Invariant Theory of Quiver Settings and Complete Intersections

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Köszönetnyilvánítás

Köszönöm témavezetőmnek Domokos Mátyásnak, hogy bevezetést nyújtott a matematika eme izgalmas fejezetébe, és hogy fáradhatatlanul segítette ennek a szakdolgozatnak az elkészülését.

Köszönöm továbbá Fehér Borbálának a szakdolgozat megszerkesztésében és ellenőrzésében nyújtott segítségét, valamint hasznos észrevételeit.

Contents

1 Introduction

Representations of a quiver with a fixed dimension vector (quiver setting) are parametrized by a vector space together with a linear action of a product of general linear groups such that two point belong to the same orbit if and only if the corresponding representations are isomorphic. Therefore the quotient constructions of algebraic geometry can be applied to give an approximation to the problem of classification of isomorphism classes of representations. The simplest quotient varieties, the so called affine quotients are defined in terms of invariant polynomial functions on the representation spaces. In this way we obtain a wealth of natural invariant theory situations. It is traditional in invariant theory to try to describe those situations where the corresponding ring of invariants has good commutative algebraic properties (eg. is a polynomial ring, or is a complete intersection). This is the main theme of the present report.

Although these affine quotient varieties turn out to reflect faithfully the class of semisimple representations only, the study of them is motivated by more sophisticated quotient constructions as well. Geometric invariant theory has been applied by A. King to construct non-trivial projective quotients (even in cases when the affine quotient is a single point). It was shown by Adriaenssens and Le Bruyn [1] that the study of the local structure (say singularities) of these projective quotients (moduli spaces) can be reduced to the study of affine quotients of other quiver settings. (See [6] for an application illustrating the power of this method.) Therefore the results in this report on the affine quotient varieties of representation spaces of quivers have relevance also for the study of more general moduli spaces of quiver representations.

The new results in the report concern the special case when the values of the dimension vector are all one. Though this is a strong restriction from the point of view of representation theory, it still covers a rather interesting class, since the corresponding affine quotients are toric varieties. The question when a toric variety is a complete intersection received considerable attention in the literature, see for example [8] or [9]. The study of toric ideals has become an active area of research in recent years, and our work can be viewed from this perspective as well.

2 Basic Properties of Quivers

2.1 Path algebras and quiver representations

A *quiver* $Q = (V, A, s, t)$ is a quadruple consisting of a set of vertices V, a set of arrows A, and two maps $s, t : A \to V$ which assign to each arrow its starting and terminating vertex (loops and multiple arrows are possible). A *non-trivial path* x in Q is a sequence $\rho_1, \rho_2 ... \rho_n$ of arrows which satisfies $t(\rho_i) = s(\rho_{i+1})$ for $1 \leq i < n$, $s(\rho_1)$ and $t(\rho_n)$ are called the starting and terminating vertex of the path and are noted by $s(x)$ and $t(x)$ respectively. For each vertex v in V we also define a *trivial path,* denoted by e_v, which contains of no arrows and starts and terminates in v. If x consisting of $\rho_1, \rho_2, ..., \rho_n$ and y consisting of $\tau_1, \tau_2, ..., \tau_m$ are two paths which satisfy $t(x) = s(y)$ then we define their *composition* to be the path $\rho_1, \rho_2, ..., \rho_n, \tau_1, \tau_2, ..., \tau_m$.

For a field k, the *path-algebra* kQ is the k-algebra with basis the paths in Q, and with the product of two paths x, y given by:

$$xy = \begin{cases} composition & if\ (t(x) = s(y)) \\ 0 & otherwise \end{cases}$$

It is easy to see that this is an associative algebra. For example if Q consists of one vertex and r loops then kQ is the free associative algebra on r letters.

A *representation* X of Q is given by a vector space X_v for each $v \in V$, and a linear map $X_\rho : X_{s(\rho)} \to X_{t(\rho)}$ for each $\rho \in A$.

A *morphism* $\theta : X \to X'$ is given by linear maps $\theta_v : X_v \to X'_v$ for each $v \in V$ satisfying $\theta_{t(\rho)} X_\rho = X'_\rho \theta_{s(\rho)}$ for each $\rho \in A$. θ is an isomorphism if all of its components are isomorphisms.

The category of representations will be denoted by $Rep(Q)$.

There is a natural correspondence between representations of Q and right-modules over kQ. For a representation X, $\oplus X_v$ can be regarded as a right kQ-module, defining $x\rho = X_\rho(x)$ for each $\rho \in A$ and $x \in X_{s(\rho)}$. Conversely if M is a right kQ-module a representation can be defined by $X_v = M * e_v$ and $X_\rho(x) = x * \rho * e_{t(\rho)}$ for $x \in X_{s(\rho)}$. It is easy to verify that we get functors this way between $Rep(Q)$ and $Mod - kQ$ and that these are inverses of each other (see [5, Page 6]), resulting in the following lemma:

Lemma 2.1. *The category $Rep(Q)$ is equivalent to $Mod - kQ$.*

Throughout the rest of this report we will assume that k is an algebraically closed field of characteristic zero, and we will denote it by $\mathbb{C}$. This is convenient since we will use several results of LeBruyn and Procesi [4], and Raf Bocklandt [3, 2], who worked with this assumption. However as it was shown by Domokos and Zubkov in [7] many of the results extend to fields with positive characteristic as well. For example the classification of quivers with genuine simple representation we will recall below, holds over an arbitrary field.

The *dimension vector* $\alpha : V \to \mathbb{N}$ of a representation X is defined by $\alpha(v) = dim(X_v)$. The pair (Q, α) is called a *quiver setting*, and $\alpha(v)$ is referred to as the dimension of the vertex v. A quiver setting is called *genuine* if no vertex has dimension zero.

The Ringel form of a quiver is

$$\chi_Q(\alpha, \beta) = \sum_{v \in V} \alpha(v) * \beta(v) - \sum_{\rho \in A} \alpha(s(\rho)) * \beta(t(\rho)).$$

A representation is called *simple* if the only collection of subspaces $V_v \subseteq X_v$ with the property $\forall \rho \in A : X_\rho V_{s(\rho)} \subseteq V_{t(\rho)}$ are the trivial ones. This is the same as the corresponding kQ module being simple. A representation equivalent to the direct sum of simple representations is called *semisimple*.

If Q contains no oriented cycles then the only simple representations of Q are the ones where for some $v_0 \in V$:

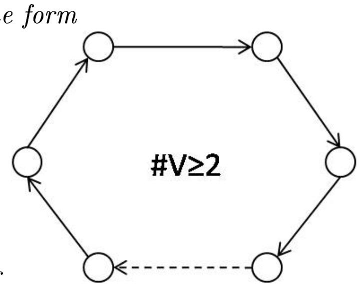
$$\alpha(v) = \begin{cases} 1 & v = v_0 \\ 0 & v \neq v_0 \end{cases}$$

and $X_\rho = 0$ for all $\rho \in A$. Obviously, it is only genuine if Q has one vertex.

When Q is allowed to have oriented cycles, a result by Le Bruyn and Procesi [4, Theorem 4] gives us a characterization of the dimension vectors for which Q has simple representations.

Theorem 2.2. *Let (Q, α) be a genuine quiver setting. There exist simple representations of dimension vector α if and only if*

- Q is of the form

, , *or* 

and $\alpha(v) = 1$ for all $v \in V$.

- Q is not of the form above, but strongly connected and

$$\forall v \in V: \quad \chi_Q(\epsilon_v, \alpha) \leq 0 \quad and \quad \chi_Q(\alpha, \epsilon_v) \leq 0$$

where $\epsilon_v(u) = \begin{cases} 1 & if\ v = u \\ 0 & otherwise. \end{cases}$

If (Q, α) is not genuine, the simple representations classes are in bijective correspondence to the simple representations classes of the genuine quiver setting obtained by deleting all vertices with dimension zero.

2.2 Quotient spaces

2.2.1 The action of $GL(\alpha)$ on $Rep_\alpha Q$

Let $Rep_\alpha Q$ denote the set of representations of Q with dimension vector α. Since a representation in $Rep_\alpha Q$ is completely determined by the linear morphisms assigned to the arrows, we have:

$$Rep_\alpha Q = \bigoplus_{\rho \in A} Mat_{\alpha(t(a)) \times \alpha(s(a))}(\mathbb{C}).$$

To a dimension vector α we will also assign the reductive linear group:

$$GL_\alpha := \bigoplus_{v \in V} GL_{\alpha(v)}(\mathbb{C}).$$

GL_α can be viewed as the group of base changes in the respective vector spaces, thus it has a natural action on $Rep_\alpha Q$. For an element $g = (g_{v_1}, ..., g_{v_n})$ in GL_α

and a representation $X \in Rep_\alpha Q$:

$$X^g_\rho = g_{t(\rho)} X_\rho g^{-1}_{s(\rho)}.$$

Clearly the GL_α orbits in $Rep_\alpha Q$ under this action are the isomorphism classes of representations. It is known [4, 5] that the orbit of an $X \in Rep_\alpha Q$ element is closed if and only if X is semisimple. Moreover it was shown in [12] that every $X \in Rep_\alpha Q$ can be (not necessarily uniquely) written as $X = X_s + X_n$, where X_s is a semisimple representation and X_n is such that the zero representation lies in the closure of its orbit under the action of the stabilizer subgroup of X_s in GL_α. Therefore the study of the orbit structure of GL_α on $Rep_\alpha Q$ breaks down into the study of closed orbits which correspond to semi-simple representations, and the study of certain linear subgroups of GL_α acting on the nilpotent representations. In this report we will only be concerned with the sooner.

To better understand the orbit structure (and thus the isoclasses of representations) of $Rep_\alpha Q$ under the described group action, one wants to construct a quotient space that parametrizes the orbits. The difficulty with this arises from the fact that the set theoretic quotient usually does not have a good structure. To obtain a quotient space with better properties we must allow it to identify some of the orbits.

2.2.2 The geometric quotient

Any algebraic variety X can be regarded as a topological space equipped with a sheaf of functions, whose section algebra over an open set U consists of the rational functions of X that are regular on U (noted by $k[U]$). If an algebraic group G acts on X we can regard the quotient space X/G which as a set will consist of the orbits under the action of G on X, will be equipped with the quotient topology and the sheaf that is the direct image of the sheaf of invariant functions of X. The quotient map from X to X/G will be denoted by $\pi_{X/G}$. The space X/G is not, in general, an algebraic variety. A necessary condition for it to be an algebraic variety is that all the orbits are closed and (if X is irreducible) have the same dimension (see [16]). This does not hold in the case of $Rep_\alpha Q$, except for some trivial cases (if there is no arrows at all, or there is only one loop on a vertex with dimension 1). When X/G is an algebraic variety, the pair $(X/G, \pi_{X/G})$ is called the *geometric quotient* for the action $G : X$. The geometric quotient can also be defined axiomatically (as described in [16]):

Definition 2.3. A pair $(Y,\ \pi_Y)$ where Y is an algebraic variety and π_Y is a morphism of X into Y is called a *geometric quotient* for the action $G : X$ if the following conditions are satisfied:

1) the morphism π_Y is surjective

2) the morphism π_Y is open

3) its fibers are precisely the orbits of G

4) for each open subset $U \subseteq Y$ the homomorphism $\pi_Y* : k[U] \to k[\pi^{-1}(U)]^G$ is an isomorphism.

2.2.3 The affine quotient

X/G defined above can be characterized (up to isomorphism) by having the universal property in the category of topological spaces with sheaves of functions: if Y is a topological space with a sheaf of functions and $\pi_Y : X \to Y$ is a morphism that is constant on the orbits of G then there exists a unique morphism $\varphi : X/G \to Y$ such that $\pi_Y = \varphi \circ \pi_{X/G}$. When the geometrical quotient exists (so X/G is an algebraic variety) it has this same property in the category of algebraic varieties. However, even when the geometrical quotient does not exist it is possible that an object in the category of affine algebraic varieties will have this property. If such an object exists we will call it the *affine quotient* for the action $G : X$. When X is an affine variety and G is a reductive group (so in all of the cases we are interested in) this affine quotient always exists and has many useful properties.

If G is reductive, the algebra $k[x]^G$ of G invariant regular functions is finitely generated and we can consider the affine variety $Spec(k[X]^G)$. It will be denoted by $X//G$ and the morphism $X \to X//G$ defined by the embedding $k[X]^G \to k[X]$ by $\pi_{X//G}$. $\pi_{X//G}$ is surjective and constant on the orbits of G. It can be shown that the pair $(X//G, \pi_{X//G})$ is the affine quotient for the action $G : X$ (see [16, Theorem 4.9]). Moreover if $f_1, ..., f_m$ generate the ring of invariants for the action $G : X$, $X//G$ can be interpreted as the image of the morphism:

$$X \to \mathbb{C}^m \quad x \to (f_1(x), ..., f_m(x)).$$

An important property of $\pi_{X//G}$ is that every fiber contains exactly one closed orbit, which in the case of the action described in Section 2.2.1 means that $X//G$ parametrizes the isomoprhism classes of semisimple representations. For $X = Rep_\alpha Q$ and $G = GL_\alpha$ this quotient space will be denoted by $iss_\alpha Q$, and the ring of invariant polynomials (which is the coordinate ring of $iss_\alpha Q$) will be denoted by $\mathbb{C}[iss_\alpha Q]$.

2.3 The coordinate ring

As noted above, the map $\pi : Rep_\alpha Q \to iss_\alpha Q$ can be realized in coordinate form with the help of a generator system of the ring of invariants. A cycle c in Q is a sequence of arrows $\rho_1, ...\rho_m$ for which $t(\rho_i) = s(\rho_{i+1})$ and $t(\rho_m) = s(\rho_1)$ holds (so we allow a cycle to run through a vertex more than once). For a cycle $c = (\rho_1, ..., \rho_m)$ consider the polynomial:

$$f_c : Rep_\alpha Q \to \mathbb{C} \quad X \to Tr(X_{\rho_1} ... X_{\rho_n}).$$

Clearly f_c is GL_α invariant. Moreover if c_1 and c_2 are cyclic permutations of each other then $f_{c_1} = f_{c_2}$. A cycle is called *primitive* if it does not run through any vertex more than once. Any cycle can be decomposed into primitive cycles, it is however not true that the corresponding invariant polynomial decomposes to a product of invariants corresponding to primitive cycles. We call a cycle *quasi-primitive* for a dimension vector α if the vertices that are run through more than once have dimension bigger than 1. If c is not quasi-primitive then for some cyclic permutation of its arrows $X_{\rho_1}...X_{\rho_n}$ will be a product of 1x1 matrices and $Tr(X_{\rho_1}...X_{\rho_n})$ will be the product of the traces of these matrices, so we will be able to write f_c as a product of polynomials corresponding to quasi-primitive cycles. We recall a result of LeBruyn and Procesi [4], that shows us that quasi-primitive cycles of a bounded length generate all of the invariant polynomials.

Theorem 2.4. *$\mathbb{C}[iss_\alpha Q]$ is generated by all f_c where c is a quasi-primitive cycle with length smaller than α^2+1. We can turn $\mathbb{C}[iss_\alpha Q]$ into a graded ring by giving f_c the length of its cycle as degree.*

3 Tools

For the rest of the report we will be interested in the classification of quiver settings based on some geometrical properties of $iss_\alpha Q$. A quiver setting (Q, α) is said to be *coregular*, if $iss_\alpha Q$ is an affine space. This is the same as $iss_\alpha Q$ being smooth at 0 (see [2, Theorem 2.1]).

Definition 3.1. An affine variety V of dimension n is called a complete intersection if

$$\mathbb{C}[V] \cong \mathbb{C}[X_1, ..., X_k]/(f_1, ..., f_l),$$

such that $k - l = n$.

This is also called an *ideal theoretic complete intersection* (a *set theoretic complete* intersection is defined similarly, replacing the ideal generated by the polynomials f_i by the generated radical ideal). A quiver setting (Q, α) is called a complete intersection if $iss_\alpha Q$ is a complete intersection. We will abbreviate the name of this property into C.I..

The aim is to classify quiver settings with these two properties. The classification has been done for the coregular quiver settings and the symmetric C.I. quiver settings by Raf Bocklandt in [2] and [3]. In this report we will also show a classification for non-symmetric C.I. quiver settings when all of the vertices have dimension 1. For these purposes we will introduce some methods of simplifying the structure of a quiver while preserving the above properties.

3.1 Subquivers

Definition 3.2. A quiver $Q' = (V', A', s', t')$ is a *subquiver* of the quiver $Q = (V, A, s, t)$ if (up to graph isomorphism) $V' \subseteq V$, $A' \subseteq A$, $s' = s|_{A'}$ and $t' = t|_{A'}$.

If Q' is a subquiver of Q and $\alpha' = \alpha|_{Q'}$, then if Q is coregular then Q' is coregular and if Q is a C.I. then Q' is a C.I., so to show that a quiver is not coregular (resp. C.I.) it is satisfactory to show a subquiver that is not coregular (resp. C.I.). The first statement can be found in [2, Lemma 2.3], and the second one in [3, Lemma 4.3], although the proof for the second statement in that article is not clear. However a similar statement for the ring that is invariant under the action of $SL_\alpha \subset GL_\alpha$ is proven in [6, Lemma 3.3], and the same argument can be applied in the case of GL_α.

3.2 Strongly connected components and connected sums

Two vertices v and w are said to be *strongly connected* if there is a path from v to w and a path from w to v. Clearly this is an equivalence relation. The subquivers consisting of the set of vertices of an equivalence class and all arrows in between them are called the *strongly connected components* of Q. All cycles of a quiver will run inside one of the strongly connected components which leads to a result that we recall from [2, Lemma 2.4]:

Lemma 3.3. *If (Q, α) is a quiver setting then*

$$\mathbb{C}[iss_\alpha Q] = \bigotimes_i \mathbb{C}[iss_{\alpha_i} Q_i],$$

where Q_i are the strongly connected components of Q and $\alpha_i = \alpha|_{Q_i}$.

It follows that Q is coregular (resp. C.I.) if all of its strongly connected components are coregular (resp. C.I.).

If a quiver can be decomposed into subquivers that have no arrows running in between them and only intersect each other in vertices of dimension one, then it is easy to see that every quasi-primitive cycle has to run inside one of these subquivers. This inspires the following definition:

Definition 3.4. A quiver $Q = (V, A, s, t)$ is said to be the *connected sum* of 2 subquivers $Q_1 = (V_1, A_1, s_1, t_1)$ and $Q_2 = (V_2, A_2, s_2, t_2)$ at the vertex v, if the two subquivers make up the whole quiver and only intersect in the vertex v. So in symbols $V = V_1 \cup V_2$, $A = A_1 \cup A_2$, $V_1 \cap V_2 = v$, and $A_1 \cap A_2 = \{\emptyset\}$. We note this by $Q = Q_1 \#_v Q_2$. The connected sum of three or more quivers can be defined similarly, for sake of simplicity we will write $Q_1 \#_v Q_2 \#_w Q_3$ instead of $(Q_1 \#_v Q_2) \#_w Q_3$.

Since the ring of invariants is generated by the polynomials associated to quasi-primitive cycles, a similar result to Lemma 3.3 can be said about connected sums in vertices of dimension one [3, Lemma 3.2]:

Lemma 3.5. *Suppose $Q = Q_1 \#_v Q_2$ $\alpha(v) = 1$, then*

$$\mathbb{C}[iss_\alpha Q] = \mathbb{C}[iss_{\alpha_1} Q_1] \otimes \mathbb{C}[iss_{\alpha_2} Q_2],$$

where $\alpha_{1,2} = \alpha|_{Q_{1,2}}$.

We will call a quiver *prime* if it can not be written as a non-trivial connected sum in vertices of dimension one. Based on the previous two lemmas we can conclude that it is satisfactory to classify coregular or C.I. quiver settings that are prime and strongly connected.

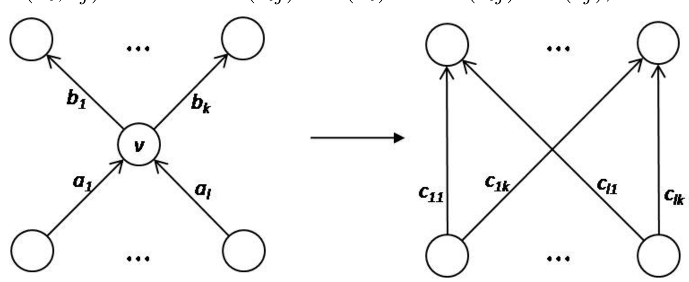
3.3 Reduction steps

In [2] Raf Bocklandt introduces some methods of reducing the number of arrows and vertices in a quiver setting (Q, α), so that the ring of invariants of the new quiver will be the same or closely related to $\mathbb{C}[iss_\alpha Q]$. We recall these reduction steps. (Once again ϵ_v is the dimension vector that is 1 in v and 0 elsewhere)

Lemma 3.6. *(Reduction RI: removing vertices) Suppose (Q, α) is a quiver setting and v is a vertex without loops such that*

$$\chi_Q(\alpha, \epsilon_v) \geq 0 \quad \textit{or} \quad \chi_Q(\epsilon_v, \alpha) \geq 0.$$

Let $(i_1, ..., i_l)$ denote the vertices from which arrows point to v and $(u_1, ..., u_v)$ denote the vertices to which arrows point from v. Construct a new quiver setting (Q', α') by removing the vertex v and all of the arrows $(a_1, ..., a_l)$ pointing to v and the arrows $(b_1, ..., b_k)$ coming from v, and adding a new arrow c_{ij} for each pair (a_i, b_j) such that $s'(c_{ij}) = s(a_i)$ and $t'(c_{ij}) = t(b_j)$, as illustrated below:



These two quiver settings now have isomorphic ring of invariants.

Lemma 3.7. *(Reduction RII: removing loops of dimension 1) Suppose that (Q, a) is a quiver setting and v a vertex with k loops and $\alpha(v) = 1$. Take Q' the corresponding quiver without the loops of v, then the following identity holds:*

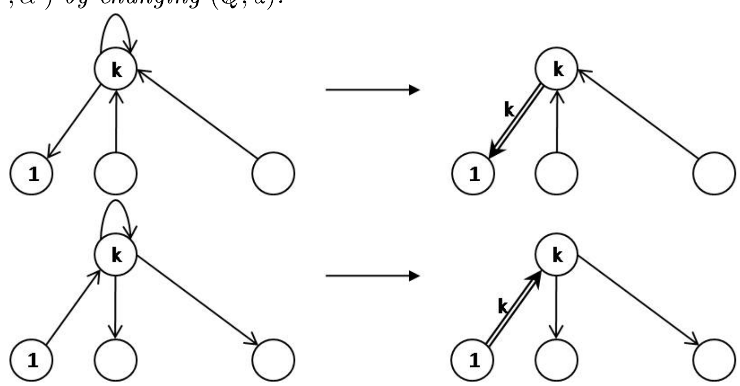
$$\mathbb{C}[iss_\alpha Q] = \mathbb{C}[iss_\alpha Q'] \otimes \mathbb{C}[X_1, ..., X_k],$$

where X_i are the polynomials that correspond to the loops of v.

Lemma 3.8. *(Reduction RIII: removing a loop of higher dimension). Suppose (Q, a) is a quiver setting and v is a vertex of dimension $k \geq 2$ with one loop such that*

$$\chi_Q(\alpha, \epsilon_v) = -1 \quad \text{or} \quad \chi_Q(\epsilon_v, \alpha) = 1.$$

(In other words, aside of the loop, there is either a single arrow leaving v and it points to a vertex with dimension 1, or there is a single arrow pointing to v and it comes from a vertex with dimension 1). Construct a new quiver setting (Q', α') by changing (Q, a):



We have the following identity:

$$\mathbb{C}[iss_\alpha Q] = \mathbb{C}[iss_\alpha Q'] \otimes \mathbb{C}[X_1, ..., X_k].$$

Since the tensor product with a polynomial ring does not change the property of being coregular or C.I., we can summarize:

Proposition 3.9. *If (Q, α) can be transformed into (Q', α') by the above steps or their inverses, then (Q, α) is coregular (resp. C.I.) if and only if (Q', α') is coregular (resp. C.I.).*

We will call a quiver setting *reduced* if none of the above steps can be applied to it, we can conclude that it is satisfactory to classify coregular and C.I. quiver settings that are reduced.

3.4 Local quivers

Both smoothness and being a C.I. are properties that can be interpreted locally. Smoothness of a variety by definition means that it is smooth in every point (being smooth in a point means that the rank of the Jacobian of the defining polynomials is maximal locally around that point). We say that a variety is a C.I. in a point x if the ideal of the variety is a C.I. in the local ring of that point in the ambient affine space. It can be shown that if the variety is a C.I. than this holds for every point (see for ex. [13]). This means that in order to prove that $iss_\alpha Q$ is not coregular or a C.I. for some quiver setting it satisfies to show that it does not have these properties locally around a given point.

It is also known (see for example the proof of Lemma 3.3 in [6]) that a homogeneous ideal in a polynomial ring is a complete intersection if its localization by the ideal of positively graded elements is a complete intersection. Since the ideal of relations for a quiver is a homogeneous ring (if we give the generators the grade equal to the length of the corresponding cycles), to see that a quiver is a C.I. it suffices to see that its localization around the zero representation is a C.I..

To understand the local structure of $iss_\alpha Q$ we recall some results from Luna [14] and LeBruyn and Procesi [4]. For these we will need some definitions.

An *étale morphism* of affine varieties is a smooth (in the analytical sense) morphism of relative dimension 0. (The analogue of the notion of submersion for complex manifolds.) These morphisms are useful for us since if there is an étale isomorphism from an open neighborhood of $p \in Q$ to an open neighborhood of $p' \in Q'$ then Q will be smooth (resp. locally C.I.) in p if and only if Q' is smooth (resp. locally C.I.) in p'. (The first statement is true because the morphism is smooth in the analytical sense for the latter see [10].)

If X is an affine variety over the field k, the *tangent space* T_xX of a point x of the affine variety X is the k-vector space of k-derivations of $k[X]$, i.e. linear maps $D : k[X] \to k$ such that $D(fg) = f(x)(Dg) + (Df)g(x)$. A morphism $\varphi : Y \to X$ defines a map of tangent spaces $(d\varphi)_y : T_yY \to T_xX$, which is called the *differential* of φ at y. If Y is a subvariety of X and y is a point on Y then the tangent space T_yY can be regarded as a subspace of the tangent space T_yX (formally we can regard $(d\iota)T_yY \subseteq T_yX$, where ι is the inclusion map). The *normal space* of a subvariety at a point x is direct complement of the tangent space of the subvariety.

If G is a reductive group acting on the affine variety X, and x is a point whose orbit is closed, let N_x denote the normal space of the orbit of x at the point x. The stabilizer group G_x acts linearly on N_x, so we can consider the quotient variety $N_x//G_x$. It follows from Luna's étale slice theorem [14] that there is an étale isomorphism between a neighborhood V of $0 \in N_x//G_x$ and a neighborhood U in $\pi_{X/G}(x) \in X/G$.

In the case of GL_α acting on $iss_\alpha Q$ a theorem of Le Bruyn and Procesi [4, Theorem 5] showed that for every point $p \in iss_\alpha Q$ corresponding to a semi-simple representation, we can build a quiver setting (Q_p, α_p) which will be isomorphic as a GL_α representation to the normal space of the orbit of p.

Theorem 3.10. *For a point $p \in iss_\alpha Q$ corresponding to a semisimple representation $V = S_1^{\oplus a_1} \oplus ... \oplus S_k^{\oplus a_k}$, there is a quiver setting (Q_p, α_p) called the* local quiver setting *such that we have an étale isomorphism between an open neighborhood of the zero representation in $iss_{\alpha_p} Q_p$ and an open neighborhood of p in $iss_\alpha Q$.*

Q_p has k vertices corresponding to the set $\{S_i\}$ of simple factors of V and between S_i and S_j the number of arrows equals

$$\delta_{ij} - \chi_Q(\alpha_i, \alpha_j),$$

where α_i is the dimension vector of the simple component S_i and χ_Q is the Ringel form of the quiver Q. The dimension vector α_p is defined to be $(a_1, ..., a_k)$, where the a_i are the multiplicities of the simple components in V.

Remark 3.11. Due to our earlier note on étale isomorphisms preserving the properties of smoothness and being C.I., to show that a quiver setting is not coregular (resp. C.I.) it is satisfactory to find a local quiver setting that is not coregular (resp. C.I.).

The structure of the local quiver setting only depends on the dimension vectors of the simple components. So to find all local quivers of a given quiver setting we have to decompose α into a linear combination of dimension vectors $\alpha = \sum a_i * \beta_i$ ($a_i \in \mathbb{N}$ and the β_i-s are not necessarily different) and check if there is a semi-simple representation corresponding to this decomposition. This depends on two conditions: there has to be a simple representation corresponding to each β_i which we can check using Theorem 2.2, and if some of the β_i-s are the same there has to be at least as many different simple representation classes with dimension vector β_i. For checking the latter condition we recall from [4, Theorem 6] that in all of the cases described in Theorem 2.2 the dimension of $iss_\alpha Q$ is given by $1 - \chi_Q(\alpha, \alpha)$, which is bigger than zero except for the one vertex without loops, so in all the other cases there are infinitely many classes of semi-simple representations, and in the case of one vertex without loops there is a unique simple representation.

3.5 Quivers with one dimensional vertices

We will briefly overview what the above results mean when all the vertices have dimension 1. For a quiver $Q = (V, A, s, t)$ and $\alpha = (1, ..., 1)$, $1 - \chi_Q(\epsilon_v, \alpha)$ and $1 - \chi_Q(\alpha, \epsilon_v)$ are the in-degree and the out-degree of the vertex v, and $\chi_Q(\alpha, \alpha) = |V| - |A|$. According to Theorem 2.2 there is a simple representation with dimension vector α if and only if Q is strongly connected ($\chi_Q(\epsilon_v, \alpha) \leq 0$ and $\chi_Q(\alpha, \epsilon_v) \leq 0$ holds automatically in this case). Applying Theorem 3.10 we can see that to construct a local quiver (Q', α') we have to decompose Q to strongly connected complete subquivers, then the vertices of Q' will correspond to these subquivers, and the number of arrows between two vertices will equal to the number of arrows between the corresponding subquivers of Q. Since each simple component is listed once in the decomposition, we have $\alpha' = (1, ..., 1)$. We will say that Q' is the local quiver we get by *gluing* the vertices in some strongly connected subquivers.

Also for a strongly connected Q, $dim(iss_\alpha Q) = 1 - \chi_Q(\alpha, \alpha) = 1 + |A| - |V|$. The quasi-primitive cycles and the primitive cycles are the same, and they generate the ring of invariants. It is also clear that all of these cycles are needed to generate that ring. Let C denote the set of primitive cycles in Q, $iss_\alpha Q$ is embedded in a $|C|$ dimensional affine space, so

$$codim(iss_\alpha Q) = |C| + |V| - |A| - 1.$$

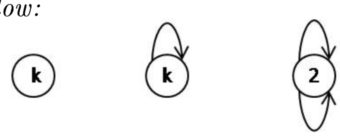
For an arbitrary quiver Q we will use the notation $F(Q) = |C| + |V| - |A| - 1$. (It is worth noting that we now have a geometrical proof for the combinatorial fact that $F(Q) \geq 0$ for any strongly connected quiver Q.) For a quiver setting in which all vertices are 1 dimensional, $iss_\alpha Q$ being smooth (so an affine space) is equivalent to $F(Q) \geq 0$, and $iss_\alpha Q$ being a C.I. is equivalent to the ideal of $iss_\alpha Q$ being generated by $F(Q)$ elements.

We also note that RIII can never be applied on a quiver with one dimensional vertices, so being reduced in this case means, that there is no loops in the quiver and all the vertices have in-degree and out-degree greater than or equal to 2, or that the quiver consists of a single vertex with no loops.

4 Coregular Quiver Settings

Raf Bocklandt in [2] gives a complete classification of all quiver settings that are coregular.

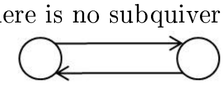
Theorem 4.1. *Let (Q, α) be a genuine strongly connected reduced quiver setting. Then (Q, α) is coregular if and only if it is one of the three quiver settings below:*



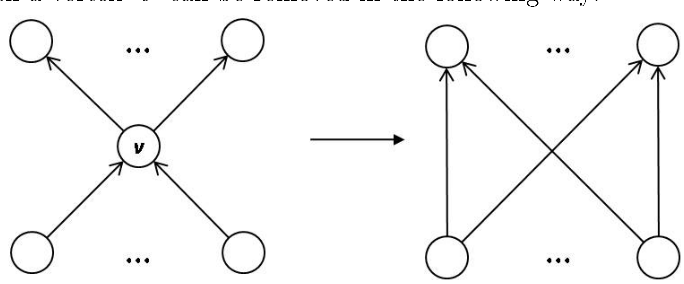
.

4.1 Proof of Theorem 4.1

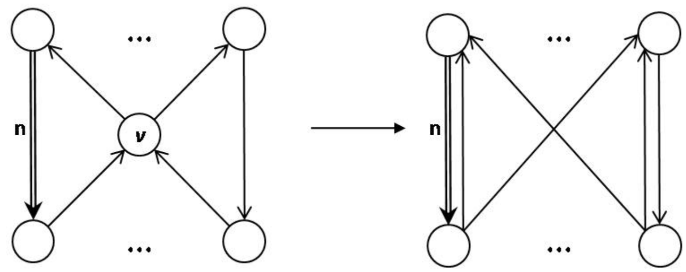
There is an error in [2], in the proof for the above theorem. When the author discusses the case $\alpha = (1, ..., 1)$, he argues on the bottom of page 312 that when there is no subquiver of form



then a vertex 'v' can be removed in the following way:



without changing the number of primitive cycles. This is however not true since non-primitive cycles, that run through v multiple times, but do not run through any other vertex more then once, will become primitive cycles in the new quiver. The number of new cycles can be arbitrarily large as demonstrated on the example below:



The quiver on the left has $n+1$ primitive cycles, while the one on the right has $2n+1$.

As it is explained in Section 3.5, to prove Theorem 4.1 in the case $\alpha = (1, ..., 1)$, we have to see that the only reduced quiver, for which $F(Q) = 0$ holds, is the one consisting of a single vertex with no loops. This follows from the lemma below:

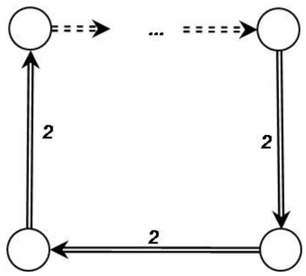
Lemma 4.2. *If Q is a strongly-connected quiver without loops, and for every vertex the in-degree and the out-degree are both at least 2, then $F(Q) \geq 1$.*

Proof. We prove the theorem by induction on the number of vertices. For one vertex the statement is true, since there is no such quiver with one vertex at all. Lets suppose we already saw that the statement is true for quivers with at most k vertices. It then follows that the following stronger statement is true for quivers with at most k vertices:

(*) *If Q is a strongly-connected quiver,* ***with at least two vertices****, without loops, and for every vertex,* ***with the possible exception of one vertex,*** *the in-degree and the out-degree are both at least 2, then $F(Q) \geq 1$.*

We prove this by induction as well. (*) is obviously true if there is only two vertices since if one of them has in-degree and out-degree two or bigger then so does the other. Lets suppose (*) is true for some $l < k$, and lets regard a quiver Q with $l+1$ vertices that has at most one vertex whose in- and out-degrees are not both at least two. If it has no such vertex then $F(Q) \geq 1$ follows from $k \geq l+1$ and the induction hypothesis on the original lemma. If it has exactly one such vertex than we apply the reduction step RI, and then RII to remove all possible loops, and get a quiver Q' that has again at most one vertex whose in- and out-degrees are not both at least two. So applying the induction hypothesis on Q' we get $F(Q') \geq 1$ and because neither RI, nor RII can change this property, $F(Q) \geq 1$ holds.

Now we proceed with the induction on the original lemma. Lets suppose Q is a quiver with $k+1$ vertices for which every vertex has in- and out-degrees 2 or greater. If all primitive cycles of Q are $k+1$ long than Q has a subquiver of form:



for which $F(Q) \geq 1$.

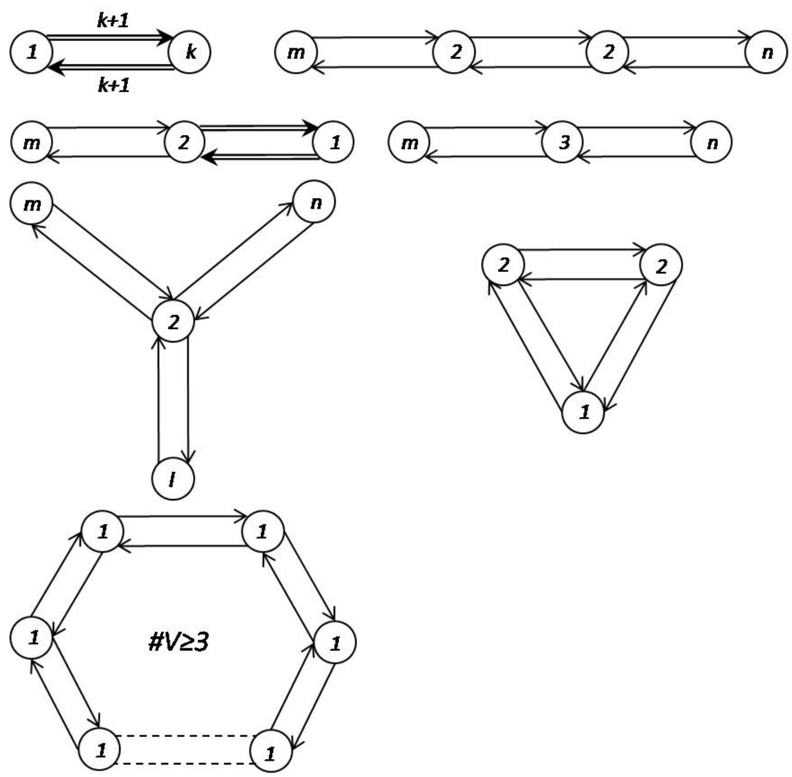
If there is a primitive cycle shorter than $k+1$ then let Q' be the local quiver of Q we get by gluing the vertices of this cycle. Let Q'' be the quiver we get from Q' by removing all loops. Q'' has at most one vertex that can have in- or out-degree 1 (namely the new vertex we created by gluing the cycle), it is strongly connected and has at least 2 vertices but no more than k, so we can apply (*) and see that $F(Q'') \geq 1$. Since we got Q'' by applying RII on a local quiver of Q according to Proposition 3.9 and Remark 3.11, $F(Q) \geq 1$ follows. $\square$

5 Complete Intersections

5.1 The symmetric case

A *symmetric quiver setting* is one, in which for any two vertices v_1 and v_2 the number of arrows pointing from v_1 to v_2 equals the number of arrows pointing from v_2 to v_1. In [3] Bocklandt classified all the symmetric prime reduced quiver settings.

Theorem 5.1. *Let (Q, α) be a symmetric prime reduced quiver setting without loops. If $iss_\alpha Q$ is a complete intersection then (Q, α) is either coregular or is one of the following list.*



(The last pictures shows a quiver with at least three vertices whose arrows form two oppositely directed cycles that both go through all the vertices.)

5.2 The one dimensional case

Giving a list of all C.I. quiver settings that are reduced (in the sense described in 3.3) seems hopeless even in the $\alpha = (1, ..., 1)$ case. Here we will introduce a new reduction step that preserves both the property of being C.I. and not being C.I., and proceed to show that a strongly connected, prime quiver setting which can not be reduced by either this new reduction step or the steps RI and RII is C.I. if and only if it is the quiver consisting of a single vertex and no loops. (We remind that the reduction step RIII is never applicable to quivers with one dimensional vertices, so we will not need it in this section.)

We will call a pair of vertices a *connected pair* if there is arrows both ways between them. Also note that by *path* we always mean a directed path that does not run through the same vertex twice.

Theorem 5.2. *Let (Q, α) be a quiver with $\alpha = (1, ..., 1)$, and (v_1, v_2) a connected pair in Q. Let (Q', α') denote the local quiver of Q we get by gluing the vertices v_1 and v_2. Suppose at least one of the following holds:*

a) There are exactly two paths from v_1 to v_2 and exactly two paths from v_2 to v_1.

b) There is exactly one path from v_1 to v_2.

c) There is exactly one path from v_2 to v_1.

Then we have: (Q, α) is a complete intersection if and only if (Q', α') is a complete intersection.

Proof. Since (Q', α') is a local quiver of (Q, α), we only have to prove that if (Q', α') is a C.I. then (Q, α) is a C.I..

The arrow from v_1 to v_2 will be noted by a_1 and the the arrow from v_2 to v_1 will be noted by a_2 (Note that we never exclude the case in the theorem that there is more arrows running between v_1 and v_2 but we will not use any special notations for those.) Primitive cycles running through both v_1 and v_2 are formed by disjoint paths from v_1 to v_2 and v_2 to v_1. In case a) the paths between the two vertices are a_1, a_2 and one more path both ways which will be denoted by Γ_1 and Γ_2. It follows that there are 3 or 4 primitive cycles running through the two vertices depending on whether Γ_1 and Γ_2 intersect each other or not (anywhere else than in the vertices v_1 and v_2). The element of $\mathbb{C}[iss_\alpha Q]$ that corresponds to the cycle (a_1, a_2) will be noted by p, the elements that correspond to (a_1, Γ_2) and (a_2, Γ_1) will be noted by q_1 and q_2, and if Γ_1 and Γ_2

form a primitive cycle the corresponding element will be noted by s. We will say we are in case a1) when Γ_1 and Γ_2 intersect each other, and case a2) when they do not. In case b) and c), the paths from v_1 and v_2 will be denoted by $\Gamma_{1,...,j}$ and the primitive cycles running through the two vertices will be denoted by p and $q_{1,...,j}$ as above. The rest of the primitive cycles in Q will be denoted by $r_1, ... r_k$.

Since all dimension vectors in the proof are $(1, ..., 1)$ we will write Q instead of (Q, α). $\mathbb{C}[Rep_\alpha Q]$ is a polynomial ring in $|A|$ variables, and the GL_α invariants that correspond to the primitive cycles are monomials in this ring. So, according to Theorem 2.4, $\mathbb{C}[iss_\alpha Q]$ is isomorphic to a monomial subring of a polynomial ring. Let n denote the number of primitive cycles in Q and $f_1, ..., f_n$ denote the monomials corresponding to the primitive cycles. $iss_\alpha Q$ is then embedded in an n dimensional affine space, and we have a morphism

$$\varphi : \mathbb{C}[x_1, ... x_n] \to \mathbb{C}[iss_\alpha Q] \quad \varphi(x_i) = f_i$$

for which $Ker(\varphi)$ is the ideal of the variety $iss_\alpha Q$. We will refer to $Ker(\varphi)$ as the ideal of relations, and to its elements as relations on Q. In our case the ring $\mathbb{C}[x_1, ... x_n]$ will be denoted by R, and its variables will be noted by x_p, x_{q_i}, x_{r_i}, x_s.

It is important to note that, since $\mathbb{C}[iss_\alpha Q]$ is a monomial ring, the ideal $Ker(\varphi')$ (also referred to as the *toric ideal* of the monomial subring) is generated by binomials (see for ex., [15, Proposition 7.1.2]). Moreover some binomial $m_1 - m_2$ is in $Ker(\varphi)$ if and only if the multisets of arrows corresponding to m_1 and m_2 are the same (this is clear if we regard $\mathbb{C}[iss_\alpha Q]$ as a subring of the polynomial ring $\mathbb{C}[Rep_\alpha Q]$).

Note that when we glue together some vertices in a quiver there will be a natural graph-homomorphism between the old and the new quiver, so it makes sense to talk about the image and pre-image of vertices, arrows, paths and cycles. The image of a path remains a path if it did not run through the glued subquiver twice, and it will become a cycle if it started from and ended in the glued subquiver. The image of a cycle will always be a cycle, but the image of a primitive cycle will only remain primitive if it did not run through the glued subquiver twice. The pre-image of a path (resp. a primitive cycle) is always a path (resp. a primitive cycle).

In our case the images of the arrows a_1 and a_2 will be loops on the glued vertex w, which (by the reduction step RII) can be removed without changing the C.I. property of the quiver. For sake of simplicity Q' will denote the quiver from which these cycles have already been removed. With these loops removed the only case when the image of a primitive cycle in Q will not be primitive is a2), the image of the primitive cycle corresponding to s will be formed by the images of the cycles corresponding to q_1 and q_2. So it makes sense to denote the the primitive cycles in Q' by q'_i, r'_i and the variables of the corresponding polynomial ring R' by $x_{q'_i}$, $x_{r'_i}$. As above we have a morphism $\varphi' : R' \to \mathbb{C}[iss_{\alpha'}Q']$ for which $Ker(\varphi')$ is the ideal of relations on Q'.

The graph-homomorphism between Q and Q' induces an epimorphism $\Psi : \mathbb{C}[Rep_\alpha Q] \to \mathbb{C}[Rep_{\alpha'}Q']$, whose restriction to $\mathbb{C}[iss_\alpha Q]$, which will be denoted by ψ, is an epimorphism to $\mathbb{C}[iss_{\alpha'}Q']$ (since the pre-images of primitive cycles of Q' are primitive cycles of Q.) We have:

$$\psi(p) = 1 \quad \psi(q_i) = q'_i \quad \psi(r_i) = r'_i$$

and in the case a2) $\psi(s) = q'_1 q'_2$.

We can also define a morphism:

$$\theta : R \to R' : \quad \theta(x_p) = 1 \quad \theta(x_{q_i}) = x_{q'_i} \quad \theta(x_{r_i}) = x_{r'_i}$$

and in the case a2) $\theta(x_s) = x_{q'_1} x_{q'_2}$.

We have:

$$\psi \circ \varphi(x_p) = \varphi' \circ \theta(x_p) = 1 \quad \psi \circ \varphi(x_{q_i}) = \varphi' \circ \theta(x_{q_i}) = q_i \quad \psi \circ \varphi(x_{r_i}) = \varphi' \circ \theta(x_{r_i}) = r'_i$$

and in the case a2) $\psi \circ \varphi(x_s) = \varphi' \circ \theta(x_s) = q'_1 q'_2$. To sum this up the diagram below is commutative and all the morphisms in it are epimorphisms.

$$\begin{array}{ccc} & \varphi & \\ R & \to & \mathbb{C}[iss_\alpha Q] \\ & & \\ \downarrow \theta & & \downarrow \psi \\ & \varphi' & \\ R' & \to & \mathbb{C}[iss_{\alpha'} Q'] \end{array}$$

Clearly $Ker(\theta) =< 1 - x_p >$ in cases a1), b) and c), and $Ker(\theta) =< 1 - x_p, x_{q_1}x_{q_2} - x_s >$ in case a2).

Lemma 5.3. $Ker(\psi) =< 1 - p >$.

Proof. Let us suppose $f \in Ker(\psi)$, and let $g \in R$ be a polynomial for which $\varphi(g) = f$. Clearly $\theta(g) \in Ker(\varphi')$, so $\theta(g) = \sum t_i b_i$ for some binomials $b_i \in Ker(\varphi')$. R' can be regarded as a subring of R and the injection map $\iota : R \to R'$ is a right inverse of θ. $g - \iota\theta(g) \in Ker\theta$, so

$$g = \iota(\sum t_i b_i) + \tau_1 * (1 - x_p)$$

in cases a1), b) and c), and

$$g = \iota(\sum t_i b_i) + \tau_1(1 - x_p) + \tau_2 * (x_{q_1}x_{q_2} - x_s)$$

in case a2). Note that in case a2) $s = pq_1q_2$, since the multisets of arrows corresponding to the two sides of the equation are the same, so $q_1q_2 - s = (1 - p) * q_1q_2$. It follows that

$$f = \varphi\iota(\sum t_i b_i) + \varphi(\tau_1) * (1 - p)$$

in cases a1), b) and c), and

$$f = \varphi\iota(\sum t_i b_i) + \varphi(\tau_1) * (1 - p) + \varphi(\tau_2) * q_1q_2 * (1 - p)$$

in case a2).

So it is satisfactory to prove the lemma for binomials $m_1 - m_2 \in Ker(\psi)$. $\psi(m_1) =\psi(m_2)$ means that the multisets of arrows corresponding to $\psi(m_1)$ and $\psi(m_2)$ are the same in Q' so the multisets of arrows corresponding to m_1 and m_2 in Q only differ in the arrows a_1 and a_2. However these multisets are both unions of primitive cycles, and in such a union for each vertex the number of arrows leaving the vertex and going into the vertex are equal. Thus if for example the multiset corresponding to m_1 has a_1 in it k more times than the multiset corresponding to m_2 then it also must have a_1 in it k more times than m_2, which means $m_2 = p^k * m_1$, and it follows that

$$m_1 - m_2 = (1 - p^k) * m_1 \in< 1 - p > .$$

□

Lemma 5.4. $\theta(Ker(\varphi)) = Ker(\varphi')$.

Proof. Clearly $\theta(Ker(\varphi)) \subseteq Ker(\varphi')$. For surjectivity lets suppose that $t \in Ker(\varphi')$. Because θ is surjective there is a $y \in R$, such that $\theta(y) = t$. Since $\varphi' \circ \theta = \psi \circ \varphi$, $\varphi(y) \in Ker(\psi)$, and by the previous lemma $\varphi(y) = (1-p) * f$ for some $f \in \mathbb{C}[iss_\alpha Q]$. Since φ is surjective there is a $g \in R$ such that $\varphi(g) = f$. It follows that

$$y - (1 - x_p) * f \in Ker(\varphi)$$

and

$$\theta(y - (1 - x_p) * f) = t.$$

□

Denoting by V', A', C' the set of vertices, arrows and primitive cycles in Q', we have $|V'| = |V| - 1$ and $|A'| = |A| - 2$. In cases a1), b), c) $|C'| = |C| - 1$ so (as explained in Section 3.5)

$$codim(iss_{\alpha'}Q')) = F(Q') = |C'| + |V'| - |A'| - 1 = codim(iss_\alpha Q).$$

In case a2) $|C'| = |C| - 2$ and $codim(iss_{\alpha'}Q')) = codim(iss_\alpha Q)) - 1$.

As it has already been noted $Ker(\varphi)$ is generated by binomials. The elements of R and thus $Ker(\varphi)$ can be graded so that each variable has grade equal to the length of the corresponding primitive cycle. With this grading all of the binomials in $Ker(\varphi)$ are homogeneous, so $Ker(\varphi)$ is a homogeneous ideal. It is then known that all minimal homogeneous systems of generators of $Ker(\varphi)$ have the same number of elements, and that a generating set with minimal number of elements can be chosen to be homogeneous. So to find out the number of elements needed to generate $Ker(\varphi)$ we only have to find a minimal binomial generating set.

In order to see the relation between the minimal number of binomials needed to generate $Ker(\varphi)$ and $Ker(\varphi')$, lets take a look at how a minimal set of binomials generating the ideal of relations can be selected for an arbitrary quiver. Let U denote a multiset of arrows, in which each vertex has the same in-degree as out-degree (meaning that U is a union of directed cycles), we will say that the monomial m is a partition of U if U is a union of the cycles represented by

m. As we have noted earlier a binomial, $m_1 - m_2$ is in the ideal of relations if and only if m_1 and m_2 are partitions of the same multiset. For a relation $m_1 - m_2$ to be generated by some other binomials we need an equation

$$m_1 - m_2 = r_1(n_1 - l_1) + r_2(n_2 - l_2) + \dots r_k(n_k - l_k)$$

to hold for some monomials n_i, l_i and polynomials r_i. For the sake of simplicity we can suppose that no partial sum on the right hand side equals zero and that $m_1 = r_1 * n_1$ and $m_2 = r_k * l_k$ and for $1 \leq i \leq k-1 : r_i * l_i = r_{i+1} * n_{i+1}$ (we can achieve that by reordering the sum on the right hand side). Clearly the n_i-s and l_i-s correspond to partitions of subsets of U. This gives us a chance to find a minimal set of generators recursively.

If a minimal set A generating the relations on all multisets strictly smaller than U has already been found we will have to chose a minimal set of relations B on U, such that $A \cup B$ generate the relations on U. We can define a relation among the partitions of U: $m_1 \sim_U m_2$ if and only if $m_1 - m_2$ is generated by relations on multisets strictly smaller than U (meaning that we have an equation as above, with $deg(r_i) > 0$ for each i). Clearly this is an equivalence relation. If there are n equivalence classes of $\sim_U$, then we will need at least $n-1$ relations in B. For example if $m_1, m_2, \dots, m_n$ are representatives of each equivalence class of $\sim_U$, then $A \cup \{m_1 - m_2, m_2 - m_3, \dots, m_{n-1} - m_n\}$ minimally generate all the relations on U. Thus if we want to compare the sizes of minimal generating sets in $Ker(\varphi)$ and $Ker(\varphi')$, we only have to compare the sum of the number of equivalence classes for each multiset in the two quivers. Note that $\sim_U$ only depends on the quiver and the multiset and not how the generators in smaller multisets were chosen, so it makes sense to use the notation

$$E(U) := |\{equivalence\,classes\,of\ \sim_U\}| - 1.$$

Note that $E(U) = 0$ with finitely many exceptions since, the ideal of relations is finitely generated.

In case a1) it is satisfactory to prove that $\sum E(U) \leq \sum E(U')$, where the left hand sum runs over all the arrow multisets of Q and the right hand sum runs over all the arrow multisets in Q'. If U is a multiset of arrows in Q than by slight abuse of notation we can write $\theta(U)$ for its image in Q'. Obviously if U and V only differ in the arrows a_1 and a_2 then $\theta(U) = \theta(V)$. The left hand sum can be written as

$$\sum_{a_1 \in U \, or \, a_2 \in U} E(U) + \sum_{a_1 \notin U \, and \, a_2 \notin U} E(U)$$

and the right hand sum can be written as

$$\sum_{a_1 \notin U \, and \, a_2 \notin U} E(\theta(U)).$$

Lets now look at $\sum_{a_1 \in U \, or \, a_2 \in U} E(U)$. If $a_1 \in U$ but $a_2 \notin U$ then all partitions of U will be of form $q_1 * t$ (where t is a partition of $U \backslash k_1$), since the only cycle containing a_1 but not a_2 is q_1. These partitions are all equivalent since $q_1 * t_1 - q_1 * t_2 = q_1 * (t_1 - t_2)$ and $t_1 - t_2$ is a relation on a proper subset of U. So for such a U we have $E(U) = 0$, and the same can be said when $a_1 \notin U$ and $a_2 \in U$. Let U_q be the multiset formed by the arrows of q_1 and q_2. A partition of U_q is either $q_1 * q_2$ or of form $p * t$ (where t is a monomial corresponding to some partition of $U_q \backslash \{a_1, a_2\}$), clearly the latter are all equivalent with each other but can not be equivalent with $k_1 * k_2$ (noteworthily no partition consisting of exactly two cycles can be equivalent to some other partition since that would require one of the cycles to be written as a union of strictly smaller cycles which is of course impossible). So we have $E(U_q) = 1$. Let us now suppose that U contains both $a_1$and a_2 but it is not U_q. A partition of U in this case is either of form $p * t$ or of form $q_1 * q_2 * \varrho$ (t and ϱ are monomials corresponding to partitions of the remaining arrows). The ones that are of form $p * t$ are clearly equivalent with each other. If U has a partition of form $q_1 * q_2 * \varrho$ then $U_q \subset U$, so using relations on U_q we can show that $q_1 * q_2 * \varrho \sim_U p * \varrho'$ for some ϱ'. So in this case we also get $E(U) = 0$. Thus $\sum_{e_1 \in U \, or \, e_2 \in U} E(U) = 1$. So now we have to prove

$$\sum_{e_1 \notin U \, and \, e_2 \notin U} E(U) + 1 \leq \sum_{e_1 \notin U \, and \, e_2 \notin U} E(\theta(U)).$$

If U is a multiset that does not contain a_1 and a_2, and m_1 and m_2 are two partitions of U for which $\theta(m_1) \sim_{\theta(U)} \theta(m_2)$, then m_1 and m_2 are also equivalent in U. To see this, regard the equation

$$\theta(m_1) - \theta(m_2) = r_1(n_1 - l_1) + r_2(n_2 - l_2) + ...r_k(n_k - l_k).$$

Let r_i^*, n_i^*, l_i^* note the pre-images of the monomials on the right hand side that does not contain x_p, in this case

$$m_1 - m_2 = r_1^*(f_1^* - g_1^*) + r_2^*(f_2^* - g_2^*) + \ldots r_k^*(f_k^* - g_k^*)$$

holds in Q since the difference of the two sides could only be some multiple of $1 - x_p$, which can only be zero, since neither sides contain x_p. So we get $E(U) \leq E(\theta(U))$. Let $U_0 = U_q \setminus \{a_1, a_2\}$. $\theta(U_0)$ has the partition $x_{q_1'} * x_{q_2'}$ which is not the image of any partition of U_0 and is not equivalent to any other partition of $\theta(U_0)$ (once again, since it is a partition with exactly two cycles). So we get $E(U_0) + 1 \leq \theta(E(U_0)$ and with this

$$\sum_{e_1 \notin U \, and \, e_2 \notin U} E(U) + 1 \leq \sum_{e_1 \notin U \, and \, e_2 \notin U} E(\theta(U))$$

is proven.

We can use a similar argument in case a2) except that now it is not true anymore that $\theta(q_1) * \theta(q_2)$ is not the image of any partition of U_0, in fact it will be the image of the only partition of U_0 namely x_s (we remind x_s corresponds to the cycle formed by the arrows of $q_1$and q_2 without the arrows $a_1$and a_2). So this time we get

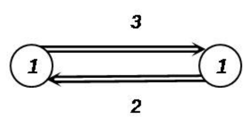
$$\sum E(U) \leq \sum E(U') + 1$$

which is exactly what we need to prove.

In case b) (case c) can be done similarly) we know that the only cycle going through the arrow a_2 is p. If for some $U_1 \subseteq U_2$ we have $\theta(U_1) = \theta(U_2)$ then $U_2 = U_1 \cup \{a_1, a_2\}^n$ (since both multisets correspond to unions of cycles). Since a_2 is then contained in U_2 at least n times, all partitions of U_2 can be written as $p^n * s$ and these are clearly equivalent with each other, so we have $E(U_2) = 0$. So every multiset in Q' has at most one pre-image that has more than one equivalence class. Also $E(U) \leq E(\theta(U))$ can be proven the same way as above, so $\sum E(U) \leq \sum E(U')$ follows. $\square$

The reduction step introduced in the above theorem will be referred to as *RIV*. Now we will proceed to show that the reduction steps RI-IV, and decomposition into prime components will reduce all strongly connected C.I. quivers to a single vertex without loops. First we will show two examples of non-C.I. quivers, that will play an important role in the proof of the upcoming lemmas.

Proposition 5.5. *The quiver setting*



is not a C.I..

Proof. There are 6 primitive cycles in the above quiver so

$$codim(iss_{\alpha}Q) = 6 + 2 - 5 - 1 = 2$$

Denoting the variables corresponding to the cycles by c_{ik} $1 \leq i \leq 3$, $1 \leq k \leq 2$, the following relations generate $Ker(\varphi)$:

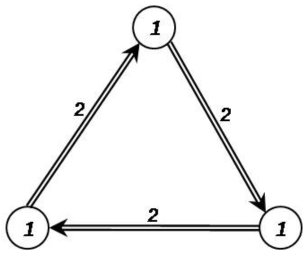
$$c_{11}c_{22} - c_{21}c_{12}$$

$$c_{11}c_{32} - c_{31}c_{12}$$

$$c_{21}c_{32} - c_{31}c_{22}.$$

Clearly none of these relations are generated by the others. The argument in the second part of Theorem 5.2 could be used to see that they indeed generate $Ker(\varphi)$, though to prove the fact that the quiver is not a C.I. it is satisfactory to see that any generating set consists of at least 3 elements. It is easy to see that $iss_{\alpha}Q$ in this case is isomorphic to the variety of 3x2 matrices with rank 1 or 0. $\square$

Proposition 5.6. *The quiver setting*



is not a C..I.

Proof. There are 8 primitive cycles in this quiver so

$$codim(iss(Q)) = 8 + 3 - 6 - 1 = 4.$$

Let a_{ik} $1 \leq i \leq 3$, $1 \leq k \leq 2$ denote the edges and c_{ikj} denote the cycle consisting of a_{1i}, a_{2k}, a_{3j}. Arrow multisets of type $\{a_{11,}a_{12}, a_{21}, a_{22}, a_{31}, a_{31}\}$ can be partitioned to directed cycles exactly two different ways, namely $c_{111} * c_{221}$ and $c_{121} * c_{211}$, giving us one relation for each of these multisets. These relations can not be generated by relations on smaller multisets, since the only non-trivial subsets of the above multiset that is a union of cycles consists of only one cycle, which does not yield any non-trivial relations. Thus the following six relations will be part of any minimal binomial generator system in the ideal of relations:

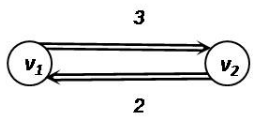
$$\begin{array}{ll} c_{111} * c_{221} = c_{121} * c_{211} & c_{112} * c_{222} = c_{122} * c_{212} \\ c_{111} * c_{212} = c_{112} * c_{211} & c_{121} * c_{222} = c_{122} * c_{221} \\ c_{111} * c_{122} = c_{112} * c_{121} & c_{211} * c_{222} = c_{212} * c_{221}. \end{array}$$

So the quiver setting is not a complete intersection. (It can be easily verified that a minimal generating set of $Ker(\varphi)$ consists of 9 elements in this case, but we will not need this fact.) □

Now we will prove two lemmas, that in essence will show us that whenever RIV can not be applied on a reduced, prime quiver setting, that quiver setting can not be a C.I.. (We remind that by "a reduced quiver setting" we still mean a quiver that can not be reduced with the original reduction steps RI-III.)

Lemma 5.7. *If (Q, α) is a quiver setting with $\alpha = (1, ..., 1)$ in which there is a connected pair (v_1, v_2), and there are at least three paths from v_1 to v_2 and at least two paths from v_2 to v_1 then (Q, α) is not a C.I..*

Proof. We prove by induction on the number of vertices in Q. If Q has two vertices, then it contains the sub-quiver:



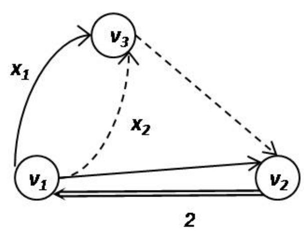
(I)

which is not a C.I. by Proposition 5.5.

We have an arrow a_1 from v_1 to v_2 and an arrow a_2 from v_2 to v_1, let x_1 and x_2 denote the paths from v_1 to v_2 that are not $\{a_1\}$, and y denote the path from v_2 to v_1 that is not $\{a_2\}$. Let us regard the sub-quiver Q' that is made of these three paths and the arrows between v_1 and v_2. If Q' has a vertex with in- or out-degree 1, then we can apply RI (which will not change the number of paths between v_1 and v_2) and by the induction hypothesis Q' can not be a C.I.. So we only have to look at the cases where all vertices have in- and out-degrees of at least 2 (so Q' is reduced).

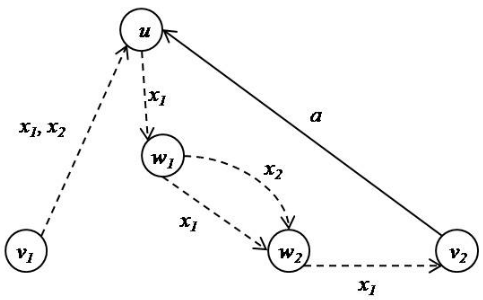
First we discuss the case when y consists of a single arrow from v_2 to v_1. Let us look at the vertex where the first arrow of x_1 points to. If it is v_2 then, since Q' is reduced, it can only be the quiver (I).

If it is some other vertex v_3 then since Q' is reduced there has to be another arrow pointing to v_3, which can only be part of x_2. If we delete the arrows that are in x_1 but not in x_2 except for the first arrow of x_1 we get a quiver of form:



(dashed arrows indicate a single directed path with arbitrarily many vertices) which once again can be reduced to (I).

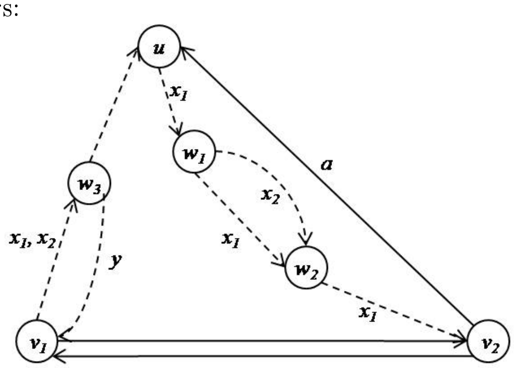
If y contains more than one arrow, let a denote its first arrow and u the vertex a points to. Since Q' is reduced x_1 or x_2 has to contain the vertex u as well. We can suppose x_1 does. The arrow a and the part of x_1 that is between u and v_2 forms a directed cycle c. Now we can regard the local quiver Q'' of Q' that we get by gluing the vertices of c. Q'' has less vertices than Q' so if it satisfies the conditions in the proposition we can apply the induction hypothesis and conclude that Q' is not a C.I.. In other words, for Q' to be a C.I. there can only be one directed path both ways between the images of v_1 and v_2 in Q'', meaning that there is only one directed path both ways between the cycle c and v_1 in Q'. It follows then that the segment of x_1 that is between v_1 and u is also contained in x_2. Let Q_0 be the subquiver of Q' that consists of x_1, a, and the part of x_2 that starts from the first arrow in which it differs from x_1 and ends in the first vertex which is also on x_1 (this vertex can be v_2). Q_0 is of form:

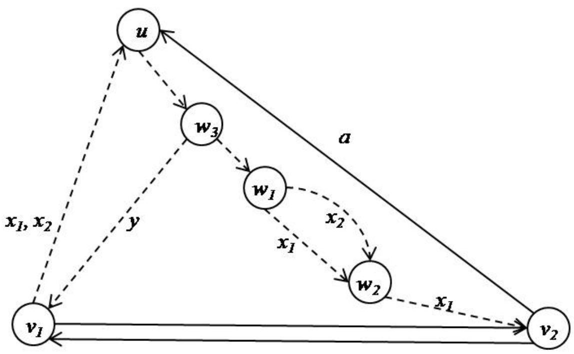


(Including the special cases when w_1 is u and w_2 is v_2.)

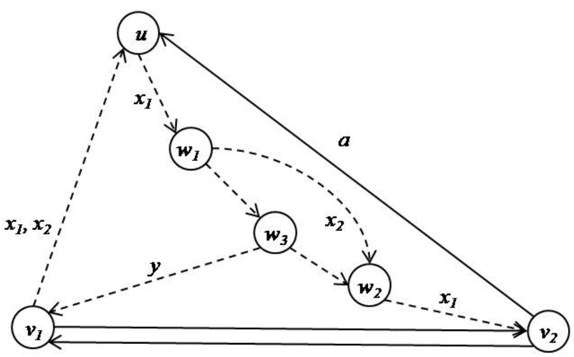
Let Q_1 be the subquiver of Q' we get by adding to Q_1 the part of y that is between the last vertex of y that is part of Q_0 (there is at least one such vertex: u) and v_1, as well as adding the two arrows between v_1 and v_2.

Depending on where y departs from Q_0, Q_1 will be one of the following quivers:

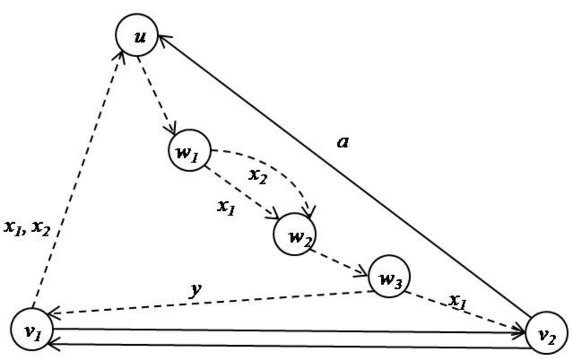
(II) 



(III)



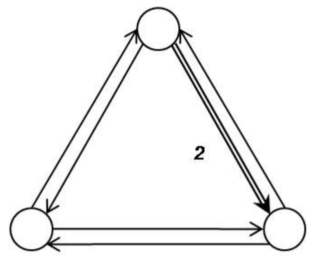
(IV)



(V)

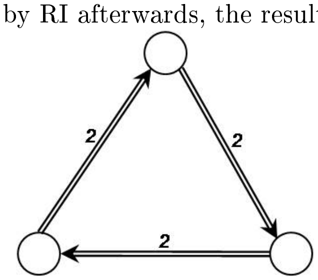
Case (II): The segments of x_1 and y between v_1 and w_3 form a cycle, and in the local quiver we get by gluing the vertices of this cycle there will still be at least two paths from v_1 to v_2 and a path from v_2 to v_1 so we can apply the induction hypothesis and conclude that Q' is not a C.I..

Case (III) and (IV): In both cases if we reduce by RI, we get the following quiver:



The local quiver we get by gluing the bottom two vertices will be (I) (or more precisely: (I) with two extra loops on one of the vertices) which is not a C.I..

Case(V): If we take the local quiver we get by gluing v_1 and v_2 , and reduce it by RI afterwards, the resulting quiver will be of form:



Which is not a C.I. according to Proposition 5.6, so neither can be Q'. $\square$

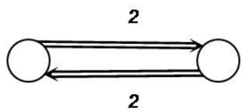
For technical purposes we will state the next lemma in a weaker and stronger form and prove them simultaneously, using parallel induction somewhat similarly to the proof of Lemma 4.2.

Lemma 5.8. *Let (Q, α) be a reduced (by RI, RII, and RIII), strongly connected, prime quiver setting with $\alpha = (1, ..., 1)$, that has at least two vertices and does not contain any connected pairs. Then (Q, α) is not a C.I..*

Lemma 5.9. *Let(Q, α) be a strongly connected, prime quiver setting with at least three vertices and no loops, and with $\alpha = (1, ..., 1)$. If there is a vertex v in Q such that v is a member of every connected pair of Q and any vertex except v has in-degree and out-degree at least 2, then (Q, α) is not a C.I..*

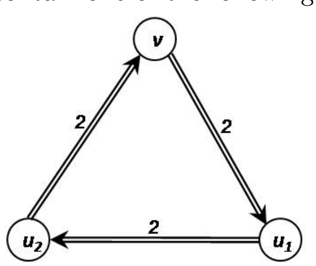
Proof. We will prove by induction on the number of vertices in Q. Supposing that Lemma 5.8 holds for quivers with at most k vertices and Lemma 5.9 holds for quivers with at most $k-1$ vertices, we will first show that Lemma 5.9 holds for quivers with k vertices as well. Then we will use this result to show that Lemma 5.8 holds for quivers with $k+1$ vertices.

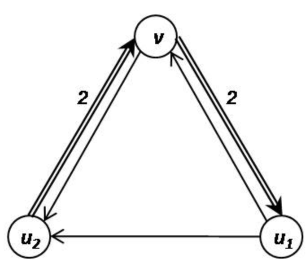
In the case Q is a quiver with two vertices Lemma 5.8 holds trivially since a strongly connected quiver with 2 vertices contains a connected pair. Lemma 5.9 does not hold for two vertices since the quiver:

(I) 

satisfies the conditions, and is a C.I.. Noteworthily it is the only exception (since one of the vertices need to have in- and out-degree 2 or greater and any further arrows would result in the quiver not being a C.I.).

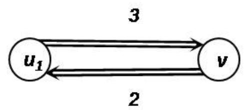
We will show the lemmas directly in the case of three vertices (it suffices to show Lemma 5.9). Let the vertices be v, u_1, u_2 with u_1 and u_2 having in- and out-degrees 2 or greater, and u_1 and u_2 not forming a connected pair. For the sake of simplicity we can suppose that there is no arrow from u_2 to u_1. In this case there has to be at least 2 arrows going from u_2 to v, and 2 arrows going from v to u_1 (because of the condition on the in- and out-degrees), and at least one arrow going from u_1 to u_2 (because the quiver is not prime). Moreover if there is only one arrow going from u_1 to u_2 then there also has to be at least one arrow going from v to u_2 since the in-degree of u_2 is at least 2. So the quiver will contain one of the following subquivers:

(II) 

(III) 

We have already seen in Proposition 5.6 that (II) is not a C.I..

For (III) if we delete the arrow from v to u_2 and apply RI on u_2 we get:

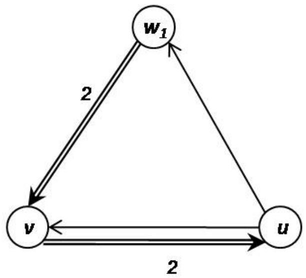


(IV)

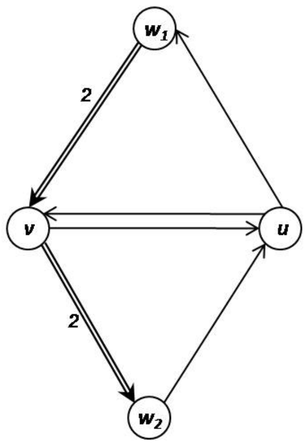
which is not a C.I..

Now let us suppose that Q has $k > 3$ vertices, satisfies the conditions in Lemma 5.9 and is a C.I.. If there is neither connected pairs nor a vertex with in- or out-degree less than 2 in Q we have a contradiction since we supposed that Lemma 5.8 holds for k vertices.

If there is a connected pair in Q, this pair has to contain the vertex v and some other vertex u. Let Q' denote the local quiver of Q we get by gluing the vertices of this connected pair. Q' is strongly connected but not necessarily prime. However if Q' has non-trivial prime components $Q'_1, \ldots, Q'_k$ then it must be the connected sum $Q' = Q'_1 \#_{v'} Q'_2 \#_{v'} \ldots \#_{v'} Q'_k$ since if two prime components would meet in some vertex $w' \neq v'$ then there would be two vertices x', y' in Q' such that all paths between them run through w', and since w' has a unique pre-image in Q the same would have to hold in Q for the pre-images of x', y', w' which would contradict with Q being a prime quiver. We can conclude that all vertices in the prime components, except v', have as many in- and out-degrees as they have in Q', which is the same as their pre-images have in Q. Also v' will be contained in all connected pairs of Q' and thus in all connected pairs of the prime components. If Q is a C.I. then so is Q' and its prime components, and due to the induction hypothesis on Lemma 5.9 this means that the prime components of Q' have to be quivers with two vertices that are of form (I) (otherwise they would satisfy the conditions in Lemma 5.9). This means that every vertex in Q aside of v and u have exactly two arrows going to either u or v and exactly two arrows arriving from either u or v, and no other arrows. Since Q is prime, there has to be an arrow between u and some other vertex $w_1 \neq v$. We can suppose this arrow is pointing from u to w_1. Since u and w_1 are not a connected pair the two arrows leaving w_1 are both pointing to v. Since u has in-degree of at least 2 there has to be either two arrows from v to u, or an arrow pointing to u from some other vertex w_2. In the latter case there also has to be two arrows pointing from v to w_2. This means Q contains one of the following sub-quivers:



(V)



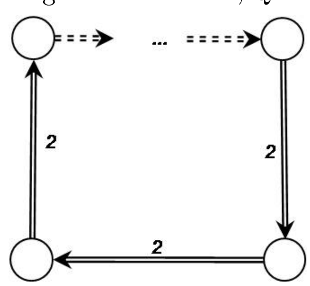
(VI)

neither of which are C.I.-s. (By applying RI on w_1 in the first case and both w_1 and w_2 in the second case we get quivers that contain a subquiver of form (IV).)

If Q contains no connected pairs but it has a vertex v with out-degree 1, then let u denote the vertex where the only arrow leaving v points to, and Q' denote the quiver we get by applying RI on v. Applying the same argument as above on Q' we can conclude that $Q' = Q'_1 \#_{v'} Q'_2 \#_{v'} \ldots \#_{v'} Q'_k$ where the Q'_i-s are all of form (I). Since u has in-degree of at least 2 in Q there has to be an arrow entering u from some vertex $w_2 \neq v$. Since Q is strongly connected it must contain an arrow that points to w_2, however due to our result on Q' this arrow can only leave from u or v. Since there is only one arrow leaving v and that points to u, we can conclude that there is an arrow pointing from u to w_2 contradicting with the supposition that Q contains no connected pairs.

Now we are left to prove Lemma 5.8 on k vertices, supposing that we already know that both lemmas are true for quivers with at most $k-1$ vertices. Let c be a cycle of length l in Q , going through the vertices $v_1, v_2, ..., v_l$. For the sake of simplicity v_i and v_j will denote the same vertex if $i \equiv j$ $(mod\, l)$. If c is a cycle of minimum length in Q then any arrow between the vertices of c has to point from some v_i to v_{i+1}. Since if an arrow pointed to any vertex at least two steps away in the cycle it would form a shorter cycle than l along with some of the original arrows of c. Also no arrow can point from v_i to v_{i-1} since Q contains no connected pairs. For Q to be a C.I. there can not be more than two extra arrows going between the vertices of c otherwise we could reduce it to (II) which is not a C.I.. Also if we look at the local quiver Q' we get by gluing the vertices of this cycle, using the same argument as above, we can see that the prime components in Q' satisfy the conditions of Lemma 5.9 except for having at least three vertices, so Q' has to either consist of a single vertex (if $l = k$) or be a connected sum of quivers of form (I). This means that any vertex in Q other then $v_1, v_2, ..., v_l$ will have in- and out-degree 2 and all of its arrows will point to a vertex in c or come from a vertex in c.

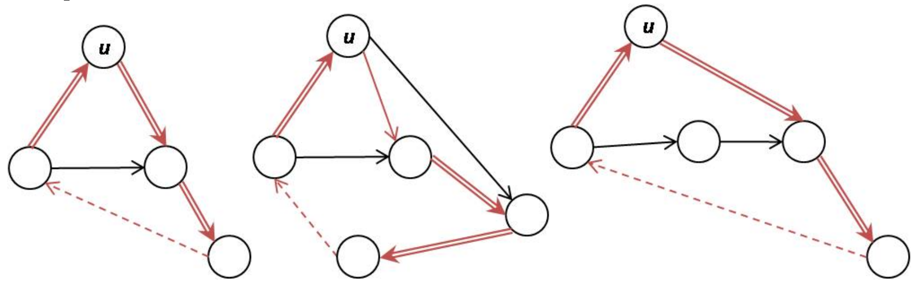
If the minimal cycle length in Q is k, then, since all vertices have in- and out-degree of at least 2, Q will contain the subquiver:



By repeatedly deleting an arrow and applying RI this quiver can be reduced to (II) which is not a C.I..

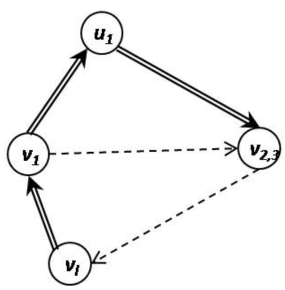
If the minimal cycle length in Q is $k-1$ then let c be a minimal cycle with vertices $v_1, v_2, ..., v_{k-1}$ and u be the only vertex of Q that is not in c. As noted above there is two arrows going from u to c and two arrows going from c to u. If an arrow points from v_i to u and another arrow points from u to v_j then $j - i \equiv 1 \ (mod\ k-1)$ or $i - j \equiv 1 \ (mod\ k-1)$, otherwise the cycle $u, v_j, v_{j+1} \dots v_i, u$ would be shorter than $k-1$ contradicting that c is a cycle of minimal length. For this condition to be satisfied either the two arrows entering u have to point to the same vertex or the two arrows leaving u have to come from the same vertex. We can suppose the latter holds. Also note that if there is no arrow going from u to some v_i then there has to be at least two arrows going from v_{i-1} to v_i, and similarly if there is no arrow going from some

v_i to u then there has to be at least two arrows going from v_i to v_{i+1} due to all vertices having in- and out-degrees of at least 2. So depending on where the arrows departing from u point to, Q will contain one of the following three sub-quivers:

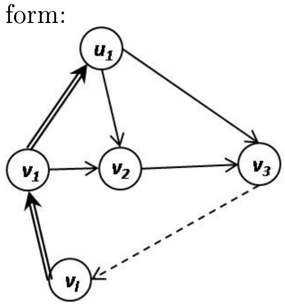


All of these contain a cycle with three double vertices (as indicated with red lines on the pictures) which reduce to (II) and are therefore not C.I.-s.

If the minimal cycle length in Q is smaller than $k-1$, there is a minimal cycle c in Q with vertices $v_1, v_2, ..., v_l$ and at least two vertices u_1 and u_2 outside this cycle. Applying the same argument as above we can suppose that there is two arrows going from v_1 to u_1. There has to be at least two arrow pointing to v_1 and none of these can be leaving from u_1. If both arrows entering v_1 are leaving from v_l then we can regard the subquiver of Q consisting of the cycle c the vertex u_1 and the four arrows that are incident to u_1. This will be of form:

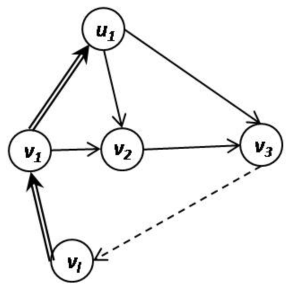


or of form:

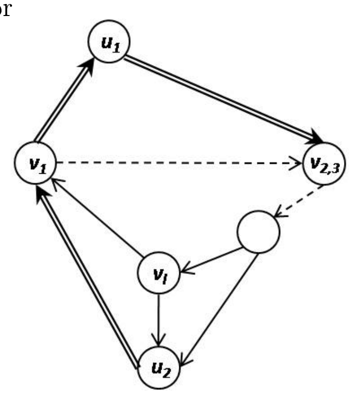


The first case already contains a cycle with three double arrows, and the second case can be reduced to the first case by applying RI on v_2.

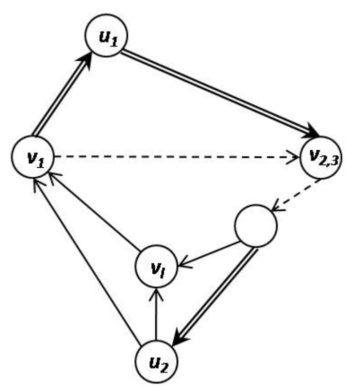
If one of the arrows entering v_1 are leaving from some vertex u_2 outside the cycle c, let us regard the subquiver Q' that consists of the cycle c, the vertices u_1, u_2 and the eight arrows that are incident to u_1 or u_2. If there is an arrow from u_1 to v_3 then there can be no arrow between u_2 and v_2 (either direction) otherwise the local quiver we get by gluing the vertices of the cycle $u_1, v_3, v_4...v_1, u_1$ would not be a connected sum of quivers of form (I) (since the images of u_2 and v_2 would be separate vertices in that quiver with an arrow between them) and due to the induction hypothesis on Lemma 5.9 it could not be a C.I.. Thus if there is an arrow from u_1 to v_3 then there is only one arrow leaving v_2 in Q' and thus we can apply RI, and in the resulting quiver we will have a double arrow from u_1 to c and a double arrow from c to u_1. Depending on how the arrows incident to u_2 are arranged, Q' will be one of the following quivers:



or



or



all of which contain a cycle with three double arrows, therefore Q' (and Q) can not be C.I.-s. □

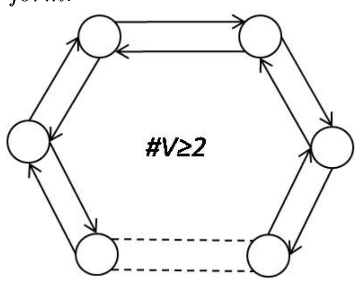
Theorem 5.10. *If (Q,α) is a strongly connected, prime quiver setting with $\alpha = (1,...,1)$ and none of the reduction steps RI-RIV can be applied, then (Q,α) consists of a single vertex with no loops.*

Proof. Follows immediately from Lemmas 5.7 and 5.8. □

5.2.1 Hypersurfaces

We will use the above results to give a list of all reduced quiver settings with one dimensional vertices whose quotient varieties are hypersurfaces.

Theorem 5.11. *Let (Q,α) be a strongly connected, reduced quiver setting with $\alpha = (1,...,1)$. Then $iss_\alpha Q$ is a hypersurface if and only if Q is coregular or of the form:*



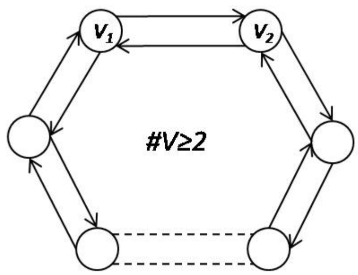
(eg. a quiver with at least two vertices whose arrows form two oppositely directed cycles that both go through all the vertices.)

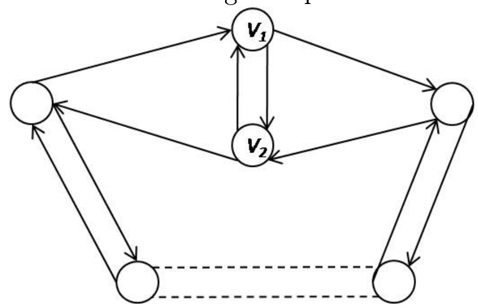
Proof. Note that in the theorem we do not require Q to be prime. It would be unnecessary because for the codimension of a connected sum of two quiver settings we have $F(Q_1\#_v Q_2) = F(Q_1) + F(Q_2)$, so applying the stronger (*) version of Lemma 4.2 we can conclude that a strongly connected, reduced quiver setting with $F(Q) = 1$ is automatically prime. Also note that hypersurfaces are C.I.-s, so we can apply Lemmas 5.7 and 5.8 and see that there is always a connected pair of vertices in Q and RIV is always applicable on this pair.

If $Q = (V, A, s, t)$ is of the above form then it is easy to see that $|A| = 2|V|$ and $|C| = |V| + 2$, so $F(Q) = |C| + |V| - |A| - 1 = 1$, meaning that $iss_\alpha Q$ is indeed a hypersurface.

We prove the opposite direction by induction on the number of vertices. We remind that if Q' is a quiver we got by applying RIV on a connected pair of Q then in the cases a1), b) and c) we saw that $F(Q') = F(Q)$ and in the case a2) we saw that $F(Q') = F(Q) - 1$ (for details, and the explanation of the cases see Theorem 5.2). For $|V| = 2$ there can not be more than two arrows running from either vertex to the other, since otherwise $iss_\alpha Q$ would not even be a complete intersection, and even less so a hypersurface (and for $|V| = 1$ Q is always coregular).

Let us suppose that there is a connected pair (v_1, v_2) in Q with exactly one arrow pointing from each vertex to the other. In this case if we apply RIV on this connected pair the resulting quiver Q' will be reduced as well, so applying the induction hypothesis Q will be one of the following two quivers:



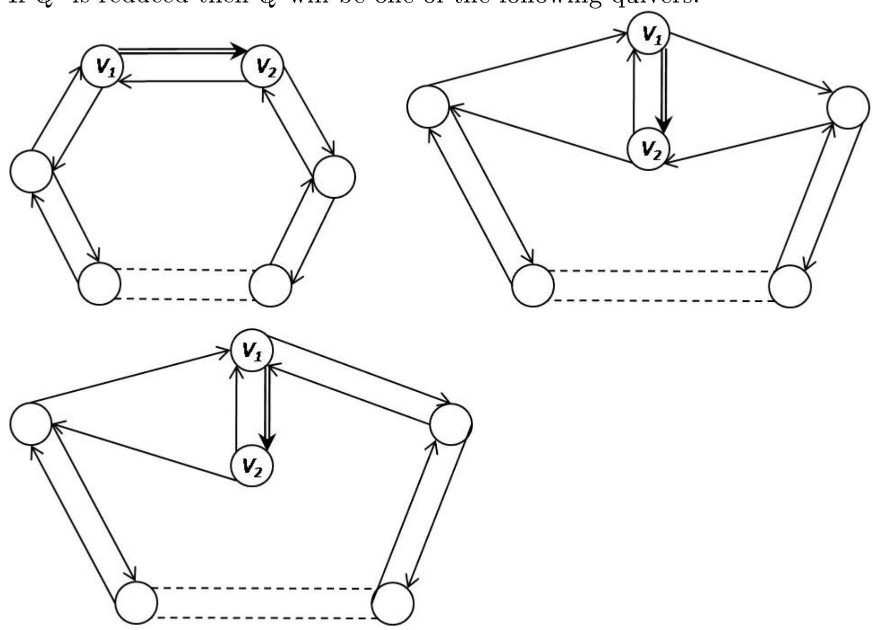


The first case is the one in the theorem, and for the second case it is easy to check that $|C| = |V| + 3$ and $F(Q) = 2$.

Now let us suppose there is a connected pair (v_1, v_2) with exactly 2 arrows pointing from each vertex to the other. If there is no other vertex in the quiver then this is one of the quivers described in the theorem. If there is a third vertex v_3 then, since Q is prime, we either have a path from v_1 to v_3 that does not go through v_2 or a path from v_3 to v_1 that does not go through v_2. Let us suppose the first one holds and call this path P_1. If there is a path P_2 from v_3 to v_2 that does not go through v_1 then P_1 and P_2 form a directed path from v_1 to v_2 so by Lemma 5.7 we can see that Q can not be a C.I., so it can neither be a hypersurface. If all the paths from v_3 to v_2 go through v_1 then, since Q is prime, there has to be a path P_3 from v_2 to v_3 that does not go through v_1. Also since Q is strongly connected there has to be a path from v_3 to the pair (v_1, v_2), if this path reaches (v_1, v_2) at v_1 then along with P_3 it forms a path from v_2 to v_1 , similarly if it reaches (v_1, v_2) at v_2 then along with P_1 it forms a path from v_1 to v_2, in both cases we can apply Lemma 5.7 and see that Q can not be a hypersurface.

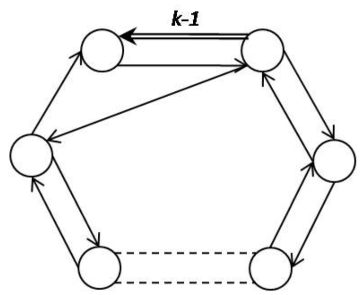
Now if there is no connected pairs in Q with exactly 1 or exactly 2 arrows going both ways, then by Lemmas 5.7 and 5.8, we can conclude that there has to be a connected pair (v_1, v_2) with exactly 1 arrows pointing from v_1 to v_2, which will be noted by a, and exactly 2 arrows pointing from v_2 to v_1. Let Q' denote the quiver we get by applying RIV on (v_1, v_2) and removing all loops by RII, and w denote the image of (v_1, v_2) in Q'. By the stronger (*) version of Lemma 4.2 we can see that $F(Q') = 1$. Therefore there can not be any paths from v_1 to v_2 in Q other than (a) or we would be in case a2) of Theorem 5.2, and we would have $F(Q) = F(Q') + 1 = 2$.

If Q' is reduced then Q will be one of the following quivers:

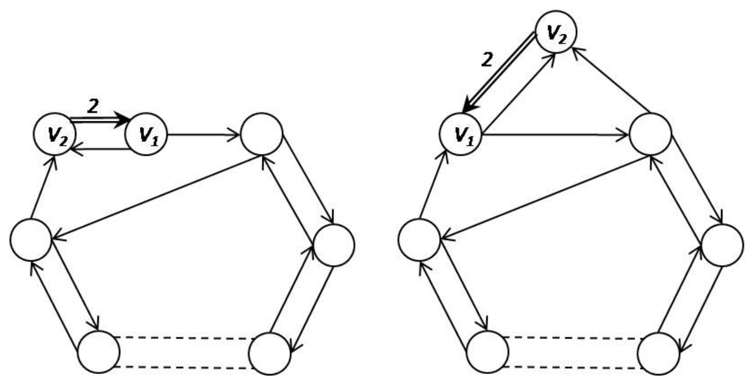


It is easy to check that for the first one $|C| = |V| + 4$ and $F(Q) = 2$, for the second one $|C| = |V| + 5$ and $F(Q) = 3$, and for the third one $|C| = |V| + 4$ and $F(Q) = 2$, so none of these are hypersurfaces.

If Q' is not reduced then w has in- or out-degree 1 (all of the other vertices have the same in- and out-degrees as in Q). For sake of simplicity let us suppose that it has out-degree 1 and in-degree k. Let Q'' denote the quiver we get by applying RI on w and removing all loops by RII. If Q'' is reduced then by the induction hypothesis it is of the form described in the theorem, it follows then that Q' is of form:



Consequently Q has to contain on of the following subquivers:



Again it is easy to check that $F(Q) > 1$ in both cases.

Let now suppose that Q'' is not reduced. Let u denote the vertex the single arrow leaving from w in Q' points to, and w' denote the vertex in Q'' we got by gluing u and w with RI. Noting the number of arrows pointing from u to w by l it is easy to see that:

$$in(w') = in(w) + in(u) - l - 1 \quad \text{and} \quad out(w') = out(u) - l$$

(since loops have been removed from Q''). So if Q'' is not reduced we have $l > 1$, meaning that (w, u) is a connected pair in Q'. By slight abuse of notation, we can note the pre-image of u in Q by u as well. Since (w, u) is a connected pair in Q' there has to be an arrow pointing from either v_1 or v_2 to u and an arrow pointing from u to either v_1 or v_2 in Q. We have three cases:

i) If there is an arrow pointing from u to v_1 and an arrow pointing from v_1 to u then there has to be at least one more arrow between the two vertices since we supposed that there is no connected pair in Q with exactly one arrow going both ways. This arrow can not point from v_1 to u otherwise we would have

$in(v_1) \geq 3$ and $out(v_1) \geq 3$ and since it is easy to check that:

$$in(w) = in(v_1) + in(v_2) - 3 \quad \text{and} \quad out(w) = out(v_1) + out(v_2) - 3,$$

the vertex w would have in- and out-degrees 2 or greater and Q' could not be reduced, contradicting our supposition. So the extra arrow has to point from u to v_1. Since Q is prime there has to be either a path from u to v_2 that does not go through v_1 or a path from v_2 to u that does not go through v_1. In the first case by adding the arrow pointing from v_1 to u to this path, we get a path from v_1 to v_2 that is different from the path consisting of the single arrow a so due to our earlier note we have $F(Q) \geq 2$. Similarly in the second case by adding the arrow a to the path from v_2 to u we get a path from v_1 to u that is different from the arrow pointing from v_1 to u so applying the same argument on the connected pair (v_2, u) we can see that $F(Q) \geq 2$.

ii) If there is an arrow pointing from v_1 to u and an arrow pointing from u to v_2, then these two arrows form a path from v_1 to v_2 and once again we are in case a2) of Theorem 5.2 and can conclude that $F(Q) \geq 2$.

iii) If there is an arrow pointing from v_2 to u and an arrow pointing from u to v_1 then we have $in(v_1) \geq 3$ and $out(v_2) \geq 3$, also because Q is reduced we have $out(v_1) \geq 2$ and $in(v_1) \geq 2$, so just as in case i) we can conclude that $in(w) \geq 2$ and $out(w) \geq 2$, meaning that Q' is reduced contradicting our supposition. $\square$

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