Uniform infinite triangulations
on the sphere and on the torus

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Abstract

In [1] Angel and Schramm defined a distribution over infinite planar triangulations which they called the uniform infinite planar triangulation (UIPT). They did this by considering the uniform distribution on triangulations of the sphere with a fixed, finite number of vertices and by taking a weak limit of these distributions as the number of vertices tends to infinity.

In this thesis we examine what happens when we repeat this process on the torus. To do this we have to consider a slightly different kind of triangulation as in [1] in that we allow loops and multiple edges. We show that if the limit exists both on the sphere and on the torus then in both cases it is the exact same distribution on infinite triangulations of the plane.

The key to showing this is the new asymptotic enumeration result that the number of triangulations of the torus with a hole with k vertices on the boundary and n vertices in total is

$$\frac{k\binom{2k}{k}}{2^{2k+2} \cdot 3^{\frac{k-1}{2}}}(12\sqrt{3})^n \text{ as } n \to \infty$$

Contents

1 Introduction

In [1] Angel and Schramm defined a distribution over infinite planar triangulations which they called the uniform infinite planar triangulation (UIPT). Their motivation was to examine what a generic planar geometry could look like. A triangulation is in fact a lattice and a lattice is a possible discrete representation of a planar geometry. They also list connections in physics, most importantly the relation of triangulations of surfaces to 2-dimensional quantum gravity.

The most important result of Angel and Schramm in [1] is that they create the foundation of a rigorous study of infinite triangulations as opposed to previous results which focused on asymptotic properties of finite triangulations. Their method is to consider the uniform distribution on triangulations of the sphere with a fixed, finite number of vertices and they take a kind of local limit of these distributions as the number of vertices tends to infinity. They show that the limit exists and that it is in fact a distribution over infinite triangulations of the plane. They also deduce several properties of the limiting distribution which show that this distribution is also uniform in some sense.

The purpose of this thesis is to examine what happens if we repeat this process on the torus. Such a study can be interesting for the same reasons as that in [1]. However, this thesis has further motivation. In [4] Benjamini and Lovász showed a random process, which

enables one to tell the genus of a surface if given a map on the surface. Since this process only uses local steps it can be regarded as a way to deduce global information about the map from local observation.

Our question is similar. Indeed, if the local limit of uniform distributions over triangulations of the torus were different from that on the sphere then we could distinguish with high probability a large random triangulation of the torus from a triangulation of the sphere by only looking at the neighbourhood of some vertices. In this case global information could be deduced from local observation. On the other hand, if the limit is the same then a large triangulation of the sphere is indistinguishable from one of the torus by looking at neighbourhoods of vertices.

The basis of the study of the UIPT was a collection of previous enumeration results of the number of triangulations of the sphere and of the disk (the sphere with a hole). For our purposes we need the asymptotic number of triangulations of the sphere with zero, one and two holes and of the torus with and without hole. All but the last estimation were known previously. However, the last one is essential since it determines the exact distribution we are looking for. Hence a new result about the asymptotic number of triangulations of the torus with a hole is the centrepiece of this thesis.

It is important to note that in [1] triangulations without loops and triangulations without loops and multiple edges were examined. However, some of the considerations there cannot be repeated easily on the torus if we do not allow loops and multiple edges. Thus as well as examining the torus we will repeat the same considerations as in [1] on the sphere but with loops and multiple edges allowed.

Unfortunately, a result that the limiting distribution exists for this slightly more general type of triangulation is still missing. According to personal communication with Omer Angel the limit does exist as such local differences in what type of maps we allow do not make a difference in the global structure. However, as of the writing of this thesis the author could not get hold of a proof of this fact. This will not stop us from determining what the limiting distributions on the sphere and on the torus are like supposing they exist.

In Section 2 we will introduce definitions and in Section 3 we will state our main results. In Section 4 we will list all previous enumeration results that we need and transform them to the form in which we will use them. We will also state the new asymptotic result about the number of triangulations of the torus with a hole. In Section 5 we will show both for the sphere and for the torus that if the limit exists then it is a distribution over infinite

triangulations of the plane.

In Section 6 we will deduce our most important result: If the limit exists on the sphere and on the torus then it is the exact same distribution over infinite triangulations of the plane. Thus a large random triangulation of the sphere is indistinguishable from one of the torus by making local observations.

What remains is the deduction of our new asymptotic enumeration results. In Section 7 we will show an exact but not closed formula for the number of triangulations of the torus with a fixed number of vertices and a fixed size hole. For this we will use a technique based on the Lagrange theorem for implicit functions that is described in [3]. In Section 8 we will estimate this formula as the number of vertices tends to infinity. This will be based on instructions given in [5] on how to estimate sums arising in combinatorics.

2 Definitions

2.1 Triangulations

We start by defining finite triangulations on surfaces without boundaries.

Definition 2.1. Let us consider a connected compact orientable surface $\mathbb{S}$ without boundaries. (We will mainly be concerned with $\mathbb{S}^2$ and $\mathbb{T}^2$, the sphere and the torus, respectively). Suppose a finite connected multigraph G, that is, a graph with loops and multiple edges allowed, is embedded in $\mathbb{S}$. A *face* is then a connected component of $\mathbb{S}\backslash G$. A face is called a *triangle* if its interior is homeomorphic to a disk and is incident to exactly three edges of G, counted with multiplicity.

An *embedded triangulation T on* $\mathbb{S}$ is then such a graph G with some of its triangular faces selected. An *embedded triangulation of* $\mathbb{S}$, also referred to as a *complete embedded triangulation*, is an embedded triangulation on $\mathbb{S}$, where all faces are triangular and are selected.

The *support* $S(T) \subseteq \mathbb{S}$ of an embedded triangulation T is the union of G and the selected triangular faces. The selected faces are called *inner faces*, other faces are called *outer faces*. Vertices and edges in the interior of $S(T)$ are called *inner vertices* and *inner edges*, respectively. Vertices and edges on the boundary of $S(T)$ are called *boundary vertices* and *boundary edges*, respectively.

Two embedded triangulations can have the same combinatorial structure. In this case we do not want to consider them to be different.

Definition 2.2. Two embedded triangulations T_1, T_2 on $\mathcal{S}$ are *equivalent* if there is a homeomorphism between $S(T_1)$ and $S(T_2)$ that preserves all vertices, edges and inner faces.

Note that two equivalent embedded triangulations can have very different outer faces. A cycle in G might be embedded as a null-homotopic cycle in one embedded triangulation and as a non-trivial cycle in another and the two embedded triangulations could still be equivalent. Thus the structure of outer faces is not well-defined for equivalence classes of embedded triangulations. However, all other notions introduced in Definition 2.1 are preserved by a homeomorphism of the support and are thus well-defined for equivalence classes of embedded triangulations as well.

A fundamental problem that arises when we examine these triangulations is that a few of them can have non-trivial combinatorial symmetries. To get rid of this problem we will introduce a slightly improved definition.

Definition 2.3. A *rooted embedded triangulation* on $\mathbb{S}$ is an embedded triangulation on $\mathbb{S}$ where a vertex of G is selected to be the *root vertex*, an edge of G incident to the root vertex is selected to be the *root edge* and is directed such that its source is the root vertex (arbitrarily if it is a loop) and a side of the root edge is also selected. The face incident to the root edge on the selected side is the *root face*. These selected objects altogether are called the *root*.

Two embedded rooted triangulations are *equivalent* if there is a homeomorphism between their supports that preserves vertices and edges and preserves the selection of the root. A *triangulation* is an equivalence class of embedded rooted triangulations.

We will also need the notion of triangulations of surfaces with boundaries. We will not use partial triangulations only complete triangulations of such surfaces.

Definition 2.4. Let us take a compact orientable surface $\mathbb{S}$ with l disjoint circular boundaries $H_1, H_2, \ldots H_l$, also called *holes*. A *triangulation of $\mathcal{S}$* is defined just as in Definitions 2.1 and 2.3 except that the holes are covered by disjoint proper cycles of G and the root is selected in such a way that the root vertex and the root edge are located on the boundary of H_1 and the side of the root edge selected is that which is incident to the hole and opposite to the interior of $\mathbb{S}$.

The breakthrough in the work of Angel and Schramm in [1] is that they also consider infinite triangulations. Note that if we embed an infinite graph G in a compact surface

and select some of the triangular faces then the support of the selected object cannot be compact. In particular in cannot be the whole surface. Thus we cannot talk about infinite triangulations of a compact surface only partial triangulations on the surface.

Definition 2.5. Let us take a compact orientable surface $\mathbb{S}$. An *infinite triangulation on* $\mathbb{S}$ is defined just as in Definitions 2.1 and 2.3 except that the graph G is infinite and the embedding is required to be *locally finite*. That is, any point in the support of the triangulation must have a small enough neighbourhood that intersects only a finite number of vertices, edges and inner faces of the triangulation. In particular, the degree of any vertex in the graph must be finite.

The plane, however, is not compact. We can thus talk about triangulations of the plane.

Definition 2.6. A *(complete, infinite, rooted) triangulation of the plane* is defined just as in Definition 2.5 except that $\mathbb{S} = \mathbb{R}^2$ and the support of the triangulation is the whole plane.

Those triangulations on a compact orientable surface without boundaries which have a support homeomorphic to an open disk can also canonically be identified with triangulations of the plane.

Most of the results introduced in this thesis are based on counting triangulations. Thus we will use the following notation:

Definition 2.7. Let S_n and T_n denote the number of triangulations of the sphere and of the torus, respectively, with n vertices in total. Let $S_{n,k_1,k_2,\ldots,k_h}$ and $T_{n,k_1,k_2,\ldots,k_h}$ be the number of triangulations of the sphere and of the torus, respectively, with h holes with $k_1, k_2, \ldots, k_h$ vertices (and the same number of edges) on each boundary and n vertices in total, such that the root edge lies on the boundary of the first hole.

2.2   The space of triangulations

We will want to create probability distributions on triangulations. Thus we need to examine the space of triangulations more closely. (Here we are only going to consider surfaces without boundaries. Triangulations of surfaces with boundaries are only used in counting finite triangulations.)

Definition 2.8. For a surface $\mathbb{S}$ let the space of (finite or infinite) triangulations on $\mathbb{S}$ be denoted by $\mathcal{T}_{\mathbb{S}}$.

$\mathcal{T}_{\mathbb{S}}$ is a metric space: two triangulations are close to each other if they agree on a large combinatorial ball around the root.

Definition 2.9. Let us take a triangulation T on an orientable surface $\mathbb{S}$ without boundaries. The *combinatorial ball* of T of radius r is denoted by $B_r(T)$ and is a triangulation in $\mathcal{T}_{\mathbb{S}}$. It is defined recursively: $B_0(T)$ is just the root vertex of T. $B_r(T)$ consists of $B_{r-1}(T)$, all triangles of T incident to vertices of $B_{r-1}(T)$ and all vertices and edges incident to these triangles.

The above definition might seem somewhat technical. But that is to ensure the following properties, which all follow directly from the definition:
Suppose $T \in \mathcal{T}_{\mathbb{S}}$, its dual graph is connected and each of its edges is incident to a triangle on both sides (in particular, T can be a triangulation of $\mathbb{S}$ or an infinite triangulation on $\mathbb{S}$ with support homeomorphic to the plane). Then

- any edge of $B_r(T)$ is incident to at least one triangle,

- the dual graph of $B_r(T)$ is connected,

- the vertices of $B_r(T)$ are simply the vertices at a graph-theoretic distance of at most r away from the root,

- the vertices of $B_{r-1}(T)$ are inner vertices of $B_r(T)$, and

- the endpoints of the boundary edges of $B_r(T)$ are exactly at a distance of r away from the root.

Definition 2.10. The distance of triangulations $T_1, T_2 \in \mathcal{T}_{\mathbb{S}}$ is

$$d(T_1, T_2) = \frac{1}{\sup\{r \in \mathbb{N} | B_r(T_1) = B_r(T_2)\}}$$

(If we want to avoid that some distances are infinite we can define $d(T_1, T_2) = 2$ if the above supremum is zero. All other distances are at most 1.)

Before we can prove some basic properties of this distance function we need a technical proposition:

Proposition 2.1. *Suppose $T_1, T_2, T_3 \ldots$ is an infinite sequence of finite triangulations on an orientable surface $\mathbb{S}$ without boundaries such that T_i is a sub-triangulation of T_{i+1}*

$(i = 1, 2, 3, \ldots)$. *That is, we can delete some vertices, edges and triangles of T_{i+1} such that the same embedding leads to a triangulation equivalent to T_i.*

Then there is a unique triangulation $T \in \mathcal{T}_\mathbb{S}$ such that $T_1, T_2, \ldots$ can be realized as a chain of sub-triangulations of T and the union of their support covers the support of T.

Proof (sketch). Any embedding of a finite triangulation defines the boundary components of the triangulation, which are closed walks on the graph that use each edge at most twice altogether. For a fixed triangulation there is a finite number of ways these boundary components can be selected.

The embedding also defines the topology of the outer faces. The interior of each outer face can be an orientable surface with a number of holes, where each hole is attached to a boundary component of the triangulation. With some consideration about Euler-characteristic we can deduce that for a fixed selection of boundary components there are only a finite number of ways we can select the topology of the outer faces.

Thus, for any finite triangulation there is a finite number of ways its realizations can define the combination of boundary components and the topology of the outer faces. Thus, using König's lemma, we can make a selection of the above structures for each T_i such that the selection for T_{i+1} is compatible to the selection for T_i. That is, there is a realization of T_{i+1} with the given properties such that the same embedding restricted on T_i fulfills the selection for T_i.

Furthermore, if two realizations of a finite triangulation define the above structures in the same way then the homeomorphism between their support can be extended to a homeomorphism of the whole surface $\mathbb{S}$.

Thus, we can define embeddings of T_i $(i = 1, 2, 3, \ldots)$ one after the other such that each embedding conforms to our selection of combinatorial and topological properties and the embedding of T_{i+1} restricted to T_i is the same as the embedding of T_i.

This process defines T as requested.

For uniqueness suppose T' has the same properties as T and take a realization of both. We will give a homeomorphism between the support of T and T' that preserves the combinatorial structure thus showing they are equivalent.

For any i there is a homeomorphism between the support of the T_i in T and the T_i in T'. In fact, there might be several that differ in how they map the combinatorial structure, that is, which vertex, edge and face they map into each vertex, edge and face. But since T_i is finite, there is only a finite number of ways they can map the combinatorial structure. Thus, again, we can use König's lemma to select a homeomorphism for each i such that the

homeomorphisms for each i are compatible in how they map the combinatorial structure. We can now define the homeomorphism for each i one after the other such that consecutive ones are compatible not only in how they map the combinatorial structure but in the exact homeomorphism on the smaller support.

To do that all we need to know is that if we are given a homeomorphism between two triangulations that preserves the combinatorial structure then we can slightly modify the homeomorphism so that it still maps the combinatorial structure the same way but each edge is mapped to the corresponding edge in any prescribed way.

Let's suppose we fixed the homeomorphism ϕ_i between T_i in T and in T'. There is a homeomorphism between the T_{i+1} in T and in T' that is compatible with ϕ_i in how it maps the combinatorial structure. We can modify it such that it maps the boundary edges of T_i in the same way as ϕ_i. If this mapping defines ϕ_{i+1} on $S(T_{i+1})\backslash S(T_i)$ it connects well with ϕ_i.

This process defines a homeomorphism between $S(T)$ and $S(T')$ that preserves the combinatorial structure thus showing they are equivalent. $\qquad\square$

Lemma 2.2.   *1. d is a metric on $\mathcal{T}_{\mathbb{S}}$.*

    2. $(\mathcal{T}_{\mathbb{S}}, d)$ is complete.

    3. $(\mathcal{T}_{\mathbb{S}}, d)$ is separable.

    4. $(\mathcal{T}_{\mathbb{S}}, d)$ is not compact.

Proof. Symmetry of d is trivial. The triangle inequality follows from the special structure of $(\mathcal{T}_{\mathbb{S}}, d)$:

$$d(T_1, T_3) \leq \max(d(T_1, T_2), d(T_2, T_3))$$

Indeed, all three triangulations agree on the ball of radius $\frac{1}{\max(d(T_1,T_2),d(T_2,T_3))}$.

We need to show that if $d(T_1, T_2) = 0$ then $T_1 = T_2$. Indeed, $d(T_1, T_2)$ means that there is an equivalence between an arbitrarily large ball of T_1 and T_2. It follows from the uniqueness part of Proposition 2.1 that $T_1 = T_2$.

For completeness suppose we are given a Cauchy-sequence $T_1, T_2, \dots$. For any fixed r the balls $B_r(T_i)$ are the same for large enough i. This is a chain of r-balls and so from the existence part of Proposition 2.1 there is a triangulation T with these r-balls. The sequence $T_1, T_2, \dots$ converges to T.

To see separability notice that any triangulation can be arbitrarily approximated by its r-balls. For any fixed r and n there is a finite number of possible r-balls with n vertices. The number of selections of r and n is countable thus the total number of possible r-balls is countable.

$(\mathcal{T}_\mathbb{S}, d)$ is not compact since for example there is an infinite number of possible 2-balls. Any two of the balls are triangulations at a distance at least 1 away from each other which contradicts compactness. $\qquad\square$

To us the metric balls of $\mathcal{T}_\mathbb{S}$ will be of great importance.

Definition 2.11. Let us consider the space of triangulations $\mathcal{T}_\mathbb{S}$ on an orientable surface $\mathbb{S}$ without boundaries. The *metric balls* in $\mathcal{T}_\mathbb{S}$ are its subsets of the form $\{T \in \mathcal{T}_\mathbb{S} | B_r(T) = B_r(T_0)\}$ where T_0 is a fixed triangulation on $\mathbb{S}$. This is the set of all triangulations at most $\frac{1}{r}$ away from the triangulation T_0.

The metric d on $\mathcal{T}_\mathbb{S}$ induces a topology on $\mathcal{T}_\mathbb{S}$, which is generated by the metric balls of $\mathcal{T}_\mathbb{S}$. In this topology the set of all finite triangulations on $\mathbb{S}$ is discrete, their accumulation points are infinite triangulations.

We can also consider the set $\mathcal{B}_\mathbb{S}$ of Borel-sets of $(\mathcal{T}_\mathbb{S}, d)$. $(\mathcal{T}_\mathbb{S}, \mathcal{B}_\mathbb{S})$ is then a measurable space generated by the metric balls. It is on this measurable space that we will define probability distributions. We will be interested in uniform distributions over triangulations of $\mathbb{S}$.

2.3 Probability distributions

Definition 2.12. For a compact orientable surface $\mathbb{S}$ without boundaries let $\tau_n^\mathbb{S}$ be the uniform measure over (complete) triangulations of $\mathbb{S}$ with n vertices in total.

We will be interested in the weak limit of $\tau_n^\mathbb{S}$ as $n \to \infty$.

Definition 2.13. A sequence of measures ν_n on $(\mathcal{T}_\mathbb{S}, \mathcal{B}_\mathbb{S})$ weakly converges to a measure ν as $n \to \infty$ if for every bounded continuous function $f : \mathcal{T}_\mathbb{S} \to \mathbb{R}$

$$\lim_{n\to\infty} \int_{\mathcal{T}_\mathbb{S}} f d\nu_n = \int_{\mathcal{T}_\mathbb{S}} f d\nu$$

Because of the unusual properties of $\mathcal{T}_\mathbb{S}$ the characteristic functions of metric balls of $\mathcal{T}_\mathbb{S}$ are continuous (and bounded). Thus if $\mathcal{B}$ is a metric ball then

$$\nu_n(\mathcal{B}) \to \nu(\mathcal{B})$$

When trying to prove a property of the limit of a weakly convergent sequence of measures we will often succeed by describing the property as a countable disjoint union of metric balls:

Proposition 2.3. *Let $\nu_n \to \nu$ be a weakly convergent sequence of probability measures on $\mathcal{T}_\mathbb{S}$ where $\mathbb{S}$ is a compact orientable surface without boundaries. Let $\mathcal{A}$ be any event on $\mathcal{T}_\mathbb{S}$ that can be described as a disjoint union of a countable set of metric balls. Then*

$$\nu_n(\mathcal{A}) \to \nu(\mathcal{A})$$

Proof. ν_n converges to ν on each metric ball. For the sum over the metric balls to converge it is enough to note that the sum is at most 1 for each measure since they are all probability measures. □

Since the metric balls of $\mathcal{T}_\mathbb{S}$ generate the Borel-sets of $\mathcal{T}_\mathbb{S}$ the values $\nu(\{T|B_r(T) = B_r(T_0)\})$ determine the measure ν.

Thus, supposing the limit $\tau_n^\mathbb{S} \to \tau^\mathbb{S}$ exists for some surface $\mathbb{S}$ we can describe $\tau^\mathbb{S}$ by calculating the limits

$$\lim_{n \to \infty} \tau_n^\mathbb{S}(\{T|B_r(T) = B_r(T_0)\})$$

2.4 Rigidity

To calculate probabilities of the form $\tau_n^\mathbb{S}(\{T|B_r(T) = B_r(T_0)\})$ we will want to calculate the number of triangulations of a surface with a fixed r-ball. We will do that by calculating the number of ways its outer faces can be triangulated with a fixed number of vertices. That means firstly enumerating the number of ways we can select the topology of the outer faces and secondly enumerating the number of ways we can triangulate these outer faces that are surfaces with boundaries.

But is this method valid? If we triangulate the outer faces differently are we always going to get different triangulations? This is not true for any partial triangulation. Using the following definition it is shown in [1] that this problem does not occur for cases that we examine.

Definition 2.14. A finite, partial triangulation $T \in \mathcal{T}_\mathbb{S}$ is *rigid* if for any finite, complete triangulation $T' \in \mathcal{T}_\mathbb{S}$ triangulation T can be realized as a sub-triangulation of T' in at most one way. In other words we can only choose the topology of outer faces of T and completely triangulate these outer faces to get T' in at most one way.

The following proposition, which is stated in [1] and is easy to show, gives a sufficient criterion for rigidity.

Proposition 2.4. *If every vertex and edge of T is incident to at least one triangle of T and the dual graph of T (the vertices of which are triangles of T) is connected then T is rigid.*

Corollary 2.5. *The r-balls of a finite triangulation of a compact, orientable surface $\mathbb{S}$ without boundaries are rigid. Furthermore, if we take a possible r-ball of triangulations of $\mathbb{S}$, choose the topology of some outer faces and completely triangulate them then the resulting triangulation is still rigid.*

Remark. We have shown that triangulating outer faces of an r-ball differently we get different triangulations. To be precise, we also need that any triangulation of the outer faces as surfaces with boundaries defines a valid triangulation of the whole surface. This is where we use the fact that we allowed loops and multiple edges. If we did not allow them then a triangulation of an outer face as a surface with boundaries might for example have an edge between two vertices of the hole that get attached to the same vertex in the r-ball, which would create a loop.

3 Main results

In this thesis we will examine the limit of the measures $\tau_n^{\mathbb{S}^2}$ and $\tau_n^{\mathbb{T}^2}$ on triangulations on the sphere and on the torus, respectively.

According to personal communication with Omer Angel these limits do exist. However, as of the writing of this thesis the author could not get hold of a proof of this fact.

From now on we will assume the existence of both limits:

$$\tau_n^{\mathbb{S}^2} \to \tau^{\mathbb{S}^2} \text{ and } \tau_n^{\mathbb{T}^2} \to \tau^{\mathbb{T}^2}$$

(It is important to note that instead of supposing that these limits exist we could state our results for any subsequential limit of $\tau_n^{\mathbb{S}^2}$ and $\tau_n^{\mathbb{T}^2}$ for which our proofs will be perfectly valid. This form of our results shows that if one proves that the limits exists then our results will automatically hold for the limit. Furthermore, results about subsequential limits might also be helpful when proving that the limit of the whole sequence exists.)

In Section 5 we will show that $\tau^{\mathbb{S}^2}$ and $\tau^{\mathbb{T}^2}$ are both distributions on triangulations of the plane. That is, their support is the set of triangulations on $\mathbb{S}^2$ and on $\mathbb{T}^2$, respectively,

with a support homeomorphic to an open disk. A big portion of this proof closely follows the proof that appears in [1]. But additional considerations are necessary for the torus and we will give more details of the whole proof than in [1].

In Section 6 we will show that

$$\tau^{\mathbb{S}^2} = \tau^{\mathbb{T}^2}$$

if we identify their support in the natural way.

The basis of the above result is the asymptotic enumeration formula, proved in Sections 7 and 8 that

$$T_{n,k} \sim \frac{k\binom{2k}{k}}{2^{2k+2} \cdot 3^{\frac{k-1}{2}}} (12\sqrt{3})^n \text{ as } n \to \infty$$

4 Auxiliary enumeration results

For our purposes we need an estimation as $n \to \infty$ of S_n and T_n, the number of triangulations of the sphere and of the torus, respectively, with n vertices in total, and $S_{n,k}, S_{n,k_1,k_2}$ and $T_{n,k}$, the number of triangulations of the sphere with one and two holes and of the torus with a hole, respectively, with k or k_1 and k_2 vertices (and edges) on each boundary and n vertices in total.

In some cases it is easier to enumerate triangulations by edges instead.

Definition 4.1. Let $\widetilde{S}_{m,k_1,k_2,\ldots,k_h}$ and $\widetilde{T}_{m,k_1,k_2,\ldots,k_h}$ denote the number of rooted triangulations of the sphere and of the torus, respectively, with h holes with $k_1, k_2, \ldots, k_h$ edges (and vertices) on each boundary and m edges in total, such that the root edge lies on the boundary of the first hole.

Proposition 4.1. *From Euler's formula it is easy to show that*

$$S_{n,k_1,k_2,\ldots,k_h} = \widetilde{S}_{(3n-\sum k_i+3h-6),k_1,k_2,\ldots,k_h} \text{ and } T_{n,k_1,k_2,\ldots,k_h} = \widetilde{T}_{(3n-\sum k_i+3h),k_1,k_2,\ldots,k_h}$$

In [6] Gao showed that

$$S_n \sim \frac{1}{2^{\frac{7}{2}} \cdot 3^{\frac{5}{2}} \cdot \sqrt{\pi}} n^{-\frac{5}{2}} (12\sqrt{3})^n \text{ and } T_n \sim \frac{1}{8}(12\sqrt{3})^n \tag{1}$$

In Sections 7 and 8 we will show that

$$T_{n,k} \sim \frac{k\binom{2k}{k}}{2^{2k+2} \cdot 3^{\frac{k-1}{2}}} (12\sqrt{3})^n \text{ as } n \to \infty \tag{2}$$

In [2] Krikun showed that if $m \not\equiv 2\sum k_i \pmod 3$ then $\widetilde{S}_{m,k_1,\ldots,k_h} = 0$ and otherwise

$$\widetilde{S}_{m,k_1,k_2,\ldots,k_h} = \frac{4^{\frac{m-2\sum k_i}{3}}(m-2)!!}{\left(\frac{m-2\sum k_i}{3} - h + 2\right)! \left(2\sum k_i + \frac{m-2\sum k_i}{3}\right)!!} k_1 \prod_{i=1}^{h} \binom{2k_i}{k_i} \tag{3}$$

Thus from Proposition 4.1.

$$S_{n,k_1,k_2,\ldots,k_h} = \frac{4^{n-\sum k_i+h-2}(3n - \sum k_i + 3h - 8)!!}{(n - \sum k_i)! \, (n + \sum k_i + h - 2)!!} k_1 \prod_{i=1}^{h} \binom{2k_i}{k_i} \tag{4}$$

We now need to estimate this as $n \to \infty$

Stirling's formula states that

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$$

From Stirling's formula it easy to show that

$$n!! \sim \begin{cases} \sqrt{\pi n} \left(\frac{n}{e}\right)^{\frac{n}{2}} & \text{if } n \text{ is even} \\ \sqrt{2n} \left(\frac{n}{e}\right)^{\frac{n}{2}} & \text{if } n \text{ is odd} \end{cases}$$

Since $3n - \sum k_i + 3h - 8$ and $n + \sum k_i + h - 2$ are of the same parity

$$\frac{(3n - \sum k_i + 3h - 8)!!}{(n + \sum k_i + h - 2)!!} \sim \sqrt{\frac{3n - \sum k_i + 3h - 8}{n + \sum k_i + h - 2}} \cdot \frac{\left(\frac{3n - \sum k_i + 3h - 8}{e}\right)^{\frac{3n - \sum k_i + 3h - 8}{2}}}{\left(\frac{n + \sum k_i + h - 2}{e}\right)^{\frac{n + \sum k_i + h - 2}{2}}}$$

and thus

$$S_{n,k_1,\ldots,k_h} \sim 4^{n-\sum k_i+h-2} \sqrt{\frac{3n - \sum k_i + 3h - 8}{2\pi(n - \sum k_i)(n + \sum k_i + h - 2)}}$$

$$\cdot \frac{\left(\frac{3n - \sum k_i + 3h - 8}{e}\right)^{\frac{3n - \sum k_i + 3h - 8}{2}}}{\left(\frac{n - \sum k_i}{e}\right)^{n - \sum k_i} \left(\frac{n + \sum k_i + h - 2}{e}\right)^{\frac{n + \sum k_i + h - 2}{2}}} \cdot k_1 \prod_{i=1}^{h} \binom{2k_i}{k_i}$$

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$$\sim 4^{n-\sum k_i+h-2}\cdot e^{3-h}\cdot \sqrt{\frac{3}{2\pi n}}\cdot \frac{(3n)^{\frac{3n-\sum k_i+3h-8}{2}}}{n^{n-\sum k_i}n^{\frac{n+\sum k_i+h-2}{2}}}$$

$$\cdot \frac{\left(1-\frac{\sum k_i-3h+8}{3n}\right)^{\frac{3n-\sum k_i+3h-8}{2}}}{\left(1-\frac{\sum k_i}{n}\right)^{n-\sum k_i}\left(1+\frac{\sum k_i+h-2}{n}\right)^{\frac{n+\sum k_i+h-2}{2}}}\cdot k_1\prod_{i=1}^{h}\binom{2k_i}{k_i}$$

$$\sim 4^{n-\sum k_i+h-2}\cdot e^{3-h}\cdot \sqrt{\frac{3}{2\pi n}}\cdot \frac{3^{\frac{3n-\sum k_i+3h-8}{2}}}{n^{3-h}}\cdot \frac{\left(1-\frac{\sum k_i-3h+8}{3n}\right)^{\frac{3n}{2}}}{\left(1-\frac{\sum k_i}{n}\right)^{n}\left(1+\frac{\sum k_i+h-2}{n}\right)^{\frac{n}{2}}}\cdot k_1\prod_{i=1}^{h}\binom{2k_i}{k_i}$$

$$\sim \frac{1}{2^{\frac{4\sum k_i-4h+9}{2}}3^{\frac{\sum k_i-3h+7}{2}}\sqrt{\pi}}\cdot e^{3-h}\cdot n^{h-\frac{7}{2}}(12\sqrt{3})^n\cdot \frac{e^{-\frac{\sum k_i-3h+8}{2}}}{e^{-\sum k_i}\cdot e^{\frac{\sum k_i+h-2}{2}}}\cdot k_1\prod_{i=1}^{h}\binom{2k_i}{k_i}$$

$$= \frac{k_1\prod_{i=1}^{h}\binom{2k_i}{k_i}}{2^{\frac{4\sum k_i-4h+9}{2}}3^{\frac{\sum k_i-3h+7}{2}}\sqrt{\pi}}\cdot n^{h-\frac{7}{2}}(12\sqrt{3})^n$$

In particular

$$S_{n,k}\sim \frac{k\binom{2k}{k}}{2^{\frac{4k+5}{2}}3^{\frac{k+4}{2}}\sqrt{\pi}}\cdot n^{-\frac{5}{2}}(12\sqrt{3})^n \text{ and } S_{n,k_1,k_2}\sim \frac{k_1\binom{2k_1}{k_1}\binom{2k_2}{k_2}}{2^{\frac{4k_1+4k_2+1}{2}}3^{\frac{k_1+k_2+1}{2}}\sqrt{\pi}}\cdot n^{-\frac{3}{2}}(12\sqrt{3})^n \quad (5)$$

5 Planarity

In this section we will show that $\tau^{\mathbb{S}^2}$ and $\tau^{\mathbb{T}^2}$ (supposing they exist) are distributions over planar triangulations. There are two things we have to prove to see this.

Firstly, we will show that the probability by $\tau_n^{\mathbb{T}^2}$ of seeing a certain r-ball in a finite triangulation of the torus such that the r-ball contains a topologically non-trivial cycle tends to zero as $n\to\infty$. This statement has some subtleties to it. In general it is not well-defined whether a triangulation contains a non-trivial cycle. A partial triangulation might have one embedding where it does and another that doesn't. However, in a finite, complete triangulation of the torus we can determine about each cycle whether it is null-homotopic or not. This is because the equivalence of two finite triangulations of the torus is shown by a homeomorphism of the whole torus which maps non-trivial cycles into non-trivial cycles.

Thus, it makes sense to talk about the probability by $\tau_n^{\mathbb{T}^2}$ of seeing a certain r-ball with a non-trivial cycle. We will conclude from this first property that by $\tau^{\mathbb{T}^2}$ the support of infinite triangulations can almost surely be embedded in the plane.

Secondly, we have to show that infinite triangulations on the sphere and on the torus almost surely have no additional holes, we can cover the whole plane with them. To show that there are no finite external faces, in fact no external faces incident to any edge, is easy. We also need to show that there is only one infinite external face. The way to do this is to prove one-endedness:

Definition 5.1 (see [1]). A triangulation $T \in \mathcal{T}_{\mathbb{S}}$ is *one-ended* if no finite sub-triangulation cuts it into more than one infinite component.

The proof of one-endedness of the UIPT appears in [1]. We will closely follow that proof for our type of triangulation and both for the sphere and the torus but we will give slightly more details.

And finally, combining the two properties above we will conclude that the support of triangulations is almost surely homeomorphic to an open disk (that is, to the plane) both for $\tau^{\mathbb{S}^2}$ and $\tau^{\mathbb{T}^2}$.

5.1   Tools

In what follows we will use $C, C_i, C(k), C_i(k,l)$ and similar notations for constants or constants that depend only on some parameters. In different formulas these constants might be different. We will only use subscripts to distinguish within one formula.

From the asymptotic formulas for $S_n, T_n, S_{n,k}, S_{n,k_1,k_2}, T_{n,k}$ we can deduce that there are constants $C_1, C_2, C_3(k), C_4(k_1,k_2), C_5(k)$ such that

$$
\begin{aligned}
S_n &\geq C_1 n^{-\frac{5}{2}} (12\sqrt{3})^n \\
T_n &\geq C_2 \cdot (12\sqrt{3})^n \\
S_{n,k} &\leq C_3(k) n^{-\frac{5}{2}} (12\sqrt{3})^n \\
S_{n,k_1,k_2} &\leq C_4(k_1,k_2) n^{-\frac{3}{2}} (12\sqrt{3})^n \\
T_{n,k} &\leq C_5(k)(12\sqrt{3})^n
\end{aligned}
\tag{6}
$$

We will make use of the following technical lemma when proving that a triangulation chosen by $\tau^{\mathbb{T}^2}$ almost surely has no non-trivial cycles:

Lemma 5.1. *Let $\alpha \in \mathbb{R}, \alpha > 1$, $n, h \in \mathbb{Z}^+$. Let*

$$S(n, h, \alpha) = \sum_{\substack{n_i \geq 1 \ (i=1...h) \\ n_1 + \cdots + n_h = n}} \prod_{i=1}^{h} n_i^{-\alpha}$$

. Then $S(n, h, \alpha) \leq C(h, \alpha) n^{-\alpha}$

Proof.

$$\sum_{\substack{n_i \geq 1 \ (i=1...h) \\ n_1 + \cdots + n_h = n}} \prod_{i=1}^{h} n_i^{-\alpha} \leq h! \sum_{\substack{n_1 \geq \cdots \geq n_h \geq 1 \\ n_1 + \cdots + n_h = n}} \prod_{i=1}^{h} n_i^{-\alpha}$$

$$\leq h! \sum_{\substack{n_1 \geq \cdots \geq n_h \geq 1 \\ n_1 + \cdots + n_h = n}} \left(\frac{n}{h}\right)^{-\alpha} \prod_{i=2}^{h} n_i^{-\alpha}$$

$$\leq h! \left(\frac{n}{h}\right)^{-\alpha} \sum_{\substack{n_i \geq 1 \ (i=2...h)}} \prod_{i=2}^{h} n_i^{-\alpha}$$

$$= C_1(h, \alpha) n^{-\alpha} \prod_{i=2}^{h} \sum_{n_i \geq 1} n_i^{-\alpha}$$

$$= C_2(h, \alpha) n^{-\alpha}$$

where at the end we used the fact that $\sum_{m=1}^{\infty} m^{-\alpha}$ is a finite constant if $\alpha > 1$. $\square$

A modified version of the above lemma appears in [1] and will be used to prove that the probability of an r-ball having two outer faces with more than a vertices is small. This lemma goes as follows:

Lemma 5.2. *Let $\alpha \in \mathbb{R}, \alpha > 1$, $n, h, a \in \mathbb{Z}^+$. Let*

$$S(n, h, \alpha, a) = \sum_{\substack{n_i \geq 1 \ (i=1...h) \\ n_1, n_2 > a \\ n_1 + \cdots + n_h = n}} \prod_{i=1}^{h} n_i^{-\alpha}$$

. Then $S(n, h, \alpha, a) \leq C(h, \alpha) a^{-(\alpha-1)} n^{-\alpha}$

Proof.

$$\sum_{\substack{n_i \geq 1 \ (i=1...h) \\ n_1,n_2 > a \\ n_1 + \cdots + n_h = n}} \prod_{i=1}^{h} n_i^{-\alpha} \leq h! \sum_{\substack{n_1 \geq \cdots \geq n_h \geq 1 \\ n_2 > a \\ n_1 + \cdots + n_h = n}} \prod_{i=1}^{h} n_i^{-\alpha}$$

$$\leq h! \sum_{\substack{n_1 \geq \cdots \geq n_h \geq 1 \\ n_2 > a \\ n_1 + \cdots + n_h = n}} \left(\frac{n}{h}\right)^{-\alpha} \prod_{i=2}^{h} n_i^{-\alpha}$$

$$\leq h! \left(\frac{n}{h}\right)^{-\alpha} \sum_{\substack{n_2 > a \\ n_i \geq 1 \ (i=3...h)}} \prod_{i=2}^{h} n_i^{-\alpha}$$

$$= C_1(h,\alpha) n^{-\alpha} \left(\sum_{n_2 > a} n_2^{-\alpha}\right) \left(\prod_{i=3}^{h} \sum_{\substack{n_i \geq 1 \\ (i=3...h)}} n_i^{-\alpha}\right)$$

$$= C_2(h,\alpha) a^{-(\alpha-1)} n^{-\alpha}$$

where at the end we used the facts that if $\alpha > 1$ is fixed then $\sum_{m=1}^{\infty} m^{-\alpha}$ is a finite constant and $\sum_{m=a+1}^{\infty} m^{-\alpha} = O(m^{-(\alpha-1)})$. $\qquad\square$

We will also need the following simple facts:

Proposition 5.3. *Let us suppose that T is a finite triangulation on the sphere, its dual graph is connected and all of its edges and vertices are incident to at least one triangle. Then*

1. *the boundary components of T are well defined (do not depend on the embedding),*

2. *each are proper cycles (do not repeat edges and vertices),*

3. *they can only intersect each other in vertices*

4. *and all outer faces are disks.*

Proof. The boundary edges of T are well defined: those that are incident to exactly one triangle (since there are none that are incident to zero). This is a subgraph of the graph G underlying T. The number of boundary edges incident to each node is even since in an arbitrary embedding of T the outer faces touch two edges each time they traverse the node along their boundary and each edge can be used only once.

Thus this subgraph is an edge-disjoint union of proper cycles. Let us take any proper cycle C in the subgraph. Let us take an arbitrary embedding of T. Then C divides the sphere into to two components, both are disks. The root is on one side. When walking on the dual graph we cannot cross C since any of the edges of C only have a triangle on one side. Thus all triangles are on the same side of C as the root. Furthermore, since all vertices and edges are incident to at least one triangle they are also on the same side of C. Hence there are no vertices, edges or triangles on the other side of C, that disk is a connected component of $\mathbb{S}^2 \backslash G$, it is an outer face and its boundary is C.

But the embedding was arbitrary thus C is a boundary component in any embedding. And C was an arbitrary proper cycle among boundary edges thus each cycle is always a boundary and the corresponding outer face is a disk. Since all boundary edges are in at most one cycle this covers all boundary edges. Since each boundary edge can be used at most once there are no more boundaries and outer faces than the ones described so far. This proves all that we claimed. □

5.2   Non-trivial cycles

What can the topology of outer faces of an r-ball B_r of a finite triangulation of $\mathbb{T}^2$ look like? The boundary components of B_r are closed walks that can only repeat vertices but no edges (if they repeated an edge then that edge would not be incident to any triangle of B_r.) Similarly, two boundary components might intersect in a vertex but not in an edge. The interior of outer faces is homeomorphic to a surface with a number of holes and each of these holes is attached to a boundary component of B_r.

With some considerations about Euler-characteristic we can see that the outer faces must mostly be disks (spheres with a hole), but there can be a single outer face with the topology of a tube (a sphere with two holes) attached to two boundary components of B_r or alternatively there might be a single outer face with the topology of a torus with a hole attached to a single boundary component.

In the last case B_r does not contain a non-trivial cycle, otherwise it does.

Let's fix a possible r-ball B_r and its boundary components. Let the number of boundary components be h and the number of vertices on each (counted with multiplicity) be $\underline{k} = (k_1, k_2, \ldots, k_h)$. We denote the total number of vertices of B_r by l.

Lemma 5.4. *Let's suppose that closing each boundary component with a disk completes B_r to a torus. Then the probability by $\tau_n^{\mathbb{T}^2}$ of seeing B_r as the r-ball, with the selected*

boundary components, and all outer faces having interior homeomorphic to an open disk tends to 0 as $n \to \infty$.

Proof. The number of finite triangulations of the torus with n vertices in total, with r-ball B_r, with the given boundary components and all outer faces of B_r being disks is

$$\sum_{\substack{n_i \geq k_i \ (i=1...h) \\ n_1 - k_1 + \cdots + n_h - k_h + l = n}} S_{n_1,k_1} \cdot S_{n_2,k_2} \ldots S_{n_h,k_h} \leq \sum_{\substack{n_i \geq k_i \ (i=1...h) \\ n_1 + \cdots + n_h = n + \sum k_i - l}} \prod_{i=1}^{h} C(k_i) n_i^{-\frac{5}{2}} (12\sqrt{3})^{n_i}$$

$$\leq C_1(\underline{k}, l)(12\sqrt{3})^n \sum_{\substack{n_i \geq 1 \ (i=1...h) \\ n_1 + \cdots + n_h = n + \sum k_i - l}} \prod_{i=1}^{h} n_i^{-\frac{5}{2}}$$

$$\leq C_2(\underline{k}, l)(12\sqrt{3})^n \left(n + \sum k_i - l \right)^{-\frac{5}{2}}$$

$$\leq C_3(\underline{k}, l)(12\sqrt{3})^n n^{-\frac{5}{2}}$$

where in the first inequality we used (6) and in the third inequality we used Lemma 5.1 with $\alpha = \frac{5}{2}$.

Thus the probability by $\tau_n^{\mathbb{T}^2}$ of seeing B_r as the r-ball, with the selected boundary components and with all outer faces having interior homeomorphic to an open disk is

$$\tau_n^{\mathbb{T}^2}\{T | B_r(T) = B_r, \text{ outer faces are disks}\} \leq \frac{C_1(\underline{k}, l)(12\sqrt{3})^n n^{-\frac{5}{2}}}{T_n}$$

$$\leq \frac{C_1(\underline{k}, l)(12\sqrt{3})^n n^{-\frac{5}{2}}}{C_2 \cdot (12\sqrt{3})^n}$$

$$\leq C_3(\underline{k}, l) n^{-\frac{5}{2}} \to 0 \text{ as } n \to \infty$$

$\square$

Lemma 5.5. *Let's suppose that connecting two boundary components with a tube and closing others with a disk completes B_r to a torus. Then the probability by $\tau_n^{\mathbb{T}^2}$ of seeing B_r as the r-ball, with the selected boundary components, with one outer face connecting two boundary components with a tube and all other outer faces having interior homeomorphic to an open disk tends to 0 as $n \to \infty$.*

Proof. Let's first consider the case when the first two boundary components are connected with a tube. The number of finite triangulations of the torus with n vertices in total, with

r-ball B_r, with the selected boundary components, with the first two boundary components being connected by a tube and all other outer faces of B_r being disks is

$$\sum_{\substack{m \geq k_1 + k_2 \\ n_i \geq k_i \ (i=3...h) \\ m-k_1-k_2+\sum_{3 \leq i \leq h}(n_i-k_i)+l=n}} \left(S_{m,k_1,k_2} \prod_{i=3...h} S_{n_i,k_i} \right)$$

$$\leq \sum_{\substack{m \geq k_1 + k_2 \\ n_i \geq k_i \ (i=3...h) \\ m+\sum_{3 \leq i \leq h} n_i = n + \sum k_i - l}} C_1(k_1, k_2) m^{-\frac{3}{2}} (12\sqrt{3})^m \prod_{i=3...h} C_2(k_i) n_i^{-\frac{5}{2}} (12\sqrt{3})^{n_i}$$

$$\leq C_3(\underline{k}, l)(12\sqrt{3})^n \sum_{\substack{m \geq 1 \\ n_i \geq 1 \ (i=3...h) \\ m+\sum_{3 \leq i \leq h} n_i = n + \sum k_i - l}} m^{-\frac{3}{2}} \prod_{i=3}^{h} n_i^{-\frac{5}{2}}$$

$$\leq C_3(\underline{k}, l)(12\sqrt{3})^n \sum_{\substack{m \geq 1 \\ n_i \geq 1 \ (i=3...h) \\ m+\sum_{3 \leq i \leq h} n_i = n + \sum k_i - l}} m^{-\frac{3}{2}} \prod_{i=3}^{h} n_i^{-\frac{3}{2}}$$

$$\leq C_4(\underline{k}, l)(12\sqrt{3})^n \left(n + \sum k_i - l \right)^{-\frac{3}{2}}$$

$$\leq C_5(\underline{k}, l)(12\sqrt{3})^n n^{-\frac{3}{2}}$$

where in the first inequality we used (6) and in the fourth inequality we used Lemma 5.1 with $\alpha = \frac{3}{2}$.

Since there are at most $\binom{h}{2}$ possibilities for choosing the boundary components that we connect by a tube the total number of ways we can triangulate the torus with n vertices such that B_r is the r-ball, boundary components are as prescribed, one outer face is a tube and the others are disks is still at most

$$C(\underline{k}, l)(12\sqrt{3})^n n^{-\frac{3}{2}}$$

Thus the probability by $\tau_n^{\mathbb{T}^2}$ of this event is

$\tau_n^{\mathbb{T}^2} \{T | B_r(T) = B_r,$ one outer face is tube, others are disks$\}$

$$\leq \frac{C_1(\underline{k}, l)(12\sqrt{3})^n n^{-\frac{3}{2}}}{T_n}$$

$$\leq \frac{C_1(\underline{k}, l)(12\sqrt{3})^n n^{-\frac{3}{2}}}{C_2 \cdot (12\sqrt{3})^n}$$

$$\leq C_3(\underline{k}, l) n^{-\frac{3}{2}} \to 0 \text{ as } n \to \infty$$

$\square$

From Lemmas 5.4 and 5.5 we can prove

Corollary 5.6. *For a fixed r*

$$\lim_{n \to \infty} \tau_n^{\mathbb{T}^2} \left(\{T \in \mathcal{T}_{\mathbb{T}^2} | B_r(T) \text{ is embedded w. torus w. a hole and disks}\} \right) = 1$$

Remark. This statement is a bit sloppy since for arbitrary triangulations the above condition is not well-defined. But $\tau_n^{\mathbb{T}^2}$ only selects finite, complete triangulations of the torus and for such triangulations the outer faces of its r-balls are well-defined.

Proof. We will show the complement:

$$\lim_{n \to \infty} \tau_n^{\mathbb{T}^2} \left(\{T \in \mathcal{T}_{\mathbb{T}^2} | B_r(T) \text{ is not embedded w. torus w. a hole and disks}\} \right) = 0$$

The set of possible r-balls is countable. For each possible r-ball the number of ways we can select its boundary components is a finite number that only depends on B_r itself. The limit is zero for each possible r-ball and each selection of boundary components thus it is zero for the sum over all of these selections (taking into account that for each n the sum is at most 1). $\square$

This leads to the proof of the first property of infinite triangulations that we need for planarity:

Definition 5.2. Let $\mathcal{T}_{\mathbb{T}^2}^0$ denote the set of triangulations T that can be embedded in the torus such that for all values of r one outer face of $B_r(T)$ is a torus with a hole and the others are disks.

Remark. These are in fact the triangulations that can be embedded in the plane but we will get back to that later. For now the above form will be more practical.

Lemma 5.7. $\tau^{\mathbb{T}^2}(\mathcal{T}_{\mathbb{T}^2}^0) = 1$

Proof. For a fixed r

$$\tau^{\mathbb{T}^2}\left(\{T \in \mathcal{T}_{\mathbb{T}^2}|\ B_r(T) \text{ can be embedded w. torus w. a hole and disks}\}\right)$$
$$= \lim_{n \to \infty} \tau_n^{\mathbb{T}^2}\left(\{T \in \mathcal{T}_{\mathbb{T}^2}|\ B_r(T) \text{ can be embedded w. torus w. a hole and disks}\}\right)$$

since the above event is a countable disjoint union of metric balls of the same radius and thus we can use Proposition 2.3

We restrict the above set if we take only to those triangulations of the torus with n vertices where the r ball is actually embedded as wished. The above is thus at least

$$\lim_{n \to \infty} \tau_n^{\mathbb{T}^2}\left(\{T \in \mathcal{T}_{\mathbb{T}^2}|\ B_r(T) \text{ is embedded w. torus w. a hole and disks}\}\right) = 1$$

because of Corollary 5.6.

Taking the intersection of the examined events for all r we get:

$$\tau^{\mathbb{T}^2}\left(\{T \in \mathcal{T}_{\mathbb{T}^2}|\ \forall r\ B_r(T) \text{ can be embedded w. torus w. a hole and disks}\}\right) = 1$$

And the above set of triangulations is $\mathcal{T}_{\mathbb{T}^2}^0$ itself. Indeed, suppose that $\forall r\ B_r(T)$ can be embedded as wished. Similarly to the proof of Proposition 2.1, using König's lemma we can select the boundary components of each $B_r(T)$ and the outer face of $B_r(T)$ that is a torus with a hole such that the selections are compatible. Just as in Proposition 2.1 we can then create the embeddings of each $B_r(T)$ one by one such that the embeddings extend each other. This embeds the whole support of T as wished showing that it is in $\mathcal{T}_{\mathbb{T}^2}^0$. $\qquad \square$

5.3   One-endedness

We will first prove one-endedness on the sphere.

Lemma 5.8. $\tau^{\mathbb{S}^2}\left(\{T \in \mathcal{T}_{\mathbb{S}^2}|\ T \text{ is one-ended}\}\right) = 1$

Proof. Let's suppose that $\tau^{\mathbb{S}^2}\left(\{T \in \mathcal{T}_{\mathbb{S}^2}|\ T \text{ is one-ended}\}\right) < 1$. That means that if we choose triangulation T randomly by $\tau^{\mathbb{S}^2}$ then with positive probability there is a finite sub-triangulation T' of T such that T' cuts T into more than one infinite components. Then if we take a large enough r then $B_r(T)$ contains T' and thus $B_r(T)$ cuts T into more than one infinite components.

$$\tau^{\mathbb{S}^2}\left(\{T \in \mathcal{T}_{\mathbb{S}^2} | \ \exists r \text{ s. t. } B_r(T) \text{ cuts } T \text{ into} > 1 \ \infty \text{ component}\}\right) > 0$$

and since the set of possible values of r is countable there is a single r such that

$$\tau^{\mathbb{S}^2}\left(\{T \in \mathcal{T}_{\mathbb{S}^2} | \ B_r(T) \text{ cuts } T \text{ into} > 1 \ \infty \text{ component}\}\right) > 0$$

Furthermore since the set of possible r-balls for a fixed r is countable there is a fixed r-ball B_r such that

$$\tau^{\mathbb{S}^2}\left(\{T \in \mathcal{T}_{\mathbb{S}^2} | \ B_r(T) = B_r \text{ and it cuts } T \text{ into} > 1 \ \infty \text{ component}\}\right) > 0$$

We can overestimate this value by fixing an $a \in \mathbb{Z}^+$ and looking at the probability that B_r cuts T into more than one component with more than a vertices. Let us fix a value of a. We want to use

$$\lim_{n \to \infty} \tau_n^{\mathbb{S}^2}\left(\{T \in \mathcal{T}_{\mathbb{S}^2} | \ B_r(T) = B_r \text{ and it cuts } T \text{ into} > 1 \text{ comp. with} > a \text{ vert.}\}\right)$$
$$= \tau^{\mathbb{S}^2}\left(\{T \in \mathcal{T}_{\mathbb{S}^2} | \ B_r(T) = B_r \text{ and it cuts } T \text{ into} > 1 \text{ comp. with} > a \text{ vert.}\}\right)$$

Indeed, whether T is in the above subset can be determined from a large enough ball, for example from $B_{r+a+1}(T)$. This is because on the sphere all outer faces of $B_r(T)$ are disks and are thus only connected to one boundary component of $B_r(T)$. Thus two components of $B_{r+a+1}(T) \backslash B_r(T)$ cannot get connected in $T \backslash B_r(T)$. Furthermore, if the number of vertices of a component of $B_{r+a+1}(T) \backslash B_r(T)$ is not more than a then it will not increase any more if we look at greater balls. All we need to check is whether there are at least two components of $B_{r+a+1}(T) \backslash B_r(T)$ with more than a vertices.

Thus the above set is a disjoint countable union of metric balls, those of the form $\{T \in \mathcal{T}_{\mathbb{S}^2} | \ B_{r+a+1}(T) = B_{r+a+1}\}$ where B_{r+a+1} is an $(r + a + 1)$-ball such that $B_{r+a+1} \backslash B_r(B_{r+a+1})$ has more than one components with more than a vertices. Thus, using Proposition 2.3 we get that $\tau_n^{\mathbb{S}^2}$ converges to $\tau^{\mathbb{S}^2}$ on the union of the balls.

Let the number of boundary components of B_r be h (boundary components are now well defined because of Proposition 5.3) and the number of vertices on each (counted with multiplicity) be $\underline{k} = (k_1, k_2, \ldots, k_h)$. We denote the total number of vertices of B_r by l. Then the number of triangulations of the sphere with n vertices in total, with r-ball B_r and the first two outer faces of B_r being triangulated with more then a inner vertices is

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$$\sum_{\substack{n_1 > a+k_1, n_2 > a+k_2 \\ n_i \geq k_i \ (i=3...h) \\ n_1-k_1+\cdots+n_h-k_h+l=n}} S_{n_1,k_1} \cdot S_{n_2,k_2} \ldots S_{n_h,k_h}$$

$$\leq \sum_{\substack{n_1 > a, n_2 > a \\ n_i \geq k_i \ (i=3...h) \\ n_1+\cdots+n_h=n+\sum k_i - l}} \prod_{i=1}^{h} C(k_i) n_i^{-\frac{5}{2}} (12\sqrt{3})^{n_i}$$

$$\leq C_1(\underline{k}, l)(12\sqrt{3})^n \sum_{\substack{n_1 > a, n_2 > a \\ n_i \geq 1 \ (i=3...h) \\ n_1+\cdots+n_h=n+\sum k_i - l}} \prod_{i=1}^{h} n_i^{-\frac{5}{2}}$$

$$\leq C_2(\underline{k}, l)(12\sqrt{3})^n a^{-\frac{3}{2}} \left(n + \sum k_i - l\right)^{-\frac{5}{2}}$$

$$\leq C_3(\underline{k}, l)(12\sqrt{3})^n a^{-\frac{3}{2}} n^{-\frac{5}{2}}$$

where at the end we used Lemma 5.2.

There are at most $\binom{h}{2}$ possible selections of the outer faces of B_r that we fix to have more than a inner vertices and h is also determined by $\underline{k}$. Thus the number of triangulations of the sphere with n vertices in total, with r-ball B_r and at least two outer faces of B_r being triangulated with more then a inner vertices is still at most $C_4(\underline{k}, l)(12\sqrt{3})^n a^{-\frac{3}{2}} n^{-\frac{5}{2}}$ with some constant $C_4(\underline{k}, l)$ that only depends on B_r. Thus

$$\tau_n^{\mathbb{S}^2} \left(\{T \in \mathcal{T}_{\mathbb{S}^2} | \ B_r(T) = B_r \text{ and it cuts } T \text{ into} > 1 \text{ comp. with} > a \text{ vert.}\}\right)$$
$$\leq \frac{C_1(\underline{k}, l) n^{-\frac{5}{2}} (12\sqrt{3})^n a^{-\frac{3}{2}}}{C_2 n^{-\frac{5}{2}} (12\sqrt{3})^n} \leq C_3(\underline{k}, l) a^{-\frac{3}{2}}$$

But this means that this holds for the limit too:

$$\tau^{\mathbb{S}^2} \left(\{T \in \mathcal{T}_{\mathbb{S}^2} | \ B_r(T) = B_r \text{ and it cuts } T \text{ into} > 1 \text{ comp. with} > a \text{ vert.}\}\right) \leq C(\underline{k}, l) a^{-\frac{3}{2}}$$

And thus

$$0 < \tau^{\mathbb{S}^2} \left(\{T \in \mathcal{T}_{\mathbb{S}^2} | \ B_r(T) = B_r \text{ and it cuts } T \text{ into} > 1 \ \infty \text{ component}\}\right) \leq C(\underline{k}, l) a^{-\frac{3}{2}}$$

But this should hold for all a, which is not possible. This proves that $\tau^{\mathbb{S}^2}$ is almost surely one-ended. $\qquad\square$

Let us now prove the same result for the torus.

Lemma 5.9. $\tau^{\mathbb{T}^2}\left(\{T \in \mathcal{T}_{\mathbb{T}^2} | \ T \ is \ one\text{-}ended\}\right) = 1$

Proof. Much of the proof is the same as in Lemma 5.8. Again, we suppose the above does not hold and this means that there is a fixed r-ball B_r such that

$$\tau^{\mathbb{T}^2}\left(\{T \in \mathcal{T}_{\mathbb{T}^2} | \ B_r(T) = B_r \text{ and it cuts } T \text{ into} > 1 \ \infty \text{ component}\}\right) > 0$$

Again, we can overestimate this value by fixing an $a \in \mathbb{Z}^+$ and looking at the probability that B_r cuts T into more than one component with more than a vertices. We fix a value of a.

We again want to use

$$\lim_{n \to \infty} \tau_n^{\mathbb{T}^2}\left(\{T \in \mathcal{T}_{\mathbb{T}^2} | \ B_r(T) = B_r \text{ and it cuts } T \text{ into} > 1 \text{ comp. with} > a \text{ vert.}\}\right)$$
$$= \tau^{\mathbb{T}^2}\left(\{T \in \mathcal{T}_{\mathbb{T}^2} | \ B_r(T) = B_r \text{ and it cuts } T \text{ into} > 1 \text{ comp. with} > a \text{ vert.}\}\right)$$

The reasoning is similar except that on the torus it is possible that some components of $B_{r+a+1}(T) \backslash B_r(T)$ get connected in T if one of the outer faces of $B_r(T)$ is a tube. But using Lemma 5.7 we can still say

$$\tau^{\mathbb{T}^2}\left(\{T \in \mathcal{T}_{\mathbb{T}^2} | \ B_r(T) = B_r \text{ and it cuts } T \text{ into} > 1 \text{ comp. with} > a \text{ vert.}\}\right)$$
$$= \tau^{\mathbb{T}^2}\left(\{T \in \mathcal{T}_{\mathbb{T}^2} | \ B_{r+a+1}(T) \backslash B_r(T) \text{ has} > 1 \text{ comp. with} > a \text{ vert.}\}\right)$$

since the two conditions are equivalent on $\mathcal{T}_{\mathbb{T}^2}^0$ and the remaining cases are negligible. Indeed, since $T \in \mathcal{T}_{\mathbb{T}^2}^0$ can be embedded such that all outer faces of $B_r(T)$ are only connected to one boundary component, the components of $B_{r+a+1}(T) \backslash B_r(T)$ cannot get connected in T.

Thus the above event is still a disjoint, countable union of metric balls, we can switch to the limit.

Furthermore, we know that the boundary components of B_r are well defined. This is because we know by Lemma 5.7 that $\tau^{\mathbb{T}^2}\left(\{T \in \mathcal{T}_{\mathbb{T}^2} | \ B_r(T) = B_r\}\right)$ can only be positive if B_r can be embedded such that one of its outer faces is a torus and others are disks. If we

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cut the torus with a hole and replace it with a disk we get an embedding of $B_r(T)$ into the sphere. We can thus apply Lemma 5.3 to say that the boundary components of $B_r(T)$ are well defined.

We want to estimate

$$\lim_{n\to\infty} \tau_n^{\mathbb{T}^2}\left(\{T \in \mathcal{T}_{\mathbb{T}^2}|\ B_r(T) = B_r \text{ and it cuts } T \text{ into} > 1 \text{ comp. with} > a \text{ vert.}\}\right)$$

From Corollary 5.6 we know that in the limit the probability of cases other than the outer faces of $B_r(T)$ being a torus with a hole and disks is zero. Furthermore, if the outer face that is a torus with a hole is triangulated with at most a vertices then the outer faces of a slightly larger ball with fixed radius, $B_{r+a+1}(T)$ for example, are all disks. Again, from Corollary 5.6

$$\lim_{n\to\infty} \tau_n^{\mathbb{T}^2}\left(\{T \in \mathcal{T}_{\mathbb{T}^2}|\ \text{ outer faces of } B_{r+a+1}(T) \text{ are disks}\}\right) = 0$$

thus the limit we want to estimate is in fact

$$\lim_{n\to\infty} \tau_n^{\mathbb{T}^2}\left(\{T \in \mathcal{T}_{\mathbb{T}^2}|\ B_r(T) = B_r,\right.$$

$$\left. \text{outer faces are torus w. hole and disks, torus w. hole and a disk has} > a \text{ vert.}\}\right)$$

Again, let the number of boundary components of B_r be h and the number of vertices on each (counted with multiplicity) be $\underline{k} = (k_1, k_2, \ldots, k_h)$. We denote the total number of vertices of B_r by l. Then the number of triangulations of the torus with n vertices in total, with r-ball B_r, the first outer face of B_r being a torus with a hole, the second being triangulated with more then a inner vertices is

$$\sum_{\substack{n_1>a_1+k_1,n_2>a+k_2 \\ n_i\geq k_i \ (i=3...h) \\ n_1-k_1+\cdots+n_h-k_h+l=n}} T_{n_1,k_1}\cdot S_{n_2,k_2}\ldots S_{n_h,k_h}$$

$$\leq \sum_{\substack{n_1>a,n_2>a \\ n_i\geq k_i \ (i=3...h) \\ n_1+\cdots+n_h=n+\sum k_i-l}} C_1(k_1)(12\sqrt{3})^{n_1}\prod_{i=2}^{h}C_2(k_i)n_i^{-\frac{5}{2}}(12\sqrt{3})^{n_i}$$

$$\leq C_3(\underline{k},l)(12\sqrt{3})^n \sum_{\substack{n_1,n_2>a \\ n_i\geq 1 \ (i=3...h) \\ n_1+\cdots+n_h=n+\sum k_i-l}} \prod_{i=2}^{h}n_i^{-\frac{5}{2}}$$

$$= C_3(\underline{k},l)(12\sqrt{3})^n \sum_{\substack{n_1,n_2>a \\ n_i\geq 1 \ (i=3...h) \\ n_1+\cdots+n_h=n+\sum k_i-l}} n_1^{\frac{5}{2}}\prod_{i=1}^{h}n_i^{-\frac{5}{2}}$$

$$\leq C_3(\underline{k},l)(12\sqrt{3})^n n^{\frac{5}{2}} \sum_{\substack{n_1,n_2>a \\ n_i\geq 1 \ (i=3...h) \\ n_1+\cdots+n_h=n+\sum k_i-l}} \prod_{i=1}^{h}n_i^{-\frac{5}{2}}$$

$$\leq C_4(\underline{k},l)(12\sqrt{3})^n n^{\frac{5}{2}} a^{-\frac{3}{2}}\left(n+\sum k_i-l\right)^{-\frac{5}{2}}$$

$$\leq C_5(\underline{k},l)(12\sqrt{3})^n a^{-\frac{3}{2}}$$

where at the end we used Lemma 5.2.

There are at most $h(h-1)$ possible selections of the outer face that is a torus with a hole and of the disk that is triangulated with more than a vertices and h is also determined by $\underline{k}$. Thus the number of triangulations of the torus with n vertices in total, with r-ball B_r and with one outer face of B_r being a torus with a hole with more than a vertices and others being disks, one being triangulated with more then a inner vertices is still at most $C_6(\underline{k},l)(12\sqrt{3})^n a^{-\frac{3}{2}}$ with some constant $C_6(\underline{k},l)$ that only depends on B_r. Thus

$$\lim_{n\to\infty}\tau_n^{\mathbb{T}^2}\left(\{T\in\mathcal{T}_{\mathbb{T}^2}|\ B_r(T)=B_r \text{ and it cuts } T \text{ into } >1 \text{ comp. with } >a \text{ vert.}\}\right)$$

$$\leq \frac{C_1(\underline{k},l)(12\sqrt{3})^n a^{-\frac{3}{2}}}{C_2(12\sqrt{3})^n}\leq C_3(\underline{k},l)a^{-\frac{3}{2}}$$

But this means that this holds for the limit too:

$$\tau^{\mathbb{T}^2}\left(\{T \in \mathcal{T}_{\mathbb{T}^2}|\ B_r(T) = B_r \text{ and it cuts } T \text{ into } > 1 \text{ comp. with } > a \text{ vert.}\}\right) \leq C(\underline{k}, l)a^{-\frac{3}{2}}$$

And thus

$$0 < \tau^{\mathbb{T}^2}\left(\{T \in \mathcal{T}_{\mathbb{T}^2}|\ B_r(T) = B_r \text{ and it cuts } T \text{ into } > 1 \infty \text{ component}\}\right) \leq C(\underline{k}, l)a^{-\frac{3}{2}}$$

But this should hold for all a, which is not possible. This proves that $\tau^{\mathbb{T}^2}$ is almost surely one-ended.

$\square$

5.4 Conclusion

Theorem 5.10. *A triangulation chosen by $\tau^{\mathbb{S}^2}$ is almost surely a triangulation of the plane. That is, $\tau^{\mathbb{S}^2}\left(\{T \in \mathcal{T}_{\mathbb{S}^2}|\ S(T) \approx \mathbb{R}^2\}\right) = 1$*

Proof. First of all, a triangulation chosen by $\tau^{\mathbb{S}^2}$ almost surely has a triangle incident to both sides of each edge, that is, it has no embedding with an outer face incident to an edge. This is because if T has an edge that is not incident to a triangle on both sides then this can be seen from a large enough r-ball. The probability of triangulations with such r-balls is zero by $\tau_n^{\mathbb{S}^2}$ thus it is zero by $\tau^{\mathbb{S}^2}$ as well.

It is similarly easy to see that a triangulation chosen by $\tau^{\mathbb{S}^2}$ is almost surely infinite.

Thus, a triangulation chosen by $\tau^{\mathbb{S}^2}$ is almost surely infinite, one-ended (Lemma 5.8) and has a triangle incident on both sides of each edge. We claim that such a triangulation is a triangulation of the plane.

Let us take such a triangulation T and we define $B'_r(T)$ as follows: by Lemma 5.3 the outer faces of $B_r(T)$ are well defined, they are all disks and their boundary components are well-defined proper cycles. Since T is one-ended, all except one outer face has a finite number of vertices. Since both sides of each edge are incident to a triangle the finite outer faces are completely triangulated. Let $B'_r(T)$ be the triangulation we get from $B_r(T)$ if we also take the triangulation of all its finite, completely triangulated outer faces. The boundary of $B'_r(T)$ is a single proper cycle, the boundary of the single infinite outer face of $B_r(T)$. Thus the support $S(B'_r(T))$ is homeomorphic to a closed disk. Furthermore, all the boundary vertices of $B'_r(T)$ are r away from the root, thus the boundary of $B'_r(T)$ and

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$B'_{r+1}(T)$ are completely disjoint.

The support $S(T)$ is thus the union of a chain of closed disks with disjoint boundaries. That is homeomorphic to a plane. $\qquad\square$

Theorem 5.11. *A triangulation chosen by $\tau^{\mathbb{T}^2}$ is almost surely a triangulation of the plane. That is, $\tau^{\mathbb{T}^2}\left(\{T \in \mathcal{T}_{\mathbb{T}^2} \mid S(T) \approx \mathbb{R}^2\}\right) = 1$*

Proof. The proof is almost the same as that of Theorem 5.10. All we have to add is that a triangulation chosen by $\tau^{\mathbb{T}^2}$ is almost surely in $\mathcal{T}_{\mathbb{T}^2}^0$ (see Lemma 5.7) (as well as fulfilling all other requirements in the previous proof). That is, it can be embedded such that all of its r-balls have an outer face homeomorphic to a torus with a hole and the others are disks.

Thus, when defining $B'_r(T)$ the single, infinite outer face must be a torus with a hole, otherwise for a large enough q the ball $B_q(T)$ that contains the whole of $B'_r(T)$ would only have outer faces homeomorphic to a disk. If we cut the outer face of $B'_r(T)$ and replace it with a disk, we get a triangulation on the sphere, thus, using Lemma 5.3 the single boundary of $B'_r(T)$ is well defined, is a proper cycle and the support $S(B'_r(T))$ is homeomorphic to a closed disk.

The rest of the proof goes as that of Theorem 5.10. $\qquad\square$

6 Taking the limit

Theorem 6.1. *$\tau^{\mathbb{S}^2}$ and $\tau^{\mathbb{T}^2}$ define the same distribution on planar triangulations.*

Proof. From Theorems 5.10 and 5.11 we know that both distributions are essentially defined on planar triangulations. Since the measurable space on planar triangulations is generated by metric balls it is enough to show that for each possible combinatorial r-ball of a triangulation of the plane B_r

$$\tau^{\mathbb{S}^2}\left(\{T \in \mathcal{T}_{\mathbb{S}^2} \mid B_r(T) = B_r\}\right) = \tau^{\mathbb{T}^2}\left(\{T \in \mathcal{T}_{\mathbb{T}^2} \mid B_r(T) = B_r\}\right)$$

We know that the boundary components of B_r are well-defined and almost surely all finite outer faces of $B_r(T)$ must be disks. There is a finite number of ways we can select the boundary component that the infinite face is attached to and a countable number of ways we can finitely triangulate all other faces. If we triangulate the finite outer faces of B_r we get a triangulation with a single boundary that is a proper cycle. Thus, the above event

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can be subdivided into a countable number of disjoint events such that it is enough to show that

$$\tau^{\mathbb{S}^2}\left(\{T \in \mathcal{T}_{\mathbb{S}^2} | \, D \subset T \text{ (as rooted triang.)}\}\right) = \tau^{\mathbb{T}^2}\left(\{T \in \mathcal{T}_{\mathbb{T}^2} | \, D \subset T \text{ (as rooted triang.)}\}\right)$$

where D is a finite triangulation on the plane (or equivalently on the sphere or on the torus) with support homeomorphic to a closed disk. $D \subset T$ can be decided from a large enough r-ball, where r depends only on D. Thus we can get $\tau^{\mathbb{S}^2}$ and $\tau^{\mathbb{T}^2}$ from the limit. Let's denote the number of inner vertices of D by l and the number of boundary vertices by k. The number of triangulations T of the sphere with n vertices in total such that $D \subset T$ is the number of ways we can triangulate the single outer face, a disk, with k vertices on the boundary and $n-l$ vertices in total. Similarly, the number of triangulations T of the torus with n vertices in total such that $D \subset T$ is the number of ways we can triangulate the single outer face, a torus with a hole, with k vertices on the boundary and $n-l$ vertices in total.

Thus, using the enumeration results in Section 4 we get that

$$\tau^{\mathbb{S}^2}\left(\{T \in \mathcal{T}_{\mathbb{S}^2} | \, D \subset T\}\right) = \lim_{n \to \infty} \tau_n^{\mathbb{S}^2}\left(\{T \in \mathcal{T}_{\mathbb{S}^2} | \, D \subset T\}\right) = \lim_{n \to \infty} \frac{S_{(n-l),k}}{S_n}$$

$$= \lim_{n \to \infty} \frac{\dfrac{k\binom{2k}{k}}{2^{\frac{4k+5}{2}} 3^{\frac{k+4}{2}} \sqrt{\pi}} \cdot (n-l)^{-\frac{5}{2}} (12\sqrt{3})^{n-l}}{\dfrac{1}{2^{\frac{7}{2}} \cdot 3^{\frac{5}{2}} \cdot \sqrt{\pi}} \cdot n^{-\frac{5}{2}} (12\sqrt{3})^n} = \frac{k\binom{2k}{k}}{2^{2k-1} 3^{\frac{k-1}{2}} (12\sqrt{3})^l}$$

and

$$\tau^{\mathbb{T}^2}\left(\{T \in \mathcal{T}_{\mathbb{T}^2} | \, D \subset T\}\right) = \lim_{n \to \infty} \tau_n^{\mathbb{T}^2}\left(\{T \in \mathcal{T}_{\mathbb{T}^2} | \, D \subset T\}\right) = \lim_{n \to \infty} \frac{S_{(n-l),k}}{S_n}$$

$$= \lim_{n \to \infty} \frac{\dfrac{k\binom{2k}{k}}{2^{2k+2} \cdot 3^{\frac{k-1}{2}}} \cdot (12\sqrt{3})^{n-l}}{\dfrac{1}{8} \cdot (12\sqrt{3})^n} = \frac{k\binom{2k}{k}}{2^{2k-1} 3^{\frac{k-1}{2}} (12\sqrt{3})^l}$$

$\square$

7 The number of triangulations of the torus with a hole

In this section we aim to determine $\widetilde{T}_{m,k}$, the number of triangulations of the torus with a hole with k edges on the boundary of the hole and m edges in total. We define a formal multivariate power series, the generating function of $\widetilde{T}_{m,k}$:

Definition 7.1.
$$\widetilde{T}(x,y) = \sum_{m,k \geq 0} \widetilde{T}_{m,k} x^m y^k$$

The following result and the key elements of its proof are presented in [2].

Theorem 7.1.
$$\widetilde{T}(x,y) = \frac{(1 - 16h^5 y)h^5 y}{(1 - 4h^3)^2 (1 - 4h^2 y)^{5/2}}$$

where h is a formal power series in x satisfying

$$h(x) = x\sqrt{1 + 8h^3(x)}$$

Remark. In fact the explicit formula for h is also calculated in [2].

$$h(x) = \sum_{k \geq 0} \frac{4^k (3k-1)!!}{k!(k+1)!!} x^{3k+1}$$

All we need from the exact coefficients is that $[x^m]h(x) = 0$ for all $m \not\equiv 1 \ (3)$ and $[x]h(x) \neq 0$.

We will apply the above theorem, which involves the implicitly defined power series h, to extract coefficients of $\widetilde{T}(x,y)$ using the Lagrange theorem, which appears in Section 1.2 of [3].

Theorem 7.2 (Lagrange Theorem for Implicit Functions). *Let $\phi(\lambda)$ be a formal power series in λ with $\phi(0) \neq 0$. Then there is a unique formal power series $w(t)$ with $w(0) = 0$ that satisfies $w(t) = t\phi(w(t))$.*

Furthermore, if $f(\lambda)$ is any other formal power series in λ with λ^k being the smallest term with a nonzero coefficient then we can express the coefficients of $f(w(t))$ in the following way:

$$[t^n]f(w(t)) = \begin{cases} \frac{1}{n}[\lambda^{n-1}]\{f'(\lambda)\phi^n(\lambda)\} & \text{for } n \neq 0, n \geq k \\ [\lambda^0]f(\lambda) & \text{for } n = 0 \end{cases}$$

Remark. In [3] a slightly more general version of the theorem is stated since f is allowed to be any Laurent-series.

We first have to find $[y^k]\widetilde{T}(x,y)$ in terms of h. We will start that by expressing $\frac{1}{(1-4h^2y)^{5/2}}$ as a power series in y.

$$
\begin{aligned}
\frac{1}{(1-4h^2y)^{5/2}} &= \sum_{k\geq 0} \binom{-\frac{5}{2}}{k}(-4)^k \cdot h^{2k} \cdot y^k \\
&= \sum_{k\geq 0} \frac{(-1)^k \cdot (2k+3)!!}{3 \cdot 2^k \cdot k!} \cdot (-4)^k \cdot h^{2k} \cdot y^k \\
&= \sum_{k\geq 0} \frac{(2k+3)!}{2^{(k+1)} \cdot (k+1)! \cdot 3 \cdot 2^k \cdot k!} \cdot 4^k \cdot h^{2k} \cdot y^k \\
&= \sum_{k\geq 0} \frac{1}{3} \cdot \frac{(2k+3)(2k+2)(2k+1)}{2(k+1)} \cdot \binom{2k}{k} \cdot h^{2k} \cdot y^k \\
&= \sum_{k\geq 0} \frac{(2k+3)(2k+1)}{3} \cdot \binom{2k}{k} \cdot h^{2k} \cdot y^k
\end{aligned}
$$

Thus we can express $[y^k]\widetilde{T}(x,y)$ in terms of h.

$$
\widetilde{T}(x,y) = \sum_{k\geq 0} \frac{(2k+3)(2k+1)}{3} \cdot \binom{2k}{k} \cdot \left[\frac{h^{2k+5}}{(1-4h^3)^2} \cdot y^{k+1} - \frac{16 \cdot h^{2k+10}}{(1-4h^3)^2} \cdot y^{k+2}\right] =
$$

$$
\sum_{k\geq 1} \frac{1}{3}\left[(2k+1)(2k-1)\binom{2k-2}{k-1}\frac{h^{2k+3}}{(1-4h^3)^2} - 16(2k-1)(2k-3)\binom{2k-4}{k-2}\frac{h^{2k+6}}{(1-4h^3)^2}\right]y^k
$$

$$(7)$$

Now we need to find the coefficients $[x^m]\frac{h^l(x)}{(1-4h^3(x))^2}$ for $l \in \mathbb{N}$.

Since $[x^m]h(x) = 0$ for all $m \not\equiv 1\ (3)$ and $[x]h(x) \neq 0$, the substitution $h(x) = x\sqrt{1+\zeta(x^3)}$ suggested in [2], is allowed, and with $t = x^3$

$$
\zeta(t) = 8t(1+\zeta(t))^{3/2}
$$

Remark. It would also be possible to get a formula for $[x^m]\frac{h^l(x)}{(1-4h^3(x))^2}$ using the Lagrange theorem with $w = h$ and $f(\lambda) = \frac{\lambda^l}{(1-4\lambda^3)^2}$ but the calculations turn out to be simpler with the above substitution.

$$[x^m]\frac{h^l}{(1-4h^3)^2} = [x^m]\frac{x^l(1+\zeta)^{\frac{l}{2}}}{(1-4x^3(1+\zeta)^{\frac{3}{2}})^2} = [x^{m-l}]\frac{(1+\zeta)^{\frac{l}{2}}}{(1-4t(1+\zeta)^{\frac{3}{2}})^2} = [x^{m-l}]\frac{(1+\zeta)^{\frac{l}{2}}}{(1-\frac{\zeta}{2})^2}$$

This is 0 if $m \not\equiv l$ (3), otherwise let $p = \frac{m-l}{3}$

$$[x^m]\frac{h^l}{(1-4h^3)^2} = [t^p]\frac{(1+\zeta)^{\frac{l}{2}}}{(1-\frac{\zeta}{2})^2}$$

We can now apply the Lagrange theorem with $w = \zeta$, $\phi(\lambda) = 8(1+\lambda)^{\frac{3}{2}}$ and $f(\lambda) = \frac{(1+\lambda)^{\frac{l}{2}}}{(1-\frac{\lambda}{2})^2}$.
For $p \geq 1$

$$
\begin{aligned}
[t^p]\frac{(1+\zeta)^{\frac{l}{2}}}{(1-\frac{\zeta}{2})^2} &= \frac{1}{p}[\lambda^{p-1}]\left[\left(\frac{(1+\lambda)^{\frac{l}{2}}}{(1-\frac{\lambda}{2})^2}\right)' \cdot 8^p \cdot (1+\lambda)^{\frac{3p}{2}}\right] \\
&= \frac{1}{p}[\lambda^{p-1}]\left[\frac{(1-\frac{\lambda}{2})^2\frac{l}{2}(1+\lambda)^{\frac{l}{2}-1} + (1+\lambda)^{\frac{l}{2}}(1-\frac{\lambda}{2})}{(1-\frac{\lambda}{2})^4} \cdot 8^p \cdot (1+\lambda)^{\frac{3p}{2}}\right] \\
&= \frac{8^p}{p}[\lambda^{p-1}]\left[\frac{l}{2}\frac{(1+\lambda)^{\frac{3p+l-2}{2}}}{(1-\frac{\lambda}{2})^2} + \frac{(1+\lambda)^{\frac{3p+l}{2}}}{(1-\frac{\lambda}{2})^3}\right]
\end{aligned}
$$

The coefficients in the above formula can be expressed as a sum and thus we get that if $m \equiv l$ (3) then with $p = \frac{m-l}{3}$

$$[x^m]\frac{h^l}{(1-4h^3)^2} = \begin{cases} \frac{8^p}{p}\left[\frac{l}{2}\sum_{i=0}^{p-1}\frac{i+1}{2^i}\binom{\frac{3p+l-2}{2}}{p-i-1} + \frac{1}{2}\sum_{i=0}^{p-1}\frac{(i+1)(i+2)}{2^i}\binom{\frac{3p+l}{2}}{p-i-1}\right] & \text{if } m > l \\ 1 & \text{if } m = l \end{cases} \qquad (8)$$

This combined with (7) gives a formula for $\widetilde{T}_{m,k}$ when $m \equiv 2k$ (3) and $m \geq 2k+3$. Otherwise $\widetilde{T}_{m,k} = 0$.

Remark. The arguments of the binomial terms in the above formula are not necessarily integers. However, for $0 \leq k \leq n \in \mathbb{R}$ we can define $\binom{n}{k}$ as

$$\frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)}$$

and (8) still holds.

8 Asymptotic estimation of the number of triangulations of the torus with a hole

In this section we aim to give an asymptotic estimation of $\widetilde{T}_{m,k}$ as $m \to \infty$. Since $\widetilde{T}_{m,k}$ can only be positive for $m \equiv 2k$ (3) we are only estimating these values.

Since k is now fixed we need to examine the asymptotic behaviour of

$$[x^m]\frac{h^l}{(1 - 4h^3)^2}$$

That is, considering (8), we have to estimate

$$\frac{8^p}{p}\left[\frac{l}{2}\sum_{i=0}^{p-1}\frac{i+1}{2^i}\binom{\frac{3p+l-2}{2}}{p-i-1} + \frac{1}{2}\sum_{i=0}^{p-1}\frac{(i+1)(i+2)}{2^i}\binom{\frac{3p+l}{2}}{p-i-1}\right] \tag{9}$$

as $p \to \infty$.

We are going to use some simple tools, taken from [5]:

Firstly, let's suppose we are given

$$\sum_i a_i^{(n)}$$

a sequence of finite series with $a_i^{(n)} \geq 0$ and we manage to find an estimation in the form $a_i^{(n)} \sim f_i^{(n)}$ $\forall i$ as $n \to \infty$. (We consider $a_i^{(n)}$ to be 0 when i is outside the range of summation.) This does not imply that

$$\sum_i a_i^{(n)} \sim \sum_i f_i^{(n)}$$

as $n \to \infty$.

However the implication does hold if $a_i^{(n)} \sim f_i^{(n)}$ uniformly as $n \to \infty$:

Proposition 8.1. *If*

$$a_i^{(n)} = f_i^{(n)}(1 + o(1))$$

where the estimation of the $o(1)$ term depends on n only and not on i then

$$\sum_i a_i^{(n)} \sim \sum_i f_i^{(n)} \text{ as } n \to \infty$$

Secondly, from Stirling's formula it follows that for $0 \leq k \leq n \in \mathbb{R}$

$$\binom{n}{k} = \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)}$$
$$= \left(\frac{n}{2\pi k(n-k)}\right)^{\frac{1}{2}} \left(\frac{n}{k}\right)^k \left(\frac{n}{n-k}\right)^{n-k} \left(1 + O\left(\frac{1}{k} + \frac{1}{n-k}\right)\right) \tag{10}$$

Thirdly, the Taylor series of $\log(1+x)$ in 0 gives that

$$\log(1+\varepsilon) = \varepsilon - \frac{\varepsilon^2}{2} + O(\varepsilon^3) \text{ as } \varepsilon \to 0 \tag{11}$$

And finally, the following proposition can be deduced easily from (10) and (11).

Proposition 8.2. *Let's suppose that s_n is a function of n such that $0 < s_n < n$. We want to estimate the binomial terms around $\binom{n}{s_n}$. If t_n is small enough, that is $t_n^2 = o(s_n)$ and $t_n^2 = o(n - s_n)$, then*

$$\binom{n}{s_n + t_n} \sim \binom{n}{s_n} \left(\frac{n - s_n}{s_n}\right)^{t_n}$$

Let's now start estimating (9) by examining

$$\sum_{i=0}^{p-1} \frac{(i+1)(i+2)}{2^i} \binom{\frac{3p+l}{2}}{p-i-1} \text{ as } p \to \infty$$

How do the terms behave for a fixed p? Let's look at the ratio of subsequent terms.

$$\frac{\frac{(i+2)(i+3)}{2^{i+1}}\binom{\frac{3p+l}{2}}{p-i-2}}{\frac{(i+1)(i+2)}{2^i}\binom{\frac{3p+l}{2}}{p-i-1}} = \frac{(i+3)(p-i-1)}{2(i+1)\left(\frac{p+l+2i+4}{2}\right)} = \frac{(i+3)(p-i-1)}{(i+1)(p+l+2i+4)}$$

Thus the terms are increasing from index i to index $i+1$ if and only if

$$(i+1)(p+l+2i+4) < (i+3)(p-i-1)$$
$$ip + p + il + l + 2i^2 + 2i + 4i + 4 < ip + 3p - i^2 - 3i - i - 3$$
$$3i^2 + (l+10)i - (2p - l - 7) < 0$$
$$i < \frac{-(l+10) + \sqrt{(l+10)^2 + 12(2p - l - 7)}}{6} \sim \sqrt{\frac{2p}{3}}$$

Thus the terms are unimodal for each p and the peak is asymptotically at $\sqrt{\frac{2p}{3}}$. Let's now estimate only a range $i \in [0, p^s]$ in the summation. If $\frac{1}{2} < s < 1$ then this includes all the biggest terms for large enough values of p. Using (10)

$$\sum_{i=0}^{p^s} \frac{(i+1)(i+2)}{2^i} \binom{\frac{3p+l}{2}}{p-i-1} \sim$$

$$\sum_{i=0}^{p^s} \frac{(i+1)(i+2)}{2^i} \sqrt{\frac{\frac{3p+l}{2}}{2\pi(p-i-1)(\frac{p+l}{2}+i+1)}} \left(\frac{\frac{3p+l}{2}}{p-i-1}\right)^{p-i-1} \left(\frac{\frac{3p+l}{2}}{\frac{p+l}{2}+i+1}\right)^{\frac{p+l}{2}+i+1} \tag{12}$$

This holds because the estimation given by (10) is uniform in i:

$$O\left(\frac{1}{p-i-1} + \frac{1}{\frac{p+l}{2}+i+1}\right) = O\left(\frac{1}{p-p^s-1} + \frac{1}{\frac{p}{2}}\right)$$

Furthermore, (12) can be rewritten as

$$\sum_{i=0}^{p^s} \left[\frac{(i+1)(i+2)}{2^i} \sqrt{\frac{3}{2\pi}} \sqrt{\frac{p+\frac{l}{3}}{(p-i-1)(p+l+2i+2)}} \left(\frac{3}{2}\right)^{p-i-1} 3^{\frac{p+l}{2}+i+1}. \right.$$

$$\left. \cdot \left(\frac{p+\frac{l}{3}}{p-i-1}\right)^{p-i-1} \left(\frac{\frac{p}{2}+\frac{l}{6}}{\frac{p+l}{2}+i+1}\right)^{\frac{p+l}{2}+i+1} \right] \tag{13}$$

Since

$$\sqrt{\frac{p+\frac{l}{3}}{(p-i-1)(p+l+2i+2)}} \sim \sqrt{\frac{1}{p}}$$

uniformly in i,

$$(13) \sim \sqrt{\frac{3}{2\pi p}} \frac{3^{\frac{3p+l}{2}}}{2^{p-1}} \sum_{i=0}^{p^s} (i+1)(i+2) \left(1+\frac{i+\frac{l}{3}+1}{p-i-1}\right)^{p-i-1} \left(1-\frac{i+\frac{l}{3}+1}{\frac{p+l}{2}+i+1}\right)^{\frac{p+l}{2}+i+1}$$

$$= \sqrt{\frac{3}{2\pi p}} \frac{3^{\frac{3p+l}{2}}}{2^{p-1}} \sum_{i=0}^{p^s} (i+1)(i+2)e^{\left((p-i-1)\log\left(1+\frac{i+\frac{l}{3}+1}{p-i-1}\right)+\left(\frac{p+l}{2}+i+1\right)\log\left(1-\frac{i+\frac{l}{3}+1}{\frac{p+l}{2}+i+1}\right)\right)} \tag{14}$$

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From (11).

$$(p - i - 1) \log \left(1 + \frac{i + \frac{l}{3} + 1}{p - i - 1} \right) = \left(i + \frac{l}{3} + 1 \right) - \frac{\left(i + \frac{l}{3} + 1 \right)^2}{2(p - i - 1)} + O \left(\frac{\left(i + \frac{l}{3} + 1 \right)^3}{(p - i - 1)^2} \right)$$

and

$$\left(\frac{p + l}{2} + i + 1 \right) \log \left(1 - \frac{i + \frac{l}{3} + 1}{\frac{p+l}{2} + i + 1} \right)$$

$$= - \left(i + \frac{l}{3} + 1 \right) - \frac{\left(i + \frac{l}{3} + 1 \right)^2}{2 \left(\frac{p+l}{2} + i + 1 \right)} + O \left(\frac{\left(i + \frac{l}{3} + 1 \right)^3}{\left(\frac{p+l}{2} + i + 1 \right)^2} \right)$$

If $s < \frac{2}{3}$ then the O terms above are $o(1)$ uniformly and thus

$$(14) \sim \sqrt{\frac{3}{2\pi p}} \frac{3^{\frac{3p+l}{2}}}{2^{p-1}} p^s \sum_{i=0}^{} (i+1)(i+2) e^{\left(-\frac{\left(i + \frac{l}{3} + 1 \right)^2}{2(p - i - 1)} - \frac{\left(i + \frac{l}{3} + 1 \right)^2}{2 \left(\frac{p+l}{2} + i + 1 \right)} \right)} \tag{15}$$

We will now approximate the exponent:

$$\frac{\left(i + \frac{l}{3} + 1 \right)^2}{2(p - i - 1)} = \frac{(i + 1)^2}{2p} + \left(\frac{\left(i + \frac{l}{3} + 1 \right)^2}{2(p - i - 1)} - \frac{(i + 1)^2}{2p} \right)$$

$$= \frac{(i + 1)^2}{2p} + \frac{p \left(i + \frac{l}{3} + 1 \right)^2 - (p - i - 1)(i + 1)^2}{2p(p - i - 1)}$$

$$= \frac{(i + 1)^2}{2p} + \frac{O(pi)}{2p^2 + o(p^2)} = \frac{(i + 1)^2}{2p} + o(1)$$

uniformly in i. Similarly

$$\frac{\left(i + \frac{l}{3} + 1 \right)^2}{2 \left(\frac{p+l}{2} + i + 1 \right)} = \frac{(i + 1)^2}{p} + o(1)$$

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uniformly in i. Thus

$$(15) \sim \sqrt{\frac{3}{2\pi p}} \frac{3^{\frac{3p+l}{2}}}{2^{p-1}} \sum_{i=0}^{p^s} (i+1)(i+2)e^{-\frac{3(i+1)^2}{2p}}$$

$$= \sqrt{\frac{3}{2\pi p}} \frac{3^{\frac{3p+l}{2}}}{2^{p-1}} \left(\sum_{i=0}^{p^s} (i+1)^2 e^{-\frac{3(i+1)^2}{2p}} + \sum_{i=0}^{p^s} (i+1)e^{-\frac{3(i+1)^2}{2p}} \right)$$

$$= \sqrt{\frac{3}{2\pi p}} \frac{3^{\frac{3p+l}{2}}}{2^{p-1}} \left(p^{\frac{3}{2}} \frac{1}{\sqrt{p}} \sum_{i=0}^{p^s} \left(\frac{i+1}{\sqrt{p}} \right)^2 e^{-\frac{3}{2}\left(\frac{i+1}{\sqrt{p}}\right)^2} + p \frac{1}{\sqrt{p}} \sum_{i=0}^{p^s} \left(\frac{i+1}{\sqrt{p}} \right) e^{-\frac{3}{2}\left(\frac{i+1}{\sqrt{p}}\right)^2} \right) \quad (16)$$

These sums are integral approximation sums with step size $\frac{1}{\sqrt{p}}$ and thus

$$(16) \sim \sqrt{\frac{3}{2\pi p}} \frac{3^{\frac{3p+l}{2}}}{2^{p-1}} \left(p^{\frac{3}{2}} \int_0^{\frac{p^s+1}{\sqrt{p}}} x^2 e^{-\frac{3}{2}x^2} \, \mathrm{d}x + p \int_0^{\frac{p^s+1}{\sqrt{p}}} x e^{-\frac{3}{2}x^2} \, \mathrm{d}x \right)$$

$$\sim \sqrt{\frac{3}{2\pi p}} \frac{3^{\frac{3p+l}{2}}}{2^{p-1}} \left(p^{\frac{3}{2}} \int_0^\infty x^2 e^{-\frac{3}{2}x^2} \, \mathrm{d}x + p \int_0^\infty x e^{-\frac{3}{2}x^2} \, \mathrm{d}x \right)$$

$$\sim \sqrt{\frac{3}{2\pi p}} \frac{3^{\frac{3p+l}{2}}}{2^{p-1}} \left(p^{\frac{3}{2}} \int_0^\infty x^2 e^{-\frac{3}{2}x^2} \, \mathrm{d}x \right)$$

$$= \sqrt{\frac{3}{2\pi}} \frac{3^{\frac{3p+l}{2}}}{2^{p-1}} p \left(\left[x \left(-\frac{1}{3} e^{-\frac{3}{2}x^2} \right) \right]_0^\infty - \int_0^\infty \left(-\frac{1}{3} \right) e^{-\frac{3}{2}x^2} \, \mathrm{d}x \right)$$

$$= \sqrt{\frac{3}{2\pi}} \frac{3^{\frac{3p+l}{2}}}{2^{p-1}} p \frac{1}{3\sqrt{3}} \left(\sqrt{3} \int_0^\infty e^{-\frac{(\sqrt{3}x)^2}{2}} \, \mathrm{d}x \right)$$

$$= \sqrt{\frac{3}{2\pi}} \frac{3^{\frac{3p+l}{2}}}{2^{p-1}} p \frac{1}{3\sqrt{3}} \int_0^\infty e^{-\frac{x^2}{2}} \, \mathrm{d}x$$

$$= \sqrt{\frac{3}{2\pi}} \frac{3^{\frac{3p+l}{2}}}{2^{p-1}} p \frac{1}{3\sqrt{3}} \sqrt{\frac{\pi}{2}} = 3^{\frac{l-2}{2}} \left(\frac{3^{\frac{3}{2}}}{2} \right)^p p$$

Hence

$$\sum_{i=0}^{p^s} \frac{(i+1)(i+2)}{2^i} \binom{\frac{3p+l}{2}}{p-i-1} \sim 3^{\frac{l-2}{2}} \left(\frac{3^{\frac{3}{2}}}{2} \right)^p p \text{ as } p \to \infty \text{ whenever } \frac{1}{2} < s < \frac{2}{3} \quad (17)$$

We will show that the rest of the terms are insignificant, ie. they do not change the asymptotic behaviour of the sum. We know already that the terms after $i = p^s$ are

monotonically decreasing. The first few terms after $i = p^s$ do not make a difference since (17) implies that for $\frac{1}{2} < s < \frac{2}{3}$ and $0 < t < s$

$$\sum_{i=0}^{p^s+p^t} \frac{(i+1)(i+2)}{2^i} \binom{\frac{3p+l}{2}}{p-i-1} \sim 3^{\frac{l-2}{2}} \left(\frac{3^{\frac{3}{2}}}{2}\right)^p p \text{ as } p \to \infty$$

We will show that the terms for $p^s + p^t < i < p$ are very small even compared to the single term $i = p^s$. We will estimate the term $i = p^s + p^t$ using Proposition 8.2. with $n = \frac{3p+l}{2}$, $s_n = p - p^s - 1$, $t_n = -p^t$. For Proposition 8.2. to be applicable we need $t_n^2 = o(s_n)$ and $t_n^2 = o(n - s_n)$, that is, $p^{2t} = o(p)$ thus we need to choose t such that $t < \frac{1}{2}$. With this choice

$$\frac{(p^s + p^t + 1)(p^s + p^t + 2)}{2^{p^s+p^t}} \binom{\frac{3p+l}{2}}{p - p^s - p^t - 1}$$

$$\sim \frac{(p^s + p^t + 1)(p^s + p^t + 2)}{2^{p^s+p^t}} \binom{\frac{3p+l}{2}}{p - p^s - 1} \left(\frac{\frac{p+l}{2} + p^s + 1}{p - p^s - 1}\right)^{p^t}$$

$$\sim \frac{(p^s + 1)(p^s + 2)}{2^{p^s}} \binom{\frac{3p+l}{2}}{p - p^s - 1} \left(\frac{1}{4}\right)^{p^t} \left(\frac{p + l + 2p^s + 2}{p - p^s - 1}\right)^{p^t}$$

$$= \frac{(p^s + 1)(p^s + 2)}{2^{p^s}} \binom{\frac{3p+l}{2}}{p - p^s - 1} o\left(\frac{1}{2^{p^t}}\right)$$

$$= \frac{(p^s + 1)(p^s + 2)}{2^{p^s}} \binom{\frac{3p+l}{2}}{p - p^s - 1} o\left(\frac{1}{p}\right)$$

Since all the terms with $p^s + p^t < i < p$ are less than the above and there is at most p of them

$$\sum_{i=p^s+p^t+1}^{p-1} \frac{(i+1)(i+2)}{2^i} \binom{\frac{3p+l}{2}}{p-i-1} = o\left(\frac{(p^s + 1)(p^s + 2)}{2^{p^s}} \binom{\frac{3p+l}{2}}{p - p^s - 1}\right)$$

$$= o\left(\sum_{i=0}^{p^s+p^t} \frac{(i+1)(i+2)}{2^i} \binom{\frac{3p+l}{2}}{p-i-1}\right)$$

So eventually we get that

$$\sum_{i=0}^{p-1} \frac{(i+1)(i+2)}{2^i} \binom{\frac{3p+l}{2}}{p-i-1} \sim 3^{\frac{l-2}{2}} \left(\frac{3^{\frac{3}{2}}}{2}\right)^p p \text{ as } p \to \infty \tag{18}$$

With very similar calculations we can show that the first sum in (8) is insignificant since it turns out that

$$\sum_{i=0}^{p-1} \frac{i+1}{2^i} \binom{\frac{3p+l-2}{2}}{p-i-1} \sim c(l) \left(\frac{3^{\frac{3}{2}}}{2}\right)^p \sqrt{p} \text{ as } p \to \infty \tag{19}$$

Putting (18) and (19) into (8) we get that with $m \equiv l$ (3) and $p = \frac{m-l}{3}$

$$[x^m] \frac{h^l}{(1-4h^3)^2} \sim \frac{8^p}{p} \left[\frac{1}{2} 3^{\frac{l-2}{2}} \left(\frac{3^{\frac{3}{2}}}{2}\right)^p p\right] = \frac{3^{\frac{l-2}{2}}}{2} \left(4 \cdot 3^{\frac{3}{2}}\right)^p$$

$$= \frac{1}{2^{\frac{2l}{3}+1} \cdot 3} \left(4 \cdot 3^{\frac{3}{2}}\right)^{\frac{m}{3}} = \frac{1}{2^{\frac{2l}{3}+1} \cdot 3} \left(\sqrt[3]{4} \cdot \sqrt{3}\right)^m \text{ as } m \to \infty \tag{20}$$

So finally we can estimate $\widetilde{T}_{m,k}$ as $m \to \infty$. From (7) we know that for $k \geq 1$

$$\widetilde{T}_{m,k} = \frac{1}{3} \left[(2k+1)(2k-1) \binom{2k-2}{k-1} [x^m] \frac{h^{2k+3}}{(1-4h^3)^2}\right.$$
$$\left. -16(2k-1)(2k-3) \binom{2k-4}{k-2} [x^m] \frac{h^{2k+6}}{(1-4h^3)^2}\right]$$

Thus for $m \equiv 2k$ (3)

$$\widetilde{T}_{m,k} \sim \frac{1}{3}\left[(2k+1)(2k-1)\binom{2k-2}{k-1}\frac{1}{2^{\frac{4k}{3}+3}\cdot 3}\left(\sqrt[3]{4}\cdot\sqrt{3}\right)^m \right.$$

$$\left. -16(2k-1)(2k-3)\binom{2k-4}{k-2}\frac{1}{2^{\frac{4k}{3}+5}\cdot 3}\left(\sqrt[3]{4}\cdot\sqrt{3}\right)^m \right]$$

$$= \frac{1}{2^{\frac{4k}{3}+1}\cdot 9}\left(\sqrt[3]{4}\sqrt{3}\right)^m\left[\frac{(2k+1)(2k-1)}{4}\binom{2k-2}{k-1} - (2k-1)(2k-3)\binom{2k-4}{k-2}\right]$$

$$= \frac{1}{2^{\frac{4k}{3}+1}\cdot 9}\left(\sqrt[3]{4}\sqrt{3}\right)^m\binom{2k}{k}\left[\frac{(2k+1)k^2}{4\cdot(2k)} - \frac{k^2(k-1)^2}{(2k)(2k-2)}\right]$$

$$= \frac{1}{2^{\frac{4k}{3}+1}\cdot 9}\left(\sqrt[3]{4}\sqrt{3}\right)^m\binom{2k}{k}\left[\frac{(2k+1)k}{4\cdot 2} - \frac{k(k-1)}{2\cdot 2}\right]$$

$$= \frac{1}{2^{\frac{4k}{3}+1}\cdot 9}\left(\sqrt[3]{4}\sqrt{3}\right)^m\binom{2k}{k}\left[\frac{(2k+1)k - 2k(k-1)}{8}\right]$$

$$= \frac{1}{2^{\frac{4k}{3}+1}\cdot 9}\left(\sqrt[3]{4}\sqrt{3}\right)^m\binom{2k}{k}\frac{3k}{8}$$

Theorem 8.3. *If $m \not\equiv 2k$ (3) then $\widetilde{T}_{m,k} = 0$. Otherwise*

$$\widetilde{T}_{m,k} \sim \frac{k\binom{2k}{k}}{2^{\frac{4k}{3}+4}\cdot 3}\left(\sqrt[3]{4}\sqrt{3}\right)^m \quad \text{as } m \to \infty$$

Using Proposition 4.1. this theorem yields an approximation for $T_{n,k}$ as well:

Corollary 8.4. *As $n \to \infty$*

$$T_{n,k} = \widetilde{T}_{3n-k+3,k} \sim \frac{k\binom{2k}{k}}{2^{\frac{4k}{3}+4}\cdot 3}\left(\sqrt[3]{4}\sqrt{3}\right)^{3n-k+3} = \frac{k\binom{2k}{k}}{2^{2k+2}\cdot 3^{\frac{k-1}{2}}}(12\sqrt{3})^n$$

Acknowledgments

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