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# Generalized stable matchings: theory and applications

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# 1 Introduction

The stable marriages problem was first introduced by [Gale-Shapley (1962)]. In America, the deferred acceptance algorithm was used in the National Intern Matching Problem since 1952, even before this mathematical model was made.

Another application of the stable matchings is the Hungarian college admissions system. [Gale-Shapley (1962)] also gave a model to college admissions, but the Hungarian system are different, because here can be ties between the applicants, and if there is a tie, the university must accept all of the tied students or none of them.

This can be captured with scoring choice functions. [Fleiner (2003)] and [Hatfield-Milgrom (2005)] discovered that the fixed point theorem of [Tarski (1955)] is a very effective tool to find the stable outcomes.

Using choice functions, stability can be defined in many ways. Here we introduce the so called three-part, four-part and dominating stability concepts for every comonotone choice functions, and show that these three are equivalent, if preferences has the path independence property.

We use score-stability and two-sided score stability for the college admissions problem. In [Knuth (1976)] they show that stable marriages form a distributive lattice, dedicated to John Conway. Using Tarski's fixed point theorem we prove the lattice property of university admissions too, and generalize Blair's theorem about the lattice of stable cores.

In the second part of the thesis, we study abstract models of supply chains. Consider a directed graph, where the nodes are the trading companies, and the edges correspond to the possible trades. In an article [Fleiner (2009)] every edge has a maximal capacity, the traded volumes form a flow, except from the source (manufacturer) and sink (final consumer), everyone buys as much as they sell. This model differs from the traditional flows in such a way that, all agents have a preference order over his input and output edges, and he wants to trade with his preferred partners.

In the stable marriage problem an edge is blocking, if both participants are single or prefer the other to his/her current partner. Similarly an edge in the trading network is blocking, if they prefer to sell/buy from the other, and they have free capacity for it, or decrease other trades for its sake. But now we define blocking coalitions as walks, where the first and last agent would like to make new trades, and the intermediate companies pass on the sufficient quantities. We allow cycles in the graph.

In the article [Ostrovsky (2008)] the flow does not need to satisfy Kirchhoff's law, but if they buy more, they want to sell more, and they choose the best possible set of edges. These conditions are described with the so called *same side substitutable, cross side*

*complementary* and *path independent* properties. In this second model, we have an upstream-downstream market, there are basic inputs, intermediates and final outputs, so the graph is acyclic. Each edge is associated with a finite set of possible prices. The edge-price pairs are called *contracts*, they can be represented as parallel edges in the graph.

In this present work, in Section 3, we generalize Ostrovsky's theorem to the case of cyclic graphs, first with the assumption that each agent's choice is discrete, whether he chooses to trade over an edge or not. Then we consider continuous choices, where they can decide how much quantity they trade. A supply chain is stable if there is no blocking walk where all participants would like to sell/forward/buy extra volume.

These three models' common background is the Tarski fixed point theorem. The fixed points can be assigned to the stable sets.

A further generalization of the model is described in the article [Hatfield et al. (2011)]. Here the prices of the trades are continuous, every  $p \in \mathbb{R}$  is allowed. The price is added to the utility function, so if a firm prefers selling  $a$  to  $b$ , its choice can be reversed if  $b$  will pay more.

Thus the players can compensate each other for a less useful selection, so the problem is related to maximum weight matchings instead of / in addition to the stable matchings.

In this article [Hatfield et al. (2011)] they define the *fully substitutable* property, which is, because of the prices, a stronger requirement than the SSS and CSC used in the third chapter. This stronger requirement leads to a stronger claim for the existence of a stable solution, not only where no blocking walks, but there is no blocking subset of the edges.

The authors prove the existence of competitive equilibrium first, then they show that all equilibrium solutions are stable. However, they do not use the Tarski fixed-point theorem, they reduce the problem to the many-to-one stable matchings by [Kelso and Crawford (1982)].

Our goal is to generalize it to the case of continuous quantity trading.

In Section 2, we introduce basic definitions in the theory of stable matchings. We show the connection between the path independent property, and a preference ordering over all subsets.

In Section 3, we compare five stability definitions, introduce the college admissions model, show the lattice property of stable score vectors using one generalization of Blair's theorem.

Here the connection between first three stability concepts (Theorems 19,20 and 21), the statements about score-stability and the generalization of Blair’s theorem are new results. We also show counterexamples for cases where one kind of stability does not imply the another.

In Section 4, we capture the preferences in the firm-worker assignment of [Kelso and Crawford (1982)] with comonotone choice functions.

In Section 5, we solve the problem of supply chains without money in the discrete or continuous quantities case using Tarski’s fix point theorem.

The new results in this section are the generalization of Ostrovsky’s theorem (Theorem 36), Lemma 38 and Theorem 39, and the similar theorem for the existence of a chain-stable network in case of continuous quantities, Theorem 43.

In Section 6, we search for equilibrium and stable sets in a supply chain market with money. We suppose Conjecture 54 and show that if it is true, there exists an equilibrium in the continuous price-continuous quantity market.

We conclude in Section 7.

## 2 Preliminaries

In the stable marriage model, there are  $n$  men:  $M = m_1, \dots, m_n$  and  $n$  women:  $W = w_1, \dots, w_n$ , each of them having a strict preference order on the members of the other gender. Let  $G$  be a bipartite graph with color classes  $M$  and  $W$ , and let set  $E$  of edges of  $G$  denote the possible marriages. We use the notation  $w <_m w'$ , if man  $m$  prefers woman  $w'$  to  $w$ . If an  $S$  set of marriages is a matching in  $G$ , then  $S \subseteq E$  can also be described as an involutive function:  $s : M \cup W \rightarrow M \cup W$ . For a married pair  $(m, w) \in S$ , let  $s(m) = w$ ,  $s(w) = m$ , and we define  $s(a) = a$ , if  $a \in M \cup W$  remains single.

With these notations, a marriage scheme  $S$  is *stable*, if for any  $(m, w) \notin S$  pair,  $s(m) >_m w$ , or  $s(w) >_w m$ .

The men’s preference systems give us a partial order on stable marriage schemes:  $S \geq_M S'$ , if  $s(m_i) \geq_{m_i} s'(m_i)$ , for all  $m_i \in M$ , and  $S >_M S'$ , if  $S \geq_M S'$  and there exists a man  $m_i$  such that  $s(m_i) >_{m_i} s'(m_i)$ . In the similar way, there is an other ordering  $>_W$  defined by women.

**Theorem 1** (Gale-Shapley). *[Gale-Shapley (1962)] There always exists a stable matching, and it can be found with the deferred acceptance algorithm.*

In the deferred acceptance algorithm, in each round every unengaged man proposes to the most-preferred woman to whom he has not yet proposed. Each woman then

considers all her suitors and keeps her most preferred man, and refuses the others. Next round, the men who were not accepted, continue proposing to the next best girl. The algorithm stops in at most  $2n^2$  steps since every man propose to every women at most once.

The algorithm guarantees that everyone gets married and the marriages are stable.

**Theorem 2** (Knuth). *[Knuth (1976)] If preference orders are strict, the common orderings for men and women are the opposite of each other. Say, if  $S$  and  $S'$  are two stable matchings, then  $S \geq_M S' \Leftrightarrow S \leq_W S'$*

**Definition 1.** We call a stable matching  $S$  *male-optimal* (*female-optimal*) if there is no stable matching  $S'$ , such that  $S <_M S'$  ( $S <_W S'$ ).

A stable matching  $S$  is *male-pessimal* (*female-pessimal*) if there is no stable matching  $S'$ , such that  $S >_M S'$  ( $S <_W S'$ ).

**Theorem 3.** *[Gale-Shapley (1962)] The stable marriage scheme given by the Gale-Shapley algorithm is male-optimal and female-pessimal.*

In the stable marriage problem, we can use choice functions to define the preference orders for the two sides of the market.

**Definition 2.** Set function  $F : 2^E \rightarrow 2^E$  is called a *choice function* if  $F(A) \subseteq A$  holds for any subset  $A$  of ground set  $E$ .

For example, if a man wants to choose between two girls  $a$  and  $b$ , and prefers  $a$  (we use the notation  $a > b$ ), his choice function will be  $F : 2^{\{a,b\}} \rightarrow 2^{\{a,b\}}$

$$F(\emptyset) = \emptyset$$

$$F(\{a\}) = \{a\}$$

$$F(\{b\}) = \{b\}$$

$$F(\{a, b\}) = \{a\}$$

As we will define later, this choice function is comonotone and path independent.

**Definition 3.** A set function  $F : 2^E \rightarrow 2^E$  is *monotone* if  $F(A) \subseteq F(B)$  whenever  $A \subseteq B \subseteq E$  holds.

**Definition 4.** A choice function  $F : 2^E \rightarrow 2^E$  is *comonotone* (sometimes it is called *substitutable*) if  $\overline{F}(A) \subseteq \overline{F}(B)$  for any  $A \subseteq B$ . (That is, if  $\overline{F}$  is a monotone function.)

The stable marriage problem can be represented with a  $G = (M, W, E)$  bipartite graph, where  $M$  is the set of the men,  $W$  is the set of women, and edges  $E$  is the set of possible marriages between them. Let  $F$  be the common choice function for men

over the edges of the bipartite graph, where every man choose the best of the edges he is incident with (i.e. the marriages he can participate in). Let  $G$  be the similar choice function for women. A set of marriages  $S \subseteq E$  is stable, if any not married pair  $mw = e$ ,  $e \notin S$ ,  $m$  or  $w$  do not want to choose  $e$  so  $e \notin F(S + e)$  (we say  $e$  is  $F$ -dominated) or  $e \notin G(S + e)$  ( $e$  is  $G$ -dominated). Let  $A_0$  be the set of  $F$ -dominated edges, and  $B_0$  is the set of  $G$ -dominated edges. Now  $A_0$ ,  $B_0$  and  $S$  are disjoint sets,  $S \cup A_0 \cup B_0 = E$ , so for  $A = A_0 \cup S$ ,  $B = B_0 \cup S$ , we get  $A \cup B = E$  and  $A \cap B = S$ . If  $F$  and  $G$  are comonotone, the edges dominated one by one will be dominated altogether, so  $F(A) = S, G(B) = S$ . We can generalize this stability to any kind of choice functions.

**Definition 5.** Subset  $S$  of  $E$  is *(three-part) stable*, if  $F(S) = S = G(S)$ , and there exists subsets  $A$  and  $B$  of  $E$ , such that  $F(A) = S = G(B)$  and  $A \cup B = E$ ,  $A \cap B = S$ . Pair  $(A, B)$  with this property is called an *FG-stable pair*, and  $S$  is an *FG-stable core*. We can define a partial order on *FG-stable pairs*:  $(A', B') \leq (A, B)$ , if  $A' \subseteq A$  and  $B' \supseteq B$ .

The following theorem is the key to the Gale-Shapley algorithm that finds a stable matching. Recall that a *lattice* is a partially ordered set  $L$  with the property that any two elements  $x, y$  of  $L$  have a greatest lower bound  $x \wedge y$  and a least upper bound  $x \vee y$ . A lattice  $L$  is complete if any subset  $X$  of  $L$  has a greatest lower bound  $\bigwedge X$  and a least upper bound  $\bigvee X$ . Function  $f : L \rightarrow L'$  from poset  $L$  to poset  $L'$  is *monotone* if  $x \leq y$  implies  $f(x) \leq f(y)$  for any elements  $x, y$  of  $L$ .

**Theorem 4** (Tarski's fixed point theorem [Tarski (1955)]). *Let  $L$  be complete lattice, and  $f : L \rightarrow L$  be a monotone function on  $L$ . Then  $L_f$  is a nonempty, complete lattice on the restricted partial order where  $L_f = \{x \in L : f(x) = x\}$  is the set of fixed points of  $f$ .*

Theorem 4 implies the following corollary.

**Theorem 5.** [Fleiner (2003)] *If  $F, G : 2^E \rightarrow 2^E$  are comonotone choice functions, than the *FG-stable pairs* form a nonempty complete lattice for partial order  $(\leq)$ .*

Define function  $f : 2^E \times 2^E \rightarrow 2^E \times 2^E$  by

$$f(A, B) := (E \setminus (\overline{G(B)}), E \setminus (\overline{F(A)}) = (E \setminus (B \setminus G(B)), E \setminus (A \setminus F(A)))$$

It is straightforward to see that *FG-stable pairs* are exactly the fixed points of  $f$ . Using Tarski's fixed point theorem, *FG-stable pairs* form a lattice.



**Definition 6.** A choice function  $F$  is *path independent*, if it has the property

$$F(A) \subseteq B \subseteq A \Rightarrow F(A) = F(B)$$

for every  $A, B$  subsets of  $E$ .

Sometimes path independence is defined with:  $F(A \cup B) = F(F(A) \cup F(B))$  for every  $A, B$ .

**Theorem 6.** [Fleiner (2002)] *If  $F$  is a comonotone choice function, these two definitions are equivalent.*

*Proof.* Suppose that  $F(A \cup B) = F(F(A) \cup F(B))$  for every  $A, B$ .

Then  $F(A) = F(A \cup A) = F(F(A) \cup F(A)) = F(F(A))$ .

If  $F(A) \subseteq B \subseteq A$  then  $F(B) = F(F(A) \cup B) = F(F(F(A) \cup F(B))) = F(F(A) \cup F(B)) = F(A \cup B) = F(A)$ .

For the opposite direction:  $A, B$  are arbitrary subsets of  $E$ , now

$$F(A \cup B) = F(A \cup B) \cap (A \cup B) = (F(A \cup B) \cap A) \cup (F(A \cup B) \cap B) \subseteq F(A) \cup F(B) \subseteq A \cup B \\ F(A \cup B) \subseteq F(A) \cup F(B) \subseteq A \cup B \Rightarrow F(F(A) \cup F(B)) = F(A \cup B) \quad \square$$

If  $F$  is not comonotone, we will use the first definition.

**Theorem 7.** *If there exist a well-ordering of all subsets of  $E$ , such that  $F(A)$  is the best choice of subsets of  $A$ , the choice function  $F$  is path independent (with definition  $F(A) \subseteq B \subseteq A \Rightarrow F(A) = F(B)$ ).*

Note that a well-ordering means every set has a smallest element. Because we defined preferences the way that  $a > b$  is  $a$  is better than  $b$ , what we need is every set has a biggest element, so an opposite well-ordering.

*Proof.* If there is a well-ordering above all subsets,  $F(A) \subseteq B \subseteq A$  means that the best of all  $X \subseteq A$  is also in  $B$ . Since  $B$  is smaller than  $A$ , the all subsets of  $B$ ,  $\mathcal{P}(B) \subseteq \mathcal{P}(A)$ , so  $B$  cannot have a better subset,  $F(A)$  is also the best in  $B$ , so  $F(A) = F(B)$ . □

The theorem is not true in the opposite direction: there exists a choice function which is path independent, but there is no corresponding ordering above the subsets. For example, let  $F$  be the following choice function defined on three elements ( $\{a, b, c\}$ ):

$$F(\emptyset) = \emptyset$$

$$F(\{a\}) = \{a\}$$

$$F(\{b\}) = \{b\}$$

$$F(\{c\}) = \{c\}$$

$$F(\{a, b\}) = \{a\}$$

$$F(\{a, c\}) = \{c\}$$

$$F(\{b, c\}) = \{b\}$$

$$F(\{a, b, c\}) = \{a, b, c\}$$

This function satisfies  $F(A) \subseteq B \subseteq A \Rightarrow F(A) = F(B)$ . Suppose that there is a good ordering. Since  $F(ab) = a$ ,  $F(ac) = c$  and  $F(bc) = b$ , so  $a > b > c > a$  must stand. This is not transitive, so it is not an ordering.

But this counterexample is not comonotone. If we add the comonotone property, we can prove this direction too.

**Theorem 8.** *If  $F$  is comonotone and path independent over a finite set  $E$ , then there exists an ordering over all subsets of  $E$  for which  $F(A) = \max_{X \subseteq A} X$ .*

*Proof.* We say that  $A$  is an *independent* set, if  $F(A) = A$ . If for any  $A, B$  sets,  $F(A) = B$ , then  $F(A) = B \subseteq A$ , so  $F(B) = B$ , which means only independent sets can be chosen. If  $F(A) \neq A$ , then in our ordering  $A < \emptyset$ . We don't bother with defining the ordering between non-independent sets, since they are never chosen.

If  $A$  and  $B$  are independent sets:

If  $F(A \cup B) = A$ , then  $A > B$ .

If  $F(A \cup B) = B$ , then  $B > A$ .

If  $F(A \cup B) = C$  is neither  $A$  nor  $B$ , and there exist a chain  $A_1, A_2, \dots, A_n$  such that  $F(A \cup A_1) = A$ ,  $F(A_1 \cup A_2) = A_1, \dots, F(A_{n-1} \cup A_n) = A_{n-1}, F(A_n \cup B) = A_n$  then  $A > B$ .

Where none of these occurs, we can pick the better freely. We need to prove that this ordering is transitive.

Indirectly, suppose there exist  $A_1, A_2, \dots, A_n$  such that  $F(A_1 \cup A_2) = A_1, F(A_2 \cup A_3) = A_2, \dots, F(A_{n-1} \cup A_n) = A_{n-1}, F(A_n \cup A_1) = A_n$ . Then for every  $i$ ,  $\overline{F}(A_i \cup A_{i+1}) = A_{i+1} \setminus A_i \subseteq \overline{F}(\bigcup_{i=1}^n A_i)$ , so  $B = F(\bigcup_{i=1}^n A_i) \subseteq F(\bigcap_{i=1}^n A_i)$ . With the path independent property, from  $B = F(\bigcup_{i=1}^n A_i) \subseteq A_i \cup A_{i+1} \subseteq \bigcup_{i=1}^n A_i$ , it follows that  $F(A_i \cup A_{i+1}) = B$  for every  $i$  which contradicts our assumption.

So the ordering is transitive, and since we decided something in the uncertain cases, the ordering is trichotomous.

What is left behind to show this ordering defines the same function what we want. Indirect: suppose there is  $A$  subset of  $E$ , and from all subsets of  $A$ , our ordering gives  $B$  as maximal, but the choice function chooses  $F(A) = C$ . Then  $B > C$  so

$F(B \cup C) \neq C$  (since if it were  $C$ , we would defined the order in the opposite direction). From  $B \subseteq A, C \subseteq A$ , we get  $B \cup C \subseteq A$ . Since  $C = F(A) \subseteq B \cup C \subseteq A$ , from path independence  $F(B \cup C) = C$ . We reached a contradiction. □

## 3 Stability concepts

### 3.1 Score stability

The stable matchings can be used for the Hungarian university admissions system. There, every applicant has a strict preference order over the colleges she applies to, and each college assigns some score (an integer between 1 and  $M$ ) to each of its applicants. Moreover, each college  $C$  has a quota  $q(C)$  on admissible applicants. According to the law, no college can accept more applicants than its quota, moreover if an applicant with a certain score is not acceptable to some college than any applicant with the same score has to be unacceptable.

To determine the admissions after all information is known, each college has to declare a score limit. Assume we have  $n$  applicants  $A_1, A_2, \dots, A_n$  and  $m$  colleges  $C_1, C_2, \dots, C_m$  with score limits  $t_1, t_2, \dots, t_m$ , respectively. Each applicant will become a student on her most preferred college where she has high enough score. More precisely, applicant  $A_i$  is assigned to college  $C_j$  if  $S(i, j) \geq t_j$  (i.e. score  $S(i, j)$  of  $A_i$  at  $C_j$  is not less than threshold  $t_j$  for  $C_j$ ) and  $S(i, j') < t_{j'}$  for  $j' < j$  (i.e. score  $S(i, j')$  of  $A_i$  at  $C_{j'}$  is less than the score limit  $t_{j'}$ ). The vector of declared score limits  $(t_1, t_2, \dots, t_m)$  is called a *score vector*. Below we define the stability of a score vector according to the Hungarian law.

**Definition 7.** Score vector  $(t_1, t_2, \dots, t_m)$  is *valid* if no college exceeds its quota with these score limits.

Score vector  $(t_1, t_2, \dots, t_m)$  is *critical* if for every college  $C_j$  either  $t_j = 0$  or score vector  $(t_1, t_2, \dots, t_{j-1}, t_j - 1, t_{j+1}, \dots, t_m)$  would assign more than  $q(C_j)$  students to  $C_j$ . (That is, no college can lower its quota without exceeding its quota.)

A score vector  $\underline{s}$  is *stable* if  $\underline{s}$  is valid and critical.

The above college admissions model determines a natural choice function for the students and another one for the colleges. Let  $E$  be the set of all applications of all applicants. (It is convenient to think that  $E$  is the set of edges of the bipartite graph with color classes  $\{A_1, \dots, A_n\}$  and  $\{C_1, \dots, C_m\}$  where each edge  $A_i C_j$  of the graph corresponds to an application of  $A_i$  to  $C_j$ .) So for subset  $X \subseteq E$  of applications  $F(X)$  denotes the set of most preferred applications of each applicant. Similarly,  $G(X)$

denotes the set of applications that colleges would choose if they can select freely. More precisely, let  $X_j$  denote the set of applications to  $C_j$  from  $X$ , and let  $C_j$  declare a score limit  $t_j$  such that no more than  $q(C_j)$  applications of  $X_j$  has score at least  $t_j$ , but either  $t_j = 0$  or more than  $q(C_j)$  applications has score at least  $t_j - 1$ . Define choice function  $G : 2^E \rightarrow 2^E$  for the colleges such that  $G(X)$  is the set of applications with score at least the corresponding score limit. By definition,  $G(X)$  cannot assign more than  $q(C_j)$  applications to  $C_j$  for any college  $C_j$ .

*It is easy to see that choice function  $F$  of the applicants is path independent, but  $G$  for the universities is not.*

**Definition 8.** We call a  $G$  choice function *scoring choice function* over the set  $E$ , if there exist scores for every edge and quotas for the universities what define  $G$  with the previous method.

Every scoring choice functions are comonotone, but not necessary path independent.

**Definition 9.** An subset  $S$  of  $E$  of applications is an *assignment* if each applicant  $A_i$  has at most one application in  $S$  and each college  $C_j$  has at most  $q(C_j)$  applications in  $S$ .

An assignment  $S$  is *score stable* if for any application  $e$  of (say)  $A_i$  to  $C_j$  at least one of the following alternatives holds. Either  $e \in S$  or there is an application  $s$  of  $A_i$  in  $S$  such that  $s \succeq_{A_i} e$  (i.e.  $A_i$  prefers  $s$  to  $e$ ) or  $|\{A_l : S(l, j) \geq S(i, j) \text{ and } C_j \leq_{A_l} S(A_l)\}| > q(C_j)$ , where  $S(A_l)$  denotes the college that  $S$  assigns to applicant  $A_l$ . This third condition says that there are more than  $q(C_j)$  applicants that  $C_j$  should accept if  $C_j$  accepts application  $e$  of  $A_i$ .

Application  $e$  is *score blocking* assignment  $S$ , if the above condition does not hold for  $e$ . (In particular, a score blocking assignment of  $S$  cannot belong to  $S$ .)

We can define college-optimality/pessimality, applicant-optimality/pessimality in the same way as we did with stable marriages.

**Theorem 9.** *For any finite set of applicants, colleges and set of applications, for arbitrary positive scores of the applications there always exists a stable score vector.*

However, the score-stable solutions are not equivalent with the three-part stability in Section 2, so we need to use a new stability concept.

First, we introduce the dominating function:

**Definition 10.** For choice function  $F : 2^X \rightarrow 2^X$  let *dominating function*

$$\mathcal{D}_F(A) := \{x \in X : x \notin F(A \cup \{x\})\}$$

denotes the set of dominated choices.

**Theorem 10.** *If  $F$  is a comonotone choice function then  $\mathcal{D}_F$  is monotone.*

*Proof.* Let  $A \subseteq B$ . If  $x \in \mathcal{D}_F(A)$  then  $x \notin F(A \cup \{x\}) \Rightarrow x \in \overline{F}(A \cup \{x\})$ . From  $F$  is comonotone,  $x \in \overline{F}(B \cup \{x\}) \Rightarrow x \notin F(B \cup \{x\})$  so  $x \in \mathcal{D}_F(B)$ . □

**Definition 11.** Subset  $S$  of  $E$  is *four-part stable*, if  $F(S) = S = G(S)$ , and there exists subsets  $A$  and  $B$  of  $E$ , such that  $F(A) = S = G(B)$  and  $A \cap B = S$ , and  $\mathcal{D}_F(A) = E \setminus B, \mathcal{D}_G(B) = E \setminus A$ .

**Theorem 11.** *The score-stability is equivalent with four-part stability.*

*Proof.* Suppose that subset  $S$  of  $E$  is score stable. Let  $B = \{(a, c) : c \geq_a s(a)\}$ . Now  $G(A)=S$ , because all student-application pair, where the student would like to change his university, is in  $B$ , so if by decreasing its score limit by one, some university would fulfil its quota from all edges, it would also fulfil by the edges in  $B$ .

Let  $\mathcal{D}_G(B) = B \setminus S \cup D$ , whereby  $D = \mathcal{D}_G(B) \setminus (B \setminus S)$ . Let  $A = E \setminus \mathcal{D}_G(B)$

This means  $A \cup D = \{(a, c) : c \leq_a s(a)\}$  so  $F(B \cup D) = S$ , because the students like the other edges less than their university in assignment  $S$ .

Choice function  $F$  is path independent, so  $S \subseteq A \subseteq (A \cup D)$  is followed by  $F(A) = S$ . From lemma 18,  $\mathcal{D}_F(A) = \mathcal{D}_f(S) = \mathcal{D}_F(A \cup D) = (A \cup D) \setminus S = E \setminus A$ , so  $S$  is four-part stable.

If  $S$  is four-part stable, with  $A, B$ , and  $D = E \setminus (A \cup B)$ : Similarly with the earlier part,  $\mathcal{D}_F(A \cup D) = (A \cup D) \setminus S$ , so for all  $x \notin A \cup D, x \in F(A \cup D \cup x)$ .

$$(a, c) \in B \setminus S \Rightarrow c >_a s(a)$$

If  $c >_a s(a)$ ,  $(a, c)$  can not be in  $(A \cup D) \setminus S$ , otherwise student  $a$  would choose it. This means  $(a, c) \in (A \cup D) \setminus S \Rightarrow c <_a s(a)$

$$(a, c) \in S \Rightarrow c =_a s(a)$$

$$(a, c) \in B \setminus S \Rightarrow c >_a s(a)$$

Assignment  $S$  is score-stable: it is valid because  $F(S) = G(S) = S$ , so no one has more students than its quota. And if college  $c$  decreases its score limit, all of the applicants, who would like to change is in  $A$ . But  $F(A) = S$ , so  $c$  can not lower among the applications in  $A$ , so also can't do this with all applications. □

### 3.2 Generalization of Blair's theorem

**Definition 12.** Define a partial order on stable matchings: if  $S$  and are  $S'$   $FG$ -stable cores,  $S' \leq_F S$  if  $F(S \cup S') = S$ .

Blair proved the lattice property of  $FG$ -stable cores assuming the path independent property of the choice functions [Blair (1988)].

**Theorem 12** (Blair). [Blair (1988)] *If  $F, G : 2^E \rightarrow 2^E$  are comonotone path independent choice functions then the  $FG$ -stable cores form a lattice for partial order  $<_F$ .*

Now, we don't require both side's path independency, just on one side:

**Theorem 13** (Generalization of Blair's theorem). *If  $F$  and  $G$  are comonotone choice functions and  $F$  is path independent, then the  $FG$ -stable cores form a lattice for partial order  $<_F$*

*Moreover, if  $S$  is stable, then there is only one corresponding  $FG$ -stable  $(A, B)$  pair, and  $S \leq_F S' \Leftrightarrow (A, B) \leq (A', B')$*

*Proof. (i) For given stable core  $S$  there is a unique  $(A, B)$  pair.*

Suppose that there are two different stable pairs for  $S$ :  $(A, B)$  and  $(A', B')$ .

We can assume that there exists a  $b$  for which  $b \in B$ , but  $b \notin B'$ . Since  $S \subseteq B'$ , it follows that  $b \notin S$ . Moreover  $b \in F(A \cup b)$  but  $b \notin F(A' \cup b)$  because  $b \in F(A \cup b) \Leftrightarrow b \notin B$ . Because  $A' \setminus S = \overline{F}(A') \subseteq \overline{F}(A' \cup b)$ , we get  $F(A' \cup b) \subseteq S \subseteq A' \cup b$ , hence  $F(A' \cup b) = S$ . We know that  $(A' \cup b) \setminus S = \overline{F}(A' \cup b)$  and  $A \setminus S = \overline{F}(A)$ .

Since  $F$  is comonotone  $\overline{F}(A' \cup A \cup b)$  is bigger than both,  $(A' \cup A \cup b) \setminus S \subseteq \overline{F}(A' \cup A \cup b)$ , so  $F(A' \cup A \cup b) \subseteq S$ .

From  $F(A' \cup b) = S$  we get  $F(A \cup b) \subseteq F(A \cup b) \cup F(A' \cup b) \subseteq A \cup b$ , so  $F(A \cup b) = F(F(A \cup b) \cup F(A' \cup b))$

Using that  $F$  is path independent,  $b \in F(A \cup b) = F(F(A \cup b) \cup F(A' \cup b)) = F(A \cup b) \cup (A' \cup b) = F(A' \cup A \cup b) \subseteq S$ . Therefore  $b \in S$ , which leads to contradiction.

*Let  $S$  and  $S'$  be two different stable sets. The stable pair corresponding to  $S$  is  $(A, B)$ , and for  $S'$  there is  $(A', B')$ .*

*(ii)  $(A, B) \leq (A', B') \Rightarrow S \leq_F S'$ .*

From the pairs ordering,  $S \subseteq A \subseteq A'$  and  $S' \subseteq A'$ , so  $S \cup S' \subseteq A'$ .

Since  $F$  is path independent, from  $S' = F(A') \subseteq S \cup S' \subseteq A'$  we get that  $F(S \cup S') = S'$ .

*(iii)  $S \leq_F S', \Rightarrow (A, B) \leq (A', B')$ .*

Suppose that  $B' \not\subseteq B$ . Consequently  $\exists b$  such that  $b \in B$ , but  $b \notin B'$ . From lemma 18 we get  $B \in \mathcal{D}_F(S)$ , but  $B \notin \mathcal{D}_F(S')$ . Now  $b \in F(S' \cup b)$ ,  $b \notin f(S \cup b)$ ,  $F(S \cup S') = S'$  therefore  $F(S \cup S' \cup b) \subseteq S' \cup b \subseteq S \cup S' \cup b$ . Because  $F$  is path independent,  $F(S \cup S' \cup b) = F(S' \cup b) \ni b$ , hence  $b \in F(S \cup b)$ , and it is a contradiction.

Accordingly  $B' \subseteq B$ , so  $\mathcal{D}_F(B') \subseteq \mathcal{D}_F(B)$ ,  $E \setminus A' \subseteq E \setminus A$ , hence  $A' \supseteq A$ .

(iv) *The stable sets form a lattice.*

From the previous points, we can see that there is a bijection between the stable  $S$  sets and stable  $(A, B)$  pairs, and it keeps the ordering. Because we know the lattice property for the stable pairs, we get the same lattice for stable sets. □

**Corollary 14.** *The score-stable sets form a lattice, because it is equivalent with 4-part stability, and the applicants choice function is path independent, so we can use the generalized Blair theorem.*

**Remark 15.** *If  $F$  and  $G$  are comonotone choice functions but none of them is path independent, then the lattice property doesn't hold. It is also false that for stable set  $s$  there is only one corresponding  $(A, B)$  pair.*

*Consider the following example: We have two applicants,  $s$  and  $s'$ . The choice function is what we named as  $q = 1$  for both sides:*

$$F(s) = s \quad G(s) = s$$

$$F(s') = s' \quad G(s') = s'$$

$$F(s, s') = \emptyset \quad G(s, s') = \emptyset$$

*Both of the choice functions are comonotone, but none is path independent. In this situation we have four stable  $(A, B)$  pairs:*

$$A = \emptyset, B = (s, s') \quad S = \emptyset$$

$$A = s, B = s \quad S = s$$

$$A = s', B = s' \quad S = s'$$

$$A = (s, s'), B = \emptyset \quad S = \emptyset$$

*We can say that  $\emptyset \leq_F s$  and  $\emptyset \leq_F s'$ , but because the symmetric construction, we can't define ordering between  $s$  és  $s'$ . So these two points don't have a supremum.*

### 3.3 Two-sided score limits

In this model, not just the colleges, but the applicants also can choose more than one place to go. Every applicant  $A$  has a quota  $q(A)$ , and assign a score to the universities she applies. As before, each college  $C$  assigns some score (an integer between 1 and  $M$ ) to each of its applicants, and declare a  $t_C$  score limit. The applicant  $A$  also has a  $t_A$  score limit. Assume we have  $n$  applicants  $A_1, A_2, \dots, A_n$  and  $m$  colleges  $C_1, C_2, \dots, C_m$  with score limits  $t_{A_1}, t_{A_2}, \dots, t_{A_n}, t_{C_1}, \dots, t_{C_m}$ ,

Those applications realize, where both of the sides exceeds the other side's score limit.

A score vector is *valid*, if no one gets more students/ goes to more college than its quota.

Score vector  $t_{A_1}, t_{A_2}, \dots, t_{A_n}, t_{C_1}, \dots, t_{C_m}$ , is *critical* if for every  $X \in C \cup A$  either  $t_X = 0$  or score vector  $t_{A_1}, t_{A_2}, \dots, t_X - 1, \dots, t_{C_m}$ , would assign more than  $q(X)$  students/ colleges to  $X$ . (That is, no college or applicant can lower its quota without exceeding its quota.)

A score vector  $\underline{s}$  is *stable* if  $\underline{s}$  is valid and critical.

The ordering we use between score vectors:  $t \leq_2 t'$  if  $t_C \leq t'_C$  for every colleges, and  $t_A \geq t'_A$  for every applicants.

**Theorem 16.** [Jankó (2009)] *The stable score vectors form a nonempty lattice for the partial order  $\leq_2$*

**Definition 13.** An admission  $S \subseteq E$  is *two sided score-stable*, if there exist  $\underline{t}$  stable score vector, for which the set of realized applications is  $S$ .

Differently from one-sided scores, now for a given score-stable admission  $S$  there can exist more than one corresponding score-stable  $\underline{t}$  vector.

Example: one college, one applicant, every point is 0, every quota is 0. Score vectors (0,1) and (1,0) are both score-stable and give the stable admission  $S = \emptyset$ .

### 3.4 Connections between different kinds of stability

Given a bipartite graph with color classes  $M$  and  $W$ , and the set of edges is  $E$ . Let  $F$  denote the common choice function of the men,  $G$  is the women's choice function. In this model, stability can be defined in more ways. In this subsection we show what is the difference between these definitions.

We can define five kinds of stability:

1. Subset  $S$  of  $E$  is *three-part stable*, if  $F(S) = S = G(S)$ , and there exists subsets  $A$  and  $B$  of  $E$ , such that  $F(A) = S = G(B)$  and  $A \cup B = E$ ,  $A \cap B = S$ .
2. Subset  $S$  of  $E$  is *four-part stable*, if  $F(S) = S = G(S)$ , and there exists subsets  $A$  and  $B$  of  $E$ , such that  $F(A) = S = G(B)$  and  $A \cap B = S$ , and  $\mathcal{D}_F(A) = E \setminus B$ ,  $\mathcal{D}_G(B) = E \setminus A$ .
3. Subset  $S$  of  $E$  is *dominating stable*, if  $F(S) = S = G(S)$  and  $\mathcal{D}_F(S) \cup \mathcal{D}_G(S) = E \setminus S$ .



4. If  $F$  is given by a strict ordering,  $G$  is a scoring choice function, we can define *score-stability* as in Subsection 3.1.
5. If  $F$  and  $G$  are scoring choice functions, we can define *two-sided score-stability*.

In the stable marriage problem, the original definition gives the dominating stability: if an edge is not in the marriage scheme, it must be dominated by the men or by the women.

We use the notation  $A+x$  for adding a set with only one element, so  $A+x = A \cup \{x\}$ .

**Lemma 17.** *If  $F$  is comonotone and path independent:*

$$F(A) = S \Leftrightarrow \mathcal{D}_F(S) \cap A = A \setminus S.$$

*Proof.*  $\Rightarrow$

For every  $x \in A \setminus S$ , because  $S \subset S+x \subset A$  and  $F$  is path independent,  $F(S+x) = S$ , so  $\mathcal{D}_F(S) \supseteq A \setminus S$ .

$\Leftarrow$

For every  $x \in A \setminus S$ ,  $\overline{F}(S+x) = x$ , so from comonotony,  $x \in \overline{F}(A)$ . From this,  $\overline{F}(A) \supseteq A \setminus S$ , witch means  $F(A) \subseteq S$ .

From the path independent property,  $F(A) \subseteq S \subseteq A \Rightarrow F(A) = F(S) = S$  □

**Lemma 18.** *If  $F$  is comonotone, path independent and  $F(A) = S$ , then  $\mathcal{D}_F(A) = \mathcal{D}_F(S)$ .*

*Proof.* We saw that  $\mathcal{D}_F$  is monotone, hence  $\mathcal{D}_F(S) \subseteq \mathcal{D}_F(A)$ .

If  $x \in \mathcal{D}_F(A)$ , then  $F(A+x) \subseteq A \subseteq A+x$ , and because of the path independency,  $F(A+x) = F(A) = S$ .

Therefore  $F(A+x) \subseteq S+x \subseteq A+x$ , so  $F(S+x) = F(A+x) = S$ . It means  $S$  dominates  $x$ , so  $\mathcal{D}_F(A) = \mathcal{D}_F(S)$ . □

**Theorem 19.** *If  $F$  and  $G$  are comonotone and path independent, three-part, four-part and dominating stability are equivalent.*

*Proof.* *three-part*  $\Rightarrow$  *dominating*

There are  $A, B$  such that  $F(A) = S = G(B)$ . From Lemma 17  $\mathcal{D}_F(S) \supseteq A \setminus S$ . The same goes to  $G(B)$ , so  $\mathcal{D}_G(S) \supseteq B \setminus S$ . Their union is

$$\mathcal{D}_F(S) \cup \mathcal{D}_G(S) \supseteq ((A \setminus S) \cup (B \setminus S)) = E \setminus S.$$

And  $S$  do not dominate itself, so  $\mathcal{D}_F(S) \cup \mathcal{D}_G(S) = E \setminus S$ .

*dominating  $\Rightarrow$  four-part*

We know that  $\mathcal{D}_F(S) \cup \mathcal{D}_G(S) = E \setminus S$ . Let  $A = E \setminus \mathcal{D}_G(S)$   $B = E \setminus \mathcal{D}_F(S)$ .

$A \subseteq S \cup \mathcal{D}_F(S)$  so  $F(A) = S$ . From lemma 18  $\mathcal{D}_F(A) = \mathcal{D}_F(S) = E \setminus B$ . Similarly,  $\mathcal{D}_G(B) = \mathcal{D}_G(S) = E \setminus A$ . With this  $A, B, S$  is four-part stable.

*four-part  $\Rightarrow$  three-part*

There exists subsets  $A$  and  $B$  of  $E$ , such that  $F(A) = S = G(B)$  and  $A \cap B = S$ , and  $\mathcal{D}_F(A) = E \setminus B, \mathcal{D}_G(B) = E \setminus A$ .

Let  $D = E \setminus (A \cup B)$  and  $A' = A \cup D, B' = B$

Now  $A' \cup B' = E, A' \cap B' = A \cap B = S$  and from lemma 18  $\mathcal{D}_F(S) = \mathcal{D}_F(A) = E \setminus B = (A \setminus S) \cup D = A' \setminus S$ . From lemma 17,  $F(A') = S, G(B') = G(B) = S$ , so with  $A', B', S$  is three-part stable.

□

**Theorem 20.** *If  $F, G$  are comonotone choice functions,  $F$  is path independent, but  $G$  is not, then every four-part stable set is three-part stable, but none of the other directions in Theorem 19 are true.*

*Proof.* Note that in the third section of the previous proof, we have not used that  $G$  should be path independent.

See counterexamples for the other directions later.

□

**Theorem 21.** *If  $F, G$  are comonotone choice functions, but not path independent, none of the first three property (three-part, four-part, dominating) follows from any of the others.*

Compared to other stability concepts, the three-part, four-part and dominating stability are defined on every comonotone functions  $F, G$ , for the one sided score stability we need strict preference ordering on one side, and a scoring choice function on the other side. Two-sided score stability needs scoring choice functions on both sides.

As we showed in Theorem 11, score stability is equivalent with four-part stability.

**Theorem 22.** *The two-sided score stability is not equivalent with the one-sided score stability. Moreover, it is not equivalent with any of the other stability definitions.*

**Theorem 23.** *If  $F$  and  $G$  are scoring choice functions, then every two-sided score stable set is also three-part stable.*

*Proof.* Let  $S$  be a two-sided score stable set, with a corresponding score vector  $t$ . If  $e \notin S$ , then  $e$  didn't reach the score limit on one side. If it was rejected by the university (choice function  $F$ ) then let  $e \in A \setminus S$ , and if  $e$  was rejected by the student ( $G$ ) let  $e \in B \setminus S$ . If it was rejected by both sides,  $e$  can be anywhere outside of  $S$ . The score vector  $t$  is stable, so if a university or a student  $a$  lowers its quota by one, it will get too many applicants/universities. If  $e = ab$  was blocked on both ends, it will not be realized even after  $a$  lowers its score limit. So the people who cause problem to  $a$  are those who was only blocked by  $a$ , so they are all in  $A \setminus S$ . With score limit  $t$ , applications chosen from  $A$  is  $S$ , and with score limit  $(t_1, \dots, t_a - 1, \dots, t_n)$ , applications chosen from  $A$  exceeds the quota of  $a$  so by definition of choice  $F$ ,  $F(A) = S$ . Similarly  $G(B) = S$ , and by the construction  $A \cup B = E$ ,  $A \cap B = S$ . Therefore  $S$  is three-part stable.  $\square$

We define some examples for choice functions:

$$F = q1$$

There are two students applying for the same university, with equal points, but the quota is one. If someone applies alone he is accepted, but if both of them apply the university refuses both.

$$F(\emptyset) = \emptyset$$

$$F(a) = a$$

$$F(b) = b$$

$$F(ab) = \emptyset$$

$$F = q2$$

There are two students applying for the same university, with equal points, the quota is two, so everybody is accepted.

$$F(\emptyset) = \emptyset$$

$$F(a) = a$$

$$F(b) = b$$

$$F(ab) = ab$$

$$F = (a > b)$$

There are two students applying for the same university,  $a$  is better than  $b$ , the quota is one. So the university chooses  $a$ .

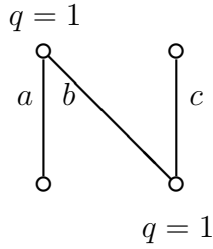
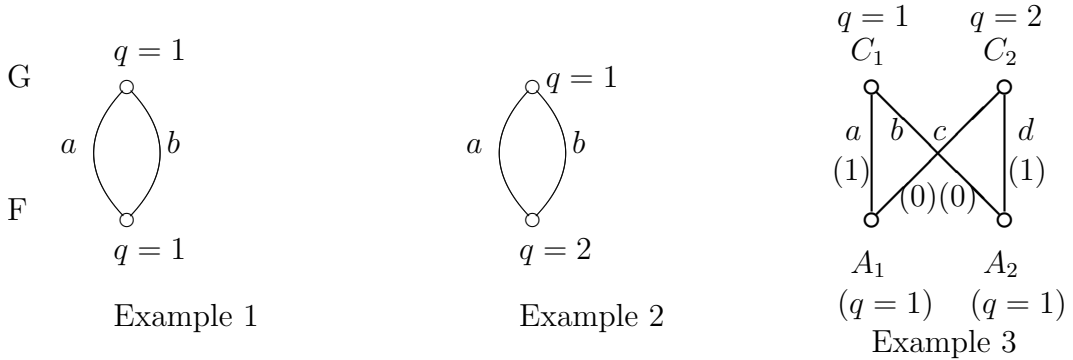
$$F(\emptyset) = \emptyset$$

$$F(a) = a$$

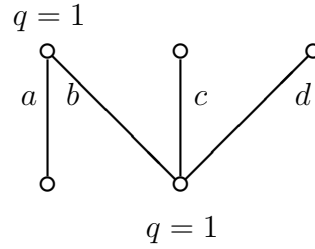
$$F(b) = b$$

$$F(ab) = a$$

Counterexamples for Theorems 20 and 21, and that three-part stability does not imply two-sided score stability in the general case:



Example 4



Example 5

In the first, second, fourth and fifth example, every student (or college) gets score 0 everywhere.

To show that the one-sided and two-sided score stability is not equivalent, we use Example 3:

There are two colleges and two applicants: Both applicants got the same score (0 point) on the entrance exam, and applicant  $A_1$  prefers college  $C_1$ , applicant  $A_2$  prefers college  $C_2$ . The quota of  $C_1$  is 1, for  $C_2$  it is 2.

If we consider the situation as a two-sided scoring system,  $C_1$  receives 1 point at  $A_1$ , 0 at  $A_2$ , and the reverse for  $C_2$ . Both of the applicants have a quota of 1.

For one-sided score stability, score vector  $(0,0)$  is stable,  $(1,0)$  is valid, but not stable.)

The stable allocation is  $S = \{C_1A_1, C_2A_2\} = \{a, d\}$

For two-sided score limits  $(t_{C_1}, t_{C_2}, t_{A_1}, t_{A_2},) = (1, 0, 0, 0)$  is stable, since if  $C_1$  decreases its limit to  $(0,0,0,0)$ , all four applications realizes, an  $C_1$  gets 2 students instead of one. (And students get 2 colleges instead of one.) So the stable allocation is  $S' = \{C_1A_2, C_2A_2\} = \{b, d\}$

example	$F$	path ind?	$G$	path ind?
1	$q1$	no	$q1$	no
2	$q2$	yes	$q1$	no
3	$a > c + b > d$	yes	$q1(a, b) + q2(c, d)$	no
4	$a + q1(bc)$	no	$q1(a, b) + c$	no
5	$a + q1(b, c, d)$	no	$q1(a, b) + c + d$	no

In the second and third example,  $G$  is path independent, so these are also good counterexamples for Theorem 20.

Here "+" means that what  $F$  choose from  $a$  and  $b$  is independent from what he choose from  $c$  and  $d$  (The choices are given on two different vertex). More generally, if  $X \cap Y = \emptyset$  and  $F_1 : 2^X \rightarrow 2^X$ ,  $F_2 : 2^Y \rightarrow 2^Y$ ,  $F : 2^{X \cup Y} \rightarrow 2^{X \cup Y}$ ,  $F = F_1 + F_2$  means for every  $A \subseteq X \cup Y$ ,  $F(A) = F_1(A \cap X) \cup F_2(A \cap Y)$ . We can say that  $F$  is a direct sum of  $F_1$  and  $F_2$ .

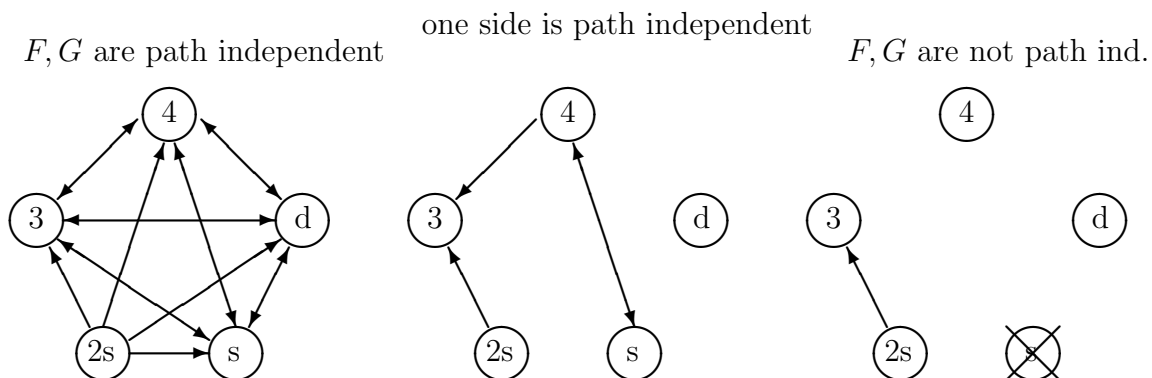
The stable sets will be:

Example	3-part	4-part	dominating	score	2 score
1	$\emptyset$	$\emptyset, \{a\}, \{b\}$	$\{a\}, \{b\}$	X	$\emptyset$
2	$\emptyset$	$\emptyset$	$\{a\}, \{b\}$	$\{a\}$	$\emptyset$
3	$\{a, d\}, \{c, d\}$	$\{a, d\}$	$\{a, d\}$	$\{a, d\}$	$\{a, d\}, \{c, d\}$
4	$\{a\}, \{c\}$	$\{a, c\}$	$\{a, c\}$	X	$\{a\}, \{c\}$
5	$\emptyset, \{a\}$	$\{a\}$	$\{b\}, \{a, c\}, \{a, d\}$	X	$\{a\}$

X means the one-sided score stable matching is not defined in these examples, because none of  $F$  and  $G$  are path independent.

Graphs of the connections:

Notations: 3=three-part stable, 4=four-part stable, d=dominating stable, s=score stable, 2s= two-sided score stable



## 4 The Kelso-Crawford model

In the Kelso-Crawford model [Kelso and Crawford (1982)], there are  $m$  workers and  $n$  firms. Each firm hires as many workers as it wishes, but a worker is allowed to work only for one firm. So it is a many-to-one matching. The utility function of worker  $i$ , if he works for firm  $j$  at salary  $p_{ij}$  is  $u^i(j, p_{ij})$ , where  $u^i$  is strictly increasing and continuous in its second argument.

The firm  $j$ 's utility with the set of workers  $C^j$  and salary vector  $p^j$  is  $\pi(C^j, p^j) = y^j(C^j) - \sum_{i \in C^j} p_{ij}$ .

Let  $\sigma_{ij}$  defined by  $u^i(j, \sigma_{ij}) = u^i(0, 0)$ , thus  $\sigma_{ij}$  is the lowest salary at which worker  $i$  would ever consider working for firm  $j$ .

With a given salary vector  $p$ , the choice function of firm  $j$  is the following:

$$F^j(C^j) = \{C : C \subseteq C^j \text{ and } \pi^j(C, p^j) = \max_{C \in C^j} \pi^j(C, p^j)\}$$

The set of optimal worker sets, where the  $\pi$  profit of the firm is maximal. Differently from the previous sections, now the choice function is not uniquely defined. We need the following assumptions about the firms' and workers' preferences:

(MP) (marginal product)  $y^j(C \cup \{i\}) - y^j(C) - \sigma_{ij} \geq 0$  for every  $i, j$ .

This means it is worth for a firm to hire a worker on his lowest possible salary, because the marginal product is bigger than the salary paid.

(NFL) No free lunch  $y^j(\emptyset) = 0$  for all  $j$ .

If nobody works to firm  $j$ , its utility is zero.

The most important property is the third one:

Consider two vectors of salaries  $p^j$  and  $\tilde{p}^j$  facing firm  $j$ . Let  $T^j(C^j) = \{i : i \in C^j \text{ and } p_{ij} = \tilde{p}_{ij}\}$  be the set of workers who were chosen under salary vector  $p$ , and their salary have not increased.

The *gross substitutes* (GS) property :

(GS) for every firm  $j$ , if  $C^j \in F^j(p^j)$  and  $\tilde{p}^j \geq p^j$ , then there exists  $\tilde{C}^j \in \tilde{F}^j(\tilde{p}^j)$  such that  $T^j(C^j) \subseteq \tilde{C}^j$

We search for some stable allocations in this model.

**Definition 14.** [Kelso and Crawford (1982)] An *individually rational* allocation is an assignment of workers to firm together with a salary schedule such that, if  $f : \{1, \dots, m\} \rightarrow \{1, \dots, n\}$  is the function that represents the assignments (so that  $f(i)$  is the firm to which worker  $i$  is assigned) and  $C^j = \{i | j = f(i)\}$  so  $C^j$  is the set of workers hired by

firm  $j$ . It satisfies the following properties:

$$p_{if(i)} \geq \sigma_{if(i)}$$

$$\pi^j(C^j, p^j) \geq y^j(C^j) - \sum_{i \in C^j} p_{ij} \geq 0.$$

**Definition 15.** [Kelso and Crawford (1982)] A *(discrete) strict score allocation* is an individually rational allocation  $(f, p_{1f(1)}, \dots, p_{mf(m)})$  such that there are no coalition of a firm and a set of workers  $(j, C)$  and (integer) salaries  $r^j = (r_{1j}, \dots, r_{mj})$  that satisfy:

$$u^i(j, r_{ij}) \geq u^i(f(i), p_{if(i)}) \quad \text{for all } i \in C$$

$$\pi^j(C^j, r^j) \geq \pi^j(C^j, p^j)$$

with strict inequality holding for at least one member of  $C \cup \{j\}$ .

**Definition 16.** A *(discrete) core allocation* is defined in the same way as a (discrete) strict core allocation, except that it is required that there is no firm-set of workers combination and (integer) salaries that satisfy all equations with strict inequality.

In the stable marriage model, stable matchings were exactly the core of the game.

Now the Kelso-Crawford algorithm finds the discrete core. It is similar to the Gale-Shapley algorithm, with proposal and rejection rounds, but now we involve money, and since  $F$  multi-valued, this algorithm can not be that easily described with Tarski's theorem.

The steps of Kelso-Crawford algorithm [Kelso and Crawford (1982)], page 1488:

"R1. In round zero, every firm propose to every worker with salaries  $p_{ij}(0) = \sigma_{ij}$ . (In our special case,  $\sigma_{ij} = 0$ .)

R2. On each round, each firm makes offers to the members of one of its favourite set of workers, given the schedule of permitted salaries  $p^j(t) = [p_{1j}(t), \dots, p_{mj}(t)]$ . That is, firm  $j$  makes offers to the members of  $C^j[p^j(t)]$ , where  $C^j[p^j(t)]$  maximizes  $\pi^j[C, p^j(t)]$ . Firms may break ties between sets of workers however they like, with the following exception: Any offer made by firm  $j$  in round  $t - 1$  that was not rejected must be repeated in round  $t$ .

R3. Each worker who receives more than one offers rejects all but his or her favourite (taking salaries into account) which he or she tentatively accepts. Workers may break ties at any time however they like.

R4. Offers not rejected in previous periods remain still in force. If worker  $i$  rejected an offer from firm  $j$  in round  $t - 1$ ,  $p_{ij}(t) = p_{ij}(t - 1) + 1$ , otherwise  $p_{ij}(t) = p_{ij}(t - 1)$ . Firms continue to make offers to their favourite sets of workers, taking into account

their permitted salaries.

R5. The process stops when no rejections are issued in same period. Workers then accept the offers that remain in force from the firms they have not rejected.”

**Lemma 24.** [Kelso and Crawford (1982)] *The process converges to a discrete core allocation in the discrete market for it is defined.*

We study the connection between comonotonicity in Section 2, and the gross substitutes property here. We claim that they are the same property in some sense.

**Lemma 25.** [Fleiner (2003)] *The choice function  $F$  is comonotone if and only if for every  $A \subseteq B$  sets  $F(A) \supseteq A \cap F(B)$ .*

*Proof.* If  $F$  is comonotone and  $A \subseteq B$ , then  $\overline{F}(A) \subseteq \overline{F}(B)$ , so  $A \setminus F(A) \subseteq B \setminus F(B)$ , therefore  $F(A) \supseteq A \setminus (B \setminus F(B)) = A \cap F(B)$ .

In the other direction, if  $F(A) \supseteq A \cap F(B)$ , then  $A \setminus F(A) \subseteq A \setminus F(B) \subseteq B \setminus F(B)$ , so  $\overline{F}(A) \subseteq \overline{F}(B)$ .  $\square$

**Lemma 26.** [Hatfield et al. (2011)] *If we demand gross substitutability only for prices  $\tilde{p} \geq p$ , where  $|F^j(\tilde{p})| = |F^j(p)|$ , this is equivalent with the original (GS) definition that is for every good  $C^j \in F^j(p)$  there exists a good  $\tilde{C}^j \in F^j(\tilde{p})$ .*

Actually they proved this for fully substitutable preferences, what we will define later. But gross substitutability is a special case of fully substitutability in a supply chain market where firms are buyers and they don't sell anything, and workers are the sellers. So this lemma is a corollary of (DFS)  $\Leftrightarrow$  (DCFS) in Appendix A of [Hatfield et al. (2011)].

Call a firm-worker-salary triplet a *contract*, like in the article [Hatfield-Milgrom (2005)].

**Theorem 27.** *If we restrict the choice function in Kelso-Crawford model to the  $p$  values, where there is a unique best set of workers, the gross substitutes property is equivalent with comonotonicity, defined on contracts.*

This is also a special case of the (CFS)  $\Leftrightarrow$  (DFS) theorem in [Hatfield et al. (2011)].

*Proof.* Let  $\tilde{p}^j \geq p^j$  be two vectors of salaries involving firm  $j$ , and let  $C^j \in F^j(p^j)$ .

As we defined before,  $T^j(C^j) = \{i : i \in C^j \text{ and } p_{ij} = \tilde{p}_{ij}\}$  is a subset of worker-firm edges. Let  $E_j$  be the set of all edges incident with firm  $j$ . ( $E_j = \{jw, w \text{ is a worker}\}$ ).

Let  $M = \{i : i \in E_j \text{ and } p_{ij} = \tilde{p}_{ij}\}$  be the set of edges where the salary remains the same, and  $V = \{i : i \in E_j \text{ and } p_{ij} < \tilde{p}_{ij}\}$  the set of edges where the price increases.



From this  $T^j(C^j) = C^j \cap M$ .

The set of contracts is  $E_j \times \mathbb{R}^{|E_j|}$ . Introduce the following subsets:

$$A = \{(e, p) | e \in M, p = p\}$$

$$B = \{(e, p) | e \in V, p = p\}$$

$$\tilde{B} = \{(e, p) | e \in V, p(e) = \tilde{p}\}$$

If more salaries are available for the same worker, the firm's choice function chooses the lowest salary.

Comonotone  $\Rightarrow$  Gross Substitutes:

Since only the lowest salary matters,  $F(B \cup A) = F(B \cup \tilde{B} \cup A)$ . The gross substitutes property means  $F(B \cup A) \cap A \subseteq F(\tilde{B} \cup A)$ , because with salary vector  $p$ , firm chose from  $B \cup A$ , so  $C^j = F(B \cup A)$ , and the contracts where price remained the same is  $T^j = F(B \cup A) \cap A$ . What we need is the elements of  $T^j$  is also chosen under salaries  $\tilde{p}$ , so  $T^j \subseteq F(\tilde{B} \cup A)$ .

From Lemma 25,  $F(B \cup \tilde{B} \cup A) \cap (\tilde{B} \cup A) \subseteq F(\tilde{B} \cup A)$ . The firm doesn't choose a contract from  $\tilde{B}$  if  $(B \cup \tilde{B} \cup A)$  is available, so  $F(B \cup \tilde{B} \cup A) \cap A \subseteq F(\tilde{B} \cup A)$ . From  $F(B \cup A) = F(B \cup \tilde{B} \cup A)$  we get  $F(B \cup A) \cap A \subseteq F(\tilde{B} \cup A)$  and this is what we wanted.

Gross Substitutes  $\Rightarrow$  comonotone

Let  $X \subseteq Y$  two arbitrary contract-sets, and let  $p(e)$  for every edge the cheapest prize, where  $(e, p(e))$  is in  $Y$ .  $p(e) = \min\{p, (e, p) \in Y\}$ . If  $e$  is not represented in  $Y$ , with any salary, let  $p(e)$  be infinitely high, so firm  $j$  will never going to choose it in any coalition, with any salaries of the other workers. We cannot leave out edges, because the definition of GS applies for the whole edge-set. Let  $\tilde{p}(e)$  be the cheapest salary of  $e$  in  $X$ . Similarly if  $(e, p) \notin X$ , we use infinite price. Since  $Y$  is bigger than  $X$ ,  $\tilde{p}^j \geq p^j$ .

$$\text{Let } M = \{i : ij \in E_j \text{ and } p_{ij} = \tilde{p}_{ij}\} \quad V = \{i : ij \in E_j \text{ and } p_{ij} < \tilde{p}_{ij}\}$$

$$\text{Legyen } A = \{(e, p) | e \in M, p = p\}$$

$$B = \{(e, p) | e \in V, p = p\}$$

$$\tilde{B} = \{(e, p) | e \in V, p(e) = \tilde{p}\}$$

$$D = \{(e, p) | e \in V, p < p(e) < \tilde{p}\}$$

$$C = Y \setminus (A \cup B \cup \tilde{B} \cup D) = \{(e, p) | e \in V, p(e) > \tilde{p} \text{ or } e \in M, p(e) > p\}$$

$$\text{With these subsets } X = A \cup \tilde{B} \cup C$$

$$Y = A \cup B \cup \tilde{B} \cup C \cup D$$

$F$  choose the cheapest price everywhere, so  $F(Y) = F(A \cup B)$  and  $F(X) = F(A \cup \tilde{B})$ . From gross substitutes,  $F(B \cup A) \cap A \subseteq F(\tilde{B} \cup A)$  so  $F(Y) \cap A \subseteq F(X)$ .  $F(Y) \cap X = F(Y) \cap A \subseteq F(X)$  because  $F(Y)$  doesn't choose from  $\tilde{B}$  and  $C$ . From Lemma 25,  $F$  is comonotone.  $\square$

An interesting question is for what  $y^j$  utilities will the preferences of the firms satisfy the gross substitutes criterium (comonotony).

**Theorem 28.** [Kelso and Crawford (1982)] *If a firm  $j$  can choose only from two workers  $i_1$  and  $i_2$  the utilities are finite and  $y^j(\emptyset) = 0$ , then its preferences are gross substitutable if and only if  $y^j$  is subadditive i.e.,  $y^j(\{i_1, i_2\}) \leq y^j(\{i_1\}) + y^j(\{i_2\})$ .*

If there are more than two workers, subadditivity is not sufficient. [Kelso and Crawford (1982)] shows a counterexample for three workers, where the technology firm  $j$  is subadditive, but GS fails.

**Definition 17.** An utility function  $y : 2^E \rightarrow \mathbb{R}$  is submodular if for all  $A, B \subseteq E$   $y(A) + y(B) \geq y(A \cup B) + y(A \cap B)$ .

The class of gross substitutes utility functions is a subclass of submodular utility functions, moreover gross substitutes is equivalent with the *single improvement property* introduced by [Gul-Stacchetti (1999)].

It was shown by [Fujishige-Yang (2003)] that a utility function  $y : 2^E \rightarrow \mathbb{R}$  satisfies the gross substitutes condition if and only if  $y$  is an  $M^{\natural}$ -concave function.

Workers are identical if the  $j$ 's utility of a group workers depends only on the number of workers it hire, so there exists a  $\tilde{y}^j$  function such that  $y^j(C) = \tilde{y}^j(|C|)$ .

[Kelso and Crawford (1982)] introduced the nonincreasing returns property:

(DR)  $\tilde{y}^j(w+1) - \tilde{y}^j(w) \leq \tilde{y}^j(w) - \tilde{y}^j(w-1)$  for integer values of  $1 \leq w \leq m-1$ , where  $w$  is the number of identical workers firm  $j$  hires.

**Theorem 29.** [Kelso and Crawford (1982)] *If workers are identical, (DR) is equivalent with (GS).*

We will use this theorem in the last section.

## 4.1 Maximum weighted matchings

Let  $G = (V, E)$  be a bipartite graph, where there exists a perfect matching in  $G$ , and every edge  $e$  has a nonnegative weight  $c(e)$ . Our aim to find a maximal weighted perfect matching.

We call a function  $\pi$  over the vertices a *weighted-covering* if for every  $u, v$  vertices  $\pi(u) + \pi(v) \geq c(uv)$ . An edge  $e = uv$  is *exact* for vertex cover  $\pi$ , if  $\pi(u) + \pi(v) = c(uv)$ . The weight of  $\pi$  is  $\pi(V) = \sum_{v \in V} \pi(v)$ .

**Theorem 30.** [Egerváry (1931)]

Let  $G = (S, T, E)$  be a bipartite graph with  $|S| = |T|$  and let  $c : E \rightarrow \mathbb{R}_+$  a nonnegative weight function. The maximum weight of a perfect matching of  $G$  is equal to the minimum weight of nonnegative, integer-valued weighted-covering of  $c$ .

If  $G$  is a complete bipartite graph, the optimal weighted-covering can be chosen as nonnegative, and if  $c$  is integer-valued, the optimal  $\pi$  can be integer-valued too.

It can be formalized as a linear programming problem:

$$\max_{M \text{ is complete matching}} \sum_{uv \in M} c(uv) = \min_{\pi \text{ is a weighted-covering}} \sum_{v \in V} \pi(v)$$

Let  $A$  be the incidence matrix of graph  $G$ , so the rows represent the vertices, the columns represent the edges, and  $a_{ve} = 1$ , if  $e = uv$  for some  $u$ , and  $a_{ve} = 0$  otherwise. Since  $G$  is bipartite this matrix is totally unimodular.

**Definition 18.** A matrix  $A$  is totally unimodular if every subdeterminant of  $A$  is either 1, 0 or  $-1$ .

The maximal weighed matching is:

$$\max cx, Ax = \mathbf{1}, x \geq 0$$

The dual problem:  $\min \mathbf{1}y, yA \geq c$

Where  $\mathbf{1}$  is the everywhere one vector. The dual problem is the same as the minimum weighted-covering.

From the duality theorem  $\max xc = \min \mathbf{1}y$ , and since  $A$  is totally unimodular, there exist optimal integer-valued solutions for  $x$  and  $y$ . Here  $x$  will be the characteristic vector of the matching, and  $y = \pi$  is the weighted-covering.

**Theorem 31** (Duality theorem). Suppose that polyhedron  $R = \{x = (x_0, x_1) : Px_0 + Ax_1 = b_0, Qx_0 + Bx_1 \leq b_1, x_1 \geq 0\}$  is not empty,  $\{cx : x \in R\}$  has an upper bound. Then the dual polyhedron of  $R$  is  $R^* = \{y = (y_0, y_1) : Py_0 + Qy_1 = c_0, Ay_0 + By_1 \geq c_1, y_1 \geq 0\}$  which is also nonempty and the maximum of the primal problem equals to the minimum of the dual problem:

$$\max\{cx : x \in R\} = \min\{by : y \in R^*\}.$$

We show that a maximal weighed matching can be found by the Kelso-Crawford algorithm, as the K-C algorithm searches a discrete core, we only use it for integer-valued maximal weighted matchings.

Let  $G = (S, T, E)$  be a bipartite graph with  $|S| = |T| = n$ . It is sufficient to prove the case where  $G$  is complete and bipartite, since if  $G$  is not complete, we add a big enough constant  $K$  to every  $c(e)$  weight, (let  $K > n * \max_{e \in E} c(e)$ ), and we add the missing edges with 0 weight. Now the graph is complete bipartite, the weights are still nonnegative, and the weight of every perfect matching were increased by  $nK$ , the non-perfect matchings' weight were increased by  $< nK$ .

With the original weights, the max weighted non-perfect matching is at most  $(n - 1) \max_{e \in E} c(e)$ , and a weight of a perfect matching is at least 0. If we add  $K$  to the weights,  $(n - 1) \max_{e \in E} c(e) + (n - 1)K < nK$ , so the new maximal matching was a perfect matching in the original graph.

In this special case of the Kelso-Crawford model, the workers utility is linear in the salaries, so  $u^i(j, p_{ij}) = u^i(j, 0) + p_{ij}$ .

Working for any firm worths 0 utility to every worker, so they decide only on the salary. So  $u^i(j) = 0 \forall i, j$ , and the utility of being unemployed is also 0,  $u^i(\emptyset) = 0 \forall i$ . It follows that  $\sigma_{ij} = 0 \forall i, j$ .

Every firm is acceptable to every worker,  $G = (F, W, E)$  is a complete bipartite graph, and for a weight function  $c : E \rightarrow \mathbb{R}_+$ , let the utility of the firms be

$y^j(C) = \max_{i \in C} c(ij), y^j(\emptyset) = 0$ . By definition,  $y^j$  has the no free lunch property. Since  $\sigma_{ij} = 0$ , the marginal product property is reduced to  $y^j(C \cup \{i\}) - y^j(C) \geq 0$ , i.e. the  $y^j$  utility function is monotone. Choosing from a bigger set, the maximum cannot decrease, so this is true.

The choice function of firm  $j$  chooses one worker  $i$  for which  $c_{ij} - p_{ij}$  is maximal and add some other worker, whose salary is 0, because it does not affect its utility. If  $j$  would fire more than one worker with positive salary, and the maximum of  $c_{ij} - p_{ij}$  is reached at  $i'$ ,

$$\pi^j(C) = y^j(C) - \sum_{i \in C} p_{ij} = \max_{i \in C} c(ij) - \sum_{i \in C} p_{ij} = c_{i'j} - \sum_{i \in C} p_{ij} < c_{i'j} - p_{i'j}.$$

The other workers do not change the maximum of  $c$ , but decrease the utility of the firm with their salary.

For this choice function, gross substitutes holds, because if  $\tilde{p} \geq p$  and  $\tilde{p}_{ij} = p_{ij}$ , and  $i$  were the one chosen with maximal  $c_{ij} - p_{ij}$  value, the others only could change for worse, so  $i$  is still maximal. If  $i$ 's salary was 0, it remains zero. So there is a  $\tilde{C}^j \in F^j(\tilde{s}^j)$  which chooses every worker who were chosen before and did not change their salaries.

In the first step of the Kelso-Crawford algorithm, every firm proposes to every

worker with salary 0. The workers are indifferent between the firms, so they accept one however they like, for example they can use lexicographic ordering, and everybody accepts the first firm's proposal. They break ties in such way they like working to any firm more than being unemployed. Every firm keeps the offers which was not rejected, and increase the salary by  $\Delta$  if it was rejected, completing it to a favourite set under this new salary vector. Using GS, there is a favourite set which contains all not rejected offers. Let  $\Delta$  be a small positive value,  $\Delta < 1/n$ .

Iterating these steps, denote the allocation given at the end of the algorithm with  $f$ , where  $f(i)$  is the firm where worker  $i$  goes.

This algorithm also gives us the dual vector, the weighted-covering. For a worker  $i$ , let  $\pi(i)$  be the salary he gets firm  $f(i)$ , so  $\pi(i) = p_{if(i)}$ . If  $j$  is a firm, let  $\pi(j)$  be his utility at the end:  $\pi(j) = \max_{f(i)=j} (c_{ij} - p_{ij})$ .

We show that this  $\pi$  is "nearly covering", which means  $\pi(i) + \pi(j) \geq c_{ij} - \Delta$ . If  $i$  works for firm  $j$ , there are two cases:

Firm  $j$  reached its maximal  $c_{ij} - p_{ij}$  utility at worker  $i$ .  $\pi(i) = p_{ij}$  and  $\pi(j) = c_{ij} - p_{ij}$ , so  $\pi(i) + \pi(j) = c_{ij}$ , that is  $ij$  is an exact edge.

In the other case,  $i$  gets 0 salary, so  $\pi(i) = 0$ , and  $\pi(j) = \max_{f(i)=j} (c_{ij} - p_{ij}) \geq c_{ij} - 0$ . Therefore  $\pi(i) + \pi(j) = 0 + \pi(j) \geq c_{ij}$ .

If  $i$  does not work for firm  $j$ :

We can imagine the  $ij$  relationship as many parallel edges in the graph, where every edge has a possible salary  $0, \Delta, 2\Delta, \dots, c_{ij}$ . (The salary cannot be less than zero because it starts from zero and increases, and cannot be more than  $c_{ij}$ , because that would mean negative utility for the firm, so the firm rather chooses the empty set than offering a salary higher than  $c_{ij}$ .)

Some of these parallel edges were refused by the worker, let  $k$  be the biggest salary of these. Worker  $i$  rejected this offer, because he got an at least  $k$  salary offer for another firm, so  $\pi(i) \geq k$ .

In this cases  $\pi$  is not just nearly covering, but a proper weighted-covering.

Salaries bigger than  $k$  between firm  $j$  and worker  $i$  did not realize, because firm  $j$  did not offer it. By offering  $k + \Delta$ , the firm could achieve  $c_{ij} - k - \Delta$  utility, so  $j$  did not offer it because the firm reached this utility with another worker-set:  $\pi(j) \geq c_{ij} - k - \Delta$ . From these two inequalities we get that  $\pi(i) + \pi(j) \geq k + c_{ij} - k - \Delta = c_{ij} - \Delta$ , so  $\pi$  is nearly covering.

One problem is that the firm-worker assignment is not always a matching. If at least two workers work for firm  $j$ , all but one get salary 0. We call worker  $i$  who gets positive salary from firm  $j$ , "j's favourite". If  $i'$  works for  $j$ , but gets 0 salary, and firm  $j'$  has no workers, then  $\pi(i') = 0, \pi(j') = 0$ , and  $0 = \pi(i') + \pi(j') \geq c_{i'j'} - \Delta$ , so  $\Delta \geq c_{i'j'} \geq 0$ . Since  $c_{ij}$  is integer,  $c_{ij} = 0$ , so the edge  $i'j'$  is exact.

Change the assignment such that every firm keeps its favourite worker, and the non-favourite workers go to some firms with workers. Since the number of firms and workers are equal, this gives a perfect matching, call it  $M$ . Every edge in  $M$  is exact to  $\pi$ . Therefore the weight of  $\pi$  equals to the weight of  $M$ .  $\sum_M c(e) = \sum_{v \in V} \pi(v)$  Every  $c(e)$  is integer, so  $\sum_{v \in V} \pi(v)$  is also integer.

We claim that this is a maximal weighted perfect matching.

Suppose there exist a better  $M'$  matching,  $\sum_{e \in M'} c(e) > \sum_{e \in M} c(e)$ . Since for every edge  $\pi(i) + \pi(j) \geq c_{ij} - \Delta$ , the weight of the new matching is  $c(M') = \sum_{M'} c(e) \leq \sum_{v \in V} \pi(v) + n\Delta < \sum_{v \in V} \pi(v) + 1$  This  $c(M')$  is also integer, so it cannot be bigger than  $c(M)$ , therefore  $c(M)$  is maximal.

The number of the proposal steps are at least  $n^2 \frac{\max c_{ij} + \Delta}{\Delta} \approx n^3 \max c_{ij}$ , because each firm propose to each worker at one given price at most once, and prices can go from 0 to  $c_{ij}$ . So the number of the steps is polynomial in  $n$ . When firm  $j$  chooses the best subset, in the general case of the K-C algorithm  $j$  should check exponentially many subsets of  $E_j$ , but with this special utility function, firm only has to find the workers with maximal  $c(e) - p(e)$  value, and choose the one who was not rejected in former rounds. If every best worker rejected firm  $j$  in previous rounds, firm  $j$  just chooses one of them arbitrary, and add some zero-salary workers. It can be done in linear time, and the workers choose the best salary proposal in linear time, so the algorithm runs in polynomial,  $O(n^4)$  time.

**Remark 32.** *If weights can be any real values, and we know that the minimal nonzero difference between the weights of two different perfect matching is  $\varepsilon$ , then with  $\Delta < \frac{\varepsilon}{n}$ , this method finds the maximum weighted matching.*

## 5 Supply chains without money

Consider a directed graph  $G = (V, E)$ , where vertices represent people, and the arcs are trades between suppliers and buyers. In this way,  $G$  is a supply chain. Everybody in the market has some preference order on whom he would like to sell to or buy from.

A model is described in [Fleiner (2009)], where every node has a strict ordering over the incoming and outgoing arcs. For two arcs  $vu$  and  $vw$ , we say  $vu \succeq_v vw$ , if node  $v$  prefers selling to  $u$  instead of  $w$ . A *network* is a quadruple  $(G, s, t, c)$ , where  $G$  is a directed graph,  $s$  and  $t$  are different nodes of  $G$  and  $c : E \rightarrow \mathbb{R}_+$  is a function that determines the capacity  $c(a)$  of arc  $a$ .

The special nodes  $s$  and  $t$  are terminal points: source and target. Just like in the usual setting, we allow arcs to enter  $s$  or leave  $t$ .

A *flow* is a function  $f : E \rightarrow \mathbb{R}$  such that  $0 \leq f(a) \leq c(a)$  and every vertex different from  $s$  and  $t$  satisfies Kirchhoff's law:  $\sum_{uv \in E} f(uv) = \sum_{vw \in E} f(vw)$  The amount of the incoming flow equals the amount of outgoing flow for  $v$ .

We call an arc  $a$  *f-unsaturated*, if  $f(a) < c(a)$ .

A *blocking walk* of flow  $f$  is an alternating sequence of incident vertices and arcs  $P = (v_1, a_1, \dots, v_k)$  such that all the following properties hold:

arc  $a_i$  points from  $v_i$  to  $v_{i+1}$

$P$  has no terminal inner point

each arc  $a_i$  of  $P$  is *f-unsaturated*

$v_1 = s$  or  $v_1 = t$ , or there is an arc  $a' = v_1u$  such that  $f(a') > 0$  and  $a_1 \succ_{v_1} a'$  ( $a_1$  dominates  $a'$  in  $v_1$ )

$v_k = s$  or  $v_k = t$ , or there is an arc  $a'' = wv_k$  such that  $f(a'') > 0$  and  $a_{k-1} \succ_{v_k} a''$  ( $a_{k-1}$  dominates  $a''$  in  $v_k$ )

A flow  $f$  is *stable* if no blocking walk to  $f$  exists.

**Theorem 33** ([Fleiner (2009)]). *If network  $(G, s, t, c)$  and preference orders  $\prec_v$  describe a stable flow problem then there always exist a stable flow  $f$ . If capacity function  $c$  is integral then there exists an integral stable flow.*

In another model from [Ostrovsky (2008)], every vertex  $v$  has a  $Ch_v$  choice function over the arcs adjacent with  $v$ . Let  $D(v)$  be the set of arcs starting from  $v$  (contracts where  $v$  is the seller), and  $U(v)$  is the set of arcs ending at  $v$  (where  $v$  is the buyer).

Let  $s_c$  denotes the seller and  $b_c$  the buyer of arc  $c$ , then

$$D(v) = \{c \in E | v = s_c\} \text{ and } U(v) = \{c \in E | v = b_c\}$$

For a set of edges  $X$  incident with  $v$ , let  $D(X) = D(v) \cap X$ ,  $U(X) = U(v) \cap X$

**Definition 19.** [Ostrovsky (2008)] Preferences of agent  $a$  are *same-side substitutable* (SSS) if for any two sets of contracts  $X$  and  $Y$  such that  $D(X) = D(Y)$  and  $U(X) \subseteq$

$U(Y), U(X) \setminus U(Ch(X)) \subseteq U(Y) \setminus U(Ch(Y))$ , and for any two sets  $X$  and  $Y$  such that  $U(X) = U(Y)$  and  $D(X) \subseteq D(Y), D(X) \setminus D(Ch(X)) \subseteq D(Y) \setminus D(Ch(Y))$ . That is, preferences are same-side substitutable if, choosing from a bigger set of contracts on one side, the agent does not accept any contracts on that side that he rejected when he was choosing from the smaller set.

Preferences of agent  $a$  are *cross-side complementary* (CSC) if for any two sets of contracts  $X$  and  $Y$  such that  $D(X) = D(Y)$  and  $U(X) \subseteq U(Y), D(Ch(X)) \subseteq D(Ch(Y))$ , and for any two sets  $X$  and  $Y$  such that  $U(X) = U(Y)$  and  $D(X) \subseteq D(Y), U(Ch(X)) \subseteq U(Ch(Y))$ . That is, preferences are cross-side complementary if, when presented with a bigger set of contracts on one side, an agent does not reject any contract on the other side that he accepted before.

**Statement 34.** *If all nodes have strict preferences over arcs, the capacities are integers, and they choose the best buys and sells such that it satisfies Kirchhoff's law, this defines a SSS and CSC choice function.*

*Proof.* For a terminal node  $v = s$  or  $t$ , it choose every possible trades, so  $Ch_v(A) = A$  for all  $A \subseteq E$  and set of refused trades are  $\overline{Ch}_v(A) = \emptyset$  everywhere, so it satisfies SSS and CSC properties.

If  $v$  is not a terminal point, let  $|D(X)| = k, |U(X)| = l$ , where  $k \leq l$ . The choice function chooses all (outgoing) edges in  $D(X)$ , and the  $k$  best (incoming) edges from  $U(X)$ . If  $Y$  is a bigger set such that  $D(X) = D(Y)$  and  $U(X) \subseteq U(Y)$ , then  $k = |D(Y)| \leq |U(Y)|$ , so  $Ch_v$  chooses the whole  $D(Y)$ , therefore CSC is true, and it chooses the best  $k$  edges from  $U(Y)$ . If some edge weren't in the best  $k$  in  $D(X)$ , is neither in best  $k$  of a bigger set  $D(Y)$ , so the set of rejected edges is expanding, SSS is true.

If  $Y$  satisfies  $D(X) \subseteq D(Y)$  and  $U(X) = U(Y)$ , let  $|D(Y)| = k'$ . There are two cases:

If  $k' \leq l$ , so  $|D(Y)| \leq |U(Y)|$ , the choice function choose the whole  $D(Y)$ , the set of rejected outgoing edges remains  $\emptyset$  so it is SSS. And it chooses the best  $k'$  from  $U(Y) = U(X)$  i.e. more elements from the same set, so CSC is also true.

In the case if  $k' > l$ ,  $Ch_v$  chooses the best  $l$  from  $D(Y)$  the set of rejected outgoing edges expands from  $\emptyset$  to something else, so it is SSS. And chooses all  $U(Y)$  instead of the best  $k$ , so the set of selected ingoing edges expands, it is also CSC.

□

We say that a network is chain stable if there is no sequence of agents who could become better off by forming new contracts among themselves and possibly dropping



some of their current contracts.

We call a collection of bilateral relationships between the nodes in a market a network, i.e., a network is simply a set of contracts. Let  $M(a)$  denote the set of contracts involving vertex  $a$  in network  $M$ . Network  $M$  is *individually rational* if, for any agent  $a$ ,  $Ch_a(M(a)) = M(a)$  so no agent would like to unilaterally drop any of his contracts.

A chain is a sequence of contracts,  $c_1, \dots, c_n$ , such that for any  $i < n$ ,  $b_{c_i} = s_{c_{i+1}}$ , i.e., the buyer in contract  $c_i$  is the same node as the seller in contract  $c_{i+1}$ .

For notational convenience, let  $b_i = b_{c_i}$  and  $s_i = s_{c_i}$ . For a network  $M$ , a *chain block* is a chain  $c_1, \dots, c_n$  such that

$$\begin{aligned} \forall i \leq n, c_i &\notin M, \\ c_1 &\in Ch_{s_1}(M(s_1) \cup c_1), \\ c_n &\in Ch_{b_n}(M(b_n) \cup c_n), \\ \text{and } \forall i < n, \{c_i, c_{i+1}\} &\subseteq Ch_{b_i=s_{i+1}}(M(b_i) \cup c_i \cup c_{i+1}). \end{aligned}$$

A network is *chain stable* if it is individually rational and has no chain blocks.

Ostrovsky defined the chain stability only for acyclic graphs: upstream-downstream networks, where there are suppliers of basic inputs (sources), intermediaries and consumers of final outputs (targets).

**Theorem 35** ([Ostrovsky (2008)]). *If every vertex's choice function is SSS, CSC and path independent, and  $G$  is acyclic, then there exists a chain stable network.*

Now we allow circles in the graph.

**Definition 20.** A *blocking walk* of network  $M$  is an alternating sequence of incident vertices and arcs  $P = (v_1, a_1, \dots, a_{k-1}, v_k)$  such that all the following properties hold:

$$\begin{aligned} \text{arc } a_i &\text{ points from } v_i \text{ to } v_{i+1} \\ \text{each arc } a_i &\notin M \\ a_1 &\in Ch_{v_1}(M(v_1) \cup a_1), \\ a_{k-1} &\in Ch_{v_k}(M(v_k) \cup a_{k-1}), \\ \text{and } \forall i < n, \{a_i, a_{i+1}\} &\subseteq Ch_{v_{i+1}}(M(v_{i+1}) \cup a_i \cup a_{i+1}). \end{aligned}$$

Nodes  $v_1, \dots, v_k$  do not have to be different.

A network is *chain stable* if it is individually rational and has no blocking walks.

**Theorem 36.** (*Generalization of Ostrovsky's theorem*) *If every vertex's choice function is SSS, CSC and path independent, then there exists a chain stable network.*

*Proof.* Transform  $G = (V, E)$  to a bipartite graph such way that we divide each arc to two arcs by a new vertex. From arc  $e = uv$  we get  $e_1 = ux$  and  $e_2 = xv$ . Denote the new graph  $G' = (V', E')$ . Let be set of the beginning of the arcs  $D$ ,  $D = \{ux|u \in V\}$ , the end of the arcs is  $U$ .  $U = \{xv|v \in V\}$  Therefore  $E' = D \cup U$ .

We call the original edges full edges, and the new ones half edges.

Every new vertex has the following choice function:

The  $x$  node has exactly two arcs,  $e_1 = ux$  és  $e_2 = xv$ .

$$G(\{e_1, e_2\}) = \{e_1, e_2\}$$

$$G(\{e_1\}) = \emptyset$$

$$G(\{e_2\}) = \emptyset$$

$$G(\emptyset) = \emptyset$$

Note that this choice function is also SSS, CSC and path independent (the subset ordering  $\{e_1, e_2\} > \emptyset > \text{all other sets}$  defines it).

Define a new partial ordering over the subsets of  $E'$ : for  $X, Y \subseteq E'$ , we say  $X \preceq Y$  if  $X \cap D \subseteq Y \cap D$  and  $X \cap U \supseteq Y \cap U$ .

**Lemma 37.** *If choice function  $F$  is SSS and CSC, then  $F$  is comonotone for the new partial ordering, that is  $\bar{F}$  is monotone:  $X \preceq Y \Rightarrow \bar{F}(X) \preceq \bar{F}(Y)$ .*

If  $X \preceq Y$ , let  $Z = (Y \cap D) \cup (X \cap U)$ . Now  $X \preceq Z \preceq Y$ . The  $X$  and  $Z$  set has the same intersection with  $U$ , and  $X \cap D \subseteq Z \cap D = Y \cap D$ , so from the SSS property  $D(X) \setminus D(F(X)) \subseteq D(Z) \setminus D(F(Z))$ , from the CSC property  $U(F(X)) \subseteq U(F(Z))$ , and because  $U(X) = U(Z)$ , we get  $U(X) \setminus U(F(X)) \supseteq U(Z) \setminus U(F(Z))$ . Therefore  $F(X) \preceq F(Z)$ .

In the similar way,  $Z$  and  $Y$  has the same intersection with  $D$ , so we get the same statements with opposite roles of  $U$  and  $D$ .  $F(X) \preceq F(Z)$ ,  $F(Z) \preceq F(Y)$  so  $F(X) \preceq F(Y)$ .

Since the original vertices have disjoint arcs, we can define a common choice function. Let  $F(M) = \bigcup Ch_a(M(a))$ .

And let  $G$  be a common choice function for the new vertices.

Functions  $F$  and  $G$  are comonotone for the new  $\preceq$  ordering. So using Theorem 5 there exist a three-part stable solution, i.e.  $\exists A, B$  for which  $F(A) = S = G(B)$ , and  $A \cap B = S$ ,  $A \cup B = E$ .

**Lemma 38.** *If subset  $S$  of  $E$  is three-part stable, then it is chain stable.*

Suppose that  $S$  three-part stable. Since  $G(S) = S$ ,  $S$  contains only full edges. From the edges of  $E \setminus S$ , set  $B \setminus S$  cannot contain a full edge, because then  $G$  would choose it. Therefore  $A \setminus S$  has at least one half from every edges in  $E \setminus S$ .

Color the half edges in  $A \setminus S$  to red.

Consider an arbitrary  $P = (e_1, \dots, e_{2k})$  directed walk (where  $e_i$ -s are half edges) outside of  $S$ . ( $P \cap S = \emptyset$ ). A directed path on the original edges is also a directed path on the new edges. If the first or last half-edge of  $P$  is red, then  $P$  can't be a blocking walk, since  $e_1 \in A \setminus S$ , so  $S \subseteq S \cup e_1 \subseteq A$ , and from the path independent property  $F(S \cup e_1) = S$  which means  $e_1$  is dominated. The same applies to  $e_{2k}$ .

Suppose that neither the first, nor the last half-edge is red. Since every edge has at least one red half, there exists an original vertex, where two red half-edges meet. Call them  $e_{2j}$  and  $e_{2j+1}$ . From  $S \subseteq S \cup \{e_{2j}, e_{2j+1}\} \subseteq A$ , we get  $F(S \cup \{e_{2j}, e_{2j+1}\}) = S$ . Therefore  $P$  is not a blocking walk.

We showed that no walk can block, so  $S$  is chain stable.

There always exists a three-part stable solution, and every three-part stable is chain stable, so there exists a chain stable solution. □

**Theorem 39.** *The opposite direction is also true: every chain stable  $S$  is three-part stable with the new ordering  $\prec$ .*

*Proof.* Color a half edge  $e \notin S$  outside of  $S$  to red if  $e \notin F(S + e)$ , so  $S$  dominates them with the  $F$  choice function. Color  $e \notin S$  to blue, if  $e \in F(S + e)$ , i.e. it is not dominated. If there exists a blocking path, the first and last half edge must be blue, hence the players would choose them.

Since there is no blocking edge, both halves of an edge cannot be blue, so every edge has at least one red part. Therefore  $G(S \cup \text{blue edges}) = S$ .

It would be simple to place the red edges in  $A \setminus S$ , and the blue ones in  $B \setminus S$ . But it may occur that  $a$  and  $b$  half edges are red, yet  $\{a, b\} \subseteq F(S \cup \{a, b\})$ . Because of the path independent property, from  $a$  and  $b$  the choice  $F(S \cup \{a, b\})$  contains both or none, since if  $a \notin F(S \cup \{a, b\})$ , then  $F(S \cup \{a, b\}) \subseteq S \cup b \subseteq S \cup \{a, b\}$  so

$F(S \cup \{a, b\}) = F(S \cup b)$  and  $b \notin F(S \cup b)$ , therefore  $b \notin F(S \cup \{a, b\})$

If  $\{a, b\} \subseteq F(S \cup a, b)$ , we draw a new edge from the beginning of  $a$  to the end of  $b$ . We color these new edges to green.

The set of the middle vertices is  $V_h$ . Let  $D_1$  be set of vertices in  $V_h$  who has a blue incoming edge, and  $U_1$  is the set of  $V_h$ -vertices with blue outgoing edge. As we said before, there is no blue full edge, hence  $D_1 \cap U_1 = \emptyset$ .

If there is a directed walk on only green edges from  $D_1$  to  $U_1$ , there exist a blocking walk, the first and last vertices choose the blue edge, and the middle vertices choose the two red edges connected with the green edge.

Since  $S$  is chain stable, there is no such walk, we cannot go from  $D_1$  to  $U_1$ . Let  $Z_1$  be the set of  $V_h$ -vertices available from  $D_1$  on green edges, and let  $Z_2 = V_h \setminus Z_1$ .

If  $v \in Z_1$ , no blue edge can leave it. Recolor the incoming red edges to blue. If  $v \in Z_2$ , no blue edge can go into it. Recolor the outgoing red edges to blue. With this method, there is still no full blue edge.

Now put the red edges into  $A \setminus S$  and the blue ones into  $B \setminus S$ .

Every edge outside of  $S$  has at least one red half, so  $G(B) = S$

For the original vertices, if  $v \in V$ , let  $a = xv$  be an incoming half edge, and  $b = vy$  is an outgoing half edge, which were originally red.

If  $\{a, b\} \subseteq F(S \cup ab)$ , then one of  $a$  and  $b$  was changed to blue. If it did not, then  $x$  changes the edge before it, so  $x \in V_1$ , and  $y$  changes the edge after it, so  $y \in V_2$ . Moreover there is a green edge from  $x$  to  $y$ . This contradicts the definition of  $V_1, V_2$ .

Consider the set of remaining red edges incident with  $v$ . By the definition of red edges,

$F(S \cup a) = S$ , and from the former argument, for every incoming-outgoing pair,  $F(S \cup ab) = S$ .

**Lemma 40.** *If the choice function  $F$  of vertex  $v$  is SSS, CSC and path independent, and the incoming edges are  $a_1, \dots, a_k$ , the leaving edges are  $b_1, \dots, b_l$ , where for all  $a_i, b_j$  pairs,  $F(S \cup \{a_i, b_j\}) = S$ , then  $F(S \cup \{a_1, \dots, a_k, b_1, \dots, b_l\}) = S$ .*

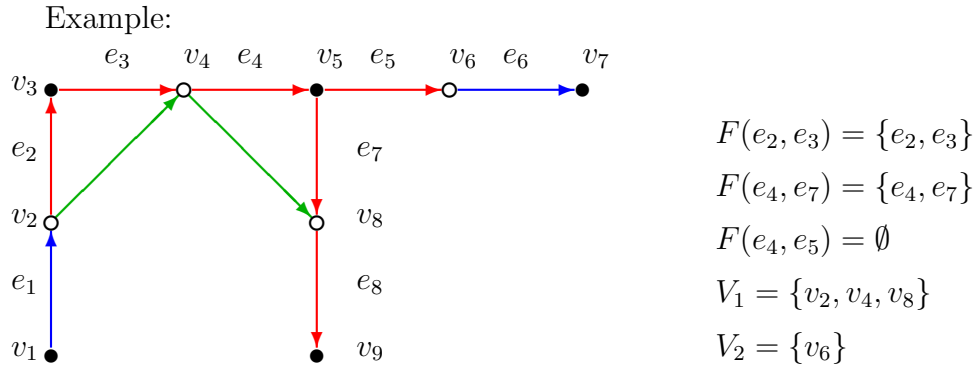
Since  $F(S \cup \{a_i, b_j\})$  refuses  $b_j$ , from the SSS property  $F(S \cup \{a_i, b_1, \dots, b_l\})$  refuses every  $b_j$ . Since  $F(S \cup \{a_i, b_1, \dots, b_l\}) \subseteq S \cup \{a_i \subseteq S \cup a_i, b_1, \dots, b_l\}$  from path independence  $F(S \cup \{a_i, b_1, \dots, b_l\}) = S$  for every  $i$ .

The same can be said for the opposite side, so  $F(S \cup \{a_1, \dots, a_k, b_j\}) = S$  for every  $j$ .

Using SSS again,  $F(S \cup \{a_i, b_1, \dots, b_l\})$  rejects  $a_i$ , so  $F(S \cup \{a_1, \dots, a_k, b_1, \dots, b_l\})$  also reject  $a_i$ -t, for every  $i$ .  $F(S \cup \{a_1, \dots, a_k, b_j\}) = S$  rejects  $b_j$ , so  $F(S \cup \{a_1, \dots, a_k, b_1, \dots, b_l\})$  rejects  $b_j$ -t, for every  $j$ .

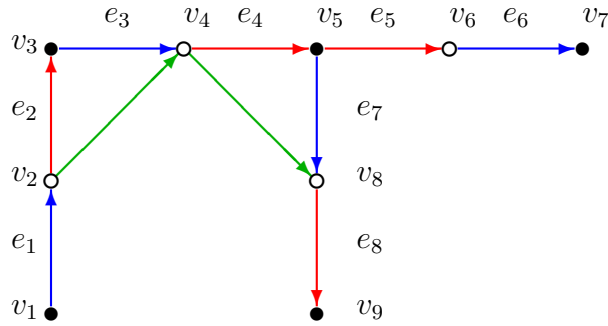
Therefore  $F(S \cup \{a_1, \dots, a_k, b_1, \dots, b_l\}) \subseteq S \subset S \cup \{a_1, \dots, a_k, b_1, \dots, b_l\}$   
so  $F(S \cup \{a_1, \dots, a_k, b_1, \dots, b_l\}) = F(S) = S$ .

Using this lemma to the red edges,  $F(A) = S$ , and from the construction  $G(B) = S$  and  $A \cap B = S, A \cup B = E$ , therefore  $S$  is three-part stable. □



Here  $S = \emptyset$  is stable, there is no blocking walk.

After the color changing:



In this model, we only know that there are no blocking walks, where the first and last edges are not chosen by the players separately. But if the walk is a circle, it can happen that  $v_1 = v_k$  would like to choose  $e_1$  and  $e_{k-1}$  together.

**Definition 21.** A *blocking circle* of network  $M$  is an alternating sequence of incident vertices and arcs  $C = (v_1, a_1, \dots, a_{k-1}, v_k)$  such that all the following properties hold:

$$v_1 = v_k$$

arc  $a_i$  points from  $v_i$  to  $v_{i+1}$

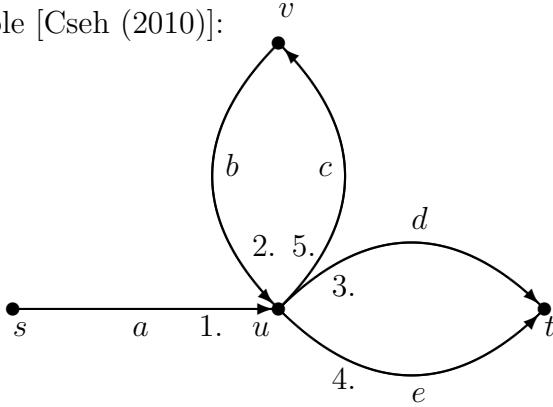
each arc  $a_i \notin M$

$$\{a_1, a_{k-1}\} \subseteq Ch_{v_1}(M(v_1) \cup a_1 \cup a_{k-1}),$$

$$\forall i < n, \{a_i, a_{i+1}\} \subseteq Ch_{v_{i+1}}(M(v_{i+1}) \cup a_i \cup a_{i+1}).$$

A network is *fully stable* if it is individually rational, and there is no blocking circle and no blocking walk.

Unfortunately, a fully stable network not always exist. Consider the following example [Cseh (2010)]:



This model is described as in [Fleiner (2009)]. The agents  $u$  and  $v$  want to satisfy Kirchhoff's law, and  $u$ 's preference ordering is  $a > b > d > e > c$ , he wants to buy and sell equal quantities to his best possible partners. Agent  $v$  buys and sells both  $b$  and  $c$  or none of them.

The source is  $s$ , the target is  $t$ , they want to trade everything they can.

With choice functions,  $Ch_s(a) = a$

$Ch_v(b) = \emptyset, Ch_v(c) = \emptyset, Ch_v(b, c) = b, c$

$Ch_t(d) = d, Ch_t(e) = e, Ch_t(d, e) = d, e$

$Ch_u(a) = \emptyset, Ch_u(b) = \emptyset, \dots Ch_u(a, b, c, d, e) = a, b, d, e$

The network  $\{a, d\}$  is chain stable, since adding one of  $b$  or  $c$  is blocked at  $v$ , adding the  $b, c$  circle, we can decide where to start it. If  $v_1 = u = v_3, v_2 = v$  it is blocked at  $u$ , because  $Ch_u(a, b, d) = a, d, Ch_u(a, c, d) = a, d$ . If  $v_1 = v = v_3, v_2 = u$  it is blocked at  $u$ , because  $Ch_v(b) = \emptyset, Ch_v(c) = \emptyset$ .

If we try  $e$ , the edge  $d$  is better, so  $Ch_u(a, d, e) = a, d$

If we add  $b, c, e$  together,  $Ch_u(a, b, d) = a, d, Ch_u(a, c, d, e) = a, d$ , so it is not blocking.

But  $b, c$  is a blocking circle,  $Ch_u(a, b, c, d) = a, b, c, d, Ch_v(b, c) = b, c$  so  $a, d$  is not fully stable. Adding this circle to the network,  $a, b, c, d$  is not stable, since  $e$  edge is blocking:  $Ch_u(a, b, c, d, e) = a, b, d, e$  and  $Ch_t(d, e) = d, e$ .

**Remark 41.** For a given  $G$  graph with preferences, it is an open problem to decide whether there exists a fully stable network.

## 5.1 Continuous quantities

In the previous model, one trading is realized or not, but we did not talk about how much they trade. Now, assign a volume to every trade,  $q : E \rightarrow \mathbb{R}$  describes the supply chain. If  $E$  is finite, we can consider it as a vector in  $\mathbb{R}^{|E|}$ . Instead of Kirchhoff's law, every node makes some transformation on the goods. (For example: a firm  $v$  buys 100 units of wood, and sells 20 chairs) For every vertex, an SSS, CSC choice function describes the preferences.

In the continuous case, a choice function is  $F : \mathbb{R}^{|E|} \rightarrow \mathbb{R}^{|E|}$  for which  $F(A) \leq A$  for every  $A \in \mathbb{R}^{|E|}$ . Here  $\leq$  means  $F(A)(e) \leq A(e)$  for every edge. If for a subset  $A$  of the edges, we use the characteristic vector  $A(e) = 1$  if  $e \in A$  and  $A(e) = 0$  if  $e \notin A$ , we get the original definition.

Denote the set of unselected elements with  $\bar{F}(A) = A - F(A)$ .

A choice function  $F$  is comonotone, if for every  $A \leq B$ ,  $A - F(A) \leq B - F(B)$ , so  $\bar{F}(A) \leq \bar{F}(B)$ .

Cut each edge in half,  $E'$  is the set of the half edges. We can generalize the  $q(e')$  quantity of products for every  $e' \in E'$ , now the function  $q$  maps from  $E'$  to  $\mathbb{R}$ . The edge set  $E' = D \cup U$  where  $D$  is the set of the beginner half of the edges, and  $U$  is the ending half of the edges. Let  $q_D$  be the value of function  $q$ , on the half edges in  $D$ , and  $q_U$ , is the value of  $q$  restricted to  $U$ .

The  $F$  choice function is SSS and CSC, if for any two quantity functions  $q$  and  $q'$ , such that  $q_D = q'_D$ ,  $q_U \leq q'_U$  then  $\bar{F}(q_U) \leq \bar{F}(q'_U)$  and  $F(q_D) \leq F(q'_D)$ . And similarly, for every  $q_U = q'_U$ ,  $q_D \leq q'_D$  we get  $\bar{F}(q_D) \leq \bar{F}(q'_D)$  and  $F(q_U) \leq F(q'_U)$ .

In the new partial order  $q \preceq q'$ , if  $q_U \leq q'_U$ ,  $q_D \geq q'_D$ . As we have seen in the discrete case, the choice function  $F$  is SSS and CSC if and only if  $F$  is comonotone for this partial order, i.e.  $q \preceq q' \Rightarrow \bar{F}(q) \preceq \bar{F}(q')$ .

We use the same method as in [Fleiner (2009)]. Each edge has a capacity  $c(e)$ . Let  $L$  be a set of nonnegative mappings  $l : E \rightarrow \mathbb{R}$  such that  $0 \leq l(e) \leq c(e)$  holds for any edge  $e$  of  $E$ . Observe that  $L$  forms a complete lattice under partial order

$\leq$ . If  $l_i : E \rightarrow \mathbb{R}$  are elements of  $L$  for  $i \in I$  then  $\bigvee_{i \in I} l_i(e) = \sup\{l_i(e) : i \in I\}$  and  $\bigwedge_{i \in I} l_i(e) = \inf\{l_i(e) : i \in I\}$ , since the  $l_i(e)$  elements are bounded, infimum and supremum always exist.

We define two choice functions on  $L$ , for the two sides of the market, now  $F$  for the original nodes, and  $G$  for the middle nodes.

Define mappings on  $L$  by  $F^*(l) = c - \overline{F}(l)$  and  $G^*(l) = c - \overline{G}(l)$  that is for each edge  $e$ ,  $F^*(l)(e) = c(e) - \overline{F}(l)(e)$  and  $G^*(l)(e) = c(e) - \overline{G}(l)(e)$ . For any element  $l$  of  $L$ , we have  $F^*(l), G^*(l) \in L$  and since  $\overline{F}$  is monotone for the ordering  $\preceq$ , if  $l \preceq l'$ , then  $F^*(l) \succeq F^*(l')$  and  $G^*(l) \succeq G^*(l')$ . So if we compose these functions, then we get a monotone function  $H = F^* \circ G^*$

$H(l)(e) = c(e) - \overline{F}(c - \overline{G}(l))(e)$ . From Tarski's fixed point theorem, there is some function  $l$  of  $L$  such that

$$H(l)(e) = c(e) - \overline{F}(c - \overline{G}(l))(e) = l(e)$$

Define  $s = G(l)$ ,  $s_G = l - s$ ,  $s_F = c - l$ , so  $s = s_G + s + s_F$ .

Rearranging the sides, we get  $s_F = c - l = \overline{F}(c - \overline{G}(l)) = \overline{F}(s + s_F)$ .

Extending the definition of three-part stability,  $s = G(l)$  is three-part stable, since for  $A = s + s_F$ ,  $B = l = s + s_G$ ,  $F(A) = s = G(B)$  and  $A + B = c + s$ .

An arc  $e$  is  $q$ -unsaturated, if  $q(e) < c(e)$ .

**Definition 22.** A blocking walk of network  $q$  is an alternating sequence of incident vertices and arcs  $P = (v_1, a_1, \dots, v_k)$  such that all the following properties hold:

arc  $a_i$  points from  $v_i$  to  $v_{i+1}$

each arc  $a_i$  of  $P$  is  $q$ -unsaturated

there exist a  $\varepsilon_i$  for every  $v_i$  such that

$$\varepsilon_1 a_1 \leq F(q + \varepsilon_1 a_1)$$

$$\varepsilon_k a_{k-1} \leq F(q + \varepsilon_k a_{k-1})$$

$$\varepsilon_i a_{i-1} + \varepsilon_i a_i \leq F(q + \varepsilon_i a_{i-1} + \varepsilon_i a_i)$$

**Definition 23.** A network  $l$  is chain-stable, if it is individually rational:  $F(l) = l = G(l)$ , and there is no blocking walk.

**Theorem 42.** A network is chain stable if and only if it is three-part stable.

*Proof.* Similarly to the proof of Lemma 38 and Theorem 39. Now we only prove the three-part stable  $\Rightarrow$  chain stable direction, because this is what we need for the next theorem.

Suppose that  $s$  is three-part stable. It mean there exist  $A, B : E' \rightarrow \mathbb{R}$  such that  $A = s + s_F$ ,  $B = l = s + s_G$ ,  $F(A) = s = G(B)$  and  $A + B = c + s$ . From path



independency,  $s$  is also individually rational,  $F(s) = s = G(s)$ .

Color an unsaturated half edge  $e$  to red, if  $A(e) > s(e)$ , so  $A - s$  has some positive quantity of  $e$ . If  $A(e) = s(e)$ , color  $e$  to blue.

Since  $G(s) = s$ , for two halves of an original edge  $e_1, e_2$ ,  $s(e_1) = s(e_2)$ . If  $e_1, e_2$  are unsaturated,  $B(e_1) = s(e_1)$  or  $B(e_2) = s(e_2)$  otherwise  $G$  would choose a bigger volume. So at least one of the two halves is red. This is true for every unsaturated edge.

Consider an arbitrary  $P = (e_1, \dots, e_{2k})$  directed walk of unsaturated half edges. If the first or last half-edge of  $P$  is red, then  $P$  can't be a blocking walk, since  $\varepsilon e_1 \in A - s$ , so  $s \leq s + \varepsilon e_1 \leq A$ , and from the path independent property  $F(S + \varepsilon e_1) = S$  which means  $e_1$  is dominated. The same applies to  $e_{2k}$ .

Suppose that neither the first, nor the last half-edge is red. Since every edge has at least one red half, there exists an original vertex, where two red half-edges meet. Call them  $e_{2j}$  and  $e_{2j+1}$ . From  $s \leq s + \varepsilon_{2j} e_{2j} + \varepsilon_{2j+1} e_{2j+1} \leq A$ , we get  $F(s + \varepsilon_{2j} e_{2j} + \varepsilon_{2j+1} e_{2j+1}) = s$ . Therefore  $P$  is not a blocking walk.

We showed that no walk can block, so  $S$  is chain stable.

□

**Theorem 43.** *If preferences are SSS, CSC and path independent, there always exists a chain stable solution.*

*Proof.* Similarly to the discrete case, we cut each edge in half. For the middle points, we define choice function  $G$ . From edge  $e = uv$  we created  $e_1 = ux$  and  $e_2 = xv$ . If in a network  $q$ , the volume of solded products on the edges are  $a = q(e_1)$  and  $b = q(e_2)$ , the choice of  $x$  is  $G(a, b) = (\min(a, b), \min(a, b))$  which means  $x$  sells exactly what he buys, with the biggest possible amount.

We show that  $G$  is comonotone for  $\preceq$ .

If  $a < b$   $G(a, b) = (a, a)$ , so  $\overline{G}(a, b) = (0, b - a)$ .  $(a, b) \preceq (a', b')$  means  $a \leq a', b \geq b'$ . Suppose that  $a < b$ .

If  $a$  decreases,  $b$  increases  $\overline{G}(a', b') = (0, b' - a')$ , and  $b' - a' > b - a$ , so  $\overline{G}(a, b) \succeq \overline{G}(a', b')$ .

If  $a$  increases,  $b$  decreases, and still  $a' < b'$   $\overline{G}(a', b') = (0, b' - a')$ , and  $b' - a' < b - a$ , so  $\overline{G}(a, b) \preceq \overline{G}(a', b')$

If  $a$  increases,  $b$  decreases, and  $a' > b'$   $\overline{G}(a', b') = (a' - b', 0)$ , and  $a' - b' > 0$ ,  $0 < a - b$  so  $\overline{G}(a, b) \preceq \overline{G}(a', b')$ .

Choice function  $G$  is also path independent, if  $(\min(a, b), \min(a, b)) \leq (a', b') \leq (a, b)$ , one of  $a', b'$  must be  $\min(a, b)$ , so  $G(a', b') = (\min(a, b), \min(a, b))$

There always exists a three-part stable solution, and every three-part stable is chain stable, so there exists a chain stable solution. □

## 6 Supply chains with money

In the paper [Hatfield et al. (2011)], they introduce a model with continuous prices. There is a finite set  $I = 1, 2, \dots, n$  of agents in the economy. They are represented with vertices. The agents can participate in bilateral trades, which correspond to the edges of the directed graph  $G = (I, E)$ . So  $E$  is the set of all possible trades. Every trade  $e$  is associated with buyer  $b(e)$  and a seller  $s(e)$ . In this model, circles in the graph are allowed.

The following definitions are all from [Hatfield et al. (2011)]:

We call a pair  $(e, p_e)$  a *contract* where  $e$  is a trade and  $p_e$  is the price at which the trade occurs. The set of available contracts is  $X = E \times \mathbb{R}$ .

For any set of contracts  $Y$ , the set of trades involved in  $Y$  is denoted with  $\tau(Y)$ , so  $\tau(Y) = \{e \in E : (e, p_e) \in Y \text{ for some } p_e \in \mathbb{R}\}$ . For an agent  $i$  and a set of contracts  $Y \subseteq X$ , denote the contracts coming into  $i$  with  $Y_{i-}$  so  $Y_{i-} = \{y \in Y : i = b(y)\}$ , and the set of contracts in  $Y$  leaving  $i$  is  $Y_{i+}$  so  $Y_{i+} = \{y \in Y : i = s(y)\}$

Let  $Y_i$  denote the set of all contracts in  $Y$  incident with  $i$ , so  $Y_i = Y_{i-} \cup Y_{i+}$ , and  $E_i$  the set of edges incident with  $i$ .

We say the set of contracts  $Y$  is *feasible*, if there is no trade  $e$  and prices  $p_e \neq p'_e$  such that both contracts  $(e, p_e)$  and  $(e, p'_e)$  are in  $Y$ .

An *outcome*  $A \subseteq X$  is a feasible set of contracts. So it is a set of edges with a price for every edge in the set, but we don't have prices for the edges outside the set.

An *arrangement* is a pair  $[\Psi, p]$  where  $\Psi \subseteq E$  is a set of trades and  $p \in \mathbb{R}^{|E|}$  is a vector of prices for all trades in the economy.

Each player  $v_i$  has a valuation function  $u_i(A)$  over all subsets of edges  $A \subseteq E$  it is incident with. The valuation  $u_i$  gives rise to an utility function  $U_i$  over sets of contracts.

$$U_i(Y) = u_i(\tau(Y)) + \sum_{(e, p_e) \in Y_{i+}} p_e - \sum_{(e, p_e) \in Y_{i-}} p_e$$

There are two kind of choice function here: The *choice correspondence* of agent  $i$

from the set of contracts  $Y \subseteq X$  is

$$C_i(Y) = \operatorname{argmax}_{Z \subseteq Y, Z \text{ is feasible}} U_i(Z)$$

This  $C_i$  is a choice function over the contracts, but it differs from a regular choice function, in a way that it can be multiple valued, if maximum is taken on more than one set. So  $C_i(Y)$  is the set of all best choices from  $Y$ . It may be empty-valued too, for example if  $Y = \{(e_0, p), 0 < p < 1\}$ .

The *demand correspondence* of agent  $i$  with a given price vector  $p \in \mathbb{R}^{|E|}$  is defined as the set of trades maximizing agent  $i$ 's utility under prices  $p$ :

$$D_i(p) = \operatorname{argmax}_{A \subseteq E} U_i([A, p])$$

The demand correspondence function chooses from the edges, not from the contracts, and always from the whole set of the edges. The optimal set is not uniquely defined,  $D_i(p)$  is the set of all optimal edge-sets.

The *indirect utility function* of agent  $i$  is

$$V_i(p) = \max_{\Psi \subseteq E_i} U_i([\Psi, p])$$

Let  $V(p) = \sum_{i \in I} V_i(p)$ .

Let  $a(Z)$  denote the of agents involved in contracts in  $Z$  as buyers or sellers.  
 $a(Z) = \bigcup_{z \in Z} \{b(z), s(z)\}$

**Definition 24.** An outcome  $A$  is *stable* if it is:

1. *Individually rational*:  $A_i \in C_i(A)$  for all  $i$ ,
2. *Unblocked*: There is no feasible nonempty blocking set of contracts  $Z \subseteq X$  such that  $Z \cap A = \emptyset$  and for all  $i \in a(Z)$ , for all of  $i$ 's choices  $Y \in C_i(Z \cup A)$  we have  $Z_i \subseteq Y$ .

**Definition 25.** An agreement  $[\Psi, p]$  is a *competitive equilibrium* if for all  $i \in I$

$$\Psi_i \in D_i(p)$$

Based on the SSS, CSC definitions by [Ostrovsky (2008)], Hatfield et al. introduce the fully substitutable property. It is similar to the "comonotone for ordering  $\prec$ " definition in Section 3, but now they define monotonicity over contracts, not edges, and we need some modifications because of the multiple valued definition of  $C_i$ . They introduce three equivalent definitions of full substitutability: choice-language fully substitutable, demand-language fully substitutable and indicator-language fully substitutable preferences.

**Definition 26.** [Hatfield et al. (2011)] Agent  $i$ 's preferences are *choice-language fully substitutable*(CFS) if:

1. For all sets of contracts  $Y, Z \subseteq X_i$ , such that  $|C_i(Z)| = |C_i(Y)| = 1$ , if  $Y_{i+} = Z_{i+}$  and  $Y_{i-} \subseteq Y_{i-}$  then for the unique  $Y^* \in C_i(Y)$  and  $Z^* \in C_i(Z)$ , we have  $(Y_{i-} - Y_{i-}^*) \subseteq (Z_{i-} - Z_{i-}^*)$  and  $Y_{i+}^* \subseteq Z_{i+}^*$
2. For all sets of contracts  $Y, Z \subseteq X_i$ , such that  $|C_i(Z)| = |C_i(Y)| = 1$ , if  $Y_{i-} = Z_{i-}$  and  $Y_{i+} \subseteq Y_{i+}$  then for the unique  $Y^* \in C_i(Y)$  and  $Z^* \in C_i(Z)$ , we have  $(Y_{i+} - Y_{i+}^*) \subseteq (Z_{i+} - Z_{i+}^*)$  and  $Y_{i-}^* \subseteq Z_{i-}^*$

It means that when attention is restricted to sets for which  $C_i$  is single-valued, and the set of option to  $i$  on one side expands,  $i$  rejects a larger set of contracts on that side (SSS), and select a larger set of contracts on the other side.

**Definition 27.** Agent  $i$ 's preferences are *demand-language fully substitutable* (DFS) if:

1. for all price vectors  $p, p' \in \mathbb{R}^{|E|}$  such that  $|D_i(p)| = |D_i(p')| = 1$ ,  $p_e = p'_e$  for all  $e \in E_{i->}$ , and  $p_e \geq p'_e$  for all  $e \in E_{->i}$ , for the unique  $A \in D_i(p)$  and  $A' \in D_i(p')$ , we have  $\{e \in A'_{->i} : p_e = p'_e\} \subseteq A_{->i}$  and  $A_{i->} \subseteq A'_{i->}$
2. for all price vectors  $p, p' \in \mathbb{R}^{|E|}$  such that  $|D_i(p)| = |D_i(p')| = 1$ ,  $p_e = p'_e$  for all  $e \in E_{->i}$ , and  $p_e \geq p'_e$  for all  $e \in E_{i->}$ , for the unique  $A \in D_i(p)$  and  $A' \in D_i(p')$ , we have  $\{e \in A'_{i->} : p_e = p'_e\} \subseteq A_{i->}$  and  $A_{->i} \subseteq A'_{->i}$

The demand correspondence  $D_i$  is fully substitutable if, when attention is restricted to prices for which demands are single-valued, a decrease in the price of some inputs for agent  $i$  leads to the decrease in his demand for other inputs and to an increase in his supply of outputs, and an increase in the price of some outputs leads to the decrease in his supply of other outputs and an increase in his demand for inputs. For each agent  $i$ , for any set of trades  $A \subseteq E_i$  define the (*generalized*) *indicator function*  $\chi(A) \in \{-1, 0, 1\}^{|E_i|}$  to be the vector with component  $\chi_e(A) = 1$  for each upstream trade  $e \in A_{i-}$ , and  $\chi_e(A) = -1$  for each downstream trade  $e \in A_{i+}$  and  $\chi_e(A) = 0 = 0$  for each trade  $e \notin A$ . The interpretation of  $\chi(A)$  is that an agent buys a strictly positive amount of a good if he is the buyer in a trade in  $A$ , and "buys" a strictly negative amount if he is the seller of such a trade.

**Definition 28.** [Hatfield et al. (2011)] Agent  $i$ 's preferences are *indicator-language fully substitutable* (IFS) if for all price vectors  $p, p' \in \mathbb{R}^{|E|}$  such that  $|D_i(p)| = |D_i(p')| = 1$ , and  $p_e \geq p'_e$  for the unique  $A \in D_i(p)$  and  $A' \in D_i(p')$ , we have  $\chi_e(A) \leq \chi_e(A')$  for each  $e \in E_i$  such that  $p_e = p'_e$ .

**Theorem 44.** [Hatfield et al. (2011)] *If all agents' preferences are fully substitutable, there always exists a competitive equilibrium.*

They proved this theorem by reducing it to many-to-one stable matchings in a bipartite graph, a model from [Kelso and Crawford (1982)].

For an arrangement  $[\Psi, p]$ , let  $\kappa([\Psi, p]) = \bigcup_{e \in \Psi} \{(e, p_e)\}$  denote the set of contracts induced by the arrangement. So  $\kappa([\Psi, p])$  is an outcome.

**Theorem 45.** [Hatfield et al. (2011)] *Suppose  $[\Psi, p]$  is a competitive equilibrium. Then  $\kappa([\Psi, p])$  is stable.*

These two theorems together prove that there always exists a stable solution.

**Definition 29.** [Hatfield et al. (2011)] The set of trades  $\Psi$  is *efficient* if  $\sum_{i \in V} u_i(\Psi) \geq \sum_{i \in V} u_i(\Psi')$  for any other  $\Psi' \in E$ , that is where the sum of utilities is maximal.

Denote this maximum with

$$U^* = \max_{\Psi \in E} \sum_{i \in V} u_i(\Psi).$$

Therefore  $\Psi$  is efficient if and only if  $U^* = \sum_{i \in I} u_i(\Psi)$

**Theorem 46.** [Hatfield et al. (2011)] *Suppose agents' preferences are fully substitutable. Then for any competitive equilibrium  $[\Phi, p]$  and efficient set of trades  $\Psi$ ,  $[\Psi, p]$  is also a competitive equilibrium. Furthermore, the set of competitive equilibrium prices forms a lattice.*

**Lemma 47.** [Hatfield et al. (2011)] *A price vector  $p'$  is a competitive equilibrium price vector if and only if  $p' \in \operatorname{argmin}_p V(p)$*

They proved this lemma using the existence of competitive equilibrium.

The search for a competitive equilibrium is easier after these theorems. We only have to check the efficient edge sets, and the  $V(p)$ -minimizing price vectors.

We can conclude it in a theorem, which is weaker than the former statements, but sufficient for showing the existence of a competitive equilibrium.

**Theorem 48.** *If  $p' \in \operatorname{argmin}_p V(p)$  there exists an efficient set  $\Psi$ , where  $[\Psi, p']$  is a competitive equilibrium.*

This theorem is a corollary of Theorem 44, Theorem 46 and Lemma 47.

For any subset of edges  $\Phi \in E$  and arbitrary price vector  $p$ , let  $Y = \kappa([\Phi, p])$  denote the set of contracts formed from the edges of  $\Phi$  with corresponding prices from  $p$ . For

every  $i \in I$ ,  $U_i(Y) \leq V_i(p)$  from the definition of  $V_i(p)$ . If we sum the utilities for all agents, each price appear twice, with different signs, so they cancel out.

$$\sum_{i \in I} U_i(Y) = \sum_{i \in I} (u_i(\tau(Y)) + \sum_{(e, p_e) \in Y_{i+}} p_e - \sum_{(e, p_e) \in Y_{i-}} p_e) = \sum_{i \in I} u_i(\tau(Y))$$

So they have the same maximum:

$$U^* = \max_{\tau(Y) \subseteq E} \sum_{i \in I} u_i(\tau(Y)) = \max_{Y \subseteq X} \sum_{i \in I} U_i(Y)$$

The indirect utility function maximizes the utilities one by one, and  $U^*$  maximizes the utilities together, so  $V(p)$  is greater.

$$U^* = \max_{Y \subseteq X} \sum_{i \in I} U_i(Y) \leq \sum_{i \in I} \max_{Y \subseteq X} U_i(Y) = \sum_{i \in I} V_i(p) = V(p)$$

Let  $V^* = \min_p V(p)$ , that is  $V^* = V(p')$  Since  $U^* \leq V(p)$  for any  $p$ ,  $U^* \leq \min_p V(p) = V^*$ .

**Lemma 49.** *If  $U^* = V^*$  then there exists a competitive equilibrium.*

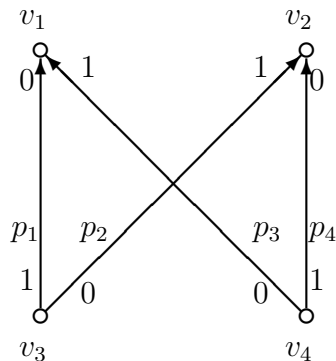
*Proof.* If  $U^* = V^*$ , then

$$U^* = \max_{Y \subseteq X} \sum_{i \in I} U_i(Y) = \sum_{i \in I} \max_{Y \subseteq X} U_i(Y) = \sum_{i \in I} V_i(p') = V(p')$$

so there is a  $Y^*$  set of contracts, where all agents have maximal utility,  $U^* = \sum_{i \in I} U_i(Y^*)$  and  $\max_{Y \subseteq X_i} U_i(Y) = U_i(Y_i)$ , so  $Y_i \in D_i(p)$ , therefore  $[Y^*, p']$  is a competitive equilibrium. □

An example:

There are four players: two buyers ( $v_1, v_2$ ) and two sellers ( $v_3, v_4$ )  
Buyer  $v_1$  prefers buying from  $v_4$ , and  $v_2$  prefers  $v_3$ . This gives them utility of 1, every other scenario's utility is zero. Similarly  $v_3$  would like to sell to  $v_1$ , and  $v_4$  prefers  $v_2$ .



Let the edges be  $e_1 = v_3v_1$ ,  $e_2 = v_3v_2$ ,  $e_3 = v_4v_1$ ,  $e_4 = v_4v_2$ , and the corresponding prices are  $p_1, p_2, p_4, p_5$ . The utilities of firm  $v_1$  over the set edges it is incident with it are the followings (without prices):

$$u_1(\emptyset) = 0$$

$$u_1(e_1) = 0 \quad u_1(e_3) = 1 \quad u_1(e_1, e_3) = 0$$

With prices:

$$U_1(\emptyset) = 0$$

$$U_1(e_1, p_1) = 0 - p_1 \quad U_1(e_3, p_3) = 1 - p_3 \quad U_1(e_1, e_3, p_1, p_3) = 0 - p_1 - p_3$$

Therefore the  $D_1$  demand correspondence function is  $D_1(p_1, p_3) = \operatorname{argmax}(0, -p_1, 1 - p_3, -p_1 - p_3)$

If we look only at the preference ordering, the market can be considered as a stable marriage problem with two men and two women. There the stable solutions are  $\{e_1, e_4\}$  and  $\{e_2, e_3\}$ .

Under what price will  $\{e_1, e_4\}$  be a competitive equilibrium? We are looking for a price vector  $p$  which satisfies:

$$\max(0, -p_1, 1 - p_3, -p_1 - p_3) = -p_1$$

$$\max(0, -p_4, 1 - p_2, -p_4 - p_2) = -p_4$$

$$\max(0, 1 + p_1, p_2, p_1 + p_2) = 1 + p_1$$

$$\max(0, 1 + p_4, p_3, p_3 + p_3) = 1 + p_4$$

From these inequalities  $-1 \leq p_1 \leq 0, 0 \leq p_2 \leq 1, 0 \leq p_3 \leq 1, -1 \leq p_4 \leq 0$ . Moreover:

$$1 - p_3 \leq -p_1$$

$$1 - p_2 \leq -p_4$$

$$p_2 \leq 1 + p_1$$

$$p_3 \leq 1 + p_4$$

Summing these four inequalities the prices cancel out and we get  $2 \leq 2$ . Since  $2 = 2$ , we have equality in all of the four inequalities. The  $p$  prices satisfying these conditions are  $p_1 = x, p_2 = 1 + x, p_3 = 1 + x, p_4 = x$  where  $-1 \leq x \leq 0$ . With this  $p$  vector,  $\{e_1, e_4\}$  is an competitive equilibrium.

Note that with this  $p$ , the  $1 - p_3, 1 - p_2, p_2, p_3$  are also maximal utilities for  $v_1, v_2, v_3, v_4$ , therefore  $\{e_2, e_3\}$  is also a competitive equilibrium.

The lesson of this example is that we cannot modify the  $D_i$  choice function to a unique-valued one. For example if we use the rule that in case of equal utility, a firm chooses the lexicographically first best set, then under price vector  $p = (x, 1 + x, 1 + x, x)$

(for example  $p = (-1/2, 1/2, 1/2, -1/2)$ ) player  $v_1$  chooses  $e_1$ , player  $v_2$  chooses  $e_2$ ,  $v_3$  chooses  $e_1$  and  $v_4$  chooses  $e_3$ , and we do not find the competitive equilibrium. If they use in a smarter method, everyone choose  $e_1$  or  $e_4$  if possible, then we find the competitive equilibrium  $\{e_1, e_4\}$ , but lose the other solution  $\{e_2, e_3\}$ .

## 6.1 Continuous prices and quantities

Denote the set of trades with  $E$ , and to every  $e \in E$  edge belongs a quantity  $q_e$  and a price  $p_e$ . We denote the price vector with  $p$ , the quantity vector with  $q$ . If  $q(e) \leq q'(e)$  for every  $e \in E$ , we say  $q \leq q'$ .

Each edge has a maximal capacity  $c(e)$ , so for every  $i$   $0 < q(e) < c(e)$ . Let  $T = \prod_{i \in I} [0, c_e]$ , the set of possible quantity vectors. The prices can take any real value,  $p \in \mathbb{R}^{|E|}$ .

Now we call a quantity vector-price vector pair an *agreement*. The set of possible arrangements is  $X = T \times \mathbb{R}^{|E|}$

Valuation function  $u_i$  is defined on  $T$  for every agent  $i$ . Suppose that  $u_i$  is continuous and  $u_i(0) = 0$ . Let  $E_{i+}$  denotes the set of edges leaving vertex  $i$ , and  $E_{i-}$  is the set of edges coming to  $i$ . A player  $i$ 's utility over the agreements is

$$U_i(q, p) = u_i(q) + \sum_{e \in E_{i+}} p_e q(e) - \sum_{e \in E_{i-}} p_e q(e).$$

We can change the definition of *choice correspondence* function as follows:

If  $Y \subseteq X$  is a set of agreements

$$C_i(Y) = \operatorname{argmax}_{[q,p] \in Y} U_i(q, p)$$

If in a special case where we only need to choose from a feasible set, where each edge has only one possible price, and the quantities has some upper bound, the  $C_i$  function can be written as

$$C_i(q, p) = \operatorname{argmax}_{q' \leq q} U_i(q, p)$$

The *demand correspondence* function will be:

$$D_i(p) = \operatorname{argmax}_{q \in T} U_i(q, p)$$

The *indirect utility function* of agent  $i$  is

$$V_i(p) = \max_{q \in T} U_i(q, p)$$



Let  $V(p) = \sum_{i \in I} V_i(p)$ , and  $V^* = \min_p V(p)$ .

Let  $U^* = \max_{q \in T} \sum_{i \in V} u_i(q)$  the maximal common utility over the networks. We say  $q$  is *efficient*, if  $U^* = \sum_{i \in V} u_i(q)$ .

**Definition 30.** An agreement  $[q, p]$  is a *competitive equilibrium* if for all  $i \in I$

$$q \in D_i(p)$$

We would like to show the continuous version of Theorem 44:

**Conjecture 50.** *If all preferences are fully substitutable there exist  $p, q$  such that  $[q, p]$  is a competitive equilibrium.*

We use the same  $U^*$  maximal utility as before:

$$U^* = \max_q \sum_{i \in I} U_i(p, q) \leq \sum_{i \in I} \max_q U_i(p, q) = \sum_{i \in I} V_i(p) = V(p)$$

Let  $V^* = \min_p V(p)$ , that is  $V^* = V(p')$  Since  $U^* \leq V(p)$  for any  $p$ ,  $U^* \leq \min_p V(p) = V^*$ .

We can state the same as we did in the discrete case:

**Lemma 51.** *If  $U^* = V^*$  then there exists a competitive equilibrium.*

*Proof.* If  $U^* = V^*$ , then

$$U^* = \max_q \sum_{i \in I} U_i(Y) = \sum_{i \in I} \max_q U_i(Y) = \sum_{i \in I} V_i(p') = V(p')$$

so there is a  $Y^*$  set of contracts, where all agents have maximal utility,  $U^* = \sum_{i \in I} U_i(Y^*)$  and  $\max_{Y \subseteq X_i} U_i(Y) = U_i(Y_i)$ , so  $Y_i \in D_i(p)$ , therefore  $[Y^*, p']$  is a competitive equilibrium.  $\square$

An interesting property of  $D_i(p)$  is the following one:

**Lemma 52.** *Agent  $i$ 's preferences are fully substitutable, and for agreements  $[q_0, p]$ ,  $[q_1, p]$ ,  $[q_2, p]$  the price vector is the same, and for each  $e \in E_{i-}$   $q_0(e) = q_1(e) = q_2(e)$ , and for each  $e \in E_{i+}$   $q_0(e) \leq q_1(e) \leq q_2(e)$ . If  $q_0, q_2 \in D_i(p)$ , then  $q_1 \in D_i(p)$ .*

*Similarly, if for each  $e \in E_{i+}$   $q_0(e) = q_1(e) = q_2(e)$ , and for each  $e \in E_{i-}$   $q_0(e) \leq q_1(e) \leq q_2(e)$ , we get  $q_0, q_2 \in D_i(p) \Rightarrow q_1 \in D_i(p)$ .*

*Proof.* We prove the first part:

Suppose that  $q_1 \notin D_i(p)$ , then  $U_i(q_2, p) = U_i(q_0, p) > U_i(q_1, p)$ , because  $U(q_0, p)$  and  $U(q_2, p)$  are maximal and  $U(q_1, p)$  is not.

Choose an edge  $e \in E_{i+}$  for which  $q_2(e) > q_0(e)$ . Let  $\varepsilon$  be a small positive number, such that  $0 < \varepsilon q_1(e) < U(q_0, p) - U(q_1, p)$ . Increase  $e$ 's price by  $\varepsilon$ , so  $p'_e = p_e + \varepsilon$ ,  $p'_f = p_f$  for any  $f \neq e$  edge

$$\text{Now } U_i(q_2, p') = U_i(q_2, p) + q_2(e)\varepsilon$$

$$U_i(q_1, p') = U_i(q_1, p) + q_1(e)\varepsilon$$

$$U_i(q_0, p') = U_i(q_0, p) + q_0(e)\varepsilon$$

Since  $U_i(q_2, p) = U_i(q_0, p)$  and  $q_2(e) > q_0(e)$ , for the new utilities  $U_i(q_2, p') \geq U_i(q_0, p')$ , and  $\varepsilon$  is so small that  $q_1$  can't reach the other's utility, so  $U_i(q_1, p') \geq U_i(q_0, p')$ .

$U_i(q_2, p)$  has the maximal utility of all agreements  $[q, p] : q \in T, p$  is fixed, so  $U_i(q_2, p')$  has the maximal utility of all agreements  $[q, p] : q \in T, q(e) \leq q_2(e)p'$  is fixed. Therefore  $q_2 \in C_i(q_2, p')$ . If  $|C_i(q_2, p)| \neq 1$ , we can perturbate the prices with smaller  $\varepsilon_j$ -s in a way that the ordering of the utilities remain, but  $U_i(q, p'')$  is different for all  $q \in T$ . So  $\{q_2\} = C_i(q_2, p'')$

For  $q_0$ , there is a smaller flow with bigger utility, so  $C_i(q_0, p') \neq q_0$ . It means  $\overline{C}_i(q_2, p') = 0$  and  $\overline{C}_i(q_0, p') > 0$  which contradicts same-side substitutability.

The second part is analogous with changing the roles of  $E_{i+}$ , and  $E_{i-}$  and we decrease  $p_e$  by  $\varepsilon$  instead of increasing. □

**Definition 31.** An one-dimensional real function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is *concave* if for every  $x, y \in \mathbb{R}$  and  $0 \leq \lambda \leq 1$ :

$$\lambda f(x) + (1 - \lambda)f(y) \leq f(\lambda x + (1 - \lambda)y).$$

**Definition 32.** The *hypograph* of a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is the set of points lying on or below its graph:

$$\text{hyp}f = \{(x, \mu) : x \in \mathbb{R}^n, \mu \in \mathbb{R}, \mu \leq f(x)\} \subseteq \mathbb{R}^{n+1}$$

A function is concave if and only if its hypograph is a convex set.

Let  $[a, b]$  a closed interval.

**Definition 33.** For a function  $f : [a, b] \rightarrow \mathbb{R}$ , the *upper convex hull* of  $f$  is the following function  $\hat{f} : [a, b] \rightarrow \mathbb{R}$ :

$$\hat{f}(x) = \sup_{a \leq x_1, x_2 \leq b, 0 \leq \lambda \leq 1} \{y | y = \lambda f(x_1) + (1 - \lambda)f(x_2), x = \lambda x_1 + (1 - \lambda)x_2\}$$

If  $x = \lambda x_1 + (1 - \lambda)x_2$  then  $x_2 = \frac{x - \lambda x_1}{1 - \lambda}$ , so the definition of  $\hat{f}$  can be written as

$$\hat{f}(x) = \sup_{a \leq x_1 \leq b, 0 \leq \lambda < 1} \left\{ y \mid y = \lambda f(x_1) + (1 - \lambda) f\left(\frac{x - \lambda x_1}{1 - \lambda}\right) \right\}$$

It is correct only if  $\lambda \neq 1$ , but if  $\lambda = 1$ , let  $x_2$  take the role of  $x_1$ , and for some other  $x_3 \geq x_2$ , let  $\lambda = 0$ , because they both give the same point  $f(x_2)$ .

We can think of  $\hat{f}$  as the upper border of the convex hull of  $\text{hyp}f$ . If  $f$  is continuous and has an upper bound, then  $\text{hyp}f$  is a closed set, so its convex hull is also closed, and since the hypograph has an upper bound,  $\hat{f}$  is also bounded from above, moreover  $\hat{f}$  is concave and continuous.

A property of the utility function:

**Lemma 53.** *For a player  $i \in V$ , the set of edges incident with  $i$  is  $E_i$ . Pick an edge  $e' \in E_i$ . Let  $T_{-e'} = T = \prod_{e \in E, e \neq e'} [0, c_e]$  the set of possible quantities for every edge in  $E_i$  but  $e'$ . We claim that if the preferences of  $i$  are path independent, SSS and CSC, then for  $q_{-e'} = 0$  the function  $f(q'_e) = u_i(q'_e, q_{-e'}) = u_i(q'_e, 0)$  is concave.*

*Proof.* The domain of  $f$  is the  $[0, c_{e'}]$  interval.

Suppose  $f$  is not concave, then  $f \neq \hat{f}$ , so there exist a point  $y$  where  $f(y) < \hat{f}(y)$ . Let  $x_1$  and  $z_1$ , the end of the section which defines  $\hat{f}(y)$ , so  $x_1 < y < z_1$  and  $y = \lambda x_1 + (1 - \lambda)z_1$ ,  $f(y) = \lambda_1 f(x_1) + (1 - \lambda_1)f(z_1)$

Consider the following linear function:  $l(q) = f(x_1) + \frac{q - x_1}{z_1 - x_1}(f(z_1) - f(x_1))$

Here  $f(x_1) = l(x_1)$ ,  $f(y) < l(y)$  and  $f(z_1) = l(z_1)$ .

Note that  $l$  is an upper bound of  $f$ . If there would be a  $z$  such that  $f(z) > l(z)$ , consider the case when  $z > x_1$ . Then  $l_z(x) = f(x_1) + \frac{x - x_1}{z - x_1}(f(z) - f(x_1))$  linear function is bigger than  $l$  for values bigger than  $x_1$ , therefore we have a better value for  $\hat{f}(y)$ . If  $z < x_1$  then we use the linear combination of  $z$  and  $z_1$ .

Let  $p = \frac{f(z_1) - f(x_1)}{z_1 - x_1}$  be the slope of the a string  $[(x_1, f(x_1)), (z_1, f(z_1))]$ . Let the price of edge  $e'$  be  $p_{e'} = p - \varepsilon$ , if  $i$  is the buyer in trade  $e'$  (i.e.  $e' \in E_{i-}$ ) and let the price of edge  $e'$  be  $p_{e'} = p + \varepsilon$ , if  $i$  is the seller in  $e'$  ( $e' \in E_{i+}$ ).

The utility function of firm  $i$  is:

$$\begin{aligned} U_i(q) &= u_i(q_{e'}, 0) + \sum_{e \in E_{i+}} p_e q(e) - \sum_{e \in E_{i-}} p_e q(e) = \\ &= f(q_{e'}) \pm q(e') p(e') = \\ &= f(q_{e'}) - q(e') \frac{f(z) - f(x)}{z - x} + q(e') \varepsilon \end{aligned}$$

$$f(q) - qp = f(q) - l(q) + f(x_1) - x_1p$$

$f(x_1) - x_1p$  is a constant.

$U_i(q_{e'}, q_{-e'}) = f(q) - l(q) + f(x_1) - x_1p + q\varepsilon = f(q) - l(q) + K + q\varepsilon$  where  $K = f(x_1) - x_1p$  is also a constant for  $q(e')$ .

$$U_i(x_1, 0) = f(x_1) - l(x_1) + K + x_1\varepsilon = K + x_2\varepsilon \quad U_i(y, 0) = f(y) - l(y) + K + y\varepsilon < K + y\varepsilon$$

$$U_i(z_1, 0) = f(z_1) - l(z_1) + K + z_1\varepsilon < K + z_2\varepsilon$$

Let  $0 < \varepsilon < \frac{l(y)-f(y)}{y-x_1}$ , then  $U_i(x_1, 0) > U_i(y, 0)$  but  $U_i(z_1, 0) > U_i(x_1, 0)$ .

Since  $U_i(x_1, 0) > U_i(y, 0)$  we get that  $C_i((y, 0), p)$  chooses on edge  $e'$  a quantity smaller than  $y$ , so it rejects a positive value of  $e'$ .

Since  $f(q) \leq l(q)$ , and  $U_i(q_{e'}, 0) = f(q) - l(q) + K + q\varepsilon \leq K + q\varepsilon$ , if  $z' < z$  then  $U_i(z', 0) \leq K_2 + z'\varepsilon \leq K + z'\varepsilon = U_i(z, 0)$ , therefore under prices  $p$ ,  $z_1$  has the maximal utility of all quantity vectors where  $q(e') \leq z_1$  and  $q_{-e'}$  is fixed. It means that  $C_i((z_1, 0), p)$  chooses the whole  $z_1$  quantity on edge  $e'$ , so it rejects zero value of  $e'$ . This contradicts the SSS property, so  $f$  must be concave. □

**Definition 34.** A  $(q, p)$  arrangement is stable, if:

It is individually rational:  $C_i(q, p) = q$  for every  $i$ .

There is no blocking quantity vector  $z$ :  $z$  blocks if  $z \geq 0$ ,  $q + z \leq c$  and with some price vector  $p'$  firm  $i$  would like to add  $z$  quantity to the trades where  $z$  is positive:

$$C_i(q + z, p') \geq \chi_{z>0}(q + z)$$

We generalize the indicator-language fully substitutable property by [Hatfield et al. (2011)] For an agent  $i$ , on the set of edges incident with  $i$ , and a given quantity vector  $q$ , let a modified quantity vector be:

$$\tilde{q}(e) = q(e) \text{ if } e \in E_{- \rightarrow i}$$

$$\tilde{q}(e) = -q(e) \text{ if } e \in E_{i - \rightarrow}$$

**Definition 35.** With continuous quantities, we say that agent  $i$ 's preferences are *fully substitutable*, if for all price vectors  $p, p' \in \mathbb{R}^{|E|}$  such that  $|D_i(p)| = |D_i(p')| = 1$ , and  $p \leq p'$  for the unique  $q \in D_i(p)$  and  $q' \in D_i(p')$  we have  $\tilde{q}(e) \leq \tilde{q}'(e)$  for each  $e$  such that  $p(e) = p'(e)$

It means, if some prizes increase, on the trades where the prize remained the same,  $i$  wants to buy more and sell less quantity.

Let  $N$  be a sufficiently large integer. Replace every edge in the graph  $G$  with  $N$  parallel edges, if the capacity of  $e$  was  $c(e)$ , the new edges have  $\frac{c(e)}{N}$  capacity. Denote

the new graph with  $G' = (V, E')$ , and define a discrete market on  $G'$  where in every new trade they sell the whole  $\frac{c(e)}{N}$  quantity or none.

The utility function of firm  $i \in V$  is defined for  $A \subseteq E'$  as  $u_i(A) = u_i(q)$ , where  $q(e)$  is the sum of capacities of edges in  $A$ .

Looking at one group of parallel edges between nodes  $i$  and  $j$ , the trades are identical in the sense it was defined in Section 4. Using Theorem 29, an utility function defines substitutable preferences if  $u(w + 1) - u(w) \leq u(w) - u(w - 1)$ , so the differences are decreasing, which means  $u$  is a concave function. And this will be true, since we proved in the previous Lemma 53 that a one-dimensional cut of the utility function is concave.

**Definition 36.** Firm  $i$ 's preferences are *very fully substitutable* (VFS), if it is fully substitutable for the new  $G'$  graph.

**Conjecture 54.** *If the original preferences were fully substitutable, they are also fully substitutable in the new model, so every FS function is VFS too.*

If we suppose that preferences are VFS, we can use Theorem 44 for the new supply chain to prove the existence of a competitive equilibrium

Let  $U_N^* = \max \sum u_i$  in the new market.  $V_N^* = \sum \max u_i$

The existence of a competitive equilibrium is equivalent with  $U_N^* = V_N^*$

The utility function  $u_i$  is continuous for every player, so the maximum of  $u_i$  is also continuous.

If  $N \rightarrow \infty$  then  $U_N^* \rightarrow U^*$  and  $V_N^* \rightarrow V^*$ . Since we have competitive equilibrium for every discrete market,  $U_N^* = V_N^*$  for every  $N$ , therefore  $U^* = V^*$ . From Statement 51 there exists a competitive equilibrium on the continuous-quantities market.

Similarly to the proof of Theorem 45 in [Hatfield et al. (2011)], we show that:

**Theorem 55.** *If  $[s, p]$  is a competitive equilibrium then  $[s, p]$  is stable.*

*Proof.* Suppose that  $[s, p]$  is a competitive equilibrium de but it is not stable. There are two cases: it is not individually rational, or there is a blocking quantity vector  $z$ . If it is not individually rational  $\exists i$  such that  $s \notin C_i(s)$ , which means  $s \notin D_i(p)$  and this contradicts that  $[s, p]$  is a competitive equilibrium.

If there exists a blocking vector  $z$ , let  $Z$  be the set of edges where  $z > 0$ , and  $a(Z)$  is the set of people involved in trades of  $Z$ .

Since  $z$  blocks, there is a price  $\tilde{p}$  for edges in  $Z$ , such that for every  $j \in a(Z)$  and for every  $y_j \in C_j(s + z)$  they choose  $y_j(e) = z(e) + s(e)$  on the edges where  $z > 0$ .

They can choose anything on the edges outside of  $Z$ . Since  $s$  is not in the chosen best vectors, the utility of  $s$  is smaller. For every  $j \in a(Z)$

$$\begin{aligned} U_j(s) &< U_j(y_j) = u_j(y_j) + \sum_{e \in Z_{j+}} \tilde{p}(e)(z(e) + s(e)) - \sum_{e \in Z_{j-}} \tilde{p}(e)(z(e) + s(e)) \\ &\quad + \sum_{(E \setminus Z)_{j+}} p(e)y_j(e) - \sum_{(E \setminus Z)_{j-}} p(e)y_j(e) \end{aligned}$$

Summing these to every vertices in  $a(Z)$ , the  $\tilde{p}$  prices cancel out.

$$\begin{aligned} \sum_{j \in a(Z)} U_j(s) &< \sum_{j \in a(Z)} U_j(y) = \sum_{j \in a(Z)} \left[ u_j(y) + \sum_{(E \setminus Z)_+} p(e)y_j(e) - \sum_{(E \setminus Z)_-} p(e)y_j(e) \right] \\ &= \sum_{j \in a(Z)} \left[ u_j(y) + \sum_{e \in Z_{j+}} p(e)y_j(e) - \sum_{e \in Z_{j-}} p(e)y_j(e) + \sum_{(E \setminus Z)_+} p(e)y_j(e) - \sum_{(E \setminus Z)_-} p(e)y_j(e) \right] \\ &= \sum_{j \in a(Z)} \left[ u_j(y) + \sum_{e \in E_{j+}} p(e)y_j(e) - \sum_{e \in E_{j-}} p(e)y_j(e) \right] = \sum_{j \in a(Z)} U_j(y_j) \end{aligned}$$

Therefore for some  $j$   $U_j(s, p) < U_j(y_j, p)$  so  $s \notin D_j(p)$  and this contradicts that  $[s, p]$  is a competitive equilibrium.

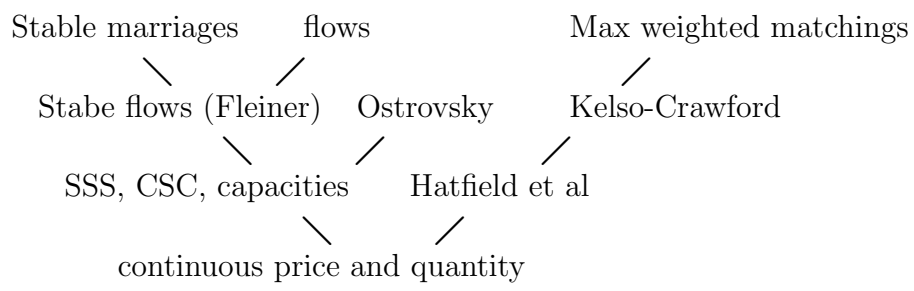
□

So if Conjecture 54 is true, then using Theorem 44 for every new graph with multiplied edges, from the continuity of  $U^*$  and  $V^*$  and from Lemma 51 we get there exists a competitive equilibrium for the marked with divisible goods, therefore Conjecture 50 is also true. Moreover this solution is stable.

## 7 Conclusion

We had an overview on many different stability concepts: Abstract definitions like three-part, four-part stability, and applications like the stability of Hungarian college admissions. In the two-sided market with money the gross substitutes property is the special case of fully substitutability in supply chain market. We introduced a common generalization of Fleiner's and Ostovsky's model in Section 5.2, moreover, we searched for a common generalization of this, and Hatfield et al.'s model. Since the model of [Hatfield et al. (2011)] contains the Kelso-Crawford model, and since Kelso-Crawford is good for finding a maximal weighted matching, we can conclude the family of models

in the following diagram:



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