NYILATKOZAT

Név: Schweitzer Ádám

ELTE Természettudományi Kar, szak: Matematika BSc.

NEPTUN azonosító: EFZY81

Szakdolgozat címe:
Divisor theories in complex surfaces and graphs

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Budapest, 2021.05.30.

Schweitzer Ádám

a hallgató aláírása

Divisor theories in complex surfaces and graphs

Schweitzer Ádám

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2021

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Acknowledgements

I am grateful to my supervisor, András Némethi for his help in introducing me to this topic and supporting my work.

Chapter 1

Introduction

1.1 Introduction

This thesis attempts to serve as an introduction and a side-by-side comparison of divisors on Riemann surfaces and graphs. As the first theory is more vast the beginning aims to introduce the definitions and theorems necessary to understand divisor theory on complex manifolds. In the end, we compare divisors on graphs and Riemann surfaces and finally prove the famous Riemann-Roch theorem for both cases.

In algebraic geometry, divisors are a generalization of hypersurfaces, this notion proved fruitful in the research of complex manifolds and their holomorphic and meromorphic functions. In their introduction we largely follow Chapter 3 in [15].

Riemann surfaces are 1-dimensional complex manifolds. Unlike real manifolds with dimension 1 these have a much richer structure. Its study often resulted in theorems that were later generalized for arbitrary complex manifolds. Our introduction of this notion will be based on Chapter 4 in [15].

The study of divisors of graphs is a recently developed field of study centered around the chip-firing games, certain combinatorial games played on graphs. Its introduction is based on the divisors in algebraic geometry, and surprisingly shares many of their properties. We will follow [3] in their introduction.

Riemann-Roch theorem is central in the study of both types of divisors. It implies numerous powerful theorems in both cases. We will follow [21] in the proof of the version for Riemann surfaces, and follow [3] in the case of graphs, ultimately proving some important theorems for chip firing games as corollaries.

As the scope of this thesis is rather deep, we expect the reader to be somewhat familiar with complex analysis and linear algebra and topology, and know the basic definitions in commutative algebra.

Now we will give a short description of the contents of each chapter.
In Chapter 1 we give an introduction to the thesis, and introduce the notations used.

In Chapter 2 we state the theorems used from complex analysis in several variables, and define complex manifolds.
In Chapter 3 we introduce analytic sets, and we study their local and global properties.
In Chapter 4 we define fiber bundles and line bundles.
In Chapter 5 we study the properties of previous definitions in the case of projective manifolds.
In Chapter 6 we introduce divisors on complex manifolds, and study their properties.
In Chapter 7 we study the previous definitions on Riemann surfaces and introduce holomorphic 1-forms.
In Chapter 8 we discuss the variants of the chip-firing game along with some of the most important related results.
In Chapter 9 we define divisors for graphs, and study them both in the case of Riemann surfaces and graphs.
In Chapter 10 we prove the Riemann-Roch theorem in both cases, and use the graph theoretic version to prove various results about the chip-firing game.

1.2 Notations

In this section we establish some basic notation.
We will denote the natural numbers (including zero) as $\mathbb{N}$, and the integers as $\mathbb{Z}$. We define $\mathbb{R}^+$ as the set of positive real numbers.
In this thesis we will work with loopless graphs (graphs with no edges from a vertex to itself) with the possibility of multiple edges between two vertices (these graphs are referred to as *multigraphs*). We refer to the vertex set of a graph G as $V(G)$, and to its edge set as $E(G)$. We will denote any edge between vertices $v, u \in V(G)$ as $vu \in E(G)$. We define the degree of a vertex, as the number of edges containing it. We denote the degree of a vertex v as $\deg(v)$. We define a subgraph of a graph as a graph with a subset of the edges and all of the vertices. We define a graph as connected, if for any pair of vertices u, v there is a sequence of vertices starting with u and ending with v such that for any two adjacent vertices in the sequence there is an edge between them in the graph. We call a subgraph a spanning tree, if it is connected, but if we remove any edge, it will not be connected anymore.
We will denote vectors as $x = (x_1, x_2, \ldots, x_n)$.
We will denote the $n \times n$ matrix ring on field k as $M_n(k)$.
We define the adjacency matrix $A(G) \in M_{|V(G)|}(\mathbb{C})$ of a graph G as the matrix that contains the number of edges between i, j in the entry in the i-th row, and j-th column. We define the degree matrix $I(G) \in M_{|V(G)|}(\mathbb{C})$ of a graph G as the matrix with only diagonal non-zero entries where the entry in the i-th row and column is the degree of the i-th vertex. We define the laplacian matrix $L(G) \in M_{|V(G)|}(\mathbb{C})$ of a graph G as $L(G) = I(G) - A(G)$.
For a formal sum D with coefficients in $\mathbb{Z}$ on a set A we denote the coefficient corresponding to an element $a \in A$ as $D(a)$.
We will denote the Jacobian matrix of a function $F : \mathbb{C}^n \to \mathbb{C}^m$ as $\frac{\partial F}{\partial x}$.

We define a complex polydisc as $\{z : |z_1| \leq t_1, |z_2| \leq t_2, \ldots, |z_n| \leq t_n\} \subset \mathbb{C}^n$ for $t_1, t_2, \ldots, t_n \in \mathbb{R}^+$.
For a function $f : A \to B$ we will denote the function we get if we constrain it to $S \subset A$ as $f|_S$. For a function $f : A \to B$ we denote its graph as $\Gamma(f) \subset A \times B$.
For a product space $M \times N$ we will denote the canonical projections as π_M and π_N.
When no confusion is possible we may use the word divisor in the context of a ring.

Chapter 2

Complex manifolds

In this section we introduce complex manifolds along with some of their most important properties and corresponding theorems.

2.1 Complex analysis in several variables

First we state some well known theorems about holomorphic functions in $\mathbb{C}^n$. For a deeper introduction to complex analysis and the proofs of the following theorems see [11]. The first theorem we state simplifies the requirements of holomorphism.

Theorem 2.1.1 (Hartog's theorem) A function $f : \mathbb{C}^n \to \mathbb{C}$ is holomorphic, if and only if, it is holomorphic in each variable.

Our next theorem is the Cauchy integral formula in several variables. This result follows easily from its 1-dimensional variant.

Theorem 2.1.2 (Cauchy integral formula) Let $f : D \subset \mathbb{C}^n \to \mathbb{C}$ be holomorphic on the polydisc $\{w : |w_1| = r_1, |w_2| = r_2, \ldots |w_n| = r_n\}$ for $r_1, r_2, \ldots, r_n \in \mathbb{R}^+$, and let $c = (c_1, c_2, \ldots, c_n)$ be inside the polydisc. The following holds:

$$f(c) = \left(\frac{1}{2\pi i}\right)^n \int_{|w_1|=r_1, |w_2|=r_2, \ldots |w_n|=r_n} \frac{f(w)}{\prod_{i=1}^n (c_i - w_i)} \mathrm{d}w.$$

Furthermore, if $m_1, m_2, \ldots m_n \in \mathbb{N}$ and $\sum_{i=1}^n m_i = M$:

$$\frac{\partial^M f(c)}{\partial c_1^{m_1} c_2^{m_2} \ldots c_n^{m_n}} = \prod_{i=1}^n m_i! \left(\frac{1}{2\pi i}\right)^n \int_{|w_1|=r_1, |w_2|=r_2, \ldots |w_n|=r_n} \frac{f(w)}{\prod_{i=1}^n (c_i - w_i)^{m_i+1}} \mathrm{d}w$$

The following theorem is the Gauss's mean principle is several variables, this result follows easily both from its 1-dimensional variant, and the previous theorem. Intuitively, it states that at any point a holomorphic function is equal to the average of its values on a polydisc around the original point.

Theorem 2.1.3 (Gauss's mean principle) Let $f : D \subset \mathbb{C}^n \to \mathbb{C}$ be holomorphic on the polydisc $\{w : |w_1| = r_1, |w_2| = r_2, \ldots |w_n| = r_n\}$ for $r_1, r_2, \ldots, r_n \in \mathbb{R}^+$. Then the following holds:

$$f(0) = \left(\frac{1}{2\pi}\right)^n \int_{|w_1|=r_1, |w_2|=r_2, \ldots |w_n|=r_n} f(w) |\mathrm{d}w|$$

The following principle can be proved from its original variant, but we give a proof relying on our previous theorem.

Theorem 2.1.4 (Maximum principle) Let a non-constant $f : D \subset \mathbb{C}^n \to \mathbb{C}$ be holomorphic on D, where D is a connected open set. There can be no local maxima of f in D.

Proof. Let us suppose that there is a local maxima c in D. Let us use the previous theorem, with small enough r_i such that on the polydisc $\{w : |w_1| = r_1, |w_2| = r_2, \dots |w_n| = r_n\}$ it holds that $f(w) \leq D$.

$$f(c) = \left(\frac{1}{2\pi i}\right)^n \int_{|w_1|=r_1,|w_2|=r_2,\dots|w_n|=r_n} f(w)|\mathrm{d}w|$$

This implies that $f(w) = f(c)$ for all w on our polydisc, as f is continuous, and the right side is an average on the polydisc. Therefore, there exists a small neighborhood around each local maxima, where f is constant, therefore, the set of local maximas are both open and closed in D, therefore, either f is constant or there is no local maxima. □

The following statement is especially important, as it gives a connection between the ring $\mathbb{C}\langle z_1, z_2, \dots, z_n\rangle$ (the ring of convergent power series) and holomorphic functions, allowing us to draw conclusions about the behaviours of the latter from the algebraic properties of the former.

Proposition 2.1.5 Let a non-constant $f : D \subset \mathbb{C}^n \to \mathbb{C}$ be holomorphic on D, where D is an open set, and let $c = (c_1, c_2, \dots, c_n) \in D$. Then there exists a neighborhood around c where f can be written as a power series uniquely, that is

$$f(z) = \sum_{m_1\geq 0, m_2\geq 0,\dots,m_n\geq 0} a_{m_1,m_2,\dots,m_n}(z_1 - c_1)^{m_1}(z_2 - c_2)^{m_2} \dots (z_n - c_n)^{m_n}$$

This property is sometimes referred to as f being complex analytic.

The Cauchy integral formula allows us to calculate the coefficients in this power series from the values of f around c.

The following theorem implies that a holomorphic function's global properties are implied from its local structure.

Theorem 2.1.6 (Principle of analytic continuation) Let $f : D \subset \mathbb{C}^n \to \mathbb{C}$ and $g : D \subset \mathbb{C}^n \to \mathbb{C}$ be two holomorphic functions on the connected open set D. If $f = g$ on an open non-empty subset of D, then they equal everywhere on D. Or equivalently, if any holomorphic function h defined on the open connected set D vanishes on an open non-empty set, then $h \equiv 0$.

The following theorem enables us to locally parameterize the vanishing points of a holomorphic function around regular points, allowing us to better analyze this local structure.

Theorem 2.1.7 (Implicit mapping theorem) Let $f : D_1 \times D_2 \subset \mathbb{C}^{m+n} \to \mathbb{C}^m$ be holomorphic on $D_1 \times D_2$ where $D_1 \times D_2$ is the product of two open sets, and let

$$(x', y') = (x'_1, x'_2, \ldots, x'_m, y'_1, y'_2, \ldots, y'_n) \in \mathbb{C}^{m+n}.$$

If the Jacobian matrix of $\frac{\partial f}{\partial x}$ is invertible in the point (x', y'), then there is a neighborhood of (x', y') where the set $\{f(x, y) = 0\}$ can be constructed uniquely as the graph of a holomorphic function $\Phi : \mathbb{C}^n \to \mathbb{C}^m$ (that is $f(\phi(y), y) = 0$). Furthermore the jacobian of ϕ will be of the form

$$\frac{\partial \phi}{\partial y}(y') = \left[\frac{\partial f}{\partial x}(y', \phi(y'))\right]^{-1} \left[\frac{\partial f}{\partial y}(y', \phi(y'))\right].$$

Our next theorem, the Riemann extension theorem, seems far stronger in the multivariable version. It describes when can we extend a holomorphic function to a thin set. We first define thin sets in $\mathbb{C}^n$.

Definition 2.1.8 (Thin set) We refer to a set $S \in \mathbb{C}^n$ as *thin*, if there is a neighborhood around any point $p \in \mathbb{C}^n$, such that on that neighborhood there are holomorphic functions such that the intersection of their roots are exactly the points of S in that neighborhood.

Note that in the 1-dimensional case, thin sets are exactly sets containing only isolated points, thus the statement of the next theorem gives us the usual statement in 1 dimension.

Theorem 2.1.9 (Riemann extension theorem) Let an $f : D \setminus S \subset \mathbb{C}^n \to \mathbb{C}$ be holomorphic on $D \setminus S$, where D is an open set, and S is a thin subset in D. If around any point of S it holds that f is bounded, then f can be extended uniquely to D.

2.2 Complex manifolds

Now we define complex manifolds and state the generalization of some of these results.

Definition 2.2.1 (Complex manifold) Let M be a connected Hausdorff space with an open covering U_α and functions $f_\alpha : U_\alpha \to \mathbb{C}^n$ such that for any α, β it holds that $f_\alpha \circ f_\beta^{-1}$ is biholomorphic on $f_\beta(U_\alpha \cap U_\beta)$. We refer to (U_α, f_α) as *charts* on M, and we refer to their sets as an *atlas* on M. We call two atlases equivalent if their union is an atlas as well. We refer to M along with an equivalence class of atlases as a *complex manifold*.

It can be seen that the n in the definition of f is constant, and it is referred to as the *dimension of a complex manifold* and denoted as $\dim(M)$. It can be seen that a complex manifold of dimension n has topological dimension $2n$ (as a real

differentiable manifold). We refer to a 1-dimensional complex manifold as a *Riemann surface*.
With this definition we gave manifolds a holomorphic structure and we can generalize some concepts to these manifolds.

Definition 2.2.2 (Holomorphic map) We refer to a map $f : M \to N$ where N and M are complex manifolds as a *holomorphic map* if for any chart (U_α, g_α) and (V_β, h_β) corresponding to M and N, the map $h_\beta \circ f \circ g_\alpha^{-1}$ is holomorphic everywhere where it is defined.

We can define $\mathbb{C}^n$ as a complex manifold, where the functions in the charts are all the identity function. We can also define the holomorphic structure of products of complex manifolds with the product of atlases.

Definition 2.2.3 (Biholomorphic map) We refer to a map $f : M \to N$ where N and M are complex manifolds as a *biholomorphic map* if it is bijective, and its inverse is holomorphic as well. We call two subsets of complex manifolds biholomorphic if there is a biholomorphism between them.

Biholomorphic equivalence describes the holomorphic structure. We state the following beautiful theorem describing the biholomorphic equivalence class of the unit circle in $\mathbb{C}$.

Theorem 2.2.4 (Riemann mapping theorem [19]) Let G be a simply-connected non-empty open subset of $\mathbb{C}$. If $G \neq \mathbb{C}$, then it is biholomorphic to the open unit circle

$$\mathbb{D} = \{c \in \mathbb{C} : |c| < 1\}.$$

Any subset of a complex manifold that is a complex manifold with the atlas we get from restricting the original charts is called *submanifold*. It can be seen easily that any open subset of a complex manifold is a submanifold (the converse need not be true, for example in the case of $\mathbb{C} \subset \mathbb{C}^2$).
Now we prove some results analogous to the ones at the beginning of the chapter on complex manifolds.

Theorem 2.2.5 (Maximum principle on complex manifolds) Let a non-constant $f : M \subset \mathbb{C}^n \to \mathbb{C}$ be holomorphic on M, where M is a complex manifold. There can be no local maxima of f in M.

Proof. Let us suppose that there exists a local maxima p. Then, there is a chart (U_α, g_α) such that $p \in U_\alpha$. By the original Maximal Principle, it must hold that $f \circ g_\alpha^{-1}$ is constant for some c. Let $Z = \{p \in M | f(p) = c\}$. If $U_\alpha \cap Z \neq \emptyset$, then $U_\alpha \subset Z$, as M is connected, and Z is not empty it must hold that $Z = M$. □

With this theorem we can prove the following statement about holomorphic maps on compact complex manifolds.

Proposition 2.2.6 Any holomorphic map $f : M \to C^n$, where M is a compact complex manifold, is constant.

Proof. By the compactness of M it also follows that there exists a point $p \in M$ where $|f(p)|$ is maximal. If f is non-constant, then by the previous theorem we are in contradiction, therefore, it follows that f is indeed constant. □

Now we prove the principle of analytic continuation for complex manifolds. This will allow us later to completely define a holomorphic function's local properties by its local behaviour through a chart. This is useful to us, as this means that by studying the local properties of holomorphic functions on $\mathbb{C}^n$ we in fact reveal a lot about the local properties of all holomorphic functions.

Theorem 2.2.7 (Principle of analytic continuation on complex manifolds) Let $f : M \to \mathbb{C}$ and $g : M \to \mathbb{C}$ be two holomorphic functions on the complex manifold M. If $f = g$ on an open non-empty subset of M, then they equal everywhere on M. Or equivalently, if any holomorphic function h defined on the complex manifold M vanishes on an open non-empty set, then $h \equiv 0$.

Proof. We will prove the second form of the statement. Let $Z = \text{int}(V(h))$, we know that Z is not empty. Let us take any (U_α, f_α) chart, where $U_\alpha \cap Z \neq \emptyset$. Then $h(f_\alpha^{-1}(z)) = 0$ on $f_\alpha(Z \cap U_\alpha)$, which is open, therefore, by the original version of this theorem, $h(f_\alpha^{-1}(z)) \equiv 0$, therefore $h(U_\alpha) = 0$. This implies that for any U_α in the open covering, if $Z \cap U_\alpha$ is not empty, then $Z \cap U_\alpha = U_\alpha$. As Z is not empty, and M is connected, this implies that $Z = M$, therefore $h \equiv 0$. □

Chapter 3

Analytic sets

3.1 Analytic sets

Analytic sets are the generalization of thin sets to manifolds.

Definition 3.1.1 (Analytic set) Let a set $A \subset D$ be an *analytic set*, if for any $p \in M$ there is a neighborhood U_p of p such that there are holomorphic functions f_i from U_p such that $A \cap U_p = \bigcap_i V(f_i)$

Theorem 3.1.2 (Riemann extension theorem on complex manifolds) Let an $f : M \setminus S \subset \mathbb{C}^n \to \mathbb{C}$ be holomorphic on $M \setminus S$, where M is a complex manifold, and S is an analytic proper subset in M. If around all points of M it holds that f is bounded, then f can be extended uniquely to M.

Proof. The preimage of S through a chart will be a thin set. This implies that on every chart f can be extended uniquely to U_α. By the principle of analytic continuation on complex manifolds these extensions must be compatible. This implies that we can extend f to M. □

Now we inspect the local behaviour of analytic sets. For this we have to better understand the local structure of holomorphic functions first.

Definition 3.1.3 Let two holomorphic functions from two neighborhoods of $p \in M$ be equivalent if on the connected component of the intersection of the two neighborhoods that contain p they are equal.

Proposition 3.1.4 The above defined equivalence is an equivalence relation.

Proof. The reflexivity and symmetry is clear in this case, therefore we only have to show the transitivity of this relation. We will prove that if there is an open set (such as a neighborhood of p) where the two holomorphic functions are equal, then they are equal on the intersection of their domain (if this is a connected set). This guarantees transitivity as on the intersections of the three open subsets, all three functions are equal, thus pairwise they have to equal on the intersections as well. This follows instantly from the principle of analytic continuation on the submanifold that is the connected component of the intersection. □

Definition 3.1.5 (Germ) We define *germs* as the equivalence classes of the above defined equivalence on the holomorphic functions defined in a point.

Analytic sets can be defined as the sets that around any point are the intersection of the roots of several germ.

The study of germs reveals a lot about the local properties of an analytic set.

3.2 Ring of convergent power series

Definition 3.2.1 (Ring of convergent power series) We define the *ring of convergent power series* in n variable as the subring of the ring of power series, where the elements converge on a neighborhood of 0. This forms a ring with the usual operations. We will denote this ring as $\mathbb{C}\langle x_1, x_2, \dots x_n \rangle$.

Proposition 3.2.2 The ring of germs around any point is isomorphic to the ring of convergent power series in n variables where n is the dimension of the manifold.

Proof. Through a chart with the domain containing p we can obtain a convergent power series around the image of p which is equal to the function's image through the chart, thus creating an isomorphism of the rings. □

Now we analyze this ring to better understand its properties following [15] Chapter 3.

Definition 3.2.3 (Regular of order *k*) A germ is *regular of order* k in x_n if

$$f(0, 0, \dots, x_n) = c_k x_n^k + c_{k+1} x_n^{k+1} \dots$$

where $c_k \neq 0$. We call f *regular*, if it is regular of order k with $k > 0$.

Remark 3.2.4 With a suitable change of coordinates, any non-constant f which vanishes at the origin can be taken to a regular element in x_n.

Proposition 3.2.5 A $f \in \mathbb{C}\langle x_1, x_2 \dots, x_n \rangle$ is a unit exactly when it is regular of order 0 (or in other words, it does not vanish in $(0, 0, \dots, 0)$).

Proof. A convergent power series not satisfying the condition above can not be a unit, as if it would have an inverse, their product in $(0, 0, \dots, 0)$ would be 1, which can't be expressed as a product of zero.

For an f satisfying the condition we can construct its inverse as the expansion of $1/f$. This can be expressed as a convergent power series around $(0, 0, \dots, 0)$, as there is a small enough polydisc around the point where f is not zero and defined (as its vanishing points constitute a closed set not containing the point), thus it is defined. The product of the two functions will be 1 in a small neighborhood around the origin, thus it must be the unit in the ring. □

Definition 3.2.6 (Weierstrass polynomial) An element $f \in \mathbb{C}\langle x_1, x_2 \ldots, x_{n-1}\rangle[x_n]$ is called a *Weierstrass polynomial* in x_n with order k if $f|_{x_i=0 \quad \forall_{i \leq n-1}} = x_n^k$.

Theorem 3.2.7 (Weierstrass Preparation theorem) If $f \in \mathbb{C}\langle x_1, x_2, \ldots x_n\rangle$ is regular of order k in x_n, then it can be written uniquely as $f = ug$, where u is a unit in $\mathbb{C}\langle x_1, x_2 \ldots x_n\rangle$, and g is a Weierstrass polynomial in x_n with order k.

Proof. We study f around 0 in x_n in the disc $|x_n| < r$. If we constrain $x' = (x_1, x_2, \ldots, x_{n-1})$ to a small enough neighborhood, such that

$$|f(0, 0, \ldots, x_n) - f(x_1, x_2, \ldots, x_n)| < \epsilon$$

for any $|x_n| < r$, then by the Rouche theorem applied for a fix x' vector we get that there is exactly k roots of f in $|x_n| < r$ for any such fixed vector.
Let α_i be the functions of the k roots from x'. Note that $\alpha_i(0) = 0$. Let us take the polynomials $\sum_{i=0}^{k} \alpha_i^k$. These are holomorphic as they can be calculated with the following integral:

$$\sum_{i=0}^{k} \alpha_i^p = \frac{1}{2\pi i} \int_{|t_n - x_n| = r} \frac{\partial f(x', t_n)}{\partial t_n} \frac{t_n^p}{f(x', t_n)} \mathrm{d}t.$$

From this, it follows that the elementary symmetric functions of α_i are holomorphic as well, as they can be obtained as the polynomials of these. Thus we can construct a Weierstrass polynomial g with order k which will have these symmetric polynomials as its coefficients in x_n. This indeed is a Weierstrass polynomial, as in $x_n = 0$ the only root is 0.
We only have to prove that f/g is a unit, or due to 3.2.5, that f/g is holomorphic and does not vanish in the origin. At any point where g is not 0 we can bound f/g from above by the maximum principle in x_n by taking a small enough polydisc in the neighborhood and taking the quotient of the supremum of f on the border in x_n and the infimum of g on the same border, for a small enough $x_n < r_n$ such that this border does not contain a zero of the function. Due to this our function is bounded, thus it can be extended due to the Reimann extension theorem.
The uniqueness of g and u comes from the fact that g is defined by its zeros. □

Theorem 3.2.8 (Weierstrass division theorem) For any h Weierstrass polynomial in x_n of order k, and any $f \in \mathbb{C}\langle x_1, x_2 \ldots x_n\rangle$, there are uniquely determined $g \in \mathbb{C}\langle x_1, x_2 \ldots x_n\rangle$ and $r \in \mathbb{C}\langle x_1, x_2 \ldots x_{n-1}\rangle[x_n]$ such that:

$$f = gh + r$$

and the degree of r in x_n is less than k.

Proof. Taking a small enough polydisc of the origin we can suppose that both f and h are holomorphic, and h is not zero when $x_n = r_n$ where x_n is bounded by r_n in the polydisc . In this case we can construct g as follows:

$$g(x', x_n) = \frac{1}{2\pi i} \int_{t_n = r_n} \frac{f(x', t_n)}{h(x', t_n)} \cdot \frac{\mathrm{d}t_n}{(t_n - x_n)}$$

If $r = f - hg$, then it can be expressed as:

$$\frac{1}{2\pi i}\int_{t_n=r_n} f(x', t_n)\cdot\frac{\mathrm{d}t_n}{(t_n - x_n)} - h(x', x_n)\frac{1}{2\pi i}\int_{|t_n|=r_n}\frac{f(x', t_n)}{h(x', t_n)}\cdot\frac{\mathrm{d}t_n}{(t_n - x_n)}$$

$$\frac{1}{2\pi i}\int_{|t_n|=r_n}\frac{f(x', t_n)}{h(x', t_n)}\cdot(h(x', t_n) - h(x', x_n))\cdot\frac{\mathrm{d}t_n}{(t_n - x_n)}$$

As $(h(x', t_n) - h(x', x_n))\cdot\frac{1}{(t_n - x_n)}$ is a polynomial in x_n, the entire expression will be one as well. □

Remark 3.2.9 If $f \in \mathbb{C}\langle x_1, x_2 \dots x_{n-1}\rangle[x_n]$, then our construction gives a g in $\mathbb{C}\langle x_1, x_2 \dots x_{n-1}\rangle[x_n]$ as well.

With these fundamental theorems we can prove various properties of the ring of germs.

Proposition 3.2.10 $\mathbb{C}\langle x_1, x_2 \dots x_n\rangle$ is

1. UFD
2. noetherian

Proof. (1) We will prove the statement by induction on n.
From the Gauss lemma it follows that $\mathbb{C}\langle x_1, x_2 \dots x_{n-1}\rangle[x_n]$ is an UFD. Any non-unit $f \in \mathbb{C}\langle x_1, x_2 \dots x_n\rangle$ can be taken to a regular f_0 by a change of coordinates due to 3.2.4. By 3.2.7 it follows that $f_0 = ug$, where $g \in \mathbb{C}\langle x_1, x_2 \dots x_{n-1}\rangle[x_n]$ is a Weierstrass polynomial in x_n. We will now prove that $f_0 = ug = ug_1g_2 \dots g_n$ is a unique factorization in $\mathbb{C}\langle x_1, x_2 \dots x_n\rangle$. This will be an irreducible factorization as $g = g_1g_2 \dots g_n$ is an irreducible factorization in $\mathbb{C}\langle x_1, x_2 \dots x_{n-1}\rangle[x_n]$, and if $g_i = r_1r_2$ where $r_1, r_2 \in \mathbb{C}\langle x_1, x_2 \dots x_n\rangle$, then there are Weierstrass polynomials r_1', r_2' and units u_1, u_2 such that $r_1 = u_1r_1', r_2 = u_2r_2'$, thus $g_i = r_1'r_2'u_1u_2$, and this implies, that $g_i = r_1r_2$ as the factorization in the Weierstrass preparation theorem is unique and $g \cdot 1 = (r_1'r_2')(u_1u_2)$. This would imply that g is not irreducible in $\mathbb{C}\langle x_1, x_2 \dots x_{n-1}\rangle[x_n]$, thus we would reach a contradiction.
Now we will prove that this factorization is unique as well. Let's suppose that there is another factorization of $f_0 = vf_1f_2 \dots f_m$ where v is a unit. From 3.2.7 it follows that $f_i = v_ih_i$. This gives that $f_0 = (u\prod_{i=1}^m v_i)(\prod_{i=1}^m h_m)$. From the uniqueness part of 3.2.7 it is clear that $g_1g_2 \dots g_n = h_1h_2 \dots h_m$, but this implies that they are the same factorization in $\mathbb{C}\langle x_1, x_2 \dots x_{n-1}\rangle[x_n]$, thus $f_0 = vf_1f_2 \dots f_m$ is equivalent to $f_0 = ug = ug_1g_2 \dots g_n$. □

Proof. (2) We will prove the statement by induction on n.
From Hilbert's basis theorem it follows that $\mathbb{C}\langle x_1, x_2 \dots x_{n-1}\rangle[x_n]$ is noetherian.
Let us suppose that $\mathbb{C}\langle x_1, x_2 \dots x_n\rangle$ is not noetherian. In this case there are elements $f_1, f_2 \dots$ such that no elements are contained in the ideal generated by the previous ones. As a non-unit f_i is only not general in x_n if f_i vanishes on the axis corresponding to x_n, and any f_i vanishes on finitely many lines starting from the origin (this can

be seen as if a series vanishes on a line, then every homogeneous part vanishes on it as well, therefore there are only countably many such lines). As all of the f_i vanishes on countably many lines, there is a linear change of coordinates such that every f_i is general in x_n. After the change, due to the preparation theorem (3.2.7), every $f_i = g_i u_i$. As $g_i \in \mathbb{C}\langle x_1, x_2 \dots x_{n-1}\rangle[x_n]$, it holds that, there is a $g_n \in (g_1, g_2 \dots g_{n-1})$, therefore $g_n = \sum_{i=1}^{n-1} g_i x_i$, as an ideal in $\mathbb{C}\langle x_1, x_2 \dots x_n\rangle$ is closed to multiplication, it contains, g_i, therefore $g_n \in (f_1, f_2 \dots f_{n-1})$, and $f_n \in (f_1, f_2 \dots f_{n-1})$, thus we reached a contradiction. □

As the ring of germs has the UFD property, we can define *irreducible germs* and *irreducible factorizations*.

Definition 3.2.11 We can define the local equivalence of analytic sets around a point p following the definition of germs. Let two analytic sets S and G be equivalent if there is an open neighborhood U of p where $S \cap U = G \cap U$. We call the equivalence classes the *germs of a set*.

Every analytic set around a point is described by the roots of some germs. This defines a ring of germs $I(S)$ vanishing on the germ of a set S. This ideal is generated by finitely many germs of functions. Conversely, to every ideal, we can define a set $V(I)$ where every germ of the ideal vanishes. The following famous theorem of Hilbert establishes their exact relationship.

Theorem 3.2.12 (Hilbert's nullstellensatz [16](p. 43)) If $f \in \mathbb{C}\langle x_1, x_2, \dots, x_n\rangle$ vanishes on $V(I)$, then $f \in \sqrt{I}$, or equivalently $I(V(I)) = \sqrt{I}$ (we know that $V(I(V)) = V$ where V is analytic set). Where $\sqrt{I}$ is the radical of I, that is the ideal $\sqrt{I} = (f|\exists_k f^k \in I)$.

A germ of an analytic set S is said to be *irreducible*, if $I(S)$ is a prime ideal. As $I(S) = \sqrt{I(S)}$, it holds that $I(S)$ can be written uniquely as the intersection of prime ideals P_j (see [17]). By the nullstellensatz, there are S_j branches of the germ of the analytic set, such that $I(S_j) = P_j$. This gives the irreducible decomposition of a germ of an analytic set to *branches*.

3.3 Properties of analytic sets

We can also define the irreducibility of analytic sets. Let an analytic set be *reducible*, if it can be written as the non-trivial union of two analytic sets. We refer to an analytic set as *irreducible* if it is not reducible. Every analytic set can be decomposed to irreducible parts, as any intersection of analytic sets is analytic and the set of germs is noetherian. Furthermore on a compact manifold, there can be only finitely many irreducible parts (this can be seen from the fact, that such a manifold has a finite covering with open subsets not containing the analytic set and with small enough neighborhoods of the points of the set, such that it can be decomposed locally into finitely many branches.)

Definition 3.3.1 (Singular points) If S is an analytic set, and $p \in S$, then there are f_i germs such that the germ of S is the intersection of $V(f_i)$. Let us examine the rank of the Jacobian matrix corresponding to $(f_1, f_2 \ldots, f_n)$. If this rank is constant in a neighborhood of p, we refer to this point as *regular*, else we refer to it as *singular*. It can be shown that the choice of germs does not matter.

We refer to the set of singular points as $\text{Sing}(S)$

It can be shown that $\text{Sing}(S)$ will be analytical, and nowhere dense in S (see [16](p. 58)).

Now we turn our attention to defining the dimension of an analytic set locally around a point p. One approach is to define the dimension locally, based on the algebraic structure of the germs of holomorphic functions defining the set.

Definition 3.3.2 (Krull dimension) For a commutative ring R we define its Krull dimension as the length of the longest chain of prime ideals. We define the length of a chain of prime ideals $R_0 \subsetneq R_1 \subsetneq \ldots \subsetneq R_n$ as n.

Every germ of an analytic set corresponds to a radical I by the Nullstellensatz. We define the Krull dimension of that germ as the Krull dimension of $\mathbb{C}[[x_1, x_2 \ldots x_n]]/I$. Note that the Krull dimension of $\mathbb{C}[[x_1, x_2 \ldots x_n]]$ will be exactly n (see Theorem 15.4 in [12]). This definition behaves as expected for trivial examples. The dimension around an interior point will be n, and the dimension of a point will be 0.

Proposition 3.3.3 [16](p. 58) For any analytic set S, it holds that $S \setminus \text{Sing}(S)$ is a complex manifold. It will be connected exactly when S is irreducible.

With this we can define the *dimension of an irreducible analytic set* as the dimension of $S \setminus \text{Sing}(S)$. We define the *dimension of an analytic set* as the maximal dimension of its irreducible components. The dimension of an irreducible germ of an analytic set will be the minimal dimension among all the elements in the germ as an equivalence class. The dimension of a germ of an analytic set will be the maximal dimension among its irreducible branches. We will refer to the dimension of the germ of S around p as the dimension of an analytic set S at point p, we will denote it with $\dim_p(S)$. Note that $\dim(S) = \max_p \dim_p(S)$

We will now prove the following intuitive statements:

Proposition 3.3.4 If M is a complex manifold, and $S \subset M$ is an analytic subset, such that $S \neq M$, then

1. the interior of S is empty
2. $M \setminus S$ is path connected.

Proof. (1) Around any point S is biholomorphic to the set of common roots of some holomorphic functions around 0 in $\mathbb{C}^{\dim(M)}$. If there is an interior point of the set, then these functions must be the constant 0 functions. This implies that $S = M$ as $\text{int}(S) \cap U_p = U_p$ or $\text{int}(S) \cap U_p = \emptyset$ for all points p, therefore $\text{int}(S) = M$ as it is not empty. □

Proof. (2) We will first prove that around any point $p \in M$ there is a neighborhood U_p such that $(M \setminus S) \cap U_p$ is connected, and then we will prove that this implies our statement.

Let $n := \dim(M)$. It is enough to prove the statement in $\mathbb{C}^n$, as it is locally biholomorphic to our manifold. Let $Z \subset \mathbb{C}^n$ be an analytic set of dimension at most $\dim(M) - 1$ (we can suppose this as the dimension bounds the local dimension). Let us suppose that $Z \subset \mathbb{C}^n$ is not path connected, so there are two points $p, q \in \mathbb{C}^n \setminus Z$, such that there is no path in $\mathbb{C}^n \setminus Z$ between them. Let us take the line L that contains both p, and q. $L \cap Z$ cannot have a limit point in L, as then all of the holomorphic functions that define Z must vanish on L entirely, therefore $L \cap Z$ consist of isolated points, therefore $L \setminus Z$ is path connected, and therefore there exists a path between p and q.

Now we can prove our original statement. By the previous statement we know that S does not have interior points, and it is closed. Furthermore, we have proved that $M \setminus Z$ is locally connected in M. Let us suppose that $M \setminus Z$ can be partitioned to two disjoint parts A and B that are both open and closed in $M \setminus Z$. As Z is nowhere dense, we know that $M = \overline{A \cup B} = \overline{A} \cup \overline{B}$. However we know that A and B cannot have a common boundary point, as M is locally connected, and then $A \cup B$ would be connected as well. Therefore $\overline{A} \cup \overline{B}$ will partition M into two disjoint closed sets, but M is connected, therefore we have reached a contradiction. □

Now we state the following important theorem of Remmert and Stein without proof (see [16](p. 123)).

Theorem 3.3.5 (Remmert-Stein Continuation Theorem) For any $S \subset M$ analytic set, if $T \subset M \setminus S$ is analytic in $M \setminus S$, and $\dim(S) < \dim_p(T)$ for any point $p \in M \setminus S$, then $\overline{T}$ is analytic in M, where $\overline{T}$ denotes the closure of T in M.

3.4 Meromorphic maps

Now we will turn our attention to meromorphic maps on complex manifolds. We will give a rather general definition following [18], but later on we will give a more intuitive equivalent definition for the case of projective manifolds.

First we need to introduce some other definitions.

Definition 3.4.1 (Proper map) A continuous map $f : M \to N$ between Hausdorff spaces is called *proper*, if for any compact set $K \subset N$, it holds that $f^{-1}(K)$ is compact in M. Note that if M is compact, then every map has this property.

We state the following fundamental theorem about proper maps without proof (see [16](p. 129)).

Theorem 3.4.2 (Proper mapping theorem) Let $f : M \to N$ be a holomorphic map between complex manifolds. Let $S \subset M$ be an analytic set. If $f|_S : S \to N$

is proper, then $f(S) \subset N$ is analytic as well. If S is irreducible, then $f(S)$ will be too.

The dimension of $f(S)$ can be given as follows. Let $r_p(f) = \dim_p(S) - \dim_p(S \cap f^{-1}f(p))$. Then $\dim(f(S)) = \max_p r_p(f)$.

Definition 3.4.3 (Proper modification) Let M, N be complex manifolds, and $X \subset M$-be an irreducible analytic set. Let $f : M \to N$ be a holomorphic map. $f|_X$ is called *proper modification* if it satisfies the following:

- $f|_X$ is surjective
- $f|_X$ is proper
- There exists an analytic set $S \subset M$ such that $\mathrm{Sing}(X) \subset S \subset X$ and there exists an analytic set $T \subset N$ such that f induces a biholomorphic map from $X \setminus S$ to $N \setminus T$.

Now we can define a meromorphic map. As we will see this notion generalizes the usual meromorphic functions, and for Riemann surfaces the two definitions coincide.

Definition 3.4.4 (Meromorphic map) Let $f : M \to \mathcal{P}(N)$, where $\mathcal{P}$ represents the power set of N. Let us take the graph of f, or in other words, the set

$$\Gamma(f) = \{(x, y) \in M \times N | y \in f(x)\}.$$

Such a function f is called meromorphic, if $\Gamma(f)$ is irreducible analytic, and the natural projection of $\Gamma(f)$ onto M is a proper modification.

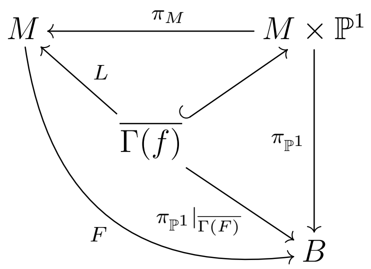
The points $p \in M$ where $f(p)$ has more than one element are called its *points of indeterminacy*. In [18] it is proven that this set is analytic with dimension at most $\dim(M) - 2$. Now we will define meromorphic functions, and prove the equivalence of the definitions.

Definition 3.4.5 (Meromorphic function) Let $f : M \setminus S \to \mathbb{C}$ where S is an analytic set, and $S \neq M$. Then f is called a *meromorphic function* if around any point there is an open neighborhood, and holomorphic functions g, h, such that $f(x) = \frac{g(x)}{h(x)}$ at all point in the neighborhood, where f is defined.

Proposition 3.4.6 Every meromorphic map onto $\mathbb{P}^1$ uniquely induces a meromorphic function.

Proof. Let for a meromorphic map $f : M \to \mathcal{P}(\mathbb{P}^1)$, the points of indeterminacy form an analytic set S_1, therefore on $M \setminus S_1$ we can view f as a conventional function. Due to the definition of meromorphic maps, the natural projection $\Gamma(f) \to M$ is a biholomorphism restricted to $\Gamma(f) \setminus T \to M \setminus S_2$ for some analytic sets S_2, T, and the natural projection is holomorphic, therefore $f|_{S_1 \cup S_2} = \pi_{\mathbb{P}^1} \circ (\pi_M|_{S_1 \cup S_2})^{-1}$ is holomorphic. Let $f(p) = (f_1(p) : f_2(p))$, this way, f induces a map onto $\mathbb{C}$ as $(1 : \frac{f_2(p)}{f_1(p)})$, where $f_1(p) \neq 0$. Let $S_3 = \{p | f_1(p) = 0\}$. As f_1 is holomorphic, this is an analytic set, and therefore $\frac{f_2(p)}{f_1(p)}$ will induce meromorphic function from $M \setminus (S_1 \cup S_2 \cup S_3)$.

An f meromorphic function defined on $M \setminus S$ defines a meromorphic map as follows: If around a point p it holds that $f = \frac{g}{h}$ (where g and h are relatively prime), then let $F(x) = (g(x) : h(x))$ where it does not hold that both $h(x) = 0$ and $g(x) = 0$. This extends f to some points of S where locally $h(x) = 0$ but $g(x) \neq 0$. This still must be a holomorphic map. The set G where locally $g(x) = h(x) = 0$ has at most dimension $\dim(M) - 2$ as g and h are relative prime. Let us examine $\Gamma(F) \subset M \times \mathbb{P}^1$. As F is holomorphic on $M \setminus G$, $\Gamma(F)$ is an analytic set in $M \times \mathbb{P}^1 \setminus G \times \mathbb{P}^1$. The dimension of $\Gamma(F)$ is the same as the dimension of M as the natural projection is a biholomorphism between them. Therefore, we can apply the Remmert-Stein continuation theorem (3.3.5) on $\Gamma(F) \subset M \times \mathbb{P}^1 \setminus G \times \mathbb{P}^1$ to get that $\overline{\Gamma(F)}$ is an analytic set. $\overline{\Gamma(F)}$ is connected, as the original manifold was connected, and $\dim(G) < \dim(M) - 1$, therefore $M \setminus G$ will be connected as well by 3.3.4, therefore it is an irreducible analytic set, and so is its image under a biholomorphism. Let $L(x) = \pi_M|_{\overline{\Gamma(F)}}(x)$



□

As $\mathbb{P}^1$ is compact, it holds that L is surjective, $\overline{\Gamma(F)}$ is irreducible (as $\overline{\Gamma(F)}$ is connected thus by 3.3.3 it will be irreducible), $\pi_M|_{\Gamma(F)}$ is biholomorphic, therefore, it is only left to prove that it is proper.

Let us suppose that it is not proper. That means that there is a compact set $K \subset M$, such that $L^{-1}(K)$ is not compact. $K \cap G$ is a closed subset of a compact set, therefore it is compact. Therefore we know that $(K \cap G) \times \mathbb{P}^1$ is compact as well. Let an open covering U_α be given on $L^{-1}(K)$. We will create a finite subcovering, thus proving that $L^{-1}(K)$ is compact. $L^{-1}(K) \cap ((K \cap G) \times \mathbb{P}^1)$ is compact, therefore there is a finite subcovering V_i of it in U_α. Let us take the covering of K consisting of all the sets $L(U_\alpha)$ for any U_α. This has a finite subcovering of it. The union of the corresponding sets of this subcovering, and V_i covers $L^{-1}(K)$ as V_i covers $L^{-1}(K \cap G)$, and on the rest of $L^{-1}(K)$ the function L is biholomorphic onto $K \setminus G$, therefore the original sets cover the image of $L^{-1}(K \setminus G)$, thus the two subcoverings cover the entire set.

Remark 3.4.7 It can be proved in a similar way, that for meromorphic functions f_i, the map $(1 : f_1 : f_2 \ldots : f_n)$ induces a meromorphic map.

Chapter 4

Line bundles

The study of meromorphic maps and functions on complex manifolds often uses line bundles, or more generally fiber bundles. In this chapter we give a brief introduction to these.

4.1 Fiber bundles

First we must define a complex Lie group.

Definition 4.1.1 (Complex Lie group) Let G be a complex manifold with a group structure. We call such a structure a complex Lie group if the usual group operations are holomorphic.

Examples of such structures would be $\mathrm{GL}(\mathbb{C}, n), \mathrm{SL}(\mathbb{C}, n)$. We call a biholomorphism from a complex manifold to itself an *automorphism*. Due to the next theorem the automorphisms of a compact complex manifold form a complex Lie group.

Theorem 4.1.2 (Bochner-Montgomery [6]) The automorphisms of a complex manifold M form a complex Lie group $\mathrm{Aut}(M)$, that acts on M holomorphically.

Now we can define fiber bundles.

Definition 4.1.3 (Fiber bundles) Let B, M, F be complex manifolds, and let $\pi : B \to M$ be a surjective holomorphic map between them. Let $G \subset \mathrm{Aut}(F)$ be a complex Lie group. Then π is called a *fiber bundle* on M with standard fiber F and structure group G, if there is an open covering U_α on M, such that $\pi^{-1}(U_\alpha)$ is biholomorphic through a function f_α to $F \times U_\alpha$ such that if the canonical projection to the second coordinate is π_2, then the following diagram commutes for all α.

$$\begin{array}{ccc} B & \xrightarrow{f_\alpha} & F \times U_\alpha \\ & {\scriptstyle \pi}\searrow & \downarrow {\scriptstyle \pi_2} \\ & & U_\alpha \end{array}$$

Furthermore, for every α, β, there exists a $g_{\alpha,\beta}(p) : U_\alpha \cap U_\beta \to G$ holomorphic function such that in $U_\alpha \cap U_\beta$ it holds that $g_{\alpha,\beta}(p) \cdot f_\beta(p) = f_\alpha(p)$ where $g_{\alpha,\beta}$ only acts on the first coordinate.

The definition implies the following about $g_{\alpha,\beta}(p)$.

$$g_{\alpha,\alpha}(p) = 1 \quad \text{for all } \alpha$$

$$g_{\alpha,\beta}(p)g_{\beta,\gamma}(p) = g_{\alpha,\gamma}(p) \quad \text{for all } \alpha, \beta, \gamma \text{ where all three are defined}$$

These maps already define the fiber bundle. For every such M, F and G, we can create a fiber bundle $\pi : B \to M$ by taking the disjoint union of $F \times U_\alpha \times \{\alpha\}$, and factoring the manifold by $(g_{\alpha,\beta}(p) \cdot w, p, \alpha) \sim (w, p, \beta)$.

4.2 Line bundles

A fiber bundle, where F is a complex vector space and where G is the group of its complex linear automorphisms is called a *vector bundle*.

Definition 4.2.1 (Homomorphism of vector bundles) We define a homomorphism f between line bundles $\pi_1 : B_1 \to M$ and $\pi_2 : B_2 \to M$ on M, as a holomorphic function between B_1 and B_2 such that:

- the following diagram commutes,

$$\begin{array}{ccc} B_1 & \xrightarrow[f]{} & B_2 \\ {\scriptstyle \pi_1}\downarrow & & {\scriptstyle \pi_2}\downarrow \\ M & = & M \end{array}$$

- for any point $p \in M$ it must hold by the previous property that $f(\pi_1^{-1}) \subset \pi_2^{-1}(p)$, we require that f acts linearly between these sets equipped with the linear structure from the functions $f_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^n$ (as the transition functions were linear, these are well-defined).

We may define the isomorphism as bijections that are homomorphisms in both directions.
If the dimension of the vector space is 1, it is referred to as a *line bundle*. We define the *canonical line bundle* with the same open covering U_α as in the charts of M, and define $g_{\alpha,\beta} = \det(\frac{f_\alpha \circ f_\beta^{-1}}{\partial z})$. This line bundle will be independent of the choice of charts from an atlas.

For any line bundles we can take a common open covering (by taking the intersections of their covering).

Definition 4.2.2 For two line bundles we define their tensor product as follows. We take a common open covering of them, and we take their transition functions. We define the product with the product of transition functions of the two line bundles.

Definition 4.2.3 (Picard group) For a complex manifold M we define its Picard group $\mathrm{Pic}(M)$ as the group of isomorphism classes of line bundles with the tensor product.

Definition 4.2.4 (Holomorphic section) We call a holomorphic function $g : M \to B$ a *holomorphic section* of π, if $g \circ \pi = id$.

There is a natural bijection between holomorphic sections, and collections of holomorphic maps $\eta_\alpha : U_\alpha \to F$ with $\eta_\alpha = g_{\alpha,\beta}\eta_\beta$. In this way we can regard holomorphic sections of line bundles as a generalization of holomorphic functions. If we choose the transition functions as identical, we get exactly the holomorphic functions as the sections. We refer to the line bundles with constant 1 as their transition function as *trivial*.
For a line bundle L on M we define the vector space of holomorphic sections as $\Gamma(L, M)$, or if there is no possibility for confusion $\Gamma(L)$.
Through an f holomorphic function between complex manifolds, we can pull back both line bundles and their holomorphic sections.
Similarly to holomorphic sections we can define meromorphic sections.

Definition 4.2.5 (Meromorphic section) A collection of meromorphic maps $\eta_\alpha : U_\alpha \to F$ with $\eta_\alpha = g_{\alpha,\beta}\eta_\beta$ is called a *meromorphic section* of the line bundle $g_{\alpha,\beta}$.

As in the case of holomorphic sections, we can regard meromorphic sections as a generalization of meromorphic functions in the case of line bundles. Again, we get back the usual meromorphic functions, if the transition functions are identical

Chapter 5

Projective manifolds

5.1 Projective algebraic sets

In this section we discuss the behaviour of analytic sets in projective spaces.
We define protective algebraic sets in the following way.

Definition 5.1.1 (Projective algebraic set) Let $f_i : \mathbb{C}^n \to \mathbb{C}$ be homogeneous polynomials. We define the set $V(f_1, f_2, \ldots, f_m) \subset \mathbb{P}^{n-1}$ as the set of points where a every point is a root of all f_i polynomials in $p \in \mathbb{P}^{n-1}$ taken as an equivalence class in $\mathbb{C}^n \setminus \{0\}$. As all f_i is homogeneous it holds that either all points in the class p are roots, or none of them. We may refer to projective algebraic sets as projective varieties.

It is clear to see, that any projective algebraic set is analytic, as we can take $f_j(1, x_2/x_1, \ldots, x_n/x_1)$ on $U_1 = \{x | x_1 \neq 0\}$ for all j to generate the set locally, and similarly, for all other i.
Surprisingly the converse also holds by the following theorem.

Theorem 5.1.2 (Chow's theorem) All analytic sets in a projective space are projective algebraic sets.

Proof. (Proof. (Cartan)) Let us define the canonical projection $\pi : \mathbb{C}^n \setminus \{0\} \to \mathbb{P}^{n-1}$. For any analytic set $S \in \mathbb{P}^{n-1}$, it is easy to see that $V' = \pi^{-1}(S)$ will be analytical in $\mathbb{C}^n \setminus \{0\}$ and of dimension at least 1. Therefore by the Remmert-Stein continuation theorem, it must hold that $V := 0 \cup V'$ will be analytic in $\mathbb{C}^n$. We inspect the defining functions around 0, let them be f_i. We will prove that all of them are homogeneous polynomials, and they define S on $\mathbb{P}^{n-1}$. We write f_i as the sum of homogeneous polynomials $f_{i,j}$ as

$$f_i = \sum_{0 \leq j} f_{i,j}$$

where $\deg(f_{i,j}) = j$. It is only left to prove that there is exactly one non-zero $f_{i,j}$ in that sum. Note that we can take a small enough polydisc $\{x : \forall_i x_i \leq c\}$ around 0 such that it must hold that any constant multiple of a root of f_i in the polydisc is a root as well. It is clear that

$$f_i(pt) = \sum_{0 \leq j} t^j f_{i,j}(p),$$

therefore for any p root, this function will be constant 0 in t, therefore all the coefficients must be 0 as well, in p. Therefore, if p is a root of f_i, it is the root of all $f_{i,j}$. Therefore, the roots of f_i are exactly the intersection of the roots of $f_{i,j}$. As the ring of holomorphic functions around 0 is noetherian, it follows that there are finitely many $f_{i,j}$ that generate these roots. Therefore for all i we take these finitely many homogeneous polynomials, and the intersection of the roots of these will be exactly V, therefore it will be S in $\mathbb{P}^{n-1}$. □

Now we can freely restrict ourselves to the study of projective algebraic sets. We can define their reducibility and dimensions as analytic sets. We can define their ideal $I(V)$ as the ideal generated by the generating homogeneous polynomials. We refer to an ideal generated this way as homogeneous. The radical of such a polynomial will be homogeneous as well. For any projective algebraic set it is clear that $V(I) = V(\sqrt{I})$. Furthermore, as in the case of the germs of holomorphic functions, the following theorem holds in this case as well.

Theorem 5.1.3 (Projective nullstellensatz [14](p. 22)) If a homogeneous polynomial f vanishes on $V(I)$ for a homogeneous ideal I, then $f \in \sqrt{I}$, or equivalently $I(V(I)) = \sqrt{I}$. Where $\sqrt{I}$ is the radical of I, that is the ideal $\sqrt{I} = (f | \exists_k f^k \in I)$.

This allows us to inspect the properties of a projective algebraic set by studying its corresponding radical. By studying the local ring around 0, in the proof of Chow's theorem, we can deduce additional properties, that projective algebraic sets, and their defining polynomials inherit from the local study of analytic sets.
The irreducible decomposition corresponds again to a prime decomposition of the corresponding radical. Furthermore, it also holds that exactly when $\sqrt{I}$ is prime, will $V(I)$ be irreducible.

Now we turn our attention to the geometric properties of projective algebraic sets.
First we can establish a bijection between homogeneous polynomials generating projective algebraic sets in $\mathbb{P}^n$, and arbitrary polynomials in $\mathbb{C}^n$, in the usual way. If $F(x_1, x_2, \ldots, x_{n+1})$ is a homogeneous polynomial, then on the affine subset $U_1 = \{x : x_1 \neq 0\}$, we can take the polynomial $f(X_1, X_2, \ldots, X_{n+1}) := F(1, \frac{x_2}{x_1}, \ldots, \frac{x_{n+1}}{x_1})$. In the other direction, we can take the homogenization of f to get back F. We can define *affine algebraic sets* as the intersection of root sets of a polynomial. The restriction of projective algebraic sets to an affine subset results in a bijection between projective algebraic sets not containing $\{x_i = 0\}$ and affine algebraic sets.

Now we will turn our attention to the famous theorem of Bézout describing the number of intersections of projective algebraic sets. First we have to define the multiplicity of an intersection. This can be approached from many directions. The

one stated here follows Fulton in [8]. For a geometric approach for algebraic curves see [7].
We define the multiplicity of an intersection the following way:

Definition 5.1.4 We define $\mathbf{I}_p(S_1, S_2, \ldots, S_n) := \dim(\mathbb{C}\langle x_1, x_2, \ldots, x_n\rangle/S)$ where S is the radical ideal generated by the generating functions of S_i in p for all i, and where $\dim(\mathbb{C}\langle x_1, x_2, \ldots, x_n\rangle/S)$ is taken as the dimension as a vector space over $\mathbb{C}$.

This satisfies the intuitive requirements of intersection multiplicity.

1. If there is no common non-trivial analytic set of dimension at least 1 containing the point of intersection, then the intersection number is a non-negative integer, otherwise its value is defined as ∞.

2. The intersection number in a point p is zero exactly when it is not contained in the intersection of the varieties, and it is determined by the components of the sets passing through p.

3. The intersection number is invariant to linear changes of the coordinates.

4. The intersection number is invariant to the permutations of the intersecting sets.

5. The intersection number of irreducible hypersurfaces in a point p should be 1 when the intersections of the kernels of the Jacobians of the sets are trivial (this is referred to as intersecting transversally).

Now we can state the famous theorem of Bézout.

Theorem 5.1.5 (Bézout [9](p. 54)) The sum of intersection multiplicities of n hypersurfaces in $\mathbb{P}^n$ without common divisors, is either the product of their degrees as polynomials or there is a non-trivial projective algebraic set in the intersection.

For a simple proof for the case algebraic curves with the definitions above see [22](p. 15).
This result is fundamental in the study of intersection theory. For a detailed discussion of intersection theory see [8].

5.2 Projective manifolds

Now we turn our attention to projective manifolds.

Definition 5.2.1 (Projective manifold) We define a projective manifold as a complex manifold that has a holomorphic imbedding into a projective space.

In our study we will only be examining the following class of projective manifolds.

Definition 5.2.2 (Projective algebraic manifold) We define a projective algebraic manifold as an irreducible non-singular projective algebraic set.

Projective algebraic manifolds behave particularly nicely. We can define the following topology on them.

Definition 5.2.3 (Zariski topology) For a projective algebraic manifold $M \subset \mathbb{P}^n$ we define the closed sets in the Zariski topology as the projective algebraic sets contained in $\mathbb{P}^n$.

Proposition 5.2.4 This defines a topology on M.

Proof. The intersection of arbitrarily many such sets will be projective algebraic as their defining polynomials around any point (attained by taking an affine part of $\mathbb{P}^n$) will have a finite set which generates the same ideal, as $\mathbb{C}[x_1, x_2, \ldots, x_n]$ is noetherian. The intersecting set will be analytic, as suitable ones of the previous finitely many polynomials generate them around any point. By Chow's theorem this implies that this set is projective algebraic as well.

The union of finitely many such sets must be a projective algebraic set as well, as if we take the generating set of homogeneous polynomials of each set, and take all of these polynomials and multiply them each by the product of polynomials of all other sets, we get polynomials that must define the union of the original sets. □

For any homogeneous prime ideal P corresponding to the irreducible projective algebraic set V we can define $\Gamma_h[V] = \mathbb{C}[x_1, x_2, \ldots, x_{n+1}]/P$, the *homogeneous coordinate ring* of V. For an element $F + P$ we can define $\deg(F + P) = \deg(F)$ as P is a homogeneous ideal. Elements of this ring are not functions of V, but for homogeneous polynomials F, G of the same degree on a Zariski open subset of V where G doesn't vanish we have the function

$$\left(\frac{F+P}{G+P}\right)(p) = \frac{F(p)}{G(p)}.$$

We can assume that F, G are coprime. We refer to these elements of the quotient field $\Gamma_h(V)$ of the homogeneous coordinate ring as *rational functions.*

Proposition 5.2.5 ([1](p. 168)) Every meromorphic function f on a projective algebraic manifold $M \subset \mathbb{P}^n$ can be extended, to $\mathbb{P}^n$ meromorphically, or in other words, every such function is rational.

The field of rational functions $\mathbb{C}(V)$ on a projective algebraic manifold is extensively studied, and its structure reveals a lot about the properties of the manifold. The following proposition is an example of this.

For a projective space $\mathbb{P}^{n+1}$ let us examine $\mathbb{C}(\mathbb{P}^{n+1})$.

It is clear that functions $X_1/X_{n+1}, X_2/X_{n+1}, \ldots, X_n/X_{n+1}$ are elements of this field. They must generate every rational function on $\mathbb{P}^n$, as any such function can be written as the quotient of homogeneous polynomials. These functions must be algebraically independent over $\mathbb{C}$, as there can not be a non-trivial polynomial that has a root in them, as $X_1/X_{n+1}, X_2/X_{n+1}, \ldots, X_n/X_{n+1}$ attains every value in $\mathbb{C}^n$, and therefore such a function would have to vanish on every point, thus it must be trivial. This implies that

$$\mathbb{C}(\mathbb{P}^{n+1}) = \mathbb{C}(X_1/X_{n+1}, X_2/X_{n+1}, \ldots, X_n/X_{n+1}).$$

For a field extension $F : G$, we refer to the minimal amount of algebraically independent elements such that F is a finite extension of the field generated by them as the *transcendence degree* of F over G.

Proposition 5.2.6 ([14](p 36.)) For a projective algebraic variety V, the transcendence degree of $\mathbb{C}(V)$ over $\mathbb{C}$ is dim V.

Proof. Let the dimension of V be $n-k$. Let us choose a projective subspace N of dimension $n-k$ and another T of dimension of dimension $k-1$ such that T does not intersect V. It follows that any (k-dimensional) subspace K_p spanned by one point p of N and T intersects M in m-points (counting with multiplicity). Note that K_p for all p spans the entire projective space. Let us change the coordinates of $\mathbb{P}^n$ in the following way: Let the plane spanned by the first $n-k$ coordinates be N, and the plane spanned by the rest be T. We will prove that $\mathbb{C}(V)$ is a finite extension of the field corresponding to the projective subspace of the first $n-k$ coordinates. Therefore, it is enough to prove that any rational function $f : V \to \mathbb{C}$ is the root of a polynomial in the coordinate functions corresponding to the first $n-k$ coordinates. If given the values of the first $n-k$ coordinates, there is only finitely many possible values for f. This implies that for any point p on N there is a polynomial which's roots are the possible values for f. The coefficients of these polynomials must form a rational function in p as the values of f change rationally in p on a Zariski open set and so are their symmetrical polynomials, therefore they must be polynomials of the first $n-k$ coordinate functions (as any rational function on P_{n-k} can be expressed as such a polynomial). This implies, that this is a $G(y)$ polynomial in $\mathbb{C}(X_1/X_n, X_2/X_n, \ldots X_{n-k}/X_n)[Y]$, that has $f = y$ as its root, therefore the extension is finite, and the proof is finished. □

For any point p we can define the ring of rational functions defined at p. This ring is local with the unique maximal ideal of functions vanishing at p.
For a set U we define the set $\Gamma(U)$ of *regular functions* on U as the functions defined on every point of U. Now we can define rational and regular maps.

Definition 5.2.7 (Regular map) Let V, W be projective varieties. Let $f : U \subset V \to W$ be a function, where U is Zariski open. We call f a *regular map* of U onto W, if it satisfies the following:

- f is continuous in the Zariski topology
- For all $W' \subset W$ Zariski open subset, and all $h \in \Gamma(h)$ it holds that $h \circ f \in \Gamma(f^{-1}(W'))$.

For projective algebraic manifolds this is equivalent to f being holomorphic.
We can define the equivalence of regular maps similarly as in the case of holomorphic map. We refer to two regular map and their domains as *equivalent* if they are equal on the intersection of their domain (which is a Zariski open set).

Proposition 5.2.8 This defines an equivalence relation.

Proof. Regular maps are defined to be continuous on the Zariski topology. We can prove that any open set in the Zariski open set is dense. This follows from the fact that any Zariski closed set is nowhere dense in the euclidean topology by 3.3.4. It is also easy to see that Zariski open sets must be open in the usual topology. Therefore, it must follow that there can be no two disjoint non-empty Zariski open sets, as the complement of one is nowhere dense in the euclidean topology, and the other is non-empty and open in the euclidean topology. □

We refer to an equivalency class as a *rational map* from V to W. We call the union of the domains in the equivalency class the domain of the class. We refer to the complement as the *points of indeterminacy.*

Proposition 5.2.9 Every rational map $f : U \subset V \subset \mathbb{P}^n \to W \subset \mathbb{P}^{r+1}$ can be written as

$$f(p) = (1 : f_1(p) : f_2(p) : \ldots : f_n(p))$$

where f_i are rational functions, and every such function is rational. Furthermore the set of rational and meromorphic maps defined on the same set are the same.

Proof. For the proof of the first statement let us take the functions X_i/X_{r+1} on W. Let us take $(X_i/X_{r+1}) \circ f$, by the definition of rational functions, these will be a rational functions in U, and this implies the first statement if we take f_i to be $(X_i/X_{r+1}) \circ f$.
Such a function must be rational. The first property is guaranteed by the fact that f_i are continuous in the Zariski topology. Any rational function on W must be the polynomial of X_i/X_{n+1} by the proof of 5.2.6. Therefore for any rational function on W we must get a polynomial of rational functions in V which will again be rational, therefore the second property is satisfied as well.
The second part of the statement comes from the fact that for meromorphic maps f_1, f_2 it must hold that $f_1 \circ f_2$ is meromorphic as well. As rational functions are the same as meromorphic function on V by 5.2.5, it must hold here that $(X_i/X_{r+1}) \circ f$ is a meromorphic function by 3.4.6 for any meromorphic map f. By this it again must hold that any such function is of the form

$$f(p) = (1 : f_1(p) : f_2(p) : \ldots : f_n(p)),$$

furthermore every such function is meromorphic, therefore the two sets of functions coincide. □

Examine that by multiplying with the denominators every rational function can be expressed as $f(p) = (F_1(p) : F_2(p) : \ldots : F_{n+1}(p))$, where F_i are coprime homogeneous polynomials of the same degree.
Now we define regular maps.

Definition 5.2.10 (Regular maps) Any $f : V \to W$ rational map is a *regular map*, if its domain is exactly V (in other words, it is defined on every point of V).

Any regular map is holomorphic, as any rational map is holomorphic where it is defined. On the other hand holomorphic functions must be regular, as they are meromorphic and defined everywhere on the manifold.

These correspondences of projective algebraic and analytic objects (meromorphic and holomorphic maps correspond to rational and regular ones, analytic sets correspond to projective algebraic sets) are the examples of the *GAGA principle* referring to the paper "Géometrie Algébrique et Géométrie Analytique" by Serre (1956) [20].

We define *biregular functions* as bijective regular functions whose inverse is regular as well. We call two manifolds *isomorphic* or *biregular* if there is a biregular function between them. We call two manifolds *birational* if there are Zariski open subsets of them that have a biregular function between them. This is a coarser equivalence relation than biregularity, but it still describes a lot about projective varieties' structure. The following theorem is an important example of this.

Theorem 5.2.11 Projective varieties V and W are birational exactly, when $\mathbb{C}(V)$ and $\mathbb{C}(W)$ are isomorphic over $\mathbb{C}$

Proof. It can be easily seen that a rational function creates an isomorphism between the function fields.
If there is an isomorphism of $\mathbb{C}(V)$ and $\mathbb{C}(W)$ over $\mathbb{C}$, then that implies, that the image of the coordinate functions $X_i/X_{n+1} \in \mathbb{C}(W)$ can be written as the polynomial f_i of the coordinate functions in $\mathbb{C}(V)$. Therefore the function $(f_1 : f_2 : \ldots : f_n : 1)$ will be a suitable birational function. □

Theorem 5.2.12 (Hironaka [10]) For any projective variety S it holds that there is a projective algebraic manifold M such that it has a birational map to S such that it is biregular on $S \setminus \mathrm{Sing}(S)$.

The above mentioned M projective algebraic manifold is referred to as the *resolution of singularity* of V.

Chapter 6

Divisors on manifolds

In this chapter we assume that any manifold M is compact and projective algebraic of dimension n.

6.1 Weil and Cartier divisors

First we define our main object of study.

Definition 6.1.1 (Weil divisor) We define a Weil divisor D to be a finite formal sum of the form

$$\sum_{i=1}^{r} a_i D_i$$

where $a_i \in \mathbb{Z}$ and D_i is an irreducible analytic set of dimension $\dim(M) - 1$ in M. We refer to the set of divisors over a manifold M as $\mathrm{Div}(M)$.

Note that $\mathrm{Div}(M)$ is an abelian group with the formal sum, we refer to this as the *divisor group*.
We say that a Weil divisor is *effective* (or positive) if $a_i \geq 0$ for all i. We denote this by $D \geq 0$. This also defines a partial ordering on $\mathrm{Div}(M)$ A divisor where the number of summands is one, and the coefficient is 1 is called a *prime divisor*. We refer to the sum of coefficients of a divisor D as its degree denoted with $\deg(D)$. We denote the sum of positive coefficients as $\deg^+(D)$.

Now we provide an alternative definition of divisors, which in our case coincides with the first.
Let us take any open covering U_α on M and meromorphic functions $f_\alpha : U_\alpha \to \mathbb{C}$ such that on $U_\alpha \cap U_\beta$ the function $\frac{f_\alpha}{f_\beta}$ is non-vanishing holomorphic. It can be seen that any such function is a meromorphic section of the line bundle defined by $f_{\alpha,\beta} = \frac{f_\alpha}{f_\beta}$, furthermore, to any meromorphic section there is such a collection.
We define two such function sets $\{f_\alpha\}, \{g_\beta\}$ with $f_{\alpha,\beta} = g_{\alpha,\beta}$ to be equivalent, if on a common open covering it holds that on any set U'_α, the function $\frac{f_\alpha}{g_\alpha}$ is non-vanishing holomorphic.

Definition 6.1.2 (Cartier divisor) We define a Cartier divisor to be an equivalence class of meromorphic sections of line bundles.

Now we prove that the two definitions coincide in our case.

Proposition 6.1.3 There is a bijection between Weil and Cartier divisors.

Proof. Let us take a Weil divisor D and an open covering U_α such that in any of them D_i is given by a holomorphic function $h_{\alpha,i} : U_\alpha \to \mathbb{C}$ such that it is not properly divided by any other function that would generate D_i. We define

$$f_\alpha = h_{\alpha,1}^{a_1} h_{\alpha,2}^{a_2} \dots h_{\alpha,r}^{a_r}.$$

Note that for any α, β it holds that $\frac{f_\alpha}{f_\beta}$ is non-vanishing holomorphic on $U_\alpha \cap U_\beta$, as locally around a point it holds that if we take the quotient $\frac{h_{\alpha,i}}{h_{\beta,i}}$ then at any point p it must be non-zero, as the fraction must be a unit.

In the other direction if we take f_α and write it as the fraction of coprime holomorphic functions, then we get a factorization $f_\alpha = h_{\alpha,1}^{a_1} h_{\alpha,2}^{a_2} \dots h_{\alpha,r}^{a_r}$. By the analytic nullstellensatz it follows that $h_{\alpha,1}$ are distinct, and as such the hypersurface they define is suitable for D_i. It is only left to prove that the corresponding a_i is constant. On U_α this holds from the definition. On the intersections the root locus of $h_{\alpha,i}$ must be covered by a $h_{\beta,j}$ and their exponent must be the same, as in $U_\alpha \cap U_\beta$ by the analytic nullstellensatz the two must be the same disregarding multiplication by units. □

From now on we will use the two notions interchangeably.

For any Cartier divisor D we can define $[D]$ to be the line bundle corresponding to D.

Definition 6.1.4 (Linear equivalence) We define two divisors to be linearly equivalent if and only if their corresponding line bundles are the same. We may denote it for divisors D, D' as $D \sim D'$.

Note that the equivalence of $D_1 = \{f_\alpha\}$ and $D_2 = \{g_\alpha\}$ is equivalent to there being a meromorphic function $l : M \to \mathbb{C}$ that satisfies, that $f_\alpha \cdot l = g_\alpha$. This follows from the fact that if the two divisors are from the same line bundle, then:

$$\frac{\frac{f_\alpha}{g_\alpha}}{\frac{f_\beta}{g_\beta}} = \frac{f_{\alpha,\beta}}{f_{\alpha,\beta}} = 1$$

Therefore $\frac{f_\alpha}{g_\alpha}$ is a meromorphic section of the trivial bundle, and as such it is a meromorphic function on M.

We refer to divisors generated by meromorphic functions as *principal divisors*. It is clear that these divisors are exactly the ones linear equivalent to the trivial divisor (the empty formal sum). We denote their group as $\mathrm{Prin}(M) \leq \mathrm{Div}(M)$.

The function $D \mapsto [D]$ defines a homomorphism from $\mathrm{Div}(M)$ to $\mathrm{Pic}(M)$. This leads us to the following interesting statement.

Theorem 6.1.5 ([1] p.161) The following sequence is exact.

$$1 \longrightarrow \mathbb{C}^* \longrightarrow \mathbb{C}^*(M) \longrightarrow \mathrm{Div}(M) \longrightarrow \mathrm{Pic}(M) \longrightarrow 1$$

The function taking $\mathbb{C}^*$ to $\mathbb{C}^*(M)$ is the one taking any constant to the constant function. It is clear that any such function is taken to the trivial divisor. The image of $\mathbb{C}^*(M)$ is exactly the divisors generated by meromorphic functions, and as such they are taken to the trivial bundle in the Picard group. The fact that every line bundle has a corresponding divisor, or in other words it has a non-trivial meromorphic section is the hard part of the theorem above proved in the cited work of Griffiths and Harris.

6.2 Linear systems

For a divisor D we define $L(D)$ to be the meromorphic functions f such that the sum of the divisor generated by f and D is effective, that is $D + (f) \geq 0$. Note that this set along with the constant 0 function forms a vector space.

We will prove that this vector space is isomorphic to $\Gamma(M, [D])$ (remember that $\Gamma(M, [D])$ denotes the holomorphic sections of $[D]$ on M). For the sake of convenience we denote $\Gamma(M, [D])$ as $\Gamma(D)$.

Proposition 6.2.1 There is an isomorphism between $L(D)$ and $\Gamma(D)$.

Proof. For any element $g \in L(D)$ let us take the functions $g \cdot f_\alpha$ where $\{f_\alpha\} = D$. This function is a holomorphic section, as it is still a section in $[D]$ as the transition functions of f are constant 1, and it is holomorphic, as the corresponding divisor is effective, therefore in the factorization there can't be a holomorphic $h_{\alpha,i}$ with negative exponent.

In the other direction, for any holomorphic section g_α if we take the section $\frac{g_\alpha}{f_\alpha}$ we get a suitable meromorphic function as the transition functions must be 1 as the transition functions of g_α and f_α coincide. □

It surprisingly holds that $L(D)$ is finite dimensional. For the proof of this see [14](p. 103).

For a divisor D we denote by $|D|$ the set of effective divisors that are linearly equivalent to D. We refer to this set as the *complete linear system* determined by D. For the canonical line bundle K_M we refer to $|K_M|$ as the canonical linear system. It can be seen that $|D|$ is isomorphic to $\mathbb{P}(L(D))$, as we can take the linear subspace of f to $D + (f)$, and this determines (f) therefore this is indeed a bijection.

Definition 6.2.2 (Linear system) We refer to subspaces of $|D| = \mathbb{P}(L(D))$ as *linear systems.*

We define the dimension of a linear system as the dimension of this subspace. For a linear system Λ we define its *fixed part* as the largest divisor such that it is smaller than all of the divisors in Λ. As all the divisors in Λ are positive, the fixed component is non-negative. We call a linear system fixed component free, if its free part is the trivial divisor. For a fixed component free linear system Λ we define its *base locus* $\mathrm{Bs}(\Lambda) \subset M$ as the set of points contained in the hypersurface corresponding to each element $D \in \Lambda$.

Proposition 6.2.3 If D, D' are divisors, then it holds that

$$\dim(|D|) + \dim(|D'|) \leq \dim(|D + D'|).$$

Furthermore if $D + D'$ is fixed component free, then equality only holds if and only if there exists a positive divisor D_0 such that

- $|D_0|$ has no fixed component
- $\dim |jD_0| = j$ for $1 \leq j \leq |\dim(D + D')|$
- D is linearly equivalent to kD_0 and D' is linearly equivalent to jD_0 for some $k, j > 0$.

We only prove the first statement here. For the proof of the second statement see [15](p. 258).

Proof. For any pair of elements $(D_1, D_2) \in |D| \times |D'|$ let us choose the element $D_1 + D_2 \in |D + D'|$. This is a holomorphic mapping from $|D| \times |D'|$ to $|D + D'|$. Note that any $D_l \in |D + D'|$ can be written only finitely many ways as the sum of effective divisors, and therefore this map can only take finitely many pairs to one point. This implies that the dimension of the image is the same as of the domain (this follows from the fact that this map is proper, as its domain is compact, and we can use the proper mapping theorem). Therefore the image has dimension $\dim(|D|) + \dim(|D'|)$, therefore

$$\dim(|D|) + \dim(|D'|) \leq \dim(|D + D'|),$$

and the statement is proven. □

To any r-dimensional linear system Λ there corresponds a projective subspace of dimension r, and to that there corresponds a linear subspace in $\Gamma(D)$ of dimension $r + 1$. Let us take a basis $\eta_1, \eta_2, \ldots, \eta_{r+1}$. Let us define the functions

$$f_{\Lambda,\alpha} = (\eta_{1,\alpha} : \eta_{2,\alpha} : \ldots : \eta_{r,\alpha}) \in \mathbb{P}^r.$$

Remember that $\eta_{i,\alpha} \cdot f_{\alpha,\beta} = \eta_{i,\beta}$ for all i, therefore in the intersections $U_\alpha \cap U_\beta$ the functions are all multiplied by the same function when changing from a $f_{\Lambda,\alpha}$ to $f_{\Lambda,\beta}$, therefore we can define f_Λ on M as a meromorphic map.
Note that this function changes only by a projective automorphism when defined by another basis (this follows from the fact that the basis transformation defines the change in $\eta_{1,\alpha}, \eta_{2,\alpha}, \ldots, \eta_{r,\alpha}$, therefore its projectivization defines the change in $f_{\Lambda,\alpha}$, thus in f_Λ as well).
We refer to f_Λ as the *meromorphic map associated with* Λ. We refer to the function $f_{|K_M|}$ as the *canonical map.*
Now we prove some properties of these maps.

Proposition 6.2.4 Let $\Lambda \in |D|$ be a r-dimensional linear system and F_0 be its free part.

1. $f_\Lambda = f_{\Lambda - F_0}$
2. The points of indeterminacy of f_Λ is exactly $\mathrm{Bs}(\Lambda - F_0)$
3. $f_\Lambda(M)$ is not contained in any hyperplane.

Proof. (Proof. (1)) Let us take

$$f_\Lambda = (\eta_{1,\alpha} : \eta_{2,\alpha} : \ldots : \eta_{r-1,\alpha}).$$

As $\Lambda > F_0$ it must hold, than $\{\eta_{2,\alpha}\} > F_0$, therefore if $F_0 = \{g_\alpha\}$, then

$$f_\Lambda = \left(\frac{\eta_{1,\alpha}}{g_\alpha} : \frac{\eta_{2,\alpha}}{g_\alpha} : \ldots : \frac{\eta_{r-1,\alpha}}{g_\alpha}\right)$$

is still a properly defined meromorphic map, and is equivalent to the first one. However $\frac{\eta_{r-1,\alpha}}{g_\alpha} \in \Lambda - F_0$, therefore this function is $f_{\Lambda - F_0}$, and the statement is proven. □

Proof. ((2)) This follows from the fact that $\eta_{i,\alpha}$ is a holomorphic section, and as such the only point where the function is not defined, is where all of $\eta_{i,\alpha}$ vanishes. As these sections form a basis, the points where all of them vanish are the same, where every function in the linear space behind Λ vanishes, that is the points of the base locus. □

Proof. ((3)) If there would be such a hyperplane, then by a projective transformation we could achieve a constantly 0 basis element, but that can not be the part of any basis, thus we have reached a contradiction. □

Remark 6.2.5 Remember that the points of indeterminacy of a meromorphic map has dimension at most $n - 2$, and by the second statement it follows that

$$\dim(\mathrm{Bs}(\Lambda)) \leq \dim(M) - 2.$$

Let us take any meromorphic map $h : M \to \mathbb{P}^r$ and transform it to the form

$$(F_1(p) : F_2(p) : \ldots : F_{r+1}(p))$$

where F_i are homogeneous polynomials of the same degree m as per 5.2.9. Let us take any hyperplane H corresponding to $a \in \mathbb{P}^{r*}$ in $\mathbb{P}^r$. We can create a homogeneous polynomial on M as follows. Let

$$F_H(p) = F_1(p)a_1 + F_1(p)a_2 + \ldots + F_{r+1}(p)a_{r+1}$$

Then we can take the affine restrictions of this:

$$g_j\left(\frac{p_1}{p_j}, \frac{p_2}{p_j}, \ldots, \frac{p_n}{p_j}\right) = F_H\left(\frac{p_1}{p_j}, \frac{p_2}{p_j}, \ldots, \frac{p_n}{p_j}\right).$$

These functions define a Cartier divisor, as on the intersection of the affine spaces corresponding to the i-th and j-th coordinates in the projective space the transition function is

$$\frac{g_j(p)}{g_i(p)} = \left(\frac{p_i}{p_j}\right)^m$$

which is non-vanishing, and holomorphic. Let us refer to this divisor as D_H.

Proposition 6.2.6 The set of all D_H generates a fixed point free linear system, and this gives a bijection between r-dimensional fixed component free linear systems, and meromorphic maps to $\mathbb{P}^r$.

Proof. We only have to prove that the holomorphic sections of the form $g_j\left(\frac{p_1}{p_j}, \frac{p_2}{p_j}, \dots, \frac{p_n}{p_j}\right)$ are closed to addition. This holds trivially on every open set, thus on the entire $\mathbb{P}^r$ as well. It follows that $\Lambda_h = \{D_H : \forall H\}$ is a linear system. This linear system must be of dimension r as the function $a \in \mathbb{P}^{r*} \mapsto D_H$ is an isomorphism between the spaces (as it is linear on the underlying vector spaces).
It only remains to prove that $\Lambda_{f_\Lambda} = \Lambda$ and $f_{\Lambda_h} = h$ up to a projective transformation. Let us call H_i the hyperplane corresponding to the i-th coordinate. Let us inspect $D_{H_i} \in \Lambda_{f_\Lambda}$.

$$f_\Lambda = (\eta_{1,\alpha} : \eta_{2,\alpha} : \dots : \eta_{r-1,\alpha}) = (F_1(p) : F_2(p) : \dots : F_{r+1}(p))$$

Then $F_{H_i}(p) = F_i(p)$ and on U_α it will be exactly $\eta_{i,\alpha}$ with transition functions $f_{\alpha,\beta}$, therefore we get Λ as the projective space.
In the other direction, $h = (F_1(p) : F_2(p) : \dots : F_{r+1}(p))$ if we take H_i then $D_{H_i} \in \Lambda_h$ is exactly $F_i(p)$ therefore if we take the basis corresponding to H_i we get exactly

$$(F_1(p) : F_2(p) : \dots : F_{r+1}(p)) = h,$$

thus the statement is proven. □

We define a linear subsystem as a projective subspace $\Lambda' \subset \Lambda$.

Proposition 6.2.7 Let Λ be a r-dimensional linear system, and Λ' be its k-dimensional subsystem. There is a subspace S and a projection π_S to that, such that the following diagram commutes (where all the functions are rational).

$$\begin{array}{ccc} M & \xrightarrow{f_\Lambda} & \mathbb{P}^r \\ & \underset{f_{\Lambda'}}{\searrow} & \downarrow \pi_S \\ & & \mathbb{P}^k \end{array}$$

Proof. Let us choose a basis from Λ' and extend it to a basis of Λ. Let us construct f_Λ and $f_{\Lambda'}$ according to this basis (and respectively the subbasis). In this case

$$f_\Lambda = (F_1(p) : F_2(p) : \dots : F_{r+1}(p))$$

$$f_{\Lambda'} = (F_1(p) : F_2(p) : \dots : F_{k+1}(p)),$$

therefore the projection to the first $k+1$ basis element will be a suitable such projection. □

Now we state a consequential theorem about these meromorphic maps corresponding to linear systems. For its proof see [15](p. 255).

Theorem 6.2.8 (Bertini) Let Λ be a fixed point free linear system of dimension $r \geq 1$.

1. If $\dim(f_\Lambda(M)) > 1$ then it holds that a general member of Λ is a prime divisor. If $\dim(f_\Lambda(M)) = 1$ a general member is the sum of a fixed number (at least r) of distinct prime divisors.

2. A general member of Λ has its singularity contained in the base locus of Λ.

Chapter 7

Riemann surfaces

7.1 Riemann surfaces

Riemann surfaces are, by definition, 1-dimensional complex manifolds. The goal of this thesis is to showcase the analogous theories of divisors in Riemann surfaces and graphs. However, Riemann surfaces have some properties that we must state before we can get to divisor theories. We will only discuss compact Riemann surfaces.

Holomorphic mappings between Riemann surfaces behave especially nicely, their local properties can be described explicitly.

Proposition 7.1.1 For any non-constant map $f : M \to N$ where M, N are Riemann surfaces, and for any point $p \in M$ there are biholomorphisms $g_p : \mathbb{C} \to U_p$, $g_{f(p)} : \mathbb{C} \to U_{f(p)}$ of the neighborhoods of p and $f(p)$ to $\mathbb{C}$ such that $g_{f(p)}^{-1} \circ f \circ g_p(z) = z^k$ for some $k \geq 1$ integer. This implies that f is an open map.

Proof. Through the charts to any holomorphic function $f : M \to N$ we can identify a holomorphic function $f^* : U_p \to U_{f(p)}$ around p, such that 0 corresponds to p. This implies, that we only have to prove the theorem for f^*. Let us take the series expansion around 0.

$$f^*(z) = \sum_{i=k}^{\infty} a_i z^i = z^k \sum_{i=0}^{\infty} a_{i+k} z^i \quad (a_k \neq 0)$$

Let $h(z) := \sum_{i=0}^{\infty} a_{i+k} z^i$. As $a_k \neq 0$, there must be a neighborhood of 0 where h does not vanish. This implies that there exists a k-th root of h around zero. Therefore the function $z(h(z))^{1/k}$ takes z^k to $z^k h(z)$ such that around 0 it is biholomorphism (as $(h)^{1/k} \neq 0$ in 0). This gives a new local coordinate for U_p, and this proves our claim. □

Corollary 7.1.2 Any holomorphic function $f : M \to N$, where M, N are Riemann surfaces, is either constant or surjective.

Proof. If f is non-constant, then the image must be open, as it is an open mapping, and compact, thus closed, therefore it must be the entire N.

We refer to the maps g_p as the *local uniformizing parameters* around p. We refer to any point p where $k \geq 2$ as a *point of ramification*, and to k as the *index of ramification*. We denote the latter as e_p for any point p. The set of all of these points must be discrete as the derivative at them must be zero through any chart, and therefore if there is a limit point of them, then at that point the derivative's series expansion is zero, thus it must be zero everywhere, thus the function must be constant.

We may refer to such non-constant holomorphisms as *ramified covering maps* as aside from the points of ramification and their images these functions are covering maps. We refer to the degree of these covering maps as the degree of the ramified covering map.

Riemann surfaces are of topological dimension 2. Therefore we may study the relation of its topological properties to its complex structure.

The complex structure on a Riemann surface ensures that is directed, thus its genus is defined.

Theorem 7.1.3 (Riemann-Hurwitz formula) Let M, N be compact Riemann surfaces with genus g_M, g_N respectively. For any non-constant map $f : M \to N$ of degree d it holds that

$$2g_M - 2 = d(2g_N - 2) + \sum_p e_p$$

Proof. Any compact surface can be triangulated. Let us take a triangulation on N that contains the points of ramification. Let the number of points be s_0, edges be s_1, faces be s_2. The number of edges and faces must change by a factor of d through f if we take its pull-back. For points however, it only holds that for any point $q \in N$ the sum of the ramification indices of its pre-images is d.

Therefore

$$s_0' = ds_0 - \sum_p (e_p - 1), s_1' = ds_1, s_2' = ds_2.$$

From these results the genus can be calculated as

$$g_N = \frac{-s_0 + s_1 - s_2 - 2}{2}, g_M = \frac{-ds_0 + \sum_p (e_p - 1) + ds_1 - ds_2 - 2}{2}$$

and the result follows. □

Now we take a look at meromorphic maps. As the dimension in our case is 1, the dimension of points of indeterminacy can be at most -1, thus their set is empty. This implies that all meromorphic maps are in fact holomorphic, and furthermore, that every meromorphic function can be interpreted as a holomorphic map to $\mathbb{P}^1$.

7.2 Analytic curves

We refer to an analytic set with dimension 1 as an analytic curve.

The following definition allows us to better analyze the local properties of analytic curves in their singularities.

Definition 7.2.1 (Local uniformizing parameter) For any $C \subset \mathbb{C}^n$ analytic curve that contains 0 and is irreducible at it, we call a map $\delta(z)$ from a disc K in $\mathbb{C}$ to the curve around zero, as a *local uniformizing parameter* if:

1. $\phi(0) = 0$,
2. The image of the disc is C restricted to a neighborhood U_0 around 0,
3. ϕ is a homeomorphism,
4. ϕ is biholomorphic when restricted to $K \setminus 0$ and $C \cap U_0 \setminus 0$.

Surprisingly there exists such a parameterization in all cases. For the proof see [15](p. 275).

Proposition 7.2.2 For any $C \subset \mathbb{C}^n$ analytic curve that contains 0 and is irreducible at it, there exists a local uniformizing parameter, unique up to holomorphic isomorphisms.

The theorem of Hironaka 5.2.12 allows us to define the genus of an arbitrary curve as the genus of its resolution of singularity.
The theory of analytic curves (up to bimeromorphic equivalence), compact Riemann surfaces, and algebraic function fields in one variable are surprisingly equivalent. This fundamental result may be regarded as another example of the GAGA principle.

Theorem 7.2.3 (The trinity [15](p. 288)) The following categories are equivalent:

1. compact Riemann surfaces
2. the dual of the category of algebraic function fields in one variable
3. non-singular irreducible projective algebraic curves

Corollary 7.2.4 Any compact Riemann surface is projective algebraic.

Note that this theorem does not hold for manifolds in larger dimensions.

7.3 Meromorphic 1-forms on Riemann manifolds

Now we turn our attention to the analytic properties of Riemann manifolds. We can define line integrals on Riemann surfaces, by defining corresponding 1-forms. Many of the results from conventional complex analysis generalize to the case of Riemann manifolds. We define them the following way.

Definition 7.3.1 (Meromorphic 1-form) For every meromorphic section ω of the canonical bundle we can create a *meromorphic 1-form* as follows.

$$g_\alpha : U_\alpha \to \mathbb{C}$$

$$g_\alpha = g_\beta \frac{\mathrm{d}f_\beta}{\mathrm{d}f_\alpha}$$

Where $\mathrm{d}f_\alpha$ is the differential form of f_α. Let $\omega_\alpha = g_\alpha \mathrm{d}f_\alpha$. Therefore ω is a properly defined 1-form on $M \setminus S$ where S is the set where the section is not defined (or it has ∞ as its value on $\mathbb{P}^1$).

These forms are sometimes referred to as *Abelian differentials of the second kind*, while the first kind being the holomorphic 1-forms.
This implies that we can define for a path $u : [0,1] \to M$ the line integral $\int_u \omega(z)\mathrm{d}f(z)$ the conventional way.

We know that by 6.1.5 that every meromorphic section of K_M can be obtained from any other by multiplying with a meromorphic function, and thus this holds for the meromorphic 1-forms as well.

Now we will prove the generalization of the following well-known theorem on the projective plane.
We define the poles of a meromorphic 1-form as the points where the value of the corresponding meromorphic form is not defined. We can define any points *residue* as the integral on a small circle around the point on any chart containing the point.
Now we can state our theorem:

Theorem 7.3.2 The sum of residues on a compact Riemann surface M is zero, for any meromorphic 1-form ω.

Proof. Let us take any triangulation of M such that the poles of ω are contained in separate faces, and each triangle is contained in a chart's image (by subdividing triangles this is always achievable). Let us calculate the sum of integrals on all triangles clockwise (this is well defined as charts can't change the direction). This is clearly zero, as on every edge we integrated both ways exactly once. On the other hand, on the triangles containing the poles, the integral is exactly the residue of the point, and anywhere else, the integral must be zero, as the form is defined inside the triangle, and its image through a chart must have integral 0. □

Let us look at the local properties of residues. Let us take a meromorphic function f with a pole p with multiplicity k and a local coordinate t around it, and a holomorphic 1-form ω. Let us write ω in t. It is clear that the residue on $\mathbb{C}$ with the resulting holomorphic 1-form will be the same. Let us suppose that

$$h(t) = a_{-k}t^{-k} + a_{-k+1}t^{-k+1} \ldots + a_0 + \ldots$$

$$\omega = (w_0 + w_1 t + w_2 t^2 \ldots)dt$$

this implies that

$$Res_p(h(t)\omega) = w_0 a_{-1} + w_1 a_{-2} + \ldots + w_{k-1} a_{-k}.$$

Thus it is clear that the residue only depends on the negative coefficients when ω is holomorphic.
Let us denote $P_h(t)$ the $\sum_{i=-k}^{-1} a_{-i} t^i$.
Surprisingly a converse of 7.3.2 can now be stated as well.

Theorem 7.3.3 Let us take a set of points $p_1, p_2, \ldots, p_l$ and positive integers $k_1, k_2, \ldots, k_l$, and around every point a chart t_i. For any set of vectors

$$\underline{a_1} \in \mathbb{C}^{k_1}, \underline{a_2} \in \mathbb{C}^{k_2}, \ldots, \underline{a_l} \in \mathbb{C}^{k_l}$$

we give sufficient (and necessary) conditions to the existence of a meromorphic function that has poles of order k_i in p_i for every i, has no other poles, and in the local coordinate system around p_i, the vector of negative coefficients of its Laurent series is exactly the vector $\underline{a_i}$ that is locally $P_{h_i}(t_i)$.
Such a function exists, if and only if if for any holomorphic 1-form ω the sum of the residues of $\omega P_{h_i}(t_i)$ is zero.

We won't prove this theorem, for its proof see [13](p. 188).

Chapter 8

Chip-firing games

Chip-firing games are an important object of study of combinatorics. This thesis aims to showcase the connection of one of its notable variants to algebraic geometry, or more precisely the Riemann-Roch theorem. First we describe some famous variants and state some corresponding results.

8.1 The chip-firing game of Björner-Lovász-Shor

The following 1-player game is the chip-firing game of Björner-Lovász-Shor studied in [5]. It is sometimes referred to as the chip-firing game, as it predates the other variants. Let G be a loopless connected graph. The game is played the following way. To every vertex we assign a non-negative number representing the number of *chips* on the vertex. A move consists of the following. First, we pick a vertex with a larger number of chips than its degree. Then, we move one chip to every adjacent vertex with multiplicity (that is we move one for each edge between the two vertices) from the chosen one. The game is finished when the player can not make any moves (such states are referred to as *terminal states*). Note that the game may go on infinitely.
Now we state some of the results in [5].

Theorem 8.1.1 (Theorem 2.1 in [5]) Whether the game finishes in finite steps, is independent of the sequence of moves. Furthermore, if it finishes, then the number of steps taken, and the terminal state is also independent of the taken moves.

Theorem 8.1.2 (Theorem 3,3 in [5]) Let N be the total number of chips in the game.

1. Then, if $N > 2|E(G)| - |V(g)|$, the game is infinite.

2. For $|E(G)| \leq N \leq 2|E(G)| - |V(g)|$ there exists an initial state resulting in a finite game and there exists an initial state which results in an infinite game.

3. If $N < |E(G)|$ then the game finishes in finite moves.

Though we will mainly study another variant (Baker-Norine) we will relate the obtained results to this as well.

8.2 The chip-firing game of Biggs

In the variant of Biggs in [4], there is a single vertex designated as the *bank*. When we take our move we cannot choose the bank even if it has sufficient chips. If there is no other vertex with more chips than neighbours we move chips from the bank to every one of its neighbours may resulting in a negative number of chips left at the bank. The set of states where only the bank can move are called *stable states*.

Interestingly the stable states form an abelian group in the following way. For the product take the state we get as the sum of two stable states a, b, and take the (uniquely determined) stable c state that we can reach from this, then $a * b = c$.

It is also proven that the order of this group is the number of spanning trees of the graph (surprisingly such a result is proven for certain other structures in other variants). It is also related to other groups corresponding to the graph.

8.3 The chip-firing game of Baker-Norine

This is the variant we will mainly study in this thesis. We will refer to this from now as the chip-firing game and refer to other variants by name. This variant was studied in [3] by Baker and Norine, and in this thesis we will compare their results using notions from algebraic geometry in a graph theoretic setting to the originals.

In this variant we allow negative values of chips. Our goal is to reach a state where the number of chips is positive on every vertex. The move changes in the following way. We may choose any vertex, and choose to either move one chip to every adjacent vertex with multiplicity from it, or move one chip from every adjacent vertex with multiplicity to it.

It is clear that the initial state has to contain a non-negative number of chips in total for the game to be winnable.

We will give the proof for the following result of Baker and Norine following [3]. This theorem can be looked at as an analogue of 8.1.2.

Theorem 8.3.1 Let N be the total number of chips in the game.

1. If $N \geq |E(G)| - |V(G)| + 1$ then the game is winnable.
2. For $N < |E(G)| - |V(G)| + 1$ there exists an initial construction such that the game is not winnable.

Chapter 9

Divisor theories on Riemann surfaces and graphs

In this chapter all graphs will be undirected loopless finite graphs (multi-edges are allowed), Riemann surfaces will refer to compact Riemann surfaces (these will be projective algebraic as well by 7.2.4).

Before introducing divisors to graphs we must define the analogue of some of the notions we defined on complex manifolds in graph theory.
We formally define *holomorphic functions* as functions from $V(G)$ to $\mathbb{N}$, and *meromorphic functions* as functions from $V(G)$ to $\mathbb{Z}$. Though we will refer to them as holomorphic and meromorphic functions but they correspond more clearly to $\log(|f|)$ for a function $f : M \to \mathbb{C}$. This correspondence indicates a firm connection to tropical algebraic geometry. We define the abelian group $\mathbb{M}(G)$ of meromorphic functions with the operation of pointwise addition (this corresponds to $\mathbb{C}(M)^{\times}$).
We will also use the definition of the genus of a graph G. Let $g = |E(G)|-|V(G)|+1$. This also agrees with the genus of the graph as a CW complex.
The introduction of graph divisors in this chapter loosely follows [3]. The discussion of divisor theory loosely follows [15] Chapter 4.

9.1 Divisors

Divisors on Riemann surfaces

Divisors on Riemann surfaces have a significantly simpler structure than in the case of a general complex manifold. As the hypersurfaces in Riemann surfaces are actually the points of the surface, every divisor is the formal sum of points of the surface.

Any non-constant meromorphic function is a holomorphic mapping to $\mathbb{P}^1$ restricted to one affine space. Therefore it is also a covering map apart from its ramification points. Note that if a ramification point is a root, then the multiplicity of the root is exactly the ramification index. This implies that the number of points which go to a point for a meromorphic function is exactly the degree of the function. This

implies that the same number of points go to zero, as to infinity in $\mathbb{P}^1$, therefore the degree of any meromorphic functions divisor is zero.
By 7.2.4, and 6.1.5, it is clear that $\text{Pic}(M) = \frac{\text{Div}(M)}{\text{Prin}(M)}$.

Divisors on graphs

Divisors in graphs are by definition, formal integer sums of vertices, that is:

$$\sum_{i=1}^{n} a_i v_i = D.$$

A divisor is called *effective*, if it has no negative coefficients. Their group with the operation of formal addition is denoted as $\text{Div}(G)$. Its degree is the sum of the coefficients. We denote this value as $\deg(D)$ for any divisor D. We denote the sum of positive coefficients as $\deg^+(D)$.
We say that a meromorphic function $f : V(G) \to \mathbb{Z}$ *induces* the following divisor.

$$\delta(f) = D = \sum_{v \in V(G)} v(\deg(v) f(v) - \sum_{u \in N(v)} f(u))$$

We refer to the operator $\delta : \mathbb{M}(G) \to \text{Div}(G)$ as the laplacian operator. Note that if we represent the elements of $\mathbb{M}(G)$ and $\text{Div}(G)$ as vectors, then the operator acts as a multiplication with the laplacian matrix of G. In the theory of Riemann surfaces there exists a laplacian operator not discussed in this thesis, that recovers the divisor corresponding to a meromorphic function f from $\log(f)$. Note that similarly to the case of Riemann manifolds, here it also holds that the divisors corresponding to any meromorphic function have degree 0.

The divisors corresponding to meromorphic functions $\delta(\mathbb{M}(G))$ are referred to as *principal divisors*. We will use the notations $(f) = \delta(f)$. Their group is denoted as $\text{Prin}(G)$. Following our observations on Riemann surfaces we denote $\frac{\text{Div}(G)}{\text{Prin}(G)}$ as $\text{Pic}(G)$ and refer to it as the *Picard group* of a graph G. The group $\text{Div}^0(G)$ will denote the elements $\text{Div}(G)$ with degree 0. We will call $\frac{D^0(G)}{\text{Pic}(G)} \leq \frac{D(G)}{\text{Pic}(G)}$ as $\text{Jac}(G)$. The order of this group is exactly the number of spanning trees of G.

Proposition 9.1.1 The order of $\text{Jac}(G)$ is the number of spanning trees of G.

Proof. By the matrix-tree theorem, if we take a $(n-1) \times (n-1)$ minor of the laplacian matrix, then its determinant is exactly the number of spanning trees in G. Remember that the effect of the laplacian operator corresponds to a multiplication by the laplacian matrix. In $\text{Div}^0(G)$ all vectors have sum 0, therefore it is sufficient to study only the first $n-1$ of its coordinates. This implies that $\text{Div}^0(G) = \mathbb{Z}^{n-1}$. We factor this by the image of $\Delta(\mathbb{M}(G))$. Notice that the subgroup generated by $\Delta(\mathbb{M}(G))$ have a basis of $n-1$ elements, namely the first $n-1$ rows of the laplacian matrix (as the group is generated by the rows, and it has rank $n-1$ and the n-th row is the integer linear combination of the rest). It is easy to see that the factor's order is exactly the determinant of the matrix of basis-es, as every coset will have exactly one representative in the parallelepiped generated by the basis. As we disregard the last coordinate of $\mathrm{Div}(G)$ and the basis only contains the first $n-1$ rows of the laplacian matrix this will be exactly the determinant of the first $(n-1)\times(n-1)$ minor of the laplacian matrix. □

Remark 9.1.2 Let us talk about the connection of these definitions to the chip-firing game.
In the definition of linear equivalence the transformation of a divisor D by adding (f) corresponds to a series of moves, that takes a state, where every vertex have as many chips on it as its coefficient is in D to one that has as many chips as the corresponding coefficient in $D+(f)$. To check this, it is enough to inspect the case where f is 0, on every vertex but one (let it be v), where it is ± 1. In this case $D+(f)$ is exactly the state D after a move where we chose v (the type of move depends on the sign of f in v).

The following definition is closely related to the stable states of the Biggs variant of the chip firing game.

Definition 9.1.3 (Reduced divisor) We say that D is a reduced divisor in v_0, if

1. every vertex aside v_0 has a positive amount of chips on it
2. for every subset $A \subset V(G)\setminus\{v_0\}$ there exists a vertex $v \in A$ such that it has less chips on it, then edges going from v to vertices outside of A.

Proposition 9.1.4 For every v_0 vertex and for every divisor D there exists a unique reduced divisor D_0 in v_0, such that D and D_0 are linearly equivalent.

Proof. For any vertex $v \in V(G)$, let $d(V)$ be the shortest path between v_0 and v. Let $d = \max_{v\in V(G)} d(v)$ and let S_k be the set of vertices that's distance to v_0 is exactly k.
Let us define the following two vectors.

$$\mu_1(D) = \left(\sum_{v\in S_d, D(v)<0} D(v), \sum_{v\in S_{d-1}, D(v)<0} D(v), \dots, \sum_{v\in S_1, D(v)<0} D(v)\right)$$

$$\mu_2(D) = \left(\sum_{v\in S_0} D(v), \sum_{v\in S_1} D(v), \dots, \sum_{v\in S_d} D(v)\right)$$

Let us define the ordering on these vectors as the lexicographical order, and let us change to a linearly equivalent D such that:

1. $\mu_1(D) = \max_{D\sim D'}(\mu_1(D'))$
2. $\mu_2(D) = \max_{D\sim D', \mu_1(D)=\mu_1(D')}(\mu_2(D'))$

This D is reduced in v_0. Let us prove (1), let us suppose that there is a vertex v with a negative number of chips. Then there must exist an adjacent vertex v' that is closer to v_0. If we take the divisor $D - \delta(v')$ (that is we move one chip away from v' to each adjacent vertex (with multiplicity)), then the sum of number of chips with $d(v)$ distance on vertices with a negative number of chips must have increased as $d(v') < d(v)$, and the number of chips closer the $d(v)$ must remain unchanged. This implies that $\mu_1(D - \delta(v')) < \mu_1(D)$ and we have reached a contradiction.

Now we turn to prove property (2). Let us suppose that there is a set A for which the statement does not hold, that is, the number of chips on every vertex is bigger than its number of neighbours outside of A. Let us inspect the divisor $D - \sum_{a \in A} \delta(a)$. Every vertex had a positive number of chips in D. From every vertex v we have subtracted $\deg(v)$ chips, but we also moved one to it from every adjacent vertex (with multiplicity) in A, thus we have only subtracted the number of neighbours outside of A, thus the resulting number must stay non-negative. This implies that $\mu_1(D) = \mu_1(D - \sum_{a \in A} \delta(a))$. However, there must exist a vertex v_A that is adjacent to A, but is closer to v_0, than any $a \in A$. This implies that the number of total chips with distance $d(v_A)$ must increase, and the number of chips with less distance must not decrease, thus $\mu_2(D - \sum_{a \in A} \delta(a)) > \mu_2(D)$, thus we have reached a contradiction.

Now, it is only left to prove that this is the unique corresponding reduced divisor in v_0. Let us suppose the contrary, and take two reduced divisors D, D' in v_0, such that $D \sim D'$. There exists a $f \in \mathbb{M}(G)$ such that $D' = (f) + D$. We may assume by symmetry that there exists such D, D', f that there is a vertex v with $f(v) > f(v_0)$. Let A be the set of maximal points of f. It is clear that $v_0 \notin A$. For every vertex in $v \in A$ we take away $(f(v) - f(w))$ chips for every adjacent w. This implies that we take away at least as many chips, as the number of neighbours of v outside of A. However, the resulting divisor is still effective, thus property 2 does not hold in A. We have reached a contradiction. □

9.2 Linear systems

Linear systems in Riemann surfaces

Theorem 9.2.1 The dimension of $L(D)$ for any divisor D on a compact Riemann surface is finite, furthermore it is at most $\deg(D) + 1$, if D is effective.

Proof. For any K, D divisors, if $K \leq D$, then $\dim(L(K)) \leq \dim(L(D))$, therefore it is sufficient to prove the second part of the statement.
Let us take any element $g \in L(D)$. This is a holomorphic section of $|D|$, therefore around any point p_j there exists a chart, such that through it, it is a conventional meromorphic function in a small neighborhood. This implies that we can take its Laurent series. It can only have poles in the base points of D, therefore we can take the Laurent series around each point p_j, and take its coefficients. As at a point with multiplicity d_j there can be a pole of degree at most d_j, therefore the Laurent series

is of the form:

$$g_j(z) = a_{j,-d_j} z^{-d_j} + a_{j,-d_j+1} z^{-d_j+1} + \ldots + a_{j,0} + a_{j,1} z + \ldots$$

Let us take the linear map that takes $g \in L(D)$ to $\mathbb{C}^{d_1} \times \mathbb{C}^{d_2} \times \ldots \times \mathbb{C}^{d_n}$, such that it takes g to the vector with $(a_{j,-d_j}, a_{j,-d_j+1}, \ldots, a_{j,-1})$ in the j-th product space. Note that a g that is in the kernel is holomorphic in all p_j, therefore at every point of the surface. As our surface is compact, this implies that the kernel contains only constant functions, and as such the dimension of $L(D) \leq 1 + \sum d_i = 1 + \deg(D)$.□

As all meromorphic function have degree 0, this implies that any divisor in $|D|$ has the same degree. This means that we can define the degree of a linear system as the degree of its elements.

The dimension of a linear system reveals a lot about its structure. The next theorem is an example of this.Let

$$\Lambda - E = \{D \in \Lambda : D \geq E\}.$$

Theorem 9.2.2 Let Δ be a linear system of degree d and dimension r. In this case the following hold:

1. A point p is fixed exactly if $\dim(\Delta - p) = r$
2. Δ is fixed point free, and f_Δ is injective exactly, if for all points $p \neq q$, the following holds $\dim(\Delta - p - q) = r - 2$.

Proof. The proof of the three statements rely on the following. For any point p it holds that for any holomorphic section in the affine space behind $\Lambda - p$ the value at p is zero. This follows from the fact that any divisor that is bigger than the divisor of p has a root in p.

When we take $\Lambda - p$ instead of Λ the dimension can only stay the same, if all elements of Λ are 0 in p (this proves the first statement). If the latter is not the case, then there is at least one element of every basis that is not zero at p in the affine space. It is clear that the elements that are 0 will form a hyperplane, thus in that case the dimension is $r - 1$.

If we take another point and change from $\Delta - p$ to $\Delta - p - q$ the only sections remaining are the ones that are zero in both p, and q. This again can only change the dimension by one or zero, and it is clear that if q is a fixed point, the latter will be the case. This implies that in the second case there can be no fixed points.

If $\dim(\Delta - p) = r - 1$ and $\dim(\Delta - p - q) = r - 1$, that implies that for any holomorphic section if it is zero in p it will be zero in q. If this is not the case, that means that in any basis in the affine space there must be a holomorphic section that does not agree in the points, thus p and q are not taken to the same point by f_Δ. This implies the second statement.

Now we turn our attention to the canonical bundle. Let us calculate the degree of the canonical line bundle, or in other words, the degree of its meromorphic sections (this is well-defined according to 6.1.5).
To any meromorphic function we can assign the 1-form $\mathrm{d}g$ defined as the derivative of a meromorphic function g on any chart. This is a well defined holomorphic section of K_M, as $\frac{\mathrm{d}g}{\mathrm{d}f_\alpha} = \frac{\mathrm{d}g}{\mathrm{d}f_\beta}\frac{df_\beta}{df_\alpha}$.

Lemma 9.2.3 The degree of K_M is $2g-2$.

Proof. Let us take a meromorphic function g with $n = \deg(g)$ poles of order one. This can always be done, as there are finitely many branch points of any meromorphic function $g : M \to \mathbb{P}^1$, thus there exists a projective transformation of $\mathbb{P}^1$ such that the transformed function takes only simple points to infinity. Let us inspect the meromorphic section $\mathrm{d}g$ of K_M. It has n pole of order two, as the derivative of a function with an order one pole will have an order two pole. It will have roots in exactly the places where g branches, as there its series expansion will look like

$$g_p(z) = a_0 + a_k z^k + a_{k+1} z^{k+1} + \dots$$

It is easy to see that the order of the root is exactly $e_p - 1$ in the derivative. This implies by the Riemann-Hurwitz formula applied to g (due to $\mathbb{P}^1$ having genus 0) that

$$\deg(\mathrm{d}g) = \sum_p (e_p - 1) - 2n = 2g - 2.$$

The following theorem is an important tool in the theory of Riemann surfaces, but its proof is not stated here (see [2]).

Theorem 9.2.4 (Riemann's existence theorem) For all p, q points in a compact Riemann surface there is a meromorphic function $f : M \to \mathbb{C}$ such that $f(p) \neq f(q)$.

Linear systems in Graphs

We define *linear equivalence* between divisors D, D' as in the case of manifolds, that is two are equal, if there is a meromorphic function f, such that $D + (f) = D'$.
We define the complete linear system $|D|$ as the effective divisors linearly equivalent to D.
We define the canonical divisor K_G on a graph as $\sum_{v \in V(G)} (\deg(v) - 2)v$. Clearly, $\deg(K_G) = 2|E(G)| - |V(G)| = 2g - 2$.

The dimension of complete linear systems are important notions of the theory of divisors. Regretfully, $|D|$ is not equipped with a definition of dimension that would behave analogously to the original. We define the correct analogue as follows.
Let $r(D)$ be -1 if $|D| = \emptyset$. In any other case let $r(D)$ be the highest integer k such that for any effective divisor E of degree k it holds that $|D - E| \neq 0$.
Analogous to 6.2.3 the following holds for the dimensions of complete linear systems.

Proposition 9.2.5 If D, D' are divisors, then it holds that

$$r(D) + r(D') \leq r(D + D').$$

Proof. Let us suppose that the statement does not hold. There exists an effective divisor E of degree $r(D) + r(D')$ such that $|D + D' - E| = \emptyset$. Let us write E as the sum of two effective divisors, with degree $r(D)$ and $r(D')$: $E = E_1 + E_2$. As $r(D) = \deg(E_1)$, it is clear that $|D - E_1|$ is not empty, thus there is a meromorphic function such that $D - E_1 + (f_1)$ is effective. By the same reasoning there must be f_2 such that $D' - E_2 + (f_2)$ is effective. However, this implies that

$$D - E_1 + (f_1) + D' - E_2 + (f_2) = D + D' + E + (f_1 + f_2)$$

is effective, thus $|D + D' - E|$ is not empty, and we have reached a contradiction.□

Remark 9.2.6 Note that the question of whether an initial state is winnable is equivalent to whether $r(D)$, for the divisor corresponding to the initial state in question, is -1. Furthermore, the definition of $r(D)$ can be restated as the maximal number k, such that after taking any k chips from the graph, the game is still winnable.

We define the following divisors.
Let $<_p$ be a total order on $V(G)$. We define ω_p as

$$\omega_p = \sum_{vw \in E(G), w<v} w - \sum_{v \in V(G)} v$$

Lemma 9.2.7 For any ordering, it holds that:

1. $\deg(\omega_p) = g - 1$,
2. $|\omega_p| = \emptyset$.

Proof. The first part of the statement is trivial, as we add exactly $|E(G)|$ element in the first sum, and subtract $V(G)$ in the second.
For the proof of the statement, let us take a meromorphic function f. Let us take the minimal element u of $V(G)$ according to $<_p$, where f achieves its maximal value. Now let us study the divisor $D - (f)$. The coefficient of u there must be negative, as in

$$|uv \in E(G), v < u| - 1 - \sum_{uw \in E(G)} (f(u) - f(w))$$

it holds that as $f(u)$ is maximal, that if $w < u$, then $f(u) - f(w) \geq 1$, and if $u < w$, then $f(u) - f(w) \geq 1$, that implies, that

$$\sum_{uw \in E(G)} (f(u) - f(w)) > |\{uv \in E(G), v < u\}|,$$

and this implies the statement. □

Now, we can use these special divisors.

Lemma 9.2.8 For every divisor D, there exists a $<_p$ such that exactly one of $|D|$ and $|\omega_D - D|$ is empty.

Proof. Let us choose a vertex v_0. We have proved that there is a unique reduced divisor D' in v_0, that is equivalent to D. Let us change to studying this D' from this point on. Now we define an ordering recursively. The first vertex will be v_0, and if we have defined $v_0, v_1, \ldots, v_k$, then we define v_{k+1} as follows. Let $A_k = V(G) \setminus \{v_0, v_1, \ldots, v_k\}$. Let v_k be a vertex, such that the number of chips on v_k in D is less, then its number of adjacent vertices (with multiplicity) outside of A_k. To this order there exists a divisor ω constructed as above. For every vertex other than v_0 it must hold that $D(v_k)$ is at most the number of its neighbours outside of A_{k-1} minus one, which is exactly $\omega(v_k)$. This implies that $D(v_k) \leq \omega(v_k)$. If $D(v_0) > 0$ then D is effective, therefore $|D| \neq \emptyset$. If $D(v_0) < 0$, then $D \leq \omega$, therefore $\omega - D$ is effective, and $|\omega - D| \neq \emptyset$.
Now it is only left to prove that they can't both be non-empty. If they were, then by 9.2.5 it follows, that $|\omega| \neq \emptyset$, but we have proved that this is not the case. Thus we have reached a contradiction. ☐

Corollary 9.2.9 If ω is a divisor, and $\deg(\omega) = g - 1$, and $|\omega| = \emptyset$, then there exists a $<_p$ such that $\omega \sim \omega_p$

Proof. There exists an ω_p such that $|\omega - \omega_p| \neq \emptyset$, this implies, that there is a meromorphic function f such that $\omega - \omega_p + (f) \geq 0$, but as the degree is 0 as well, this implies that $\omega - \omega_p + (f) = 0$, which implies our statement. ☐

With these properties we can get an alternative characterization of $r(D)$.

Lemma 9.2.10 For any divisor D

$$r(D) = \left(\min_{D \sim D', <_p} \deg^+(D' - \omega_p) \right) - 1$$

Proof. Let us suppose that the left side is smaller than the right. This implies, that there exists an effective divisor E with the right side as its degree such that $|D - E| = \emptyset$. This implies by the previous lemma, that there exists a $<_q$ total ordering such that $|D - E + \omega_q| \neq \emptyset$, that is $D - E + \omega_q \sim E'$ where E' is an effective divisor. Thus there is a $D' \sim D$ such that $D' - \omega_q = E - E'$.

$$\deg^+(D' - \omega_q) - 1 = \deg^+(E - E') - 1 = \deg(E) - 1$$

As we defined $\deg(E)$ to be exactly the right hand side, and this is an upper bound for the right hand side, we have reached a contradiction.
Let us suppose that the right side is smaller than the left. This implies that there exists a divisor D and ω_p such that $\deg^+(D - \omega_p) - 1 < r(D)$. Let us write $D - \omega_p = E - E'$ where E, E' are effective divisors. The previous inequality implies that $\deg(E) < r(D) + 1$. As $D - \omega_p = E - E'$, it follows that $D - E = \omega_p - E'$. However $\omega_p - E'$ can't be linearly equivalent to any effective divisor by 9.2.7. This implies, that $|D - E| = \emptyset$, thus $r(D) < \deg(E) < r(D) + 1$ which is a contradiction. ☐

Chapter 10

Riemann-Roch theorem

We have finally reached our main theorem. This theorem has wide ranging consequences in mathematics. Even the simpler version in graph theory virtually solves this version of the chip-firing games, and has powerful implications for the others.
In the case of Riemann surfaces let $r(D) = \dim(|D|)$, in the case of graphs we defined it in 9.2.

Theorem 10.0.1 (Riemann-Roch) Let K be a canonical divisor. Then the following holds for any divisor D.

$$r(D) + r(K - D) = \deg(D) - g + 1$$

10.1 Riemann-Roch for Riemann surfaces

In this section we prove the Riemann-Roch theorem for surfaces following [21].

Proof. Let $\deg(D) = d$.
First we prove the statement when $\deg(D) \geq 0$.
Let the points with positive coefficients be $p_1, p_2, \ldots, p_n \in M$, and let the coefficient of p_i be k_i. Let us take the map $A_i : L(D) \to \mathbb{C}_i^k$ that takes any meromorphic function g in $L(D)$ to the vector of its coefficients in p_i corresponding to the negative exponents, that is if we take the series expansion of g in p_i:

$$a_{-k_i} z^{-k_i} + a_{-k_i+1} z^{-k_i+1} \ldots + a_0 + \ldots$$

then $A_i(g) = (a_{-k_i}, a_{-k_i+1}, \ldots, a_{-1})$. Let the vector of all $A_i(g)$ be $A(g)$. This is clearly a linear function from $L(D)$ to $\mathbb{C}^d$.
The kernel of this function consists only of holomorphic functions, thus only constants. This implies that $\dim(\ker(A)) = 1$. It is clear that

$$\dim(\Gamma(D)) = \dim(\ker(A)) + \dim(\text{im}(A)) = 1 + \dim(\text{im}(A)).$$

Let us remind ourselves how the residue is calculated locally for holomorphic 1-forms. Let $\omega \in \Gamma(K)$ be a holomorphic 1-form. For a pole p of g let us take the series expansion of the function ω_α and g_α through a local uniformizing map t.

$$\omega_\alpha = (w_0 + w_1 t^1 + \dots)dt$$

$$g_\alpha = a_{-k_i}t^{-k_i} + a_{-k_i+1}t^{-k_i+1}\dots a_0 + \dots$$

It is clear that the -1-th coefficient of the product $\omega_\alpha g_\alpha$ is the residue. That is

$$\mathrm{Res}_p(g\omega) = a_{-k_i}w_{k_i-1} + a_{-k_i+1}w_{k_i-2} + \dots a_{-1}w_0.$$

This implies as for any $\omega \in \Gamma(K)$ the sum residues of g according to ω is zero, if we take a basis w_i, than $\mathrm{im}(A)$ is contained in the kernel of the function $B : \mathbb{C}^d \to \mathbb{C}^{\dim(\Gamma(K))}$ that calculates from the negative coefficients at the poles, the vector of residues corresponding to ω_i.
This implies that $\dim(\mathrm{im}\, A) \le \dim(\ker(B)) = d - \dim(\mathrm{im}(B))$.
However, by 7.3.3 we also know that any vector in $\ker(B)$ has also a corresponding meromorphic function, thus $\mathrm{im}(A) = \ker(B)$.
Now we calculate $\dim(\mathrm{im}(B))$. By the way the residue is calculated, it is clear that it only depends on the coefficients $w_{k_i-1}, w_{k_i-2}, \dots, w_0$ at all points. For any $\omega \in \Gamma(K)$ we assign the vector of these coefficients in $\mathbb{C}^d$, let this function's name be $C : \Gamma(K) \to \mathbb{C}^d$. The dimension of $\dim(\mathrm{im}(B))$ is the number of linearly independent such vectors in $\Gamma(K)$ (here we use the fact that the transpose of a matrix will have the same rank). For that it is enough to calculate the $\ker(C)$, as $\dim(\mathrm{im}(B)) = \dim(\mathrm{im}(C)) = \dim(\Gamma(K)) - \dim(\ker(C))$. The linear system $\ker(C)$ consists precisely of sections that vanish on each of the points with order k_i, that is they are exactly the system $\Gamma(K) - D$, thus they are in bijection to $\Gamma(K - D)$. This implies that $\dim(\ker(C)) = \dim(\Gamma(K - D))$.

$$\dim(\Gamma(D)) = 1 + \dim(\mathrm{im}(A)) = \dim(\ker(B)) + 1 = d - \dim(\mathrm{im}(B)) + 1$$

$$= d - (\dim(\Gamma(K)) - \dim(\ker(C))) + 1 = d - \dim(\Gamma(K)) + \dim(\Gamma(K - D)) + 1$$

$$\dim(\Gamma(D)) = d - \dim(\Gamma(K)) + \dim(\Gamma(K - D)) + 1$$

As $r(D) = \dim(\Gamma(D)) - 1$, the following holds

$$r(D) = \deg(D) - \dim(\Gamma(K)) + r(K - D) + 1.$$

and the statement is proven for effective D.

Now we will prove that

$$r(D) - r(K - D) \ge d - \dim(\Gamma(K)) + 1$$

for all D. For that it is enough to prove that

$$r(D - a) - r(K - D + a) \ge r(D) - r(K - D) - 1$$

for any prime divisor a which is not contained with a positive coefficient in D. It is clear, that $r(D) \ge r(D-a) \ge r(D)-1$ and $r(K-D) \le r(K-D+a) \le r(K-D)+1$ by the proof of 9.2.2. This implies, that if we suppose the contrary, then there is a

meromorphic function $f \in L(D) \setminus L(D-a)$ and $\omega \in \Gamma(K-D+a) \setminus \Gamma(K-D)$. Let us inspect $\text{Res}_{a,\omega} f$. First, it is clear, that as $D > (\omega) \geq D-a$ and $-D \leq (f) < -D+a$ it must hold, that the coefficient of a in (f) is $-D(a)$ and in (ω) it is $D(a)-1$, thus it is a pole of order one. But there can be no other pole of $f\omega$, as $(f\omega) \geq -(a)$. Therefore, the residues' sum is equal to the residue of a, that is its zero. However, that would imply that it is holomorphic at a, and that is a contradiction.
Now we have that:

$$r(D) - r(K-D) \geq d - \dim(\Gamma(K)) + 1$$

Now let us use this for $K - D$:

$$r(K-D) - r(D) \geq 2g - 2 - d - \dim(\Gamma(K)) + 1$$

$$r(D) - r(K-D) \leq \dim(\Gamma(K)) - 2g + d - 1$$

We will now prove that $\dim(\Gamma(K)) = g$.

$$r(K) \geq \deg(K) - \dim(\Gamma(K)) + r(K-K) + 1$$

by 9.2.3 it holds that:

$$2\dim(\Gamma(K)) - 1 \geq 2g - 2 + 1$$

$$\dim(\Gamma(K)) \geq g$$

If $g = 0$, then $\deg(K) = -2$, thus $\dim(\Gamma(K)) = 0$, and we are finished. In all other cases it is clear that there is a canonical divisor K' that is effective. Now we can use the positive case for the K':

$$r(K') = \deg(K') - \dim(\Gamma(K')) + r(K'-K') + 1$$

by 9.2.3 it holds that:

$$2\dim(\Gamma(K')) - 1 = 2g - 2 + 1$$

$$\dim(\Gamma(K')) = g$$

This implies the statement:

$$r(D) - r(K-D) \geq d - \dim(\Gamma(K)) + 1$$

$$r(D) - r(K-D) \geq d - g + 1$$

$$r(D) - r(K-D) \leq \dim(\Gamma(K)) - 2g + d - 1$$

$$r(D) - r(K-D) \leq -g + d - 1$$

$$r(D) - r(K-D) = d - g - 1.$$

Now we prove some easy consequences of this important theorem following [9](p. 296).

Corollary 10.1.1 Let us inspect $r(nD)$ where n is a large integer. It is clear that if $\deg(D) < 0$, then $r(nD) = -1$. If $\deg(D) = 0$, then clearly, the result can be 0 or -1 depending on if $D \sim 0$. However, in the case when $\deg(D) > 0$, then for sufficiently large n it holds that $\deg(nD) > \deg(K)$, thus $r(K - D) = -1$ and thus

$$r(D) = n\deg(D) - g.$$

Corollary 10.1.2 We call a curve elliptic, if $g = 1$. On an elliptic curve, it must hold that $\deg(K) = 0$. Furthermore $r(K) = 1$, therefore it must hold that $K \sim 0$. Furthermore, we can establish a group structure on the points of such a curve. Let $\mathrm{Div}_0(S)$ be the group of divisors with zero degree. Let us fix a point p_0. We take the map $w : p \mapsto (p) - (p_0) \in \mathrm{Div}_0(S)$. We claim that w is a bijection. Using the Riemann-Roch theorem we can prove this. Let $D \in \mathrm{Div}_0(S)$ be a divisor with degree zero, we want to prove that there exists a unique point p such that $(p) - (p_0) \sim D$. Let us take $(D + p_0)$.

$$r(D+p_0) - r(K-D-p_0) = r(D+p_0) - r(-D-p_0) = r(D+p_0) - (-1) = 1-1+1$$

$$r(D + p_0) = 0$$

Thus there is exactly one point p such that $D + p_0 \sim p$, thus $D \sim p - p_0$.

10.2 Riemann-Roch for graphs

Now we prove the Riemann-Roch theorem for graphs following [3]. Note that their proof is more general, giving equivalent conditions for Riemann-Roch type theorems, but here we will only state it for the case of graphs.

Proof. For any ω_p it must hold, that $\overline{\omega_p} := K - \omega_p$ has dimension $g-1$, and $|\overline{\omega_p}| = \emptyset$ by 9.2.7.
Now it is clear that:

$$\omega_p - D = K - D - (K - \omega_p) = K - D - \overline{\omega_p}.$$

This implies that:

$$\deg^+(D - \omega_p) - \deg^+((K - D) - \overline{\omega_p}) = \deg^+(D - \omega_p) - \deg^+(D - \omega_p) =$$

$$\deg(D - \omega_p) = \deg(D) + 1 - g$$

This means that $\deg^+(D - \omega_p) - \deg^+((K - D) - \overline{\omega_p})$ is fixed for any D and ω_p, therefore $\deg^+(D - \omega_p)$ will be minimal exactly when $\deg^+((K - D) - \overline{\omega_p})$ is. This implies, that

$$\left(\min_{D\sim D', <_p} \deg^+(D' - \omega_p)\right) - \left(\min_{(K-D)\sim D', <_p} \deg^+(D' - \omega_p)\right) = \deg(D) + 1 - g$$

$$r(D) - r(K - D) = \deg(D) + 1 - g$$

Thus the statement is true. ∎

Remark 10.2.1 This proof could be applied to the theory of Riemann surfaces, if we could prove that for any D divisor there exists a ω such that $\dim(\omega) = g - 1$ and $|\omega| = \emptyset$, and exactly one of $|D|, |\omega - D|$ is empty.

Now with this, we can prove the following powerful statement about the winnable initial states of the chip firing game.

Theorem 10.2.2 Let N be the total number of chips in the game.

1. If $N \geq |E(G)| - |V(G)| + 1$ then the game is winnable.
2. For $N < |E(G)| - |V(G)| + 1$ there exists an initial state such that the game is not winnable.

Proof. Due to 9.2.6, it is clear that it is sufficient to prove that if $\deg(D) \geq g$, then $r(D) \neq -1$, and for any $k < g$ there is a divisor D with $\deg(D) = k$, such that $r(D) = -1$.
In the first case, let us apply the Riemann-Roch theorem for graphs:

$$r(D) - r(K - d) = \deg(D) + 1 - g$$

As $\deg(D) \geq g$ and $r(K - D) \geq -1$, it follows that $r(D) \geq 0$.
In the latter case let us take a ω_p. This by 9.2.7 settles the case for $g-1$, and for any lower degree, we can subtract any effective divisor from ω_p, to get such a divisor. □

Now we present the relation with the presented variant (Baker-Norine) of the chip firing game, and the Björner-Lovász-Shor version. For that we will need the following lemma.

Lemma 10.2.3 If there is a winnable sequence of moves for an initial configuration D, then there exists one, where every move is the following: we choose a vertex with a negative number of chips, and move one chip from every adjacent vertex (with multiplicity) to it. We will refer to this way of moving as a *gather* (one vertex can only gather, if it has a negative amount of chips).

Proof. By 9.1.2 we know that $D + (f) = E$ for some meromorphic function f and effective divisor E. Let $D' = D + (f')$ be chosen so that it can be achieved from D with gatherings, such that $f' \leq f$ and $\sum_{v \in V(G)} f'(v)$ is maximal among the possible D'. We will prove that D' is effective (this would prove our claim).
Note that $f'(v) < f(v)$ implies that $E'(v) \geq 0$, as otherwise by increasing $f'(v)$ by one we would have a D' with a higher sum in f' (by increasing $f'(v)$ by one we would precisely do a gathering on the vertex v). If $f(v) = f'(v)$ it must hold that $E'(v) \geq E(v) \geq 0$ as $f(u) \leq f'(u)$ for all neighbours. This implies that E' is effective. □

Now we can relate the two versions of the game.
Let us take an initial state D and define

$$D^* := \sum_{v \in V(G)} (\deg(v) - 1)v - D.$$

Proposition 10.2.4 A game with initial state D is winnable in the Baker-Norine variant exactly, when D^* is winnable (there is a finite sequence of moves, such that no more moves can be made) in the Björner-Lovász-Shor variant.

Proof. First, note that $D(v) \geq 0$ exactly, if $D^*(v) \leq d - 1$. Note that this implies, that for any state D it holds that one can make a move in D^* in the Björner-Lovász-Shor variant on vertex v exactly, if $D(v)$ is negative, thus one can gather with vertex v. Furthermore, if in a state D we gather with a vertex v to get to a state D', then by making a move in the Björner-Lovász-Shor variant with v in state D^* we get to $(D')^*$. This implies that for any sequence of gatherings from D we can assign a sequence of moves in the Björner-Lovász-Shor variant from D^*, such that the resulting states are D' and $(D')^*$. Lastly, observe that a winning state D in the Norine-Baker variant is exactly a state where where every vertex has a non-negative number of chips, whilst in the Björner-Lovász-Shor variant the winning states are where every vertex has at most $\deg(v) - 1$ chips, and every such state is in bijection to a winning state in the Norine-Baker variant such that the two states are D^* and D. This implies that if one variant is winnable, then we can give a winning sequence of moves through this bijection, thus the statement is true. □

This property along with 10.2.2 gives a proof for the following theorem of [5].

Theorem 10.2.5 (Theorem 3,3 in [5]) Let N be the total number of chips in the game.

1. Then, if $N > 2|E(G)| - |V(g)|$ the game is infinite.
2. For $|E(G)| \leq N \leq 2|E(G)| - |V(g)|$ there exists an initial state resulting in a finite game and there exists an initial state which results in an infinite game
3. If $N < |E(G)|$ then the game finishes in finite moves.

Proof. The previous bijection will change the degree of a state of a state D^* as

$$\deg(D) = 2|E(G)| - |V(G)| - \deg(D^*).$$

We know by 10.2.2, that if

$$\deg(D) \geq |E(G)| - |V(G)| + 1$$

then the game is winnable in the Baker-Norine variant. This implies, that if

$$\deg(D^*) \leq |E(G)| - 1,$$

then

$$\deg(D) \geq |E(G)| - |V(G)| + 1,$$

thus the first case is proven.

In the second case

$$|E(G)| \leq \deg(D^*) \leq 2|E(G)| - |V(g)|,$$

thus

$$0 \leq \deg(D) \leq |E(G)| - |V(G)|,$$

thus there must be both winnable (for example the effective) and non-winnable initial states.

In the third case $|E(G)| > \deg(D^*)$, thus $\deg(D) < 0$, thus the game is unwinnable, or in other words infinite.

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