SELECTED THEOREMS IN THE GEOMETRY OF CONICS

BSC THESIS IN MATHEMATICS

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Selected theorems in the geometry of conics

A **szakdolgozat** szerzőjeként fegyelmi felelősségem tudatában kijelentem, hogy a dolgozatom önálló szellemi alkotásom, abban a hivatkozások és idézések standard szabályait következetesen alkalmaztam, mások által írt részeket a megfelelő idézés nélkül nem használtam fel.

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a hallgató aláírása

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Acknowledgments

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I would also like to say special thanks to my former English teacher Barbara Ávár in recognition of her efforts into advancing my knowledge of the English language.

1 Introduction

Conic sections have long fascinated mathematicians, forming a center of scientific study. Records of initial investigation into conics date back to as far as the ancient Greeks who instantaneously recognized their distinguished role in geometry. Their efforts were crowned by a well-structured, pioneering treatise written by Apollonius of Perga around 200 BC [5]. Major subsequent contributions to the subject were made by the likes of Blaise Pascal, Gaspard Monge, Charles-Julien Brianchon, Germinal Pierre Dandelin and Jean-Victor Poncelet.

Conics have found many applications in various fields of science. They have proven vital to understanding mechanics, optics and waves. They are crucial in the study of projectile motion and orbits of celestial bodies, transmission and reception of radio signals and reflective properties of specular surfaces.

In this piece of work, we aspire to lay some groundwork upon conics by establishing a number of their elementary yet intriguing properties being of great interest per se, then advance toward formulating more complex, perhaps lesser known, carefully selected theorems with cherry-picked proofs to provide an insight into the diverse, beautiful world of conic sections. Throughout this thesis, we attempt to build, as far as possible, purely synthetic proofs to theorems.

2 Classification of conic sections

A conic section is essentially the intersection of a two-sided infinite cone and a plane. The terms conic and conic section shall be interchangeably used throughout this thesis. A conic section is called **non-degenerate** if this intersection is not a single point or a line.

In the following, we attempt to classify conic sections according to their angle of attack, i.e. the angle spanned by the dissecting plane and the axis of the cone. To do that, we may first define three planar curves.

2.1 Definition of conic sections

Definition 2.1 *An* **ellipse** *(see Figure 1) is the locus of points on the plane for which the sum of distances from two fixed points (called foci) is a constant greater than the distance between these points. This constant is called the length of the major axis.*

The **center** *of an ellipse is the midpoint of the segment joining the two foci. Extending the line joining the two foci and intersecting it with the ellipse yields its* **major axis**. *The* **minor axis** *of an ellipse is the chord perpendicular to the major axis in the center.*

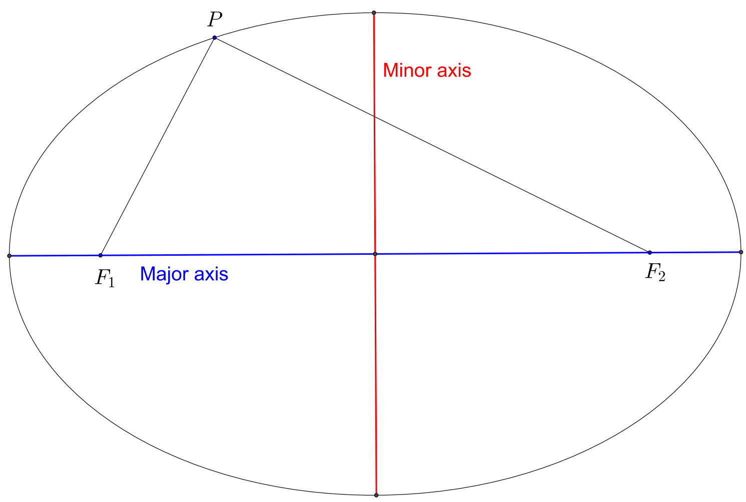

Figure 1: An ellipse

Definition 2.2 *A* **parabola** *(see Figure 2) is the locus of points on the plane which are equidistant from a fixed line and a fixed point not incident with the line. The* *line is called the directrix of the parabola and the the point is called the focus of the parabola.*

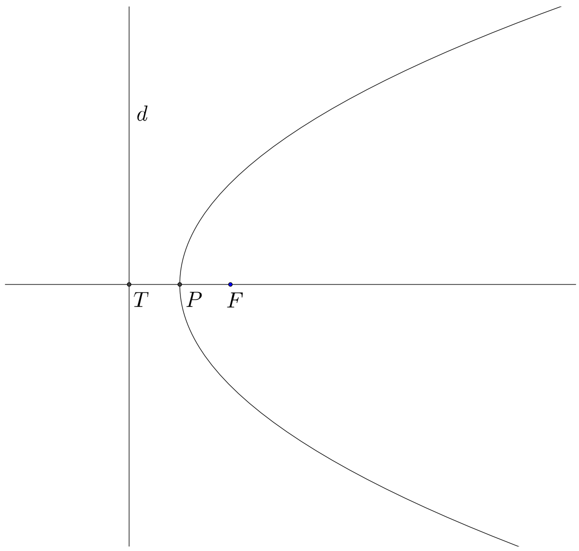

Figure 2: A parabola

Definition 2.3 *A* **hyperbola** *(see Figure 3) is the locus of points on the plane for which the absolute value of the difference of distances from two fixed points (called foci) is a constant less than the distance between these points. This constant is called the length of the real axis.*

The segment joining the two foci is the **real axis** *of the hyperbola,and the midpoint of the real axis is the* **center** *of the hyperbola.*

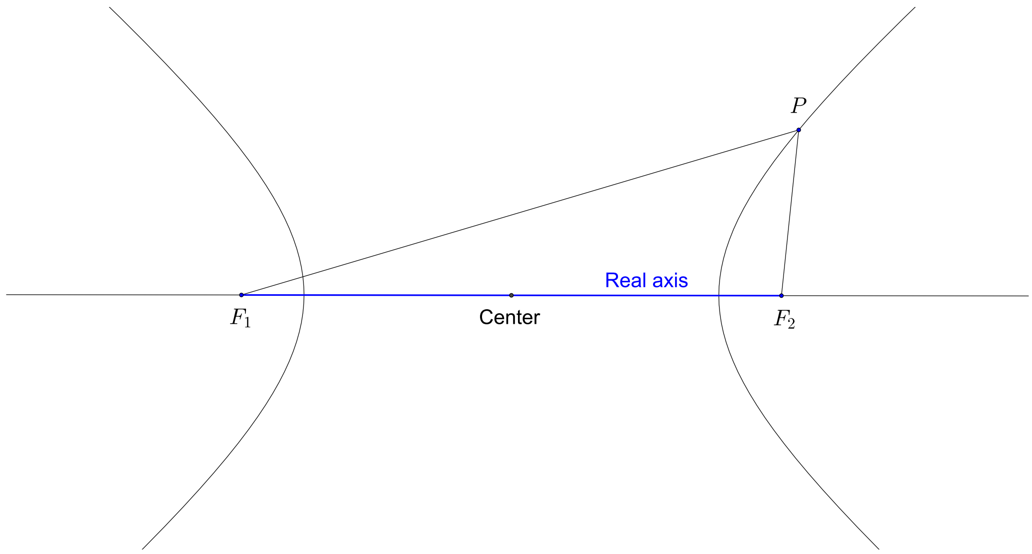

Figure 3: A hyperbola

2.2 Classification through Dandelin spheres

In this section, we will utilize tangent spheres proprietary to conic sections. The upcoming notions have been sourced from [2] and [4].

Definition 2.4 *A* **Dandelin sphere** *of a conic section is a sphere that is internally tangent to the cone and touches the conic.*

Definition 2.5 *The line joining the apex of the conic with any point on the surface of the conic except for the apex is called a* **generating line**.

Theorem 2.6 *A non-degenerate conic section is either a circle, a parabola, a hyperbola or an ellipse.*

Proof We may first consider the problem from a projective point of view by dissecting a cylinder (i.e. a cone whose apex is at infinity) with a plane that is not perpendicular to the axis of the cylinder (see Figure 4). Let π denote the mentioned plane and γ the intersection of the cylinder and the plane. Consider the two Dandelin spheres internally tangent to the cylinder in circles k_1 and k_2 and touching π from opposite sides at points F and G. Let P be an arbitrary point on γ. Drop perpendiculars from P onto circles k_1 and k_2 to cut them at points Q and P, respectively. Obviously, $PF = PQ$ and $PG = PT$ as they are tangents drawn to a sphere drawn from a common point. In light of the above, consider the sum:

$$FP + PG = PQ + PT$$

which turns out to be the distance between k_1 and k_2, a constant along the perimeter of the cylinder, independent from the choice of P. This means that all points on γ lie on an ellipse with foci F and G.

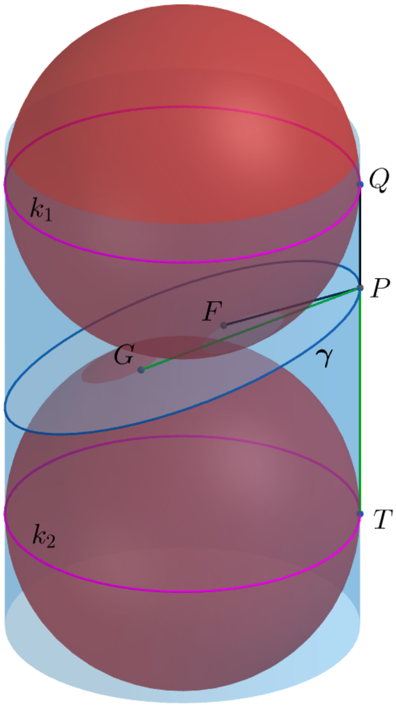

Figure 4: Dandelin spheres of an ellipse 1

If we dissect a euclidean cone with a plane not perpendicular to the axis and not parallel to any of the generating lines, it also yields an ellipse. The previous reasoning may be transmitted to this configuration by replacing the cylinder with a cone (see Figure 5).

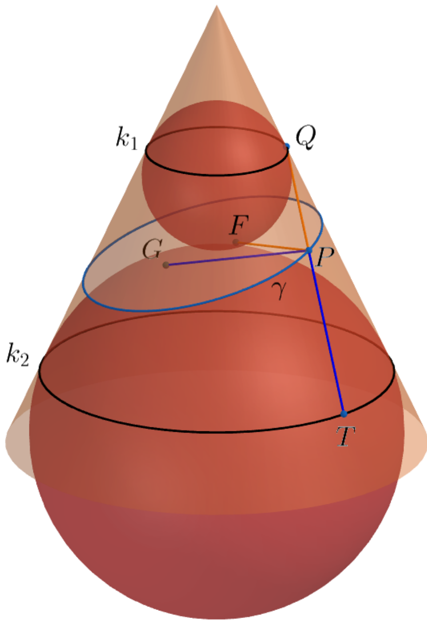

Figure 5: Dandelin spheres of an ellipse 2

Let us now cut the cone with plane π that is parallel to exactly two generating lines of the cone (see Figure 6). Let π intersect the cone in curve γ (obviously γ is not connected, it contains two branches). The two Dandelin spheres belonging to γ are tangent to the cone in circles k_1 and k_2 and touch π in points F and G. Let P be an arbitrary point on γ. Let T and Q be the intersections of the generating line going through P and circles k_1 and k_2, respectively. $PF = PQ$ and $PG = PT$ as they are tangents to a sphere drawn from a common point. Consider, in view of our previous claim, the difference

$$|PG - PF| = |PT - PQ|$$

which is a constant along the perimeter of the cone, namely the length of the segment k_1 and k_2 cut out from any generating line. Consequently, all points on γ lie on a hyperbola of foci F and G.

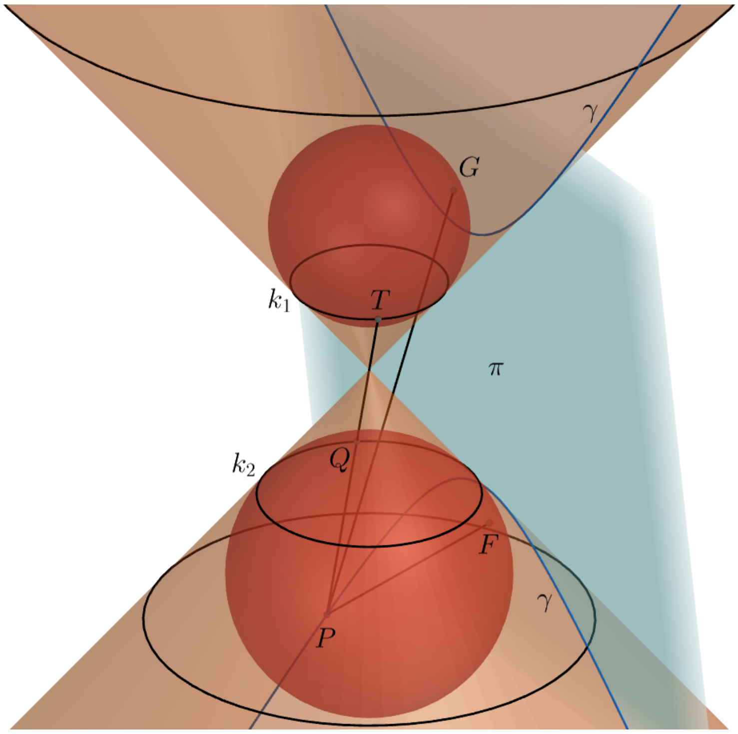

Figure 6: Dandelin spheres of a hyperbola

The last case to be examined (see Figure 7) is when the dissecting plane is parallel to exactly one generating line (denoted g). Let the dissecting plane pi intersect the cone along curve γ. The (only) Dandelin sphere associated with the conic intersects the cone in circle k and touches π in point F. Let θ be the enveloping plane of k, and s the axis of the cone, cutting through θ in a right angle. Let P be an arbitrary point on γ. Let θ intersect π in line v. Let P^* be the perpendicular projection of P onto θ and T the perpendicular projection of P onto v. Let Q be the intersection point of circle k and the generating line incident with P. As $PP^* \perp \theta$ and $s \perp \theta$, $PP^* \parallel s$. As $g \parallel \pi$ and $PT \subset \pi$, it follows that $PT \parallel g$. As a result,

$$\angle P^*PT = \angle(g, s)$$

as they are angles with parallel arms. For similar reasons,

$$\angle P^*PQ = \angle(PQ, s) = \angle(g, s).$$

Combining the equations yields:

$$\angle P^*PQ = \angle P^*PT.$$

We can now conclude that triangles $\triangle P^*PQ$ and $\triangle P^*PT$ are congruent as they have the same angles and share a common side opposite to the same pair of angles, giving us:

$$PT = PQ.$$

We can also deduce that $PQ = PF$ as they are tangents drawn to a sphere from a common point. Putting together the results yields:

$$PT = PQ = PF,$$

meaning that any point on γ has a fixed distance from point F and line v, in other words, these points lie a parabola with focus F and directrix v.

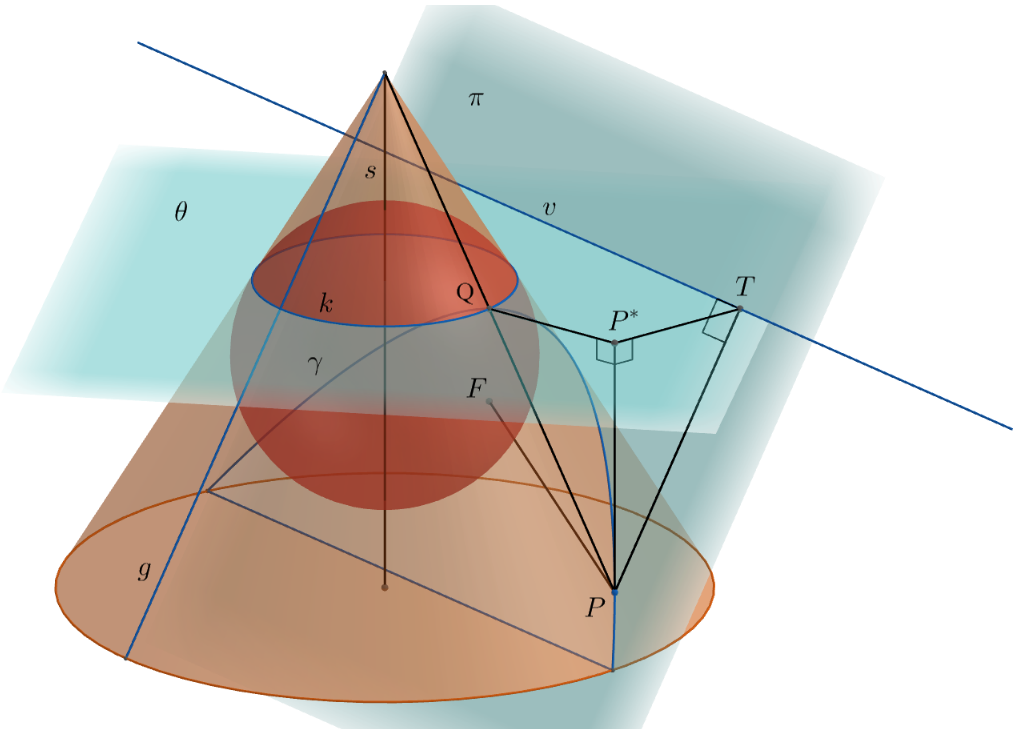

Figure 7: Dandelin sphere of a parabola

We may remark that the above three proofs are incomplete in that they only showed that the respective types of conic sections are contained in either an ellipse, a hyperbola or a parabola. To be precise, one would also have to prove inclusion in both directions, i.e. the conics are identical to one of these curves, meaning every point of the curve lies on the conic too, yet a thorough proof of this claim is beyond the limits of this thesis, it may be found in source [2] in its entirety.

3 Some basic properties of conics

It is hereby acknowledged that the following section has greatly utilized source [3].

3.1 Alternative characterization of conics

We shall introduce the subsequent classification to partition the plane.

Definition 3.1 *A non-degenerate conic divides the points of the dissecting plane into three classes as follows:*

- *The intersection of the interior of the cone and the plane, denoted* **internal points**.
- *The intersection of the exterior of the cone and the plane, denoted* **external points**.
- *The intersection of the surface of the cone and the plane, called simply the* **points of the conic**.

Theorem 3.2 *Let F be a point on the plane and v a line or a circle not passing through F. Then the centers of circles tangent to v and passing through F lie on a conic. Namely, the mentioned curve is*

- *a parabola if v is a line*
- *a circle if v is a circle and F is its center*
- *an ellipse if v is a circle and F is any internal point of v except its center*
- *a hyperbola if v is a circle and F is an external point of v.*

Proof Let c be a circle with center P, passing through F, tangent to v in T.

Suppose v is a line (see Figure 8). $PT \perp v$, as PT is a radius drawn at a point of tangency. Therefore, the distance between P and v equals PT, and $PT = PF$ as they are radii belonging to the same circle. Consequently, P lies on a parabola of focus F and directrix v.

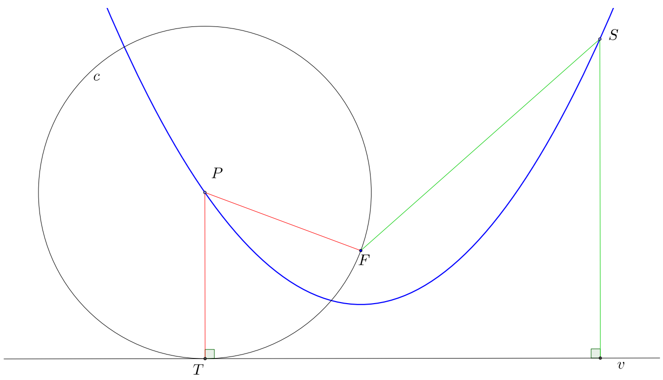

Figure 8: The directrix is a line

Suppose v is a circle of radius r, and let G denote its center (see Figure 9). If F is an internal point of v, then, as c and v are internally tangent circles,

$$r = TG = TP + PG = FP + PG.$$

As a result, if F coincides with G, then P lies on a circle of center F and radius $\frac{r}{2}$ and if they differ, then P lies on an ellipse with foci F and G, and major axis r.

Conversely, if S is an arbitrary point on this circle or ellipse, then $SF + SG = r$. Let ray SG cut v in K. Then,

$$SK = r - SG = SF$$

and therefore the circle drawn from center S with radius SF internally touches v in K.

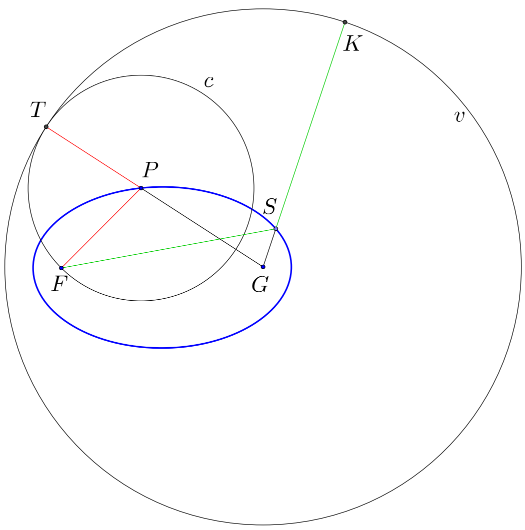

Figure 9: The directrix is a circle, F is internal

Finally, suppose F is an external point of v. Then, c is either externally or internally tangent to v as depicted beneath (see Figures 10 and 11). In both cases, the following holds: $PF = PT$, as they are radii of circle c. That yields:

$$|PF - PG| = |PT - PG| = GT = r,$$

meaning that P lies on a hyperbola with foci F and G, and real axis r.

In reverse, for an arbitrary point S on the hyperbola, either $SF = SG + r$ or $SG = SF + r$ holds. The first means that the circle drawn from center S with radius SF internally touches v, yet the second implies external tangency.

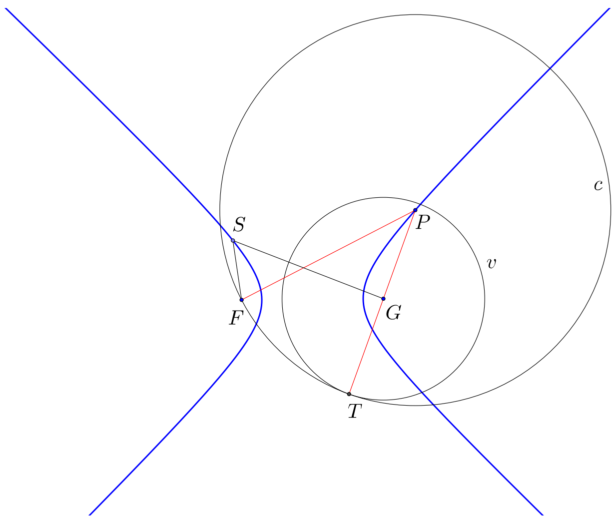

Figure 10: The directrix is a circle, F is external, case 1

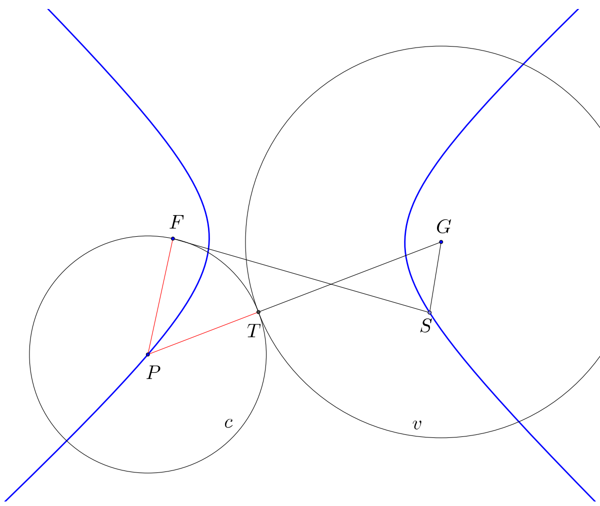

Figure 11: The directrix is a circle, F is external, case 2

Definition 3.3 *The line or circle mentioned in the above theorem is called the* **directrix** *of the respective conic (in accordance with the previous definition of the parabola), whilst the point is defined to be a* **focus** *of the conic.*

3.2 Tangential and focal properties of conic sections

Definition 3.4 *A circle and a line are said to be* **orthogonal** *if the line passes through the center of the circle or, equivalently, if the tangents drawn to the circle at the points of intersection with the line are perpendicular to the line.*

Definition 3.5 *Let P be a point on a conic γ with directrix v and focus F. Then, the* **focal rays** *belonging to P are line FP and the perpendicular dropped from P onto v. We remark that the above perpendicular is line PG when γ is an ellipse or a hyperbola with foci F and G.*

Definition 3.6 *Let γ be a hyperbola or an ellipse. A line t is said to be* **tangent** *to γ at point $P \in \gamma$ if it has no common point with γ other than P.*

Let γ be a parabola. A line t is said to be **tangent** *to γ at point P if it has no common point with γ other than P and t is not parallel to the axis of the parabola.*

The above distinction is required to distinguish tangents and axial lines cutting through the parabola at a single point. We remark that there is no need for such distinction on the real projective plane where the whole concept would be circumvented by introducing ideal points and defining tangents as lines intersecting conics at a single point, thereby avoiding the ambiguity in the case of parabolas.

Theorem 3.7 *Let P be an arbitrary point on a conic γ. Then, the tangent drawn to γ at P bisects the angle between the focal rays belonging to P.*

Proof We begin with the case of γ being either an ellipse or a hyperbola (see Figures 12 and 3). Let γ be defined by focus F and directrix v. Let P denote an arbitrary point on γ. According to Theorem 3.2, the circle with center P and radius PF is tangent to v. Let T denote the point of tangency. Let b be the angle bisector of $\angle TPF$ and K be an arbitrary point on b apart from P. As K lies on the angle bisector, $KT = KF$. Therefore, the circle drawn from center K with radius KF (denoted k) passes through T. Nevertheless, T lies on v and $P \neq K$, thus, k cannot be tangent to v. As a result, K does not lie on γ. To sum up, b has exactly one point in common with γ (namely P), thereby implying that the angle bisector is tangent to γ at P.

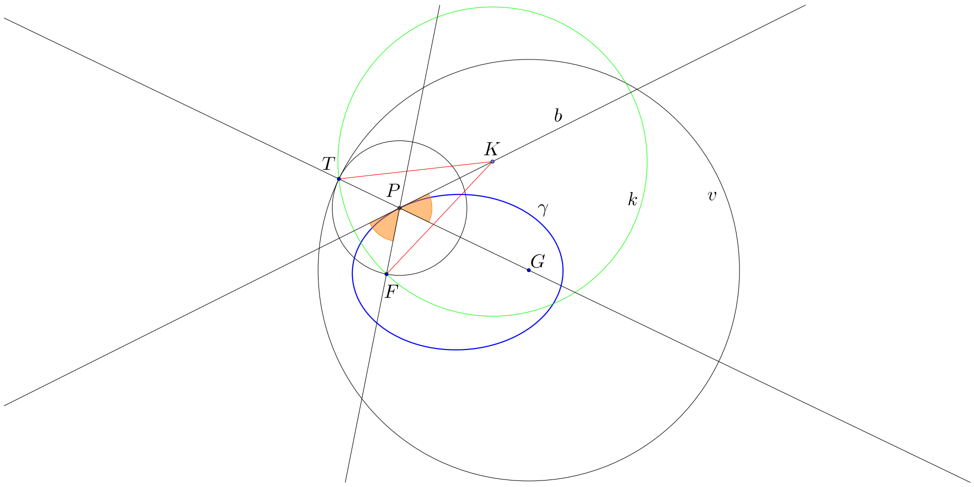

Figure 12: Tangent of an ellipse bisecting the angle of focal rays

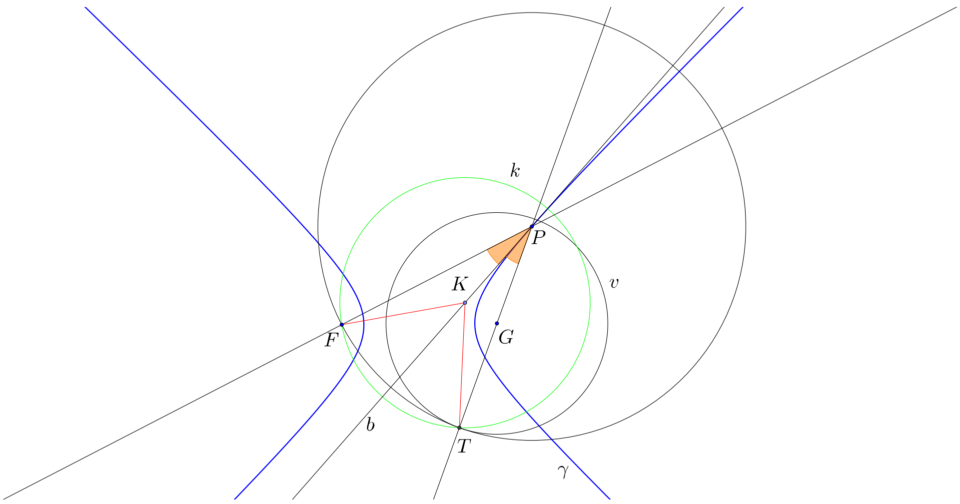

Figure 13: Tangent of a hyperbola bisecting the angle of focal rays

We may finally inspect the parabola (denoted γ) with focus F and directrix v (see Figure 14). Let s be the axis of the parabola, and K be a point different from P lying on the angle bisector of $\angle TPF$ (denoted b). The above reasoning also holds for parabolas, consequently, $b \cap \gamma = P$. For b to be tangent to γ, we have yet to prove that b and s are not parallel. By definition, $s \perp v$ and $PT \perp v$, from which it follows that $s \parallel PT$. Let us suppose the contrary, i.e. $b \parallel s$, meaning $b \parallel PT$. Then, the angle between PT and b is zero. As b bisects $\angle TPF$, we know that the angle between b and PF is also zero. As a result, lines PT and PF coincide. That yields: $F \equiv T$ which is a contradiction because T is incident with the directrix and F, by definition, is not. We have now shown that b cannot be parallel to the axis, so it has to be tangent to γ at P.

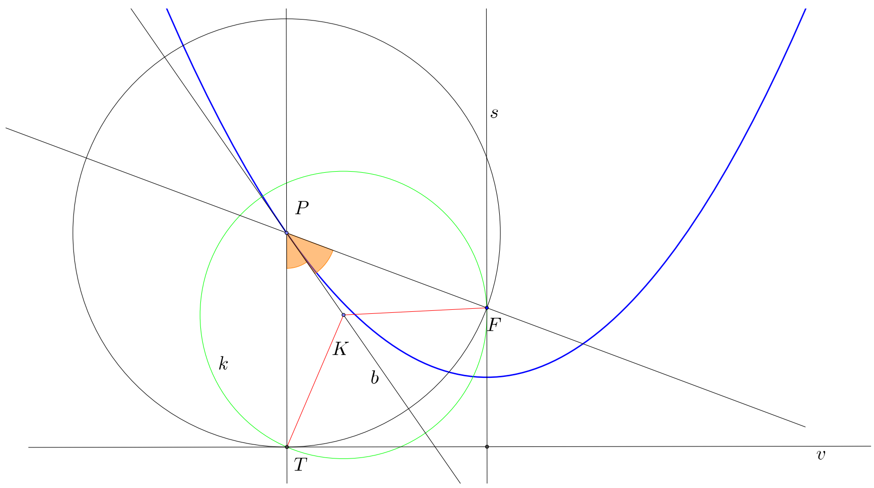

Figure 14: Tangent of a parabola bisecting the angle of focal rays

We remark that the aforementioned property is heavily exploited in applications in many fields such as optics, sound technology and and air-to-air signal reception whereby the law of wave reflection in conjunction with this property allows parabola-shaped reflective surfaces to collect waves traveling parallel to the axis in the focus where the receiver may be positioned. These instances include special mirrors, parabolic microphones used for surveillance and parabolic antennas to receive satellite signals.

From the above theorem, we can effortlessly derive an interesting corollary describing a property pertinent to all conics:

Theorem 3.8 *The reflection of the focus of a conic with respect to any of its tangents is incident with the directrix.*

Proof The statement is obvious considering elementary properties of reflection with respect to an axis, in particular the fact that the angle between the two lines connecting any point on the axis to the original and the reflected point is bisected by the axis (see Figure 15).

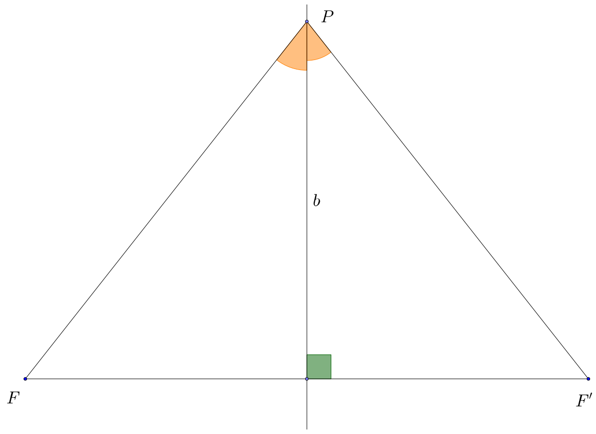

Figure 15: A property of reflection

4 Unique attributes of conic sections

In this section, we discuss certain special properties of conic sections whose description has been greatly influenced by source [1].

4.1 Tangential triangles of parabolas

We begin by proving a theorem connecting a tangent triangle of a parabola with its focus. To achieve that, let us first revisit a widely known statement concerning projections of a point onto a triangle's sides.

Lemma 4.1 (Simson's lemma) *Let ABC be a triangle and P a point outside it. The projections of P onto lines AB, BC and AC (denoted R, S and T, respectively) are collinear if and only if P lies on the circumcircle of ABC.*

Although the idea to use this lemma has been suggested by source [1], the proof in its entirety features my work.

Proof Suppose P is on the circumcircle of triangle ABC (see Figure 16). Without loss of generality, we can assume that segment BC separates points A and P. Obviously,

$$\angle PRB = \angle PSB = \pi/2,$$

so quadrilateral $PSRB$ happens to be cyclic. $\angle RSB = \angle RPB$ as they are inscribed angles subtended by the same arc.

$$\angle CTP + \angle CSP = \pi,$$

thus, quadrilateral $CTPB$ is also cyclic. $\angle TSP = \angle TCP$, as they are inscribed angles subtended by the same arc. Quadrilateral $ABPC$ is cyclic by definition, meaning that $\angle TCB = \angle PBA$. The above statements together imply that $\angle TSP = \angle PBS$. As $\angle PRB$, $\angle RPB$ and $\angle PBA$ are angles of triangle PRB, they sum up to π. Consequently, so do angles $\angle TSP$, $\angle PSB$ and $\angle RSB$, which is equivalent to the collinearity of R, S and T.

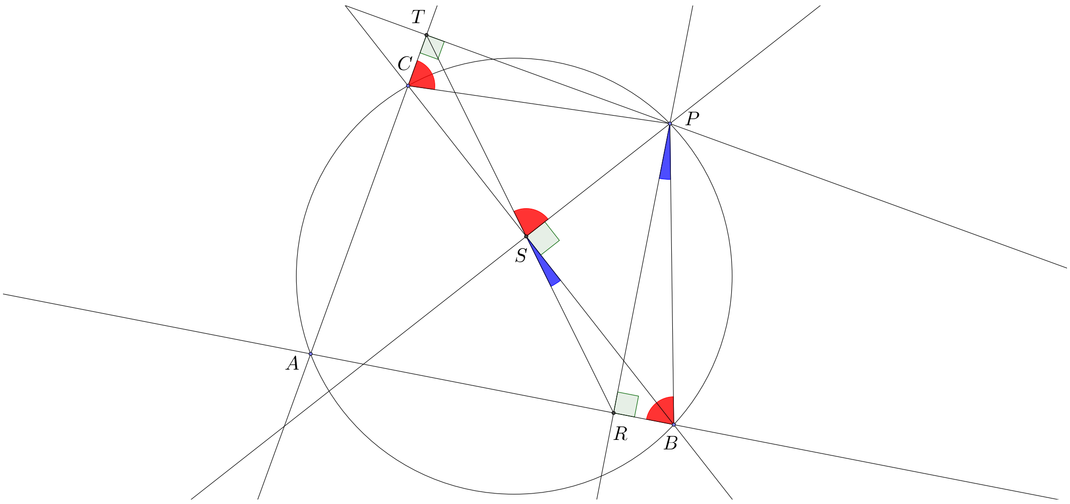

Figure 16: The Simson line

Conversely, suppose R, S and T are collinear (see Figure 17). Clearly,

$$\angle PRB = \angle PSB = \pi/2.$$

This implies that quadrilateral $PSRB$ is cyclic, from which it follows that $\angle RSB = \angle RPB$ as they are inscribed angles subtended by the same arc. Using the collinearity condition, we can deduce that $\angle TSC = \angle RSB$. For similar reasons, quadrilateral $TCSP$ is cyclic, and $\angle TSC = \angle TPC$. In conclusion, $\angle RPB = \angle TPC$, meaning that triangles TCP and RPB share two angles, so they must also share the third, i.e. $\angle TCP = \angle PBR$. This indicates that $\angle ACP$ and $\angle ABP$ are supplementary angles, which leaves quadrilateral $ABPC$ to be cyclic, therefore P lies on the circumcircle of triangle ABC.

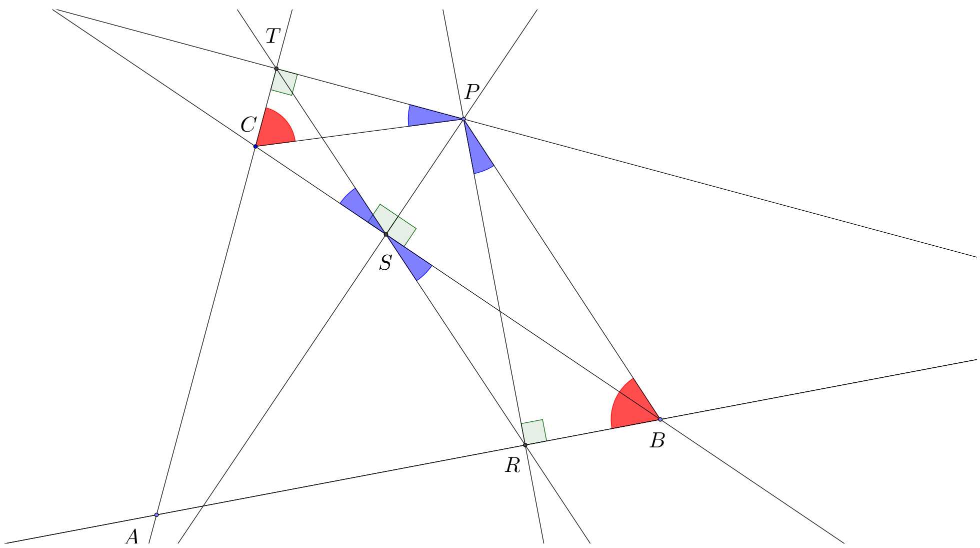

Figure 17

We would like to connect the above lemma with a claim we partially proved already, specified for parabolas.

Theorem 4.2 *The perpendicular projections of a parabola's focus onto all possible tangents lie on a straight line.*

We remark that it suffices to prove this for three arbitrary tangents which determine a so-called tangential triangle of the parabola.

Proof Let ABC be a tangential triangle of a parabola (see Figure 18). By Theorem 3.8, the reflections of the focus onto all tangents lie on the directrix. In the case of the parabola, that is a straight line (denoted d). Let the perpendicular projections of F onto tangents AB, BC and AC be named P, Q and R, respectively. Let FP, FR and FQ intersect d at points P', R' and Q' in this sequence. Due to the reflection property, $PF = PP'$, $RF = RR'$ and $QF = QQ'$ hold. Applying the midline theorem to triangles $P'FR'$ and $R'FQ'$ warrants that PR and RQ are parallel to d and incident in R. As the line parallel to d and incident with R is unique, P, Q and R are necessarily collinear.

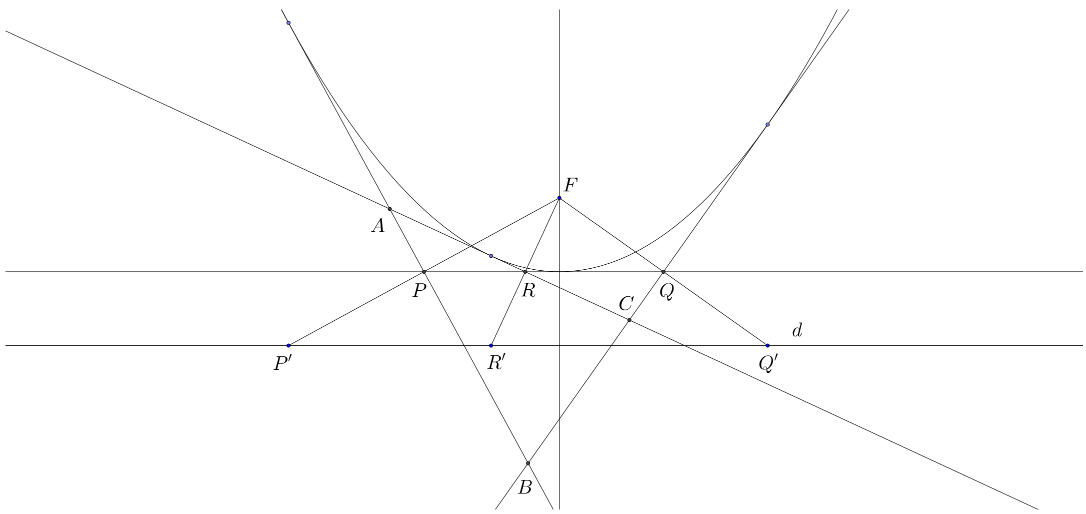

Figure 18

We can now proceed to our main theorem in the section:

Theorem 4.3 *The circumcircles of all tangential triangles of a parabola pass through the parabola's focus.*

Proof This statement is a direct consequence of the previous two. According to Simson's lemma, if the projections of F onto the sides of triangle ABC (see Figure 18) are collinear (which in this case, they are, please refer to Theorem 4.2), then F must lie on the circumcircle of triangle ABC.

We may utilize the above to derive a beautiful corollary known as **Miquel's theorem**. To do so, we are required to establish some groundwork.

There is a well-known fact about conics that has found many applications, that is, for any five points on the plane in general position (i.e. no three of them are collinear), there is a unique conic containing all of them. Proof of this statement may be omitted as it would well exceed the boundaries of this work, however, we remark that a standard proof would involve a linear algebraic approach using matrices of second-order curves. The dual of this theorem reads as follows: for any five lines on the plane in general position (i.e. no three of them are concurrent), there is a unique conic tangent to all of them.

In a projective approach, we can assume that exactly one of these lines is the line at infinity. In this configuration, the conic is tangent to this line at a point at infinity, that leaves it to be a parabola. This means that for any four euclidean lines in general position (this setting has been named a **complete quadrilateral**), there is a unique parabola tangent to them. This leads us to **Miquel's theorem**:

Theorem 4.4 (Miquel's theorem) *The circumcircles spanned by the the triangles resulting from the four lines making up a complete quadrilateral pass through a single point (called the Miquel point of the quadrilateral, see Figure 19).*

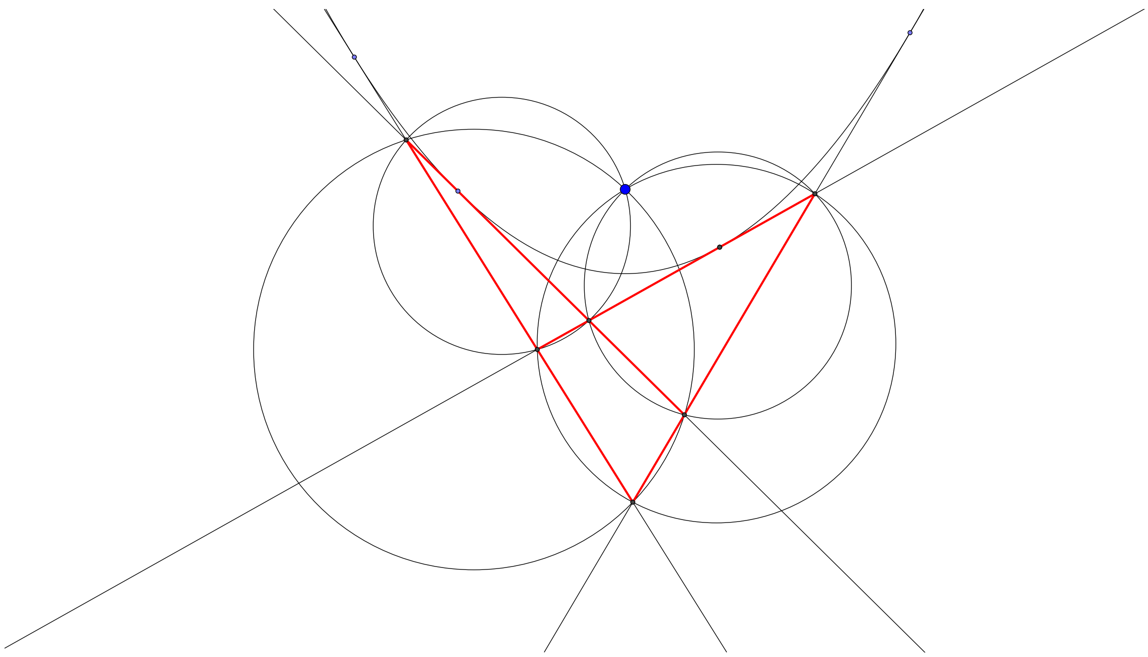

Figure 19: Miquel's theorem

Proof Note that the four triangles determined by the complete quadrilateral are tangential triangles of the inscribed parabola. Invoking Theorem 4.3 assures us that the circumcircles of these triangles contain the parabola's focus, meaning that these four circles intersect at a single point.

We would also like to mention a beautiful theorem regarding the orthocenter of a tangent triangle of a parabola. To achieve that, we go through two lemmas regarding the Simson line of a triangle.

Lemma 4.5 *Let point P lie on the circumcircle of triangle ABC. Let B' be a point also on the circumcircle such that $AC \perp PB'$. Then, the Simson line associated with P is parallel to BB' (see Figure 20).*

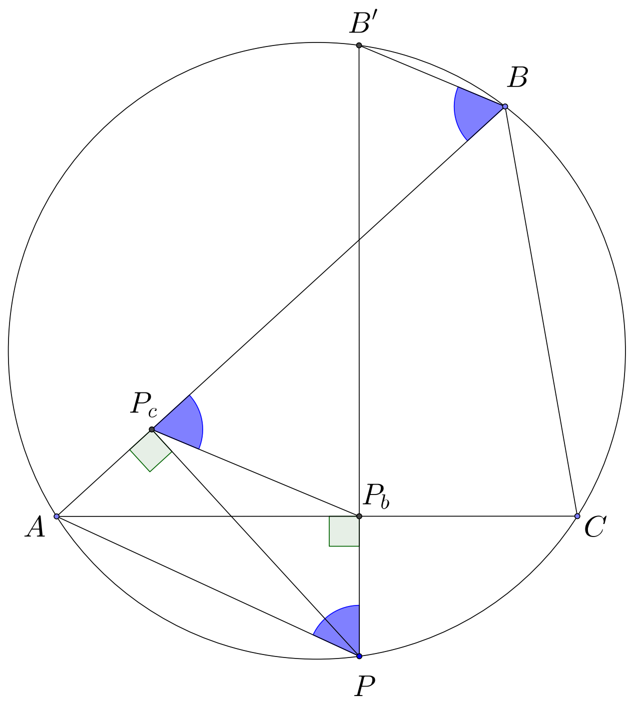

Figure 20

Proof Let P_b and P_c denote the perpendicular projections of P onto AC and AB, respectively. $\angle APB' = ABB'$ as they are angles subtended by arc AB. As $\angle AP_cP = \angle AP_bP = \frac{\pi}{2}$, quadrilateral AP_cP_bP is cyclic. Therefore,

$$\angle APB' = \angle APP_b = \pi - \angle AP_cP_b = \angle BP_cP_b,$$

implying parallelity of P_bP_c and BB'.

Lemma 4.6 *Let H be the orthocenter of triangle ABC and P a point on its circumcircle. Then, the Simson line belonging to P bisects segment PH.*

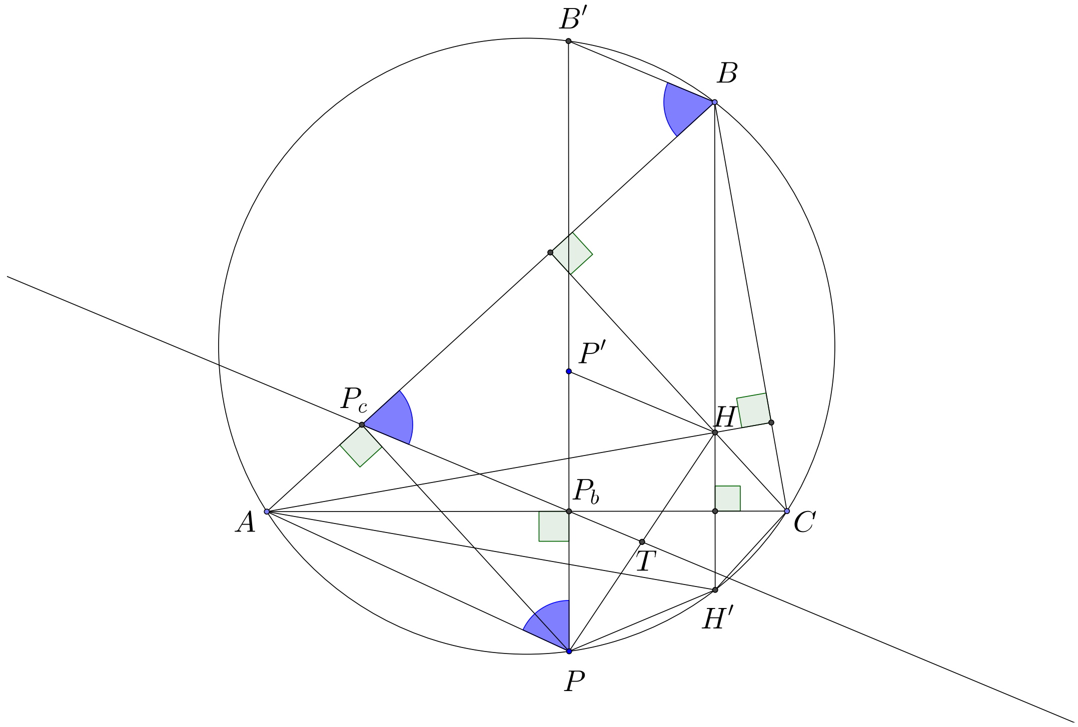

Figure 21

Proof Let us suppose that ABC is an acute triangle (see Figure 21). Then, $AB \perp HC$ and $BC \perp AH$, consequently, $\angle AHC$ and $\angle ABC$ are angles with perpendicular arms, meaning that they are either equal or supplementary angles. One is acute, the other is obtuse, this rules out the first case, yielding:

$$\angle AHC = \pi - \angle ABC.$$

Let the reflection of H with respect to AC be called H'. Due to the reflection,

$$\angle AH'C = \angle AHC = \pi - \angle ABC,$$

meaning that H' lies on the circumcircle. As $PB', BH' \perp AC$, it follows that $PB' \parallel BH'$, thus, quadrilateral $PB'BH'$ is a cyclic trapezoid, necessarily equilateral, implying symmetry. Let P' be the reflection of P with respect to AC. Due to symmetry, $P'H \parallel B'B$. In light of our previous lemma, $P'H$ is parallel to Simson line P_bP_c. Let P_bP_c intersect PH in T. As $PP_b = P_bP'$ and $P_bT \parallel P'H$, P_bT is a midline of triangle $PP'H$. This warrants $PT = TH$, completing the proof.

Assuming ABC is obtuse, the above remains true except for H and B reversing roles (see Figure 22).

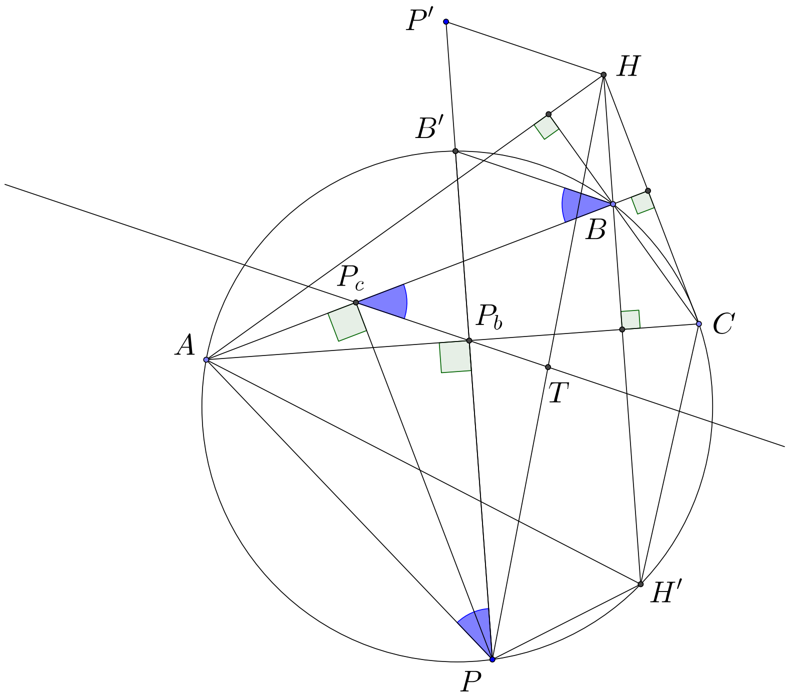

Figure 22

If ABC is a right triangle (see Figure 23), then it is easy to recognize that B corresponds with H and B' corresponds with P', the rest of the previous proof still holds.

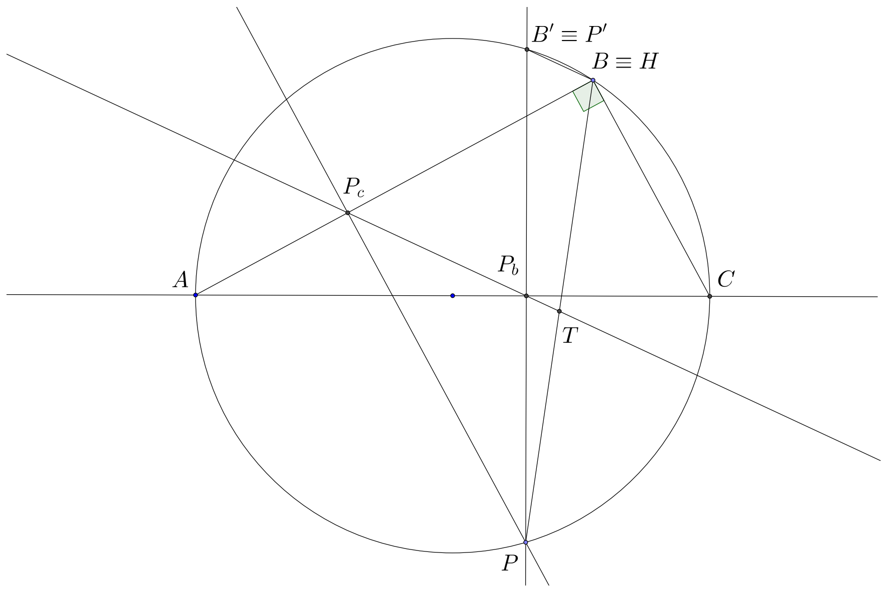

Figure 23

To crown our efforts, we return to the main theorem we intend to prove:

Theorem 4.7 *The orthocenter of a tangential triangle of a parabola lies on the directrix.*

Proof Let triangle ABC be tangent to a parabola with focus F (see Figure **??**). Let H denote the orthocenter of ABC. According to Theorem 4.3, F lies on the circumcircle of triangle ABC, meaning it has a Simson line associated with it. This Simson line bisects segment FH in point T (refer to the previous theorem). Let V be the vertex of the parabola, and D the intersection of the axis and the directrix. As we have a parabola, naturally, $FV = VD$. We have already seen that this Simson line is parallel to the directrix (Theorem 4.2). Taking the above into consideration: $FV = VD$ and $FT = TH$, applying the midline theorem to triangle DFH gives us that $VT \parallel DH$. Yet the line going through D parallel to VT is unique, this only holds for the directrix, meaning H must lie on it as desired.

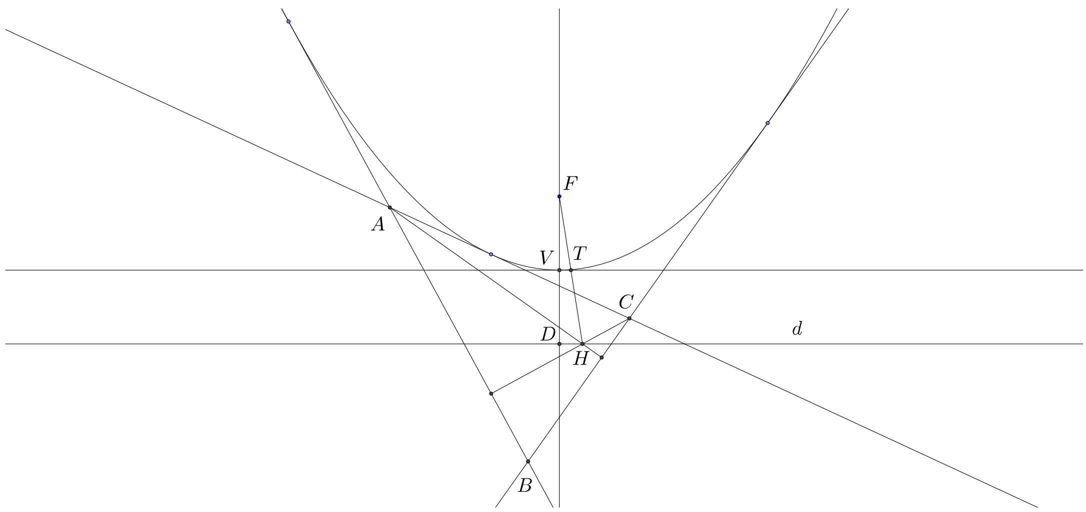

Figure 24

4.2 A remarkable property of equilateral hyperbolas

We hereby include an impressive property of equilateral hyperbolas passing through the vertices of a fixed triangle. To accomplish that, we need to make some preparations.

Definition 4.8 *The* **asymptotes** *of a hyperbola (see Figure 25) are a unique pair of lines intersecting at the hyperbola's center such that, as we move away from the center along one of these lines to infinity, the distance between the line and the hyperbola converges to* 0*. We remark that this coincides with the projective geometric definition of asymptotes, i.e. tangents drawn to a hyperbola at its points at infinity.*

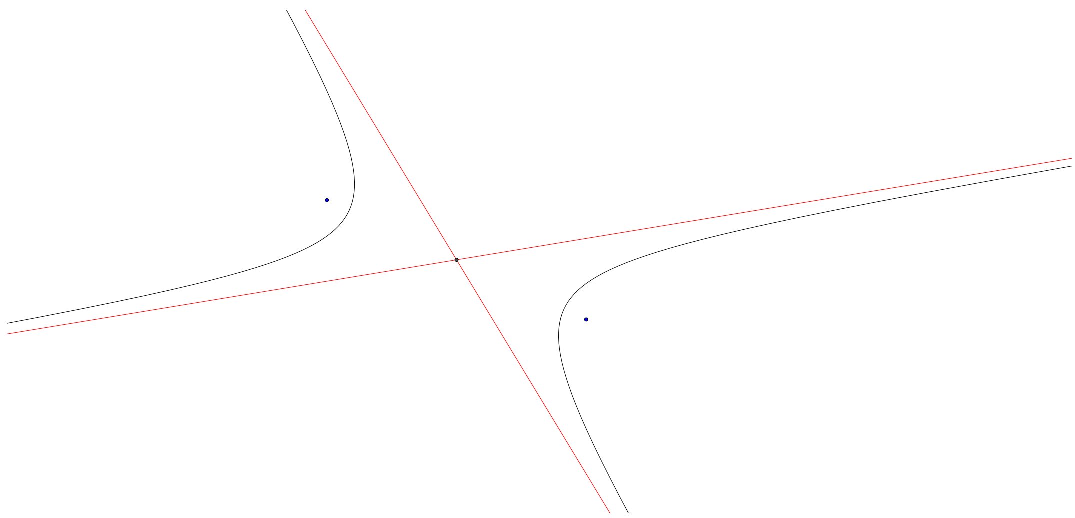

Figure 25: Asymptotes of a hyperbola

Definition 4.9 *A hyperbola is called* **equilateral** *if its asymptotes are perpendicular.*

We do not prove the following theorem as its complexity reaches out of the bounds of this thesis, but it is vital to our investigation on equilateral hyperbolas.

Theorem 4.10 *If a hyperbola passes through a triangle's vertices, then, it will only be equilateral if it also passes through the triangle's orthocenter.*

We would like to remark that this claim may be proven using Desargues's involution theorem on a pencil of conics passing through 4 fixed points (5 points determine a conic section). A proof of this theorem may be found in [1].
We may now proceed to the following lemma, of great interest on its own.

Lemma 4.11 *Let ABC be a triangle with orthocenter M. Let r denote the radius of the circumcircle of ABC. Then, $CM = 2r\cos\angle ACB$ holds.*

Proof If ABC is a right triangle, without loss of generality, we can assume that $\angle ACB = \frac{\pi}{2}$. Then, trivially,

$$2r\cos\angle ACB = AB\cos\frac{\pi}{2} = 0 = CM$$

holds as C and M correspond.

We now move forward to the case of ABC being an acute triangle (see Figure 26). Let the perpendicular line through A to AB intersect the circumcircle in T. Obviously, $AT \parallel CM$. As quadrilateral $ATCB$ is cyclic, $\angle TAB + \angle TCB = \pi$ .With this in mind, we can deduce that $\angle TCB = \frac{\pi}{2}$. That makes TC and AM parallel, and quadrilateral $ATCM$ a parallelogram. As the opposite sides of a parallelogram are of equal length, $AT = CM$.

In line with the converse of Thales's theorem, $AB = 2r$ holds. According to the law of sines, $AB = 2r \sin \angle ACB$. Applying the Pythagorean theorem to triangle TAB and substituting the above yields:

$$CM^2 = AT^2 = 4r^2 - AB^2 = 4r^2 - 4r^2 \sin^2 \angle ACB$$

$$= 4r^2(1 - \sin^2 \angle ACB) = 4r^2 \cos^2 \angle ACB$$

Taking square root results in $CM = 2r \cos \angle ACB$ as proposed.

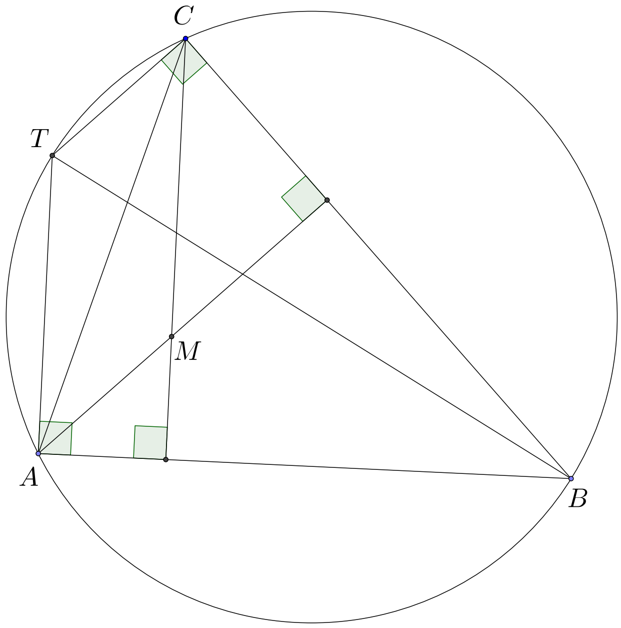

Figure 26: Distance between vertex and orthocenter, acute case

The proof also works for obtuse triangles with the following alteration in reasoning (see Figure 27):

$\angle TAB = \angle TCB = \frac{\pi}{2}$ as they are subtended by a common arc, TB. This implies that $CM \parallel TA$. The rest applies as formerly outlined.

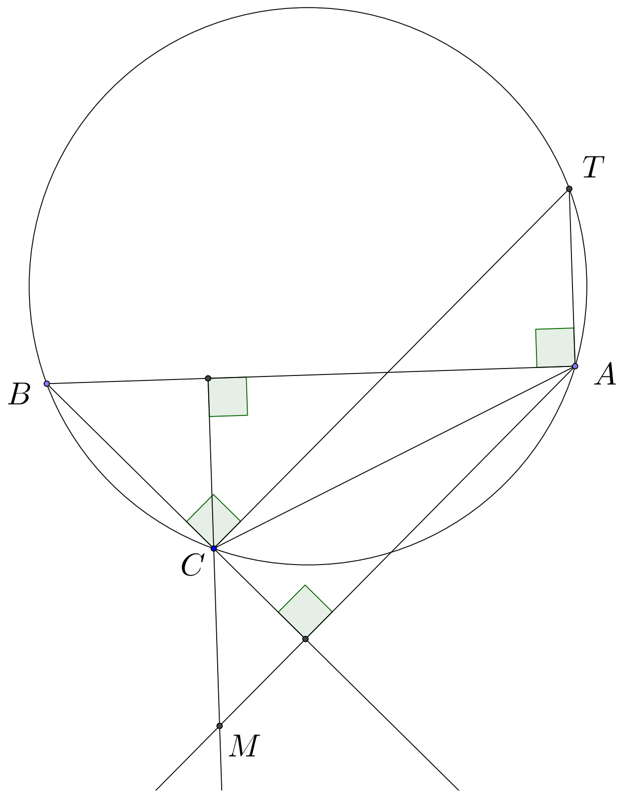

Figure 27: Distance between vertex and orthocenter, obtuse case

Definition 4.12 *It is well known that in any triangle, the midpoints of the sides, the feet of the altitudes and the midpoints of the segments connecting the triangles vertices and the orthocenter lie on a circle. This circle is called the* **Euler circle** *of the triangle, also known as the* **Feuerbach circle** *and the* **nine-point circle**.

Our work culminates in the following assertion which we only partially prove:

Theorem 4.13 *The locus of the centers of all equilateral hyperbolas passing through the vertices of a triangle ABC is the Euler circle of the triangle.*

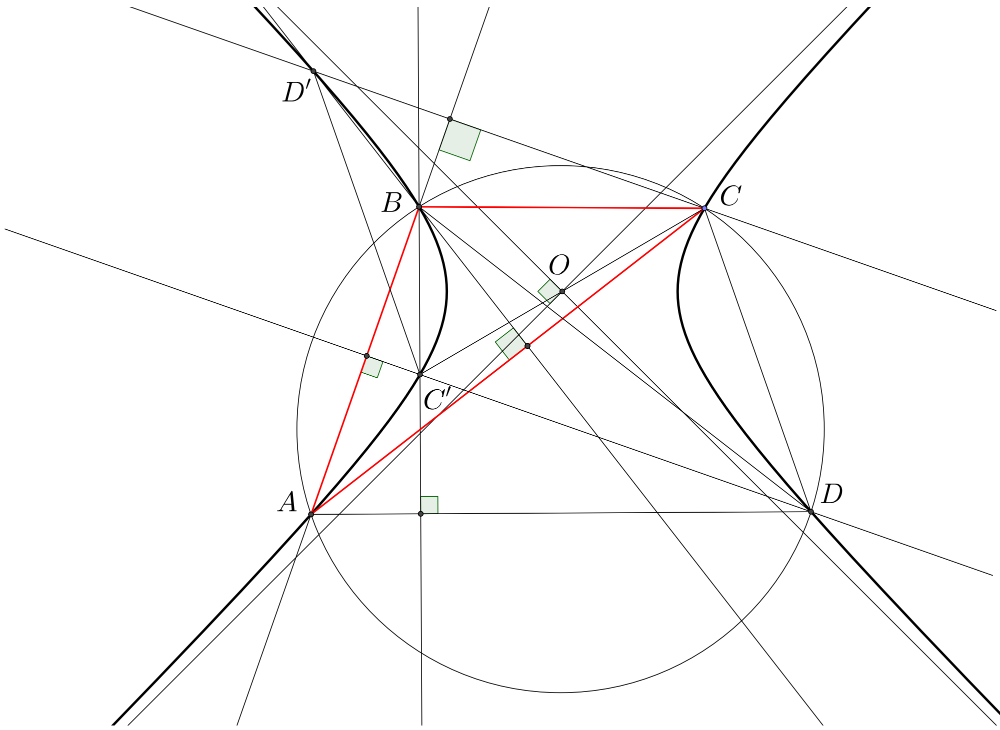

Figure 28: An equilateral hyperbola

Proof Let D denote the fourth intersection point of the circumcircle of ABC with one such hyperbola (see Figure 28). Let A', B', C' and D' be the orthocenters of triangles BCD, CDA, DAB and ABC, respectively. As quadrilateral $ABCD$ is cyclic, $\angle BCA = \angle BDA$. Invoking our previous lemma on triangles ABC and DAB, and taking into account the equality of angles yields:

$$CD' = 2r\cos\angle BCA = 2r\cos\angle BDA = DC'.$$

As $D'C \perp AB$ and $C'D \perp AB$, we can conclude that $D'C \parallel C'D$. This, in conjunction with $CD' = DC'$ warrants that quadrilateral $CDC'D'$ is a parallelogram, therefore, $C'D' \parallel CD$ and $C'D' = CD$. Let the diagonals of $CDC'D'$ intersect in O. As the diagonals of a parallelogram bisect each other, we have: $OC = OC'$ and $OD = OD'$. Following the above pattern on orthocenters A' and B', we can establish that $OA = OA'$ and $OB = OB'$. As a consequence of Theorem 4.10, one can deduce that A', B', C' and D' also lie on the hyperbola. As a hyperbola only has one center of symmetry, it must be point O. This implies that quadrilaterals $ABCD$ and $A'B'C'D'$ are centrally symmetric with respect to center O.

As O bisects the segments spanned by the vertices and the respective orthocenters of the four listed triangles, it must be incident with the Euler circle they share.

We may remark that this only proves that the centers lie on the Euler circle. It also needs proving that to every point on this circle belongs an equilateral hyperbola having this point as its center, but this claim may reach well beyond the limits of this work.

List of Figures

References

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