group G acting on X $\big(|G : \mathrm{Stab}(X_i)| = |X_i^G|\big)$, which in our case equals to $|X|$), so if $X'_{n,i}$ denotes the random variable where $X'_{n,i} = 1$ if a random permutation in S_{2^n} fixes the i-th element; then $P(X'_{n,i} = 1) = \frac{1}{2^n}$ for $i = 1, \dots, 2^n$.

Definition 4.10. Let D_n and D'_n be the **derangement number** of $\Gamma(n)$ and S_{2^n}, i.e. the number of permutations without fixed points in $\Gamma(n)$ and S_{2^n}, respectfully.

Obviously, $\frac{D_n}{|\Gamma(n)|}$ and $\frac{D'_n}{|S_{2^n}|}$ equal to the probability that a random element has no fixed points. Suppose, for a brief moment, that $(X_{n,i})_{i=1}^{2^n}$ and $(X'_{n,i})_{i=1}^{2^n}$ were independent random variables. Then, by the above results,

$$\frac{D_n}{|\Gamma(n)|} = \frac{D'_n}{|S_{2^n}|} = \left(1 - \frac{1}{2^n}\right)^{2^n} \longrightarrow \frac{1}{e}, \quad n \longrightarrow \infty$$

would hold.

However, neither $(X_{n,i})_{i=1}^{2^n}$, nor $(X'_{n,i})_{i=1}^{2^n}$ are independent variables:

$$P(X'_{n,i} = 1 \ \forall i) = \frac{1}{(2^n)!} \neq \prod_{i=1}^{2^n} \left(1 - \frac{1}{2^n}\right) = \prod_{i=1}^{2^n} P(X'_{n,i} = 1),$$

and for instance $X_{n,0} = 1 \iff X_{n,1} = 1$. Nevertheless, one could argue that they have a different "level of independence": it can be shown that in spite of $X'_{n,i}$ not being independent, $\frac{D'_n}{|S_{2^n}|} \to \frac{1}{e}$ truly holds, whereas

$$\frac{D_n}{|\Gamma(n)|} = P(\text{a Haar-random } \alpha \in \Gamma \text{ has no fixpoints at level } n) \to 1,$$

as we have proven it previously. This implies that in some sense, the fixpoint-indicators in S_{2^n} are "more independent" than in $\Gamma(n)$. Another explanation confirming this claim is that if a vertex in a Galton-Watson tree belongs to a fixed point, then removing the unique root-level path leading to it leaves the disjoint union of other Galton-Watson trees. Each of these trees have other fixed points with expected value 1. In conclusion, a fixed point in a Galton-Watson tree tends to "generate" more fixed points in the tree.

4.2 The effect of some groups on ∂T

In 2.3, we defined ∂T to be the set of all infinite rays of T. Since every automorphism of T keeps the descent relation, any automorphism in Γ maps every infinite ray to another; so every subgroup of Γ has an effect on ∂T. In this final chapter, we intend to examine this effect for some groups, in terms of randomness. Generally, the groups will be represented as the subgroups of Γ (for instance, $Z = \langle \alpha \rangle$, where the profile of α was given by switches along an infinite ray); thus we can assign them the constraint of the Haar-measure inherited from Γ.

Definition 4.11. Let G be a group assigned with Haar-measure. A **randomly evaluated word** in G is a word w in the free group F_k that we get by substituting k independent Haar-random elements of G into the generators of F_k.

Notation. The random evaluation of the word $w \in F_k$ will be denoted by $\overline{w}$.

Proposition 4.12. *A nontrivial randomly evaluated word in* Γ *has only a finite number of fixed points in* T *with probability* 1.

Proof. We use induction on the number of the random generators. When $k = 1$, the statement claims that a Haar-random element of Γ has finitely many fixed points with probability 1, which we proved in 4.4.

Now suppose $w = w_1 w_2 \dots w_k$, where w_i or its inverse is one of the generators. For $i \leq k$, let $w'_i = w_1 w_2 \dots w_i$ denote the product of the first i generators. For a vertex v, v_i will denote $v^{\overline{w}'_i}$. Suppose that v is a fixed point for $\overline{w}$, but the trajectory of v with respect to $\overline{w}$ does not intersect itself, i.e. $v_i \neq v_j$ for every $i < j, j \neq k$.

Let $g(v)$ denote the effect of $g \in \Gamma$ in the subtree of T which is rooted at v (T_v). As the trajectory of v does not intersect itself, we can give an easy formula to compute $\overline{w}(v)$:

$$\overline{w}(v) = \prod_{i=1}^{k} \overline{w}'_i(v^{\overline{w}'_{i-1}}).$$

Consequently, $\overline{w}$, as an automorphism of T_v, is the product of independent Haar-random elements; so by 4.4, $\overline{w}$ has only a finite number of fixed vertices in T_v.

Moving on to prove the proposition, one may observe that by the induction hypothesis, the evaluations of every proper subword of $w = w_1 w_2 \dots w_k$ have a finite number of fixed vertices altogether. This implies the existence of a level n in T where the evaluations of the proper subwords of w do not have fixed vertices at all. By the upper observation, for every v in this level $\overline{w}$ fixes a finite number of nodes in T_v; showing that $\overline{w}$ has at most a finite number of fixed nodes in the entire tree T.

Corollary 4.13. *Let $G = \langle g_1, g_2, \dots, g_k \rangle$ be a group generated by the Haar-random elements g_i from Γ. Then $G \simeq F_k$ and G acts freely on ∂T with probability* 1.

Proof. Every possible combination g of the generators g_i is a randomly evaluated word in a free group. As a special case of the previous proposition, $P(g = id_T) \leq P(g \text{ fixes infinitely many vertices of } T) = 0$. The other part of the claim comes from the fact that in order to fix an infinite ray of T, g must have at least an infinite number of fixed vertices in T. □

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