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How much should you trust the Least Squares Method as a Risk Manager?

MSc Thesis

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Chapter 1

Introduction

One of the most challenging problems in option pricing theory, is pricing and finding an optimal exercise strategy for options with American exercise features. The traditional valuation techniques, such as finite difference and tree-based techniques become impractical in the multi-dimensional cases.

For pricing an American-style option it is necessary to choose between exercising and holding the option at each early exercise dates. Several researchers including Francis A. Longstaff and Eduardo S. Schwartz (2001) introduced Monte Carlo simulation based methods for calculating the value of continuing to keep the option alive. Their approach includes simple least-squares regression, which follows to call their method the Least Squares Method (LSM). The LSM is a widely used and very efficient approximation technique, when one would like to compute the value of an American-style option for pricing, or future exposures in a counterparty risk setting.

Shashi Jain and Cornelis W. Oosterlee (2015) published a new simulation based technique for pricing multi-dimensional Bermudan options, it is called the Stochastic Grid Bundling Method (SGBM). This paper presents the benefits of the SGBM against the LSM from a risk management perspective.

The main objective of my thesis is to investigate the quality of these Monte Carlo simulation-based pricing methods by examining the future exposure. We are not only interested in calculating the expectation of future exposures (i.e. Expected Positive Exposure (EPE)), for risk management purposes we also heavily rely on quantiles (i.e. Potential Exposure (PE)) of the distribution. In this thesis we compare the future exposure distribution of equity options calculated by the LSM to alternatives. We consider the novel SGBM and analytical solutions if available, and investigate the exposure metric (PE, Peak PE) differences between the methods.

At first we will see how good the methods approximate the future distribution of a European option. This is a good starting point, because then we have an analytical benchmark, the Black-Scholes formula, plus in this case we do not have to determine the stopping strategy, so the implementation and the description of the methods are easier. Compared to the Black-Scholes we can come to a lot of conclusions about the two techniques.

After that we will estimate the future exposures of the LSM and SGBM for an American put option, and we will examine the difference between the two methods in the American case, similarly from risk management point of view.

The paper is organized as follows. We can see an introduction to the used Basics in Chapter 2. Chapter 3 describes the framework of the LSM pricing method, while Chapter 4 gives a description about the SGBM. After introducing the algorithms to be compared, we will move on to estimate the future distributions calculated with the LSM and alternatives. In Chapter 5 we investigate the future exposure of a European put option. In this chapter we will introduce the concept of the Potential exposure and the Peak exposure. Chapter 6 discusses the implementation for an American put option, and makes comparisons of the two used methods with the exposure metrics. Finally we will conclude in Chapter 7.

Chapter 2

Preliminaries

To understand this thesis, it can be helpful to present definitions, financial and mathematical basics that are used. This chapter summarizes these expressions and techniques.

2.1 European, American and Bermudan Options

In finance, an option is a financial derivative conveying a right - but not the obligation - to buy or sell the underlying asset at a specified strike price on a specified date. Holding a put option gives you the right to sell and holding a call option gives you the right to buy. There is a cost for acquiring an option, which is called the price of the option. At the time of maturity the holder of the option has to decide whether to exercise the option, or to leave it to expire. We are talking about a European option in that case, when the contract has only one exercise date, and it is the same as the maturity date of the option.

For a European put option we exercise the option at the maturity time, if the current stock price is smaller, than the strike price, else we leave it to expire. For a European call option we exercise at the time of maturity, if the current stock price is greater, than strike price of the option. It follows, that the payoff of a European put option at the time of maturity is

$$\max(K - S_T\,;\,0)\,,$$

and for a European call option is

$$\max(S_T - K\,;\,0)\,,$$

where K denotes the strike price of the option, and S_T denotes the stock price at the time of maturity.

Unlike European option, an American option is an exotic option, which can be exercised at any time before and including its maturity time. This exercise feature of the American option makes it difficult to find a solution to its pricing. The computation includes the solution of the optimal stopping problem, which is one of the most challenging problems in derivatives pricing.

In this thesis we estimate the value of American options with simulation-based methods, because finite differences and binomial techniques become impractical, particularly when the value of the option depends on more than one factor (Hull and Basu 2016). We will assume that the option can be exercised at a discrete set of times, and we can approximate the value of the American option by increasing the number of these times. So we approach the American option with the corresponding Bermudan option. A Bermudan option is an exotic option, which can be exercised at a discrete set of dates.

There are some important expressions in terms of the value of immediate exercise. We call an option *in-the-money* (ITM) option, if exercising the option provides income to its holder, and we call an option *out-of-the-money* (OTM) option, if its immediate exercise does not provide income. Therefore a put option is in an ITM position, if its strike price is greater, than the current stock price, respectively the call option is ITM if its strike price is smaller, than the current stock price. An option is called an *at-the-money* (ATM) option, if its strike price equals to the current stock price. (referring to Bodie, Marcus, and Kane 1999)

2.2 Simulation from Geometric Brownian Motion

In this paper we investigate simulation-based methods, so first we have to simulate paths to have values to work with. The following section gives a description of the method of the used simulation (see Huang and Huang 2009, page 20).

A standard single asset Black-Scholes model is assumed. We assume that the underlying asset S_t follows a Geometric Brownian Motion (GBM), which is a remarkable transformation of the Wiener process (also called the exponential Brownian motion). A problem of modeling the underlying asset's price with Brownian Motion is, that it can take on arbitrarily large negative values. Unlike the Wiener process, the GBM can not take negative values. An other advantage of using the GBM is, that in real life the increments of the ratios of the prices are independent, and not the prices.

In this case the risk-neutral S_t stochastic process satisfies the following stochastic differential equation :

$$dS_t = rS_t dt + \sigma S_t dW_t, \tag{2.1}$$

where W_t denotes the standard Brownian motion (or Wiener process), and r and σ are constants. With the initial S_0 value the analytic solution of the equation is

$$S_t = S_0 \exp\left\{(r - \frac{\sigma^2}{2})t + \sigma W_t\right\}, \tag{2.2}$$

which we can obtain by using the Itô-formula. If we have a discrete set of times $\{t_j : j = 0, \ldots, M; t_j \leq t_{j+1}\}$, then we can calculate the values for consecutive times with the formula below

$$S_{t_{j+1}} = S_{t_j} \exp\left\{(r - \frac{\sigma^2}{2})(t_{j+1} - t_j) + \sigma\sqrt{(t_{j+1} - t_j)} Z_{t_{j+1}}\right\}, \tag{2.3}$$

where Z_t has a standard normal distribution. It follows from that property of the Brownian motion W_t, that its increments $W_{t+\tau} - W_t$ are normally distributed with mean 0 and variance τ. With this analytic formula we can easily simulate paths, that follow Geometric Brownian Motion.

2.3 The Black-Scholes formula

Fischer Black and Myron Scholes gave an analytic solution to the price of a European option in the world described in section 2.2, assuming there are no arbitrages (Black and Scholes 1973). Before mentioning the formula, let us assume some "ideal market" conditions (referring to Duffie 1998).

- The price of the underlying asset, say stock price S_t, is assumed to follow a random walk in continuous time, thus the distribution of the possible stock prices is lognormal at the end of any finite interval.
- The short term interest rate r, and volatility of the stock σ are assumed to be known and constant through time.
- There are no transaction costs, and riskless borrowing and lending any fraction of the price of a security is possible at the constant interest rate.
- The last assumption is, that the stock pays no dividends during the option's life.

Denote the distribution function of the standard normal distribution with Φ, and the discount factor with $P = e^{-r(T-t)}$. Then the Black-Scholes formula for a European call option's price at time t is :

$$c_t = S_t * \Phi(d_1) - P * K * \Phi(d_2), \tag{2.4}$$

and for a European put option it is:

$$p_t = P * K * \Phi(-d_2) - S_t * \Phi(-d_1), \tag{2.5}$$

where

$$d_1 = \frac{\log(\frac{S_t}{PK}) + \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}} \quad \text{and} \quad d_1 = \frac{\log(\frac{S_t}{PK}) - \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}. \tag{2.6}$$

According to the Black Scholes formula under the above assumptions the price of an European option will only depend on the stock price, the constant interest rate and volatility, the given strike price K, and the given exercise date T.

2.4 The Least-squares Regression

In this paper we use the simple least-squares regression to estimate the conditional expected value of holding further the option. The least-squares method is a regression algorithm based on least-squares distances. The aim of the method is to find a curve, that fits the best on the given set of data. In this case the best-fit means, that the curve needs to have the minimal sum of squared deviations from the given values.

For instance if we have a given set of points: $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, then the best-fitting $f(x)$ curve is that function, that minimizes the

$$\sum_{i=1}^{n} d_i^2 = \sum_{i=1}^{n} (y_i - f(x_i))^2 \tag{2.7}$$

expression.

Chapter 3

Longstaff and Schwartz methodology

Longstaff and Schwartz presented an American option pricing technique, called the least squares Monte Carlo (LSM) approach (Longstaff and Schwartz 2001). The LSM algorithm is a simulation-based algorithm, which uses least-squares regression to estimate the conditional expectation of the payoff, that the optionholder can receive by keeping the option alive. The main advantage of the LSM algorithm, is that it is applicable for options with path-dependent and American exercise features and if the value of the option depends on more than one factor. It follows, that the simulation part of the method allows the underlying asset to follow complex stochastic processes like Geometric Brownian motion, or a jump diffusion process, or a Levy process. The path dependency indicates, that the payoff of the derivative depends on the price fluctuation of the underlying asset, like in the case of an Asian option, where the payoff is determined by the average price of the underlying asset over some pre-set period.

3.1 The Pricing Problem and the Notations

To understand the LSM algorithm, we start with presenting the American option pricing framework.

We assume a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is the set of the possible realizations of the stochastic economy on a specified finite time horizon $[0, T]$. The information structure in this stochastic economy is represented by the filtration $\mathcal{F}_t$, where $t \in [0, T]$. If $s \leq t$, then for the corresponding filtrations $\mathcal{F}_s \subseteq \mathcal{F}_t$, because the $\mathcal{F}_t$ σ-algebra is generated by the distinguishable events till time t. $\mathbb{P}$ is assumed to be the risk-neutral probability measure on the elements of Ω. According to the no-arbitrage pricing theory there is an equivalent martingale measure $\mathcal{Q}$.

In general, American options are continuously exercisable, but in this pricing method we assume a discrete set of exercise dates, by dividing the time interval $[0, T]$ to $M + 1$ equal pieces, and estimating the American option's price by increasing the number of M. The assumed set of exercise dates is $\{t_0, \ldots, t_M\}$, where $0 = t_0 < t_1 < \ldots < t_m < \ldots t_M = T$.

Let $h_t = h(S_t)$ be the intrinsic value of the option, i.e. the immediate exercise value is $max(h_t, 0)$. Suppose that the risk-free rate r is constant. Then let B_t be the risk-less bank account

$$B_t = e^{rt}. \tag{3.1}$$

The pricing problem is to solve the following equation, to receive the price of the option.

$$V_{t_0}(S_{t_0}) = \max_{\tau} \mathbb{E}_{\mathcal{Q}}\left(\frac{h(S_\tau)}{B_\tau}\right), \tag{3.2}$$

where τ is a stopping time taking values from $\{t_0, \ldots, t_M\}$, so that means, that we are searching for the optimal date to exercise. To find it, we have to determine the conditional expectations form continuing the option at each exercise date. Denote it with:

$$Q_{t_m}(S_{t_m}) = \frac{B_{t_m}}{B_{t_{m+1}}} * \mathbb{E}_{\mathcal{Q}}(V_{t_{m+1}}(S_{t_{m+1}})|S_{t_m}). \tag{3.3}$$

From now on we will not indicate the $\mathcal{Q}$ measure in the index of the expected values. At the time of maturity the value of the option equals the value of exercising.

$$V_T(S_T) = \max(h(S_T), 0). \tag{3.4}$$

We determine the value of the option at time t_m by comparing the immediate exercise value with the value of continuation:

$$V_{t_m}(S_{t_m}) = \max(h(S_{t_m}), Q_{t_m}(S_{t_m})). \tag{3.5}$$

The solution for the pricing problem is to obtain the $V_{t_0}(S_{t_0})$ value, i.e. the price of the option at time zero.

3.2 The LSM algorithm

The challenging part of the above described pricing framework is to determine the value of continuations. We will do that by simulating N paths according to the dynamic of the underlying asset, denote it with S_t. The conditional expected values of continuing the option are gained by a simple least-squares regression. The following subsection presents the regression step in LSM.

3.2.1 The Regression in the LSM algorithm

After simulating N paths, we will have $N * M$ simulated values. So the generated set of points is $\{S_{t_m}(n) : n = 1, \ldots, N; m = 0, \ldots, M\}$, where $S_{t_0}(n)$ are the same S_0 initial stock prices for $n = 1, \ldots, N$.

According to the backward feature of the algorithm we can assume, that when we compute the continuation value at t_m the option values at time t_{m+1} are known. These values are $V_{t_{m+1}}(S_{t_{m+1}}(n))$ for $n = 1, \ldots, N$.

In the LSM algorithm we approach the continuation value at t_m by regressing the discounted option values at t_{m+1} on K basis functions $\Phi_k(S_{t_m})$, which are highly correlated with the option's value, for $k = 1, \ldots, K$.

So we get the linear regression in the following form:

$$y_n = \sum_{k=1}^{K} \beta_k x_{nk} + \epsilon_n \tag{3.6}$$

where ϵ_n denotes the error, x_{nk} denotes the corresponding basis function as independent variable, y_n denotes the dependent variable

$$x_{nk} = \Phi_k(S_{t_m}(n)), \qquad \text{and} \qquad y_n = \frac{B_{t_m}}{B_{t_{m+1}}} V_{t_{m+1}}(S_{t_{m+1}}(n)). \tag{3.7}$$

From this regression we get the β_k coefficients, so we can use this approach in the pricing formula as below.

$$\begin{aligned} Q_{t_m}(S_{t_m}(n)) &= \mathbb{E}\,(\frac{B_{t_m}}{B_{t_{m+1}}} * V_{t_{m+1}}(S_{t_{m+1}}) \,|\, S_{t_m} = S_{t_m}(n)) \\ &\approx \mathbb{E}\,(\sum_{k=1}^{K} \beta_k(t_m)\, \Phi_k(S_{t_m}) \,|\, S_{t_m} = S_{t_m}(n)) \\ &= \sum_{k=1}^{K} \beta_k(t_m)\, \Phi_k(S_{t_m}(n)). \end{aligned} \tag{3.8}$$

With this regression step we receive an estimation of the value of continuation along each path for every exercise date. (see Maree and Du Toit n.d., page 2-3)

When pricing American options, the best choice of the basis functions depends on the contract. For example Longstaff and Schwartz like to use the Laguerre polynomials as basis functions in their paper (Longstaff and Schwartz 2001), or often used basis functions are the Hermite, the Chebysew and the Jacobi polynomials. In this paper we will use the powers of the state variables.

After including the Kth basis function we can conclude, that the result does not get any better. It follows that we don't need to use infinitely many functions. When there are more state variables, that affects the option, then the set of basis functions should contain terms in each variable and cross-products as well.

3.2.2 Optimal stopping strategy

The optimal stopping strategy can be obtained by comparing the immediate exercise value with the expected payoff from continuation along each simulated path at every exercise date, starting from the time of maturity moving backwards.

The strategy is to exercise the option, if the value of immediate exercise is greater than the estimated value of continuation. The option will be exercised at the earliest time this condition holds, since an option can be exercised only once.

As soon as we have the optimal stopping strategy, we can simply determine the corresponding cash flows along the simulated paths. Now we are able to get the option's value by discounting the cash flows to time zero and by taking the average of these values.

3.2.3 Using only in-the-money paths

LSM includes only in-the-money paths in the regression, since for the out-of-the-money paths it is never optimal to exercise the option, so we do not need to estimate the value of the continuation in that cases, because we do not have to make stopping decisions then. Using only ITM paths decreases the computational time, and leads to better estimations of the continuation values, where exercising the option is relevant.

3.3 Foresight bias of the LSM

The LSM is easy to implement, because only the simple least-squares regression is required, and it has its advantages because of the simulation step, that is why it is a simple yet powerful method. A weakness of this method, that it gives high biased estimation of the conditional expectation of continuation. (A simulation estimator X is biased high, if $\mathbb{E}(X) \geq Q$, where Q is the real option value. Likewise, the simulation estimator X is biased low, if $\mathbb{E}(X) \leq Q$. (referring to Broadie, Glasserman, et al. 2004))

After the simulation step the LSM works backwards with estimating the values of continuing the option. The optimal strategy, that we gain with the look ahead feature of the LSM method, is based on information, which is unavailable in real life. It follows, that the exercise strategy of the LSM is super optimal, i.e. better than it would be based on the theoretically available informations. (Fries 2005) (for more see Kovacs 2012, page 21)

This foresight problem can be solved for example with a path re-sampling method. It works by determining the optimal stopping strategy with LSM, and simulating a new set of paths, which is independent from the strategy, and using the received stopping rule on that. Thus, we will not use these foresight informations in the algorithm. This modification of the LSM algorithm is a low biased estimator.

Since the original LSM is superoptimal, and its previously mentioned modification is suboptimal, the two algorithms can give an optimal interval for the price.

Chapter 4

The Stochastic Grid Bundling Method

The Stochastic Grid Bundling Method (SGBM) is an algorithm, that approximates the value of a multi-dimensional Bermudan option (Jain and Oosterlee 2015). It is similar to the LSM algorithm, because it has the same pricing framework, and it is also based on simulation and contains regression. The main difference is the bundling step of SGBM, and the time of the regression. In essence, the SGBM differs from LSM in the method, with which we approach the value of continuation.

The estimated values in SGBM have lower numerical noise, than in LSM at each time step. It follows that the SGBM is an appropriate choice, if we need the option values in the intermediate time steps. For example, when we approximate future exposures, which we will examine later.

4.1 The SGBM algorithm

The pricing framework of the SGBM is the same as we presented in chapter 3. At the terminal time T the holder of the option exercises the option, if it is in ITM position, or allows it to expire otherwise. We determine the strategy backwards, and exercise the option, if exercising is worth more, than the estimated value of continuation.

As in the LSM method we simulate N sample paths starting from S_0. In every t_m exercise date we will have N generated $S_{t_m}(1), \ldots, S_{t_m}(N)$ values. We call them the grid points for every $t \in \{t_0, \ldots, t_{m-1}, t_m, \ldots, T\}$. The bundling step is to make ν partitions of the grid points in every t_m time, based on a specified distance, or technique. Note them with $\mathcal{B}_{t_m}(1), \ldots, \mathcal{B}_{t_m}(\nu)$.

4.1.1 The Regression step in SGBM

We would like to estimate the value of continuation at each $t_m \in \{t_0, \ldots, t_{M-1}\}$ time for every $n \in \{1, \ldots, N\}$ paths. After the bundling step we have to run a regression for every bundle at every t_m. With the bundling technique we can get more accurate results at the intermediate time steps. Like we did before, we can assume that the option values of time t_{m+1} are known at time t_m.

It is assumed, that in SGBM the components of the regression can be written in the following form:

$$x_{nk} = \Phi_k(S_{t_{m+1}}(n)), \qquad \text{and} \qquad y_n = V_{t_{m+1}}(S_{t_{m+1}}(n)). \tag{4.1}$$

So the following regression equation needs to be solved within every bundle. The partitioning was determined at the previous time step.

$$y_n = \sum_{k=1}^{K} \alpha_k x_{nk} + \epsilon_n \tag{4.2}$$

In this case the regression coefficients depend on time t_{m+1}.

$$V_{t_{m+1}}(S_{t_{m+1}}) \approx \sum_{k=1}^{K} \alpha_k(t_{m+1}) \Phi(S_{t_{m+1}}(n)) \tag{4.3}$$

The difference is caused by the changes in the regression. We have to pay attention to the projection between the consecutive time steps. The risk-free interest rate is still assumed to be constant.

$$\begin{aligned} Q_{t_m}(S_{t_m}(n)) &= \mathbb{E}\,(\frac{B_{t_m}}{B_{t_{m+1}}} * V_{t_{m+1}}(S_{t_{m+1}}) \,|\, S_{t_m} = S_{t_m}(n)) \\ &\approx \frac{B_{t_m}}{B_{t_{m+1}}} * \mathbb{E}\,(\sum_{k=1}^{K} \alpha_k(t_{m+1})\, \Phi_k(S_{t_{m+1}}) \,|\, S_{t_m} = S_{t_m}(n)) \\ &= \frac{B_{t_m}}{B_{t_{m+1}}} * \sum_{k=1}^{K} \alpha_k(t_{m+1}) \mathbb{E}\,(\,\Phi_k(S_{t_{m+1}}) \,|\, S_{t_m} = S_{t_m}(n)) \end{aligned} \tag{4.4}$$

Denote the conditional expectation part $\mathbb{E}(\,\Phi_k(S_{t_{m+1}}) \,|\, S_{t_m} = S_{t_m}(n))$ with $\Psi_k(S_{t_m}(n))$. This means we get

$$Q_{t_m}(S_{t_m}(n)) = \frac{B_{t_m}}{B_{t_{m+1}}} * \sum_{k=1}^{K} \alpha_k(t_{m+1}) \Psi_k(S_{t_m}(n)) \tag{4.5}$$

In SGBM the choice of the basis functions is harder, because the $\mathbb{E}(\,\Phi_k(S_{t_{m+1}}) \,|\, S_{t_m}(n))$ conditional expected values have to be known in a closed form, or there should be analytic approximation for these expressions. For example if we consider the $\Phi(S_{t_{m+1}}) = S_{t_{m+1}}$ basis function in a Black-Scholes model, then its distribution is known to be lognormal, or if we consider the $\Phi(S_{t_{m+1}}) = \log(S_{t_{m+1}})$ basis function, then its distribution is known to be normal. It follows, that the conditional expectation $\mathbb{E}(\,\Phi_k(S_{t_{m+1}}) \,|\, S_{t_m} = S_{t_m}(n))$ for these basis functions can be given in closed forms. We could use the $(S_{t_{m+1}})^2$ and the $(S_{t_{m+1}})^3$ variables as basis functions too, with the help of the variance, and third moment of the lognormal distribution.

4.1.2 The Bundling step

Using the bundling method we can approximate the distribution of $S_{t_{m+1}}$ conditional on the state $S_{t_m} = X$. Instead of simulating paths from all $S_{t_m}(n)$ grid points until time t_{m+1}, we can take the bundle, from which $S_{t_m}(n)$ starts, and we can use those paths that originate from this bundle to sample $S_{t_{m+1}}$. This method is computationally cheaper, and with increasing number of paths and bundles it approaches the true distribution.

The bundling can be implemented for example with the following two techniques:

- k-means clustering
- recursive bifurcation

k-means clustering algorithm

The k-means clustering algorithm does the partitioning by minimizing the sum of squares of the deviation from the mean within the clusters.

$$\min D(\mathcal{B}_{t_m}(1), \ldots, \mathcal{B}_{t_m}(k)) = \min \sum_{\beta=1}^{k} \Big(\sum_{S_{t_m}(n) \in \mathcal{B}_{t_m}(\beta)} \|S_{t_m}(n) - \mu_\beta\|^2 \Big), \tag{4.6}$$

where μ_β is the mean of the grid points of the $\mathcal{B}_{t_m}(\beta)$. So k-means clustering is partitioning the given data into k clusters, where every data belongs to the cluster, whose mean is nearest to it.

There are a few algorithms for this partitioning. The most popular is **Lloyd′s algorithm**. The first step of the Lloyd's algorithm is to determine the initial values $\mu_1^{(1)}, \ldots, \mu_k^{(1)}$ of the means. After that the algorithm alternates betweens two steps to reduce $D(\mathcal{B}_{t_m}(1), \ldots, \mathcal{B}_{t_m}(k))$.

1. Assignment step

   Assign the values with the cluster represented by the nearest mean point.

$$\mathcal{B}_{t_m}^{(l)}(\beta) = \left\{ S_{t_m}(n) : \|S_{t_m}(n) - \mu_\beta^{(l)}\|^2 \leq \|S_{t_m}(n) - \mu_j^{(l)}\|^2 \,, \forall\, 1 \leq j \leq k \right\}, \tag{4.7}$$

   where every S_{t_m} grid point can be assigned to only one bundle, because of the non- overlapping feature.

2. Update step

   We are ready if the assignment did not change from the previous iteration, otherwise we need to update the mean points as:

$$\mu_\beta^{(l+1)} = \frac{1}{|\mathcal{B}_{t_m}^{(l)}(\beta)|} \sum_{S_{t_m}(n) \,\in\, \mathcal{B}_{t_m}^{(l)}(\beta)} S_{t_m}(n). \tag{4.8}$$

It is easy to see, that we can only reduce $D(\mathcal{B}_{t_m}(1), \ldots, \mathcal{B}_{t_m}(k))$ with each step, and since $D(\mathcal{B}_{t_m}(1), \ldots, \mathcal{B}_{t_m}(k)) \geq 0$ the algorithm converges to a stable, fixed local minimum. So it is proven, that after a finite number of this iteration step, the structure of the partition no longer changes.

A well known problem with Lloyd's algorithm is that it is sensitive to the initial choice of the means. (referring to Slonim, Aharoni, and Crammer 2013)

The problem of converging to a local minimum is less severe with **Hartigan′s algorithm**. The Hartigan's method also starts from a given partition. The difference is the iteration step.

The algorithm picks an item $S_{t_m}(i)$ randomly, and removes it from its cluster. Then the mean of the original cluster should be updated without $S_{t_m}(i)$. Now we have to find a new cluster to $S_{t_m}(i)$, in terms of minimizing $D(\mathcal{B}_{t_m}(1), \ldots, \mathcal{B}_{t_m}(k))$, and update the mean of the new cluster.

Recursive bifurcation

For the recursive bifurcation technique please refer to Appendix B.

4.2 Bias in SGBM

The SGBM uses the same look ahead method for estimating the values of continuation. Like in the case of the LSM, it is easy to construct a low biased path estimator version of the SGBM. After the optimal stopping strategy has been obtained with the original SGBM, we generate a new set of paths, and apply the previously found stopping policy on it. As soon as the option has to be exercised along a path according to the exercise strategy, we take the immediate exercise value from that time, and discount it back to t_0. Finally we take the average of these received values.

The high biased original SGBM, and the low biased modification of it together can be a valid confidence interval for the price of the option.

Chapter 5

Future exposure of European Option

The main subject of my thesis is to compare this simulation-based and widely used pricing methods by examining the future price distributions (i.e. the future exposure) in each case. The first step should be the approximation of a European option's price, because then we do not have to determine the optimal stopping strategy and that is why the implementation is easier, plus in this case we have an analytical benchmark, the Black-Scholes formula (see in section 2.3).

To investigate the future exposures, we are interested in the price of a European option at a t time, later then t_0. We can easily do this in the analytic case, if the S_t value is known. In the LSM and SGBM cases the values at t will be the conditional expectation from continuing the option, which we get with the regression step. This chapter describes how these methods approximate the future exposures.

5.1 LSM for European Option

5.1.1 Description of the LSM for the European case

First we have to simulate N paths from Geometric Brownian Motions forward in time (as it was described in section 2.2) . To calculate the conditional expected values at t with a regression, we only need the simulated values at t and at the time of maturity T, because we do not have the chance to exercise the option between.

In these simulation algorithms we always start the regression with determining the pay-offs at the final expiration date. As it was mentioned is chapter 2 the immediate exercise value for a put option is $\max(K - S_T, 0)$ and for a call option is $\max(S_T - K, 0)$ at T. We define Y as the cash-flows at T discounted back to time t. Let X denote the simulated stock prices at t. After that we have to choose the basis functions, which are depending on X. Let $\Phi_k(X)$ be the chosen family of the basis functions for $k = 1, \ldots, K$. Now we can regress Y on the basis functions, and after we receive the regression coefficients $\beta_0, \ldots, \beta_K$, we can get the values of continuation by substituting $X = S_t$ into the regression equation with the coefficients previously obtained. The following polynomial results the future exposures.

$$\beta_0 + \beta_1 \Phi_1(X) + \ldots + \beta_K \Phi_K(X) \tag{5.1}$$

As we have seen we got the conditional expected values easily, and only one regression was involved.

5.1.2 Results

In the next subsection the application of the LSM method for European options will be demonstrated, and its accuracy will be evaluated by comparing the future exposures to the Black-Scholes formula. We implement the above described methods in *R*, which is a language and environment for data manipulation, calculation and graphical display.

For the Black-Scholes formula we only need the following inputs. The spot price of the underlying asset, note it with S_t, the strike price K, the risk-free interest rate r, the volatility of the stock price σ, and the remaining time till the maturity time: $T - t$. For the simulation we need the same values, and the number of rows N and columns M of the simulated matrix. The number of the rows indicates how many paths we simulate. The number of the columns will not play an important role in the European case, because we only need the column, which belongs to time t and the one which belongs to time T.

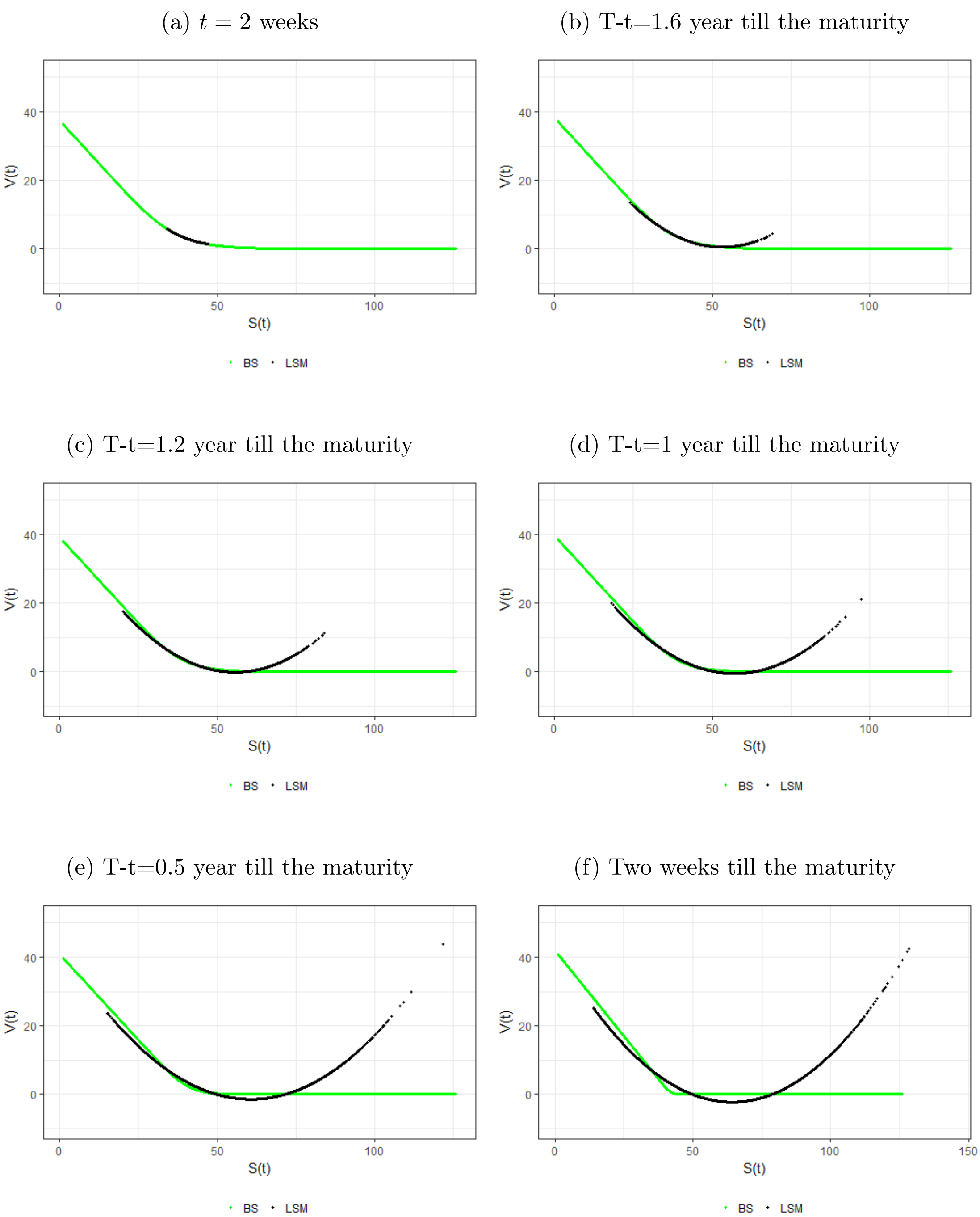
Consider a 2-year European-style put option on a non-dividend-paying stock. The short term interest rate is assumed to be 6% and constant through the lifetime of the option. The underlying stock price at the initial time step is 40, the strike price of the option is 42, the constant volatility is 20%. The Black-Scholes price of the option at the initial time step is 3.105212.

LSM based on all of the simulated paths

First we try the LSM algorithm with basis functions $\Phi_1(X) = X$, and $\Phi_2(X) = X^2$ for the previously mentioned European put option. We chose 6 different dates to illustrate the evolution of the future exposures for the examined methods. We chose one date close to the initial date, plus two more dates before the half of the option's lifetime. We plot the future exposures at the half time period, and at the time, when only half a year left till maturity, and when we are close to the maturity date.

To compare the future exposures with the Black-Scholes formula we plot the BS option values against the spot underlying asset price, and we plot the LSM values to the same figure. In this figure we have BS prices for every spot price, but we have LSM values only, where the corresponding simulated values are located. Figure 5.1 depicts the received future exposures at the 6 chosen dates, where the LSM was calculated using the basis functions X and X^2, and using 100000 simulated paths.

We can see in Figure 5.1, that as time passes and the simulated values are more scattered from the initial stock price, the regressed values of LSM fit much less to the BS curve. The figures show that the LSM approximation takes on more and more negative values, which is obviously not a good estimation of the price of the option, because you can not pay a negative price, for getting a right to do something in the future.



(a) $t = 2$ weeks

(b) T-t=1.6 year till the maturity

(c) T-t=1.2 year till the maturity

(d) T-t=1 year till the maturity

(e) T-t=0.5 year till the maturity

(f) Two weeks till the maturity

Figure 5.1: Exposures calculated with LSM (with black) for a European put option with basis functions X and X^2, and the BS curve (with green). The strike price of the put option is 42 , the stock price at the initial time was 40 , $r = 0.06$, $\sigma = 0.2$, the time of maturity is 2 years, and the LSM algorithm is based on 100000 simulated paths, and all of the paths were included in the regression.

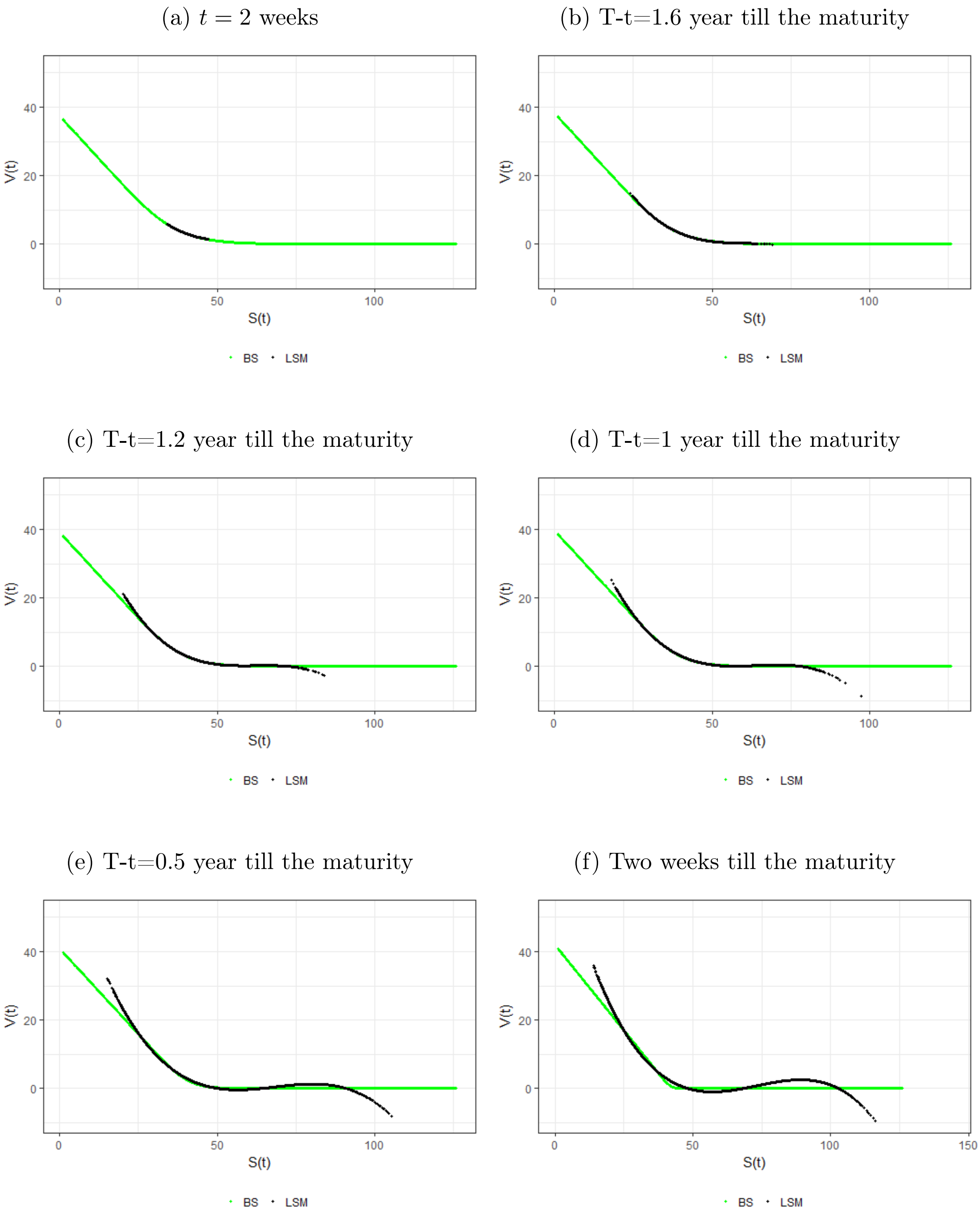


Figure 5.2: LSM for European put option with basis functions X, X^2 and X^3, and the BS curve. The parameters of the option are the same as in the $Y \sim X + X^2$ case (as it is described under Figure 5.1) , and again all of the 100000 paths were included in the regression.

Figure 5.1 shows, that for the out-of-the-money cases, where we should get zero almost everywhere, this version of the LSM gives completely false results. This proves, that suggestion of Longstaff and Schwartz to only use the in-the-money paths (Longstaff and Schwartz 2001) is a good idea.

Figure 5.2 shows what happens, when we involve the X^3 basis function in the regression. Compared to Figure 5.1 the fitting is much better to the BS curve. The approximation seems to be appropriate for the $t \leq 1$ cases. As we get closer to the expiration the LSM is increasingly different from the BS, especially in the out-of-the-money cases, where the LSM takes on negative and positive values alternatingly. In the following we will use the $\Phi_1(X) = X$, $\Phi_2(X) = X^2$, and $\Phi_3(X) = X^3$ basis functions, because it gave much better approximations compared to the $\Phi_1(X) = X$, $\Phi_2(X) = X^2$ version.

LSM only with the in-the-money paths involved

Above we showed what happens, when all of the simulated paths are involved. The LSM approach truly includes only the in-the-money paths in the regression. The Longstaff and Schwartz paper (Longstaff and Schwartz 2001) claims, that this step significantly increases the efficiency and decreases the computational time of the algorithm. In this subsection we will examine this statement with the future exposures.

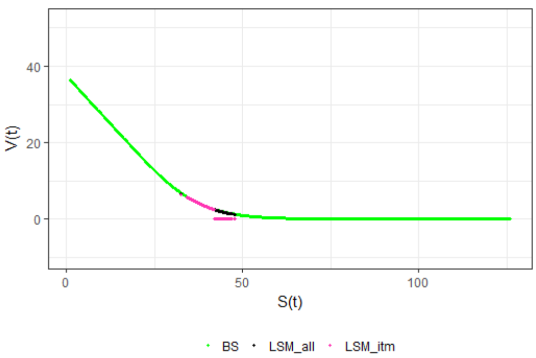
When we are in t and we have N simulated $S_t(n)$ values, we use only the in-the-money values in the regression, which means that X will include only the in-the-money elements of S_t, and Y will include the corresponding discounted pay-offs of the time of maturity. The option values along the out-of-the-money paths will get 0 values.

This modification will give a more accurate conditional expected value for the in-the-money paths. When we calculate the value of an American option, this is obviously an improvement, because in that case we need to decide at every time step and along each path, if we should exercise the option or we should hold it further. We only have to make this decision in the in-the-money paths, because the exercise is only relevant then. Since we have a more accurate conditional expectation of continuation in that case, this algorithm is more efficient. In the American option chapter we will obviously use this version of the algorithm.

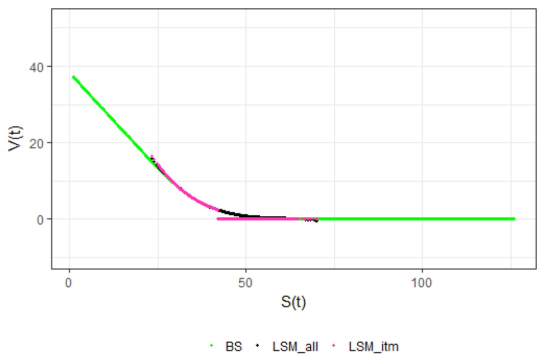
Let's see what happens when we apply this modification for European options. We consider the same European put option, as in Figure 5.1 and in Figure 5.2, to be able to compare with the LSM, when all of the paths were used in the regression. We use the $\Phi_1(X) = X$, $\Phi_2(X) = X^2$, and $\Phi_3(X) = X^3$ basis functions in both cases. The difference is, that X denotes all of the simulated values at t in the first case (depicted with black), and it denotes only the in-the-money paths of S_t in the second case (depicted with pink). (see in Figure 5.3)

Figure 5.3 shows, that in the region where exercise is relevant this modification improved the fit to the BS curve.

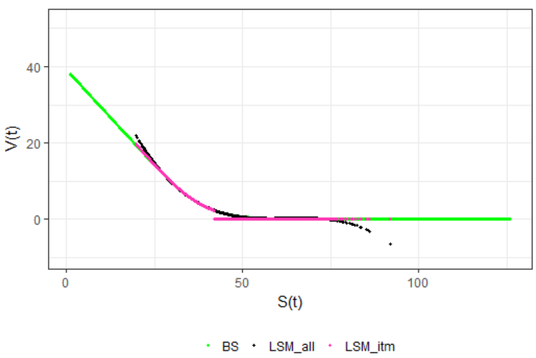
(a) $t = 2$ weeks



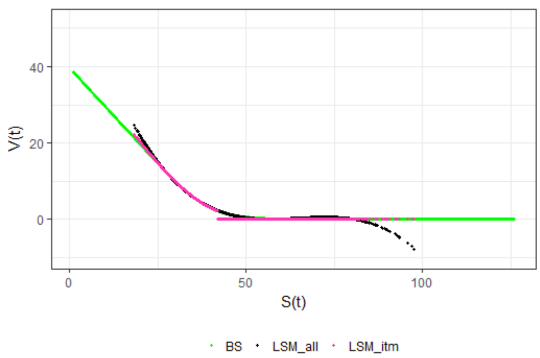
(b) T-t=1.6 year till the maturity



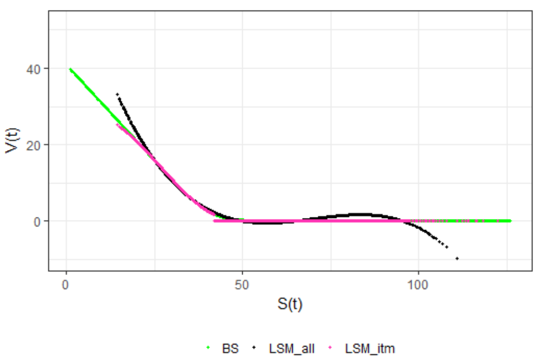
(c) T-t=1.2 year till the maturity



(d) T-t=1 year till the maturity



(e) T-t=0.5 year till the maturity



(f) Two weeks till the maturity

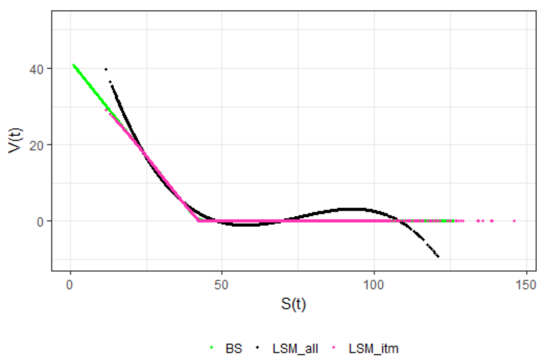


Figure 5.3: Future exposures of a European put option at 6 different times with LSM, where the pink function represents the LSM, when only the in-the-money paths are included, and the black function represents the LSM, when all of the 100000 simulated paths are included. We considered a 2-year European-style put option, with the parameters: $S_0 = 40$, $K = 42$, $r = 0.06$, $\sigma = 0.2$, $T = 2$. The green line represents the European option values against the underlying asset's spot price, calculated with the Black-Scholes formula.

5.1.3 Potential Exposure

Previously, we examined future exposures (i.e. Expected Positive Exposure=EPE) in this chapter. From a risk management perspective the upper quantiles also play an important role. The potential exposure (PE) is an important metric in risk management, which indicates a maximum amount of the expected future exposures at a specified date on a given confidence level. For example, if the level of confidence is 99%, then the $PE99$ value will be that value of future exposure, that will not be exceeded in a 99% confidence level, i.e.

$$PE99(t) = \inf \{X(t) : P(EPE(t) \geq X(t)) \leq 1 - 99\%\} \tag{5.2}$$

For the LSM we can determine the $PE99(t)$ by taking the 99% quantile of the distribution, that we got with the method at t. After calculating these measures for every columns of the simulated matrix, we can plot the $PE99(t)$ values against the time.

By LSM_{itm}, we mean that version of the method, in which we include only the ITM paths, and LSM_{all} indicates that version, when we use all of the simulated paths in the regression. Our PE confidence level exposure calculation is also based on 100000 paths. Our benchmark will be the Black-Scholes again, by calculating the BS price for each simulated value and by taking the 99% quantile of them.

Figure 5.3e shows, that for the larger option values the LSM_{all} curve is curved upward from the BS curve, and it is not true for the LSM_{itm}, when half a year is left till the expiration. That means, we except at $t = 1,5$ years, that the $PE99$ metric of the LSM_{all} will be the largest, the second largest will be the BS and the smallest will be the LSM_{itm}.

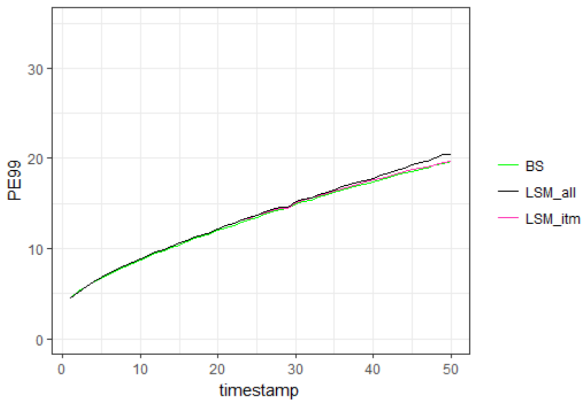


Figure 5.4: $PE99(t)$ of the LSM_{itm} (with pink), LSM_{all} (with black) and of the BS (with green) for a European put option, with the parameters: $S_0 = 40$, $K = 42$, $r = 0.06$, $\sigma = 0.2$, $T = 2$. The calculation is based on the same 100000 simulated paths for the 3 methods. (The x-axis represents the time, even though the number of the certain column of the simulated matrix is there.)

Figure 5.4 illustrates the comparison of the $PE99$ curves. The plotted curves are very similar, they move roughly together, the percentage value of the quantile is too small for the number of the paths to show clearly our expectation from the exposure figure. To see the deviations better for the larger option values, let's plot the 99.99% quantile of the methods.

Figure 5.5 shows the $PE99.99$ values. We can see in the plot, that in 99.99% confidence level the potential exposure curves are more different from each other. From this point of view the LSM_{itm} is a better choice, than the LSM_{all}. The figure shows, that LSM_{itm} estimates the option values better for the options, which are roughly in ITM positions.

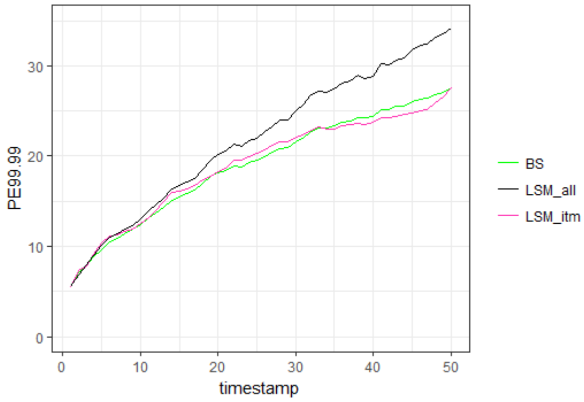


Figure 5.5: $PE99.99(t)$ of the same European put option as above, calculated with the same methods. (The x-axis represents the time, even though the number of the certain column of the simulated matrix is there.)

5.1.4 Peak Exposure

The Peak Exposure is a maximum amount of a specific exposure metric, in our case the Potential Exposure (PE_{α}), during the lifetime of the option. i.e.:

$$PeakPE = \max_{0 \leq t \leq T} \{PE_{\alpha}(t)\} \tag{5.3}$$

Calculate the Peak values for the above illustrated PE99.99(t) curves of the 3 methods (see in Figure 5.5).

$$PeakPE_{BS} = 26.45964$$

$$PeakPE_{LSM_{itm}} = 26.45964$$

$$PeakPE_{LSM_{all}} = 32.7672$$

We can see, that the $Peak\ PE99.99$ value of the LSM_{all} is much higher, than the values belonging to the other two methods. The Peak PE value of LSM_{itm} equals the Peak PE value of the BS, and they are the $PE99.99(T)$ values in both cases.

5.2 SGBM for European Option

We would like to compare our existing LSM results with the Stochastic Grid Bundling Method for the previously defined 2-year European put option.

5.2.1 Description of the SGBM for the European case

The regression part of the SGBM differs from the regression in LSM not just in the bundling step. When we would like to approximate the option values at t, we get the independent variables from the next time step. This means that the whole regression happens in the next time step. In the European case the next time step, which counts is the time of maturity, because we can not exercise the option at the earlier dates.

The Stochastic Grid Bundling Method in the European option case will contain more regressions than the LSM. Our grid points are the simulated underlying asset values at t, if we are interested in the future exposure at time t. As it was described earlier, we have to partition the $S_t(n)$ grid points, $n = 1, \ldots, N$, to get the non-overlapping sets for the regressions. The N grid points are bundled into ν partitions, denote them with $\mathcal{B}_t(1), \ldots, \mathcal{B}_t(\nu)$. Then we have to do ν regressions for the $\mathcal{B}_t(v)$, $v = 1, \ldots, \nu$, partitions separately.

As the result of the regressions we get ν kind of regression coefficient vectors. If we made the regression with K basis functions, then we should get $(K + 1)$ coefficients for each $\mathcal{B}_t(v)$ bundle. We introduce the α_k^v notation, where k reflects the corresponding basis function and v is the index of the bundle.

After we received the constant α_k^v regression coefficients, we have to determine the $\mathbb{E}(\,\Phi_k(S_T)\,|\,S_t = S_t(n))$ conditional expected values for every basis functions in every bundle. $\Psi_k(S_t^v)$ indicates the corresponding result. Then we have to substitute these calculated conditional expected values to the following equation:

$$\alpha_0^v + \alpha_1^v \Psi_1(S_t^v) + \ldots + \alpha_K^v \Psi_K(S_t^v), \tag{5.4}$$

where $S_t^v \in \mathcal{B}_t(v)$. Do not forget to discount this value back to t, because this value was calculated at T. If we do this for every $v = 1, \ldots, \nu$, then we get all values of continuation, which are exactly the required future exposures in the European case.

The aim of the bundling step is to assign similar values to the same bundle, and dissimilar ones to other bundles. For example, when we are far away from the initial time step, and the N paths of GBM are taking values from a wider interval, this bundling step separates the roughly out-of-the-money values from the in-the-money values. With this partition of the $S_t(n)$ values we expect a more accurate approach of the future distribution. When the regressions are separated, the results for the extremely out-of-the-money cases could get the expected zero values better, than if we run only one regression.

5.2.2 Results

To implement the method we will use the k-means clustering algorithm for the bundling step. In R there is an available function, called *kmeans*. With this function the Hartigan's algorithm is used by default, but with the *algorithm* variable it can be changed easily. We can choose from the algorithms given by Lloyd, Forgy and MacQueen. We will go with the Hartigan's method (referring to (Slonim, Aharoni, and Crammer 2013)). With the *nstart* command it can be set to be able to try more random starts. The maximum number of iterations can be limited with the command *iter.max*. I allowed 20 iterations and 20 opportunities to select the initial sets.

We will do the regression on the $\Phi_1(X) = X$, $\Phi_2(X) = X^2$, and $\Phi_3(X) = X^3$ basis functions, where X indicates the simulated values at T. While we assumed the Black-Scholes model, and Geometric Brownian Motion to the underlying asset's price, we know, that the distribution of S_T

is lognormal. This follows, that the conditional expected values can be written in the following closed form. (see Jain and Oosterlee 2015, page 19)

$$\begin{aligned} \mathbb{E}(\Phi_1(S_T) \mid S_t = S_t(n)) =& \mathbb{E}(S_T \mid S_t = S_t(n)) = S_t(n)\, e^{r(T-t)} \\ \mathbb{E}(\Phi_2(S_T) \mid S_t = S_t(n)) =& \mathbb{E}((S_T)^2 \mid S_t = S_t(n)) = (S_t(n))^2\, e^{(2r+\sigma^2)(T-t)} \\ \mathbb{E}(\Phi_3(S_T) \mid S_t = S_t(n)) =& \mathbb{E}((S_T)^3 \mid S_t = S_t(n)) = (S_t(n))^3\, e^{(3r+3\sigma^2)(T-t)} \end{aligned} \tag{5.5}$$

The clustering will depend on the simulated values at t. By trying the method for several clusters, we could conclude, that the greater the number of clusters, the better the result fits on the BS curve. To see this statement for concrete examples please refer to Appendix A Figure 1.

Appendix A Figure 1 shows the exposure for the 2-year European put option, when half a year left till the maturity. The SGBM was implemented with 4 different number of clusters, and the figures show how well these implementations fit the BS curve. If we use 2 clusters, the result improves compared to the LSM, when all of the paths were included. If we increase the number of clusters to 3 and to 6, the results get better and better. We can be completely satisfied with the fit of the exposure calculated with the SGBM, when the number of clusters was 10.

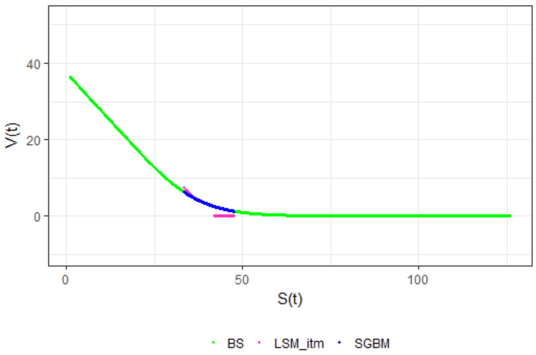
I decided to use 10 partitions in the SGBM algorithm, which means 10 regression in the European option case. If we will estimate American options, we will do 10 regression in every step between two consecutive time steps.

We estimate the future exposures of the same put option, as we did in the LSM section. The results are plotted to the same figure with the LSM to make the comparison easier. Figure shows the comparison of SGBM and that case of the LSM, when only the in-the-money paths are involved, because the real American option pricing LSM skips the out-of-the-money paths.

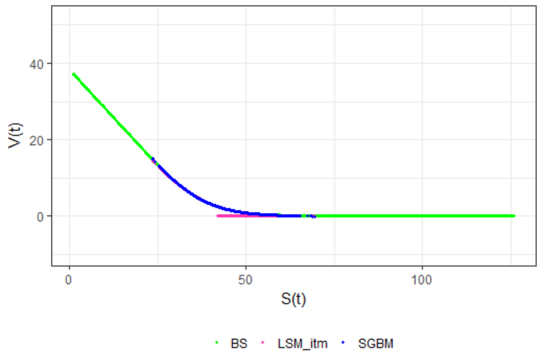
So Figure 5.6 is the most important plot of this chapter. It is clear from Figure 5.6, that the bundling step is good for the out-of-the-money paths. Overall, we can see that the shape of the function we got with SGBM is more similar to the shape of the BS curve. We can notice a higher accuracy in the case of the exposures calculated by the SGBM.

If we are interested in the future distributions, and make the out-of-the-money estimations count too, then it is obvious, that the skipping the OTM paths, and giving them the 0 values has a really bad effect on the distribution. When we depict the density functions of the future distributions, the many zero values of the LSM messes up the shape of its density function. In the case of the SGBM we do not need to miss the OTM values from the regression, so it estimates the option values well in the out-of-the-money positions too.

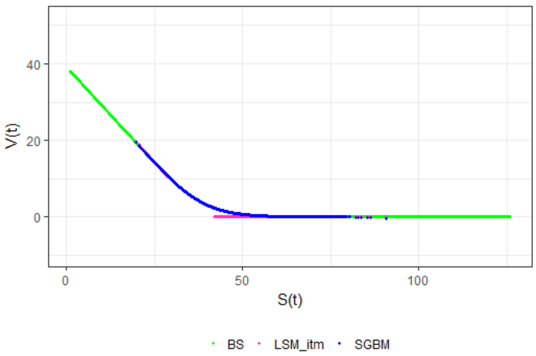
(a) $t = 2$ weeks



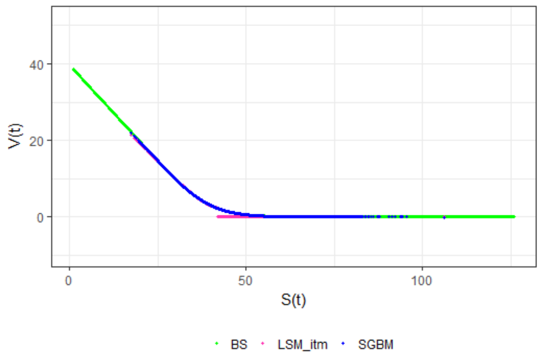
(b) $T - t = 1.6$ year till the maturity



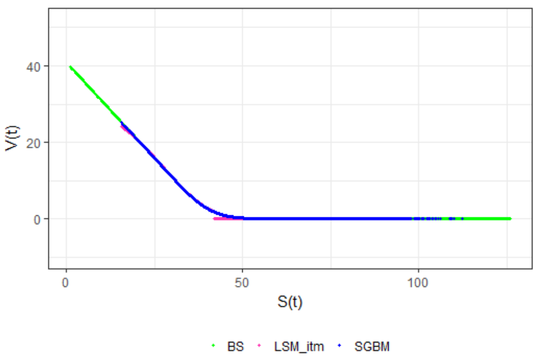
(c) $T - t = 1.2$ year till the maturity



(d) $T - t = 1$ year till the maturity



(e) $T - t = 0.5$ year till the maturity



(f) Two weeks till the maturity

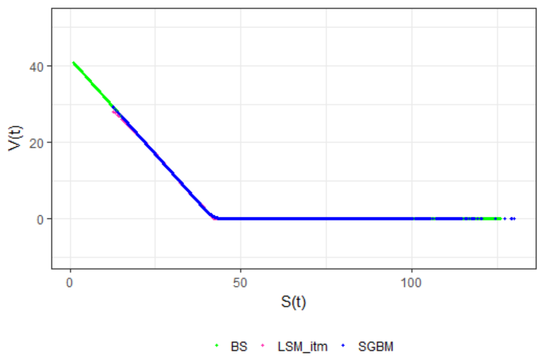
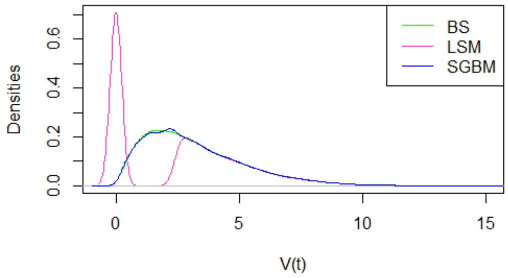
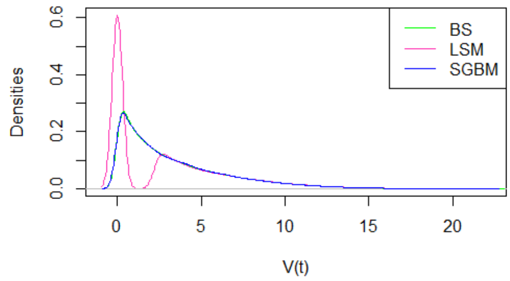


Figure 5.6: Exposures of a European put option at 6 different t times, where the blue one was calculated with SGBM, and the pink was calculated with LSM, when only the ITM paths are included. The used parameters of the option are: $S_0 = 40$, $K = 42$, $r = 0.06$, $\sigma = 0.2$, $T = 2$ and the two algorithms are based on the same 100000 paths. The green curve represents the Black-Scholes price against the underlying asset's spot price.

(a) $T - t = 0.5$ year till the maturity



(b) $T - t = 1$ year till the maturity



(c) $T - t = 1.5$ year till the maturity

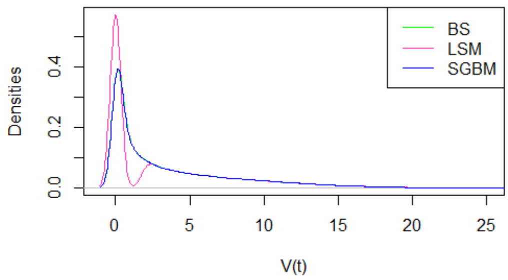


Figure 5.7: Comparison of the density functions of the BS, LSM, and SGBM for a 2-year European put option, with the strike price $K = 42$, the initial stock price $S_0 = 40$, the interest rate $r = 0.06$, and the volatility $\sigma = 0.2$.

Let's see the density functions after half a year, after one year, and after one and a half year from the initial time. Figure 5.7 shows the comparisons. We can see in the figures, that the density functions of the BS and SGBM almost equal, while the LSM results a completely different function. This proves, that the SGBM is more applicable in that case, when we are not just interested in the price of an American option, but we need to know the future distributions too. For a clearer interpretation of the density function of the LSM please refer to Appendix A Figure 2, where you can find the histogram of the LSM.

5.2.3 Potential Exposure

Finally let's examine the PE metrics of the methods. We can see in Figure 5.8, that the $PE99$ values of the three methods are very similar through the lifetime of the option, even though Figure 5.6 shows, that for the largest values the exposures differ from each other. Accordingly, the 99% quantile of the option values based on the 100000 simulated paths does not yet belong to this range, but it is good to see, that the two methods fit that much to the BS curve in their 99% quantile.

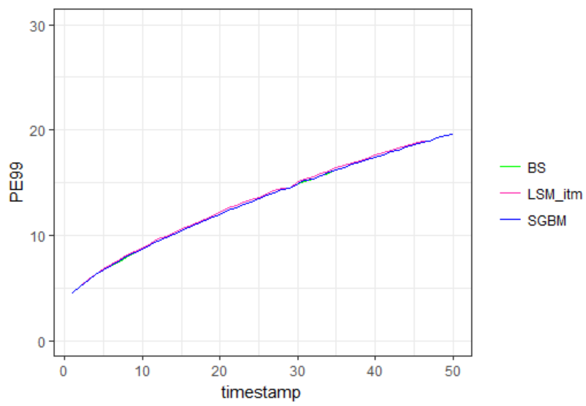


Figure 5.8: $PE99$ of the LSM_{itm} (with pink), SGBM (with blue) and of the BS (with green) for a European put option, with the parameters: $S_0 = 40$, $K = 42$, $r = 0.06$, $\sigma = 0.2$, $T = 2$ years. The calculation is based on 100000 simulated paths. (The x-axis represents the time, even though the number of the certain column of the simulated matrix is there.)

To plot the tail of the exposures, let's try to depict the $99,99\%$ quantile of the values, i.e. the $PE99.99(t)$ values through the lifetime of the option.

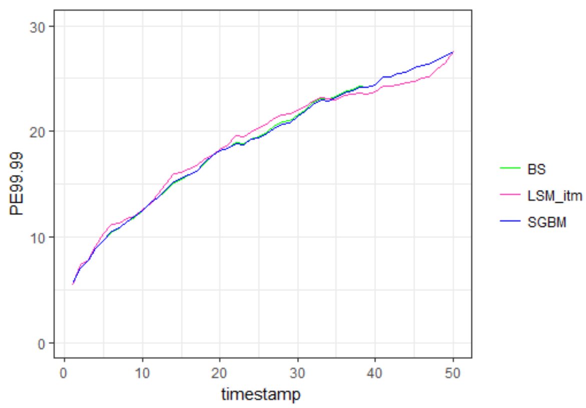


Figure 5.9: $PE99.99$ values according to the exposures plotted in Figure 5.6. (The x-axis represents the time, even though the number of the certain column of the simulated matrix is there.)

Figure 5.9 shows the discrepancies of the exposures in the 99.99% confidence level. The pink line, which represents the LSM, differs most from the other lines. From this figure we can conclude, that the SGBM is a more accurate estimation, than the LSM.

The three $PE99.99(t)$ values end at the same point, and these values equal with the Peak PE values in all cases.

$$PeakPE_{BS} = PE_{LSM_{itm}} = PeakPE_{SGBM} = 26.45964$$

Assume, that we are in a short position. Then the upper quantile will be the 1% quantile of the future distribution. The LSM does not give an estimation for the smallest values, that is why the $PE1(t)$ values of the LSM are going to be constant zero.

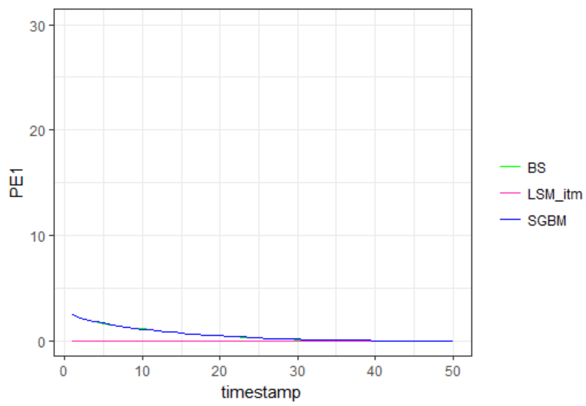


Figure 5.10: $PE1$ values of the BS, LSM_{itm} and SGBM for the same European put option as above. (The x-axis represents the time, even though the number of the certain column of the simulated matrix is there.)

Figure 5.10 illustrates the $PE1(t)$ values through the lifetime of the option. We can see, that the SGBM fits well to the BS, and the LSM is constant zero as we expected. The conclusion is, that if we have a short position of an option in our portfolio, and we would like to give the upper quantiles of the potential exposures, then the use of the SGBM is recommended in contrast to the LSM.

Chapter 6

Future exposure of American Options

This chapter gives a description of the implementation in R of the above introduced American option pricing methods. As in these simulation-based algorithms always the first step is to simulate N paths into an NxM matrix (in our case from Geometric Brownian Motion). Here M means the number of the exercise dates, so we divide the $[t_0, T]$ interval to equal distances, to get these $t_0 \leq t_1 \leq \ldots \leq t_M = T$ exercise dates. We will approximate the American option's price with the corresponding Bermudan option, with a discrete set of exercise dates: $\{t_m : m = 1, \ldots, M\}$.

6.1 Application of LSM for American Options

After we have the $N * M$ simulated values, we have to determine the pay-offs at the time of maturity t_M. For a put option these pay-offs are $V_{t_M}(S_{t_M}(n)) = \max(K - S_{t_M}(n)\,;\,0)$ for $n = 1, \ldots, N$. When we have the option values at t_M, we have to estimate the values of continuing the option at t_{M-1} with a regression. X will get those simulated $S_{t_{M-1}}(n)$ values, which are in ITM position, and Y will be the corresponding simulated values at t_M discounted back to t_{M-1}. The received values from the regression are the conditional expected values of continuation at t_{M-1}. We have to move backwards with the algorithm from the time of maturity till the initial time step.

An important step of the algorithm is, that we should use the estimated conditional expected values only once to avoid a higher error. That means, if at some time along a certain path the value of continuation is greater, than the immediate exercise value, then we do not use the continuation value in the following regression, but we use the immediate exercise value from the next time step discounted back to the following date. This is because, then we do not have to use this estimated value twice, and our result will not be so inaccurate.

This means in practice, that if $CF \in \mathbb{R}_+^{NxM}$ indicates the cash flows of immediate exercise based on the simulated matrix, and at a t time along p path the conditional expected value is greater or equal than the immediate exercise value, i.e.

$$Q_t(S_t(p)) \geq CF_t(S_t(p)), \tag{6.1}$$

then we need to use the immediate cash flow from $t + 1$ discounted back to $t - 1$ as the pth

element of Y in the calculation of the continuation value at time $t-1$. In the

$$Q_t(S_t(p)) < CF_t(S_t(p)) \tag{6.2}$$

case we use the immediate exercise value at t discounted back to $t-1$ as the corresponding element of Y. That means, that after every regression we have to make a decision along each path about what to write into the corresponding element of the next dependent variable. The *ifelse* function in R makes it possible to implement this step for the whole column.

We are moving backwards in the algorithm with these comparison steps. We can solve it with a *for* cycle from $M-1$ to 1. In the regression of the LSM the out-of-the-money paths are not used, we have to give 0 values to its continuation values. Note again, that in the American option pricing version of the LSM we do not need the values of continuation in the out-of-the-money positions. In that case we do not have to determine whether we should exercise the option or not, because the out-of-the-money expression exactly means that the immediate exercise is valueless.

If we are interested in the future exposure of an American option, then we have to determine the optimal stopping strategy first. If we have an optimal exercise date along a path, then the value of the option along this path will be the immediate exercise value at the optimal exercise date, and before this date the option's value will be the value of continuation. After we exercised the option, it will be valueless.

In the American option case we do not have an analytic benchmark like the Black-Scholes formula was in the European case, but we can compare our exposures with the BS curve now too. It is clear, that in a case of an American put option the received exposures do not have to fit in the BS curve, but we know that an American put option has to be more expensive (or the same) before the expiration, than the corresponding European put option. We have come to this claim, because the time value of a put option can be negative, and then it can be worthwhile to exercise the option before the time of maturity and with an American option we have the right to do that. (If we consider an American call option on a non-dividend stock, it will not be exercised early, it follows, that its price will equal with the equivalent European call's.)

After implementing the Least-squares Monte Carlo method for American options, we can plot the future exposures for an American option with the same parameters as we used in the European option chapter. In Figure 6.1 we can see the result of the LSM for an American put option illustrated with red. The lifetime of the option is 2 years, and the figure shows the future exposures at 6 different times gradually getting closer to the expiration date. As we excepted the red curve, which is the approximated value of an American put option, never goes under the green curve, which represents the Black Scholes price of the European put option with the same parameters. After we exercised the option along a path according to the optimal stopping strategy, the exposure of the option will not be plotted in the figures anymore.

We can also notice from Figure 6.1, that at the earlier dates the difference between the American put option price and the European put option price is bigger, and as we getting closer to the time of maturity the difference decreases. This is logical, because as we get closer to the expiration we will have less opportunities to exercise the American option. It is obvious, that the American option's price equals with the European option's price at the maturity time, and both equals with the option's immediate exercise value at T. (see in Figure 6.1f)

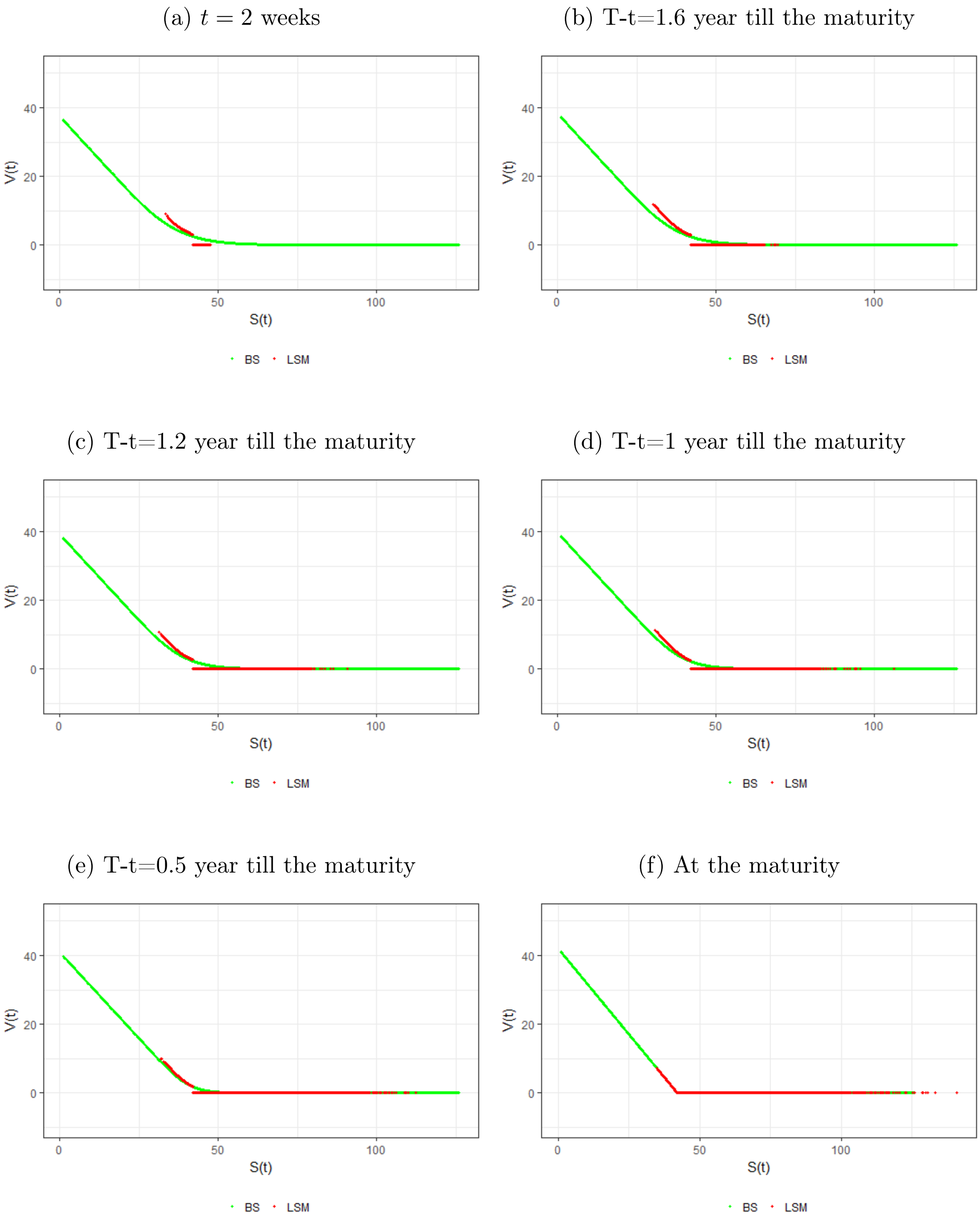


Figure 6.1: The BS curve (with green), and the future exposures by using the LSM with the basis functions X, X^2 and X^3 for an American put option (with red). The parameters of the option are: $S_0 = 40$, $K = 42$, $r = 0.06$, $\sigma = 0.2$, $T = 2$ years, and the algorithm is based on 50 exercise dates and 100000 simulated paths from GBM.

6.2 Application of SGBM for American Options

To implement the SGBM for American options we will work on the same pricing framework as in the LSM. We have to change the time of the regression, so we will take the X and Y variables from the following time step. We have the same finite set of exercise dates and then we do the regression in the nearest exercise date after t_m, denote it with t_{m+1}, when we are interested in the conditional expected values from continuation at t_m.

The bundling step will depend on the current time step (t_m). The used basis functions are $\Phi_1(S_{t_{m+1}}) = S_{t_{m+1}}$, $\Phi_2(S_{t_{m+1}}) = (S_{t_{m+1}})^2$, $\Phi_3(S_{t_{m+1}}) = (S_{t_{m+1}})^3$, and after we got the regression coefficients, we have to calculate the conditional expected values for each basis functions. We have to give closed forms for the conditional moments of a lognormal distribution, like we did in the European option case.

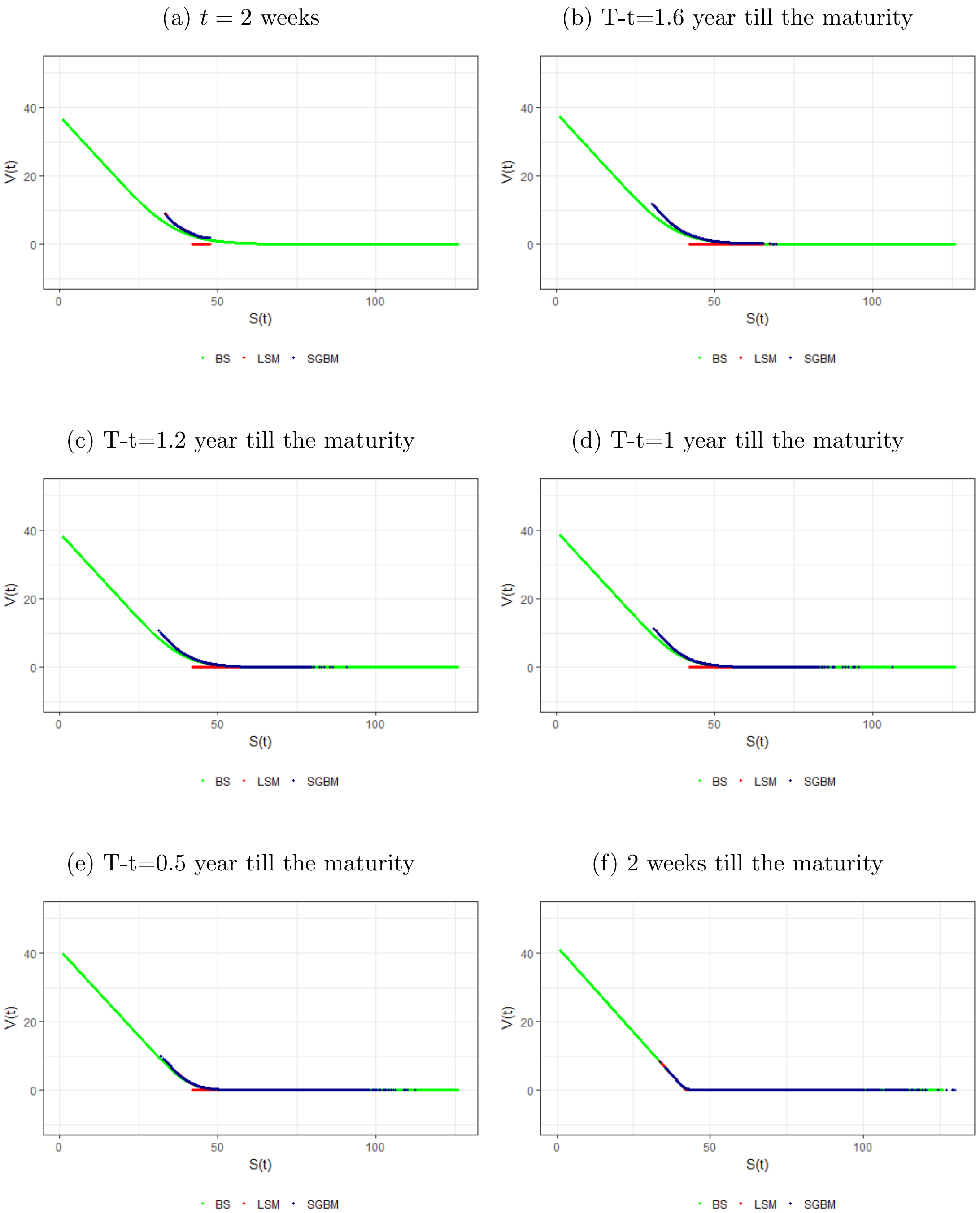
$$
\begin{aligned}
\mathbb{E}(\Phi_1(S_{t_{m+1}}) \,|\, S_{t_m} = S_{t_m}(n)) =& \mathbb{E}(S_{t_{m+1}} \,|\, S_{t_m} = S_{t_m}(n)) = S_{t_m}(n)\, e^{r(t_{m+1}-t_m)} \\
\mathbb{E}(\Phi_2(S_{t_{m+1}}) \,|\, S_{t_m} = S_{t_m}(n)) =& \mathbb{E}((S_{t_{m+1}})^2 \,|\, S_{t_m} = S_{t_m}(n)) = (S_{t_m}(n))^2\, e^{(2r+\sigma^2)(t_{m+1}-t_m)} \\
\mathbb{E}(\Phi_3(S_{t_{m+1}}) \,|\, S_{t_m} = S_{t_m}(n)) =& \mathbb{E}((S_{t_{m+1}})^3 \,|\, S_{t_m} = S_{t_m}(n)) = (S_{t_m}(n))^3\, e^{(3r+3\sigma^2)(t_{m+1}-t_m)}
\end{aligned}
\tag{6.3}
$$

When we have these values for every bundle, we have to substitute them to the polynomial, which was introduced in Chapter 4 as equation 4.5.

We repeat this procedure M times, where M is the number of the assumed exercise dates. To determine the optimal stopping strategy, we have to compare the immediate exercise values with the values of continuing the option at every intermediate exercise date along each path. Consequently the future exposure is going to be the value of continuation till the optimal exercise date along a path, it is going to be the immediate exercise value at the optimal exercise date, and after we exercised the option it will be valueless along the path. We made the bundling with k-means clustering algorithm and we set up the number of clusters to 10.

If we plot future exposures received by the SGBM for the same American put option to the same figure as we plotted the LSM, we can see, that the SGBM estimates the OTM values too, while the LSM not. (see in Figure 6.2).

The two methods differ from each other in the way, as they estimate the value of continuation, and it follows, that the optimal exercising strategies can be different too. For example in Figure 6.2f we can see, that along some paths we exercised the option earlier with the SGBM, while with the LSM we did not.



(a) $t = 2$ weeks

(b) T-t=1.6 year till the maturity

(c) T-t=1.2 year till the maturity

(d) T-t=1 year till the maturity

(e) T-t=0.5 year till the maturity

(f) 2 weeks till the maturity

Figure 6.2: Future exposures of 2-year American put option calculated with 2 different methods, the SGBM is plotted with blue, and the LSM is plotted with red. The green curve indicates the Black-Scholes prices of the European put option with the same parameters.The used parameters are: $S_0 = 40$, $K = 42$, $r = 0.06$, $\sigma = 0.2$, 50 exercise dates and 100000 simulated paths.

6.3 Potential Exposure

Figure 6.3 illustrates the 99% quantile of the future exposures based on the LSM and the SGBM methods. The two metrics almost equal through the lifetime of the option. We can see, that the shape of the $PE99(t)$ in the American put option case differs from the shape of the $PE99(t)$in the European option case.

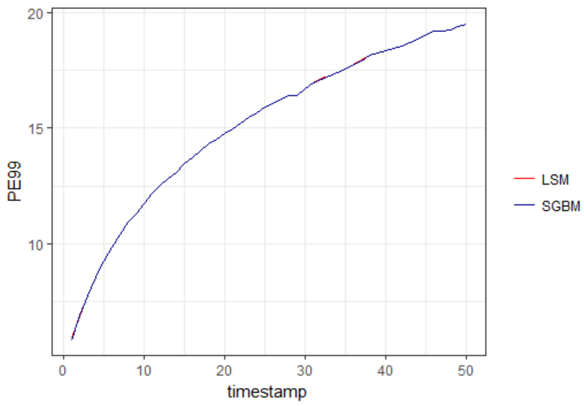


Figure 6.3: $PE99$ values of a 2-year American option on a non-dividend stock, with the parameters: $S_0 = 40$, $K = 42$, $r = 0.06$, $\sigma = 0.2$, 50 exercise dates and 100000 simulated paths. The red line represents the $PE99(t)$ of the LSM, and the blue line represents the $PE99(t)$ of the SGBM. (The x-axis represents the time, even though the number of the certain column of the simulated matrix is there.)

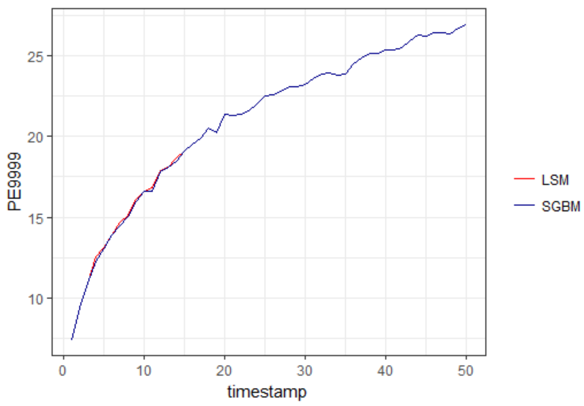
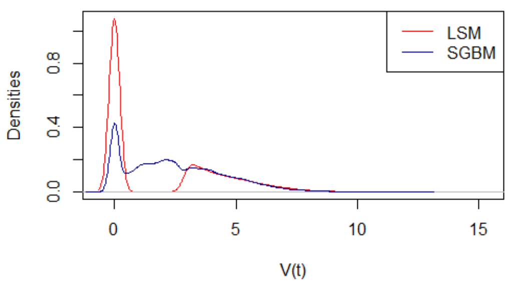


Figure 6.4: $PE99.99$ values for the same methods and for the same American put option, which is illustrated in Figure 6.3. (The x-axis represents the time, even though the number of the certain column of the simulated matrix is there.)

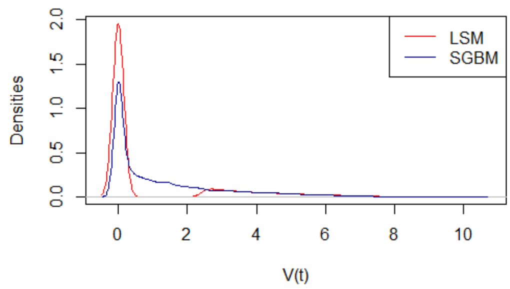
When we plot the 99.99% quantile of the future exposures for the two techniques in the American option case, we can notice a little more difference. (see in Figure 6.4)

From a counterparty perspective we have to count on to the exposures that paths, along which the option is already exercised. The value of the option is zero at a t time, if it has been exercised earlier. We can plot the density functions, which represent the value of the option from a counterparty point of view. (Figure 6.5)

(a) $T - t = 0.5$ year till the maturity



(b) $T - t = 1$ year till the maturity



(c) $T - t = 1.5$ year till the maturity

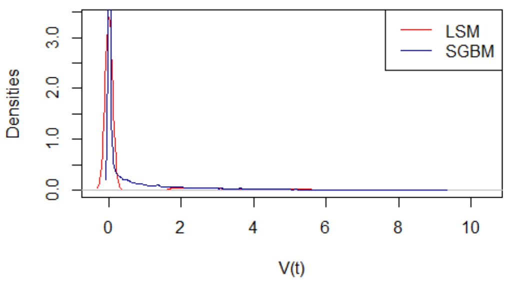


Figure 6.5: Density functions of the LSM, and SGBM for a 2-year American put option, with the strike price $K = 42$, the initial stock price $S_0 = 40$, the interest rate $r = 0.06$, and the volatility $\sigma = 0.2$.

We can see in Figure 6.5, that the LSM results a bimodal distribution. It is caused by the fact, that the LSM does not estimate the continuation values along the OTM paths. If we would like to give an estimation about the future value of the option, the SGBM is obviously a better choice.

Chapter 7

Conclusion

I have implemented two simulation-based American option pricing methods, the LSM and the SGBM. We can say, that the LSM is a simple yet powerful method. In this thesis we showed the Stochastic Grid Bundling Method, which we get by making the LSM more complex. With the SGBM we can give more accurate estimation for the future prices in the same Monte Carlo framework.

We analyzed these two methods from a risk management perspective. We wanted to decide, that which method gives better results of calculating counterparty risk metrics like $PE95$ and $PE99$. We have showed, that involving the bundling step makes it possible to estimate the option values at a certain time for every moneyness level, and not just for the ITM positions.

First we tried the two methods for a European put option. We illustrated the comparisons with plotting the future exposures calculated with the two methods for different dates during the lifetime of the option. Comparing to the Black-Scholes price we saw, that the SGBM gives a more accurate estimation. If we would like to predict the counterparty risk of the position, then to give an upper quantile of the value of the position is important. Plotting the 99% quantile of the future exposures, we showed, that the SGBM gives a better prediction. When we plotted the density functions of the future distributions we came to the same conclusion.

When we examined the methods according to the pricing of an American put option, the most spectacular difference was, that the SGBM estimates the option values to the OTM paths too. This is a advantage for example in that case, when we are in a short position. Then the $PE1$ value would play the role of the $PE99$, and it is constant zero in the LSM case.

In essence we can conclude, that the confidence level exposure (PE_α), the expected exposure and the density functions of the future distributions all suggest, that the use of the SGBM will improve the results.

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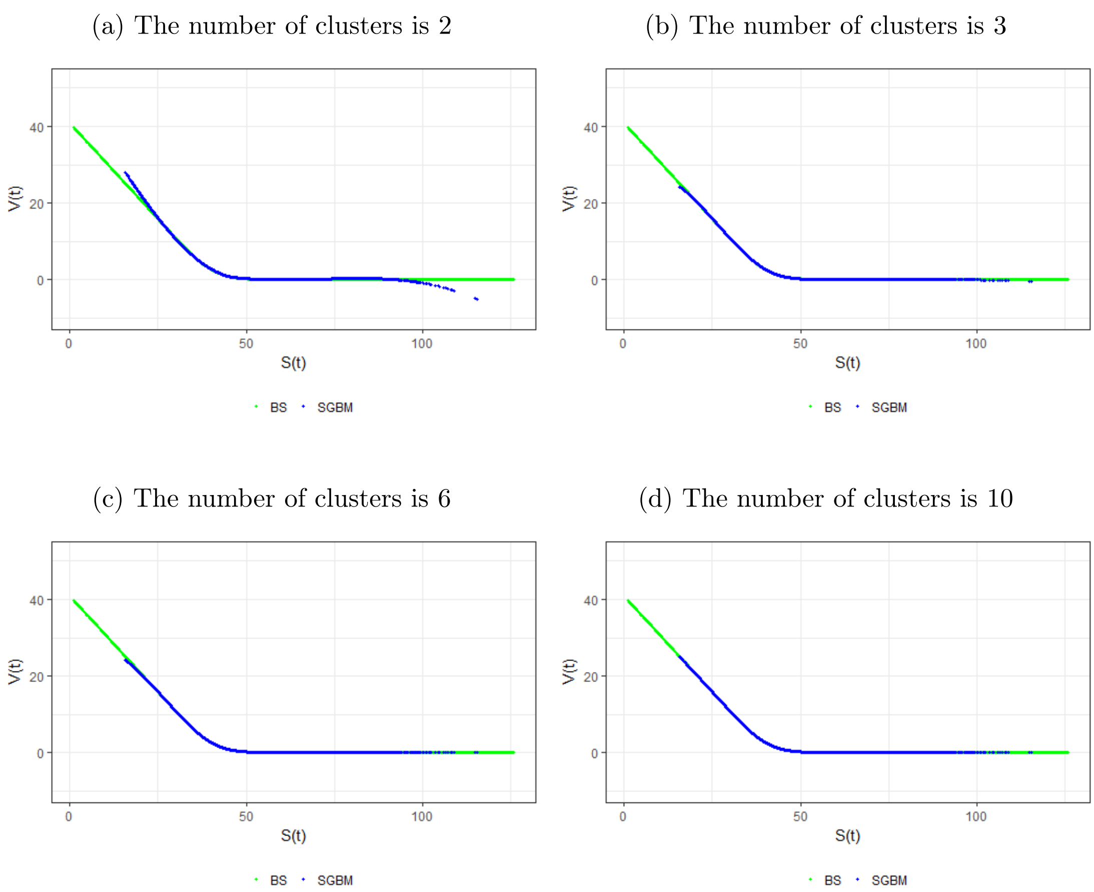
Appendix

A Figures



(a) The number of clusters is 2

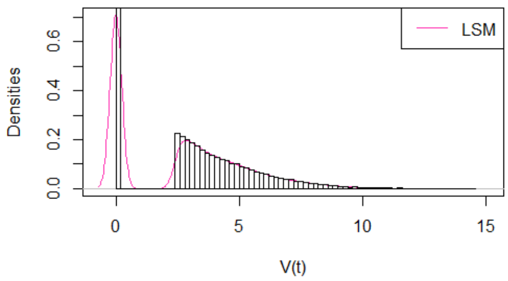
(b) The number of clusters is 3

(c) The number of clusters is 6

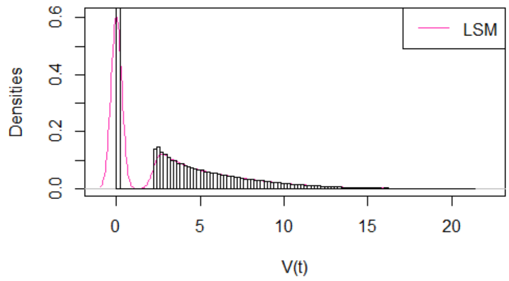
(d) The number of clusters is 10

Figure 1: SGBM with different numbers of clusters for a 2-year European put option, when half a year left till maturity ($T - t = 0.5$ year). The strike price of the option is 42, the stock price at the initial time step was 40, the risk-free interest rate is 6%, and the volatility is 0.2. The method was calculated with 100000 simulated paths.

(a) $T - t = 0.5$ year till the maturity



(b) $T - t = 1$ year till the maturity



(c) $T - t = 1.5$ year till the maturity

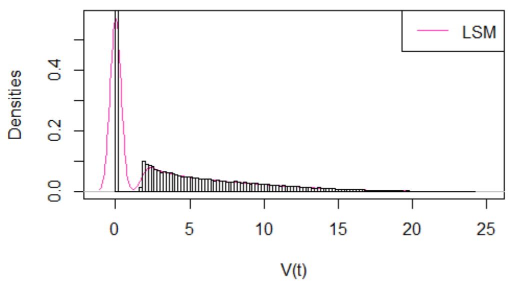


Figure 2: Histograms and density functions of LSM_{itm} for a 2-year European put option, with the strike price $K = 42$, the initial stock price $S_0 = 40$, the interest rate $r = 0.06$, and the volatility $\sigma = 0.2$. The calculation is based on 100000 simulated paths.

B Recursive bifurcation

The aim is to cluster the d-dimensional grid points $\{S_{t_m}(1), \dots, S_{t_m}(N)\}$ to 2^d non-overlapping sets. First we need to compute the mean of the grid points for every dimension, i.e.

$$\mu_\delta = \frac{1}{N} \sum_{n=1}^{N} S_{t_m}^{\delta}(n) \, , \; \delta = 1, \dots, d. \tag{1}$$

The 2^d partitions are gained by separating the grid points by their means for every d dimension

$$A_\delta = \left\{ S_{t_m}(n) : S_{t_m}^{\delta}(n) > \mu_\delta \, , \; n = 1, \dots, N \right\}$$

$$\overline{A_\delta} = \left\{ S_{t_m}(n) : S_{t_m}^{\delta}(n) \leq \mu_\delta \, , \; n = 1, \dots, N \right\}$$

and then getting the intersections of any possible d sets as:

$$\mathcal{B}_{t_m}(1) = A_1 \cap A_2 \cap \ldots \cap A_d,$$

$$\mathcal{B}_{t_m}(2) = \overline{A_1} \cap A_2 \cap \ldots \cap A_d,$$

$$\vdots$$

$$\mathcal{B}_{t_m}(2^d) = \overline{A_1} \cap \overline{A_2} \cap \ldots \cap \overline{A_d},$$

We can repeat this partitioning for the obtained bundles, and after p iterations we will gain $(2^d)^p$ non-overlapping sets.

For instance to the 2 dimensional case the given set is halved along the two dimensions, resulting in four non-overlapping sets. In the second iteration we repeat the process for every set, resulting in 16 clusters, and for the third iteration we get 64 partitions.