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Study on option price of risk target indices

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A **szakdolgozat** szerzőjeként fegyelmi felelősségem tudatában kijelentem, hogy a dolgozatom önálló szellemi alkotásom, abban a hivatkozások és idézések standard szabályait következetesen alkalmaztam, mások által írt részeket a megfelelő idézés nélkül nem használtam fel.

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a hallgató aláírása

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Contents

Chapter 1

Introduction

In my thesis I am going to present a risk management strategy which is getting more and more popular nowadays among investors. Volatility describes the instability of the market, therefore it can measure the riskiness of a financial product. Volatility targeting is a systematic strategy that helps control the risk of an investment by obtaining protection based on a method which uses the volatility to measure the risk. The strategy dynamically invests in the risky asset and in the risk-free asset rebalancing the portfolio to keep the ex-ante volatility of the portfolio at a constant predefined target level.

The advantage of the Volatility Target Strategy (also called as Risk Control Strategy) is based on that it can protect the investors from high market volatility movements. When a stressed period comes, the volatility of equities increases which implies an increase in the values of options linked to them. During the global financial crises in 2008, insurance companies suffered significant losses due to a huge increase in values of equity-linked products and poor risk management strategies. After investors had become more aware of the importance of managing risk, companies started to seek new strategies which can control the volatility of portfolios and therefore make them less affected by falling markets.

The benefits of the volatility targeting are the most significant for equities which show the strongest volatility clustering, fat tails and a negative relationship between stock returns and volatility. S&P 500 is a stock market index that measures the performance of 500 capitalization-weighted US stocks and it is the most commonly followed equity index. In the thesis I am going to focus on the S&P 500 Risk Control Index family as an example for practical application and the main theoretical question of the study is how to determine the price of an option on a Volatility Target index.

The thesis is organized as follows: In Chapter 2, I am going to present the concept of the Volatility Target Strategy and illustrate the real-world usage by highlighting the S&P 500 Risk Control Indices. In Chapter 3, after a brief technical preliminaries where the main definitions, theorems and financial models are described which are used later on in my thesis, I am going to present the precise mathematical framework and show the closed-form solution of the price of an European option linked to the Volatility Target Strategy portfolio. In Chapter 4, I am going to go into the technical and theoretical details of the implementation and provide a summary about the challenges during the simulations and the numerical results of the tests performed. Finally we conclude in Chapter 5.

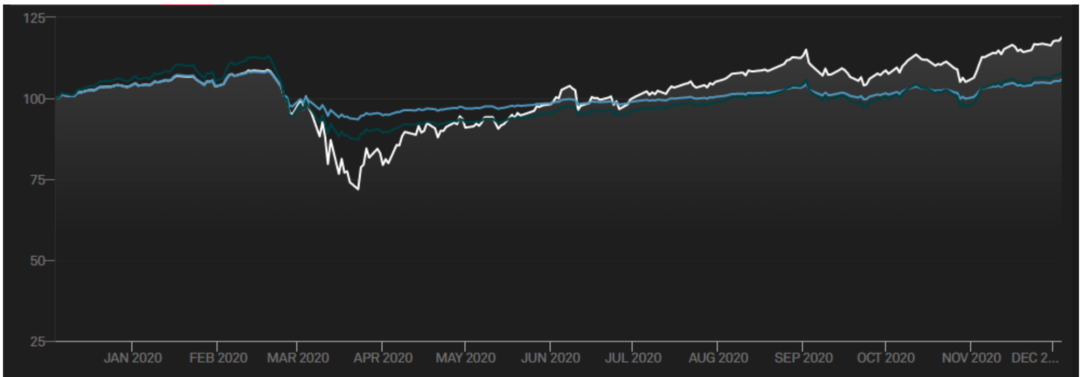
Chapter 2

Introducing the Volatility Target Strategy

During the global financial crisis in 2008 financial markets became significantly more volatile which was not present in previous years (2004-2007). This had an affect on different branches of the markets. As a real example, insurance institutions were heavily exposed to the increase of the equity volatility which caused a rise in the value of equity linked products. Variable annuities are equity linked pension contracts which values vary based on the performance of the underlying mutual funds. The benefits gained by the policyholder are exposed to the performance of these underlying funds but also there are different types of guarantee riders in order to protect the investment. These options can be bought for an extra cost for the protection but it contains, among several other factors of risk, also financial risk from the insurer's point of view. As a consequence, insurance institutions suffered unexpected losses during the crisis due to the increase in the value of the equity linked products and guarantees [1, 2, 3].

Following the crisis, investors started to look for such opportunities, which could protect their investments – in term of insurance companies such strategy which protects them from a similarly huge loss – in a volatile market environment. There are different types of dynamic asset allocation strategies which respond to the current market trends. The concept of these strategies is to shift dynamically between risk-free and risky assets to maintain a portfolio with protection against market movements. One of the most popular strategies introduced in multi asset portfolios and structured products was the Volatility Target Strategy (VTS) which determines this dynamic allocation by using the realized volatility of the underlying risky asset to maintain a controlled volatility of the portfolio. When the realized volatility of the risky asset is high, the strategy protects the investor by allocating partially into the risky asset and also partially into the risk-free asset, while the realized volatility is low, the strategy can invest completely into the risky asset. Here, the approach assumes that market volatilities are good indicators for asset allocation decisions and the aim is to maintain a portfolio with a risk level as independent of market volatilities as possible. In addition, there are other strategies, e.g. Constant Proportion Portfolio Insurance (CPPI) which also allocates dynamically between the risk-free and the risky assets, but the concept is based on market conditions and daily performances of the portfolio [9]. In the following, I am going to focus on the Volatility Target Strategy.

The S&P 500 lost 56% of its value starting with October 2007 within half a year. The correlation between asset classes increased significantly which implied the failure of portfolio constructions based on the backward-looking correlation model, as the expected diversification benefit was eliminated. S&P Dow Jones Indices has developed a risk control framework through a series of risk control indices which features the measurement of risk based on volatility to help control risk at a predefined level. These indices make it available to gain exposure to the constituents of the underling index with a proportion which dynamically changes [4]. It gives the opportunity for the investor to choose the target level based on self-preference of appetite for risk. In the next Figures 2.1 and 2.2, the performances of S&P 500 Index and S&P 500 Risk Control Indices are compared in a one-year and a three-year long perspective.



Index Name	Index Level	1 Yr Returns
S&P 500 Mar 04, 1957	3,699.12	18.84%
S&P 500 PR Daily Risk Control 10% Index (USD) Apr 15, 2019	774.21	5.79%
S&P 500 PR Daily Risk Control 18% Index (USD) Apr 15, 2019	1,214.34	8.05%

Figure 2.1: Source: S&P Dow Jones Indices LLC. Data as of December 4, 2020. Index sets to 100 on December 5, 2019. Index performance based on price return in USD. Daily rebalancing frequency.

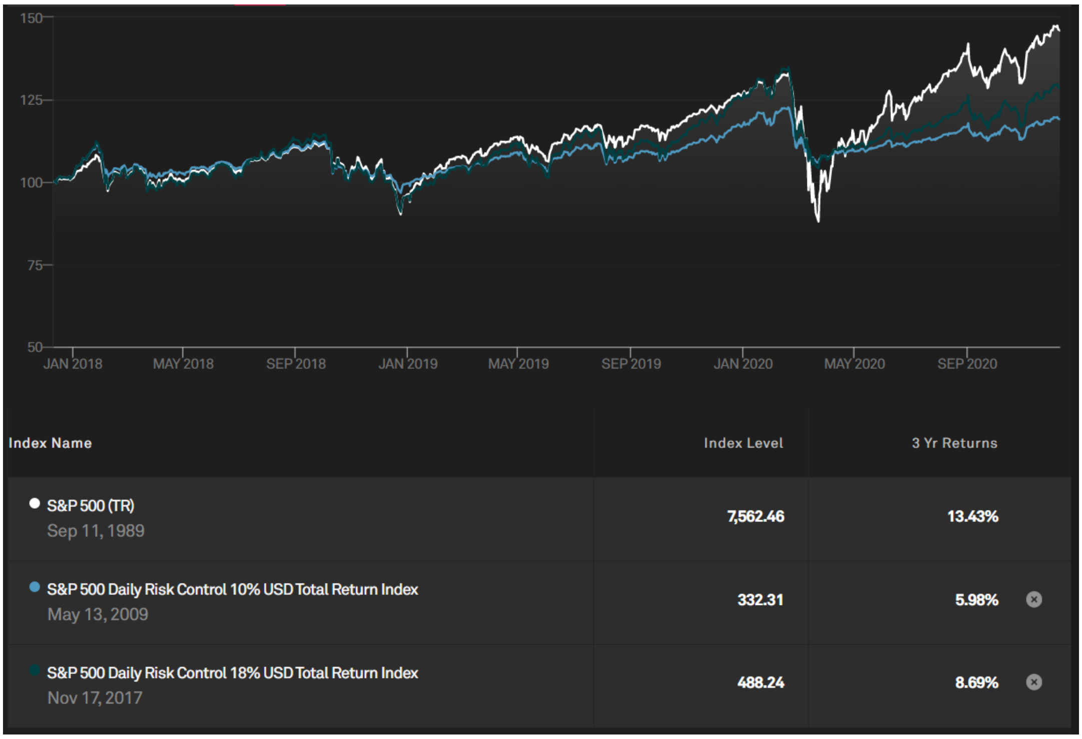


Figure 2.2: Source: S&P Dow Jones Indices LLC. Data as of December 12, 2020. Index sets to 100 on December 13, 2017. Index performance based on total return in USD. Daily rebalancing frequency.

The S&P 500 Risk Control Indices use either the S&P 500 Total Return Index or the S&P 500 Index (also referred as the price return index) as the underlying index. The difference between them is the total return index includes dividends reinvested which implies higher returns over the long-term compared to returns of the S&P 500 Index. It can be seen on the graphs above that over the past one year the performance of S&P 500 Index was more volatile than the other two risk controlled. The less volatile index was the Risk Control 10% Index, the one with the lower vol target level, in case of both perspectives. The risk control indices with target level less than 15% typically underperform the S&P 500 during a bull market, which is expected as the long-term volatility of S&P 500 Index is around 14%. This is compensated for their comparatively higher returns during the bear markets. The falling performances of the risk control indices are less relevant compared to the drop in the performance of the pure risky index in March, 2020. This is due to the lower exposure in risk control indices to the underlying risky asset as the volatility target level is lower than the volatility of the underlying index. It is interesting to look at the behavior of the Risk Control 18% Index, where the target level is higher than the long-term volatility of S&P 500. In the first two months of the year, the risk control index overperforms the S&P 500 Index, and its first drop is even higher than the drop in S&P 500 Index at the end of February, 2020. However, the volatility of the S&P 500 Index (VIX) increased in that stressed time period. This can explain why the performance of the risk control index gets moderate and the drops are lower in March 2020, compared to the drops in the performance of S&P 500.

In those cases, when the volatility of the underlying index falls below the target level of volatility, the exposure to the index increases and cash needs to be borrowed to pay for the overexposure to the underlying index. In this case, the exposure to the risky asset can be maximized in order to prevent borrowing too much from the risk-free asset. S&P 500 Risk Control Indices usually set 100% or 150% as maximum leverage, for example, the S&P 500 Daily Risk Control 18% Index permits 150% as the maximum leverage. The return of the index consists of the return on the position in the underlying index and the interest cost or gain depending on whether the position is leveraged or deleveraged. The return on the cash component is calculated using interest rates. Usually 3-months LIBOR and Overnight USD LIBOR, sometimes Rolling 3-months USD LIBOR are used. To measure the risk, these indices calculate historical volatility exponentially weighting the square of the underlying index's natural-log returns [4, 5].

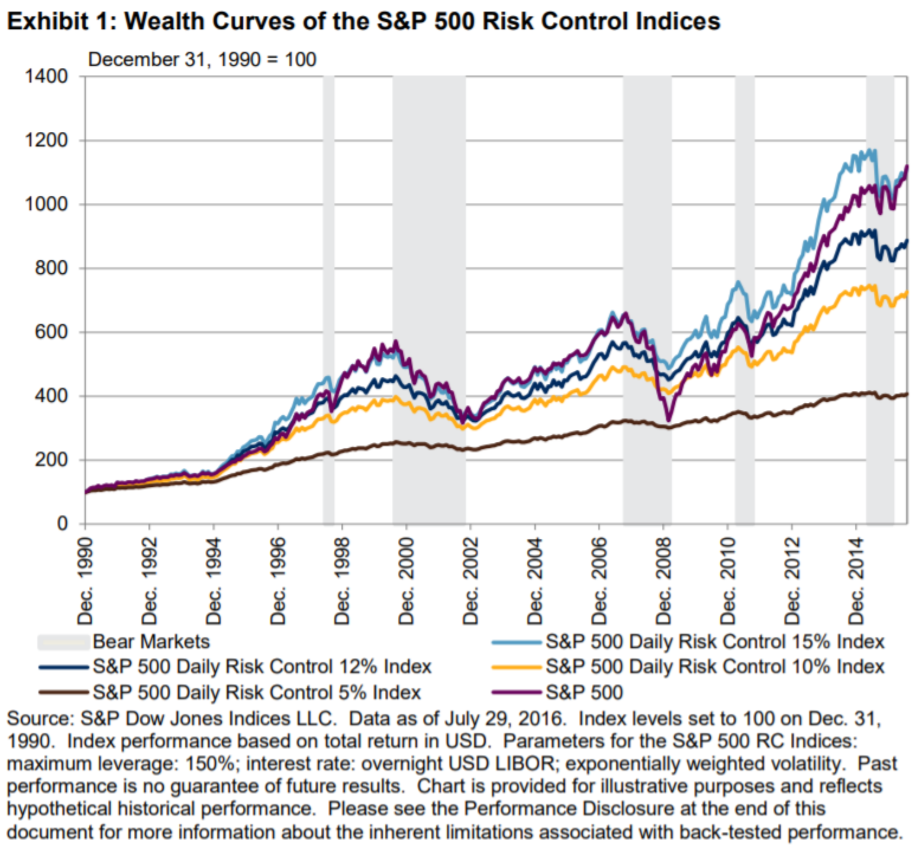


Figure 2.3: Source: S&P Dow Jones Indices LLC. Limiting Risk Exposure With S&P Risk Control Indices, 2016 [4]. Performance of S&P 500 Index and S&P 500 Risk Control Indices.

There are papers examining long-time performance of the risk control indices. In an S&P 500 study [4] Figure 2.3 is presented, reflecting hypothetical historical performances formed back-tested. We can notice that the risk control indices can overperform the S&P 500 Index, while during a bear market their exposure to the underlying index is controlled which results in a moderated drawdown in their performance.

S&P provides a detailed product description of risk control indices [5], please see the methodology presented in Section 3.4.

Furthermore, S&P also developed a new generation of risk control indices called Risk Control Indices 2.0 [7]. These strategies replace the risk-free cash portion of the investment with a liquid bond index called the reserve asset, taking into consideration the volatility of both part of the allocation (the underlying index and bond index) and the correlation between them. It takes the S&P U.S. Treasury Bond Current 5-Year Index as a bond index underling. The dynamic asset allocation works based on taking advantage of the negative relationship between stock and bond market performance. The negative correlation can contribute to lower volatility, enabling the index to maintain higher equity allocation levels than if it were using only cash. Moreover, the yield-to-maturity of the reserve asset tends to be higher than the cash money market rate. Also, allocation to the reserve asset may provide capital gains during periods of stressed market environment due to "flight to safety/quality" phenomenon, which means increasing volatility makes the investors' risk tolerance decreases which – taking under consideration even other factors like increasing inability of diversification – results in higher risk premiums [10].

As the volatility of a VTS portfolio is kept under control, the price of derivatives linked to them are less expensive compared to purely equity linked derivatives. This makes them attractive to insurance companies which usually offer variable annuity guarantees based on volatility target portfolios. By controlling the volatility, the value of guarantees is reduced which protects the insurer, but on the other hand it also means a protection to the policyholder from bearish markets. The S&P Dow Jones also has a detailed documentation on portfolio construction and performance analysis of Variable Annuities linked to risk control indices [8].

Observations show that the application of targeting constant volatility has other benefits which I just mention briefly. The strategy helps reduce tail risk, improve skewness and improves Sharpe ratio [1] by a more certain volatility forecasting of the risk control index [1]. Another advantage of a dynamic asset allocation portfolio is the diversification as the portfolio may be exposed to equities, fixed income, derivatives, index funds, mutual funds or currencies.

[1] Average return earned in excess of the risk-free rate per unit of risk. A portfolio with higher Sharpe ratio is considered to be superior to a portfolio with lower Sharpe ratio.

For equities, the success of the strategy depends on being able to take into account the negative correlation between the return and the volatility of the risky asset [11]. For this purpose, Heston model seems to be a good choice for modeling the dynamics of the underlying risky asset. Also, it can capture the attribution of fat tails which is also observable in the distribution of the returns of the risky asset. However one insufficiency of the Heston model is not being able to capture volatility clustering [14]. There are models, like GARCH model [2] and its extensions, which can capture this feature when volatility forecasting [1], but I am going to follow the S&P 500 risk control methodology which uses the exponentially weighted moving average (EWMA) method for historical volatility calculation in a way that it assigns higher weights to the most recent observations.

In the next chapter, I am going to build up the framework of the strategy following Ref. [17] and discuss the question of pricing an European option on a risk control index.

[2]A statistical model for time series data that describes the variance of the current error term as a function of the actual values of the previous time periods' error terms. GARCH models are commonly used for modeling volatility in financial markets as they can capture and model several stylized facts. Exponentially weighted moving average (EWMA) is mentioned as a possible alternative model.

Chapter 3

Theoretical framework of VTS portfolios

The theoretical framework of the Volatility Target Strategy discussed here is limited to a single target volatility level which is independent of market interest rates. The portfolio allocates between a risk-free asset and a single risky asset. We assume that the interest rate is constant, excluding the discussion of pricing the product under a hybrid stochastic volatility-stochastic interest rate model. This chapter was written based on Ref. [17, 18].

Let $(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{t\geq 0}, \mathbb{P})$ be a filtered, complete probability space where $\{\mathcal{F}(t)\}_{t\geq 0}$ is right-continuous filtration and $\mathbb{P}$ denotes the real-world measure on $(\Omega, \mathcal{F})$. The market consists two investment opportunities for investors, a riskless asset $\{B(t)\}_{t\geq 0}$, referred as bond [1] further on in the thesis and a risky underlying asset $\{S(t)\}_{t\geq 0}$, called as stock or share. The stochastic processes of the assets evolve according to the following:

$$\begin{aligned} dB(t) &= rB(t)dt, \\ dS(t) &= S(t)(\mu(t)dt + \sigma(t)dW(t)), \end{aligned} \tag{3.1}$$

where $W(t)$ is an $\mathcal{F}(t)$-adapted Wiener-process, r is a positive constant representing the risk-free rate, $\mu(t)$ and $\sigma(t)$ are $\mathcal{F}^W$-adapted stochastic processes, where $\mathcal{F}^W$ is the natural filtration generated by the Wiener process.

Before going into the details of the Volatility Target Strategy framework, I would like to mention the most important preliminaries regarding the topic. In the next section, I am going to summarize some necessary financial and mathematical expressions, theorems and models.

[1] I use the common 'bond' appellation following Ref. [17], however $\{B(t)\}_{t\geq 0}$ is actually a money market deposit account.

3.1 Preliminaries

The following stochastic calculus definitions and theorems are based on Ref. [12].

Definition 3.1.1 (Itô process). An $X(t)$ Itô process is a stochastic process of the form

$$X_t = X_0 + \int_0^t a(s, X_s)ds + \int_0^t b(s, X_s)dW_s \tag{3.2}$$

where X_0 is $\mathcal{F}_0$-measurable, $a(t, X_t)$ and $b(t, X_t)$ are $\mathcal{F}_t$-adapted processes, and $\int_0^T |a(t, X_t)|dt < \infty$ and $\int_0^T b^2(t, X_t)dt < \infty$ for $0 \leq t \leq T$. This is usually used in the following shortened form

$$dX(t) = a(t, X_t)dt + b(t, X_t)dW_t. \tag{3.3}$$

Theorem 3.1.1 (Itô's formula). *Let X_t be an Itô process described with 3.3 on $[0, T]$. If $f(t, x)$ is a continuous function with continuous partial derivatives, then the stochastic differential of the process $Y(t) = f(t, X_t)$ exists and it is given by*

$$\begin{aligned} df(t, X_t) &= \frac{\partial f}{\partial t}(t, X_t)dt + \frac{\partial f}{\partial x}(t, X_t)dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, X_t)d[X]_t \\ &= \left(\frac{\partial f}{\partial t}(t, X_t) + \frac{\partial f}{\partial x}(t, X_t)a(t, X_t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, X_t)b^2(t, X_t)\right)dt + \frac{\partial f}{\partial x}(t, X_t)b(t, X_t)dW_t, \end{aligned} \tag{3.4}$$

where $[X]_t$ is the quadratic variation of X_t. As $f(t, X_t)$ is also an Itô process, we can give the integral form of this process the same way as it is given for X_t in 3.2.

We can use the proposition called Itô Isometry, when we would like to calculate the variance of random variables given by Itô integrals.

Proposition 3.1.2 (Itô Isometry). *Suppose that the stochastic process $\varphi(t, \omega)$ is $\mathcal{F}_t$-adapted and satisfies the condition $\int_0^t E(|\varphi(s)|^2)ds \leq \infty$ for each t. Then the Itô integral $I_t(\varphi) = \int_0^t \varphi(s)dB(s)$ is a random variable with $E(I_t(\varphi)) = 0$ and*

$$E(I_t^2(\varphi)) = E\left(\int_0^t \varphi_s^2 ds\right). \tag{3.5}$$

As the main topic I am going to cover in my thesis is pricing options linked to VTS portfolios, I would like to mention the risk-neutral general option pricing formula. Let us consider the dynamics of Eq. 3.1 and assume that $\tilde{S}(t) = \frac{S(t)}{B(t)}$ is a (local) martingale with respect to the measure $\mathbb{Q}$. When markets are complete and arbitrage-free, the value of an option $\Pi_t(\Phi(S(T)))$ at time t on the contingent claim $\Phi(S(T))$ can be calculated as:

$$\Pi_t(\Phi(S(T))) = B(t)\ E^{\mathbb{Q}}\left[\frac{\Phi(S(T))}{B(T)} \mid \mathcal{F}_t\right] = e^{-r(T-t)}\ E^{\mathbb{Q}}\left[\Phi(S(T)) \mid \mathcal{F}_t\right]. \tag{3.6}$$

I am also going to present two models – and their properties – which will be used for modeling the risky underlying asset of the Volatility Target Strategy portfolio later on in my thesis. The main aspect of model selection was to be able to see the impact of stochasticity of the volatility of the underlying risky asset, therefore the choice fell on the Black-Scholes-Merton model (1973) which assumes a constant volatility and Heston's stochastic volatility model (1993). The subsections below were written based on preliminary knowledge and following Ref. [13, 14, 15, 16].

3.1.1 Black-Scholes-Merton model

Assume that the market consists of at least one risk-free asset, referred as bond and one risky asset, referred as stock which does not pay dividends. The stock price $S(t)$ follows a geometric Brownian motion, which satisfies the following stochastic differential equation

$$dS(t) = S(t)(\mu dt + \sigma dW(t)), \tag{3.7}$$

assuming that the drift μ and the volatility $\sigma > 0$ are constant. We also assume that there is no arbitrage opportunity, the market direction cannot be predicted, the market is perfectly liquid and we can sell or buy any amount of asset at any given time and these transactions do not include any commissions or costs. The price of the European call under the Black-Scholes model can be calculated by Eq.3.6 which provides the following formula at time t [13]:

$$\begin{aligned} C(S(t),t) &= S(t)N(d_1) - Ke^{-r(T-t)}N(d_2), \\ d_1 &= \frac{\log\left(\frac{S(t)}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}, \\ d_2 &= d_1 - \sigma\sqrt{T-t}, \end{aligned} \tag{3.8}$$

where

- K is the strike price,

- r is the risk-free interest rate,
- $(T-t)$ is time to maturity
- and $N(x)$ denotes the cumulative distribution function of the standard normal distribution.

By the put-call parity, the price of the European put option is calculated as

$$\begin{aligned} P(S(t),t) &= Ke^{-r(T-t)} - S(t) + C(S(t),t) \\ &= Ke^{-r(T-t)}N(-d_2) - S(t)N(-d_1). \end{aligned} \tag{3.9}$$

3.1.2 Cox–Ingersoll–Ross model

The Cox–Ingersoll–Ross (CIR) model is usually used to describe the evolution of interest rate or stochastic volatility in mathematical finance. The $R(t)$ CIR process satisfies the following differential equation [16]:

$$\begin{aligned} dR(t) &= \kappa(\vartheta - R(t))dt + \sigma\sqrt{|R(t)|}dW(t), \\ R(0) &= r_0, \end{aligned} \tag{3.10}$$

where $\kappa > 0$ corresponds to the speed of adjustment to the mean $\vartheta > 0$. If the parameters obey the Feller-condition $2\kappa\vartheta > \sigma^2$, then the process R(t) is strictly positive under continuous time, however during discretization we can get negative values.

3.1.3 Heston model

Under the Heston model, the underlying stock price $S(t)$ follows the geometric Brownian motion with a stochastic variance $\nu(t)$ which follows the CIR process. An important property of the model is that the variance process is mean-reverting. The same feature can be observed for realized volatility from market data, as well. It can also capture the correlated shocks between returns and volatility, however it cannot capture volatility clustering. Modeling variance with this process can also cause issues during model parameter calibration in a way as the calibrated parameters do not fulfill the Feller-condition.

The dynamics, which describe the evolution of the variance and stock price under the real-world measure $\mathbb{P}$, satisfy the following stochastic differential equations [14, 15]:

$$\begin{aligned} dS(t) &= \mu S(t)dt + \sqrt{\nu(t)}S(t)dW^S(t), \\ d\nu(t) &= \kappa(\theta - \nu(t))dt + \sigma\sqrt{\nu(t)}dW^\nu(t), \end{aligned} \tag{3.11}$$

where

- μ is the expected return on the underlying asset,
- $\nu(t)$ is the instantaneous variance,
- $\kappa > 0$ is the mean reversion speed of the variance process,
- $\theta > 0$ is the long-term variance,
- $\sigma > 0$ is the volatility of volatility,
- $W^S(t)$ and $W^\nu(t)$ are two Wiener processes with correlation ρ. Then their quadratic variation $[dW^S(t), dW^\nu(t)] = \rho dt$.

The correlation parameter controls the skewness [2] of the distribution of the log stock price $\log S(t)$. If $\rho > 0$, the probability densities will be positively skewed, otherwise negatively. A positive correlation implies a rise in variance when stock price increases. This implies a spread in the right tail of the distribution and a thinning in the left tail which is associated with low variance. For negative correlation, the opposite happens. The volatility parameter σ of the variance process controls the kurtosis. Kurtosis defines how heavily the tails differ from the tails of the normal distribution which determine the frequency of presence of extreme values [14, 15].

The dynamics in Eq. 3.11 are driven under the real-world measure $\mathbb{P}$, but we also need the risk neutral dynamics for pricing. The dynamics of the underlying under the risk-neutral measure $\mathbb{Q}$ are

$$dS(t) = rS(t)dt + \sqrt{\nu(t)}S(t)d\tilde{W}^S(t), \tag{3.12}$$

with the $\mathbb{Q}$-Wiener process

$$d\tilde{W}^S(t) = dW^S(t) + \frac{\mu - r}{\sqrt{\nu(t)}}dt. \tag{3.13}$$

[2] Skewness measures the asymmetry of the probability distribution of a random variable. A positive skew means more weight is assigned to the right tail of the distribution.

The risk-neutral process of the variance is obtained by using the following $\mathbb{Q}$-Wiener process

$$d\tilde{W}^{\nu}(t) = dW^{\nu}(t) + \frac{\Gamma(S(t), \nu(t), t)}{\sigma\sqrt{\nu(t)}}dt, \tag{3.14}$$

where the function

$$\Gamma(S, \nu, t) = \gamma\nu(t) \tag{3.15}$$

is called the volatility risk premium and γ is a constant. Using this Wiener process, we can obtain the risk-neutral stochastic process of the volatility as the following:

$$\begin{aligned} d\nu(t) &= [\kappa(\theta - \nu(t)) - \Gamma(S(t), \nu(t), t)]dt + \sigma\sqrt{\nu(t)}d\tilde{W}^{\nu}(t) \\ &= \kappa^*(\theta^* - \nu(t))dt + \sigma\sqrt{\nu(t)}d\tilde{W}^{\nu}(t), \end{aligned} \tag{3.16}$$

where $\kappa^* = \kappa + \gamma$ and $\theta^* = \frac{\kappa\theta}{\kappa+\gamma}$ [14, 15].

In the next section I am going to continue the discussion of the Volatility Target Strategy.

3.2 Volatility Target Strategy portfolio

First, I am going to introduce the structure of the Volatility Target Strategy portfolio (VTS portfolio) following Ref. [17]. When a portfolio is built, a positive amount of money is invested in the portfolio which is allocated between the risk-free asset and the risky asset. The investor can transfer its capitals from one asset to the other but we assume that no more money is invested in the portfolio later so that we achieve a self-financing portfolio. We also assume that there are no transaction costs. This is important as non-negligible costs could appear with continuous re-balancing which would affect the portfolio trying to maintain a fixed volatility.

Let $V(t)$ denote the value of the portfolio and $\alpha(t)$ denote the percentage of the portfolio invested in the risky asset at time t. Let us assume that $\alpha(t)$ is an adapted, predictable, right-continuous stochastic process and has left limits everywhere (càdlàg process). Let $X(t)$ and $Y(t)$ denote the amount of money invested in riskless and risky asset at time t and they evolve as:

$$\begin{aligned} dX(t) &= rX(t)dt + dM(t) - dL(t), \\ dY(t) &= \mu(t)Y(t)dt + \sigma(t)Y(t)dW(t) + dL(t) - dM(t), \end{aligned} \tag{3.17}$$

where L represents the cumulative amount of riskless asset and M represents the cumulative amount of risky asset transferred into the other asset. $V(t)$ also satisfies $V(t) = X(t) + Y(t)$, therefore $dV(t) = dX(t) + dY(t)$. Using Eq. 3.1 and 3.17 and the equation $\alpha(t) = \frac{Y(t)}{V(t)}$, $dV(t)$ can be expressed as:

$$\begin{aligned} dV(t) &= dX(t) + dY(t) \\ &= r(1-\alpha(t))V(t)dt + \mu(t)\alpha(t)V(t)dt + \sigma(t)\alpha(t)V(t)dW(t) \\ &= V(t)[(\alpha(t)(\mu(t) - r) + r)dt + \alpha(t)\sigma(t)dW(t)] \end{aligned} \tag{3.18}$$

which leads to the following dynamics of the portfolio:

$$\begin{aligned} dV(t) &= V(t)\left(\alpha(t)\frac{dS(t)}{S(t)} + (1-\alpha(t))\frac{dB(t)}{B(t)}\right), \\ V(0) &= X(0) + Y(0). \end{aligned} \tag{3.19}$$

Note that the portfolio derived by this dynamic is self-financing and it is under the real-world measure $\mathbb{P}$. To create a Volatility Target Strategy portfolio, referred as VTS portfolio from now on, we would like to determine $\alpha(t)$ so that the portfolio preserves a fixed volatility target level. First, let us define the definition of VTS portfolio and after that we determine the value of $\alpha(t)$ explicitly.

Definition 3.2.1 (Volatility Target Strategy portfolio)**.** Consider the following portfolio for any $t \geq t_0$:

$$\begin{aligned} V_{\hat{\sigma}}(t) &= X_{\hat{\alpha}}(t) + Y_{\hat{\alpha}}(t), \\ X_{\hat{\alpha}}(t) &= (1-\hat{\alpha}(t))V_{\hat{\sigma}}(t), \\ Y_{\hat{\alpha}}(t) &= \hat{\alpha}(t)V_{\hat{\sigma}}(t), \end{aligned} \tag{3.20}$$

where $X_{\hat{\alpha}}(t)$ and $Y_{\hat{\alpha}}(t)$ denote the amount of capital invested in the risk-free and risky assets and $\hat{\alpha}(t)$ denotes the proportion of portfolio value invested in the risky asset at time t. If the portfolio is self-financing, preserves a constant volatility of $\hat{\sigma}$ and evolves according to Eq. 3.17, then we say that $V_{\hat{\sigma}}$ is a *Volatility Target Strategy portfolio.*

Proposition 3.2.1. *Consider a portfolio described by Eq. 3.20 with* $\hat{\alpha}(t) = \frac{\hat{\sigma}}{\sigma(t)}$*. Then this portfolio is a Volatility Target Strategy portfolio.*

Proof. Consider the following form of the portfolio's dynamic:

$$dV_{\hat{\sigma}}(t) = V_{\hat{\sigma}}(t)[(\hat{\alpha}(t)(\mu(t) - r) + r)dt + \hat{\alpha}(t)\sigma(t)dW(t)]. \tag{3.21}$$

After substituting $\hat{\alpha}(t)$ with $\frac{\hat{\sigma}}{\sigma(t)}$, we get

$$dV_{\hat{\sigma}}(t) = V_{\hat{\sigma}}(t)\left[\left(\frac{\hat{\sigma}}{\sigma(t)}(\mu(t) - r) + r\right)dt + \hat{\sigma}dW(t)\right]. \tag{3.22}$$

□

We can see that in order to maintain a VTS portfolio, the investor needs to choose $\hat{\alpha}(t)$ inversely proportional to the actual value of the volatility.

3.3 VTS portfolio linked derivatives

In this section I am going to introduce the pricing theory of derivatives linked to VTS portfolios following Ref. [17]. We are going to consider a contingent claim with an underlying VTS portfolio and our purpose is to provide an arbitrage-free price for the claim. Also, let us consider Eq. 3.1 as the dynamics of the assets under the physical measure $\mathbb{P}$.

First, let us derive the dynamics of the risky asset under the unique equivalent martingale measure $\mathbb{Q}$, under which $\frac{S(t)}{B(t)}$ is a local martingale, by Girsanov's theorem:

$$dS(t) = S(t)(rdt + \sigma(t)dW^{\mathbb{Q}}(t)) \tag{3.23}$$

where $W^{\mathbb{Q}}(t)$ is a $\mathbb{Q}$-Wiener process. Then the dynamics of the target volatility portfolio can be expressed similarly as in Eq. 3.21 under the measure $\mathbb{Q}$:

$$\begin{aligned} dV(t) &= V(t)[(\hat{\alpha}(t)(r - r) + r)dt + \hat{\alpha}(t)\sigma(t)dW^{\mathbb{Q}}(t)] \\ &= V(t)[rdt + \hat{\alpha}(t)\sigma(t)dW^{\mathbb{Q}}(t)] = V(t)[rdt + \hat{\sigma}dW^{\mathbb{Q}}(t)], \end{aligned} \tag{3.24}$$

where $\hat{\sigma}$ denotes the volatility target level. The discounted VTS portfolio $\tilde{V}(t) = \frac{V(t)}{B(t)}$ is a martingale with respect to the measure $\mathbb{Q}$. Using the Itô's formula we can obtain:

$$\begin{aligned} d\tilde{V}(t) = d\frac{V(t)}{B(t)} &= V(t)d\frac{1}{B(t)} + \frac{1}{B(t)}dV(t) \\ &= V(t)(-r)\frac{1}{B(t)}dt + \frac{1}{B(t)}V(t)[rdt + \hat{\alpha}(t)\sigma(t)dW^{\mathbb{Q}}(t)] \\ &= \frac{V(t)}{B(t)}[\hat{\alpha}(t)\sigma(t)dW^{\mathbb{Q}}(t)]. \end{aligned} \tag{3.25}$$

Finally, the no-arbitrage price of a contingent claim $\Phi(V(T))$ at time t can be expressed by the general pricing formula 3.6:

$$\Pi_t(\Phi(V(T))) = e^{-r(T-t)}E^{\mathbb{Q}}[\Phi(V(T))|\mathcal{F}_t]. \tag{3.26}$$

3.3.1 Closed-end pricing formula for options linked to standard VTS portfolios

Let us consider the dynamics of the underlying risky asset 3.23. In case of deterministic volatility the distribution of the risky asset price can be determined, therefore a closed form can be given for the price of a European option on the pure risky asset. However, we can also give a closed formula for the price of a European option linked to a VTS portfolio in case when we do not assume the underlying risky asset to have a deterministic volatility. In the following propositions I am going to present the pricing formulas based on Ref. [17].

Proposition 3.3.1. *Assuming that the risky asset dynamics follow a generalized geometric Brownian motion with random $\mathcal{F}^W$-adapted drift and volatility, see Equation 3.1, the price at time t_0 of a call option with payoff $\Phi_{call}(V(T)) = max(0, V(T) - K)$, linked to the VTS portfolio $V_{\hat{\sigma}}$, see equation 3.22, is given by the following explicit formula:*

$$\Pi(t_0, \Phi_{call}(V_{\hat{\sigma}}(T))) = \nu N(d_1(t_0)) - Ke^{-r(T-t_0)}N(d_2(t_0)), \tag{3.27}$$

where N is the cumulative distribution function for the standard normal distribution, $\nu = V_{\hat{\sigma}}(t_0)$ is the starting value of the portfolio and we define the following parameters:

$$\begin{aligned} d_1(t_0) &= \frac{-z_{\hat{\sigma}}(t_0) + \hat{\sigma}(T - t_0)}{\sqrt{T - t_0}}, \\ d_2(t_0) &= -\frac{z_{\hat{\sigma}}(t_0)}{\sqrt{T - t_0}}, \\ z_{\hat{\sigma}}(t_0) &= \frac{1}{\hat{\sigma}} log \left(\frac{K}{\nu}\right) + \left(\frac{\hat{\sigma}}{2} - \frac{r}{\hat{\sigma}}\right)(T - t_0). \end{aligned} \tag{3.28}$$

Proof. First, let us note that the VTS portfolio described in Eq. 3.24 has a constant volatility, while the risky asset has a non constant volatility. Also, it has a constant drift under the unique equivalent martingale measure. Therefore, we can obtain an explicit solution for the VTS portfolio:

$$V_{\hat{\sigma}}(t) = \nu \exp\left(\left(r - \frac{\hat{\sigma}^2}{2}\right)(t - t_0) + \hat{\sigma} W^{\mathbb{Q}}(t - t_0)\right), \quad \text{for } t \geq t_0. \tag{3.29}$$

We would like to use the general pricing formula 3.26 to determine the fair price of the option. Note that the distribution of the logarithm of the VTS portfolio at time t follows a normal distribution with expected value of $\left(\log(\nu) + \left(r - \frac{\hat{\sigma}^2}{2}\right)(t - t_0)\right)$, and variance of $\hat{\sigma}^2(t - t_0)$. After rearranging Eq. 3.29 we have that $V_{\hat{\sigma}} > K$ if and only if

$$W^{\mathbb{Q}}(T - t_0) > \frac{1}{\hat{\sigma}} \log\left(\frac{K}{\nu}\right) + \left(\frac{\hat{\sigma}}{2} - \frac{r}{\hat{\sigma}}\right)(T - t_0) =: z_{\hat{\sigma}}(t_0). \tag{3.30}$$

Using that $e^{\hat{\sigma} y} f_{N(0,T-t_0)}(y) = \frac{1}{\sqrt{2\pi(T-t_0)}} e^{-\frac{(y-\hat{\sigma}(T-t_0))^2}{2(T-t_0)}} e^{\frac{\hat{\sigma}^2}{2}(T-t_0)}$ for the probability density function of the normal distribution, we can derive the option price similarly as the Black Scholes formula is derived:

$$\begin{aligned} \Pi(t_0, \Phi_{call}(V_{\hat{\sigma}}(T))) &= \mathbb{E}\left[e^{-r(T-t_0)}(V_{\hat{\sigma}}(T) - K)^+ | \mathcal{F}_{t_0}\right] \\ &= e^{-r(T-t_0)} \int_{z_{\hat{\sigma}}(t_0)}^{+\infty} \left(\nu \exp\left(\left(r - \frac{\hat{\sigma}^2}{2}\right)(T - t_0) + \hat{\sigma} y\right) - K\right) f_{N(0,T-t_0)}(y) dy \\ &= e^{-r(T-t_0)} \, \nu \, e^{\left(r - \frac{\hat{\sigma}^2}{2}\right)(T-t_0)} \int_{z_{\hat{\sigma}}(t_0)}^{+\infty} e^{\hat{\sigma} y} f_{N(0,T-t_0)}(y) dy - \\ &\quad - K e^{-r(T-t_0)} \int_{z_{\hat{\sigma}}(t_0)}^{+\infty} f_{N(0,T-t_0)}(y) dy \\ &= e^{-r(T-t_0)} \, \nu \, e^{\left(r - \frac{\hat{\sigma}^2}{2}\right)(T-t_0) + \frac{\hat{\sigma}^2}{2}(T-t_0)} \left(1 - N\left(\frac{z_{\hat{\sigma}}(t_0) - \hat{\sigma}(T - t_0)}{\sqrt{T - t_0}}\right)\right) - \\ &\quad - K e^{-r(T-t_0)} \left(1 - N\left(\frac{z_{\hat{\sigma}}(t_0)}{\sqrt{T - t_0}}\right)\right) \\ &= \nu N\left(\frac{-z_{\hat{\sigma}}(t_0) + \hat{\sigma}(T - t_0)}{\sqrt{T - t_0}}\right) - K e^{-r(T-t_0)} N\left(-\frac{z_{\hat{\sigma}}(t_0)}{\sqrt{T - t_0}}\right). \end{aligned}$$

□

Proposition 3.3.2. *Assuming that the risky asset dynamics follow a generalized geometric Brownian motion with random $\mathcal{F}^W$-adapted drift and volatility, see equation 3.1, the price at time t_0 of a put option with payoff $\Phi_{put}(V(T)) = max(0, K - V(T))$, linked to the VTS portfolio $V_{\hat{\sigma}}$, see equation 3.22, is given by the following explicit formula:*

$$\Pi(t_0, \Phi_{put}(V_{\hat{\sigma}}(T))) = Ke^{-r(T-t_0)}N(-d_2(t_0)) - \nu N(-d_1(t_0)), \tag{3.31}$$

where the parameters d_1, d_2 and $z_{\hat{\sigma}}$ are defined as in Proposition 3.3.1.

Proof. By the put-call parity formula, we have that the difference between the price of a call option and the price of a put option with the same strike price, time to maturity and underlying, equals the difference between the actual price of the underlying, represented by the VT portfolio in our setting, and the discounted strike price, namely:

$$\Pi(t_0, \Phi_{call}(V_{\hat{\sigma}}(T))) - \Pi(t_0, \Phi_{put}(V_{\hat{\sigma}}(T))) = v - Ke^{-r(T-t_0)}. \tag{3.32}$$

Since $N(-x) = 1 - N(x)$ for each $x \in \mathbb{R}$, after rearranging the equation above, we obtain Equation 3.31. □

3.3.2 Closed-end pricing formula for options linked to VTS portfolios with Maximum Allowed Leverage Factor

While the weights $\alpha(t)$ can take on higher values than 1, a portion of the risky investment would be financed by loans as the portfolio is self-financing. We would like to avoid this, therefore in the following, I am going to introduce the strategy which can control the value of $\alpha(t)$. Let us consider a parameter, denoted by L, determining the maximum allowed leverage of the portfolio. This parameter forces the weight process to be equal or under a specified level:

$$\tilde{\alpha}(t) := \min\left\{L; \frac{\hat{\sigma}}{\sigma(t)}\right\}. \tag{3.33}$$

The strategy using this weight process is called Volatility Target Strategy with Maximum Allowed Leverage Factor, referred as MLVTS from now on, and I will denote the corresponding weight process with $\tilde{\alpha}(t)$. We can also give an analytical solution for the value of an European call or put option linked to the MLVTS portfolio. The next propositions give these closed formulas considering a particular case of Eq. 3.1 , when

$$dS(t) = S(t)(\mu(t)dt + \sigma(t)dW(t)), \tag{3.34}$$

where $\mu, \sigma : \mathbb{R}^+ \to \mathbb{R}^+$ are deterministic functions of time [17].

Proposition 3.3.3. *Assuming that the risky asset dynamics follow a generalized geometric Brownian motion with time-dependent drift and volatility, see Equation 3.34, the price at time t_0 of a call option with payoff $\Phi_{call}(V(T)) = max(0, V(T) - K)$, linked to the MLVTS portfolio $V_{\tilde{\sigma}}(t)$, is given by the following explicit formula:*

$$\Pi(t_0, \Phi_{call}(V_{\tilde{\sigma}}(T))) = \nu N(\tilde{d}_1(t_0)) - Ke^{-r(T-t_0)}N(\tilde{d}_2(t_0)), \tag{3.35}$$

where the proportion of portfolio value invested in the risky asset is $\tilde{\alpha}(t) := \min\{L; \hat{\sigma}/\sigma(t)\}$, N is the cumulative distribution function for the standard normal distribution, $\nu = V_{\tilde{\sigma}}(t_0)$ is the starting value of the portfolio and we define the following parameters:

$$\begin{aligned}
\tilde{d}_1(t_0) &= \frac{-z_{\tilde{\sigma}}(t_0) + \varsigma(t_0, T)}{\sqrt{\varsigma(t_0, T)}}, \\
\tilde{d}_2(t_0) &= -\frac{z_{\tilde{\sigma}}(t_0)}{\sqrt{\varsigma(t_0, T)}}, \\
z_{\tilde{\sigma}}(t_0) &= log\left(\frac{K}{\nu}\right) + r(T - t_0) + \frac{\varsigma(t_0, T)}{2}, \\
\varsigma(t_0, T) &= \int_{t_0}^{T} \tilde{\sigma}(s)^2 ds, \\
\tilde{\sigma}(t_0) &= \min\{L\sigma(t); \hat{\sigma}\}.
\end{aligned} \tag{3.36}$$

Proof. In contrast what we could see for the strategy in Proposition 3.3.1, for now the $V_{\tilde{\sigma}}(t)$ portfolio has not a constant volatility and its evolution can be written as following:

$$\begin{aligned}
V_{\tilde{\sigma}}(t_0, t) &= \nu \exp\left(r(t - t_0) - \frac{1}{2}\int_{t_0}^{t} \tilde{\sigma}(s)^2 \mathrm{d}s + \int_{t_0}^{t} \tilde{\sigma}(s)\mathrm{d}W^{\mathbb{Q}}(s)\right) \\
&= \nu \exp\left(r(t - t_0) - \frac{\varsigma(t_0, t)}{2} + \int_{t_0}^{t} min(L\sigma(s), \hat{\sigma})\mathrm{d}W^{\mathbb{Q}}(s)\right),
\end{aligned} \tag{3.37}$$

where we use the notation $V_{\tilde{\sigma}}(t_1, t_2)$ to emphasize the time period we consider at each case. Let us use the notation $\tilde{W}(t-t_0) := \int_{t_0}^{t} min(L\sigma(s), \hat{\sigma})\mathrm{d}W^{\mathbb{Q}}(s) = \int_{t_0}^{t} \tilde{\sigma}(s)\mathrm{d}W^{\mathbb{Q}}(s)$, which is a normal distributed random variable with expected value of zero, and variance of $\varsigma(t_0, t)$, which is the direct consequence of Itô Isometry 3.1.2. Therefore, its probability density function is

$$f_{N(0,\varsigma(t_0,t))}(x) = \frac{1}{\sqrt{2\pi\varsigma(t_0,t)}} \exp\left(-\frac{x^2}{2\varsigma(t_0,t)}\right). \tag{3.38}$$

We can see that $V_{\tilde{\sigma}}(t) > K$ if and only if

$$\tilde{W}(T-t_0) > \log\left(\frac{K}{\nu}\right) - r(T-t_0) + \frac{\varsigma(t_0,T)}{2}. \tag{3.39}$$

Using the notation $z_{\tilde{\sigma}}(t_0) := \log\left(\frac{K}{\nu}\right) - r(T-t_0) + \frac{\varsigma(t_0,T)}{2}$, we can derive the option value as:

$$\begin{aligned}
\Pi(t_0, \Phi_{call}(V_{\tilde{\sigma}}(T))) &= \mathbb{E}\left[e^{-r(T-t_0)}(V_{\tilde{\sigma}}(T)-K)^+|\mathcal{F}_{t_0}\right] \\
&= e^{-r(T-t_0)} \int_{z_{\tilde{\sigma}}(t_0)}^{+\infty} \left(\nu \exp\left(r(T-t_0) - \frac{\varsigma(t_0,T)}{2} + x\right) - K\right) f_{N(0,\varsigma(t_0,T))}(x)dx \\
&= e^{-r(T-t_0)}\, \nu\, e^{r(T-t_0)-\frac{\varsigma(t_0,T)}{2}} \int_{z_{\tilde{\sigma}}(t_0)}^{+\infty} e^x f_{N(0,\varsigma(t_0,T))}(x)dx- \\
&\quad - Ke^{-r(T-t_0)} \int_{z_{\tilde{\sigma}}(t_0)}^{+\infty} f_{N(0,\varsigma(t_0,T))}(x)dx \\
&= \nu\, e^{-\frac{\varsigma(t_0,T)}{2}} \int_{z_{\tilde{\sigma}}(t_0)}^{+\infty} \frac{1}{\sqrt{2\pi\varsigma(t_0,T)}} e^{-\frac{(x-\varsigma(t_0,T))^2+\varsigma(t_0,T)^2}{2\varsigma(t_0,T)}} dx- \\
&\quad - Ke^{-r(T-t_0)} \int_{z_{\tilde{\sigma}}(t_0)}^{+\infty} f_{N(0,\varsigma(t_0,T))}(x)dx \\
&= \nu\left(1 - N\left(\frac{z_{\tilde{\sigma}}(t_0) - \varsigma(t_0,T)}{\sqrt{\varsigma(t_0,T)}}\right)\right) - Ke^{-r(T-t_0)}\left(1 - N\left(\frac{z_{\tilde{\sigma}}(t_0)}{\sqrt{\varsigma(t_0,T)}}\right)\right) \\
&= \nu N\left(\frac{-z_{\tilde{\sigma}}(t_0) + \varsigma(t_0,T)}{\sqrt{\varsigma(t_0,T)}}\right) - Ke^{-r(T-t_0)} N\left(-\frac{z_{\tilde{\sigma}}(t_0)}{\sqrt{\varsigma(t_0,T)}}\right).
\end{aligned}$$

□

The reason, why we assumed that the volatility is a deterministic function of time, can be seen from the formulation above. We would like $\varsigma(t_0,T)$ to be deterministic to be able to calculate the integral. As a result, we could derive a closed-end formula for the option price with this simplification easily.

Proposition 3.3.4. *Assuming that the risky asset dynamics follow a generalized geometric Brownian motion with time-dependent drift and volatility, see Equation 3.34, the price at time t_0 of a put option with payoff $\Phi_{put}(V(T)) = max(0, K - V(T))$, linked to the MLVTS portfolio $V_{\tilde{\sigma}}$, is given by the following explicit formula:*

$$\Pi(t_0, \Phi_{put}(V_{\tilde{\sigma}}(T))) = Ke^{-r(T-t_0)}N(-\tilde{d}_2(t_0)) - \nu\, N(-\tilde{d}_1(t_0)), \tag{3.40}$$

where the parameters $\tilde{d}_1(t_0)$, $\tilde{d}_2(t_0)$ *and* $z_{\hat{\sigma}}$ *are defined as in Proposition 3.3.3.*

Proof. Direct consequence of the put-call parity as in Proposition 3.3.2. □

3.4 S&P 500 Risk Control Indices mathematics methodology

I have already introduced the S&P 500 Risk Control Indices and the technical details of the strategy, so now I can present the methodology of the indices in practice. This section is written based on S&P Dow Jones Indices Index Mathematics Methodology [5].

The Risk Control Indices include the leverage factor α_t that changes based on realized volatility of the underlying asset and the maximum leverage factor L_{max}. The actual leverage factor cannot exceed the level of the maximum leverage factor which is usually set to be 100% or 150%. The return of the index consists of the return on the position in the underlying index and the interest cost or gain depending on whether the position is leveraged or deleveraged. The risk control index value $Index_t$ is calculated from this aspect following the formulation below:

$$Index_t = Index_{rb} \times (1 + IndexReturn_t), \tag{3.41}$$

where

$$\begin{aligned} IndexReturn_t = \alpha_{rb} &\left(\frac{UnderlyingIndex_t}{UnderlyingIndex_{rb}} - 1\right) \\ &+ (1-\alpha_{rb})\left[\prod_{i=rb+1}^{t}(1 + r_{i-1}\frac{D_{i-1,i}}{360}) - 1\right]. \end{aligned} \tag{3.42}$$

Combining Eq. 3.41 and 3.42 we can get

$$\begin{aligned} &Index_t = Index_{rb} \\ &\times \left[1 + \left[\alpha_{rb}\left(\frac{UnderlyingIndex_t}{UnderlyingIndex_{rb}} - 1\right) + (1-\alpha_{rb})\left[\prod_{i=rb+1}^{t}(1 + r_{i-1}\frac{D_{i-1,i}}{360}) - 1\right]\right]\right], \end{aligned} \tag{3.43}$$

where

- $UnderlyingIndex_t$ is the level of the underlying index, e.g. S&P 500 Index on day t,
- rb is the last index rebalancing date,
- VT is the target level of volatility set for the index,
- L_{max} is the maximum leverage level allowed,
- d is the lag (measured in number of days) between rebalancing date and observed historical volatility which means historical volatility as of d days prior to the rebalancing date will be used to calculate the leverage factor α_{rb},
- α_{rb} is the leverage factor set at last rebalancing date as $\min(L_{max}, \frac{VT}{RealizedVolatility_{rb-d}})$. It is calculated at each rebalancing date and held constant until the next rebalancing.
- r_t is the interest rate set for the index. A 360-day year is assumed for the interest calculations.
- $D_{i-1,i}$ is the number of calendar days between day $i-1$ and i.

The index calculates the theoretical leverage factor $\alpha_{theoretical}$ on daily basis as $\alpha_{theoretical} = \min(L_{max}, \frac{VT}{RealizedVolatility_{t-d}})$ and the index does not rebalance if the difference from the actual leverage factor is less than the Minimum Daily Allocation Change θ. This is due to rebalancing costs may occur. If the difference is bigger than the Minimum Daily Allocation Change, then day t is a rebalancing day and the value of the leverage factor is calculated as $\alpha_t = \alpha_{theoretical}$. Also, there is a potential to maximize daily allocation if it is requested. Then, if t is a rebalancing day, the value of the leverage factor can be calculated as $\alpha_t = \min(j\alpha_{rb} + \theta, j\alpha_{theoretical})$, where $j = 1$ if $\alpha_{theoretical} > \alpha_{rb}$ and $j = -1$ else. If day t is not a rebalancing day, the leverage factor does not change.

Historical volatility is calculated as the square root of annualized variance which is computed as an exponentially weighted moving average of the square of the log returns observed of the underlying index. The realized volatility of the underlying risky asset denoted by $RealizedVolatility_t$ is calculated as the maximum of two historical volatilities - one measuring the short-term and the other one measuring the long-term historical volatility.

$$RealizedVolatility_t = \max(HistoricalVolatility_{\lambda_S,t}, HistoricalVolatility_{\lambda_L,t}), \quad (3.44)$$

where

$$HistoricalVolatility_{\lambda,t} = \sqrt{\frac{252}{n} \times Variance_{\lambda,t}}, \tag{3.45}$$

where for $t > T_0$:

$$Variance_{\lambda,t} = \lambda Variance_{\lambda,t-1} + (1-\lambda)\left[\log\left(\frac{UnderlyingIndex_t}{UnderlyingIndex_{t-n}}\right)\right]^2. \tag{3.46}$$

Here

- n is the number of days inherent in the return calculation (for daily returns $n = 1$),
- $0 < \lambda < 1$ is the decaying factor used for the exponential weighting, representative λ_S and λ_L stand for the short-term and long-term decaying constants which two ones can be equivalent.

The variance is calculated at $t = 0$ as

$$Variance_{\lambda,T_0} = \sum_{i=m+1}^{T_0}\left[\frac{\beta_{\lambda,m,i}}{WeightingFactor_\lambda} \times \left[\log\left(\frac{UnderlyingIndex_i}{UnderlyingIndex_{i-n}}\right)\right]^2\right], \tag{3.47}$$

where

- m is the N$^{\text{th}}$ trading day prior to the start date T_0 of the index,
- N is the number of trading days observed for calculating the initial variance as of T_0,
- β is the weight assigned to the return on the corresponding day prior T_0 and is calculated as $\beta_{\lambda,m,i} = (1-\lambda)\lambda^{N+m-i}$. The more recent the observation is, the bigger weight is assigned as $0 < \lambda < 1$ and $N + m - i = N - (i - m)$ is decreasing.
- $WeightingFactor_\lambda = \sum_{i=m+1}^{T_0} \beta_{\lambda,m,i}$.

The interest rate, maximum leverage, target volatility and the decaying factors are defined constant throughout the life of the index.

Chapter 4

Implementation and numerical results

In the previous chapter, we could see the closed form solution for the option price linked to the VTS and MLVTS portfolios in special cases. To study the behavior of the option prices under different financial models in general, I simulated the evolution of the risky asset and the portfolio under Black Scholes model and Heston model. While the Black Scholes model assumes a constant volatility, under Heston model we can observe the effect of the stochastic volatility of the risky underlying asset. I implemented Monte Carlo simulation using Python, which was performed under the risk-neutral measure with the corresponding parameters, which is described in Chapter 3. An implementation and the results are studied and discussed in the article of Jawaid in Ref. [18] and first I am going to compare my results to these reference values. After that, I summarize the results of further tests performed based on the challenges coming up.

4.1 Technical basis and details

4.1.1 Discretization of the models

First, let me present the discretization scheme used during Monte Carlo simulation. For this, I am going to follow Ref. [14, 17].

Discretization of the process of the risky asset

Let us consider the stochastic differential equations 3.12 which describe the stock price and its volatility under the risk neutral metric $\mathbb{Q}$. Both can be written in the general form

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t. \tag{4.1}$$

X_t is simulated over a time interval $[0, T]$ equally divided into $N + 1$ points as $0 = t_0 < t_1 < ... < t_N = T$. Integrating dX_t from t to $t + dt$ produces

$$X_{t+dt} = X_t + \int_t^{t+dt} \mu(X_u, u)du + \int_t^{t+dt} \sigma(X_u, u)dW(u). \tag{4.2}$$

The Euler-Maruyama is the simplest discretization scheme, which is based on approximating the integrals using the values at the left point of the dt interval, as these values are known at time t. The first half of the integral 4.2 is approximated as

$$\int_t^{t+dt} \mu(X_u, u)du \approx \mu(X_t, t) \int_t^{t+dt} du = \mu(X_t, t)dt, \tag{4.3}$$

while the second half is approximated as

$$\int_t^{t+dt} \sigma(X_u, u)dW(u) \approx \sigma(X_t, t) \int_t^{t+dt} dW(u) = \sigma(X_t, t)\sqrt{dt}\, N(0, 1). \tag{4.4}$$

This implies that the Euler-Maruyama discretization of the Equation 3.12 under the risk neutral measure $\mathbb{Q}$ is the following

$$\begin{aligned} \nu_{t+dt} &= \nu_t + \kappa^*(\theta^* - \nu_(t))dt + \sigma\sqrt{\nu_t}\sqrt{dt}\, Z_\nu \\ S_{t+dt} &= S_t \exp\left(\left(r - \frac{\nu_t}{2}\right) dt + \sqrt{\nu_t}\sqrt{dt} Z_S\right), \end{aligned} \tag{4.5}$$

where Z_ν and Z_S are the corresponding standard normally distributed random variables with correlation ρ, and all the parameters are described in Section 3.1.3.

There is one issue which can occur during the simulation. Many simulation schemes will generate negative values for ν_t, even if the Feller condition $2\kappa^*\theta^* > \sigma^2$ is met, since the condition is valid for continuous time processes, but simulation is done in discrete time. The probability of generating a negative value in the Euler-Maruyama scheme is

$$Pr(\nu_{t+dt} < 0) = \Phi\left(\frac{-(1 - \kappa^*\, dt)\nu_t - \kappa^*\theta^*\, dt}{\sigma\sqrt{\nu_t}\sqrt{dt}}\right). \tag{4.6}$$

One way to avoid the negative variances is to floor these values at zero as $\nu_t^+ = \max(0, \nu_t)$. Also using other schemes can produce far fewer negative values, such as the next one described below.

The Equation 3.12 under the risk neutral measure $\mathbb{Q}$ is discretized according to the Milstein scheme, which is presented here without a proof (please see Ref. [14]), as following

$$\begin{aligned} \nu_{t+dt} &= \left(\sqrt{\nu_t} + \frac{1}{2}\sigma\sqrt{dt}Z_\nu\right)^2 + (\kappa^*(\theta^* - \nu_t) - \frac{1}{4}\sigma^2)dt \\ S_{t+dt} &= S_t \exp\left(\left(r - \frac{1}{2}\nu_t\right)dt + \sqrt{\nu_t}\sqrt{dt}Z_S\right). \end{aligned} \tag{4.7}$$

However, using the Milstein discretization of the variance process can reduce the frequency of the negative variances, it is still needed to apply $\nu_t^+ = \max(0, \nu_t)$, as negative values may still occur [14]. Milstein discretization of the risky asset under Heston model does not produce a more accurate approximation than Euler discretization, as we can see its dynamics is the same in Eq. 4.5 and 4.7. Under Black Scholes model the discretization of the risky asset is the same using constant volatility instead of $\sqrt{\nu_t}$.

Discretization of the process of the VTS portfolio

I approximate the path of the VTS and MLVTS portfolios by Euler-Maruyama scheme denoting the discretized portfolio process by $V_t^{\Delta t}$ at time t as following:

$$V_{t+dt}^{\Delta t} = V_t^{\Delta t}\left(1 + \alpha_t \frac{\Delta S_t}{S_t} + (1 - \alpha_t)\frac{\Delta B_t}{B_t}\right) \tag{4.8}$$

where $\Delta S_t := S(t_{t+dt}) - S(t_t)$, $\Delta B_t := B_{t+dt} - B_t$ and $V_0^{\Delta t} = \nu$. It can be shown that this numerical approximation is convergent to the V_t solution of the differential equation 3.18, describing the dynamics of a VTS or MLVTS portfolio [17] :

Proposition 4.1.1. *Let $T > 0$ be fixed. The numerical scheme 4.8 is strongly convergent to the solution of the stochastic differential equation 3.18, i.e.*

$$\lim_{\Delta t \to 0} E[|V_T - V_T^{\Delta t}|] = 0. \tag{4.9}$$

However the following modification of the Milstein scheme produces a faster convergence as it is stated in Ref. [17] [1] :

$$V_{t+dt}^{\Delta t} = V_t^{\Delta t}\left(1 + \alpha_t \frac{\Delta S_t}{S_t} + (1 - \alpha_t)\frac{\Delta B_t}{B_t} - \frac{\alpha_t(1 - \alpha_t)}{2}\left[\left(\frac{\Delta S_t}{S_t}\right)^2 - \nu_t dt\right]\right). \tag{4.10}$$

[1] The authors do not expound further details regarding this.

4.1.2 The realized volatility and the stochastic weights

During the simulation I used the Exponentially Weighted Moving Average (EWMA) method to calculate the annualized daily volatility of the risky asset. The choice falls on this method because the S&P 500 risk control indices also use the EWMA method [5, 18].

Let $R(t_i)$ denote the daily logreturns of the risky asset $S(t_i)$ as:

$$R(t_i) = \log\left(\frac{S(t_i)}{S(t_{i-1})}\right). \tag{4.11}$$

We define the realized historical volatility $U(t_i)$ using the notations as in Ref. [18]:

$$\begin{aligned} U(t_i) &= \tilde{\sigma}_{t_i}, \\ \tilde{\sigma}^2_{t_i} &= \lambda \tilde{\sigma}^2_{t_{i-1}} + (1-\lambda)\frac{1}{\Delta t} R^2_{t_{i-1}}, \end{aligned} \tag{4.12}$$

where $\Delta t = \frac{1}{252}$ and the value λ decaying constant is equivalent to the one used in the S&P 500 Methodology. During the simulation I set the historical volatility at time t_0 equal to the constant volatility of the risky asset S under the Black-Scholes model as $U(t_0) = \sigma_S$. Under the Heston model I set the same starting value of the historical volatility of the underlying risky asset. We determine the value of the stochastic weight α using the realized volatility as

$$\alpha(t_i) = \min\left(L_{max}, \frac{VT}{U(t_i)}\right), \tag{4.13}$$

where VT is the volatility target level and L_{max} denotes the maximum leverage level allowed.

4.1.3 Parameter settings

The appropriate selection of the parameters is an essential part of the simulation. Some of the parameters are related to the risk control products, such as the volatility target level, maximum leverage, decaying factor or rebalancing frequency, for them I referred the S&P Dow Jones Risk Control Indices Parameters documentation [6]. The calibration of the parameters of Heston model is beyond the scope of my thesis, for them I used the parameter settings from my reference article [18]. This decision seems to be reasonable as it is a commonly used parameter set in literature [18, 20]. Please find them in Table 4.1 below.

θ	0.22^2
κ	4.75
σ_ν	0.55
ρ	-0.569
$\nu(0)$	θ

Table 4.1: Benchmark parameters for Heston model

In order to get the parameters of Heston model under the risk-neutral measure, a transformation needs to be performed. With a minor technical change to the determination of the parameter γ used in Eq. 3.15, I am going to use $\kappa^* = (\kappa + \beta\sigma_\nu)$ and $\theta^* = \frac{\kappa\theta}{\kappa+\beta\sigma_\nu}$ according to the reference article [18]. With changing the value of the parameter β, called the market price of volatility risk, we can set the risk-neutral value of the speed of mean reversion κ^* and the long-term variance θ^*. We can see the value of these parameters with different market prices of risk in Table 4.2, which will be used during my simulations.

Market price of risk	Speed of mean reversion κ^*	Long-term variance θ^*
$\beta = 2$	5.85	0.198^2
$\beta = 0$	4.75	0.22^2
$\beta = -2$	3.65	0.251^2

Table 4.2: Parameters with different market price of risk values under risk-neutral measure

Also, there are other parameters, like the initial value of the risky asset and the VTS portfolio, the constant risk-free interest rate, time to maturity or number of simulation paths which were chosen arbitrary.

The inputs for the simulations are the following. I set the initial VTS/MLVTS portfolio value $V_0 = 100$ in each cases. $\alpha_0 = \frac{VT}{U(t_0)}$ is the initial proportion of the portfolio invested in the risky asset, where $U(t_0)$ is defined in Eq. 4.12. The constant interest rate r is set to be 2%. The number of simulation paths is usually set to be 100,000 based on convergence analysis performed (please see in Subsetion 4.2.1) and I usually examined 100 realizations of the simulations. The decaying constant is set to be 94% and the timestep Δt is equal to $\frac{1}{252}$ for a daily rebalancing frequency, according to Ref. [6] and [18] . Other parameters, such as VT level or volatility σ of the risky asset are going to be set later for testing purposes.

I am going to show later on in my thesis, what impact some of these parameter values have on the option prices. During the simulations we assume that the stock does not pay dividends and transaction costs do not occur.

4.2 Implementation results

4.2.1 Comparison analysis

First, to compare my results to the ones in my reference article [18] published by Jawaid, I performed the tests with the same parameter setting. The next tables below consists of the comparison of ATM European call options linked to MLVTS portfolio with the maturity of 1 year and with daily rebalancing frequency. The initial value of the MLVTS portfolio V_0 and the strike K are equal to 100, $L_{max} = 1$.

I performed convergence analysis to be able to choose the parameter value of the number of paths (N) within one simulation and the number of simulations (M) performed to get the statistical results. Under Black Scholes model, for $N = 10,000$ the standard deviation of 100 realizations is around 0.07, while for $N = 100,000$ it falls to around 0.02 and the running time is still acceptable [2]. It was also observed that increasing the number of realizations could not produce a smaller standard deviation, however it resulted in a significantly longer execution time.

In Table 4.3 we can see MLVTS portfolio call prices with $VT = 10\%$ with different σ_{BS} values. A call option with a constant volatility of 10% has a value of 5.01698. The closed formula given in the previous chapter gives this exactly same value for the call on the VTS portfolio in a special case, when there is no maximum leverage factor set. It is important to mention that the α_t weights used in Ref. [17] and Eq. 3.2.1 are calculated by the ratio of VT and the implied volatility (which means the volatility used to calculate α_t and dS_t at time t are equal), while in my simulations I followed the S&P methodology and used the historical volatility for α_t calculations. I also used the historical volatility as $\tilde{\sigma}(t)$ for calculating the theoretical values of call prices linked to MLVTS portfolio by Prop. 3.3.3 [17]. The reason behind this, if implied volatility was used as $\tilde{\sigma}(t)$ to calculate $\varsigma(t_0, T) = \int_{t_0}^{T} \tilde{\sigma}(s)^2 \mathrm{d}s$ in Eq. 3.36, then in case of $VT = \sigma_{BS}$ a call linked to MLVTS portfolio would have the same price as a call linked to VTS portfolio. That would show identical results which we would like to avoid, as we cap the α_t weights in one of the cases. These values are only used as reference values in the analysis for the reason above.

In the next tables, the column 'Analytical' contains these analytical prices computed in my simulation based on Ref. [17] and the previous chapter. Column 'Simulation' always contains my simulated call prices on MLVTS portfolio, while column 'Sim. in [18]' contains the reference article results of call prices published in Ref. [18]. Column 'Difference' shows the difference between my results and the reference article results [18]. As

[2] Running the codes on a computer with 4GB RAM and i3 processor.

the historical volatility is path-dependent, the columns 'Analytical' were calculated as an average of the closed formula results on the paths. My simulation prices were also taken as an average of several realizations. Therefore, the columns 'Stdev' always contain the standard deviation of the simulated values of the variable in the previous column.

In Table 4.3 as the $\frac{VT}{\sigma_{historical}}$ is lower than L_{max} and the cap effect is not dominant, the analytical MLVTS call prices approximates the theoretical BS call price with $\sigma_{BS} = 10\%$ almost within numerical precision. We can notice a max difference of approx. 0.22 between my simulated prices (which approximate the computed analytical prices from above) and the prices of the reference article [18] (where the approximation to the analytical prices is from below). For a deeper investigation on why there is an upward bias observable in my simulated prices, please see Subsection 4.2.2. Increasing the risky asset volatility does not influence the call price on MLVTS portfolio according to the results in column 'Simulation', so we can say the strategy efficiently controls the risk as we expected.

VT	σ_{BS}	Analytical	Sim. in [18]	Simulation	Stdev	Difference
10%	15%	5.0168	4.9209	5.1397	0.021	0.2188
10%	22%	5.0170	4.9625	5.1331	0.022	0.1706
10%	25%	5.0170	4.9648	5.1358	0.025	0.171

Table 4.3: The comparison of the prices of European calls linked to MLVTS portfolio under Black Scholes model I.

The next Table 4.4 provides a comparison of the prices when the VT level is equal to σ_{BS}. As the ratio of them $\frac{VT}{\sigma_{BS}}$ and L_{max} are both 1, it implies that the α_t weights are also at around 1, so we can examine the effect of capping the weights.

$VT = \sigma_{BS}$	BS call	Analytical	Stdev	Sim. in [18]	Simulation	Stdev	Difference
5%	3.1207	3.0314	0.039	3.1031	3.0546	0.0119	0.0485
10%	5.0170	4.8264	0.084	4.9766	4.8756	0.0203	0.101
15%	6.9618	6.6730	0.128	6.8669	6.7479	0.0299	0.119

Table 4.4: The comparison of the prices of European calls linked to MLVTS portfolio under Black Scholes model II.

There is an additional column 'BS call' in Table 4.4 to be able to compare the MLVTS portfolio linked call prices with the pure risky BS prices with the corresponding volatility. It can be seen that due to the cap effect the exposure to the risky asset is limited, therefore prices in column 'Simulation' are lower compared to the call prices in column 'BS call'. Overall, the simulation results support our expectation that controlled exposure to the pure risky underlying controls the option prices. A more detailed analysis can be found in Subsection 4.2.4. Also, my results compared to the reference article [18] results show an approximation within a maximum of 0.12 error.

Under Heston simulation I received similar convergence results as under Black Scholes model. I performed the following tests with $N = 100,000$ and $M = 100$. In Table 4.5 similarly to Table 4.3, the upward bias is also observable in my simulated prices under Heston model. However, we can say that the value of ν_0, which define the long-term variance θ^*, has no impact on the price of the call option linked to the MLVTS portfolio. The column 'BS (sim.)' presents the BS call prices on MLTVS portfolio simulated with the corresponding constant volatility. Comparing the Heston prices in column 'Simulation' with the BS prices found in column 'BS (sim.)', a slight difference can be observed, with higher values under Black Scholes model.

ν_0	Analytical	Stdev	Sim. in [18]	Simulation	Stdev	Difference	BS (sim.)
0.198^2	4.9911	0.053	4.8842	5.1175	0.0216	0.2333	5.1357
0.22^2	5.0007	0.042	4.9009	5.1216	0.020	0.2207	5.1331
0.251^2	5.0078	0.031	4.9147	5.1201	0.0203	0.2054	5.1372

Table 4.5: The comparison of the prices of European calls linked to MLVTS portfolio under Heston model I.

Table 4.6 provides the call prices when VT is equal to the long-term run volatility. Due to the prevailing cap effect, the exposure to the risky asset is controlled in the MLVTS portfolio which produces less expensive call options compared to calls on pure risky portfolio.

$VT = \sqrt{\nu_0}$	Analytical	Stdev	Sim. in [18]	Simulation	Stdev	Diff (BS-Heston)
5%	2.9626	0.066	3.0727	3.0160	0.010	0.0386
10%	4.5735	0.206	4.9046	4.6625	0.018	0.2131
15%	6.1885	0.417	6.6132	6.3033	0.026	0.4446

Table 4.6: The comparison of the prices of European calls linked to MLVTS portfolio under Heston model II. Last column 'Diff (BS-Heston)' compares prices in column 'Simulation' with prices in column 'Simulation' in Table 4.4.

With increasing volatility an even bigger gap can be observable between the simulated prices under the two models, the differences can be seen in the last column 'Diff (BS-Heston)'. As the volatility in Heston model is not constant, it can take higher and lower values than the constant BS volatility. Therefore it is more common to generate higher or lower returns which can imply a more volatile historical variance. If the historical volatility is high, then the α_t weights will take lower values, while if the opposite happens the α_t weights are capped. Later on in Subsection 4.2.3, more analysis will be presented to be able to provide an exact explanation on why Black Scholes prices overperform Heston prices.

Overall, it can be said that the prices of calls linked to MLVTS portfolio are less expensive compared to calls based on pure risky underlying. This is due to the volatility target strategy controls the exposure to the risky asset. Also, we could observe that the BS prices overperform the Heston prices. Comparing the results in the tables above showed that sometimes the differences are negligible, while there are cases when these differences are significant. When VT was set to be equal to the long term volatility, the cap effect could prevail better and better for higher volatility levels under both models. All my simulation results showed an upward bias compared to the analytical prices which were just reference values computed based on Propositions 3.3.3 [17]. While in Tables 4.3 and 4.5 my simulated results overperformed, when VT was equal to the long-term volatility in Tables 4.4 and 4.6, my simulated prices underperformed the prices published in Ref. [18]. In the next subsections I am going to summarize additional test results which will more highlight the behavior of the MLVTS portfolio linked options.

4.2.2 Upward bias of volatility of MLVTS portfolio

Tables 4.3 and 4.5 show an upward bias of the simulated call prices on MLVTS portfolio compared to the results of the reference article [18], to the analytical prices (calculated based on Prop. 3.3.3 [17]) and to the BS prices of European calls linked to pure risky portfolio. The approximation to the reference values was not satisfying enough, so I performed further tests to be able to see the reason behind it. I used the same parameter setting as in the previous subsection.

The tests were performed with $VT = 10\%$, $\sigma_{BS} = 22\%$ and long-term variance $\theta^* = 0.22^2$. First, I plotted out the standard deviation of the realized log-returns of the MLVTS portfolios. Figures 4.1 and 4.2 show the results with 1 year maturity.

However, the volatility target level of the portfolio is set to be 10%, we can see a slight but consistent upward bias of approx. $0.3\% - 0.35\%$ under both models. This can explain why my simulation provided higher prices for call options in Tables 4.3 and 4.5. As the portfolio composition is based on the weights, I examined the evolution of α_t. The next two figures show the α_t weights during time in the same simulation.

In Figures 4.3 and 4.4 we can observe an increase in the mean of α_t compared to the expectation that the average of the values are approximating $\frac{VT}{\sigma_{BS}} = \frac{VT}{\sqrt{\theta^*}} = 0.4545$. Going forward, in the next Figures 4.5 and 4.6, we can see the average of historical volatility

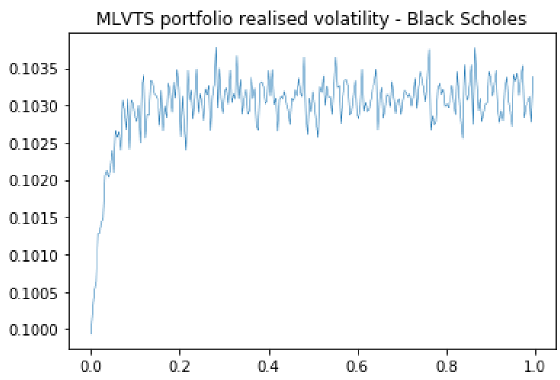


Figure 4.1: The realized volatility of MLVTS portfolio under Black Scholes model.

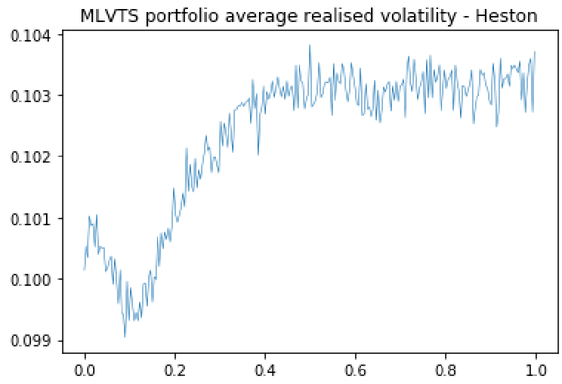


Figure 4.2: The realized volatility of MLVTS portfolio under Heston model.

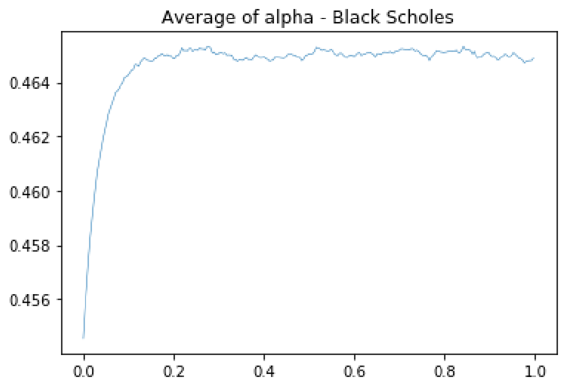


Figure 4.3: Average of α_t weights under Black Scholes model.

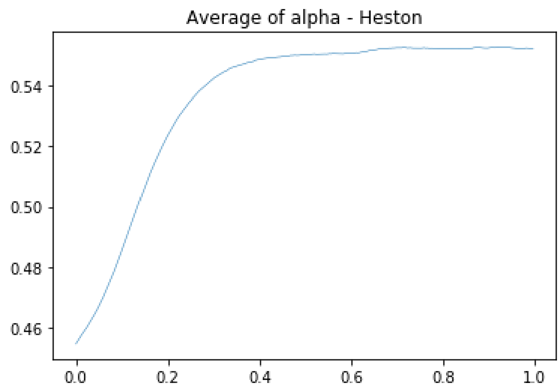


Figure 4.4: Average of α_t weights under Heston model.

$U(t_i)$ computed as the square root of exponentially weighted realized log-returns square of the risky asset. The initial value is set to the simulation volatility value of 0.22, as mentioned in Subsection 4.1.2, therefore we would expect a constant trend. However, we can observe a decrease at the beginning of the observations to a certain level under both models, however it is more moderate under Black Scholes model.

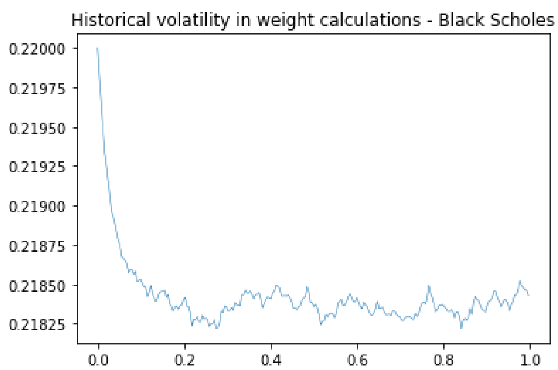


Figure 4.5: Historical volatility calculated by EWMA method under Black Scholes model.

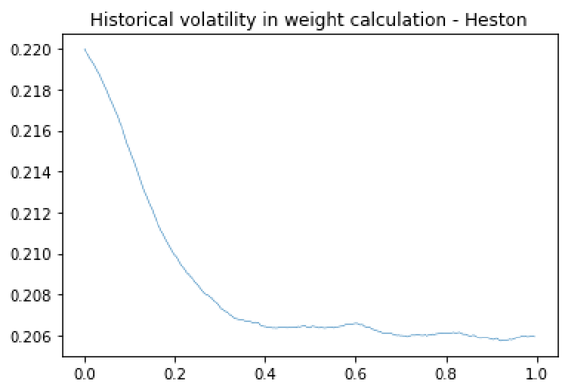


Figure 4.6: Historical volatility calculated by EWMA method under Heston model.

Figures 4.7 and 4.8 show the average of calculated historical variance $U^2(t_i)$ and the realized variance $R^2(t_i)$ of the underlying asset under Black Sholes model, while Figures 4.9 - 4.12 show the same for two different simulations under Heston model.

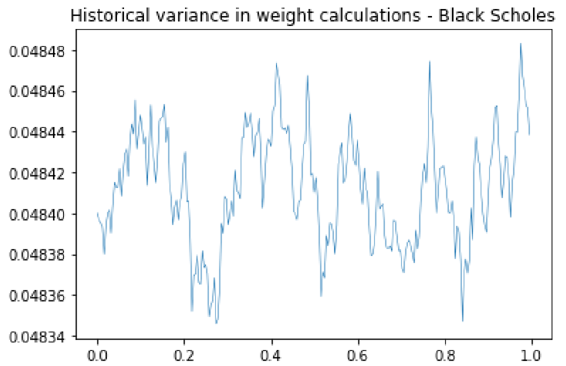


Figure 4.7: Exponentially weighted annualized historical variance under Black Scholes model.

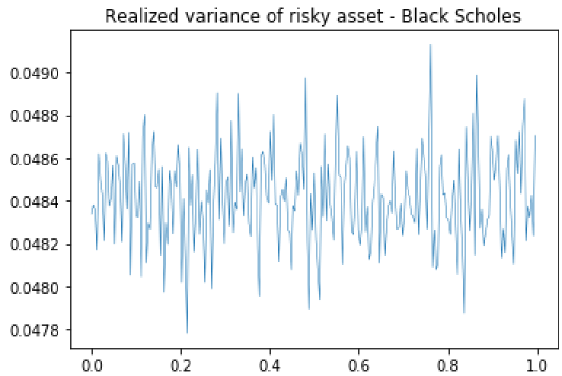


Figure 4.8: Realized variance of the risky asset log-return under Black Scholes model.

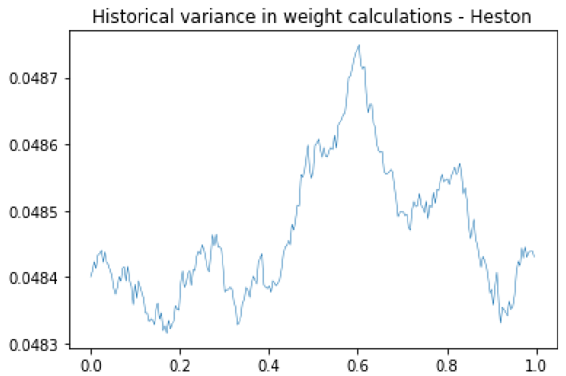


Figure 4.9: Exponentially weighted annualized historical variance under Heston model.

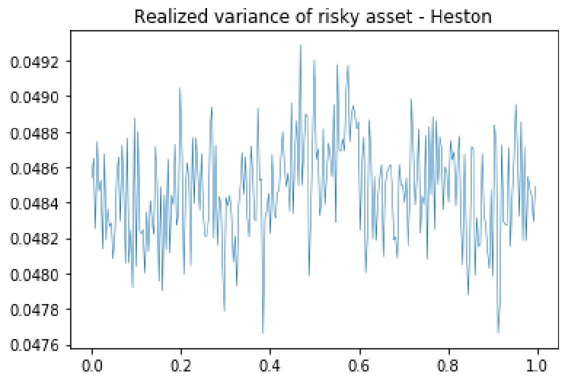


Figure 4.10: Realized variance of the risky asset log-return under Heston model.

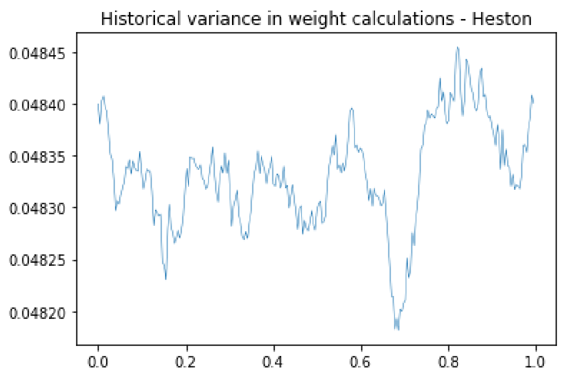


Figure 4.11: Exponentially weighted annualized historical variance under Heston model.

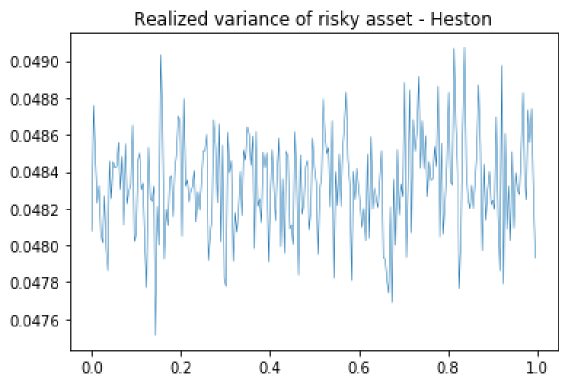


Figure 4.12: Realized variance of the risky asset log-return under Heston model.

It can be seen that there is no significant difference in Fig. 4.8, 4.10 and 4.12 from the value of the long term variance $0.22^2 = 0.0484$ of the risky asset. However under

Heston model the historical variance can show some divergence from the value of 0.0484 into both directions in Fig. 4.9 and 4.11 , the historical volatility always shows a decrease in Fig. 4.5 and 4.6 regardless this slight trend in historical variance.

The upward bias present in my simulations is also observed in the simulations of Kahl in Ref. [19] and the he suggests to calculate the expected value of the quadratic variation of the logarithm of the VTS portfolio to measure its expected realized variance. In the following I am going to follow the formulas found in Ref. [19], and the further numerical results and their relations to my figures presented here are my outcome. (Please note that in my concrete example examined in figures above, the cap effect is not dominant as weights are significantly smaller than the maximum allowed leverage level.) For this, let us consider the portfolio level expressed as in Eq. 3.24 using the notations σ_{VT} as the volatility target level instead of $\hat{\sigma}$ and ν_t^{exp} as the exponentially weighted volatility estimator:

$$\begin{aligned} E([\log V]_T) &= E\left(\int_0^T (\alpha_t\sigma_t)^2 dt\right) = \sigma_{VT}^2 E\left(\int_0^T \frac{\sigma_t^2}{\nu_t^{exp}} dt\right) \\ &\approx \sigma_{VT}^2 \sum_i E(t_{i+1}-t_i)\left[\frac{\sigma_{t_i}^2}{\nu_{t_i}^{exp}}\right] = \sigma_{VT}^2\ \Delta t\ \sigma_{t_i}^2 \sum_i E\left(\frac{1}{\nu_{t_i}^{exp}}\right). \end{aligned} \tag{4.14}$$

On one hand, we can use the Jensen's inequality $f(E(X)) \leq E(f(X))$ for the convex $f(x) = \frac{1}{x}$ function for $x > 0$ and for $\nu_{t_i}^{exp}$ as the random variable at time t_i [16, 19]. As a consequence

$$\frac{1}{E(\nu_{t_i}^{exp})} \leq E\left(\frac{1}{\nu_{t_i}^{exp}}\right). \tag{4.15}$$

Under Black Scholes model $\frac{1}{E(\nu_{t_i}^{exp})} = \frac{1}{\sigma_{BS}^2}$, therefore we can derive 4.14 further as:

$$E([\log V]_T) \approx \sigma_{VT}^2\ \Delta t\ \sigma_{BS}^2 \sum_i E\left(\frac{1}{\nu_{t_i}^{exp}}\right) \geq \sigma_{VT}^2\ \Delta t\ \sigma_{BS}^2 \sum_i \frac{1}{\sigma_{BS}^2} = \sigma_{VT}^2\ T. \tag{4.16}$$

This can partly explain the increase in the realized volatility of the MLVTS portfolio in Figures 4.1 and 4.2 as the cap effect does not have a significant effect. Similar interpretation can illustrate the decrease in historical volatility in Figures 4.5 and 4.6 applying the Jensen inequality to the volatility estimator ν_t^{exp} with the concave function $f(x) = \sqrt{x}$ for $x > 0$: the expected value of the historical volatility is not greater than the square root of the expected value of the historical variance according to $E(\sqrt{\nu_t^{exp}}) \leq \sqrt{E(\nu_t^{exp})}$.

This is in line with my observation in Figures 4.5 and 4.6 which show lower average historical volatility values than 0.22.

On the other hand, the figures above also provide the idea of studying the calculation of the volatility estimator. The daily log-returns of the risky asset are normally distributed with a mean of zero and a variance of $\sigma_t^2\ \Delta t$, therefore the square of the log-returns are chi-square distributed. We can derive the volatility estimator where $z_t = \log\left(\frac{S_t}{S_{t-1}}\right)$ as following [19]:

$$\nu_t^{exp} = \lambda\ \nu_{t-1}^{exp} + (1-\lambda)\frac{1}{\Delta t}\ z_{t-1}^2. \tag{4.17}$$

If we consider a finite lookback time horizon d and using the notation $y_t = \frac{z_t}{\sigma_t\ \sqrt{\Delta t}}$, it can be written as:

$$\nu_t^{exp} = (1-\lambda)\frac{1}{\Delta t}\ \sum_{j=1}^{d} \lambda^{j-1} z_{t-j}^2 = (1-\lambda)\frac{1}{\Delta t}\ \sigma_t^2\ \Delta t \sum_{j=1}^{d} \lambda^{j-1} y_{t-j}^2, \tag{4.18}$$

Let Y_t denote $Y_t = (1-\lambda)\sum_{j=1}^{d} \lambda^{j-1} y_{t-j}^2$ which characteristic function $\psi_{Y_t}(u)$ is known and can be used to calculate the first inverse moment of Y_t as $E(Y_t^{-1}) = \int_0^\infty \psi_{Y_t}(iu)du$. This integral has to be calculated numerically and it can be shown that $E(Y_t^{-1}) > 1$ for any $\lambda < 1$ [19]. Therefore we can continue Eq. 4.14 as

$$\begin{aligned} E([\log V]_T) &\approx \sigma_{VT}^2\ \Delta t\ \sigma_{t_i}^2 \sum_i E\left(\frac{1}{\nu_{t_i}^{exp}}\right) \approx \sigma_{VT}^2\ \Delta t\ \sigma_{t_i}^2 \sum_i E\left(\frac{1}{\sigma_{t_i}^2\ Y_{t_i}}\right) \\ &= \sigma_{VT}^2 \Delta t\ \sum_i E(Y_{t_i}^{-1}) > \sigma_{VT}^2\ T. \end{aligned} \tag{4.19}$$

We showed that the structure of historical volatility estimator causes the excess volatility and this was the reason behind the appearing upward bias of the volatility of the MLVTS portfolio in my simulation. However, Ref. [19] also shows how the λ parameter and the excess volatility of the volatility target strategy relate to each other: lower λ values produce higher excess volatility. The direct relation between the expected volatility and λ is

$$E([\log V]_T) = \sigma_{VT}^2 \beta^2\ T \sim \sigma_{VT}^2 \frac{1}{\lambda}\ T, \tag{4.20}$$

where β is the correction factor.

This connection implies that for $\lambda = 0.94$, $VT = 10\%$ and $T = 1$ the expected volatility should be around 10.314%, which is in line with my simulations in Figures 4.1 and 4.2. The next Figure 4.13 shows this relation through the call prices linked to the MLVTS portfolio with respect to the value of λ in my simulation. We can see that increasing the λ decaying constant results in decreasing call prices, which is due to the realized volatility of the MLVTS portfolio decreased according to the relation in Eq. 4.20.

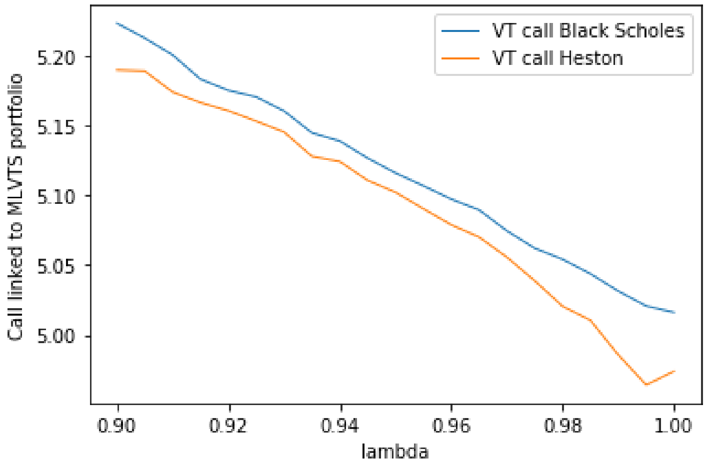


Figure 4.13: ATM call prices linked to MLVTS portfolio with respect to λ parameter.

4.2.3 Implied volatility surface

To have a better insight to the behavior of the volatility of the MLVTS portfolio, I examined the implied volatility surface of the call options. I used the Newton-Raphson method to calculate the implied volatility of the option, which derives the approximated σ_{i+1} at each step as

$$\sigma_{i+1} = \sigma_i - \frac{C(\sigma_i) - C_m}{\frac{\partial C}{\partial \sigma_i}}, \tag{4.21}$$

where σ_0 is the initial guess for the implied volatility, $C(\sigma_i)$ is the call price under Black Scholes model with a constant volatility σ_i, C_m is the market price of the option and $\frac{\partial C}{\partial \sigma_i}$ is the Vega in terms of σ_i. I used my simulated option prices under both models as market prices C_m to calculate the corresponding implied volatility σ_i. For this, we only need the value of Vega which can be achieved by taking the first partial derivative of the Black Scholes formula with respect to σ as:

$$\text{vega}(C) = \frac{\partial C}{\partial \sigma} = Se^{-r(T-t)}N'(d_1)\frac{\partial d_1}{\partial \sigma} - Ke^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial \sigma}, \tag{4.22}$$

which leads to the form of $\text{vega}(C) = S\ N'(d_1)\sqrt{T-t}$ [13].

In the next Figures 4.14 and 4.15 we can see the implied volatility surface of both the pure risky portfolio and the MLVTS portfolio under both models with respect to the strike price K ranging from 80 to 120. Volatility target level was set to be 10%, with time to maturity of 1 year. The initial value of the risky asset and the MLVTS portfolio is 100.

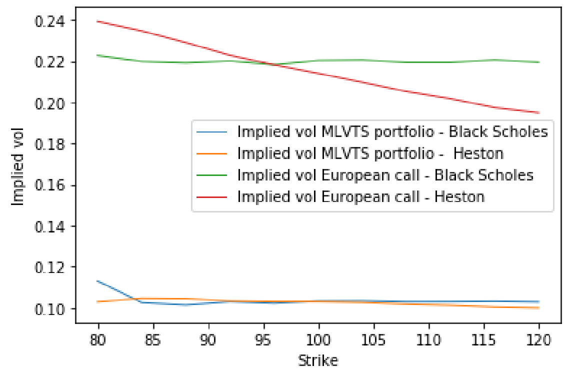


Figure 4.14: Implied volatility surface with $\sigma_{BS} = \sqrt{\theta^*} = 22\%$ and $VT = 10\%$.

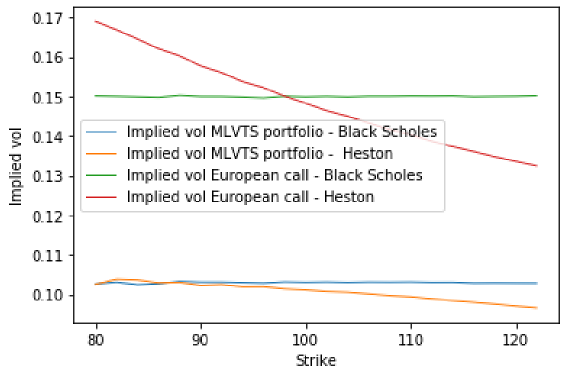


Figure 4.15: Implied volatility surface with $\sigma_{BS} = \sqrt{\theta^*} = 15\%$ and $VT = 10\%$.

Figures 4.14 and 4.15 show us the connection between the Black Scholes and Heston call prices. For $\rho = -0.569$, the graphs show Heston call prices overperform the Black Scholes call prices of pure risky portfolio for low strike values which means In-the-money (ITM) calls are more expensive under Heston model. This is due to the distribution of the log-returns is negatively skewed for $\rho < 0$ under Heston model which implies more weight is assigned to the left tail than under Black Scholes model. ITM calls have a strike price that lies in the left tail, therefore it produces more expensive deep ITM calls under Heston model. For Out-of-the-money (OTM) calls it is the opposite: less weight is assigned to the right tail where the strike price is located, as a consequence under Heston model deep OTM calls are less expensive than under Black Scholes model [14, 15].

The aim of the risk control strategy is to achieve a portfolio which targets a constant volatility level and therefore it does not depend on the volatility of the underlying risky asset. Figures 4.14 and 4.15 show us the implied volatility of the calls linked to the MLVTS portfolio approximates the 10% target level. Also, we can notice while the pure risky asset portfolio produced different option prices, the MLVTS portfolio linked calls have more nearly implied volatility surfaces and consequently more nearly prices under the two different models, however we can still observe the connection described above. The upward bias is observable when the risky asset volatility is much higher than the VT level, but when these two ones are getting closer and the α_t weights are getting bigger, therefore the exposure to the risky underlying asset as well, Heston call prices tend to decrease, which was also noticed during the comparison analysis in Subsection 4.2.1. In the following, I am going to examine the relation between the volatility target level and the long-term volatility of the underlying risky asset based on the previous observations.

4.2.4 Cap effect of the weights

I studied further the relationship between the Black Scholes and Heston prices of the call options linked to the MLVTS portfolio by performing sensitivity tests focusing on the cap effect. Previously, I performed tests mostly when the volatility of the underlying risky asset was higher than the volatility target level and it did not make it possible for the cap effect to prevail. In this subsection I am going to focus on the relation between the volatility target level and the long-term volatility and the cap effect of the α_t weights.

I examined how the option price evolves if VT level ranges from 5% to 45% while the constant and the long-term volatility of the risky underlying asset is set to be 22%. Figure 4.16 shows the ATM call option prices with respect to the volatility target level. Heston model call prices underperform the Black Scholes prices in a way when the VT level equals to or approximates the volatility σ of the underlying asset then the difference between the prices is the highest, when VT level is much lower than σ then these two prices nearly match and a constant difference can be observed between the prices for higher VT levels than σ. Also, high VT levels imply high α_t weights which are capped at 1, this can explain why the prices flatten out at the level of the price of the call option on a pure risky portfolio. Lower Heston prices can be explained by its volatility structure. The same explanation can be provided as in the previous subsection: the negative ρ parameter causes a negative skew, therefore more weight is assigned to the left tail in the distribution of the underlying log-returns.

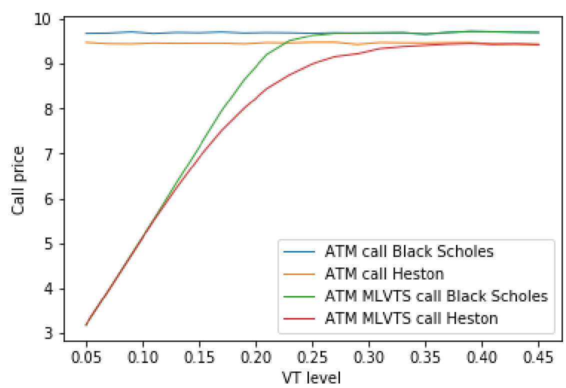


Figure 4.16: ATM call prices with respect to VT level with long-term volatility of 22%.

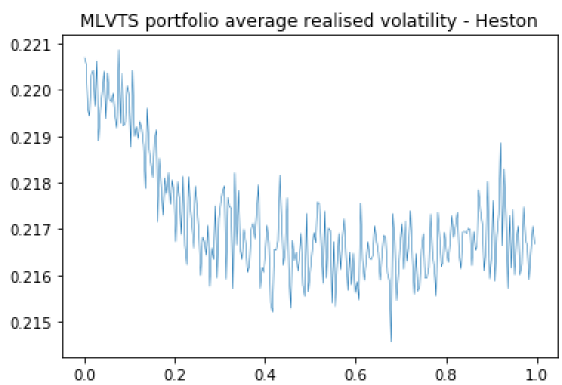


Figure 4.17: Average realized volatility of MLVTS portfolio with respect to time under Heston model. $\sqrt{\theta^*} = 22\%$ and $VT = 30\%$.

While the upward bias of the realized volatility of the MLVTS portfolio was specific to the cases when the cap effect could not prevail, the opposite happens for now when the cap effect is dominant. As not the realized volatility of the risky underlying is the driver of the asset allocation, but the cap, the realized volatility level of the MLVTS portfolio is below the VT level. This can be seen in Figure 4.17 which shows the realized MLVTS portfolio volatility under Heston model, with $\sqrt{\theta^*} = 22\%$ and $VT = 30\%$.

4.3 Summary of implementation

Overall, we can say that based on the tests performed the Heston model can capture the negative correlation between return and volatility and in many cases there is a significant difference between Heston and Black Scholes option prices. I studied the relation between the call prices under the two models and found that the general relation described in Subsection 4.2.3 is observable, however it is more noticeable when the weights and therefore the exposure to the risky asset in the MLVTS portfolio are higher. This can explain why Heston call prices underperformed Black Scholes call prices in my simulations and why the difference is getting more significant with increasing weights to the risky underlying.

Also, I could notice there is an upward bias observable in the realized volatility of the MLVTS portfolio which is due to the structure of the exponentially weighted moving average volatility estimator which is also used by the S&P 500 risk control indices. S&P Dow Jones also provides variance based risk control indices, where the target level of variance is set instead of the volatility level [5]. As the risk control indices come with different volatility target levels ranging from 5% to 18% and the underlying S&P 500 Index annual volatility is around 14%, we can say using Heston model against Black Scholes model when pricing option linked to them does mean a difference.

Furthermore, my simulation prices showed an upward bias compared to prices published by Jawaid in Ref [18] in Table 4.3 and 4.5. As an explanation could be provided in Subsection 4.2.2 using Ref. [19] which supported the phenomenon of the upward bias of the realized volatility of MLVTS portfolios using EWMA method, I found my simulation prices to be more accurate than prices in Ref. [18].

Chapter 5

Conclusion

In this thesis I presented a risk management strategy called Volatility Target Strategy and its mechanism, which is widely used in financial markets. I focused on the topic of pricing options linked to Volatility Target Strategy portfolios under different financial market models, by name Black Scholes and Heston models.

First, I introduced the S&P Dow Jones Risk Control Indices and their usage and highlighted their main properties and benefits which make them attractive to investors. After that, I presented the mathematical framework of the strategy and the closed-end pricing formulas for options linked to VTS and MLVTS portfolios in general cases. Then, the methodology of risk control indices used in real-world was also introduced.

The aim of my implementation was to investigate whether there is a difference between using a constant volatility model and a stochastic volatility model when I price an option on a portfolio targeting a constant volatility. First, I compared my simulation results to Jawaid's results published in Ref. [18] and I could see they do not match precisely. The comparison analysis raised some questions which were the base of further analysis. In my simulations, I could observe a slight upward bias of the realized volatility of the MLVTS portfolio. Kahl in Ref. [19] also investigates the reason behind this phenomenon and finds that it is the direct consequence of the structure of the stochastic weights calculated by the EWMA method. I also illustrated his statement about the relation between the memory of the lookback and excess volatility. Furthermore, I showed that the general relation described in Ref. [14] between the Black Scholes and Heston call prices is also observable even when the options are linked to the MLVTS portfolio, and results in higher Black Scholes ATM call prices. However, we could notice that the difference between the prices under the two models is more significant when the exposure to the risky underlying asset is more dominant.

List of Figures

Bibliography

[1] Perchet, R., Leote de Carvalho, R., Heckel, T., Moulin, P. (2015) *Predicting the Success of Volatility Targeting Strategies: Application to Equities and Other Asset Classes*

[2] Albeverio, S. (2013) *Investment instruments with volatility target mechanism*

[3] Jawaid, H. (2016) *An Analysis of Guaranteed Lifetime Withdrawal Benefits Linked to Target Volatility Portfolio*

[4] S&P Dow Jones Indices (2016) *Limiting Risk Exposure With S&P Risk Control Indices*

[5] S&P Dow Jones Indices (December 2020) *Index Mathematics Methodology*

[6] S&P Dow Jones Indices (September 2020) *S&P Dow Jones Risk Control Indices Methodology and Parameters*

[7] S&P Dow Jones Indices (2019) *Understanding the S&P Managed Risk 2.0 Indices*

[8] S&P Dow Jones Indices (2018) *A Performance Analysis of Variable Annuities With Risk Control*

[9] Khuman, A., Maringer, D., Constantinou, N. (2008) *Constant Proportion Portfolio Insurance: Statistical Properties and Practical Implications*

[10] Marsh, T., Pfleiderer, P. (2013) *Flight to Quality and Asset Allocation in a Financial Crisis*

[11] Bae, J., Kim, CJ., Nelson, C. R. (2004) *Why Are Stock Returns and Volatility Negatively Correlated?*

[12] Kuo, H-H. (2006) *Introduction to Stochastic Integration*

[13] Haugh, M. (2013) *Black-Scholes and the Volatility Surface*

[14] Rouah, F. D. (2013) *The Heston Model and Its Extensions in Matlab and C#*

[15] Heston, S. L. (1993) *A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options*

[16] Shreve, S. (2004) *Stochastic Calculus for Finance*

[17] Di Persio, L., Prezioso, L., Wallbaum, K. (2019) *Closed-End Formula for options linked to Target Volatility Strategies*

[18] Jawaid, H. (2015) *Pricing and Hedging of the European Option Linked to Target Volatility Portfolio*

[19] Kahl, C. (2015) *Volatility control indices*

[20] Kling, A., Ruez, F., Ruß, J. (2009) *The Impact of Stochastic Volatility on Pricing, Hedging, and Hedge Efficiency of Variable Annuity Guarantees*

Appendix A

The code of the implementation

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import norm
import random
import math
import statistics
import pandas as pd
```

A.1 Simulation of the MLVTS portfolio under Black Scholel model

```
class black_scholes:
neve= "Black Scholes"
def __init__(self, S, K, r, vol, portfolio0, lambda_star, VT, T, n, N,
statistic, graph):
    self.vol = vol
    self.stock = S
    self.strike = K
    self.portfolio0 = portfolio0
    self.r = r
    self.VT = VT
```

```
    self.lambda_star = lambda_star
    self.T = T
    self.n = n
    self.path = N
    self.statistic = statistic
    self.is_graph=graph

#option linked to voltarget portfolio with VT vol has the same
#theoretical price as a call option with VT vol:
def bs_call(self):
    d1 = (math.log(self.stock/self.strike)+(self.r+self.VT**2.0/2.0)*self.T)
/self.VT/math.sqrt(self.T)
    d2 = d1-self.VT*math.sqrt(self.T)
    df = math.exp(-self.r*self.T)
    call=self.stock*norm.cdf(d1) -df * self.strike*norm.cdf(d2)
    return(call)

def bs_call_VTP_MC_simulation(self):

    dt=1/self.n

    y=[]      #time points
    y.append(0)
    B = np.zeros((self.n*self.T+1, self.path))  #riskless
    B[0,:] = self.stock
    S = np.zeros((self.n*self.T+1, self.path))  #risky asset
    S[0,:] = self.stock

    alfa = np.zeros((self.n*self.T, self.path))  #risky proportion
    alfa[0,:] = np.minimum(1, self.VT/self.vol)
    portfolio = np.zeros((self.n*self.T+1, self.path))  #VT portfolio
    portfolio[0,:] = self.portfolio0

    #stochastic weights
    R= np.zeros((self.n*self.T+1, self.path))   #daily logreturns
    R[0,:]=self.vol   #not used
    U= np.zeros((self.n*self.T+1, self.path))   #historical vol
    U[0,:]=self.vol
```

```
    numint=np.zeros((1,self.path))   # int(portfolio_vol^2)ds

    atlag_R_sq=[]
    atlag_U=[]
    atlag_U_sq=[]
    atlag_alfa=[]
    atlag_portfolio_vol=[]
    portfolio_vol=[]
    historical_vol=[]
    historical_variance=[]
    historical_var=[]


    #############    simulation   ##############

    for i in range (1, self.n*self.T+1):
        y.append(i*dt)
        wiener_1 = np.random.normal(0,1,self.path)
        wiener_2 = np.random.normal(0,1,self.path)

        if (i == 1):
            U[i-1,:] = self.vol
            alfa[i-1,:] =  np.minimum(1, self.VT/U[i-1,:])
            numint[0,:] += self.VT**2.*dt

        else:
            U[i-1,:]= np.sqrt(self.lambda_star*U[i-2,:]**2.+
(1-self.lambda_star)*R[i-1,:]**2./dt)
            alfa[i-1,:]= np.minimum(1, self.VT/U[i-1,:])
            numint[0,:] +=  (alfa[i-1,:]*U[i-1,:])**2.*dt



        B[i,:]=B[i-1,:]*math.exp(self.r*dt)
        S[i,:]=S[i-1,:]*np.exp(((self.r-(self.vol**2.)/2.)*dt+
self.vol*dt**0.5*wiener_2))
        R[i,:]=np.log(S[i,:]/S[i-1,:])
        portfolio[i,:]=portfolio[i-1,:]*(1+alfa[i-1,:]/S[i-1,:]*(S[i,:]-S[i-1,:])+
```

```
(1-alfa[i-1,:])/B[i-1,:]*(B[i,:]-B[i-1,:]))



        atlag_R_sq.append(np.mean(R[i,:]**2./dt))
        atlag_U.append(np.mean(U[i-1,:]))
        atlag_U_sq.append(np.mean(U[i-1,:]**2.))
        atlag_portfolio_vol.append(np.mean((S[i,:]-S[i-1,:])/S[i-1,:]*
alfa[i-1,:]/np.sqrt(dt)))
        atlag_alfa.append(np.mean(alfa[i-1,:]))
        portfolio_vol.append(np.std(np.log(portfolio[i,:]/portfolio[i-1,:])
/np.sqrt(dt)))
        historical_vol.append(np.mean(U[i-1,:]))
        historical_var.append(np.mean(R[i,:]**2./dt))
        historical_variance.append(np.mean(U[i-1,:]**2.))

    call_VT_portfolio=sum(x-self.strike for x in portfolio[self.n*self.T,:]
 if x-self.strike > 0)
    call_VT_portfolio=call_VT_portfolio*math.exp(-self.r*self.T)/self.path

    call_risky=sum(x-self.strike for x in S[self.n*self.T,:] if x-self.strike > 0)
    call_risky=call_risky*math.exp(-self.r*self.T)/self.path



    ####### graph #######
    if (self.is_graph==True):
        fig1, ax1 = plt.subplots()
        fig2, ax2 = plt.subplots()
        fig3, ax3 = plt.subplots()
        fig4, ax4 = plt.subplots()
        fig5, ax5 = plt.subplots()
        timepoint2=y[:-1]

        ax2.set_title('Historical variance in weight calculations - Black Scholes')
        ax2.plot(timepoint2, historical_variance, linewidth=0.5, label='vol')
        ax3.set_title('MLVTS portfolio realized volatility - Black Scholes')
        ax3.plot(timepoint2, portfolio_vol, linewidth=0.5, label='vol')
        ax4.set_title('Average of alpha - Black Scholes')
        ax4.plot(timepoint2, atlag_alfa, linewidth=0.5, label='alpha')
```

```
        ax5.set_title('Historical volatility in weight calculations - Black Scholes')
        ax5.plot(timepoint2, historical_vol, linewidth=0.5)
        ax1.set_title('Realized variance of risky asset - Black Scholes')
        ax1.plot(timepoint2, historical_var, linewidth=.5)

        plt.show()

    ####### print out #######
    if self.statistic==True:
        print('vol négyzet: ', self.vol**2.)
        print('R^2 átlag: ', statistics.mean(atlag_R_sq))
        print('U átlag: ', statistics.mean(atlag_U), 'U^2 átlag: ',
statistics.mean(atlag_U_sq))
        print('alfa átlag: ', statistics.mean(atlag_alfa),'alfa stdev: ',
statistics.stdev(atlag_alfa))
        print('V portfolio vol: ', statistics.mean(atlag_portfolio_vol))


    ###### general analytical price for VTP linked call option ######

    z=np.zeros((1,self.path))
    d1=np.zeros((1,self.path))
    d2=np.zeros((1,self.path))
    call_VTP=np.zeros((1,self.path))

    z[0,:]=math.log(self.strike/self.portfolio0)-self.r*self.T+numint[0,:]/2
    d1[0,:] = (-z[0,:]+numint[0,:])/numint[0,:]**0.5
    d2[0,:] = -z[0,:]/numint[0,:]**0.5
    df = math.exp(-self.r*self.T)
    call_VTP[0,:]=self.portfolio0*norm.cdf(d1[0,:]) -
df * self.strike*norm.cdf(d2[0,:])
    call_VTP_atlag=np.mean(call_VTP)
    call_VTP_szoras=np.std(call_VTP)


    return(call_VT_portfolio, call_VTP_atlag, call_VTP_szoras,
statistics.mean(atlag_portfolio_vol),statistics.stdev(atlag_portfolio_vol),
call_risky)
```

```
def test_black_scholes(S, K, r, vol, portfolio0, lambda_star, VT, T, n, N, M, statist
    result_bs=[]
    bs_price=[]
    result_bs=black_scholes(S, K, r, vol, portfolio0, lambda_star, VT, T, n, N, stati
    bs_price=result_bs.bs_call()
    b=[]
    b=[result_bs.bs_call_VTP_MC_simulation() for i in range(M)]
    x=[]
    y=[]
    z=[]
    vol_mean=[]
    vol_stdev=[]
    callrisky=[]
    for i in range(0, len(b)):
        x.append(b[i][0])
        y.append(b[i][1])
        z.append(b[i][2])
        vol_mean.append(b[i][3])
        vol_stdev.append(b[i][4])
        callrisky.append(b[i][5])
    print('')
    print('Number of simulations: ', M, ' Number of paths: ', N)
    print('Theoretical price of European call with VT volatility: ', bs_price)
    print('Theoretical price of european call linked to VT portfolio:', 'mean: ',
statistics.mean(y), 'stdev: ', statistics.stdev(y))
    print('Theoretical price of european call linked to VT portfolio stdev:',
'mean: ', statistics.mean(z), 'stdev: ', statistics.stdev(z))

    print('VolTarget portfolio vol: ', 'mean: ',statistics.mean(vol_mean),
'stdev: ', statistics.stdev(vol_mean))
    print('VolTarget portfolio vol stdev: ', 'mean: ',statistics.mean(vol_stdev),
'stdev: ', statistics.stdev(vol_stdev))
    print('Simulation: European call linked to VT portfolio:', 'mean: ',
statistics.mean(x), 'stdev: ', statistics.stdev(x))
    return(statistics.mean(x), statistics.mean(callrisky))
```

A.2 Simulation of the MLVTS portfolio under Heston model

```
class Heston:
neve= "Heston"
def __init__(self, S, portfolio0, K, r, vol0, VoV, rho, voltarget, lambda_star,
T, n, N, graph, stats):
    self.stock = S
    self.portfolio0 = portfolio0
    self.vol0 = vol0
    self.VoV = VoV
    self.strike = K
    self.r = r
    self.T = T
    self.n = n
    self.path = N
    self.rho = rho
    self.VT = voltarget
    self.lambda_star = lambda_star  #decaying constant
    self.is_graph=graph
    self.is_stats=stats



def hestoncall(self):

    dt=1/self.n
    y=[]            #time points
    y.append(0)

    S=[]
    B=[]
    S = np.zeros((self.n*self.T+1, self.path))
    S[0,:] = self.stock
    B = np.zeros((self.n*self.T+1, self.path))
    B[0,:] = self.stock
    GBM = np.zeros((self.n*self.T+1, self.path))
```

```
GBM[0,:] = self.stock
V = np.zeros((self.n*self.T+1, self.path))        #variance
V[0,:] = self.vol0**2


portfolio = np.zeros((self.n*self.T+1, self.path))
portfolio[0,:] = self.portfolio0

if (self.is_graph==True):
    fig1, ax1 = plt.subplots()
    fig2, ax2 = plt.subplots()
    fig3, ax3 = plt.subplots()
    fig4, ax4 = plt.subplots()
    fig5, ax5 = plt.subplots()
    fig6, ax6 = plt.subplots()

#Heston under riskneutral measure
kappa=4.75
theta=0.22**2.
theta_star=self.vol0**2.
beta=(kappa*theta/theta_star-kappa)/self.VoV
k_star=kappa+beta*self.VoV


#stochastic weights
alfa = np.zeros((self.n*self.T+1, self.path))    #proportion of risky asset
alfa[0,:] = np.minimum(1, self.VT/self.vol0)
R= np.zeros((self.n*self.T+1, self.path))
R[0,:] = 0
U= np.zeros((self.n*self.T+1, self.path))   #U(u(k))=sigma(t(k))
U[0,:]=self.vol0
numint=np.zeros((1,self.path))

atlag_R_sq=[]
atlag_U=[]
atlag_U_sq=[]
atlag_alfa=[]
atlag_portfolio_vol=[]
```

```
    portfolio_vol=[]
    historical_vol=[]
    historical_variance=[]
    historical_var=[]

    #############    simulation using Milstein scheme   ##############

    wiener_vol = np.random.normal(0,1,(self.n*self.T, self.path))
    wiener_seged = np.random.normal(0,1,(self.n*self.T, self.path))
    wiener_stock = self.rho * wiener_vol +
math.sqrt(1-self.rho**2) * wiener_seged


    for i in range(1, self.n*self.T+1):

        y.append(i*dt)

        if (i == 1):
            U[i-1,:] = self.vol0
            alfa[i-1,:] =  np.minimum(1, self.VT/U[i-1,:])
            numint[0,:] += np.minimum(self.VT, V[0,:]**0.5)**2.*dt
        else:
            U[i-1,:]=np.sqrt(self.lambda_star*U[i-2,:]**2.+(1-self.lambda_star)*
R[i-1,:]**2./dt)
            alfa[i-1,:]= np.minimum(1, self.VT/U[i-1,:])
            numint[0,:] += (alfa[i-1,:]*U[i-1,:])**2.*dt



        B[i,:]=B[i-1,:]*math.exp(self.r*dt)
        V[i,:]=np.maximum(0,(V[i-1,:]**0.5+self.VoV/2*dt**0.5*
wiener_vol[i-1,:])**2+(k_star*(theta_star-V[i-1,:])-self.VoV**2/4)*dt)
        S[i,:]=S[i-1,:]*np.exp((self.r-V[i-1,:]/2)*dt+(np.sqrt(V[i-1,:]*dt))*
wiener_stock[i-1,:])
        R[i,:]=np.log(S[i,:]/S[i-1,:])
        portfolio[i,:]=portfolio[i-1,:]*(1+alfa[i-1,:]/S[i-1,:]*(S[i,:]-S[i-1,:])+
(1-alfa[i-1,:])/B[i-1,:]*(B[i,:]-B[i-1,:]))

        atlag_R_sq.append(np.mean(R[i,:]**2./dt))
```

```
        atlag_U.append(np.mean(U[i-1,:]))
        atlag_U_sq.append(np.mean(U[i-1,:]**2.))
        atlag_portfolio_vol.append(np.std(np.log(portfolio[i,:]/portfolio[i-1,:])
/np.sqrt(dt)))
        atlag_alfa.append(np.mean(alfa[i-1,:]))
        portfolio_vol.append(np.std(np.log(portfolio[i,:]/portfolio[i-1,:])
/np.sqrt(dt)))
        historical_vol.append(np.mean(U[i-1,:]))
        historical_var.append(np.mean(R[i,:]**2./dt))
        historical_variance.append(np.mean(U[i-1,:]**2.))


    if (self.is_graph==True):

        timepoint=y[-1:]
        timepoint2=y[:-1]
        ax1.set_title('MLVTS portfolio average realised volatility - Heston')
        ax1.plot(timepoint2, portfolio_vol, linewidth=0.5)
        ax2.set_title('Average of alpha - Heston')
        ax2.plot(timepoint2, atlag_alfa, linewidth=0.5)
        ax3.set_title('Historical volatility in weight calculation - Heston')
        ax3.plot(timepoint2, historical_vol, linewidth=0.5)
        ax4.set_title('Realized variance of risky asset - Heston')
        ax4.plot(timepoint2, historical_var, linewidth=.5)
        ax5.set_title('Historical variance in weight calculations - Heston')
        ax5.plot(timepoint2, historical_variance, linewidth=0.5, label='vol')

        plt.show()

    call_risky=sum(x-self.strike for x in S[self.n*self.T,:] if x-self.strike > 0)
    call_risky=call_risky/self.path*math.exp(-self.r*self.T)



    call_VT_portfolio=sum(x-self.strike for x in portfolio[self.n*self.T,:]
if x-self.strike > 0)
    call_VT_portfolio=call_VT_portfolio*math.exp(-self.r*self.T)/self.path

    ####### print out #######
```

```
    if(self.is_stats==True):
        print('R^2 átlag: ', statistics.mean(atlag_R_sq))
        print('U átlag: ', statistics.mean(atlag_U), 'U^2 átlag: ',
statistics.mean(atlag_U_sq))
        print('alfa átlag: ', statistics.mean(atlag_alfa),'alfa stdev: ',
statistics.stdev(atlag_alfa))
        print('V portfolio vol: ', statistics.mean(atlag_portfolio_vol))


    ###### general analytical price for VTP linked call option ######

    z=np.zeros((1,self.path))
    d1=np.zeros((1,self.path))
    d2=np.zeros((1,self.path))
    call_VTP=np.zeros((1,self.path))

    z[0,:]=math.log(self.strike/self.portfolio0)-self.r*self.T+numint[0,:]/2
    d1[0,:] = (-z[0,:]+numint[0,:])/numint[0,:]**0.5
    d2[0,:] = -z[0,:]/numint[0,:]**0.5
    df = math.exp(-self.r*self.T)
    call_VTP[0,:]=self.portfolio0*norm.cdf(d1[0,:]) -df * self.strike*norm.cdf(d2[0,:
    call_VTP_atlag=np.mean(call_VTP)
    call_VTP_szoras=np.std(call_VTP)

    return(call_VT_portfolio, call_VTP_atlag, call_VTP_szoras,
statistics.mean(atlag_portfolio_vol),statistics.stdev(atlag_portfolio_vol),call_risky


def test_heston(S, portfolio0, K, r, vol0, VoV, rho, voltarget, lambda_star,
T, n, N, M, graph,stats):
    result_heston=[]
    result_heston=Heston(S, portfolio0, K, r, vol0, VoV, rho, voltarget, lambda_star,
T, n, N, graph,stats)
    b=[]
    b=[result_heston.hestoncall() for i in range(M)]
    x=[]
    y=[]
```

```
    z=[]
    vol_mean=[]
    vol_stdev=[]
    callrisky=[]
    for i in range(0, len(b)):
        x.append(b[i][0])
        y.append(b[i][1])
        z.append(b[i][2])
        vol_mean.append(b[i][3])
        vol_stdev.append(b[i][4])
        callrisky.append(b[i][5])
    print('')
    print('Number of simulations: ', M, ' Number of paths: ', N)
    print('Theoretical price of european call linked to VT portfolio:',
'mean: ', statistics.mean(y), 'stdev: ', statistics.stdev(y))
    print('Theoretical price of european call linked to VT portfolio stdev:',
'mean: ', statistics.mean(z), 'stdev: ', statistics.stdev(z))
    print('VolTarget portfolio vol: ', 'mean: ',statistics.mean(vol_mean),
'stdev: ', statistics.stdev(vol_mean))
    print('VolTarget portfolio vol stdev: ', 'mean: ',statistics.mean(vol_stdev),
'stdev: ', statistics.stdev(vol_stdev))
    print('Simulation: European call linked to VT portfolio:', 'mean: ',
statistics.mean(x), 'stdev: ', statistics.stdev(x))
    return(statistics.mean(x),statistics.mean(callrisky))
```

A.3 Implied volatility surface analysis

```
def newton_vol_call(S, K, T, C, r, sigma):

    #S: spot price
    #K: strike price
    #T: time to maturity
    #C: Call value
    #r: interest rate
    #sigma: volatility of underlying asset
```

```
    d1 = (np.log(S / K) + (r + 0.5 * sigma ** 2) * T) / (sigma * np.sqrt(T))
    d2 = (np.log(S / K) + (r - 0.5 * sigma ** 2) * T) / (sigma * np.sqrt(T))

    fx = S * norm.cdf(d1, 0.0, 1.0) - K * np.exp(-r * T) *
norm.cdf(d2, 0.0, 1.0) - C

    vega = S * norm.pdf(d1, 0.0, 1.0) * np.sqrt(T)

    tolerance = 0.000001
    xnew=[]
    xnew  = sigma

    while abs(fx) > tolerance:
        #print(abs(fx))
        #print(xnew)
        xnew = xnew - fx/vega
        d1 = (np.log(S / K) + (r + 0.5 * xnew ** 2) * T) / (xnew * np.sqrt(T))
        d2 = (np.log(S / K) + (r - 0.5 * xnew ** 2) * T) / (xnew * np.sqrt(T))
        fx = S * norm.cdf(d1) - K * np.exp(-r * T) * norm.cdf(d2) - C
        vega = S * norm.pdf(d1, 0.0, 1.0) * np.sqrt(T)


    return (abs(xnew))


y=[]
imp_vol_BS=[]
imp_vol_Heston=[]
impl_vol_risky_h=[]
impl_vol_risky_bs=[]
for strike in range(80,124,2):
    y.append(strike)
    BS_eredmeny=test_black_scholes(S=100, K=strike, r=0.02, vol=0.15,
portfolio0=100, lambda_star = 0.94, VT=0.1, T=1, n=252, N=100000, M=50,
statistic=False,graph=False)
    call_price_VT_BS=BS_eredmeny[0]
    call_risky_BS=BS_eredmeny[1]
```

```
    heston_eredmeny=test_heston(S=100, portfolio0=100, K=strike, r=0.02, vol0=0.15,
 VoV=0.55, rho=-0.569,voltarget=0.1, lambda_star=0.94, T=1, n=252, N=100000,
M=50, graph=False,stats=False)
    call_price_VT_Heston=heston_eredmeny[0]
    call_risky_h=heston_eredmeny[1]

    imp_vol_BS.append(newton_vol_call(S=100, K=strike, T=1, C=call_price_VT_BS,
r=0.02, sigma=0.2))
    imp_vol_Heston.append(newton_vol_call(S=100, K=strike, T=1,
C=call_price_VT_Heston,r=0.02, sigma=0.2))
    impl_vol_risky_h.append(newton_vol_call(S=100, K=strike, T=1,
C=call_risky_h, r=0.02, sigma=0.2))
    impl_vol_risky_bs.append(newton_vol_call(S=100, K=strike, T=1,
C=call_risky_BS, r=0.02, sigma=0.2))

plt.plot(y, imp_vol_BS, label = 'Implied vol MLVTS portfolio - Black Scholes')
plt.plot(y, imp_vol_Heston, label = 'Implied vol MLVTS portfolio -  Heston')
plt.plot(y, impl_vol_risky_bs, label= 'Implied vol European call - Black Scholes')
plt.plot(y, impl_vol_risky_h, label= 'Implied vol European call - Heston')
plt.xlabel('Strike')
plt.ylabel('Implied vol')
plt.legend()
plt.show()
```

A.4 Sensitivity test with respect to λ decaying constant

```
y=[]
call_BS=[]
call_Heston=[]

for i in range(0,21):
    x=0.9+i*0.005
    y.append(x)
    call_BS.append(test_black_scholes(S=100, K=100, r=0.02, vol=0.22,
```

```
portfolio0=100, lambda_star = x, VT=0.1, T=1, n=252, N=100000,
M=50,statistic=False,graph=False)[0])
    call_Heston.append(test_heston(S=100, portfolio0=100, K=100, r=0.02,
vol0=0.22, VoV=0.55, rho=-0.569, voltarget=0.1, lambda_star=x, T=1, n=252,
N=100000, M=50, graph=False,stats=False)[0])

plt.plot(y, call_BS, label = 'VT call Black Scholes', linewidth=1)
plt.plot(y, call_Heston, label = 'VT call Heston', linewidth=1)
plt.xlabel('lambda')
plt.ylabel('Call linked to MLVTS portfolio')
plt.legend()
plt.show()
```

A.5 Sensitivity test with respect to VT level

```
y=[]
call_BS=[]
call_Heston=[]
call_BS_vts=[]
call_Heston_vts=[]

for i in range(0,41,2):
    y.append(i*0.01+0.05)
    test1=[]
    test2=[]
    test1=test_black_scholes(S=100, K=100, r=0.02, vol=0.22, portfolio0=100,
lambda_star = 0.94,VT=i*0.01+0.05, T=1, n=252, N=10000, M=100,
statistic=False, graph=False)
    test2=test_heston(S=100, portfolio0=100, K=100, r=0.02, vol0=0.22, VoV=0.55,
rho=-0.569, voltarget=i*0.01+0.05, lambda_star=0.94, T=1, n=252, N=10000,
M=100, graph=False,stats=False)
    call_BS.append(test1[1])
    call_Heston.append(test2[1])
    call_BS_vts.append(test1[0])
    call_Heston_vts.append(test2[0])
```

```
plt.plot(y, call_BS, label = 'ATM call Black Scholes', linewidth=1)
plt.plot(y, call_Heston, label = 'ATM call Heston', linewidth=1)
plt.plot(y, call_BS_vts, label = 'ATM MLVTS call Black Scholes', linewidth=1)
plt.plot(y, call_Heston_vts, label = 'ATM MLVTS call Heston', linewidth=1)
plt.xlabel('VT level')
plt.ylabel('Call price')
plt.legend()
plt.show()
```