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Configurations in Non-Desarguesian Planes

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Contents

Bevezetés

A projektív síkok absztrakt tanulmányozása a XIX. század első felében, J.-V. Poncelet és K. von Staudt munkásságával vette kezdetét. A geometria axiomatikus megalapozása a múlt század elején, több neves matematikus, köztük G. Fano, D. Hilbert és M. Hall közreműködésével született meg. Szintén ez idő tájt fektették le a síkok algebrai eszközökkel való tanulmányozásának és a görbeelméletnek az alapjait. Ezen alapokon számos napjainkban is népszerű matematikai terület (például lgebrai geometria) nyugszik.

Az algebrai megközelítés segítségével lehetővé vált a síkok behatóbb tanulmányozása és osztályozása. Kiemelt jelentőséggel bír e téren Hilbert 1923-as eredménye Desarguesi-síkok karakterizálásáról, mely a nem-desarguesi síkok vizsgálatában is mérföldkőnek számít.

Dolgozatom elsődleges célja klasszikus nem-desarguesi síkok bemutatása és elemzése. Desargues tételétől indulva tárgyalunk desarguesi és nem desarguesi affin és projektív síkokat. Bár a dolgozat több végtelen síkon is említ, érdeklődésünk középpontjában főként véges síkok állnak. A konstrukciók ismertetését követően különféle véges geometriai struktúrák — ívek és lefogó ponthalmazok— vizsgálatába kezdünk.

Dolgozatomban a következő tartalmi tagolást követem:

- Bevezetés.
- A második fejezet röviden összefoglalja a testtel koordinátázott affin és projektív síkokkal, Desargues tétellel és desarguesi síkokkal kapcsolatos alapvető definíciókat és összefüggéseket.
- A harmadik fejezetben több klasszikus példát mutatunk nem-desarguesi síkokra. A konstrukciók ismertetése során a szükséges algebrai fogalmak (planáris függvények, majdnemtestek) is áttekintésre kerülnek.
- A negyedik fejezetben felidézzük az ívekkel, oválisokkal, lefogó- és blokkolóhalmazokkal kapcsolatos definíciókat. Ismertetjük a testre épített síkokra vonatkozó eredményeket és belátjuk, hogy a közölt tételek nem vihetőek át tetszőleges síkra. Külön alfejezetet szentelünk Hall síkok örökölt íveinek tanulmányozására, betekintést engedve az örökölt ívek karakterizációjának jelenlegi állapotába. A még megoldatlan részesetek egyikét elemezve egy lehetséges megoldási tervet vázolunk fel. A dolgozatot Hall síkok lefogó ponthalmazira vonatkozó megjegyzésekkel zárjuk.

Configurations in Non-desarguesian Planes

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1. Introduction

The abstract study of projective geometry first arose in the work of J.-V. Poncelet and K. von Staudt in the first part of the nineteenth century. About 100 years ago, axiomatic frameworks were developed by several people, including G. Fano, D. Hilbert, and M. Hall. In the same decades the idea of examining affine and projective planes by using algebraic methods had been settled. The early results about coordinatization of planes and examination of curves are counted as sources of many even nowadays popular fields such as algebraic geometry.

In 1923, Hilbert published his landmark result about characterization of Desarguesian-planes. The proof of the theorem is based on algebraic methods and has made a remarkable contribution to the study of non-desarguesian planes.

The first objective in this thesis is to introduce and study a handful of classical non-desarguesian planes. After recalling some basic definitions and results we will turn to the study of Desargues theorem, desarguesian and non-desarguesian affine and projective planes. Although this thesis mentions infinite planes several times we will mainly focus on finite planes. We will discuss classical constructions achieved in different ways and study various finite geometric structures such as arcs and blocking sets of the introduced planes.

Throughout my thesis the different topics mentioned above will be discussed in the following order:

- Section 1 will cover Introduction.
- Section 2 will briefly summarize the basic definitions and theorems concerning affine and projective planes coordinatized with fields, Desargues theorem and desarguesian- planes as well.
- In Section 3 we will give classical examples of non-desarguesian projective planes. While introducing various constructions we will briefly discuss the involved algebraic structures such as planar functions and nearfields.

- Section 4 mainly deals with arcs, ovals and blocking sets in the previously discussed planes. After recalling basic definitions and results we will discuss a handful of known theorems concerning ovals and blocking sets of planes built on finite fields and prove that the discussed theorems cannot be extended to arbitrary planes. We will study inherited arcs in Hall-planes in details providing an insight to the current stage of the (still uncomplete) characterization of inherited arcs. A possible (rather technical) way — which may be able to solve some of the uncomplete cases — including demanding computations is also introduced in this section. Finally, we will make a few remarks concerning blocking sets of the Hall plane.

2 Desarguesian Planes

2.1 Basic Definitions

For further use we recall the definitions of affine and projective planes.

Definition 1 *An affine plane $\mathbb{A}$ is a set, the elements of which are called points, together with a collection of subsets, called lines, satisfying the following three axioms.*

A1. For every two different points there is a unique line containing them.

A2. For every line l and a point P not in l, there is a unique line containing P and disjoint from l.

A3. There are three points such that no line contains all three of them.

Definition 2 *A projective plane is a set, the elements of which are called points, together with a collection of subsets, called lines, satisfying the following three axioms.*

P1. Any two distinct points belong to exactly one line.

P2. Any two distinct lines intersect in exactly one point.

P3. There are four points such that no line contains any three of them.

Two lines are called parallel if they are equal or disjoint. Certainly, it is an equivalence-relation on the lines (transitivity follows from $A2$.) the equivalence classes are called *parallel classes*. Two affine planes $\mathbb{A}, \mathbb{A}'$ (or two projective planes $\mathbb{P}, \mathbb{P}'$) are *isomorphic* if there is a bijection $\mathbb{A} \to \mathbb{A}'$ ($\mathbb{P} \to \mathbb{P}'$) taking lines to lines.

Recall that a finite projective plane has $t^2 + t + 1$ points and $t^2 + t + 1$ lines, each line contains $t + 1$ points and each point is contained by $t + 1$ lines, where $t \in \mathbb{N}$, $t \geq 2$ is called the *order* of the plane. The notation Π_q generally refers to an arbitrary projective plane of order q.

A projective plane can be considered as a *closure* of a suitable affine plane. That is, we extend the affine plane by giving additional points, called *ideal points*, each of which corresponds to a parallel class and is contained by the lines in the corresponding class. Furthermore we define a line, called *ideal line* (or *line at infinity*), which contains exactly the ideal points.

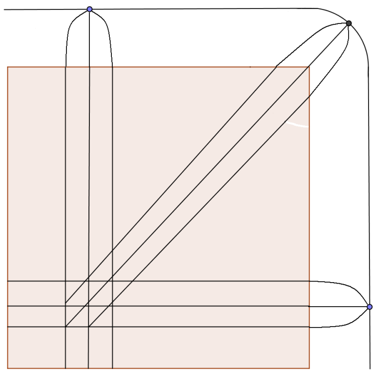

Figure 1: Projective closure

It is easy to see that the structure given in Figure 1. satisfies the axioms $P1, P2, P3$ and therefore is a projective plane. One can also define a reverse transformation which yields an affine plane by deleting an arbitrary line with its points in a projective plane. As a result, we get that a finite affine plane has t^2 points and t^2+t lines, each line contains t points and each point is contained by $t+1$ lines. We define the order in case of affine planes as the order of their closure.

2.2 Planes Coordinatized with Fields and Skewfields

We briefly recall the definitions of affine and projective planes coordinatized with fields and skewfields. Throughout this thesis — if we do not define it differently — p is a positive prime and q is a positive power of p. The notation $\mathbb{F}$ generally refers to an arbitrary field. We denote $\mathbb{F}_q = GF(q)$ that is, the Galois field of order q. In Section 4 we also use the notation $\mathbb{F} = \mathbb{F}_q$ and $\mathbb{K} = \mathbb{F}_{q^2}$.

The affine plane coordinatized with $\mathbb{F}$ is defined as the elements of $\mathbb{F}\times\mathbb{F}$ as points, with the subsets — as lines — of the following forms: $\{(x, y) : y = m \cdot x + b\}$ and $\{(x, y) : x = b\}$ for all $m, b \in \mathbb{F}$. In case of skewfields the same definition is valid. However, the famous theorem due to Wedderburn (usually called "Wedderburn's little theorem") characterizes the finite skewfields as being fields. Since we mainly focus on planes coordinatized with finite structures, it is sufficient to concentrate on planes which are coordinatized with fields. We denote the affine plane arisen by the field $\mathbb{F}$ with $AG(2, \mathbb{F})$. Furthermore, if our field is a Galois field of order q we use the notation $AG(2, q)$.

By defining the projective plane coordinatized with $\mathbb{F}$, a classical way is to associate it to the three-dimensional vectorspace V over $\mathbb{F}$. We define the relation $\sim$ in $V\setminus\{\underline{0}\} = \mathbb{F}^3\setminus\{\underline{0}\}$ on the following way: for $\underline{u}, \underline{v} \in V\setminus\{\underline{0}\}$, $\underline{u} \sim \underline{v}$ iff $\exists\lambda \in \mathbb{F}^* : \underline{u} = \lambda\cdot\underline{v}$.

Easy calculation shows that $\sim$ is an equivalence relation. We define the point-set of the projective plane $\mathcal{P}$ as $\mathbb{F}^3/\sim$, that is, the equivalence-classes of $\mathbb{F}^3$. One can easily see that each point is associated to a one-dimensional subspace of V.

The elements of $\mathcal{L}$ — the set of lines — are defined as the images of the two-dimensional subspaces of V by factoring with $\sim$. In other words, a subset of $\mathcal{P}$ is a line iff the corresponding one-dimensional subspaces are exactly the subspaces of a two-dimensional subspace in V.

As each two-dimensional subspace has a unique orthogonal space in V it can be described by giving the corresponding element of this subspace in $\mathcal{P}$. In this way the elements of $\mathcal{P}$ and $\mathcal{L}$ are both uniquely coded with the elements of $V\setminus\{\underline{0}\}$ up to scalar multiplication. For the sake of simplicity we keep the convention of writing $(\frac{a}{c}, \frac{b}{c}, 1)$ $(c \neq 0)$, $(\frac{a}{b}, 1, 0)$ and $(1, 0, 0)$ instead of $(a, b, c), (a, b, 0), (a, 0, 0) \in \mathcal{P}$, respectively. For the elements of $\mathcal{L}$ we use the notation $[\frac{a}{c}, \frac{b}{c}, 1]$ $(c \neq 0)$, $[\frac{a}{b}, 1, 0]$ and $[1, 0, 0]$. Note that the point $(a, b, c) \in \mathcal{P}$ is contained in the line $[d, e, f] \in \mathcal{L}$ if and only if $\langle (a, b, c), [d, e, f] \rangle = a \cdot d + b \cdot e + c \cdot f = 0$.

According to our previous remark deleting the line $[0, 0, 1]$ with its points, we get an affine plane with point-set $\mathcal{P}' = \{(a, b, 1) : a, b \in \mathbb{F}\}$ and line-set $\mathcal{L}' = \{[m, 1, c] : m, c \in \mathbb{F}\} \cup \{[1, 0, d] : d \in \mathbb{F}\}$. Elementary calculation shows that the function $\varphi : \mathcal{P}' \to \mathbb{F} \times \mathbb{F}$, $\varphi((a, b, 1)) = (a, b)$ is bijective and takes lines into lines, as the images of the lines $[m, 1, c]$ and $[1, 0, d]$ in $AG(2, \mathbb{F})$ are exactly the lines defined by the equations $y = -m \cdot x - c$ and $x = -d$, respectively. It means that our affine plane is isomorphic to $AG(2, q)$. $PG(2, \mathbb{F})$ denotes the projective closure of $AG(2, \mathbb{F})$. In case of Galois fields the notation $PG(2, q)$ is used.

As we have seen there are two basic concept by coordinatizing affine planes. Either we can use *euclidean-coordinates* (as we did by defining $AG(2, q)$) or consider our affine plane as embedded in its projective closure and use the previously introduced coordinate-triples called *homogeneous coordinates.* Throughout this thesis we will not prefer either to the other and choose the one which better suits the current goal. As the conversion between the two notations is straightforward we will not discuss it in details.

Example 1 *The graph of the function $y = x^2$ in affine sense contains the points $\{(a, a^2) : a \in \mathbb{F}\}$ in $AG(2, \mathbb{F})$. In projective sense, however, the same function has the form $x^2 = y \cdot z$ and contains the following points of $PG(2, \mathbb{F})$: $\{(a, a^2, 1) : a \in \mathbb{F}\} \cup \{(0, 1, 0)\}$.*

For the sake of simplicity we use the notation $(0) = (0, 0, 1)$ and $(\infty) = (0, 1, 0)$ whenever it is not confusing.

2.3 Desargues Theorem and Desarguesian Planes

The investigation of general projective planes is essentially based on the early theorem named after Desargues*. To discuss the theorem a pair of definitions about general projective planes is needed.

Definition 3 *Two triangles, with their vertices named in a particular order, are said to be perspective from a point P (or briefly, "point-perspective") if their three pairs of corresponding vertices are joined by concurrent lines.*

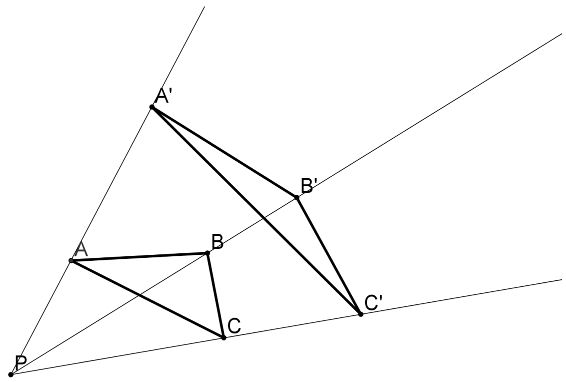

Figure 2: Point-perspective triangles with respect P

Definition 4 *Two triangles are said to be perspective from a line l ("line-perspective") if their three pairs of corresponding sides meet in collinear points.*

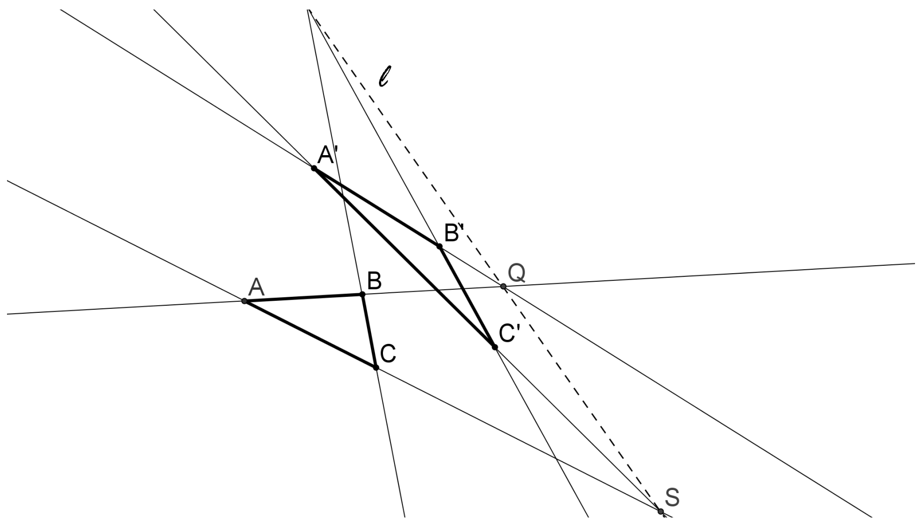

Figure 3: Line-perspective triangles with respect l

*Girard Desargues (1591-1661) French mathematician and engineer, considered as one of the founders of projective geometry.

In affine sense the definitions need additional refinement in order to translate the case when some of the intersections lie on the ideal line. Two triangles are also said to be point-perspective if the corresponding lines defined by their vertices are parallel. Two triangles are line-perspective if the pairs of corresponding sides are pairwise parallel.

The two perspectivities are joined by the following famous theorem:

Theorem 1 (Hilbert) *Let $\mathbb{F}$ be any field or skewfield. Two triangles are perspective from a point if and only if they are perspective from a line.*[†]

Desargues theorem can be formulated in higher dimensional projective spaces as well and it has been proven that in any projective space of dimension at least three Desargues theorem is a direct consequence of the space axioms. This astonishing result implied the natural question whether Desargues theorem can be also extended to an arbitrary projective plane. Moulton disproved the conjecture by constructing a projective planes and showing two triangles which do not hold Theorem 1.

Construction 1 *We construct a new plane on $\mathbb{R} \times \mathbb{R}$ by modifying the lines which have positive slopes. We define the images of the line $y = m \cdot x + b$ $(m, b \in \mathbb{R}, m > 0)$ as $\{(x, y) : y \leq 0, y = m \cdot x + b\} \cup \{(x, y) : y > 0, y = \frac{m}{2} \cdot x + b\}$. In other words one can say that the lines of positive slopes "refract" on the x-axis.*

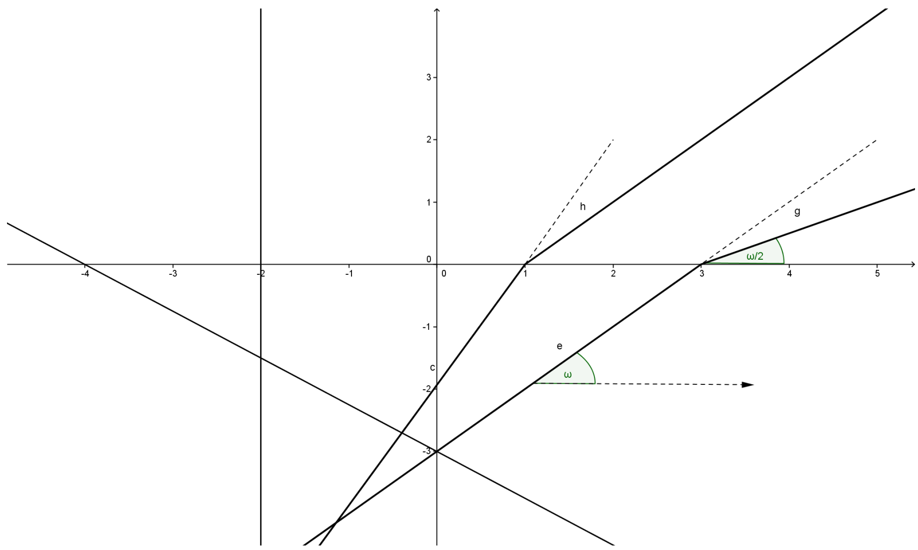

Figure 4: Moulton plane

[†]Deargues theorem in the current form was first stated by Hilbert in the early years of the nineteenth century.

One can easily prove the given structure holds axiom $A1., A2., A3.$ and so is an affine plane. To prove that Moulton's construction yields a nondesarguesian plane a possible example for point-perspective but not line-perspective triangles is given below.

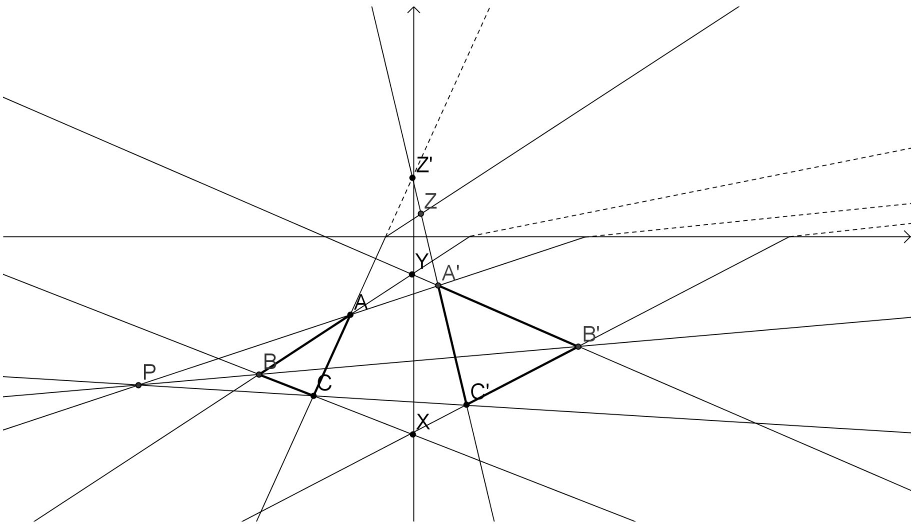

Figure 5: The triangles ABC and $A'B'C'$ are certainly point-perspective with respect P. The lines $\overline{AB}$,$\overline{AC}$, $\overline{B'C'}$ have positives slopes and so they refract on the x-axis. It yields that $\overline{AC}$ does not meet $A'C'$ on the y-axis (as it would do in $AG(2,\mathbb{R})$) and so the intersections X, Y, Z are not collinear in the Moulton plane, which disproves Desargues theorem.

A natural question arisen by Moulton's construction is for which projective planes is Theorem 1 valid. Such planes are called *desarguesian*. Many constructions for nondesarguesian planes have been developed in the early years of the 20th century. The final characterization of desarguesian planes was achieved by Hilbert.

Theorem 2 (Hilbert) *A projective plane is desarguesian if and only if it can be coordinatized with a skew-field.*

Note that as a consequence of Hilbert's theorem we get that finite desarguesian projective planes are all isomorphic to one of the $PG(2,q)$ planes for a suitable q.

Hilbert's theorem also says that whenever a coordinatization of a projective plane with a binary algebraic structure is given then the plane is desarguesian if and only if the given algebraic structure is a skewfield (see [9] for details).

3 Nondesarguesian Planes

Although the first nondesarguesian planes have been explored in the late eighties of the 19th century, it took a measurable time to invent effective methods which yield nondesarguesian planes.

Our first goal is in this section to demonstrate that searching for non-desarguesian planes may become a very demanding attempt. Afterwards we introduce general and particular methods to construct nondesarguesian affine and projective planes. Firstly, we consider a family of affine planes arisen by an elementary idea.

3.1 An Elementary Method

We define a new geometric structure on the point-set of $AG(2, \mathbb{R})$ by replacing some of the plane's lines with other algebraic curves.

Construction 2 *On the point-set* $\mathcal{P} = \mathbb{R} \times \mathbb{R}$ *we define the set of lines as the union of vertical lines in* $AG(2, \mathbb{R})$ *(i.e.* $x = c$ *for all* $c \in \mathbb{R}$*) and the graphs of the functions* $f(x) = m \cdot x^3 + b$ *for all* $m, b \in \mathbb{R}$*, that is*

$$\mathcal{L} = \{y = c : c \in \mathbb{R}\} \cup \{m \cdot x^3 + b : m, b \in \mathbb{R}\}.$$

Statement 1 $\Pi = \{\mathcal{P}, \mathcal{L}\}$ *satisfies* $A1$*,* $A2$*,* $A3$ *and* $A4$ *and so is an affine plane.*

Proof Since the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = m \cdot x^3 + b$ $(m, b \in \mathbb{R}, m \neq 0)$ is bijective, the equation-system

$$\begin{cases} y = m_1 \cdot x^3 + b_1 \\ y = m_2 \cdot x^3 + b_2 \end{cases} \tag{1}$$

has a unique solution in x for $m_1 \neq m_2$. It completes the proof of $A1$. Taking some elementary observation one can easily prove the rest which we will not discuss in details. $\square$

Corollary 1 *Assume* $f : \mathbb{R} \to \mathbb{R}$ *is a bijective function. We define the lines of* $\mathcal{P} = \mathbb{R} \times \mathbb{R}$ *as follows:*

$$\mathcal{L} = \{y = c : c \in \mathbb{R}\} \cup \{m \cdot f(x) + b : m, b \in \mathbb{R}\}.$$

In this case $(\mathcal{P}, \mathcal{L})$ *is an affine plane.*

Statement 2 $\Pi = \{\mathcal{P}, \mathcal{L}\}$ *is desarguesian. Moreover, it is isomorphic to* $AG(2, \mathbb{R})$.

Proof We define the function $\varphi : AG(2,\mathbb{R}) \to \Pi$, $\varphi((a,b)) = (a, \sqrt[3]{b})$. Certainly, φ is bijective. Since the point (a,b) is contained by the line $y = m \cdot x + b$ in $AG(2,\mathbb{R})$ iff the line $y = m \cdot x^3 + b$ in Π contains $\varphi((a,b)) = (a, \sqrt[3]{b})$, φ takes lines into lines and so is an isomorphism. □

Even using the more general form of our idea we get the same result, that is, a peculiar interpretation of $AG(2,q)$. It seems that the more complicated way of constructing a plane does not necessarily yield the more intricate structure.

Our second idea is also based on replacing the lines of $AG(2,\mathbb{R})$ with curves. This time we introduce first a general approach.

Construction 3 *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a suitable function and* $f^* \subset \mathbb{R}^2$ *the graph of* f. *We redefine the set of lines in* $AG(2,\mathbb{R})$ *as the union of vertical lines (that is,* $x = c$ *for all* $c \in \mathbb{R}$*) and the translates of* f^* *with all vectors* $\underline{v} \in \mathbb{R}^2$. *In other words,*

$$\mathcal{L} = \bigcup_{c\in\mathbb{R}} \{(c,y) : y \in \mathbb{R}\} \cup \bigcup_{a,b\in\mathbb{R}} \{(x+a, y+b) : (x,y) \in f^*\}.$$

Not surprisingly, for an arbitrary real valued function f, the associated structure given in Construction 3 is far from being an affine plane as it may harm the axioms $A1.$ and $A2.$ in many ways. To say the least, one cannot even be sure that different translations of f^* yield different lines (as they usually do not, see e.g. the constant functions).

To get an affine plane in Construction 3 a necessary and sufficient condition for f^* is the following.

Theorem 3 *Construction 3 yields an affine plane if and only if for each* $\underline{v} \in \mathbb{R}^2, \underline{v} = (v_1, v_2), v_1 \neq 0$ *there exists a unique pair* $(P_1, P_2), P_1, P_2 \in f^*$ *such that* $\overrightarrow{P_1P_2} = \underline{v}$.

A function satisfying the previous property is called *planar*. We denote the affine plane arisen by f with $I(f)$. Planar functions do exist as we will see it soon. As an example and easy exercise we state that the function $f(x) = x^2$ is planar.

For being a planar function equivalent and sufficient but not equivalent conditions are the following.

Theorem 4 $f : \mathbb{R} \to \mathbb{R}$ *is planar if and only if for all* $a \in \mathbb{R}\backslash\{0\}$ *the function* $g(x) = f(x+a) - f(x)$ *is bijective.*

Theorem 5 *If the function* $f : \mathbb{R} \to \mathbb{R}$ *is strictly convex and* $\lim\limits_{-\infty} \frac{f(x)}{x} = -\infty$, $\lim\limits_{+\infty} \frac{f(x)}{x} = +\infty$ *then* f *is planar.*

For further equivalent conditions and additional details we refer [5].

Even having a proof that we did construct an affine plane it may occur that our plane does not differ from those we have already mentioned. As an example we prove it happens by choosing the function $f(x) = x^2$.

Statement 3 *The plane $I(x^2)$ is isomorphic to $AG(2, \mathbb{R})$.*

Proof We define the function $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$, $\varphi\big((x, y)\big) = (x, x^2 + y)$. Certainly φ is bijective. In additions

$$\varphi(x, m \cdot x + b) = (x, x^2 + m \cdot x + b),$$

so the image of the line defined by the equation $y = m \cdot x + b$ is the set of points satisfying the equation $y = x^2 + m \cdot x + b = (x + \frac{m}{2})^2 + b - \frac{m^2}{4}$. It means that φ maps lines into lines and so is an isomorphism. □

By Theorem 3 the function $f(x) = x^4$ also yields an affine plane. Surprisingly the plane $I(x^4)$ reveals essential differences comparing it with $I(x^2)$ as the former one is not desarguesian. For the complete proof of the statement we refer [11]. A possible construction violating Theorem 1 is given below (note that the direct calculation of the "lines" defined by the given points may become very demanding. In [11] these overwhelming calculations are avoided).

Construction 4 *The triangles ABC and $A'B'C'$ given by the points $A(0; -1)$, $B(\frac{1}{2}; -1)$, $C(\frac{1}{2}; 3)$ and $A'(0; -2)$, $B'(1; -2)$, $C'(1; 12)$ are line perspective but not point-perspective.*

It is a natural goal to characterize the desarguesian planes arisen by real valued functions. Although the problem has not been solved yet in the general case, many particular results are known.

Theorem 6 *The continuous planar functions which yield desarguesian planes are parabolas.*

The examination of planarity can be also extended to functions over finite fields which yield various intriguing constructions. As an example we mention the function $f(x) = x^{p+1}$ is planar over $\mathbb{F}_{p^h}$. Moreover, for $h = 2$ the affine plane given by the construction is isomorphic to $AG(2, p^2)$ while for $h = 3$ it yields a proper non-desarguesian translation plane[‡]

For the deeper understanding of planar functions and affine planes built on them we refer [4],[5] and [8].

[‡] A projective plane Π_q is called translation plane with respect of the line l if for a fixed $C \in l$ and for each $A, B \notin l$ there exists an elation (a collineation fixing any point of l and mapping any line through C onto itself) which maps A to B. See further details about translation planes in [9].

3.2 Affine Planes Built on Nearfields

We are going to investigate further constructions which yield nondesarguesian planes. As we have seen, dealing with a general non-desarguesian plane may be a very tough attempt. Therefore we wish to use convenient algebraic structures to coordinatize our plane. Since according to Theorem 2 planes coordinatized with fields or skew-fields are necessarily desarguesian, we need a somehow weaker structure to succeed.

In the very early years of the 1900s Dickson had examined the independence of field axioms. As a part of his result he presented in his paper ([6]) a nine-element structure having two binary operations "+" and "·", satisfying each field axiom except (multiplicative) commutativity and right-distributivity. Two years later, Veblen and Wedderburn constructed nondesarguesian planes using Dickson's inventions. These early results are counted as the main bases of the explorations of *Nearfields*[§].

Definition 5 *The set N with the binary operations $+$ and $\cdot$ is called a (left) Nearfield if the following conditions hold:*

1. $(N, +)$ *is a group (with* 0 *as identity),*
2. $(N\backslash\{0\}, \cdot)$ *is a group,*
3. $a \cdot (b + c) = a \cdot b + a \cdot c \ \forall a, b, c \in N$.

One can define right nearfields in a similar way by modifying axiom 3. Throughout this thesis we will work with left nearfields, taking a remark that as it happens by examining rings and their opposites, one would be able to formulate the following results using right nearfields as well.

The lack of right distributivity causes weird phenomena in our structure. As an example, we mention that despite the case of fields $0 \cdot a = 0$ does not follow from the axioms as it is not even true in the general case. We disprove our assumption by introducing the unique two-element nearfield which is not isomorphic to the field $\mathbb{F}_2$.

Example 2 *Two-element nearfield.*

+	0	1
0	0	1
1	1	0

·	0	1
0	0	1
1	0	1

[§]There are even weaker algebraic structures (quasi-fields, ternary rings) which give a more general approach to the coordinatization of planes. Since calculations in the general cases (when multiplication is not even associative) may become very difficult, we focus on the special case when our quasi-field is a near-field. For additional details concerning general coordinatization we refer [9]

In some articles nearfields are defined automatically as being *commutative* ($a + b = b + a$) and *null-symmetric* ($a \cdot 0 = 0 \cdot a = 0$) as both conditions can be inferred from the axioms for $|N| \geq 3$ (commutativity is true even for $|N| = 2$). We refer [3] for details.

We are going to discuss a classical construction which yields a "proper" nearfield by "distorting" the multiplication in a field.

Construction 5 ¶ *For an arbitrary odd prime p we define a multiplication* $*$ *in* $\mathbb{F}_{q^2} = \mathbb{F}_{p^{2k}}$ *on the following way (recall that* $a \in \mathbb{F}_{q^2}$ *is a square-element if there exists* $b \in \mathbb{F}_{q^2} : b^2 = a$*):*

$$a * b = \begin{cases} a \cdot b & \text{if } a \text{ is a square,} \\ a \cdot b^q & \text{otherwise.} \end{cases} \tag{2}$$

Simple calculation yields that $(F_q, +, *)$ is a proper nearfield. A nearfield N is called *Dickson nearfield* if there is a field $\mathbb{F}$ such that N is isomorphic to $(\mathbb{F}, +, *)$ ($*$ is the distorted multiplication of $\mathbb{F}$ similarly to Construction 5).

In 1930 Zassenhaus characterized nearfields showing that they are either Dickson nearfields or belong to 8 exceptional nearfields of order $2, 5^2, 7^2, 11^2$ *(two nonisomorphic examples)*, $23^2, 29^2$ or 59^2. Hence all finite nearfields can considered to be "known".

Note that beside being very useful tools in the construction of nondesarguesian planes nearfields had been proven to play essential role in the study of sharply two-transitive permutation groups as well.

Theorem 7 ‖ *If G is a sharply 2-transitive finite group then there is a nearfield* $(N, +, \circ)$*, such that G is isomorphic to the group of all transformations* $x \longrightarrow x \circ a + b$.

Now we turn to the study of affine planes coordinatized with (Dickson's type) nearfields. Let N be a finite nearfield associated to F_{q^2} as described in Construction 5. We define the affine plane $(\mathcal{P}, \mathcal{L})$ on $\mathcal{P} = N \times N$ with the subsets (as lines):

$$\mathcal{L} = \bigcup_{m,b \in N} \{(x, y) : y = m * x + b\} \cup \bigcup_{b \in N} \{(x, y) : x = b\}.$$

¶The original construction due to Dickson is much more general, we apply it only for finite fields.

‖The theorem has a more general form concerning permutation groups of arbitrary cardinality, in which case the theorem says that for each sharply two-transitive permutation group there exists a "near-domain" (roughly spoken, an "additively nonassociative nearfield") such that G is isomorphic to the group of the mentioned transformations.

We wish to state that by choosing an arbitrary (not necessarily finite) nearfield the previously discussed construction does not necessarily yield an affine plane, as the equation $a * x = b * x + b$ may have no unique solution (or no solution at all).

Definition 6 *A (left) nearfield N is planar if the equation $a * x = b * x + c$ has a unique solution for all $a, b, c \in N$.*

Although non-planar nearfields do exists, Zemmer proved that every finite nearfield is planar. It also implies that our construction does yield an affine plane (we leave both facts as exercises). We denote the affine plane built on the Dickson's nearfield $(F_q, +, *)$ with $\sum_q$ if it is not confusing.

Our next goal is to prove that the given plane is not desarguesian. It can be easily achieved either by recalling Theorem 2 or showing a configuration which violates Theorem 1. We leave the latter way as an exercise.

Although throughout this thesis an affine plane coordinatized with a nearfield always means Construction 5 we mention that coordinatization by using nearfields can be made in numerous ways. A famous family of planes called *Hughes-planes* reveals another example of projective planes built on nearfields. In order to show a measurable difference between $\sum_q$ and Hughes-planes we mention without further discussion that the latter family does not belong to the family of translation-planes (while $\sum_q$ does).

3.3 Hall Planes

We have just seen an example for nondesarguesian planes arisen by distorting the multiplication of a finite field. In our next example we modify the structure of the affine plane $AG(2, q)$ through an operation usually called *derivation*.

We recall that every line in $AG(2, q^2)$ contains q^2 points. In addition, our plane has subplanes of order q (also called *Baer-subplane*, see Section 4) which also contains the same number of points and lines. This elementary observation had driven people to the exploration of *Hall planes*.

Definition 7 *Let q be an arbitrary power of an odd prime p and let $l_\infty = [0, 0, 1]$ be a line of $PG(2, q^2)$. A derivation set $\mathfrak{D}$ of $AG(2, q^2) = PG(2, q^2) \setminus l_\infty$ is a set of $q + 1$ points in l_∞ such that for any two affine points P and Q for which the ideal point of the line $\overline{PQ}$ is in $\mathfrak{D}$ there is a Baer subplane containing P, Q and whose ideal points coincide with $\mathfrak{D}$.*

We define a new incidence structure $\mathfrak{D}AG(2, q^2)$ as follows: the points are the points of $AG(2, q^2)$. A line is either a line of $AG(2, q^2)$ with ideal point not in $\mathfrak{D}$ or *the points of a Baer subplane in $AG(2, q^2)$ whose ideal points coincide with $\mathfrak{D}$. The incidence relation is the natural containment relation.*

It can be proven by examining numerous cases that (assuming the existence of a suitable set $\mathfrak{D}$) $\mathfrak{D}AG(2, q^2)$ is an affine plane of order q^2. We define the Hall plane of order q^2 as the projective completion of $\mathfrak{D}AG(2, q^2)$ and denote it by $Hall(q^2)$.

Certainly, the derivation set $\mathfrak{D}$ cannot be chosen arbitrarily as the condition given above does not necessarily hold. Choosing the line $[0, 0, 1]$ as l_∞, however, one can prove that the set $\mathfrak{D}_0 = \{(1, x, 0) : x \in \mathbb{F}_q\} \cup \{(0, 1, 0)\}$ is a valid derivation set, usually called the *real derivation set* of $AG(2, q^2)$. Moreover it is known that different derivation-sets yield isomorphic planes as a derivation set can be mapped to another by using an appropriate collineation. It means that a possible way of examining Hall planes is to fix a convenient derivation set (such as $\mathcal{D}_0$). However, as we will see it soon sometimes it makes more sense to take advantages of the planes symmetries on different ways.

The affine Baer-subplanes whose ideal points are those on $\mathfrak{D}_0$ are the sets $R(a, b, c) = \{(a \cdot u + b, a \cdot v + c) : u, v \in \mathbb{F}_q\}$ for all $a, b, c \in \mathbb{F}_{q^2}$, $a \neq 0$. Different choices of a, b, c do not necessarily imply different lines.

Lemma 1 *$R(a_1, b_1, c_1)$ and $R(a_2, b_2, c_2)$ are disjoint or coincident if and only if $\frac{a_2}{a_1} \in \mathbb{F}_q$ and are coincident iff $\frac{a_2}{a_1} \in \mathbb{F}_q$, $\frac{b_2 - b_1}{a_1} \in \mathbb{F}_q$ and $\frac{c_2 - c_1}{a_1} \in \mathbb{F}_q$.*

Proof If $\frac{a_1}{a_2} \in \mathbb{F}_q$ then $\{a_1 \cdot u : u \in \mathbb{F}_q\} = \{a_2 \cdot u : u \in \mathbb{F}_q\}$ and so $R(a_1, b_1, c_1) = R(a_2, b_1, c_1)$. Now if there exist $u_1, u_2, v_1, v_2 \in \mathbb{F}_q$ such that $(a_1 \cdot u_1 + b_1, a_1 \cdot v_1 + c_1) = (a_1 \cdot u_2 + b_2, a_1 \cdot v_2 + c_2)$ then $a_1 \cdot (u_1 - u_2) = b_2 - b_1$ and $a_1 \cdot (v_2 - v_1) = c_2 - c_1$. As $a_1 \neq 0$ it implies $\frac{b_2 - b_1}{a_1} \in \mathbb{F}_q$ and $\frac{c_2 - c_1}{a_1} \in \mathbb{F}_q$. Fixing the elements u_2, v_2 we get in addition that

$$\begin{cases} u_1 = u_2 + \frac{b_2 - b_1}{a_1}, \\ v_1 = v_2 + \frac{c_2 - c_1}{a_1}, \end{cases} \tag{3}$$

which are valid solutions for u_1, v_1. It means if $\frac{a_1}{a_2} \in \mathbb{F}_{||}$ and $R(a_1, b_1, c_1) \cap R(a_2, b_2, c_2) \neq \emptyset$ then $R(a_1, b_1, c_1) = R(a_2, b_2, c_2)$.

Conversely, $R(a_1, b_1, c_1) = R(a_2, b_2, c_2)$ yields that for every there $u_2, v_2 \in \mathbb{F}_q$ there exist $u_1, v_1 \in \mathbb{F}_q$ such that $(a_1 \cdot u_1 + b_1, a_1 \cdot v_1 + c_1) = (a_1 \cdot u_2 + b_2, a_1 \cdot v_2 + c_2)$. Solving the equations we get

$$\begin{cases} u_1 = \frac{a_2}{a_1} \cdot u_2 + \frac{b_2 - b_1}{a_1}, \\ v_1 = \frac{a_2}{a_1} \cdot v_2 + \frac{c_2 - c_1}{a_1}. \end{cases} \tag{4}$$

Choosing $u_2 = v_2 = 0$ yields $\frac{b_2-b_1}{a_1}, \frac{c_2-c_1}{a_1} \in \mathbb{F}_q$ and so $\frac{a_2}{a_1} \cdot u_2, \frac{a_2}{a_1} \cdot v_2 \in \mathbb{F}_q$ and therefore $\frac{a_2}{a_1} \in \mathbb{F}_q$.

The second part of the lemma follows easily from our proof.□

Statement 4 *$Hall(q^2)$ is non-desarguesian plane for all q.*

Proof With the appropriate choice of the coordinates the lines defined by the given points remain lines in $Hall(q^2)$ except the one defined by X and Y which has been converted to the real Baer subplane, no more containing Z. It certainly yields two point-perspective but not line-perspective triangles and so harms Theorem 1.

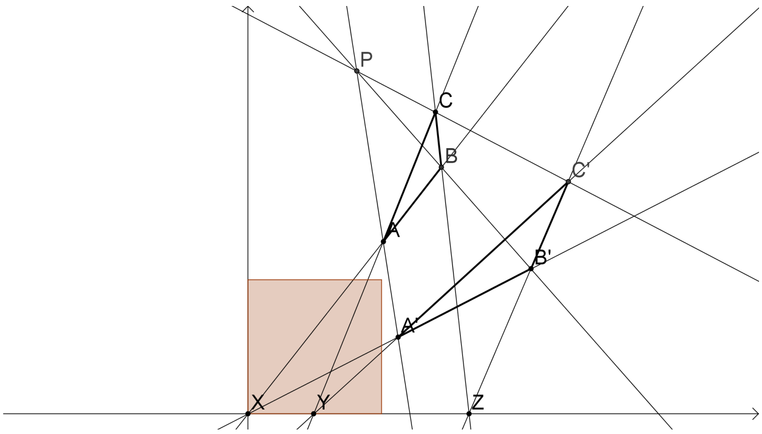

Figure 6: Triangles violating Theorem 1 in the Hall plane

Note that although it is a nice, geometric way to define Hall planes through derivation they were originally introduced algebraically via Hall quasifields.

Definition 8 *Let $f(x) = x^2 - a \cdot x - b$ be an irreducible quadratic polynomial over $\mathbb{F}_q$. The elements of the Hall quasifield $H(s, f)$ are the elements $a + i \cdot b$ ($a, b \in \mathbb{F}_q$). Addition is defined naturally, and the multiplication $\circ$ by the rule*

$$(a+i\cdot b)\circ(a'+i\cdot b') = \begin{cases} a\cdot a' + i\cdot a\cdot b & \text{if } b = 0, \\ a\cdot a' + b^{-1}b'f(a) + i\cdot\big(b\cdot a' + (1+a)\cdot b'\big) & \text{if } b \neq 0. \end{cases} \quad (5)$$

As we mentioned previously one can build an affine plane using "week" algebraic structures such as $H(s, f)$. It can be shown that with proper choice of s and f the affine plane is isomorphic to $\mathcal{D}AG(2, q)$.

Although Hall planes are not desarguesian it may happen that we have done nothing but gave a reinterpretation of the previously discussed planes built on nearfields. Fortunately it is not the case as Σ_{q^2} and $Hall(q^2)$ are non-isomorphic planes for all q. A proof of our statement through a more general approach using quasifields is presented in [9] which we neglect in this thesis.

4 Configurations in Desarguesian and Nondesarguesian Planes

The main goal of our previous section was to introduce a handful of classical constructions yielding non-desarguesian planes. In this section we analyze combinatorial properties of the mentioned planes. First and foremost we recall the basic definitions and result concerning arcs, ovals and blocking sets.

4.1 Arcs, Ovals, Blocking Sets

Definition 9 *A k-arc in a finite projective or affine plane is a set of k points no three of which are collinear. A k-arc is complete if it is not contained in a $(k+1)$-arc. A line L is secant, tangent or passant to an arc if they have 2,1 or 0 points in common, respectively.*

As the union of the lines defined by the points of an arc contains each point of the plane, it implies a k-arc cannot be complete if $q \geq \frac{k \cdot (k-1)}{2}$. On the other hand, a well-known theorem due to Bose says that a k-arc in a plane of order q contains at most $q+1$ points if q is odd and $q+2$ if q is even. $(q+1)$-arcs and $(q+2)$-arcs are called *ovals* and *hyperovals*, respectively. An oval in Π_q of even order can be uniquely completed to a hyperoval by adding a further point (the intersection of the tangent lines) called *"nucleus"* of the oval.

The existence of ovals in a general plane is a famous unsolved problem. As an early result, the ovals in $PG(2,q)$ are fully described. Later on we will introduce some classical constructions. Before doing so we recall some basic definitions.

Definition 10 *Let K be a set of k points of an arbitrary plane Π_q. A point $P \in K$ is an internal nucleus (shortly: i-nucleus) of K if every line through P meets K in at most two points (including P). The set of i-nuclei of K is denoted by $IN(K)$.*

Definition 11 *Let $H \subset \Pi_q$ and $P \in \Pi \backslash H$. P is called external point if there exists a tangent line through P, that is, a line intersecting H in exactly one point. If no such a line exists then P is called interior point.*

Let $\mathcal{O}$ be an oval on Π_q (q odd) and let $T \in \mathcal{O}$ an arbitrary point. Since $|\mathcal{O} \backslash \{T\}| = q$ and there are exactly $q+1$ lines passing through T it yields $\mathcal{O}$ has a unique tangent line through each of its points.

Now assume t is a tangent line ($T \in t$) and let $P \in t \backslash \{P\}$. Each line l through P meets $\mathcal{O}$ in at most two points, that is, $|l \cap \mathcal{O}| \in \{0,1,2\}$. As q is odd and t is tangent there must exist another tangent line t' through P.

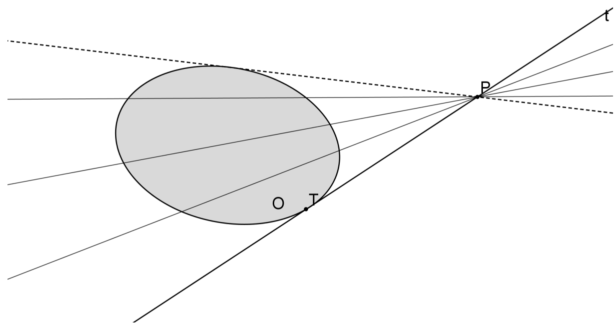

Figure 7: Tangents of an oval

On the other hand $\mathcal{O}$ possesses q tangent lines different from t each of which meets t in an external point. As each of the P's in t is contained by at least one of those lines it yields equation holds and there exist exactly two tangent lines through any P.

We have just proven if P is contained in a tangent line then P is an intersection of exactly two tangents and so we proved the following lemma.

Lemma 2 *Let q odd and $\mathcal{O}$ be an oval on Π_q. There exist either 0 or 2 tangent lines (with respect of $\mathcal{O}$) through any point $P \notin \mathcal{O}$.*

Having been proven the previous lemma elementary calculation yields the number of external and interior points.

Lemma 3 *The plane Π_q with the oval $\mathcal{O}$ has $\frac{q\cdot(q+1)}{2}$ external and $\frac{q\cdot(q-1)}{2}$ interior points. A passant or secant line l has $\frac{q+1}{2}$ external points.*

Proof The set of external point is covered by the $q+1$ tangent lines. Each of them is uniquely defined as the intersection of two of the tangents. It means the number of external points is $\binom{q+1}{2}$ and so the number of internal points is $q^2 - \binom{q+1}{2} = \frac{q\cdot(q-1)}{2}$.

If the line l has k external points and meets the $\mathcal{C}$ in m points then the — counting the number of tangents to $\mathcal{C}$ — we get $q+1 = 2\cdot k + m$. As $m = 0$ (passant) or $m = 2$ (secant) the proof has been completed. $\square$

Another intriguing topic in the study of finite affine and projective planes is the examination of point-sets which meet every line in the plane.

Definition 12 *The set of points $\mathfrak{B}$ in a projective plane is called blocking set, if does not contain a line and meets every line at least one point.*

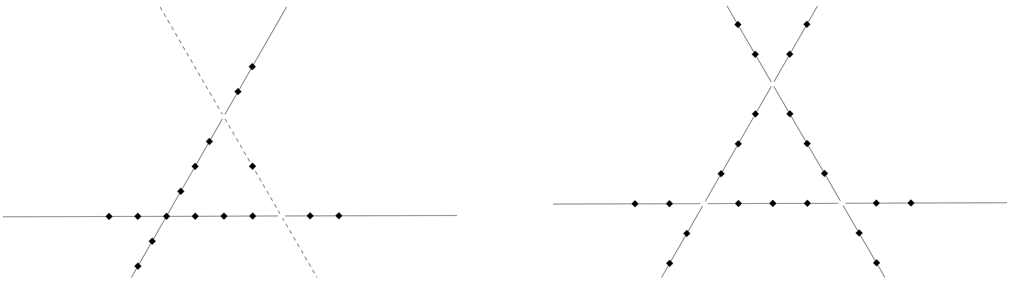

Figure 8: Minimal blocking sets of sizes $2 \cdot q - 1$ and $3 \cdot q - 3$

It is relatively easy to construct blocking sets of cardinality more than $2 \cdot q$ in an arbitrary projective plane Π_q. Two classical examples of sizes $2 \cdot q - 1$ and $3 \cdot q$ are shown below:

Both constructions have the property that taking away any element of the blocking set yields a line disjoint from the remaining elements of $\mathfrak{B}$. A blocking set with this property is called minimal. Concerning blocking sets we mainly focus on the cardinality of minimal sets. However, the question of the existence of large minimal blocking sets is also an intriguing topic offering many unsolved problems.

Since an arbitrary line intersects every other line of the plane in exactly one point, the first condition of blocking set's definition -nonexistence of lines as subset of $\mathfrak{B}$- is not negligible. A set of points meeting every line and containing at least one line is generally called *trivial blocking set* or *intersection-set*. Easy combinatorial calculation shows that an intersection-set has at least $q+1$ points in Π_q. Furthermore, equation holds if and only if the $q+1$ points form a line (see [9] for details).

According to the previously mentioned result one can easily prove that a blocking set has also cardinality at least $q+1$. The famous theorem due to Bruen and Pelikán shows that this trivial lower bound is far to be tight.

Definition 13 *A subplane of* Π_q *of order* $\sqrt{q}$ *is called Baer-subplane.*

Example 3 $PG(2,q)$ *is evidently a Baer-subplane in* $PG(2,q^2)$*.*

Theorem 8 (Bruen-Pelikán) *A Blocking set* B *in a plane of order* q *has size at least* $q + \sqrt{q} + 1$*, and equality holds iff* B *is a Baer-subplane.*

Baer-subplanes play also an important role in the topic of subplanes of an arbitrary plane:

Theorem 9 (Bruck) *If* Π_q *has a subplane* Π_s *of order* s *then either* $q = s^2$ *or* $q \geq s^2 + s$*. In the former case* Π_s *is a Baer-subplane.*

It has been a longstanding problem whether every projective plane of order q^2 has a Baer-subplane. Generally the structure of subplanes in projective planes or in other words the embedding of projective planes is a very intriguing and diversified field which offers challenging problems and astonishing results. As an example we mention that while the embedding of Fano plane is straightforward to $PG(2, 2^h)$ for each $h \in \mathbb{Z}^+$ it can be also embedded to each of the previously mentioned Hughes-planes. In addition Caliscan and Moorhouse recently showed (in [1]) an infinite family of Hughes-planes contanining subplanes of order 3. However, embedding of projective planes of higher order is still a widely open problem.

One can define blocking and intersection-sets of affine planes on a similar way. Unlike the projective case blocking sets and intersection sets of affine planes are not distinguished as they do not show significant differences.

Certainly, an affine intersection set can be completed to an intersection set in the projective sense by adding an arbitrary point of the ideal line. Not surprisingly this process does not work conversely as a projective intersection set can have much smaller cardinality in a given plane than an affine set has. In the next Section we briefly discuss the famous theorem due to Jamison and Brouver-Schrijver concerning the minimal cardinality of intersection sets in $AG(2, q)$, which reveals essential differences between intersection sets cardinalities in affine and projective sense.

4.2 Configurations in $PG(2, q)$

4.2.1 Arcs and Ovals

A landmark break-trough in the examination of the ovals in $PG(2, q)$ was Segre's theorem.

Theorem 10 (Segre) *The ovals of $PG(2, q)$ (q odd) are exactly the conics over $\mathbb{F}_q$.*

As the conics in $PG(2, q)$ are "known" up to projective equivalence Segre's theorem characterizes the ovals of $PG(2, q)$ in that sense.

Theorem 11 *Let $\mathcal{C}$ be a conic in $PG(2, q)$ and $a \in \mathbb{F}_q$.*

1. *If l_∞ is secant to $\mathcal{C}$ then $\mathcal{C}$ is projective equivalent to the graph of the (affine) function $y = \frac{a}{x}$ with its ideal points.*

2. *If l_∞ is tangent to $\mathcal{C}$ then $\mathcal{C}$ is projective equivalent to the graph of the (affine) function $y = a \cdot x^2$ with its ideal points.*

3. *If l_∞ is passant to $\mathcal{C}$ then $\mathcal{C}$ is projective equivalent to the graph of the (affine) function $y^2 - a \cdot x^2 = 1$.*

Note that in case of planes of even order the characterization of ovals in $PG(2,q)$ is far from being completed.

Even having a general overview of the ovals of $PG(2,q)$ for odd q complete arcs still remain an intriguing topic. We briefly summarize a few remarkable results concerning the possible size of a complete arc in $PG(2,q)$.

Theorem 12 *If K is a k-arc in $PG(2,q)$ and*

1. *$k \geq \frac{q+4}{2}$, if q is even,*
2. *$k \geq \frac{2q+5}{3}$, if q is odd,*

then K is contained in a unique complete arc.

Theorem 13 (Segre) *If K is a k-arc in $PG(2,q)$ and*

1. *$k > q - \sqrt{q} + 1$, if q is even,*
2. *$k > q - \frac{\sqrt{q}}{4} + \frac{7}{4}$, if q is odd,*

then K can be completed to an oval.

The proof of these theorems is based on the examination of algebraic curves associated to arcs and on Weil's theorem. In some particular cases sharper bounds have been proven by Voloch.

One of our main goals in the following Section is to show that the majority of the listed theorems cannot be extended to an arbitrary projective plane.

4.2.2 Blocking Sets

There are numerous methods to construct blocking sets which we will not discuss in details. Instead of doing so we briefly summarize the most known theorems concerning sizes of (affine or projective) blocking sets.

Theorem 14 (Blokhuis) *A blocking set B in $PG(2,p)$ contains at least $\frac{3(p+1)}{2}$ points.*

As the existence of a projective plane of order p which is not isomorphic to $PG(2,p)$ is a well known open problem, we are not able to analyze the result of the theorem in case of non-desarguesian planes. However, the theorem has a more general form due to the same author:

Theorem 15 (Blokhuis) *Let $q = p^n$, $n \geq 2$. A blocking set B in $PG(2,q)$ has at least*

1. *$q + \sqrt{q} + 1$ points, if n is even,*
2. *$q + \sqrt{p \cdot q} + 1$ points if n is odd.*

Note that in the case of even n our lower bound is the same as in Theorem 8.

We mention another famous theorem concerning blocking sets of affine planes built on fields:

Theorem 16 (Jamison,Brouwer-Schrijver) *A blocking set of B in $AG(2,q)$ contains at least $2 \cdot q - 1$ points.*

For the affine case a short and elegant proof has been given by the authors. The techniques involved in the proof can be also considered as an early stage of the later invented theorem usually known as Combinatorial-Nullstellensatz.

4.3 Configurations in Hall Planes

4.3.1 Ovals in $Hall(q^2)$

Throughout this section we consider *inherited arcs.* An arc and especially an oval of $AG(2,q^2)$ or $PG(2,q^2)$ is called inherited if it is also an arcs (oval) in $Hall(q^2)$. In other words those arcs are called inherited which remain ovals after derivation.

Inherited arcs are counted as "valuable treasures" as an arc of $AG(2,q^2)$ usually does not remain an arc in $Hall(q^2)$. In order to illustrate it we discuss two early theorems due to Szőnyi. In [13] he has proved that the conic $x \cdot y = 1$ does not remain an arc in $Hall(q^2)$, moreover it has $q^2 - q + 1$ internal nuclei and cannot be completed to an oval. Besides giving an insight to the structure of arcs in Hall planes Szőnyi's construction disproves Theorem 12 for Hall planes.

Theorem 17 (Szőnyi) *The set of points $\mathcal{C}$ defined by the function $f(x) = x^2$ together with (∞) has $q^2 - q + 1$ internal nuclei on $Hall(q^2)$ for odd q.*

Proof As $\mathcal{C}$ is an arc in $AG(2,q)$ three points cannot be contained by an old line of $\mathcal{D}AG(2,q^2)$. If the points $(x_1, x_1^2), (x_2, x_2^2), (x_3, x_3^2)$ are on a new line, then the slopes of the lines $\frac{x_i^2 - x_j^2}{x_i - x_j} = x_i + x_j$ are in $\mathbb{F}$. Since q is odd it yields $x_1, x_2, x_3 \in \mathbb{F}$.

It means that the set $\{(x, x^2) : x \in \mathbb{K}\backslash\mathbb{F}\}$ is an arc in $Hall(q^2)$. Certainly it can be extended with any pair of points of form $(x, x^2)(x \in \mathbb{F})$. Easy calculation shows that by adding a pair the arc cannot be extended with any further point and so the given points form a complete arc of size $q^2 - q + 2$. $\square$

Another construction due to the same author shows even weirder phenomena, namely the existence of complete arcs which are "almost " ovals.

Theorem 18 (Szőnyi) *For an arbitrary, fixed non-square $c \in \mathbb{K}$ the hyperbola $H = \{(x, \frac{-c}{x} | 0 \neq \in \mathbb{K})\}$ is a complete (q^2-1)-arc in $Hall(q^2)$.*

The proof of Theorem 18 can be achieved by using similar ideas and observations as previously discussed (see [13]).

Unlike the classical planes built on $\mathbb{F}_{q^2}$ the Hall plane does contain arcs of size greater than $q^2 - q$ which cannot be uniquely completed as we have seen in Theorem 17 and it also contains complete arcs of size greater than $q^2 - q + 1$. It means that neither Theorem 12 nor Theorem 13 are valid for Hall planes.

Although in the examination of arcs and ovals in the Hall planes inherited arcs have top priority it is known that there exist ovals which are not descendant of ovals lying in $AG(2,q)$. A famous theorem due to Menichetti tells us that each Hall plane of odd order does have such an oval.

Theorem 19 (Menichetti) *In every Hall plane of even square order q^2 there do exist non-inherited complete q^2-arcs.*

Similar result for odd order are not known. On the contrary, it has been drawn as a conjecture by many authors that in some particular cases (with restrictions for the value of q) the ovals of $Hall(q^2)$ are all inherited ovals. The conjecture has neither proved nor disproved in any part. From now on we turn our full attention to the inherited arcs.

The characterization of inherited arcs in $Hall(q^2)$ has begun in the early eighties and completed recently for event value of q by Cherowitzo ([2]) who finished the work of O'Keefe, Pascasio [12] as well as Korchmáros[10] and many further authors. A brief summary of the characterization with reference of the proofs is listed below.

Let $\mathfrak{D}$ be an arbitrary derivation set, $\mathcal{C}$ is the conic, l_∞ as normally. Throughout the characterization we distinguish three main cases according to l_∞ being a

I. secant line with points P and Q of the conic (*hyperbolic case*),

II. tangent line with point P of the conic (*parabolic case*),

III. an exterior line (*elliptic case*).

In addition we need to distinguish additional cases depending on the parity of q.

I.Hyperbolic Case

Theorem 20 *[12] Suppose $P, Q \in \mathfrak{D}$.*

If $q = 3$, then one of the following occurs:

i) The configuration of $\mathfrak{D}$ and $\mathfrak{C}$ is projective equivalent in $AG(2,9)$ to $\mathfrak{D}_\circ$ and the conic with equation $x \cdot y = 1$. $\mathfrak{C}$ is an incomplete $8-arc$ in $Hall(9)$ and can be completed to an oval.

ii) The configuration of $\mathfrak{D}$ and $\mathfrak{C}$ is projective equivalent in $AG(2,9)$ to $\mathfrak{D}_\circ$ and the conic with equation $x \cdot y = -d$, where d is a fixed non-square in $\mathbb{F}_9$. In this case $\mathfrak{C}$ is a complete $8-arc$ in $Hall(9)$.

If $q > 3$ odd, then one of the following occurs:

iii) The configuration of $\mathfrak{D}$ and $\mathfrak{C}$ is projective equivalent in $AG(2,q^2)$ to $\mathfrak{D}_\circ$ and the conic with equation $x \cdot y = 1$. In this case $\mathfrak{C}$ is not an arc in $Hall(q^2)$.

iv) The configuration of $\mathfrak{D}$ and $\mathfrak{C}$ is projective equivalent in $AG(2,q^2)$ to $\mathfrak{D}_\circ$ and the conic with equation $x \cdot y = -d$ where d is a fixed non-square in $\mathbb{F}_{q^2}$. In this case $\mathfrak{C}$ is a complete (q^2-1)-arc in $Hall(q^2)$.

If $q > 2$ even, the configuration of $\mathfrak{D}$ and $\mathfrak{C}$ is projective equivalent in $AG(2, q^2)$ to $\mathfrak{D}_\circ$ and the conic with equation $x \cdot y = 1$. In this case $\mathfrak{C}$ is not an arc in $Hall(q^2)$.

Hyperbolic case, q even

Theorem 21 *[2] If $q > 2$ is even and at least one of P and Q is not contained by $\mathfrak{D}$ then $\mathfrak{C}$ is not a hyperoval in $Hall(q^2)$.*

II. Parabolic case

Theorem 22 *[12] If q is odd, then $\mathfrak{C}$ is not an arc in $Hall(q^2)$.*

From now on we can assume $q > 2$ is even. In this case the nucleus N of $\mathfrak{C}$ lies on l_∞.

Theorem 23 *[12] We distinguish three cases:*

1. *If $P, N \in \mathfrak{D}$ then $\mathfrak{C}$ is not an arc in $\mathfrak{D}AG(2, q^2)$.*

2. *If $P \in \mathfrak{D}$ and $N \notin \mathfrak{D}$ then the configuration of $\mathfrak{D}$ and $\mathfrak{C}$ is projective equivalent in $AG(2, q^2)$ to $\mathfrak{D}_\circ$ and the conic with equation $y = x^2 + s \cdot x$ where $\mathbb{F}_{q^2} \backslash \mathbb{F}_q$. In this case $\mathfrak{C}$ is an incomplete q^2-arc in $\mathfrak{D}AG(2, q^2)$ and can be completed to a hyperoval in $Hall(q^2)$.*

3. *If $P \notin \mathfrak{D}$ and $N \in \mathfrak{D}$ then the configuration of $\mathfrak{D}$ and $\mathfrak{C}$ is projective equivalent in $AG(2, q^2)$ to $\mathfrak{D}_\circ$ and the conic with equation $y = x^2 + s \cdot y^2 + x$ where $\mathbb{F}_{q^2} \backslash \mathbb{F}_q$. In this case $\mathfrak{C}$ is an incomplete q^2-arc in $\mathfrak{D}AG(2, q^2)$ and can be completed to a hyperoval in $Hall(q^2)$.*

Theorem 24 *[7] Any non-degenerate conic of $PG(2, q^2)$, $q = 2^h$, with nucleus $N \in l_\infty \backslash \mathfrak{D}$ and passing through a point $P \in l_\infty \backslash \mathfrak{D}$, is an oval in $Hall(q^2)$ iff N and P are conjugate points with respect to $\mathfrak{D}$.*

III. Elliptic case, q even

Theorem 25 *If $q > 2$ is even and l_∞ does not intersect $\mathfrak{C}$ then $\mathfrak{C}$ is not a hyperoval in $Hall(q^2)$.*

Surveying the previously listed result we can conclude the following theorem:

Theorem 26 *As a main we conclude if $\mathcal{C}$ is an inherited hyperoval in $Hall(q^2)$ (q even) then it intersects the ideal line l_∞ in one point.*

As we have seen for odd q the problem is far from being fully answered. In the following pages we are going to analyze the elliptic case and suggest a program which offers a way to cope with the problem. We wish to state that in general cases our path may lead us to overwhelming calculations.

Theorem 20 repetitively states that our suspected arc is projective equivalent to a particular conic. It can be assumed without loss of generality by transforming our point-set to a convenient conic using a suitable collineation (as it had been done in the above mentioned papers). Recall that the collineation-group of $PG(2,q)$ is fully described as being a semidirect product: $Coll(PG(2,q)) = PGL(2,q) \rtimes Aut(\mathbb{F}_q)$. As we have mentioned earlier we know that this group acts transitively on the sets of possible derivation sets. It means that by analyzing a transformed conic we can assume that our current derivation set is given as the image of $\mathcal{D}_0$ under a collineation. Our goal is to gain control of the image and to investigate whether $\mathcal{K}$ (the set of external points in l_∞) and $\mathcal{D}$ can be disjoint sets. In order to explain our purpose we introduce some useful theorems due to Korchmáros.

Theorem 27 (Korchmáros) *Let $\mathcal{C}$ be a conic and r be a line of $PG(2,q)$ (q odd). For every triple $\{P_1, P_2, P_3\} \subset r \backslash \mathcal{C}$ there exists at most two triangles $\{A_1, A_2, A_3\}$ inscribed in $\mathcal{C} \backslash r$ such that $\overline{A_i A_J} \cap r \in \{P_1, P_2, P_3\}$ ($i \neq j, i, j = 1, 2, 3$).*

Theorem 28 (Korchmáros) *Let $\mathcal{C}$ be a conic in $PG(2,q)$ (q odd) and ABC a triangle inscribed in $\mathcal{C}$. Let r is a line containing neither of the vertices A, B, C and let*

$$\overline{AB} \cap r = C^*,$$
$$\overline{BC} \cap r = A^*,$$
$$\overline{CA} \cap r = B^*.$$

Then the set $\{A^*, B^*, C^*\}$ contains either one or three external points with respect of $\mathcal{C}$.

With the choice $r = l_\infty$ we get that each inscribed triangle defines a triples in l_∞, each of which has either one or three external points. As we have $\binom{q+1}{2}$ different triangles and each triple is defined by at most two triangles the number of induced triples is at least $\frac{\binom{q+1}{3}}{2}$. On the other hand elementary combinatorial calculation yields

$$\left|\Big\{\{P_1, P_2, P_3\} : P_1, P_2, P_3 \in l_\infty, \{P_1, P_2, P_3\} \cap \mathcal{K} = 1\Big\}\right| = \binom{\frac{q+1}{2}}{2} \cdot \binom{\frac{q+1}{2}}{1} \qquad (6)$$

$$\left|\Big\{\{P_1,P_2,P_3\} : P_1,P_2,P_3 \in l_\infty, \{P_1,P_2,P_3\} \cap \mathcal{K} = 3\Big\}\right| = \binom{\frac{q+1}{2}}{3} \tag{7}$$

One can easily see $\frac{\binom{q+1}{3}}{2} = \binom{\frac{q+1}{2}}{2} \cdot \binom{\frac{q+1}{2}}{1} + \binom{\frac{q+1}{2}}{3}$ which yields each triples having one or three external points is defined by exactly two triangles.

Now we are ready to prove our main theorem.

Theorem 29 *An oval remains oval in $Hall(q^2)$ iff the derivation-set consists of no external point.*

Proof If $\mathcal{C}$ has collinear triples A, B, C in $Hall(q^2)$ then they are joined by a Baer-subplane which has ideal points on l_∞. According to Theorem 28 at least one of the ideal points A^*, B^*, C^* defined by the inscribed triangle ABC is external point which also contained by $\mathfrak{D}$ and so $\mathcal{K} \cap \mathfrak{D} \neq \emptyset$.

On the contrary, if $\mathfrak{D}$ contains external point then we can complete it to a set $\{P_1, P_2, P_3\} \subset l_\infty$ which contains odd number of external points. Let A_1, A_2, A_3 a triangle associated to our triple. One can easily see that the three Baer subplanes corresponding to the pairs $\{A_1, A_2\}$, $\{A_2, A_3\}$, $\{A_3, A_1\}$ are identical (as each of them contains P_1,P_2 and P_3 as well) and so A_1, A_2, A_3 are collinear points in $Hall(q^2)$. It completes the proof of the theorem. □

We are going to illustrate the parametrization of the conic and the derivation set in the elliptic case. Our goal is to get a tool which may help to decide whether the derivation set can be disjoint from the set of external points in l_∞. Let us have an arbitrary oval of $PG(2, q)$ which does not intersect l_∞. With a proper choice of a collineation our oval can be transformed to the ellipse $\mathcal{C}$ $x^2 - k \cdot y^2 = 1$ where k is a fixed non-square in $\mathbb{K}$ (our derivation set $\mathcal{D}_0$ has been synchronously transformed to another derivation set). Certainly the point $P = (1; 0)$ is contained by C. Easy calculation shows that the tangent line to $\mathcal{C}$ through P is the vertical line $x = 1$. Let $t \in \mathbb{K}$ arbitrary element. The line of slope t through P is described by the equation $y = t \cdot x - t$ and it meets $\mathcal{C}$ in the point $Q_t\big(\big(\frac{kt^2+1}{kt^2-1}; \frac{2t}{kt^2-1}\big)\big)$. The tangent line at the point $Q(x_0; y_0) \in \mathcal{C}$, $Q \neq P$ is

$$y = \frac{x_0}{k \cdot y_0} \cdot x - \frac{1}{k \cdot y_0} \tag{8}$$

and so the tangent at Q_t is

$$y = \frac{kt^2 + 1}{2kt} \cdot x - \frac{kt^2 - 1}{2kt} \tag{9}$$

which meets l_∞ in the (ideal) point $\big(1, \frac{kt^2+1}{2kt}, 0\big)$. It yields that the set $\mathcal{K}$ of outer points of $\mathcal{C}$ is

$$\mathcal{K} = \{(1, \frac{kt^2 + 1}{2kt}, 0) : t \in \mathbb{F}_{q^2}\} \cup \{(0, 1, 0)\}$$

and so the set of outer points of $\mathcal{C}$ is parametrized with the parameter t.

Turning to the image of the derivation set we recall $\mathcal{D}_0 = \{(1, m, 0) : m \in \mathbb{F}_q\} \cup \{(0, 1, 0)\}$. Since $F_q \subset F_{q^2}$ is invariant under automorphisms of F_q^2 hence so is $\mathcal{D}_0$ as well and therefore it is sufficient to examine its image under linear transformations.

Let $A \in \mathbb{F}_{q^2}^{3\times 3}$ be invertible matrix.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Since the line l_∞ is invariant under A (as it transforms derivation set into derivation set) hence $a_{31} = a_{32} = 0$ can be easily concluded. As A is invertible it immediately yields $a_{11} \cdot a_{22} \neq a_{12} \cdot a_{21}$.

If A fixes the point $(0, 1, 0)$ then $(0, 1, 0) \in \mathcal{C} \cap \mathcal{K}$ and so the conic does not remain an oval. From now on we can assume $A(0, 1, 0)^T \neq (0, 1, 0)$, that is, $a_{21} \neq 0$. As we are focusing on images of ideal points the values of a_{13} and a_{23} are out of interest and our matrix can be written in the following form:

$$A = \begin{pmatrix} d & c & * \\ b & a & * \\ 0 & 0 & e \end{pmatrix}.$$

By the same reason as above we can assume that the equation $A\underline{x} = (0, 1, 0)$ has no solution in $\mathcal{D}_0$, which holds if and only if $\frac{d}{c} \notin \mathbb{F}_q$.

In this case:

$$\begin{cases} A \cdot (0, 1, 0)^T = (c, a, 0) \sim (1, \frac{a}{c}, 0), \\ A \cdot (1, m, 0)^T = (cm + d, am + b, 0) \sim (1, \frac{am+b}{cm+d}, 0). \end{cases} \tag{10}$$

By denoting $A = \frac{bc-ad}{c^2}$, $B = \frac{d}{c}$, $C = \frac{a}{c}$ we get

$$\begin{cases} A \cdot (0, 1, 0) \sim (1, C, 0), \\ A \cdot (1, m, 0) \sim (1, \frac{A}{m+B} + C, 0). \end{cases} \tag{11}$$

where $A, B \neq 0$, moreover $B \notin \mathbb{F}_q$. In addition, as $B \notin \mathbb{F}$ and $\mathbb{K}$ can be considered as a quadratic extension of $\mathbb{F}$: $\mathbb{K} = \mathbb{F}(k)$ ($k^2 = d \in \mathbb{F}$, a non-square element) we can assume without loss of generality that $A \cdot (1, m, 0) = (1, \frac{A}{m+k} + C, 0)$.

Our question is whether any of the following three equations has a solution in t and m ($t \in \mathbb{K}$, $m \in \mathbb{F}$) for fixed A, B, C, k parameters.

$$\frac{kt^2+1}{2kt} = \frac{A}{m+k} + C, \tag{12}$$

$$\frac{kt^2+1}{2kt} = C, \tag{13}$$

$$1 = \frac{A}{m+B} + C. \tag{14}$$

Certainly (14) is easily decidable as it only depends on the given parameters. (13) can be written in the following equivalent form: $(t-C)^2 = (-1)\cdot(C^2+\frac{1}{k})$. This equation has a solution iff $C^2+\frac{1}{k}$ is a square in $\mathbb{K}$ (in particular the equation has no solution for $C=0$).

Concerning (12) a possible way is to decompose the variables in $\mathbb{K}$ (using the base $\{1,k\}$) and look for solution in the components over $\mathbb{F}$. We therefore reformulate (12) introducing and using the following notation ($t_0, t_1, a_0, a_1, c_0, c_1 \in \mathbb{F}$)

$$t = t_0 + t_1\cdot k,$$
$$A = a_0 + a_1\cdot k,$$
$$C = c_0 + c_1\cdot k,$$

$$\big(k(t_0+t_1k)^2+1\big)\big(m+k\big) = 2k(t_0+t_1k)(a_0+a_1k)+2k(t_0+t_1k)(m+k)(c_0+c_1k) \tag{15}$$

And so we get** the following formulas for the value of m:

$$m = \frac{-dt_0^2 - dt_1^2 + 2da_0t_1 + 2da_1t_0 + 2d^2c_1t_1 + 2dc_0t_0}{2dt_0t_1 + 1 - 2dc_0t_1 - 2dc_1t_0} \tag{16}$$

$$m = \frac{-2dt_0t_1 - 1 + 2t_0a_0 + 2dt_1a_1 + 2dc_1t_0 + 2dc_0t_1}{t_0^2 + t_1^2 - 2c_0t_0 - 2dc_1t_1} \tag{17}$$

As we have seen the existence of an elliptic oval in $Hall(q)^2$ is equivalent to the solvability of equation of degree four defined by (16) and (17) in variables t_0, t_1 with parameters a_0, a_1, c_0, c_1, d over $\mathbb{F}$. Although the question of solvability may not seem much more tractable at first look we presume one can prove the following conjecture:

Conjecture 1 *The equation given above has solution for every choice of the parameters. As a consequence $Hall(q^2)$ (q odd) does not possess inherited elliptic ovals.*

** $\Big(dt_0^2 + dt_1^2 + 2dmt_0t_1 + m - 2da_0t_1 - 2da_1t_0 - 2dmc_0t_1 - 2d^2c_1t_1 - 2dmc_1t_0 - 2dc_0t_0\Big) +$
$+\Big(mt_0^2 + mt_1^2 + 2dt_0t_1 + 1 - 2t_0a_0 - 2dt_1a_1 - 2mc_0t_0 - 2dc_1t_0 - 2dmc_1t_1 - 2dc_0t_1\Big)\cdot k = 0$

4.3.2 Blocking Sets in $\mathfrak{D}AG(2, q^2)$

In this section we are going to investigate the possibility of a blocking set in the affine plane $\mathfrak{D}AG(2, q^2)$ with less than $2 \cdot q^2 - 1$ points in order to show that Theorem 16 cannot be extended to (affine) Hall planes.

Unlike arcs and ovals, blocking sets of $\mathfrak{D}AG(2, q^2)$ and $Hall(q^2)$ has been neglected in the last decades and only a handful of results are known. As an example, Bruen and de Resmini showed a construction for the case when $q = 3$ in 1983. As a matter of fact they proved that every non-desarguesian affine plane of order 9 contains a blocking set with 16 elements. The construction is mainly based on the observation that each of the mentioned affine planes contains a (projective) Fano-subplane. It can be shown by making further efforts that the given subplane can be extended to a 16-element-blocking-set.

This result shows that Theorem 16 does not hold for Hall planes of arbitrary order. Although the embedding of Fano-planes had been proven a useful tool in case of planes of order 9 it might be insufficient to prove our conjecture (that Theorem 16 can be disproved in general case). In the next pages we are going to introduce another idea as a possible way to prove the conjecture.

It is straightforward that a blocking set with less than $2 \cdot q - 1$ points in an affine plane of order q neither contains a complete line nor even $q - 1$ collinear points. In case of Hall planes, however, it makes sense to examine sets containing a line defined by the equation $y = m \cdot x + b$ where $m \in \mathbb{F}$. In the construction of Hall planes these line are replaced by Baer subplanes and therefore using them does not confront directly with the previous result.

We are going to prove that the first part of our argumentation holds even in this case.

Lemma 4 *A blocking set B of $\mathfrak{D}AG(2, q^2)$ containing a complete replaced line has at least $2 \cdot q - 1$ points.*

Proof $\mathfrak{D}AG(2, q^2)$ contains exactly $q^3 + q^2$ Baer-subplanes as new lines and each point is contained in $q + 1$ new lines. We know that each old line intersects each new in exactly one point. It is also straight that the intersection of a replaced and a new lines has either zero or q elements.

Assuming B contains a complete line L the points of L intersect $\frac{\binom{q^2}{2}}{\binom{q}{2}} = q^2 + q$ new lines. That means that the remaining $q^3 - q$ are intersected by the points of $B \backslash L$. Since for each $b \in B \backslash L$ there is exactly one new line through b and intersecting L, hence b covers q of the $q^3 - q$ remaining new lines. It yields $|B \backslash L| \geq \frac{q^3 - q}{q} = q^2 - 1$ and so $|B| \geq 2 \cdot q^2 - 1$. $\square$

Our proof directly gives strong restrictions for the second part of the argumentation.

Lemma 5 *If the blocking set B in $\mathfrak{D}AG(2,q^2)$ contains q^2-1 points of a replaced line L, then there exists q^2-1 points in B which intersects every new line. This points are pairwise connected by old lines.*

Proof Easy to see that the q^2-1 points in L intersect as many new line as the whole L, that is, q^2+q. According to the previous proof, each point b of $B\backslash$ intersects q of the remaining q^3-q new lines. Assuming $|B| \leq 2\cdot q^2-2$ implies $|B\backslash L| =\leq q^2-1 = \frac{q^3-q}{q}$. Since equality holds hence B has exactly $2\cdot q^2-2$ points no two of which are connected by a new line. □

It is a natural question whether a point-set described in Lemma 5 exists. Starting with the small possible example we show a construction with 8[††] points in $\mathfrak{D}AG(9)$.

Construction 6 *Let $\mathbb{F}_9 = \mathbb{F}_3(\alpha)$, $\alpha^2+1=0$. In this case the lines defined by the points*

$$\begin{array}{ccc} /(0,0)/ & (1,\alpha) & (2,2\alpha) \\ (\alpha,2) & (\alpha+1,2\alpha+1) & (\alpha+2,\alpha+1) \\ (2\alpha,1) & (2\alpha+1,2\alpha+2) & (2\alpha+2,\alpha+2) \end{array}$$

are all "old lines". Moreover, the given point-set is contained by the union of lines $y=\alpha\cdot x$ and $y=2\cdot\alpha\cdot x$.

The given construction may encourage us to examine special point-sets covered by the union of a pair of lines in the general case. For small value of q the examination may be achievable by involving computerized calculations.

Conjecture 2 *$\mathfrak{D}AG(2,q^2)$ has a blocking set of size $2\cdot q^2-2$ for all q.*

4.4 Configurations in Planes Built on Nearfields

As we have seen arcs and ovals in Hall planes is a popular and highly investigated topic. However, the same question seems to be neglected in case of affine planes built on Dickson's nearfields. In this section we give an example for an inherited oval in $\sum_q$ using one of the previously introduced classical examples.

[††]As the matter of fact our construction can be extended with the point (0,0) without violating the given condition.

Lemma 6 *Suppose* $x_1, x_2 \in N$ *and* $\frac{-1}{x_1 \cdot x_2}$ *is a square. If the line connecting the points* $(x_1, \frac{1}{x_1})$, $(x_2, \frac{1}{x_2})$ *is described by the equation* $y = m * x + b$, *then* m *is square in* N *and the equation can be written as* $y = m \cdot x + b$.

Proof We prove that the line $y = \frac{-1}{x_1 \cdot x_2} * x + \frac{1}{x_1} + \frac{1}{x_2}$ contains both points. As $\frac{-1}{x_1 \cdot x_2}$ is a square our equation can be written as $y = \frac{-1}{x_1 \cdot x_2} \cdot x + \frac{1}{x_1} + \frac{1}{x_2}$. It is easy to see that both substitutions are valid. As the line connecting $(x_1, \frac{1}{x_1})$ and $(x_2, \frac{1}{x_2})$ is uniquely defined, the proof is complete. $\square$

Theorem 30 *The set* $\mathcal{C} = \{(x, \frac{1}{x}) : x \in\}$ *is an inherited arc in* Σ_{q^2}. *Moreover it can be completed to an oval in the projective closure of* Σ_{q^2} *by adding the ideal points* $(1, 0, 0)$ *and* $(0, 1, 0)$.

Proof Assume $(x_1, \frac{1}{x_1})$, $(x_2, \frac{1}{x_2})$, $(x_3, \frac{1}{x_3})$ are collinear points. The x_i-s are either squares or non-squares but two of them belong to the same class and therefore for suitable $\{i, j\} \subset \{1, 2, 3\}$ $x_i \cdot x_j$ is a square and so is $\frac{-1}{x_i \cdot x_j}$ (recall that $-1 \in \mathbb{F}$ is a square). According to Lemma 6 the line defined by (x_i, x_i^2) and (x_j, x_j^2) has the form $y = m \cdot x + b$ as $m =$ is a square. As the chosen points define the same line in $AG(2, q^2)$ it means x_k, x_k^2 $(\{1, 2, 3\} \backslash \{i, j\} = \{k\})$ is contained by this line if it is contained in $AG(2, q^2)$. As $\mathcal{C}$ is an oval in $PG(2, q^2)$. As the vertical and horizontal lines are identical in $PG(2, q^2)$ and Σ_{q^2} it yields that our arc can be completed to an oval as given above. $\square$

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