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Pricing average price options

MSc Thesis

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1 Introduction

In this thesis we are going to present Asian option pricing methods. Asian options are also known as average price options (APO) and there are two types of them. Depending on the contract Asian options pay the difference between the average of the underlying asset price and the strike – that is pre-specified in the contract–, or they take the difference between the mean of the strikes and the spot price of the underlying asset at the maturity. The average is taken over the pre-set period of time and it can be either arithmetic or geometric (Mack, 2014). This thesis is mainly focusing on arithmetic average price options.

The first traded Asian option took place in Bankers Trust's Tokyo office, in Japan, in 1987. This type of exotic option is a very popular financial product in commodity markets. The high popularity is based on their benefits and these are the following. First but not least in commodity markets the transactions are related to huge quantities. So tradesmen's aim is to reduce the manipulation of the asset prices before the expiration. Hence by taking the average of the underlying asset prices makes harder to push the market up or down during the pre-set period and so it tends to be less volatile than plain vanilla option, whose payoff depend only on the asset price at the exercise day. Thus the risk of manipulation is decreased by taking the average of the assets' prices, precede wild fluctuation impacting on the trade and also the value of the option is cheaper than the plain vanilla options' values (Geman, 2009).

In contrary to European options, Asian options are hard to price. In practice traders usually use discrete monitoring and arithmetic averaging. This is a very obvious decision, since observed prices on markets are discrete, and arithmetic average is easy to count. However there is no closed form for pricing arithmetic average price options. Since, under the Black-Sholes model, where we assume that the underlying asset prices are log-normally distributed, the arithmetic average of the prices is not log-normal. This fact stimulated me to carry out research in this topic.

The rest of this thesis is organized as follows. In Section 2, we briefly introduce the theoretical framework for Asian options. It also contains a summary of the used methodologies to price Asian options and we detail four methods. In Section 3, we implement and compare the methods described before by simulation results. Finally, this thesis ends with a conclusion in Section 4.

2 Theoretical results

This section provides the theoretical basis for pricing APOs under the Black-Scholes model. It starts with a summary of the mathematical framework used for pricing APOs. Then we will give a brief overview of the pricing methods found in the literature we came across during the preparation of this thesis. Also, there will be a digression, where we will shortly present further techniques from the latest results. Later a basic pricing formula to price geometric Asian options will be given since this formula is needed for another technique. This thesis mainly focuses on the arithmetic average under discrete monitoring, in this case the option price doesn't have a closed form. At the end of this section we present some methods to price them. One of the highlighted techniques is the MC simulation. We show two variance reduction techniques as well, namely the Control variate and the Antithetic methods. The other highlighted technique is the Moment Matching method with the log-normal approximation.

2.1 Mathematical framework

Before going into details of the Asian option specialties, we will describe the necessary notations, definitions, theorems and formulas to create the mathematical framework for this thesis. The fundamental concepts of stochastic processes and financial mathematics can be found for example in Shreve (2004), Márkus (2017) or in the Stochastic Processes lecture notes by Vilmos Prokaj.

The following proposition will be used for several calculations later.

Proposition 1. Let $\alpha \in \mathbb{R}$ be a constant real number. If Z is standard normal, then

$$\mathbb{E}\left(e^{\alpha Z}\right) = e^{\frac{\alpha^2}{2}} \tag{2.1}$$

Proof.

$$\mathbb{E}\left(e^{\alpha Z}\right) = \int\limits_{-\infty}^{\infty} e^{\alpha z}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz = \int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}e^{\frac{2\alpha z - z^2}{2}}dz =$$
$$= \int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}e^{\frac{-(z-\alpha)^2+\alpha^2}{2}}dz = e^{\frac{\alpha^2}{2}}\int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}e^{\frac{-(z-\alpha)^2}{2}}dz = e^{\frac{\alpha^2}{2}}$$

□

The following propositions are needed, because techniques discussed later in this master thesis will use them. When we would like to determine the moments of the underlying asset prices S_t in Section 2.5, Itô Isometry will be useful.

Definition 1. $(\varphi_s) \in \mathcal{S}$, *if* $(\varphi_s)_{s\geq 0}$ *is a progressively measurable process and* $\mathbb{E}\left(\int \varphi_s^2 ds\right) < \infty$ *for* $\forall t$.

Proposition 2. $(\varphi_s)_{s\geq 0}$ *process is progressively measurable, if it is* $\mathcal{F}_s$*-adapted with right-continuous trajectory.*

Proposition 3. *Suppose that* $\varphi_s \in \mathcal{S}$ *and deterministic, then* $X_t = \int\limits_0^t \varphi_s dW_s$ *is*

- *a martingale,* $\mathbb{E}(X_0) = \mathbb{E}(X_t) = 0$
- *a Gaussian process*

Proposition 4 (Itô Isometry)**.** *Suppose that* $\varphi_s, \eta_s \in \mathcal{S}$, *then using the notation* $t_i \wedge t_j = min(t_i, t_j)$

$$\mathbb{E}\left(\left(\int\limits_0^t \varphi_s dW_s\right)^2\right) = \mathbb{E}\left(\int\limits_0^t \varphi_s^2 ds\right) \tag{2.2}$$

$$\mathbb{E}\left(\int\limits_0^{t_i} \varphi_s dW_s \cdot \int\limits_0^{t_j} \eta_s dW_s\right) = \mathbb{E}\left(\int\limits_0^{t_i\wedge t_j} \varphi_s \eta_s ds\right) \tag{2.3}$$

The Black-Scholes model

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q})$ be a filtered, complete probability space, where $\{\mathcal{F}_t\}_{t\geq 0}$ is a right-continuous filtration, $\mathbb{Q}$ is the risk-neutral measure on $(\Omega, \mathcal{F})$. Henceforward $\mathbb{E}$ will denote the expected value, while $\mathbb{P}$ the probability under $\mathbb{Q}$.

We define the market model as the usual Black-Scholes model, which assumes that there are no arbitrage opportunities, taxes or transaction costs, no restrictions on short selling and the market is perfectly liquid. It also considers two asset price processes with given dynamics. One is the risk-free asset $(B_t)_{t\in[0,T]}$, the other one is the risky asset $(S_t)_{t\in[0,T]}$ that follows Geometric Brownian Motion (GBM). The dynamics are given by

$$dB_t = rB_t dt \tag{2.4}$$

$$dS_t = rS_t dt + \sigma S_t dW_t \tag{2.5}$$

where r, σ are deterministic constants corresponding to the interest rate and the volatility and W_t, $0 \leq t \leq T$ is the Brownian motion under the $\mathbb{Q}$ risk-neutral measure.

It is well known that the analytical solutions for stochastic differential equations (2.4) and (2.5) are

$$B_t = B_0 e^{rt} = e^{rt}$$

$$S_t = S_0 e^{\left(r-\frac{\sigma^2}{2}\right)t+\sigma W_t} \tag{2.6}$$

where B_0 is the amount of the initial risk free asset, we consider it to be equal to 1, $S_0 \geq 0$ is the initial risky asset price. The distribution of S_t is log-normal under $\mathbb{Q}$:

$$\ln\left(\frac{S_t}{S_0}\right) \sim N\left(\left(r - \frac{\sigma^2}{2}\right)t, \sigma^2 t\right)$$

It is also known that the discounted asset price process $\frac{S_t}{B_t}$ is a martingale under $\mathbb{Q}$.

Theorem 1 (General risk-neutral pricing formula)**.** *Let V_T denote the payoff of a given derivative on the exercise day, at time T. The general risk-neutral pricing formula and the initial price of any contingent claim with V_T payoff are the following:*

$$\begin{aligned} \frac{V_t}{B_t} &= \mathbb{E}\left[\frac{V_T}{B_T}\middle| \mathcal{F}_t\right] \\ V_0 &= B_0 \cdot \mathbb{E}\left[\frac{V_T}{B_T}\middle| \mathcal{F}_0\right] = \mathbb{E}\left[\frac{V_T}{B_T}\right] = e^{-rT}\,\mathbb{E}\left[V_T\right] \end{aligned} \tag{2.7}$$

where $0 \leq t \leq T$*.* $\mathcal{F}_0 = \{\Omega, \emptyset\}$ *is the trivial* σ*-algebra.*

Throughout this thesis V_0 will denote the initial price of any contingent claim, and $c = V_0$ will stand for call options with any V_T payoff.

Fisher Black and Myron Scholes gave us a pricing formula for European options under the model and assumptions described above. The holder of an European option can only exercise it on the expiry day. The European call payoff is $V_T = (S_T - K)^+$ and the European put payoff is $V_T = (K - S_T)^+$, where S_T is the underlying asset price at time T, and K is the predefined strike price.

Theorem 2 (Black-Scholes option pricing formula)**.** *The value of the European call option on stock asset S with K strike price and T exercise date is the following:*

$$c_{BS}(r, \sigma, T) = S_0\Phi(d_1) - e^{-rT}K\Phi(d_2) \tag{2.8}$$

where $\Phi(\cdot)$ *denotes the standard normal cumulative distribution function and*

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} \qquad\qquad d_2 = d_1 - \sigma\sqrt{T} \tag{2.9}$$

r is the risk-free interest rate and σ is the volatility as defined in the Black-Scholes model.

The price of the European put option can be easily calculated from the put-call parity.

The payoff of Asian options

As mentioned in Section 1, Asian options are exotic options and also known as average price options (APO). Since V_T, the payoff depends on the average of a series of underlying prices, it can be clearly seen that Asian options are path-dependent options, which makes their pricing quite cumbersome. There are two types of Asian options:

Fixed-strike Asian options:

$$\begin{aligned} \text{call: } & (A_T - K)^+ \\ \text{put: } & (K - A_T)^+ \end{aligned} \tag{2.10}$$

Floating strike Asian option:

$$\begin{aligned} \text{call: } & (S_T - A_T)^+ \\ \text{put: } & (A_T - S_T)^+ \end{aligned}$$

where T denotes the maturity, S_T is the underlying asset price at time T, and K is the fixed strike price. A_T denotes the average of the underlying asset prices at time T. This thesis focuses on Fixed-strike Asian call options (2.10). Fixed-strike Asian put option prices can be easily calculated from the put-call parity. However, A_T the average can be of various types, which will be discussed in the next subsection 2.2.1.

2.2 Overview of pricing methods

This section contains the different types of Asian options, presents an overview of the numerous pricing methods and shows the recent directions of research activities.

2.2.1 Averaging types

In the theory of Asian options, the mean is usually either geometric or arithmetic, and the underlying asset prices are either discretely or continuously monitored. The formulas of the averages are shown in Table 1. Also, we would like to highlight that in practice financial markets usually use the arithmetic average and the prices are discretely monitored. The prices of other types are either good-to-have theoretical results or they are tools for approximating the discretely monitored arithmetic averaged APOs.

Averaging formulas	Continuously monitored	Discretely monitored
Geometric	$\exp\left(\frac{1}{T}\int\limits_0^T \ln S_u du\right)$	$\left(\prod\limits_{i=0}^{N} S_{t_i}\right)^{\frac{1}{N+1}}$
Arithmetic	$\int\limits_0^T \frac{1}{T} S_u du$	$\sum\limits_{i=0}^{N} \frac{1}{N+1} S_{t_i}$

Table 1: Averaging formulas.

The discretely monitored geometric mean can be written as the following:

$$\left(\prod_{i=0}^{N} S_{t_i}\right)^{\frac{1}{N+1}} = \exp\left(\frac{1}{N+1}\ln\left(\prod_{i=0}^{N} S_{t_i}\right)\right) = \exp\left(\frac{1}{N+1}\sum_{i=0}^{N}\ln S_{t_i}\right) \tag{2.11}$$

The discrete cases can be thought as an approximation of the continuous cases.
In case of geometric averages:

$$\exp\left(\frac{1}{N+1}\sum_{i=0}^{N}\ln S_{t_i}\right) \overset{\text{for large } N}{\approx} \exp\left(\frac{1}{T}\int\limits_0^T \ln S_u du\right) \tag{2.12}$$

In case of arithmetic averages:

$$\sum_{i=0}^{N}\frac{1}{N+1}S_{t_i} \overset{\text{for large } N}{\approx} \int\limits_0^T \frac{1}{T}S_u du \tag{2.13}$$

where $0 \leq t_0 \leq t_1 \leq \ldots \leq t_N = T$ are so-called monitoring points (usually days), T

is the exercise date.

If S_t is the Geometric Brownian Motion, then it can be seen that Asian options with geometric averaging have an explicit pricing formula. However, in practice, in futures markets the Asian options usually use arithmetic averaging that do not have an easy formula to implement, because the arithmetic average of log-normally distributed random variables is not log-normally distributed. In this master's thesis we will concentrate on pricing discretely monitored Asian options with arithmetical averaging, therefore from now on

$$A_T = \frac{1}{N+1} \sum_{i=0}^{N} S_{t_i} \tag{2.14}$$

2.2.2 Brief overview of pricing APOs under the Black-Scholes model

Since the valuation of arithmetically averaged APOs is not trivial even under the basic GBM asset price process, researchers during the last three decades have developed several exact and approximation methods to price them. The main purpose of this section is to give an overview on these methods, which can be categorized into three groups. The grouping in Table 2 is based on several books and articles I have read. The remaining part of this subsection contains a short summary of those that I have found the most valuable in Asian option pricing.

Monte Carlo simulation	Approximations	Exact formula
1. **Classical Monte Carlo**	**Moment Matching**	**for Geometric averaged**
2. Variance Reduction	**Log-normal**	**priced options** (continuous,
Antithetic variate	Edgeworth Series	**discrete**)
Control variate	Transformations	
Important sampling	Laplace	
	Fourier	
	PDE methods	
	Boundaries	
	Upper	
	Lower	

Table 2: Pricing methods under the Black-Scholes model

In Table 2, the methods taken in bold will be detailed later in Sections 2.4 and 2.5.

The related books in this topic are the following:

Clark (2014) provides the exact formula for both geometric averaged priced options under continuous and discrete monitoring. He also presents arithmetic averaged option pricing methods. For continuous averaging, he shows that the arithmetic average can be rewritten by the geometric average with a correction. For the discrete mean, he presents the moment matching technique with log-normal approximation for averaging the spot and future price averages as well. During my internship at a multinational investment bank I have found that several approaches written in this book are applied in practice for APO pricing with commodity underliers.

Eydeland and Wolyniec (2002) introduce the Vorst and Curran methods for pricing the discrete arithmetic averaged options. The Vorst method is based on giving boundaries for the price with geometric average priced option prices. The method also corrects the strike price with the difference between the expected value of arithmetic and geometric averages. They also show the basics of the Curran method, which is as well based on giving a lower boundary with the geometric mean, but provides better results in practice than the Vorst method.

Geman (2009) presents several methods to price arithmetic averaged Asian option with continuous monitoring. The author describes the formulas of the Kemna and Vorst method, which is an approximation of the arithmetic average by the geometric average considering it as the control variate. She also shows the technique of Levy (1992) (see later in this section). The writer also compares Asian and European options and shows that Asian options are not always cheaper than Europeans. It depends on the difference between the r interest rate and the y convenience yield, if $r - y$ is positive, that Asian is cheaper than European option, in the other case the relation between the prices is not clear and depends on more variables.

All the techniques shown by Privault (2013) price the continuous arithmetic averaged options. He provides the Laplace transform method, moment matching with the log-normal approximation and gives boundaries for the price and shows PDE schemes. The book includes proofs for each proposition.

Roncoroni et al. (2015) demonstrate us pricing techniques for both arithmetic Asian options with discrete and continuous averaging. We get a high-level insight into several methods. First they present the Moment matching technique by log-normal approximation (detailed in Section 2.5) and using the Edgeworth series expansion. The authors summarize the algorithm of the MC and its variance reduction techniques (detailed in Section 2.4). They also illustrate a lower boundary on the prices of the options mentioned above. This book is very practical as it describes the algorithms step by step and describes some trades in real commodity markets.

The book of Shreve (2004) provides us a general introduction to Asian options and gives us a PDE method to price both of the continuously and discretely monitored arithmetical Asian options. The shown method involves a numéraire changing technique.

The related articles in this topic are the following:

Levy (1992) is a seminal paper that approximates the discrete arithmetic averaged Asian options by matching the first two moments of the discrete arithmetic average with the log-normal distribution. He shows us that the accuracy of this approach is not much worse than Edgeworth series expansion, also it is easier to implement and it is less time-consuming. Many books and articles refer to his paper.

Lo et al. (2014) illustrate us the moment matching method for continuous arithmetic averaged Asian options. They provide the moment matching techniques for different approximate distributions such as normal, shifted gamma, shifted lognormal and shifted reciprocal gamma distribution. This approach values both of the fixed and floating strike Asian options and the results are very important, as it is well known that (due to the central limit theorem) the log-normal moment matching gets more inaccurate by increasing the number of averaging points.

Li and Chen (2016) propose a method by the Edgeworth series expansion to price continuous arithmetic averaged Asian options. Their approach also give us explicit formulas for the Greeks.

In Horvath and Medvegyev (2016) the authors compare the continuous arithmetic averaged Asian option prices calculated using the Laplace transform method with the prices simulated with MC and two variance reduction techniques for MC. They also study the efficiency of computational time. They have found that the control variate method performs better than the antithetic method in the aspect of standard errors. They also highlighted that the magnitude of $\sigma^2 T$ matters if we want to achieve time efficiency.

Chen and Lyuu (2007) describe a lower boundary both for continuously monitored arithmetic averaged Asian option with fixed and floating strike. They summarize several approaches from the literature and They also compare their results with various pricing techniques. They conclude that their approximated prices are extremely close to the real prices.

As mentioned earlier in this master thesis, the prices of geometrical APOs have a closed form under the Black-Scholes model, which is unfortunately not the case for arithmetical averaged APOs. Further in this thesis, the MC simulation and two variance reduction techniques will be detailed in section 2.4. Also, the moment matching method will be presented in section 2.5 based on the books and papers above. I have chosen to detail these methods, as during my internship I have found that in the banking industry these are the most popular techniques for APO pricing.

2.2.3 Review of the recent results in the literature

Since considerable effort was put by researchers on finding exact enough yet efficient techniques for pricing APOs under the Black-Scholes model in the past decades, now we have several valuable approaches for pricing them, which have been detailed in the previous Section 2.2.2. This subsection describes the most recently published research papers that either present important extensions such as pricing APOs under a more complicated market model, or show new mathematical pricing techniques.

Kirkby (2016) gives us a method under the exponential Lévy model for both discrete and continuous arithmetical averages. The method approximates the sum of random variables by another random variable, with any given arbitrary density function. The pricing algorithm reduces the computational cost, and therefore it is very efficient. The author generalizes this method for basket options as well.

In Kirkby and Nguyen (2020), the authors detail us a method to price discrete and continuous arithmetical average options under a general regime switching jump diffusion models. The presented, quite complicated technique is based on recursion. Their approach for pricing Asian options also works well under certain stochastic volatility jump diffusion models.

Fusai and Kyriakou (2016) show us a method for pricing both the arithmetical Asian options with discrete and continuous averaging under general asset model settings, such as under the exponential Lévy models, stochastic volatility models, and the CEV diffusion which is a family of volatility models with β elasticity parameter. The approximation is based on giving a lower bound and improving the accuracy with optimization. Their approach is competitive since its volatility is low, and the technique is easy to implement.

Mehrdoust et al. (2017) present us an efficient MC simulation under the CEV model which is a local volatility model. They also present two variance reduction techniques, namely the control variate and the antithetic variate methods.

Willems (2019) describes us a new approach for pricing continuously monitored arithmetical Asian options under the Black-Scholes model. The approach is fairly new, it approximates the price with orthogonal polynomials. The advantages of this technique are that it is explicit and numerical integration is not required. On the other hand, the main disadvantage is that it does not always guarantee convergence to the correct price.

Corsaro et al. (2019) give us a recursion based method for pricing discretely monitored arithmetical Asian options under general stochastic volatility models. The technique speeds up the pricing with parallelizing the algorithm and utilizing multiple cores, reducing its computational cost significantly.

2.3 Closed form for geometric Asian options

If the asset price process is the GBM process, then its geometric averages (see in Table 1) are log-normally distributed. This entails that there is a closed, explicit form for the price of the geometrically averaged Asian options which we are going to present in this subsection, because techniques discussed later will use these results. The formulas shown in this part will be needed for the control variate technique (see in Section 2.4.1.2) and for the moment matching method (see in Section 2.5). Clark (2014) and Roncoroni et al. (2015) contain a closed form to price the geometric Asian options with discrete averaging, which is based on the following proposition.

Proposition 5 (General pricing formula for log-normal distribution)**.** *Let us assume the Black-Scholes setting detailed earlier, and under the risk neutral measure $\mathbb{Q}$, the general payoff V_T is log-normally distributed with parameters m and ν. In this case the price of the derivative with payoff V_T at time* 0 *is the following:*

$$c = S_0 e^{m+\frac{\nu^2}{2}-rT}\Phi(d_1) - e^{-rt}K\Phi(d_2) \tag{2.15}$$

where $\Phi(\cdot)$ denotes the standard normal cumulative distribution function and

$$d_1 = \frac{\ln(\frac{S_0}{K}) + m + \nu^2}{\nu} \qquad\qquad d_2 = d_1 - \nu$$

Proof. Using the general risk-neutral pricing equation (2.7) to calculate the initial price with payoff $V_T = (U_T - K)^+$ (see in Theorem 1), the equation is the following:

$$c = e^{-rT}\,\mathbb{E}(U_T - K)^+$$

with $U_T = S_0 e^X$, where X is normally distributed with m expected value and ν^2 variance.

$$\begin{aligned} c &=^{-rT} \mathbb{E}(S_0 e^X - K)^+ \\ &= e^{-rT}\,\mathbb{E}(S_0 e^{m+\nu Z} - K)^+ \end{aligned} \tag{2.16}$$

where $Z \sim N(0,1)$. Focusing on the expected value

$$\begin{aligned} \mathbb{E}(S_0 e^{m+\nu Z} - K)^+ &= \mathbb{E}\left((S_0 e^{m+\nu Z} - K)\mathbb{1}_{\{S_0 e^{m+\nu Z} \geq K\}}\right) \\ &= \mathbb{E}\left(S_0 e^{m+\nu Z}\mathbb{1}_{\{S_0 e^{m+\nu Z} \geq K\}}\right) - \mathbb{E}\left(K\mathbb{1}_{\{S_0 e^{m+\nu Z} \geq K\}}\right) \end{aligned} \tag{2.17}$$

Let us see the first part of the Equation 2.17:

$$S_0e^m\,\mathbb{E}\left(e^{\nu Z}\mathbb{1}_{\{S_0e^{m+\nu Z}\geq K\}}\right) = S_0e^m\int\limits_{-\infty}^{\infty} e^{\nu z}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}\mathbb{1}_{\{S_0e^{m+\nu z}\geq K\}}dz = \tag{2.18}$$

$$= S_0e^m\int\limits_{y}^{\infty}\frac{1}{\sqrt{2\pi}}e^{\nu z-\frac{z^2}{2}}dz$$

$$= S_0e^m\int\limits_{y}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{(z-\nu)^2}{2}+\frac{\nu^2}{2}}dz = * \tag{2.19}$$

In (2.18) the use of the indicator function changes the integral boundary for the following:

$$\begin{aligned} S_0e^{m+\nu z} &\geq K \\ e^{m+\nu z} &\geq \frac{K}{S_0} \\ m+\nu z &\geq \ln\left(\frac{K}{S_0}\right) \\ z &\geq \frac{\ln\left(\frac{K}{S_0}\right)-m}{\nu} =: y \end{aligned}$$

Also in (2.19) $e^{\frac{v^2}{2}}$ is constant, while $\frac{1}{\sqrt{2\pi}}e^{-\frac{(z-\nu)^2}{2}}$ is the density function of a $\mathrm{N}(\nu,1)$ distributed random variable. Hence we can integrate by substitution, using $u = z-\nu$. So the integral boundary changes to the following:

$$y-\nu = \frac{\ln\left(\frac{K}{S_0}\right)-m-\nu^2}{\nu}$$

Since the density function of a standard normal distribution is integrated from $y-\nu$ to infinity, (2.19) can be expressed as

$$* = S_0e^me^{\frac{\nu^2}{2}}\int\limits_{y-\nu}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}du = S_0e^{m+\frac{\nu^2}{2}}\left(1-\Phi(y-\nu)\right) = \circledast$$

Using the $1-\Phi(x)=\Phi(-x)$ formula, where x denotes

$$-y+\nu = \frac{\ln\left(\frac{S_0}{K}\right)+m+\nu^2}{\nu} = d_1$$

we obtain

$$\begin{aligned}\circledast &= S_0 e^{m+\frac{\nu^2}{2}}\Phi(-y+\nu)\\ &= S_0 e^{m+\frac{\nu^2}{2}}\Phi(d_1)\end{aligned}\tag{2.20}$$

In the second part of Equation 2.17, K is a non-negative constant and when taking the expected value of an indicator we get the probability of the event, so

$$\begin{aligned}K\,\mathbb{E}\left(\mathbb{1}_{\{S_0e^{m+\nu Z}\geq K\}}\right) &= K\mathbb{P}(S_0e^{m+\nu Z}\geq K)\\ &= K\mathbb{P}\left(e^{m+\nu Z}\geq \frac{K}{S_0}\right)\\ &= K\mathbb{P}\left(Z\geq \frac{\ln\left(\frac{K}{S_0}\right)-m}{\nu}\right)\\ &= K\left(1-\mathbb{P}\left(Z< \frac{\ln\left(\frac{K}{S_0}\right)-m}{\nu}\right)\right)\\ &= K\left(1-\Phi\left(\frac{\ln\left(\frac{K}{S_0}\right)-m}{\nu}\right)\right)\\ &= K\Phi\left(\frac{\ln\left(\frac{S_0}{K}\right)+m}{\nu}\right)\\ &= K\Phi(d_2)\end{aligned}\tag{2.21}$$

where using notation $\frac{\ln\left(\frac{S_0}{K}\right)+m}{\nu} = d_2 = d_1-\nu$.
After putting the first (2.20) and the second (2.21) part all together and plugging into Equation 2.16, the proof is done. □

Corollary 1. If the payoff V_T is log-normally distributed with parameters m and ν parameters

$$\begin{aligned}m &= \left(r-\frac{\sigma^2}{2}\right)T\\ \nu &= \sigma\sqrt{T}\end{aligned}$$

then we get the classical Black-Scholes formula described in Theorem 2.

Proposition 6 (Parameters for log-normal distribution in case of geometric averaging and discrete monitoring)**.** *Using the general pricing formula in Proposition 5, the parameters of payoff V_T in case of discrete monitoring and geometric averaging are the following:*

$$m = \ln S_0 + \frac{1}{2}\left(r - \frac{\sigma^2}{2}\right)T \tag{2.22}$$

$$\nu = \sigma\sqrt{T}\sqrt{\frac{2N+1}{6(N+1)}} \tag{2.23}$$

Proof. Using the form of A_T such that in Equation 2.11, let us recall the logarithm of S_t from Equation 2.6.

$$\ln S_t = \ln S_0 + \left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t \tag{2.24}$$

Let us denote the discrete monitoring dates by $\{t_i : i = 0, \ldots, N, t_i < t_{i+1}\}$, where the time points are equally distributed over the time interval $[0, T]$, so $t_i = \frac{iT}{N}$. Applying $\ln S_t$ (2.24) into (2.11), the average will be the following:

$$\begin{aligned}
A_T^{geo} &= \exp\left(\frac{1}{N+1}\sum_{i=0}^{N}\ln S_{t_i}\right)\\
\ln A_T^{geo} &= \frac{1}{N+1}\sum_{i=0}^{N}\ln S_{t_i}\\
&= \ln S_0 + \frac{1}{N+1}\sum_{i=1}^{N}\left(r - \frac{\sigma^2}{2}\right)t_i + \frac{1}{N+1}\sum_{i=1}^{N}\sigma W_{t_i}
\end{aligned}$$

It can be clearly seen that A_T^{geo} is log-normally distributed, because in the exponential function W_t is Brownian motion under $\mathbb{Q}$, so the sum also stays normally distributed. The distribution's parameters are the following:
The expected value:

$$\begin{aligned}
m = \mathbb{E}(\ln A_T^{geo}) &= \mathbb{E}\left(\ln S_0 + \frac{1}{N+1}\left(r - \frac{\sigma^2}{2}\right)\sum_{i=0}^{N}t_i + \frac{1}{N+1}\sigma\sum_{i=0}^{N}W_{t_i}\right) =\\
&= \ln S_0 + \frac{1}{N+1}\left(r - \frac{\sigma^2}{2}\right)\sum_{i=0}^{N}t_i =\\
&= \ln S_0 + \frac{1}{N+1}\left(r - \frac{\sigma^2}{2}\right)\frac{T(N+1)}{2} = \ln S_0 + \frac{1}{2}\left(r - \frac{\sigma^2}{2}\right)T
\end{aligned}$$

where

- $\mathbb{E}\left(\frac{\sigma}{N+1}\sum\limits_{i=0}^{N} W_{t_i}\right)$ is zero, because W_{t_i} is a Brownian motion under $\mathbb{Q}$, therefore its expected value is zero and
- $\sum\limits_{i=0}^{N} t_i = \frac{T}{N}\sum\limits_{i=0}^{N} i = \frac{T}{N}\frac{N(N+1)}{2} = \frac{T(N+1)}{2}$

Let us concentrate on the variance:

$$\nu^2 = \mathrm{Var}(\ln A_T^{geo}) = \mathrm{Var}\left(\frac{1}{N+1}\sum_{i=0}^{N}\sigma W_{t_i}\right) = \frac{\sigma^2}{(N+1)^2}\mathrm{Var}\left(\sum_{i=0}^{N} W_{t_i}\right)$$

$$\mathrm{Var}\left(\sum_{i=0}^{N} W_{t_i}\right) = \mathbb{E}\left(\sum_{i,j=0}^{N} W_{t_i}W_{t_j}\right) = \sum_{i,j=0}^{N}\min(t_i,t_j)$$

$$\begin{aligned}
\sum_{i,j=0}^{N}\min(t_i,t_j) &= \frac{T}{N}\sum_{i,j=1}^{N}\min(i,j) = \frac{T}{N}\left(1(N+(N-1)) + 2((N-1)+(N-2)) + \dots\right)\\
&= \frac{T}{N}\left(1(2N-1) + 2(2N-3) + 3(2N-5) + \dots\right)\\
&= \frac{T}{N}\sum_{i=1}^{N}(2N-2i+1)i = \frac{T}{N}\sum_{i=1}^{N}(2(N-i)+1)i\\
&= \frac{T}{N}\sum_{i=1}^{N}((2N+1)i - 2i^2) = \frac{T}{N}\left(\sum_{i=1}^{N}((2N+1)i) - \sum_{i=1}^{N}2i^2\right)\\
&= \frac{T}{N}\left((2N+1)\frac{N(N+1)}{2} - 2\frac{N(N+1)(2N+1)}{6}\right)\\
&= T(N+1)\frac{6N+3-4N-2}{6}\\
&= T(N+1)\frac{2N+1}{6}
\end{aligned}$$

So the variance is the following:

$$\nu^2 = \mathrm{Var}(\ln A_T^{geo}) = \frac{\sigma^2}{(N+1)^2}T(N+1)\frac{2N+1}{6} = \frac{2N+1}{6(N+1)}\sigma^2 T$$

$\square$

Corollary 2 (Parameters in continuous case)**.** When N tends to infinity, the parameters in Proposition 6 reduce to

$$\begin{aligned}
m &= \ln S_0 + \frac{1}{2}\left(r - \frac{1}{2}\sigma^2\right)T\\
\nu &= \sigma\sqrt{\frac{1}{3}T}
\end{aligned}$$

which are the parameters of payoff V_T in case of continuously monitored geometric average.

Proof. The expected value does not depend on N. For variance, using the L'Hôspital's rule we get the parameter above. □

Combining Proposition 5., Corollary 1., and Proposition 6., lead us to the following assumption.

Assumption 1. We can rewrite the discretely monitored geometric averaged Asian option's pricing formula using the general Black-Scholes formula (in Proposition 2) for European call options with $\widetilde{r}$ discount rate. The formula is the following:

$$c_{gt} = e^{-rT}\,\mathbb{E}\,(A_T^{geo} - K)^+ = e^{-(r-\widetilde{r})T} S_0 \Phi(d_1) - K e^{-rT} \Phi(d_2) \tag{2.25}$$

where A_T is the discrete geometric average of the asset prices (see in Table 1) and the parameters:

$$\widetilde{r} = \frac{1}{2}\left(r - \sigma^2 \frac{N+2}{6(N+1)}\right) \qquad d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(\widetilde{r} + \frac{1}{2}\widetilde{\sigma}^2\right) T}{\widetilde{\sigma}\sqrt{T}}$$

$$\widetilde{\sigma} = \sigma \sqrt{\frac{2N+1}{6(N+1)}} \qquad d_2 = d_1 - \widetilde{\sigma}\sqrt{T}$$

Proof. Let us suppose that A_T is the discrete geometric average of the underlying asset prices. The general risk-neutral pricing formula (see in Theorem 1) for discretely monitored geometric averaged Asian call option is the following:

$$V_0 = e^{-rT}\,\mathbb{E}\left((A_T^{geo} - K)^+\right) = e^{-rT}\,\mathbb{E}\left(\left(\prod_{i=0}^{N} S_{t_i}\right)^{\frac{1}{N+1}} - K\right)^+ = *$$

with $\left(\prod_{i=0}^{N} S_{t_i}\right)^{\frac{1}{N+1}} = S_0 e^X$, where $X \sim N(m, \nu^2)$. The parameters of X (can be found in Proposition 6) are the following:

$$m = \frac{1}{2}\left(r - \frac{\sigma^2}{2}\right) T \qquad \nu = \sigma\sqrt{T}\sqrt{\frac{2N+1}{6(N+1)}} \tag{2.26}$$

By inverting the parameters in Corollary 1., we get the parameters used for Black-

Scholes formula, which are the following:

$$\widetilde{r} = \frac{1}{T}\left(m + \frac{\nu^2}{2}\right) \qquad\qquad \widetilde{\sigma} = \frac{\nu}{\sqrt{T}} \tag{2.27}$$

Plugging (2.26) into (2.27), we obtain

$$\begin{aligned}
\widetilde{r} &= \frac{1}{T}\left(\frac{1}{2}\left(r - \frac{\sigma^2}{2}\right)T + \sigma^2 T\frac{2N+1}{6(N+1)}\right) \\
&= \frac{1}{2}\left(r - \frac{\sigma^2}{2}\right) + \sigma^2\frac{2N+1}{6(N+1)} = \frac{1}{2}\left(r - \sigma^2\left(\frac{1}{2} - \frac{2N+1}{6(N+1)}\right)\right) \\
&= \frac{1}{2}\left(r - \sigma^2\frac{N+2}{6(N+1)}\right) && (2.28) \\
\widetilde{\sigma} &= \sigma\sqrt{\frac{2N+1}{6(N+1)}} && (2.29)
\end{aligned}$$

Thus we get the price of the discretely monitored geometric averaged Asian call option using the classical Black-Scholes formula (see in Theorem 2) with parameters above (see in Equation 2.28 and 2.29).
The deduction is the following:

$$* = e^{-(r-\widetilde{r})T}e^{-\widetilde{r}T}\,\mathbb{E}(V_T^{BS}(S_0, \widetilde{r}, \widetilde{\sigma})) = \circledast$$

where $V_T^{BS}(S_0, \widetilde{r}, \widetilde{\sigma}) = (S_T - K)^+$ denotes the European call option's payoff with certain parameters.

$$\begin{aligned}
\circledast &= e^{-(r-\widetilde{r})T}c_{BS}(S_0, \widetilde{r}, \widetilde{\sigma}) \\
&= e^{-(r-\widetilde{r})T}\left(S_0\Phi(d_1) - Ke^{-\widetilde{r}T}\Phi(d_2)\right) \\
&= e^{-(r-\widetilde{r})T}S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2)
\end{aligned}$$

□

In the following the c_{gt} formula in Assumption 1 will be used in Section 2.4.1.2, where c_{gt} will be the control variate.

2.4 Monte Carlo methods

This section starts with a brief overview of the classical Monte Carlo simulation technique for discretely monitored and arithmetically averaged Asian options to determine the exact price. After that, two methods will be presented how to reduce the variance of the MC, namely the Control Variate method and the Antithetic Variate method. This section is mainly based on Chapter 18 of Roncoroni et al. (2015).

MC simulation is a very popular tool to examine the properties of stochastic processes and hence it is a good choice to calculate the price of options. First, the method will be presented in general settings.

Let $g(x)$ be a measurable real function and Z_i be an independent and identically distributed random variable from an arbitrary distribution. The Strong Law of Large Numbers claims that if $E(g(Z_i)) < \infty$, then the arithmetic average of $g(Z_i)$ random variables converges to their common expected value with probability 1:

$$\frac{1}{n}\sum_{i=1}^{n} g(Z_i) \xrightarrow[a.s]{n\to\infty} \mathbb{E}(g(Z))$$

The efficiency of MC simulation techniques can be described by the convergence rate and the variance. The convergence rate of the MC is $\frac{1}{n}$ and the order of convergence is $\frac{1}{2}$. The disadvantage is that it is computing-intensive. On the other hand it is a suitable method for option pricing, as it is easy to implement, it also allows us to simulate the prices of derivatives based on a more complicated payoff (such as path dependent ones) and there are also several variance reduction techniques, which improve the accuracy of the MC, see in Section 2.4.1. Further details can be found in Horvath and Medvegyev (2016).

Applying MC for Asian options, the $g(\cdot)$ function is the following:

$$g(Z) = e^{-rT} V_T(Z)$$

where V_T is the payoff of the fixed strike Asian call option, defined in Equation 2.10. The first step of the algorithm is sampling S_t with fixed initial price $S_0 = s_0$, as follows

$$S_{t_i}^{(j)} = s_0 e^{\left(r-\frac{\sigma^2}{2}\right)t_i+\sigma W_{t_i}^{(j)}}$$

with $j = 1, 2, \ldots, n$, which is the path of the simulation and n denotes the number of simulations, t_i denote the monitoring days and $t_i = \frac{iT}{N}$ with $i = 1, 2, \ldots, N$. Since Asian options are path dependent, the paths need to be simulated such that

$$S_{t_i}^{(j)} = S_{t_{i-1}}^{(j)} e^{\left(r - \frac{\sigma^2}{2}\right)(t_i - t_{i-1}) + \sigma\left(W_{t_i}^{(j)} - W_{t_{i-1}}^{(j)}\right)} \tag{2.30}$$

Let $\Delta = t_i - t_{i-1}$ denote the time difference. The increments of the Brownian motion:

$$W_{t_i}^{(j)} - W_{t_{i-1}}^{(j)} = \sqrt{\Delta}\Phi^{-1}(U_i^{(j)}) = \sqrt{\Delta}Z_i^{(j)} \tag{2.31}$$

where $U_i^{(j)}, (i = 1, \ldots, N), (j = 1, \ldots, n)$ are uniform random variables on interval [0,1]. It is well known from probability theory that $\Phi^{-1}(U_i^{(j)}) = Z_i^{(j)} \sim N(0, 1)$. The average also needs to be calculated on each path:

$$A_T^{(j)} = \frac{1}{N+1} \sum_{i=0}^{N} S_{t_i}^{(j)} \tag{2.32}$$

Now we can calculate the arithmetic Asian option price:

$$c^{(j)} = e^{-rT} \max(A_T^{(j)} - K, 0) \tag{2.33}$$

$$c_{MC} = \frac{1}{n} \sum_{j=1}^{n} c^{(j)} \tag{2.34}$$

By the strong law of large numbers c_{MC} converges to the correct price with 1 probability. The variance and the standard error of the MC method is the following

$$\sigma_{MC}^2 = \frac{1}{n} \sum_{j=1}^{n} (c^{(j)} - c_{MC})^2 \tag{2.35}$$

$$se_{MC} = \sqrt{\frac{\sigma_{MC}^2}{n}} \tag{2.36}$$

By observing $P(\mid Z \mid \leq \Phi^{-1}(1 - \frac{\alpha}{2})) = 1 - \alpha$ for a small $\alpha > 0$ real number (typically 0.05 or 0.01). We define the $1 - \alpha$ confidence interval as

$$c_{MC} \pm \Phi^{-1}\left(1 - \frac{\alpha}{2}\right) \cdot se_{MC}$$

2.4.1 Variance reduction techniques

In this section two variance reduction techniques will be presented, namely the Control variate (CV) and the Antithetic (AV) methods. The point of variance reduction techniques is that we need fewer simulations to achieve the same accuracy as for the basic MC, thus they also enable us to reduce the running time of MC simulation.

2.4.1.1 Antithetic variate method

The basic idea behind this technique is that we would like to decrease $Y = g(Z)$ random variable's variance. For this, let us consider two identically distributed, correlated random variables Y_1 and Y_2, with common expected value as Y, hence $\frac{Y_1+Y_2}{2}$ is an unbiased estimator of $\mathbb{E}(Y)$.

$$\mathbb{E}\left(\frac{Y_1+Y_2}{2}\right) = \mathbb{E}(Y) = c_{AV}$$

The variance can be written as follows:

$$Var\left(\frac{Y_1+Y_2}{2}\right) = \frac{\sigma^2_{Y_1} + \sigma^2_{Y_2} + 2Cov(Y_1, Y_2)}{4} \tag{2.37}$$

Equation 2.37 shows that the estimator $\frac{Y_1+Y_2}{2}$ will have smaller variance than $\sigma^2_{Y_1}$ and $\sigma^2_{Y_2}$, if there is negative correlation between Y_1 and Y_2. The disadvantage of this method is that the negative correlation between them is not always assured.

Using this technique for Asian options, we generate Z_i according to (2.31) and we also take the minus of it, $-Z_i$, and the rest of the algorithm stays the same as in the MC method, but for both of Z_i and $-Z_i$. Hence the underlying asset prices are simulated also on the so-called antithetic path (denote as $\tilde{S}$):

$$\tilde{S}^{(j)}_{t_i} = \tilde{S}^{(j)}_{t_{i-1}} e^{\left(r-\frac{\sigma^2}{2}\right)(t_i-t_{i-1})+\sigma\left(\tilde{W}^{(j)}_{t_i}-\tilde{W}^{(j)}_{t_{i-1}}\right)}$$

where $\tilde{S}^{(j)}_{t_i}$ is simulated with the increment of $\tilde{W}$, which is given by

$$\tilde{W}^{(j)}_{t_i} - \tilde{W}^{(j)}_{t_{i-1}} = -\sqrt{\Delta} Z^{(j)}_i$$

where $-Z^{(j)}_i \sim N(0,1)$. The arithmetical average needed to be calculated on the original (denoted as $A^{(j)}_T$, same as in Equation 2.32) and also on the antithetic paths, which is the following:

$$\tilde{A}_T^{(j)} = \frac{1}{N+1} \sum_{i=0}^{N} S_{t_i}^{(j)}$$

So the arithmetic Asian option price according to the AV technique is the following:

$$\begin{aligned} c_{AV}^{(j)} &= e^{-rT} \max(A_T^{(j)} - K, 0) \\ c_{AV} &= \frac{1}{n} \sum_{j=1}^{n} \frac{c^{(j)} + c_{AV}^{(j)}}{2} \end{aligned} \tag{2.38}$$

where $c^{(j)}$ is calculated such as in MC simulation in Equation 2.33.
We also define the variance and the standard error of AV method:

$$\sigma_{AV}^2 = \frac{1}{n} \sum_{j=1}^{n} \left(\frac{c^{(j)} + c_{AV}^{(j)}}{2} - c_{AV} \right)^2 \tag{2.39}$$

$$se_{AV} = \sqrt{\frac{\sigma_{AV}^2}{n}} \tag{2.40}$$

The $1 - \alpha$ confidence interval of AV method is given by

$$c_{AV} \pm \Phi^{-1} \left(1 - \frac{\alpha}{2} \right) \cdot se_{AV}$$

2.4.1.2 Control variate method

This method also aims to decrease $Y = g(Z)$ random variable's variance. Instead of using the basic MC simulation to get the expected value of Y, let us focus on X, which is created as follows:

$$X = Y - \lambda(\xi - \mathbb{E}(\xi))$$

where λ is a constant and ξ is a so-called control variate, which is another random variable with known $\mathbb{E}(\xi)$ value. Taking the expected value of X, it will be equal to the expected value of Y, which was our original aim to get.

$$\mathbb{E}(X) = \mathbb{E}(Y) - \lambda(\mathbb{E}(\xi) - \mathbb{E}(\xi)) = \mathbb{E}(Y) = c_{CV} \tag{2.41}$$

Let us compute the variance of X, which should be as small as possible:

$$Var(X) = Var(Y) + \lambda^2 Var(\xi) - 2\lambda Cov(\xi, Y) \tag{2.42}$$

The optimal value of λ that minimizes the variance in Equation 2.42 is

$$\begin{aligned} 2\lambda Var(\xi) - 2Cov(\xi, Y) &= 0 \\ \lambda^* &= \frac{Cov(\xi, Y)}{Var(\xi)} \end{aligned} \tag{2.43}$$

After the optimal λ^* is plugged into Equation 2.42, it can be seen that the variance of Y is successfully reduced as follows:

$$Var(X) = Var(Y) - \frac{Cov(\xi, Y)^2}{Var(\xi)} \tag{2.44}$$

while the $Cov(\xi, Y) \neq 0$.

One question remains, how to choose the control variate ξ. In the literature (such as in Horvath and Medvegyev (2016); Roncoroni et al. (2015)) for arithmetic averaged Asian option, the usage of the geometric averaged Asian option price is proposed as the control variable, as we have an explicit formula for it (see Section 2.3 for details).

The first part of CV's algorithm remains the same as the MC method.
We simulate $S_{t_i}^{(j)}$, the underlying prices (2.30), calculate $A_T^{(j)}$, the arithmetic average

(2.32) and $c^{(j)}$, the prices of the arithmetic Asian options on each path such as in Equation 2.33.

The second part of the algorithm is the following. We take the geometric average of $S_{t_i}^{(j)}$ (see in Table 1) on each path, let us denote them as $A_T^{geo(j)}$.

$$A_T^{geo(j)} = \left(\prod_{i=0}^{N} S_{t_i}^{(j)} \right)^{\frac{1}{N+1}}$$

So the geometric Asian option prices on each path are the following:

$$c^{geo(j)} = e^{-rT} \max(A_T^{geo(j)} - K, 0)$$

The next step is that we calculate the optimal λ^* as in (2.43) for each path

$$\lambda^{*(j)} = \frac{Cov(c^{geo(j)}, c^{(j)})}{Var(c^{geo(j)})}$$

Using Equation 2.41, we get the arithmetic averaged Asian option price by the CV method as follows:

$$\begin{aligned} c_{CV}^{(j)} &= c^{(j)} - \lambda^{*(j)}(c^{geo(j)} - c_{gt}) \\ c_{CV} &= \frac{1}{n} \sum_{j=1}^{n} c_{CV}^{(j)} \end{aligned} \tag{2.45}$$

where c_{gt} is the explicit formula to price the geometric averaged Asian option as in Equation 2.25, in Assumption 1.
We also define the variance and the standard error of CV method:

$$\sigma_{CV}^2 = \frac{1}{n} \sum_{j=1}^{n} \left(c_{CV}^{(j)} - c_{CV} \right)^2 \tag{2.46}$$

$$se_{CV} = \sqrt{\frac{\sigma_{CV}^2}{n}} \tag{2.47}$$

The $1 - \alpha$ confidence interval of CV method is given by

$$c_{CV} \pm \Phi^{-1} \left(1 - \frac{\alpha}{2} \right) \cdot se_{CV}$$

2.5 Moment matching method

The basic idea behind this method is that we approximate a usually complicated distribution with an easy to handle, specific distribution by matching their first 2, 3 or 4 moments. This technique is a very popular approach for pricing Asian options on arithmetic average, as the distribution of the sum of log-normally distributed random variables is unfortunately not log-normally distributed.

The procedure of the moment matching technique contains two steps. First we compute the exact first k ($2 \leq k \in \mathbb{Z}$) moments of the complicated random variable of interest (see in Section 2.5.1), then we match them with the assigned distribution's moments (see the log-normal approximation in Section 2.5.2). In our case the complicated random variable is the sum of log-normals, the assigned distribution is the log-normal distribution and $k = 2$, i.e. we match the first 2 moments.

2.5.1 Moments of the arithmetic average in discrete case

In this section the exact moments of discretely monitored arithmetic averaged Asian options will be presented. The moment matching methodology will be exposed under a more general market model setting: when the interest rate, r_t and the volatility, σ_t depend on time, but they still remain deterministic functions. In this general case the solution of S_t, (2.6) changes to

$$S_t = S_0 e^{\int_0^t \left(r_s - \frac{\sigma_s^2}{2}\right) ds + \int_0^t \sigma_s dW_s} \tag{2.48}$$

To calculate the moments of A_T, first let us examine the distribution of the stochastic integral, $\int_0^t \sigma_s dW_s$ in Equation 2.48. As σ_t is deterministic and constant for $\forall t$, it satisfies the conditions of progressive measurability, see in Proposition 2. Also $\mathbb{E}(\int \sigma_s^2 ds) < \infty$, hence $(\sigma_s) \in \mathcal{S}$ (see Definition 1). We know from Proposition 3 that if $(\sigma_s) \in \mathcal{S}$, then (σ_s) is a Gaussian process with 0 expected value. Moreover Itô-isometry in Proposition 4 can be used. Let denote $X_t = \int_0^t \sigma_s dW_s$. Hence $X_t \sim N(0, \int_0^t \sigma_s^2 ds)$ and $X_t = Z_t \int_0^t \sigma_s ds$, where $Z \sim N(0,1)$. Using Proposition 1, we get the expected value of e^{X_t}:

$$\mathbb{E}\left(e^{X_t}\right) = \mathbb{E}\left(e^{\int_0^t \sigma_s dW_s}\right) = e^{\frac{1}{2}\int_0^t \sigma_s^2 ds}$$

The first moment of A_T

$$M_1 = \mathbb{E}(A_T) = \frac{1}{N+1}\sum_{i=0}^{N}\mathbb{E}(S_{t_i}) = \frac{1}{N+1}S_0\sum_{i=0}^{N}\exp\left(\int_0^{t_i} r_s ds\right)$$

where

$$\begin{aligned}
\mathbb{E}(S_{t_i}) &= S_0 \exp\left(\int_0^t \left(r_s - \frac{\sigma_s^2}{2}\right) ds\right)\mathbb{E}\left(\exp\left(\int_0^t \sigma_s dW_s\right)\right) \\
&= S_0 \exp\left(\int_0^t \left(r_s - \frac{\sigma_s^2}{2}\right) ds + \frac{1}{2}\int_0^t \sigma_s^2 ds\right) \\
&= S_0 \exp\left(\int_0^t r_s ds\right)
\end{aligned}$$

The second moment of A_T

$$\begin{aligned}
M_2 = \mathbb{E}(A_T^2) &= \frac{1}{(N+1)^2}\sum_{i,j=0}^{N}\mathbb{E}(S_{t_i}S_{t_j}) \\
&= \frac{1}{(N+1)^2}S_0^2\sum_{i,j=0}^{N}\exp\left(\int_0^{t_i} r_s ds + \int_0^{t_j} r_s ds + \int_0^{t_i\wedge t_j} \sigma_s^2 ds\right)
\end{aligned}$$

where

$$\begin{aligned}
&\mathbb{E}(S_{t_i}S_{t_j}) = \\
&= S_0^2 \exp\left(\int_0^{t_i}\left(r_s - \frac{\sigma_s^2}{2}\right) ds + \int_0^{t_j}\left(r_s - \frac{\sigma_s^2}{2}\right) ds\right)\mathbb{E}\left(\exp\left(\int_0^{t_i}\sigma_s dW_s + \int_0^{t_j}\sigma_s dW_s\right)\right) = **
\end{aligned} \tag{2.49}$$

Using the same arguments as above, let us examine the stochastic integrals in (2.49). We saw earlier that $(\sigma_s) \in \mathcal{S}$, so (σ_s) is a Gaussian process with 0 expected value and the variance

$$\mathbb{E}\left(\left(\int_0^{t_i}\sigma_s dW_s + \int_0^{t_j}\sigma_s dW_s\right)^2\right)$$

$$= \mathbb{E}\left(\left(\int_0^{t_i}\sigma_s dW_s\right)^2\right) + \mathbb{E}\left(\left(\int_0^{t_j}\sigma_s dW_s\right)^2\right) + 2\,\mathbb{E}\left(\int_0^{t_i}\sigma_s dW_s \cdot \int_0^{t_j}\sigma_s dW_s\right)$$

where we can use Itô-isometry (see in Equation 2.3 in Proposition 4)

$$= \int_0^{t_i}\sigma_s^2 ds + \int_0^{t_j}\sigma_s^2 ds + 2\int_0^{t_i\wedge t_j}\sigma_s^2 ds$$

Summarising the statement above

$$\left(\int_0^{t_i}\sigma_s dW_s + \int_0^{t_j}\sigma_s dW_s\right) \sim N\left(0, \int_0^{t_i}\sigma_s^2 ds + \int_0^{t_j}\sigma_s^2 ds + 2\int_0^{t_i\wedge t_j}\sigma_s^2 ds\right)$$

hence we can use Proposition 1 so

$$\mathbb{E}\left(\exp\left(\int_0^{t_i}\sigma_s dW_s + \int_0^{t_j}\sigma_s dW_s\right)\right) = \exp\left(\int_0^{t_i}\frac{\sigma_s^2}{2}ds + \int_0^{t_j}\frac{\sigma_s^2}{2}ds + \int_0^{t_i\wedge t_j}\sigma_s^2 ds\right)$$

$\mathbb{E}(S_{t_i}S_{t_j})$ (in Equation 2.49) equals to

$$** = S_0^2 \exp\left(\int_0^{t_i} r_s ds + \int_0^{t_j} r_s ds + \int_0^{t_i\wedge t_j}\sigma_s^2 ds\right)$$

When the interest rate, r and the volatility, σ are constants, the first two moments of A_T are the following:

$$M_1 = \mathbb{E}(A_T) = \frac{1}{N+1}\sum_{i=0}^{N}\mathbb{E}(S_{t_i}) = \frac{1}{N+1}S_0\sum_{i=0}^{N}e^{rt_i}$$

$$M_2 = \mathbb{E}(A_T^2) = \frac{1}{(N+1)^2}S_0^2\sum_{i,j=0}^{N}e^{r(t_i+t_j)+\sigma^2(t_i\wedge t_j)}$$

Recursive formula to calculate an arbitrary moment

Roncoroni et al. (2015) presented a recursive algorithm to calculate any moment of A_T, which is the following. Here we assume that σ and r do not depend on time. The increment of $\ln S_t$ (see in Equation 2.24):

$$\ln S_{t_k} - \ln S_{t_{k-1}} = \left(r - \frac{\sigma^2}{2}\right)\Delta + \sigma X_k^\Delta =: Z_k^\Delta$$

where $t_k = \frac{kT}{N}$ is monitoring days, $\Delta = t_k - t_{k-1} = \frac{T}{N}$, $k = 1, \ldots, N$, X_k^Δ denotes the increment of the Brownian Motion, so $X_k^\Delta \sim N(0, \Delta)$.
So $\ln S_{t_k}$ and hence S_{t_k} can be written as the following

$$\ln S_{t_k} = \ln S_0 + \sum_{j=1}^{k} \left[\ln S_{t_j} - \ln S_{t_{j-1}}\right] = \ln S_0 + \sum_{j=1}^{k} Z_j^\Delta$$
$$S_{t_k} = S_0 e^{\sum_{j=1}^{k} Z_j^\Delta}$$

Let us suppose that the initial price is fixed, such that $S_0 = s_0$. So the sum of S_t can be represented as

$$\sum_{k=0}^{N} S_{t_k} = s_0 + s_0 e^{Z_1^\Delta} + s_0 e^{Z_1^\Delta + Z_2^\Delta} + \ldots + s_0 e^{Z_1^\Delta + \ldots + Z_N^\Delta}$$
$$= s_0(1 + e^{Z_1^\Delta}(1 + e^{Z_2^\Delta}(\ldots(1 + e^{Z_N^\Delta}))))$$

Define the initial value of the recursion, which is $L_N^\Delta \equiv e^{Z_N^\Delta}$ and the recursion is the following:

$$L_k^\Delta = e^{Z_k^\Delta}(1 + L_{k+1}^\Delta)$$

while $k = N - 1, \ldots, 1$.
The expected value and the nth moment of $e^{Z_k^\Delta}$ are the following:

$$\mathbb{E}(e^{Z_k^\Delta}) = \mathbb{E}\left(e^{\left(r - \frac{\sigma^2}{2}\right)\Delta + \sigma X_k^\Delta}\right) = e^{\left(r - \frac{\sigma^2}{2}\right)\Delta + \frac{\sigma^2}{2}\Delta}$$
$$\mathbb{E}(e^{nZ_k^\Delta}) = e^{\left(r - \frac{\sigma^2}{2}\right)\Delta n + \frac{\sigma^2}{2}\Delta n^2} =: \phi_\Delta(n)$$

The nth moment of L_k^{Δ} is the following:

$$\mathbb{E}((L_k^{\Delta})^n) = \mathbb{E}\left((e^{Z_k^{\Delta}}(1 + L_{k+1}^{\Delta}))^n\right)$$

Since the increments of Brownian motions are independent

$$= \mathbb{E}(e^{nZ_k^{\Delta}})\,\mathbb{E}((1 + L_{k+1}^{\Delta})^n)$$

Using the binomial theorem

$$\begin{aligned} &= \mathbb{E}(e^{nZ_k^{\Delta}})\,\mathbb{E}\left(\sum_{q=0}^{n}\binom{n}{q}(L_{k+1}^{\Delta})^q\right) \\ &= \phi_{\Delta}(n)\sum_{q=0}^{n}\binom{n}{q}\,\mathbb{E}\left((L_{k+1}^{\Delta})^q\right) \end{aligned}$$

while the recursion starts with

$$\phi_{\Delta}(n) = \mathbb{E}(e^{nZ_N^{\Delta}}) = \mathbb{E}((L_N^{\Delta})^n)$$

Hence, we obtain A_T as

$$A_T = \frac{1}{N+1}\sum_{k=0}^{N} S_{t_k} = \frac{s_0(1 + L_1^{\Delta})}{(N+1)}$$

Calculating the nth moment of A_T

$$\begin{aligned} \mathbb{E}((A_T)^n) &= \mathbb{E}\left(\left(\frac{s_0(1 + L_1^{\Delta})}{(N+1)}\right)^n\right) \\ &= \left(\frac{s_0}{(N+1)}\right)^n \mathbb{E}\left((1 + L_1^{\Delta})^n\right) \end{aligned}$$

Using the binomial theorem

$$= \left(\frac{s_0}{(N+1)}\right)^n \sum_{j=0}^{n}\binom{n}{j}\,\mathbb{E}((L_1^{\Delta})^j)$$

2.5.2 Log-normal approximation

A very popular choice for choosing an approximate distribution is to approximate the arithmetic mean, A_T (see Table 1), by a log-normal distribution defined by its first two moments. Clark (2014) and Roncoroni et al. (2015) presented the pricing formula for log-normal approach, which is given by the following proposition.

Proposition 7 (Moment matching)**.** *Let us assume the Black-Scholes setting detailed earlier and the payoff V_T is approximated with log-normal distribution with parameters m and ν, when payoff V_T refers to a discretely monitored arithmetic average Asian option. In this case, the price of the derivative at time* 0 *is the following:*

$$c_{MM} = S_0 e^{m+\frac{\nu^2}{2}-rT}\Phi(d_1) - e^{-rt}K\Phi(d_2) \tag{2.50}$$

where $\Phi(\cdot)$ *denotes the standard normal cumulative distribution function and*

$$d_1 = \frac{\ln(\frac{S_0}{K}) + m + \nu^2}{\nu} \qquad m = 2\ln M_1 - \frac{1}{2}\ln M_2 \tag{2.51}$$

$$d_2 = d_1 - \nu \qquad \nu^2 = \ln M_2 - 2\ln M_1 \tag{2.52}$$

Proof. The pricing formula in Equation 2.50 is the same as in Equation 2.15 in Proposition 5, which is a general pricing formula for any log-normal distribution. The proof of Equation 2.50 can be found in Proposition 5.
A_T is approximated with log-normal distribution which is determined by its first two moments. So the parameters of the log-normal distribution comes from the following approximation:

$$\mathbb{E}(A_T) = M_1 \approx e^{m+\frac{\nu^2}{2}}$$
$$\mathbb{E}(A_T^2) = M_2 \approx e^{2m+2\nu^2}$$

Solving the equations above we got the approximated parameters m and ν^2 according to Equation 2.51 and 2.52 in Proposition 7. □

3 Simulation results

In the previous section, we saw different methods for pricing arithmetic averaged Asian options under discrete monitoring. This section compares the presented techniques by simulations. We implemented the pricing formulas using the statistical language R (see the implemented functions in Appendix A), and simulated several paths of Geometric Brownian motions with different parameter settings. In the following, for every case, we assume that the initial asset price equals to $S_0 = 100$.

In Section 3.1, we compare the three MC simulations with each other. We will choose one of them to compare it with the moment matching technique which will be described in Section 3.2.

3.1 Comparison of Monte Carlo simulations

Let us examine the three MC simulations introduced in Section 2.4. In Table 3, we compare the prices and standard errors of the given methods to find the best performing technique from them. Also we would like to determine the optimal simulation number with fixed parameters. We set the parameters at $r = 0.01$, $\sigma = 0.02$, $T = 1$, $N = 300$, $K = 100$, while the number of simulations m spans between 100 and 10000.

m	c_{MC}	se_{MC}	c_{CV}	se_{CV}	c_{AV}	se_{AV}
100	0.640535	0.072711	0.747978	0.000205	0.735649	0.002753
1000	0.702839	0.025566	0.747850	0.000074	0.753696	0.000316
2000	0.720403	0.018186	0.747726	0.000049	0.742714	0.000153
3000	0.740233	0.015116	0.747795	0.000044	0.749327	0.000104
4000	0.748446	0.013250	0.747815	0.000038	0.747812	0.000079
5000	0.724544	0.011666	0.747733	0.000034	0.749691	0.000063
6000	0.757631	0.010704	0.747801	0.000031	0.739119	0.000051
7000	0.743419	0.010040	0.747761	0.000029	0.748110	0.000044
8000	0.741314	0.009317	0.747784	0.000027	0.743064	0.000039
9000	0.752706	0.008860	0.747799	0.000026	0.750774	0.000035
10000	0.737109	0.008335	0.747817	0.000024	0.750744	0.000032

Table 3: Comparison of MC, CV and AV prices and standard errors with respect to the number of simulations, m.

Parameters: $S_0 = 100$, $r = 0.01$, $\sigma = 0.02$, $T = 1$, $N = 300$, $K = 100$

In Table 3, we can see that the difference between the Asian option prices are fairly small for MC repetitions over 5000, while the standard errors steadily decrease by order $\frac{1}{\sqrt{m}}$.

At 10 000 simulation runs the control variate (CV) method improves the classical MC's standard error by 347 times, while the antithetic variates (AV) technique improves it by 260 times. If we run only 1 000 simulation paths instead of 10 000, the CV improves it by 345 times, while the AV improves it by 81 times.
Figure 1 illustrates how much the variance is reduced using both the CV and AV techniques compared to MC. For these fixed parameters mentioned above, we conclude that the CV method performs slightly better for the chosen parameter setting.

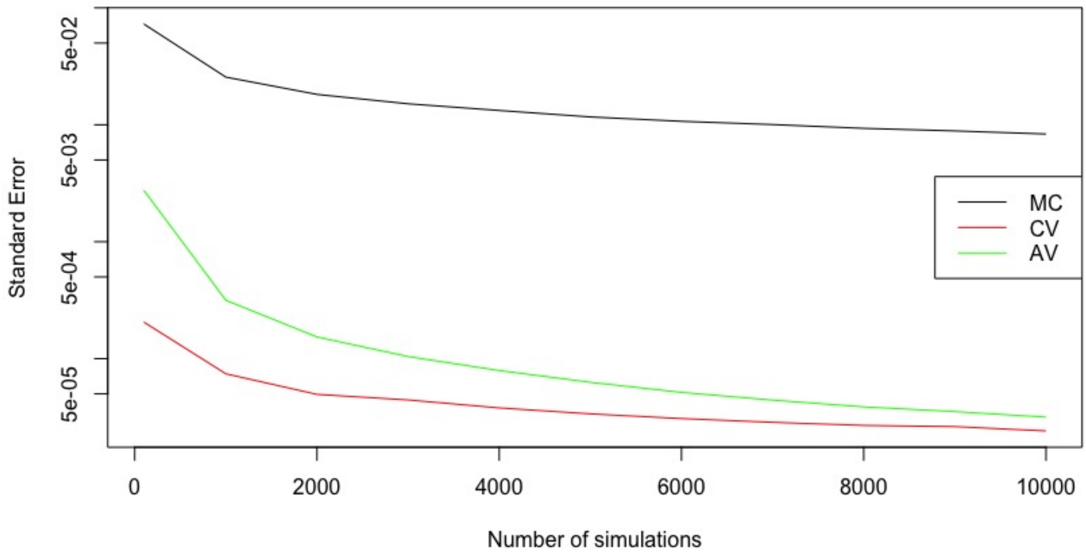


Figure 1: Comparison of MC, CV and AV standard errors.

Parameters: $S_0 = 100$, $r = 0.01$, $\sigma = 0.02$, $T = 1$, $N = 300$, $K = 100$

Since both of the variance reduction techniques performed adequately, the other aspect we compare them is the running time. The general observation is that the more times we simulate, the longer the running time is. In Figure 2, we can see the running times in seconds using the same parameters above. These running times may differ while running these simulations on other computers depending on the computer's performance. Here, we can also conclude that the CV method performs slightly better.

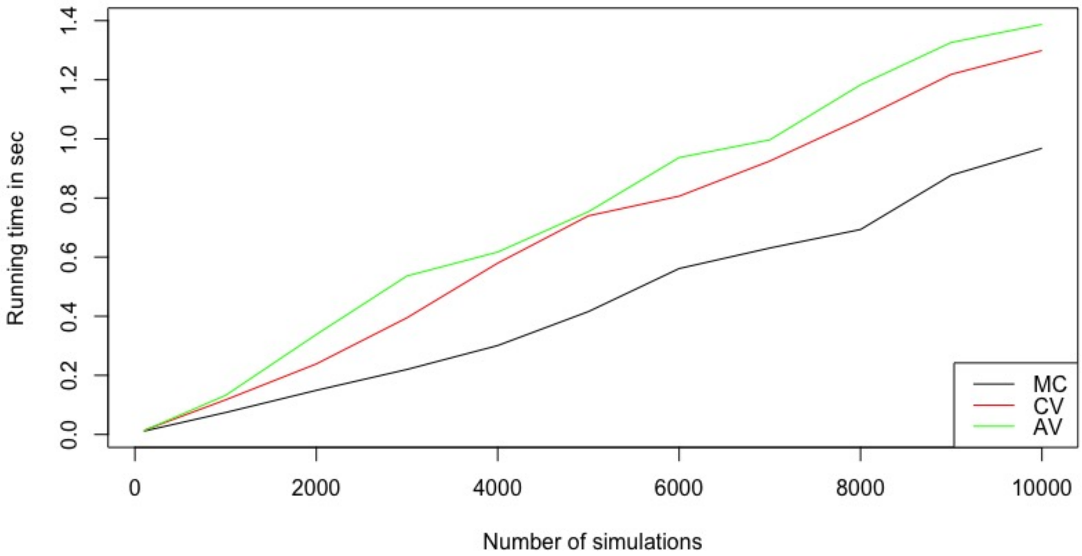


Figure 2: Comparison of MC, CV and AV running times in seconds.

Parameters: $S_0 = 100$, $r = 0.01$, $\sigma = 0.02$, $T = 1$, $N = 300$, $K = 100$

In Table 4, we can see that when we run 10 000 simulations, the CV runs 1.3 times slower than the classical MC, while the AV runs 1.4 times slower. For running only 1 000 simulations CV is 1.57, AV is 1.7 times slower. Variance reduction techniques enable us to reduce the number of simulations to get the same precision, hence they enable us to apply fewer MC repetitions, reducing the running time as well. So instead of running the classical MC with 10 000 paths for 0.968 seconds, and get 0.008335 standard error, we can choose to run it for less paths with one of the variance reduction techniques.

m	MC	CV	AV
100	0.0113	0.0138	0.0129
1000	0.0742	0.1171	0.1323
2000	0.1492	0.2386	0.3382
3000	0.2199	0.3948	0.5362
4000	0.3008	0.5796	0.6171
5000	0.4157	0.7396	0.7536
6000	0.5613	0.8061	0.9367
7000	0.6307	0.9250	0.9967
8000	0.6933	1.0666	1.1827
9000	0.8772	1.2182	1.3261
10000	0.9680	1.2987	1.3873

Table 4: Comparison of MC, CV and AV running time in seconds.

Parameters: $S_0 = 100$, $r = 0.01$, $\sigma = 0.02$, $T = 1$, $N = 300$, $K = 100$

Table 5 shows the increment of the standard error / increment of running time compared to MC. The larger number means that the performance of the method (CV or AV) is more optimal than the other. The ratio defined by the following:

$$ratio = \frac{\frac{\text{standard error by MC}}{\text{standard error by variance reduction}}}{\frac{\text{running time of variance reduction}}{\text{running time of MC}}}$$

From Table 5, we can see that for the mentioned parametrization the CV method always works better than the AV method. In contrary to the AV technique, we can also observe that increasing the simulation paths does not imply that the CV technique gets more effective than for fewer simulation paths. Hence when m is small, the CV is significantly more effective than AV, but its effectiveness decreases as m increases. However, the efficiency of AV increases while m increases. To sum up CV method works better for fewer m than AV.

m	CV	AV
100	289.530	23.016
1000	218.272	45.411
2000	229.891	52.381
3000	189.888	59.511
4000	181.422	81.597
5000	194.500	102.533
6000	242.529	124.630
7000	239.952	143.736
8000	225.560	141.216
9000	243.243	166.306
10000	257.973	184.072

Table 5: Standard error increment/Running time increment.

Parameters: $S_0 = 100$, $r = 0.01$, $\sigma = 0.02$, $T = 1$, $N = 300$, $K = 100$

Until this point we only used one parameter set for our calculations. It is particularly interesting to see whether our conclusions remain the same for smaller and larger volatility and strike or not. In Table 6, we can see the standard error increment / running time increment ratios for different σ volatility values. While σ increases, the efficiency of CV decreases. Up to this stage we can conclude that for smaller σ and fewer number of repetitions, the CV performs better than AV. However, we cannot state clearly which technique is better, since it depends on the parameters.

Let us examine the impact of the strikes. In Table 7, we can see the ratios for different K strikes. In Table 7a, when the moneyness is ITM (at time 0) and $\sigma = 0.02$ we can observe that AV performs better than CV. We can also notice that while K increases both of the CV's and AV's ratios increase. When the moneyness is OTM (at time 0) and $\sigma = 0.02$, the prices are so low (almost equal to zero) that we cannot evaluate the behaviour of the ratios. Hence we examine the effect of the strikes when $\sigma = 0.05$. From Table 7b, we can see that for higher strikes the efficiency of both of the methods decreases. When the moneyness is ITM, the AV performs better than CV. Moreover when the moneyness is OTM, the AV performs slightly better than CV.

m	$\sigma = 0.01$ CV	$\sigma = 0.01$ AV	$\sigma = 0.05$ CV	$\sigma = 0.05$ AV	$\sigma = 0.1$ CV	$\sigma = 0.1$ AV
100	365.518	35.200	117.607	20.826	70.108	19.775
1000	380.491	68.794	96.821	42.174	58.140	41.049
2000	364.330	86.122	98.289	41.645	53.385	37.200
3000	357.988	94.467	99.908	51.431	48.221	46.548
4000	365.040	133.187	88.264	61.783	44.917	58.032
5000	448.253	177.936	95.236	83.461	46.636	76.766
6000	395.889	197.189	98.050	100.336	50.066	93.231
7000	418.684	238.435	104.599	92.306	52.408	81.054
8000	395.332	202.374	93.135	107.436	49.458	98.207
9000	375.012	209.905	97.548	119.359	49.672	109.690
10000	398.839	253.344	92.287	116.211	47.418	105.830

Table 6: Standard error increment/Running time increment for different sigma parameters.

Parameters: $S_0 = 100$, $r = 0.01$, $T = 1$, $N = 300$, $K = 100$

m	$K = 75$ CV	$K = 75$ AV	$K = 90$ CV	$K = 90$ AV
100	107	233	369	812
1000	56	377	235	1756
2000	46	397	769	4608
3000	56	540	247	2656
4000	56	693	477	7196
5000	60	935	343	5900
6000	65	1156	325	6481
7000	59	1031	240	4714
8000	49	972	256	5390
9000	65	1234	245	6105
10000	55	1202	235	5550

(a) $\sigma = 0.02$

$K = 75$ CV	$K = 75$ AV	$K = 90$ CV	$K = 90$ AV	$K = 100$ CV	$K = 100$ AV	$K = 110$ CV	$K = 110$ AV
205	524	59	356	118	21	5	0
100	813	156	1202	97	42	36	41
109	1090	67	920	98	42	6	33
155	2074	106	997	100	51	3	9
121	1326	90	1094	88	62	12	25
88	1392	95	1230	95	83	23	144
77	1596	81	1155	98	100	3	17
83	1636	88	1673	105	92	15	87
96	1950	89	1722	93	107	10	70
109	2188	94	1642	98	119	5	51
117	2872	93	2299	92	116	9	85

(b) $\sigma = 0.05$

Table 7: Standard error increment/Running time increment for different strikes.

Parameters: $S_0 = 100$, $r = 0.01$, $T = 1$, $N = 300$

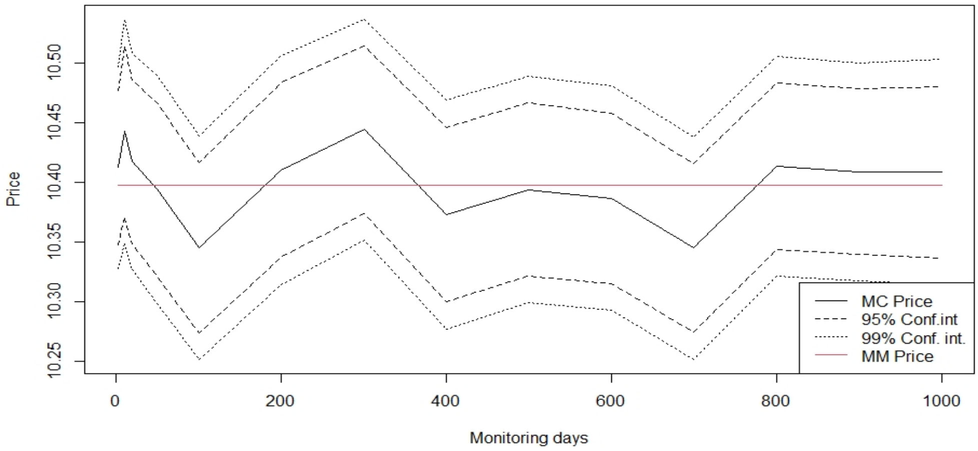
Let us make a choice to compare the moment matching technique with in the next Section 3.2. While the moneyness is ATM, according to Figure 1, Figure 2 and Table 5, we have seen that the CV technique performs slightly better for this parametrization than the AV method. Hence we choose CV to compare it with MM method, which will be discussed in the next subsection. We also choose 1 000 paths to simulate with. Since running CV with 1 000 paths still gives us accurate price and the running time is still less than running the classical MC with 10 000 paths. This seems as a good choice because CV method remains more effective for higher volatilities when the moneyness is ATM.

We would like to highlight our conclusion, which is that we cannot state clearly which technique is better. For smaller σ, less m and for near ATM moneyness, the CV performs better, which state is consistent with Horvath and Medvegyev (2016).

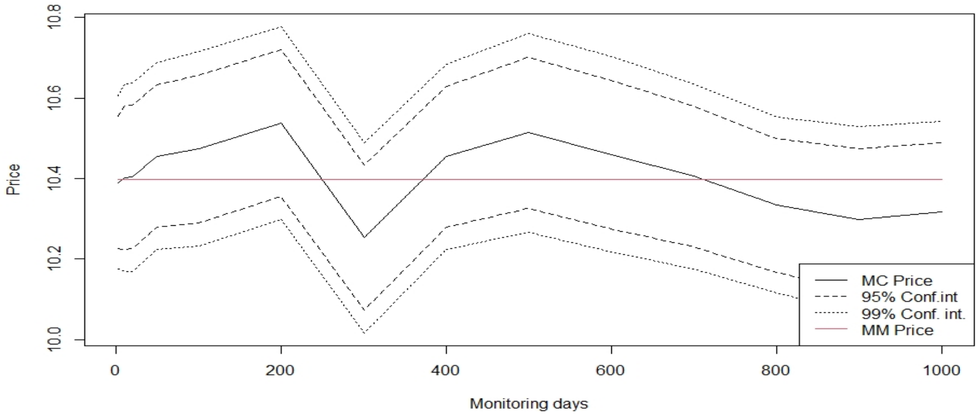
3.2 Comparison of Control Variate and Moment Matching methods

In this section we will examine how the moment matching (MM) method behaves compared to the control variate (CV) Monte Carlo technique. From the results of Lo et al. (2014), we could see that the log-normal moment matching (MM) gets more inaccurate by increasing the number of averaging points (denoted N). Hence let us first examine this phenomenon using the control variate (CV) Monte Carlo technique as a benchmark with 95% and 99% confidence intervals ($\alpha = 0.05$ and $\alpha = 0.01$).

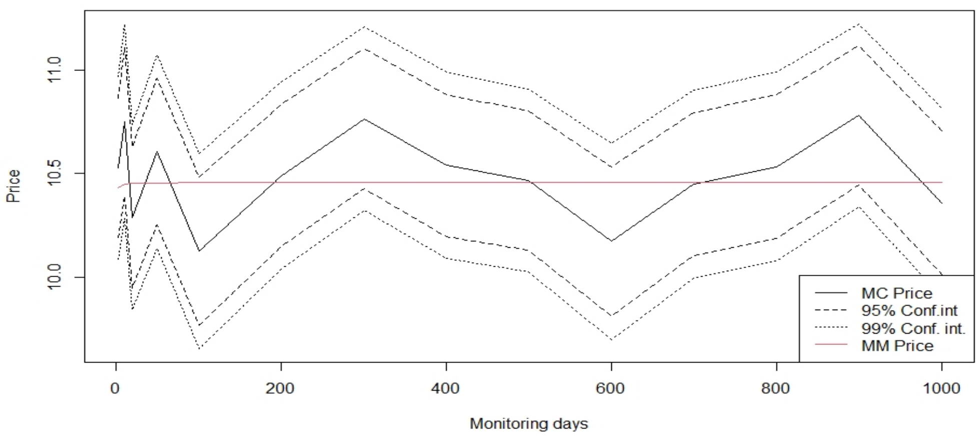
We set the parameters at $S_0 = 100$, $r = 0.01$, $T = 1$, while N spans between 10 and 1000 which is the number of the monitoring points. Let us examine more scenarios. We compared the moment matching technique with CV for varying K and σ. Figures 3, 4 and 5 show the relation through the prices using the moment matching (MM) and the control variate (CV) method with respect to the number of the monitoring points for different strikes and volatilities.



(a) $K = 90$, $\sigma = 0.02$



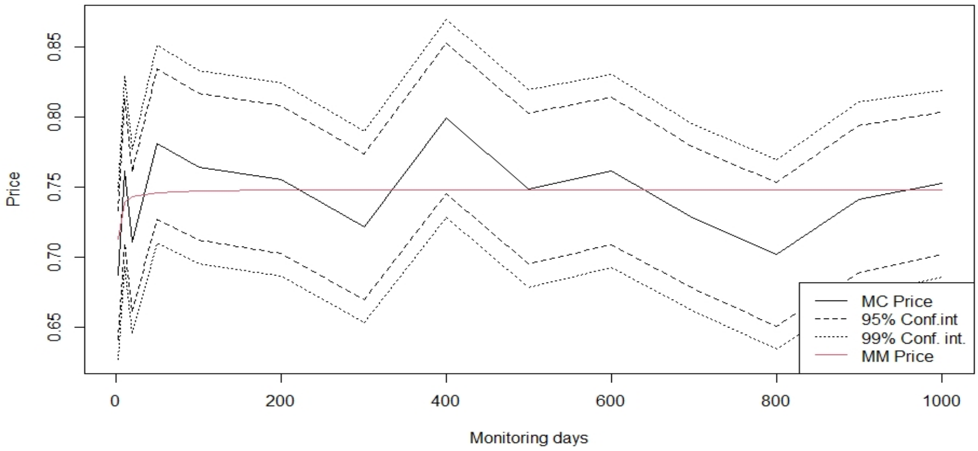
(b) $K = 90$, $\sigma = 0.05$



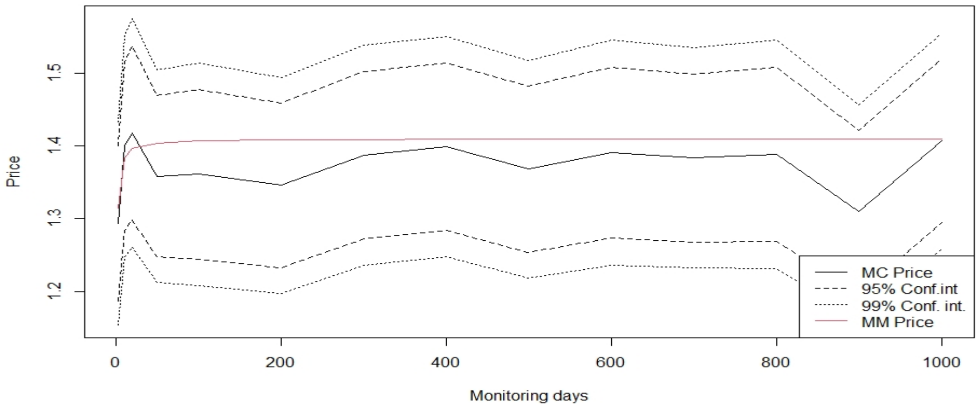
(c) $K = 90$, $\sigma = 0.1$

Figure 3: Comparison of MM and CV prices - ITM

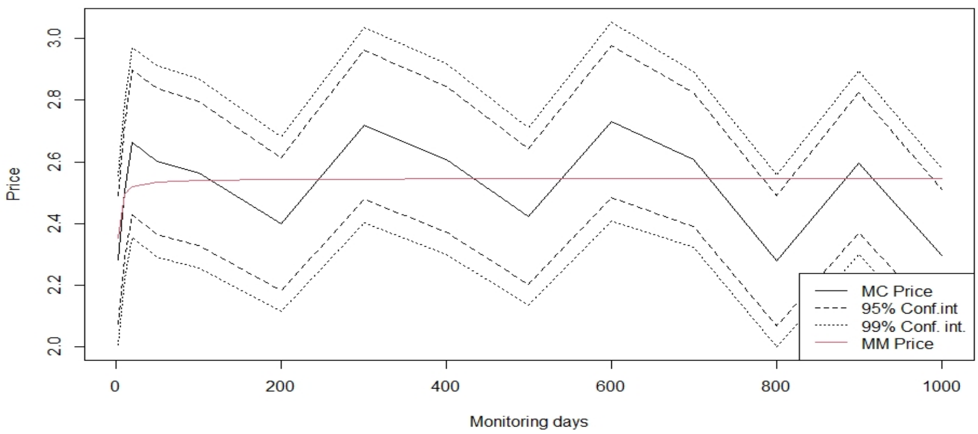
Parameters: $S_0 = 100$, $r = 0.01$, $T = 1$



(a) $K = 100$, $\sigma = 0.02$



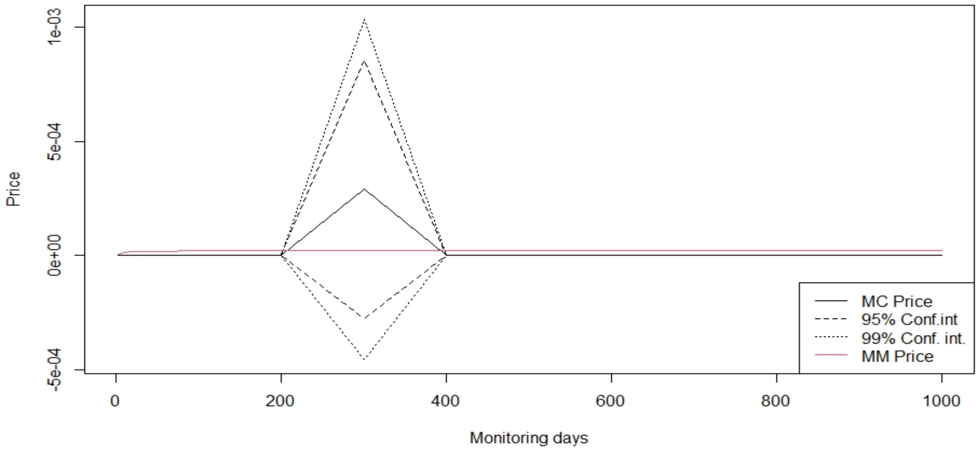
(b) $K = 100$, $\sigma = 0.05$



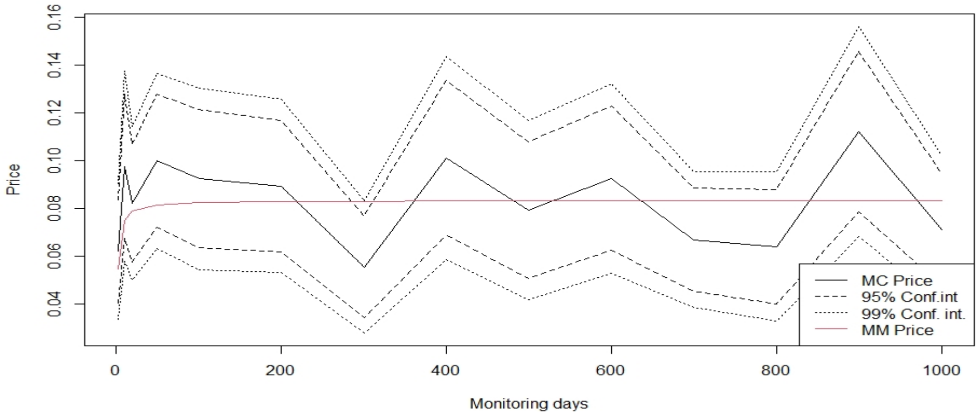
(c) $K = 100$, $\sigma = 0.1$

Figure 4: Comparison of MM and CV prices - ATM

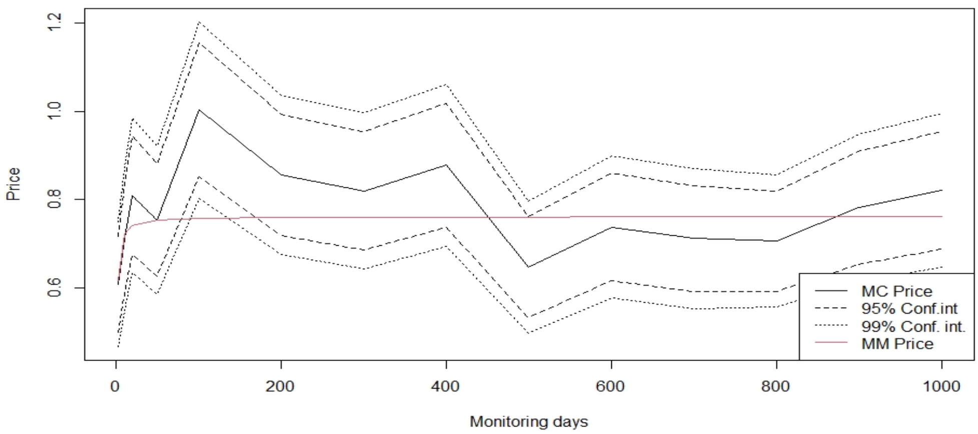
Parameters: $S_0 = 100$, $r = 0.01$, $T = 1$



(a) $K = 105$, $\sigma = 0.02$



(b) $K = 105$, $\sigma = 0.05$



(c) $K = 105$, $\sigma = 0.1$

Figure 5: Comparison of MM and CV prices - OTM

Parameters: $S_0 = 100$, $r = 0.01$, $T = 1$

From Figures 3a, 3b and 3c, we can see that for varying σ the MM method remains accurate while the moneyness is ITM. We can also observe that the prices increase as volatility increases.
From Figures 4a, 4b and 4c we can see that when the moneyness is ATM the MM performs well and the number of the monitoring points has no impact on the accuracy of the MM. In our parameter settings, the MM prices don't fall outside of the 95% confidence interval.
Figures 5a, 5b and 5c show the MM prices when the moneyness is OTM. In Figures 5a, the volatility ($\sigma = 0.02$) is so low and and $K = 105$ which combination results zero prices. For higher volatilities, in Figures 5b and 5c we can observe that the MM technique still remains adequate. In Figure 5c, the MM prices fall outside both of the 95% and 99% interval only once which we can explain with the simulation errors.

From our results and parameters, our observation is that increasing the number of the monitoring points (N) does not affect the accuracy of the MM method. In conclusion we observe that MM prices give accurate results for our mentioned parametrization until 1000 monitoring points. We can assume these monitoring points as monitoring days. Given an APO with maximum 4 years (1000 monitoring points / 250 business day) the MM method gives us punctual prices which explains its popularity in the financial industry. Longer term APOs are less common in commodity markets. Hence the MM is a suitable and accurate method to price APOs.

4 Conclusion

The subject of this thesis was the problem of pricing average price options under the Black-Scholes model. We mainly focused on pricing discretely monitored Asian options with arithmetical averaging. These types of options are highly traded products in commodity markets. Under the Black-Scholes model, where the risky asset prices follow GBM, the arithmetic average of the underlying asset prices doesn't stay log-normally distributed, hence there is no explicit form to price them.

In this thesis firstly we introduced the theoretical basis for pricing APOs under the Black-Scholes model and gave a brief overview of the pricing techniques in the literature. We presented the Monte Carlo simulation and two variance reduction techniques, namely the control variate and the antithetic methods. We used the geometric averaging Asian option price as the control variate to price them. We also described the moment matching method by log-normal approximation.

The second part of this thesis aimed to compare the methods using simulated data. We have found that the accuracy of the variance reduction techniques depend on the parameter settings. However the control variate method performed better than the antithetic variate method for the used parameters and for near ATM moneyness. We used the control variate Monte Carlo technique to check the accuracy of the moment matching technique. From our simulations, we concluded that the prices from the moment matching technique are accurate with our parameter settings.

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A Appendix

The code of the implementation

GBM simulation

```
GBM_Sim = function(s0, r, sigma, TT, N, m)
{dt = TT/N
times = seq(0, TT, dt)
out=sapply(1:m, function(i){
dWs=rnorm(N,sd=sqrt(dt))
dlogS=(r-sigma^2/2)*dt+sigma*dWs
S=c(s0,exp(log(s0)+cumsum(dlogS))) })
rownames(out) = times
return(out)}
```

Generalised Black Scholes formula for call options

```
BSCallGen = function(ss, K, TT, tt, r, m, v)
{
dd1 = ( log(ss/K) + m + v^2 ) / v
dd2 = dd1 - v
exp(-r*(TT-tt)) * (ss * exp(m+v^2/2) * pnorm(dd1) - K*pnorm(dd2))}
```

Closed formula for geometric Asian call options

```
Pricer_Geo_theo = function(s0, r, sigma, TT, N, K)
{
mm=(r-((N+2)/(6*(N+1)))*sigma^2)/2
vv=sigma*sqrt(((2*N+1)/(6*(N+1))))
d1g=(log(s0/K)+(mm+vv^2/2)*TT)/(vv*sqrt(TT))
d2g=d1g-vv*sqrt(TT)
nd1g=pnorm(d1g)
nd2g=pnorm(d2g)

geo_theo_payoff = s0*exp((mm-r)*TT)*nd1g-K*exp(-r*TT)*nd2g
return(geo_theo_payoff)
}
```

Monte Carlo method for Asian call options

```
Pricer_MC0 = function(s0, r, sigma, TT, N, m, K)
{
S = GBM_Sim(s0, r, sigma, TT, N, m)
avgs = apply(S, 2, mean)
payoff = exp(-r*TT)*pmax(avgs - K, 0)
out = c(mean(payoff), sd(payoff)/sqrt(m) )
names(out) = c("Price", "StdErr")
return(out)
}
```

Variance reduction method - Antithetic Variate

```
Pricer_MCA<-function(s0, r, sigma, TT, N, m, K)
{
dt = TT/N
times = seq(0, TT, dt)
dWs=sapply(1:m, function(i) dWs=rnorm(N,sd=sqrt(dt)) )
dWs_anti=-dWs

dlogS = (r-sigma^2/2)*dt+sigma*dWs
S = exp(log(s0) + rbind(rep(0,m), apply(dlogS,2,cumsum)))

dlogS_anti = (r-sigma^2/2)*dt+sigma*dWs_anti
S_anti = exp(log(s0) + rbind(rep(0,m), apply(dlogS_anti,2,cumsum)))

# Price of the asian option via Antithetic method
avg_mc=apply(S,2,mean)
payoff_mc=exp(-r*TT)*pmax(avg_mc - K, 0)

avg_anti=apply(S_anti,2,mean)
payoff_anti=exp(-r*TT)*pmax(avg_anti - K, 0)

antiout=c((mean(payoff_mc)+mean(payoff_anti))/2, sd((payoff_mc+payoff_anti)/2/m)
names(antiout) = c("Price", "StdErr")
return(antiout)}
```

Variance reduction method - Control Variate

```
Pricer_MCC<-function(s0, r, sigma, TT, N, m, K)
{
S = GBM_Sim(s0, r, sigma, TT, N, m)

### Price of the asian option via MC - arithmetic average /crude MC
bb2_aritm=apply(S,2,mean)
payoff_aritm=exp(-r*TT)*pmax(bb2_aritm - K, 0)

### Price of the asian option via MC - geometric average
bb2_geo=apply(S,2,geometric.mean)
payoff_geo=exp(-r*TT)*pmax(bb2_geo - K, 0)

### Theoritical price for geometic averaged option in discret case
azsian_theo_geo=Pricer_Geo_theo(s0, r, sigma, TT, N, K)

### Price of the asian option via Control variate method
bopt = ifelse(is.nan(cov(payoff_aritm, payoff_geo)/var(payoff_geo)),0,0)
payoff_control_optb=payoff_aritm-bopt*(payoff_geo-azsian_theo_geo)

contout = c(mean(payoff_control_optb, na.rm=T),sd(payoff_control_optb, na.rm=T)/s
names(contout) = c("Price", "StdErr")
return(contout)
}
```

Moment matching technique

```
Pricer_Mom_2Exact = function(s0, r, sigma, TT, N, K)
{
moms = momfun_exact(s0, r, 0, sigma, TT, N,  2) / c(s0, s0^2)

m = 2 * log(moms[1]) - log(moms[2]) / 2
v = sqrt(log(moms[2]) - 2 * log(moms[1]))

BSCallGen(s0, K, TT, 0, r, m, v)
}
```

Calculator of the exact moments

(implementation of Roncoroni et al. (2015), Section 18.2)

```
momfun_exact = function(s0, r, q, sigma, TT, N,  n)
{
delta = TT/N
phifv = function(x) exp((r-q-sigma^2/2)*delta*x+sigma^2/2*delta*x^2)
phik = phifv(1:n)

Emat = matrix(nrow=N, ncol=n+1)
Emat[,1] = 1
Emat[N,2:(n+1)] = phik
colnames(Emat) = paste0("n=", 0:n)

for (j in 1:n)
{
for (i in (N-1):1)
{
Emat[i,j+1] = phik[j]*sum( choose(j,0:j) * Emat[i+1,1:(j+1)]  )
}
}

momszamolo = function(n) (s0/(N+1))^n * sum( choose(n,0:n) * Emat[1,1:(n+1)] )
sapply(1:n, function(i) momszamolo(i) )
}
```