DUAL-CRITICAL GRAPHS

by

Sándor Kisfaludi-Bak

Advisor: Zoltán Király

Department of Computer Science

Eötvös Loránd University

Contents

Introduction

This thesis explores an interesting graph class: the class of dual-critical graphs. Studying these graphs arise many exciting questions. One of them is determining the complexity of dual-criticality test. The basic definitions show that the problem is in NP, and a result of Christian Szegedy and Balázs Szegedy [4] provides a random polynomial algorithm, which relies on formal matrix rank computing. It is unknown whether dual-criticality test can be done in deterministic polynomial time, moreover the question of being in co-NP is also open.

Since there is no written material (up to this point) that deals directly with dual-critical graphs, I collected many propositions and theorems during discussions with my advisor. The majority of these were based on his ideas, but some of them came from others, as I tried to indicate them in the text. Naturally, some other questions arose while I was proving these propositions, for example the problem of finding splitting trees and their properties.

The first chapter introduces dual-critical graphs and their basic properties. We examine connectivity conditions, splitting trees, and the background of the terminology, which lies in planar dual-critical graphs. The second chapter deals with 3-regular graphs. The main theorem of the chapter shows that dual critical graphs coincide with many graph classes when restricted to 3-regular graphs. The following sections show further equivalences, and they yield a deterministic polynomial algorithm. The final chapter is a short description of known algorithms with some necessary conditions of non-dual-criticality that were inspired by the results of my implementation of the greedy algorithm.

I hope that further investigations will be done in this area, because any step could have important consequences in complexity theory.

Acknowledgement

I would like to thank my advisor for the fruitful discussions we had, the hard work and the immense help he provided.

Chapter 1

Dual-critical graphs in general

1.1 Basic properties of dual-critical graphs

Definition 1.1.1 (Acyclic orientation)**.** An orientation $\overrightarrow{G} = (V, \overrightarrow{E})$ of a graph G is called acyclic if it does not have directed cycles.

Definition 1.1.2 (Dual-critical graph)**.** A graph G is dual-critical if it has an acyclic orientation such that all vertices except one have an odd indegree.

Remark 1.1.3. In every acyclic orientation there is a source vertex v. It has an even indegree (0). Thus an orientation of a dual critical graph that satisfies the above conditions has exactly one vertex (the source) that has an even indegree, and $|V(G)| - 1$ vertices that have odd indegrees.

It is a known fact that acyclic graphs have a topological ordering. In a topological ordering the source vertex comes first. The orientation of the edges is determined by the order of their endpoints: the source of an arc always precedes its target. Let us take a topological ordering of an orientation described above. Beginning with the second vertex, every vertex has an odd number of predecessors to which it is connected. Consequently, the class of dual-critical graphs can be characterized as the graphs that can be built by taking a single vertex, and adding new vertices that have an odd degree. Such an ordering will be called a good ordering, and the orientation defined by a good ordering will be called good orientation.

Remark 1.1.4. It is easy to see that a good orientation of a dual-critical graph is rooted connected, with the source vertex as root. Indeed, starting at the second vertex in the good ordering, every vertex has at least one incoming arc. So one could construct a backward path from any vertex to the root by going back on incoming arcs.

Corollary 1.1.5. *The vertex and edge counts in a dual-critical graph have different parity: a graph that has one vertex has this property, and the addition of odd degree vertices preserves it.*

Notation: The notation of the modulus will be omitted if the modulus of the congruence is 2, e.g. $a \equiv b$ means that a and b have the same parity, $a \not\equiv b$ means that they have different parity.

Definition 1.1.6 (Good parity, bad parity)**.** If $|V| \not\equiv |E|$ holds for a graph $G(V, E)$, we say that G has good parity. If $|V| \equiv |E|$, we say that G has bad parity.

Remark 1.1.7. Suppose that an acyclic orientation $\overrightarrow{G}$ is given for a good parity graph G. Let $w \in V(G)$ be an arbitrary vertex. If we check all indegrees in $V(G) - w$, and there is at most one even indegree, then the orientation is good: the indegree of the last vertex will be even if the others are odd, and odd if there was one even indegree in $\{\varrho(v) \mid v \in V(G) - w\}$, since $|V(G)| \not\equiv |E(G)|$.

Definition 1.1.8 (T-odd)**.** Let $T \subseteq V$. An orientation of a graph $G = (V, E)$ is T-odd if all vertices in T have odd indegree and all vertices in $V - T$ have even indegree.

Theorem 1.1.9. *The following statements are equivalent for any graph $G = (V, E)$:*

(1) G is dual-critical

(2) For all $v \in V(G)$ there is an acyclic orientation, in which all the indegrees are odd except for v.

(3) The graph has good parity and for every set $T \subsetneq V$ with $|T| \equiv |E|$, there exists a T-odd acyclic orientation.

(4) Either G is the graph that has one vertex, or it has a two class partition $A \cup B = V$ such that $G[A]$ and $G[B]$ are dual-critical, and the cut defined by A and B has an odd number of edges.

Proof.

(1)⇒(4) Take a good ordering of G. Let w be the last vertex in the ordering. Choose $A = V(G) - w$, $B = \{w\}$.

(4)⇒(3) We use induction on the number of vertices. For a vertex set $X \subseteq V(G)$ let $i(X)$ denote the number of induced edges. If $|T \cap A| \not\equiv i(A)$ and $|T \cap B| \not\equiv i(B)$, then by addition $|T| \equiv i(A) + i(B)$. After adding the number of edges between

A and B we get $|T| \not\equiv i(A) + i(B) + e(A,B) = |E|$: this is a contradiction. Without loss of generality one can suppose $i(A) \equiv |T \cap A|$. By induction we can take an orientation of $G[A]$ in which the vertices of $T \cap A$ have an odd indegree, and the vertices of $A - T$ have an even indegree. Direct the edges of $E(A,B)$ towards B.

From $|T \cap A| \equiv i(A)$ it follows that $|T \cap B| \not\equiv i(B)$. Let T'_1 be the set of vertices in $B \cap T$ that have an even number of incoming edges from A, and let T'_2 be the set of vertices in $B - T$ that have odd number of incoming edges from A. Let $T' = T'_1 \cup T'_2$.

We state that $|T'| \equiv i(B)$. The symmetric difference of T' and $B \cap T$ consists of those vertices that have an odd number of incoming edges from $E(A,B)$. Thus $|T' \oplus (T \cap B)|$ is odd, since $e(A,B)$ is odd. We have

$$\left.\begin{aligned} |T'| &\not\equiv |T \cap B| \\ |T \cap B| &\equiv i(B) \end{aligned}\right\} \Rightarrow |T'| \not\equiv i(B) \tag{1.1}$$

Now one can use induction for B and T', and fix a good orientation of $G[B]$. A and B do not span any cycles. Neither does the whole graph, since such a cycle would have to step in and out of A, but all edges in $E(A,B)$ have been oriented towards B. The vertices with an odd indegree are exactly the vertices of T.

The condition $|E(G)| \not\equiv |V(G)|$ follows from $|A| + |B| \not\equiv i(A) + i(B) + e(A,B)$.

(3)⇒(2) We may use (3) for $T = V(G) - v$.

(2)⇒(1) Obvious.

□

Proposition 1.1.10. *The following operations do not change dual-criticality, i.e. a graph is dual-critical if and only if using any of these operations make a dual-critical graph:*

(1) Deletion of two parallel edges

(2) Insertion of two parallel edges between two arbitrary vertices

(3) Division of an edge by adding a vertex in the middle

(4) Contraction of an edge that has an endpoint with degree 2.

Proof. (1) is trivial: a good orientation stays good after the operation, and if a non-dual-critical graph became dual-critical by this operation, then its good ordering would be good for the original graph as well. (2) follows from (1). (3): if G is dual critical, then the new vertex can be put anywhere between the endpoints of the divided vertex in the good ordering. If G is not dual-critical, then the new graph cannot be dual-critical. Suppose it is dual critical, then a good ordering of the new graph with a source other than the new vertex would have the new vertex between the endpoints of the divided edge, thus doing the reverse of the operation would leave a good ordering of the graph. (4) follows from (3). □

Remark 1.1.11. If a graph has loops, then it is not dual critical since all its orientations have a cycle (the loop). Also note that a graph that is not connected cannot be dual-critical: in an acyclic orientation all its components have a source vertex, which has an even indegree. Hence there are at least two vertices with even indegree. The same can be said regarding loops: a graph with a loop cannot be changed into a dual-critical graph. Using the operations (1) and (3) from 1.1.10 one can make any graph simple by dividing loops by two vertices (so a triangle is made) and eliminating parallel edge pairs. We might cut a connected graph this way, but in that case Proposition 1.1.10 states that the original graph was not dual-critical.

Definition 1.1.12 (Super-dual-critical)**.** A graph is super-dual-critical if for any vertex $v \in V(G)$ the graph $G - v$ is dual-critical.

Proposition 1.1.13. *In super-dual-critical graph all vertex degrees are odd or all vertex degrees are even. The degree parity is the same as the parity of* $|E(G)| - |V(G)|$.

Proof. For an arbitrary vertex v the graph $G - v$ has good parity, thus

$$|V(G)| - 1 \not\equiv |E(G)| - d(v) \;\Rightarrow\; d(v) \equiv |E(G)| - |V(G)|. \tag{1.2}$$

□

Corollary 1.1.14. *A super-dual-critical graph is dual-critical if and only if it has good parity, or equivalently, a super-dual-critical graph is dual-critical if and only if every vertex has odd degree.*

Proof. Let G be a super-dual-critical graph. If it has bad parity, then by Proposition 1.1.13 all degrees are even, so it cannot be dual-critical. If G has good parity, then all degrees are odd. Delete an arbitrary vertex v. The graph $G - v$ is dual-critical, hence G is dual-critical as well, since it can be obtained from $G - v$ by adding a vertex that has odd degree (v). □

Proposition 1.1.15. *The graph G has a T-odd acyclic orientation for every $T \subsetneq V(G)$ for which $|T| \equiv |E(G)|$ if and only if G is dual-critical or G is super-dual-critical.*

Proof. First we prove the existence of the orientations. If G is dual-critical, then the proposition follows from Theorem 1.1.9. If G is super-dual-critical but it is not dual-critical, then let $v \in V(G) - T$. By the same theorem, $G - v$ has a T-odd acyclic orientation since $|T| \equiv |E(G)| \equiv |E(G)| - d(v) = |E(G-v)|$. (The condition $T \neq V - v$ also holds, since $|V(G)| \equiv |E(G)| \equiv |T|$, so from $T \subsetneq V$ it follows that $|T| \leq |V(G)| - 2$.) We direct the incident edges of v towards v, and get a T-odd acyclic orientation of G.

Now we prove the other direction. If G has good parity, then the statement follows from Theorem 1.1.9. If G does not have good parity (so $|E(G)| \equiv |V(G)|$), then by Proposition 1.1.13 the graph is Eulerian. Let $v \in V(G)$ be an arbitrary vertex. We will show that $G - v$ is dual-critical. Let $w \in V(G)$ be a vertex different from v. There is a T-odd orientation $\overrightarrow{G}$ for $T = V(G) - \{v, w\}$. In this orientation, every source vertex has even in-degree (it is 0) and every sink has even indegree since G is Eulerian. In an acyclic orientation there is at least one sink and one source, so v is the source and w is the sink or vice versa. If v is a sink, then $G - v$ is dual-critical with the orientation of $\overrightarrow{G} - v$. Otherwise consider the reversed orientation $\overleftarrow{G}$. It is still T-odd, since v and w will have even indegree, and all other vertices will have odd indegree, since $\varrho_{\overleftarrow{G}}(u) = d_G(u) - \varrho_{\overrightarrow{G}}(u)$ which is odd. Hence $\overleftarrow{G} - v$ is a good orientation of $G - v$. □

We prove another equivalent description of dual-critical graphs for later use. We need to define vertex splitting first, which is the opposite of a contraction.

Definition 1.1.16. (Vertex splitting) We replace a vertex v with two vertices v' and v'' and connect them with an edge e_v. We distribute the edges incident to v among v' and v''. (So each edge will be incident to either v' or v''). This operation is a splitting of vertex v. Note that this operation can also be used on a graph that has a single vertex.

We will need a lemma to proceed.

Lemma 1.1.17. *All vertex splittings of a dual-critical graph preserve dual-criticality.*

Proof. Take a good ordering $v_1, v_2, \ldots v_n$ of G. Suppose we split vertex v to v' and v''. We call the graph obtained this way G'. If $v = v_1$, then $v', v'', v_2, v_3, \ldots v_n$ is a good ordering of G'. If $v = v_k$ $(k \neq 1)$, then v has an odd indegree, so (not counting the new edge e_v) exactly one of v' and v'' have an odd indegree. Suppose it is v'. Now $v_1, \dots v_{k-1}, v', v'', v_{k+1}, \dots v_n$ is a good ordering. □

Proposition 1.1.18. *A graph is dual-critical if and only if it can be built from a graph that has a single vertex using the following operations:*

- *Splitting of a vertex*
- *Addition of two parallel edges to an existing edge*

Proof. Item (2) in Proposition 1.1.10 and the previous lemma imply that all graphs that can be built from a single vertex using the two operations above are dual-critical.

We need to prove that all dual-critical graphs can be built. We will use induction on the number of edges. The statement is trivial if the graph has no edges. Suppose the statement is true for all graphs with at most e edges. Let G be a dual-critical graph with $|E(G)| = e + 1$. Take a good ordering of G. There is an odd number of edges connecting the first two vertices. If there are at least 3 edges, we can remove two of them: we are done by induction. If there is only one edge, then the contraction of this edge results in G', which is a dual-critical graph: the same orientation is a good orientation of G'. By induction there is a way to build G' from a single vertex using the two operations. After these operations we can make a vertex splitting that results in G. □

Remark 1.1.19. Note that there is a spanning tree that corresponds to the splitting build-up of a dual-critical graph: it is the set of edges that were introduced with vertex splittings (they were not added as parallel edges).

Finally, the following is a proposition which shows that a question about acyclic orientations with parity constraints can be viewed as dual-criticality of a slightly changed graph.

Proposition 1.1.20 (Beáta Faller)**.** *A graph $G = (V, E)$ has a T-odd acyclic orientation for some $T \subseteq V$ if and only if the graph G' obtained by adding a vertex v and connecting it to all vertices in $V - T$ is dual-critical.*

Proof. If G has such an orientation, then directing the edges away from v will give a good orientation for G'. (There are no cycles in G, and no cycle can pass through v either.) If G' is dual-critical, then it has a good orientation in which the source vertex is v. This orientation is T-odd and acyclic in G. □

1.2 Splitting trees

We would like a better understanding of splitting build-ups and the corresponding splitting-trees. We call a spanning tree T of a graph G splitting-tree if there is a splitting build-up for which the edges obtained by splittings are the edges of T.

Take a splitting tree T of a dual-critical graph G. A parallel edge pair (e_1, e_2) that is added to an edge e during the build-up has a certain property. Let f be an edge on the fundamental cycle of e when the edge pair e_1, e_2 is added (so f is an edge of the fundamental cycle of the parallel edges as well). Vertex splittings cannot change this property, so f will be a common edge of the fundamental cycles (with respect to T) of e_1 and e_2 in G.

Definition 1.2.1 (FC-matching)**.** For a spanning tree T of G there is a fundamental cycle-matching (or FC-matching) if there is a perfect matching on the edges of $G-T$ such that for every pair, the fundamental cycles of the two matched edges have a common tree-edge.

The argument above proves the following proposition:

Proposition 1.2.2. *If T is a splitting tree of a dual-critical graph then there is an FC-matching in G.* □

Definition 1.2.3 (Good cut)**.** An edge set Z is a good cut, if it is an odd proper cut for which the two components of $G - Z$ are dual-critical.

The argument below establish a connection between splitting trees and good cuts.

Proposition 1.2.4. *Every splitting tree T has an edge e such that the cut defined by the two components of $T - e$ is a good cut.*

Proof. Let e_1 and e_2 be the first edge pair added as parallel edges in a splitting build-up $\mathcal{B}$ that corresponds to the splitting tree T. They are added as parallel edges to an edge $f \in T$. Let F be the set of edges obtained by vertex splitting before f. The build-up $\mathcal{B}'$ where f is the first edge obtained by vertex splitting, and only after the addition of e_1 and e_2 do we perform the splittings that result in F is a splitting build-up of the same graph with the same splitting tree.

The cut defined by the tree edge f is clearly an odd cut, since no other tree edge is an element of it, and the later addition of parallel edge pairs preserve this property. It is easy to verify that both sides are dual-critical: the steps of the splitting build-up are performed on one side of the cut define a splitting build-up of that side. Thus both sides are dual-critical. □

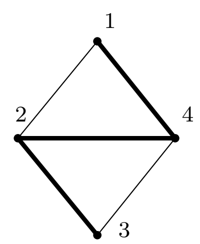
Proposition 1.2.5. *If the edge set of a tree T corresponds to a spanning arborescence in a good orientation of G, then there is a splitting build up of G where the edges obtained by vertex splitting are exactly the edges of T.*

Proof. We will use induction on the number of vertices. The statement is trivial for a graph with a single vertex. Let G be a dual-critical graph on n vertices, and let T be the edge set of a spanning arborescence in a good orientation. Choose a good ordering of G. We call the last vertex of the good ordering v. Since v has an odd degree, and v is a leaf of T, there is an even number of edges from $G - T$ that are incident to v, we call them $e_1, e_2, \dots e_{2k}$, and the edge of T incident to v will be denoted by f. The good orientation of G can be restricted to the edges of $G - v$, and $T' = T - f$ defines a spanning arborescence in it, so by induction T' is a splitting tree of $G - v$. The following build-up proves that T is a splitting tree of G:

1. We split the single vertex, the edge obtained will be $f = (wv)$.
2. We add the parallel edge pairs e_i, e_{i+1} to f for $i = 1, \dots k$.
3. We use the splitting build-up of $G - v$ beginning with vertex w.

We need to show that the vertex splittings can be done in a way that the edges added in the first two steps will have the proper endpoints when the build-up is ready. But this can be done, since the endpoint v will not be split, and all the other endpoints are obtained by splittings from the vertex w, thus we can define the vertex splittings properly. □

Remark 1.2.6. Some splitting trees cannot be constructed as spanning arborescences of good orientations. The example on the right shows a dual-critical graph: the numbers show a good ordering, and the thick edges form a splitting tree that cannot be constructed this way. In a good orientation the last vertex has an odd degree, and in every arborescence the last vertex is a leaf. Albeit in this tree both leaves have an even degree, thus it cannot be a spanning arborescence in a good orientation.

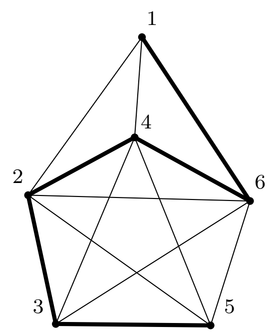


Definition 1.2.7 (Adjacency matching)**.** Let T be a spanning tree of G. An adjacency-matching is a perfect matching on the edges of $G - T$ such that in every pair the edges are adjacent (they have a common endpoint).

Proposition 1.2.8. *If G is dual-critical then it has an adjacency matching for at least one splitting tree T.*

Proof. Choose a good orientation $v_1, v_2, \dots v_n$ of G. For each vertex except v_1 we choose the incoming arc with the latest source vertex in the ordering to be an element of T. The tree is a splitting tree by Proposition 1.2.5, so we only need to show that there is an adjacency matching in $G - T$. We erase the edges of T. Now all vertices have an even indegree, since all but the first vertex had one incoming arc removed. Thus the arcs with the same endpoints can be arranged into pairs. These pairs define an adjacency matching in $G - T$. □

Remark 1.2.9. Not every spanning tree is a splitting tree. The following graph is dual-critical (the numbering of the vertices shows a good ordering), but the spanning tree formed by the thick edges is not a splitting tree. Edges (46), (23) and (35) define even cuts. Edge (16) defines a cut where one side is Eulerian (K_5), thus it cannot be dual-critical. Edge (24) defines a cut where both sides are triangles, so they are not dual-critical either. Thus the tree has no edge that defines an odd cut with both sides dual-critical, so by Proposition 1.2.4 it cannot be a splitting tree.



1.3 Dual-criticality and connectivity.

We already know that all dual-critical graphs are connected. The aim of this section is to show that deciding dual-criticality in graphs that are not 3-vertex-connected can be done by deciding dual-criticality for some 3-connected subgraphs. Maximal 2-connected subgraphs will be called blocks.

Proposition 1.3.1. *A connected graph is dual-critical if and only if all its blocks are dual-critical.*

Proof. We use induction on the number of blocks. The statement is trivial if the graph has 1 block. First we show that the blocks are dual-critical. Take an end block B (a leaf of the block tree), and the corresponding cut vertex (v). The original graph had a good ordering with a source outside B. In that ordering, no vertex of $B-v$ can precede v, since the first of those vertices would have 0 indegree. Thus all vertices in $B - v$ have an odd indegree, and their predecessors are vertices of B. Thus the orientation restricted to B is a good orientation with source v.

If the blocks are dual-critical, and B is an end block with cut vertex v, then by induction, $G - (B - v)$ is dual-critical, and so is B. Thus they both have a good orientation with source v, and this gives a good orientation of G. □

Proposition 1.3.2 (Beáta Faller)**.** *Let G be a 2-connected graph. Suppose that the deletion of the vertex set $\{v, w\}$ cuts G in $k \geq 2$ parts, their vertex sets are $V_1, V_2, \dots V_k$. Let $G_i = G[V_i \cup \{v, w\}]$ for $i = 1, 2, \dots k$. Let $\widehat{G_i} = G_i$ if G_i has good parity and $\widehat{G_i} = G_i \oplus (vw)$ otherwise. We use the shorthand notation $[k]$ for the set $\{1, 2, \dots k\}$. Then G is dual-critical if and only if it has good parity and $(\widehat{G_i})$ is dual-critical for all $i \in [k]$.*

Proof. Suppose that $\widehat{G_i}$ is dual-critical for all $i \in [k]$. Fix a good ordering of each $\widehat{G_i}$ beginning with vertex v. We can construct a good ordering of G by putting together these orderings. Let V_i^1 be the sequence of vertices in V_i that are between v and w in the good ordering of $\widehat{G_i}$ and let V_i^2 be the sequence of vertices in V_i that come after w in the good ordering of $\widehat{G_i}$. Now $v, V_1^1, V_2^1, \dots V_k^1, w, V_1^2, V_2^2, \dots V_k^2$ is a good ordering of G. We need to show that all the indegrees (except the indegree of v) are odd. The indegree of v is even, and and every vertex in V_i^1 and V_i^2 has odd indegree. Thus w has odd indegree by Remark 1.1.7.

Now suppose that G is dual-critical. G has a good ordering beginning with vertex v. Let V_i^1 be the subsequence of vertices in V_i that are between v and w in the good ordering of G and let V_i^2 be the subsequence of vertices in V_i that come after w. Now v, V_i^1, w, V_i^2 is a good ordering of $\widehat{G_i}$, because the indegree of vertices other than w remains unchanged, so the indegree of w is odd by Remark 1.1.7. □

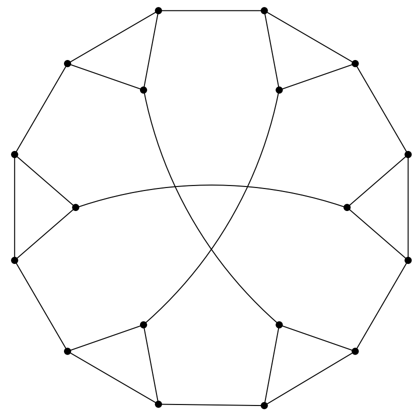
Suppose that an algorithm $\mathcal{A}$ is given that can identify 3-vertex connected dual-critical graphs. We can construct an algorithm $\mathcal{B}$ for identifying general dual-critical graphs. Let G be the input graph of $\mathcal{B}$.

First, we check if G has good parity. If it does, we check whether it is connected. (Otherwise G cannot be dual-critical). We find the block-decomposition of G. For each block B we run the algorithm again: $\mathcal{B}(G)$ will be true if for all blocks $\mathcal{B}(B)$ is true. If G is 2-connected, then we look for 2-element separating set. If we do not find such a set, then algorithm $\mathcal{A}$ can be used to decide dual-criticality. Now suppose that $\{v, w\}$ is a separating set that cuts the graphs in $k \geq 2$ parts. We run algorithm $\mathcal{B}$ recursively for each $\widehat{G_i}$. If they are all dual-critical, then G is dual-critical.

Remark 1.3.3. As we will see in the last chapter, finding interesting good parity non-dual-critical graphs is hard. Beáta Faller and Ervin Győri conjectured that any minimal 3-connected good parity graph is dual-critical. (A graph is minimal

3-connected if for any edge $e \in E(G)$ the graph $G - e$ is not 3-connected.) The graph on the right (found by my advisor) is a counterexample to this conjecture.

It is easy to see that it is minimal 3-connected, since the removal of an edge leads to a vertex with degree 2. Now suppose the graph is dual-critical. It has a vertex v which is last in a good ordering. Then $G - v$ is dual-critical. In this graph though there is no such vertex: removing any vertex, and doing simplifications (deleting vertices that have degree 1 and doing operation (1) and (4) from Proposition 1.1.10) leads to a non-connected graph.



1.4 Planar case, motivations

Before talking about dual-critical graphs in more detail, some background information should be provided about the term 'dual-critical'. In this section we are going to use matroids. Ample introductory and advanced material can be found on them. See e.g. [5]. In this section graphs may have parallel edges and loops.

Definition 1.4.1 (Factor-critical)**.** A graph G is factor-critical if for every vertex $v \in V(G)$ there is a perfect matching in $G - v$.

Definition 1.4.2 (Ear decomposition)**.** The addition of an ear to a graph is adding a path to the graph as an ear, i.e. the endpoints of the path are vertices of the original graph, the inner points are new vertices. If the the endpoints of the path are the same vertex of the original graph, then we have added a cycle. A graph has an ear decomposition if there are edge disjoint paths (or cycles with a fixed vertex) $\{P_1, P_2, \dots P_k\}$ such that the graph can be built from a single vertex by adding the paths (or cycles with a fixed vertex) as ears.

There is a well-known result about factor-critical graphs and ear decompositions.

Theorem 1.4.3. *[1] A graph is factor-critical if and only if it has an odd ear decomposition, i.e., an ear decomposition in which all ears have an odd number of edges.*

We will use the following variant of this theorem.

Theorem 1.4.4. *A graph is factor-critical if and only if it can be built from a single vertex using the following operations:*

- *Addition of an edge between two vertices or addition of a loop*
- *Division of an edge into path of length three using two new vertices*

Proposition 1.4.5. *The planar dual of a planar factor-critical graph G is always dual-critical. (So if there are multiple dual graphs depending on the planar embedding of G, then all of them are dual-critical.)*

Proof. We will show that the build-up of a factor-critical graph described in Theorem 1.4.4 is the dual of the dual-critical build-up with vertex splittings described in Definition 1.1.18. Let G be a planar factor-critical graph. Take a planar embedding of G. The statement is trivial if G has only one vertex. The addition of an edge e divides a face F in two parts. Let v be the vertex in the dual which corresponds to F. After the addition of the edge there will be two faces, let their dual vertex be v_a and v_b. Now v_a and v_b will be connected by the dual edge of e, and the incident edges of v will be either incident to v_a or v_b. Thus the addition of an edge corresponds to a vertex splitting in the dual graph. The division of an edge e into three parts corresponds to adding two parallel edges between the endpoints of the dual of e. □

Definition 1.4.6 (Blowing)**.** Let v be a vertex of a graph G. Let H be a graph. Blowing graph H into vertex v is an operation were we delete vertex v from G, we add the vertices and edges of H to G, and for every original edge vx we take a vertex $h \in V(H)$ and connect v to h.

Proposition 1.4.7. *Let G be a factor-critical graph. The graph G' that is obtained from G by blowing an odd cycle into a vertex v is factor-critical.*

Proof. We need to show that for any $w \in V(G')$ there is a perfect matching in $G'-w$. Let C be the odd cycle that we blew in. If $w \in V(C)$, then a perfect matching of $G - v$ and a perfect matching of $C - w$ together form a perfect matching of $G' - w$. If $w \notin V(C)$, then take a perfect matching M of $G - w$. There is an edge $uv \in M$. We denote by v' the new endpoint of this edge in G'. Let M_C be a perfect matching in $C - v'$. Now $(M - uv + uv') \cup M_C$ is a perfect matching of $G' - w$. □

Proposition 1.4.8. *The dual of a planar dual-critical graph G is always factor-critical. (So if there are multiple dual graphs depending on the planar embedding of G, then all of them are factor-critical.)*

Proof. We will use induction by the size of the vertex set. If $|V(G)| = 1$ then the statement is trivial. Let G be dual-critical graph, and let $v \in V(G)$ be the last vertex of a good ordering. We fix a planar embedding of G. If we remove v and the incident edges, the graph $G - v$ is dual-critical, thus its dual graph $(G - v)^*$ is factor-critical. Let f be the face in which v was (so it is the union of faces that were originally incident to v). Putting back vertex v and its incident edges corresponds to blowing in a cycle on $d(v)$ vertices into the vertex of $(G - v^*)$ that corresponds to f. Since $d(v)$ is odd, it is an odd cycle, so the dual graph G^* is factor-critical by Proposition 1.4.7. □

Since dualization and factor-criticality test can be done in polynomial time, we arrive at the following corollary.

Corollary 1.4.9. *There is a polynomial time algorithm for deciding dual-criticality if the graph is planar.* □

Proposition 1.4.5 and Proposition 1.4.8 show that dual-criticality and factor-criticality are dual concepts. An important corollary of these concerns 3-connected graphs. It is known that the dual of a planar 3-connected graph is unique up to isomorphism, which lets us to state these propositions together in the following simplified form.

Corollary 1.4.10. *A 3-connected planar graph is dual-critical if and only if its dual is factor-critical.*

Our next goal is to show that Propositions 1.4.5 and 1.4.8 can be generalized.

Remark 1.4.11. It is known that a graph is 2-connected if and only if it has an open ear-decomposition, i.e. an ear decomposition where we begin with a cycle, and the two ends of an ear cannot coincide. This statement can be ported for factor-critical graphs. A 2-connected graph is factor-critical if and only if it can be obtained from an odd cycle by adding odd length open ears.

Now we state a well-known result from matroid theory. A proof can be found in section 2.3 in [5].

Proposition 1.4.12. *The graphic matroid of the dual of a planar graph is isomorphic to the cographic matroid of the graph. (Equivalently: the dual graph's graphic matroid is the dual of the graph's graphic matroid.)*

Definition 1.4.13. [2] A sequence of circuits $\{C_0, C_1, \dots C_k\}$ of the matroid $M = (S, \mathcal{F})$ is an ear-decomposition if

(1) $C_i - (\bigcup_{j=0}^{i-1} C_j)$ is not empty for all $1 \leq i \leq k$

(2) $C_i \cap (\bigcup_{j=0}^{i-1} C_j)$ is not empty for all $1 \leq i \leq k$

(3) $C_i - (\bigcup_{j=0}^{i-1} C_j)$ is a circuit in $M/(\bigcup_{j=0}^{i-1} C_j)$ for all $1 \le i \le k$

(4) $\bigcup_{i=0}^{k} C_i = S$

An ear is a set $C_i - (\bigcup_{j=0}^{i-1} C_j)$.

This definition is the matroid equivalent of the open-ended ear-decomposition which is described in Remark 1.4.11. It follows that a 2-connected graph is factor-critical if and only if its graphic matroid has an odd ear-decomposition.

We need two basic lemmas from matroid theory. The notation M^* indicates the dual matroid of M, and M/Z is used for the contraction of the subset Z.

Lemma 1.4.14. *(Theorem 8.3 in [9]) Let M be a matroid on the set S. Then for any $Z \subseteq S$ the following holds:*

$$(M/Z)^* = M^* - Z \text{ and } M^*/Z = (M - Z)^* \tag{1.3}$$

Definition 1.4.15 (Proper cut)**.** A set Z of edges in a connected graph G is a proper cut if $G - Z$ has exactly two connected components. In a non-connected graph the edge set Z is a proper cut if there is a connected component G' for which $Z \subseteq E(G')$ and Z is a proper cut in G'.

It is known that a cut is proper if and only if it is a minimal cut.

Lemma 1.4.16 (Proposition 2.3.1 in [5])**.** *Let $M = (E(G), \mathcal{F})$ be the graphic matroid of G. The set $Z \subseteq E(G)$ is a cycle in M^* if and only if it is a proper cut in G.*

Proposition 1.4.17. *Let G be a dual-critical graph which has at least 2 vertices. For any pair of vertices $(v, w \in V(G), v \neq w)$ there is a good cut (as in Definition 1.2.3) of G that separate v and w.*

Proof. We will use induction on the number of vertices. For a graph with 2 vertices the statement is trivial. Suppose that the statement is true for all dual critical graphs with at most n vertices. Let G be a dual-critical graph, and let u be the last vertex of a good ordering of G. If $u \in \{v, w\}$ then it is easy to find a good cut: the cut consists of the incident edges of u.

If $u \notin \{v, w\}$, then we use the induction to find a good cut C in $G - u$ that separate v and w. If v or w is not a neighbour of u then C is an odd proper cut in G. If both v and w are neighbours of u, then (since $d(u)$ is odd) $e(W, \{u\})$ is odd and $e(V(G - u) - W, \{u\})$ is even or vice versa. If $e(W, \{u\})$ is odd, then the cut $W, V(G) - W$ satisfies the conditions. If $e(W, \{u\})$ is even, then the cut $W + u, V(G) - W - u$ is a good choice. □

Proposition 1.4.18. *A 2-connected graph is dual-critical if and only if its cographic matroid has an odd ear decomposition.*

Proof. Let M be the graphic matroid of G. First we prove that for any 2-connected dual-critical graph the cographic matroid has an odd ear-decomposition.

Let $C_0 \subseteq E(G)$ be a good cut in G. If both sides of $G - C_0$ have one vertex, then (C_0) is an ear-decomposition of M^*. Let G_1 be a component of $G - C_0$ that has at least two vertices. The graph G_1 is dual critical, and it has at least two distinct vertices that are incident to some edges of C_0. (If there is only one such vertex then it is a cut vertex in G.) By Proposition 1.4.17 there is an odd proper cut C_1' in G_1 that separate these vertices. Now C_1' is not a cut in G, but $C_1' \cup C_0$ is, though it is not a proper cut. We remove some edges of C_0 from $C_0 \cup C_1'$ until we get a minimal cut which is proper. (We will get a minimal cut before all edges of C_0 are removed since C_1' is not a cut in G.) Thus there is a proper cut C_1 in G for which $C_1' \subseteq C_1 \subseteq C_0 \cup C_1'$. If all components of $G - C_0 - C_1$ are isolated vertices, then we stop. Note that $C_1 - C_0 = C_1'$, thus $|C_1 - C_0|$ is odd.

Otherwise let G_2 be a connected component of $G - C_0 - C_1$. We choose a good cut C_2' that separate two vertices of G_2 that are incident to some edges in $C_0 \cup C_1$. There is a proper cut C_2 in G for which $C_2' \subseteq C_2 \subseteq C_0 \cup C_1 \cup C_2'$. We repeat this procedure until all connected components of $G - (\bigcup_{j=0}^{k} C_j)$ are isolated vertices.

In the general step let G_i be a connected component of $G - (\bigcup_{j=0}^{i-1} C_j)$ that has at least two vertices. Th graph G_i is dual-critical, thus it has an odd proper cut C_i' that separate two vertices which are incident to some edges of $(\bigcup_{j=0}^{i-1} C_j)$. (If there is no such vertex pair in G_i then G is not 2-connected.) We remove some edges of $(\bigcup_{j=0}^{i-1} C_j)$ from $(\bigcup_{j=0}^{i-1} C_j) \cup C_i'$ and find a proper cut C_i in G for which $C_i' \subseteq C_i \subseteq (\bigcup_{j=0}^{i-1} C_j) \cup C_i'$. Now C_i is a proper cut in G, thus it is a circuit in M^*. Moreover,

(1) $C_i - (\bigcup_{j=0}^{i-1} C_j) = C_i' \neq \emptyset$

(2) $C_i \cap (\bigcup_{j=0}^{i-1} C_j) \neq \emptyset$

(3) $C_i - (\bigcup_{j=0}^{i-1} C_j) = C_i'$ is a proper cut in $G - (\bigcup_{j=0}^{i-1} C_j)$, thus it is a circuit in $M^*/(\bigcup_{j=0}^{i-1} C_j)$.

Since $C_i - (\bigcup_{j=0}^{i-1} C_j) = C_i'$, the ear $C_i - (\bigcup_{j=0}^{i-1} C_j)$ is odd, and at the end of the procedure $E(G) = \bigcup_{j=0}^{k} C_j$ holds, thus $(C_0, C_1, \ldots C_k)$ is an odd ear-decomposition of M^*.

Now we prove that if M^* has an odd ear-decomposition, then G is dual-critical. Let $C_0, C_1, \ldots C_k$ be the circuits of the odd ear-decomposition. We will use induction on the number of ears. If $k = 0$ then G is a graph on two vertices and an odd number of parallel edges between them, which is clearly dual-critical. Let C'_k be the last ear, so $C'_k = C_k - (\bigcup_{j=0}^{k-1} C_j)$. Item (3) of the ear-decomposition's definition says that C'_k is a circuit in $M^*/(\bigcup_{j=0}^{k-1} C_j)$. Thus C'_k is a set of parallel edges in the graph $G - (\bigcup_{j=0}^{i-1} C_j)$. Now let $M^*_- = M^* - C'_k$. The matroid M^*_- is the cographic matroid of $G_- = G/C'_k$. Note that G_- can be obtained by deleting $|C'_k| - 1$ edges of C'_k and contracting the last edge. The circuit sequence $(C_0, C_1, \ldots C_{k-1})$ is an ear decomposition of the cographic matroid of G_- with less ears: so by induction, G_- is dual-critical. Graph G can be obtained from G_- by splitting a vertex and adding $|C'_k| - 1$ parallel edges (which is an even number), thus G is dual-critical. □

Chapter 2

3-regular dual-critical graphs

2.1 Equivalent descriptions

A 3-regular graph on n vertices have $\frac{3n}{2}$ edges, so n is even. If the graph is also dual-critical, then it must have a good parity, thus $n = 4k + 2$ for some integer k.

Definition 2.1.1 (r-rooted connected)**.** A directed graph is r-rooted connected for a vertex r if there is a path from r to v for every $v \in V - r$.

Theorem 2.1.2. [1] *The following are equivalent for any 3-regular graph $G = (V, E)$ which has $4k + 2$ vertices.*

(1) G is dual-critical.

(2) There are $k+1$ independent vertices, such that their deletion leaves a connected graph.

(3) There are $k + 1$ vertices whose deletion leaves a forest.

(4) There are some independent vertices whose deletion leaves a tree.

(5) There is a spanning tree, for which the deletion of the tree's edges makes a graph in which every component has an even number of edges.

(6) There is an r-rooted connected orientation for every vertex $r \in V$, in which all vertices but r have an odd indegree.

(7) There is an r-rooted connected orientation for a fixed root $r \in V$, in which all vertices but r have an odd indegree.

[1] Beáta Faller and Ervin Győri had the original idea that dual-criticality is equivalent to description (2) in the 3-regular case.

Proof.

(1)⇒(2) Let n_1 be the number of vertices with 1 incoming arc in the good orientation. Similarly let n_3 be the number of vertices with 3 incoming arcs. The following equations can be obtained on the number of edges and the number of vertices:

$$n_1 + n_3 + 1 = 4k + 2 \tag{2.1}$$

$$n_1 + 3n_3 = \frac{3 \cdot (4k+2)}{2} = 6k + 3 \tag{2.2}$$

The solution is $n_1 = 3k$, $n_3 = k + 1$. We observe that the vertices with 3 indegree are independent, and by deleting them the graph stays rooted connected. (The original graph was rooted connected, see remark 1.1.4)

(2)⇒(3) Delete the vertices. The graph we have left is connected, and has $6k + 3 - 3(k+1) = 3k$ edges and $4k+2-(k+1) = 3k+1$ vertices. Thus the graph is a tree.

(3)⇒(4) Let W be a vertex set with $k + 1$ vertices satisfying (3). If W spans an edge, then $G - W$ would have $3k + 1$ vertices and at least $3k + 1$ edges, so $G - W$ would span a cycle: this is a contradiction. So W is an independent set, and its deletion makes a forest that has $3k+1$ vertices and $3k$ edges, thus it is a tree.

(4)⇒(5) Let W be the set of independent vertices whose deletion leaves a tree. Take one incident edge for every vertex of W. These edges and the edge set $E(G - W)$ form a spanning tree of G. We delete the edges of T. Now the edges of the new graph are incident to W. All vertex degrees in W are 2, because all vertices of W were leaves of T. Since W is an independent set, a component C of $G - T$ has $2|V(C) \cap W|$ edges, which is even.

(5)⇒(6) By deleting a spanning tree's edges from a 3-regular graph, we get a graph in which the vertex degrees are either 0, 1 or 2. Since the components have even number of edges, they are even cycles or even paths.

Direct the tree's edges away from an arbitrary root to get an arborescence. In the arborescence the indegree of every vertex is 1 except for the root. Direct the edges of the cycles back and forth, so going around the cycle one arc points forward, the next one backward, then forward, backward, etc. The vertices of the cycle are leaves of the spanning tree, so the indegree in these vertices is 1 or 3.

We still need to define an orientation for the paths. Direct them back and forth as well, in a way that the first and last arc is directed away from the endpoints of the path. The endpoints of a path are 2 degree vertices in the original tree, so their indegree is now 1. The inner points of the paths are all leaves of the tree, so their indegree is 1 or 3. The orientation defined above satisfies all the conditions.

(6)⇒(7) Obvious.

(7)⇒(1) Suppose that the orientation given by (7) is not good (as described in the first chapter), that is, it contains a directed cycle C. Let r be the root. Since the orientation is rooted connected, there is a spanning arborescence in G rooted at r.

If $r \notin V(C)$, then let v be the vertex in C that is the nearest to r in the arborescence. (If there is more than one such vertex, we choose an arbitrary one.) The path rv in the arborescence does not contain any vertices from the cycle. The indegree of v cannot be 3, since the cycle has to leave v on some arc, so the outdegree must be at least one. Thus $\varrho(v) = 1$. It follows that the cycle passes through v's parent, a contradiction. Therefore r has to be in the cycle.

It is easy to see that r's indegree is even: there is an odd number of edges in the graph. The indegrees sum up to this odd number, and without r the sum is $4k + 1$ times an odd number which is odd.

Since r is on the cycle, its indegree cannot be 0. Thus $\varrho(r) = 2$ and $\delta(r) = 1$. Let r' be the target of the arc that leaves r. Every cycle of the graph passes through r, so all of them pass through the arc rr' as well. By reversing the orientation of rr' we get an orientation in which all vertices except r' have an odd indegree. This orientation is acyclic, since the cycles of the previous graph are not contained in the new graph (no cycle can pass through r because its new outdegree is 0), and no new cycles were formed, because such a cycle would have to use the edge $r'r$, but the outdegree of r is currently 0; we arrived at a contradiction. This completes the proof.

□

In the following sections we will prove further equivalent conditions, see Corollaries 2.2.12 and 2.3.9. They could be inserted in Theorem 2.1.2 as equivalent descriptions (8) and (9).

2.2 Allowing cycles: rooted connected orientations with parity constraints

Theorem 2.2.1 (Nebeský [4][2])**.** *A graph has an r-rooted connected orientation for a fixed $r \in V$ in which $\varrho(r)$ is even and all other vertices have an odd indegree if and only if for every partition $\mathcal{P}$ of the vertex set*

$$e(\mathcal{P}) \geq |\mathcal{P}| + \mathrm{bp}(\mathcal{P}) - 1 \tag{2.3}$$

holds, where $e(\mathcal{P})$ denotes the number of edges between different classes of $\mathcal{P}$ and $\mathrm{bp}(\mathcal{P})$ *denotes the number of classes in $\mathcal{P}$ spanning a subgraph which has bad parity.*

We will prove a stronger result. There is an even stronger version, see Theorem 1.10 in [3]. We denote symmetric difference by $\oplus$.

Theorem 2.2.2. *Let $T \subseteq V(G)$ be fixed. The graph G has an r-rooted connected orientation with a fixed root r which is $(T \oplus r)$-odd if and only if for every partition $\mathcal{P}$ of the vertex set*

$$e(\mathcal{P}) \geq |\mathcal{P}| + \mathrm{bp}_T(\mathcal{P}) - 1 \tag{2.4}$$

holds, where $\mathrm{bp}_T(\mathcal{P})$ *denotes the number of classes $C \in \mathcal{P}$ for which $i(C) \equiv |C \cap T|$.*

To prove this result, we will need two definitions and some lemmas. The proof is based on the proof of the main result in [3].

Definition 2.2.3 (Odd set, even set)**.** A vertex set C is called odd (with respect to a fixed $T \subseteq V$) if $|T \cap C| \not\equiv i(C)$, and C is called even if $|T \cap C| \equiv i(C)$. (Consequently $\mathrm{bp}_T(\mathcal{P})$ denotes the number of even classes in $\mathcal{P}$.)

Definition 2.2.4 (Tight partition)**.** A partition $\mathcal{P}$ is called tight if it satisfies (2.4) with equality.

Remark 2.2.5. For the trivial partition $\mathcal{P} = \{V\}$ the inequality (2.4) states that $0 \geq 1 - 1 + \mathrm{bp}_T(V)$, so $\mathrm{bp}_T(V) = 0$: the trivial partition is always tight if $|E(G)| \not\equiv |T|$. We say that a partition is nontrivial if it has at least two classes.

Lemma 2.2.6. *If* (2.4) *holds for the trivial partition, then for every partition $\mathcal{P}$*

$$e(\mathcal{P}) \equiv |\mathcal{P}| + \mathrm{bp}_T(\mathcal{P}) - 1 \tag{2.5}$$

[2]This theorem is stated explicitly in Corollary 1.6 in [3]

Proof. In Remark 2.2.5 we observed that $|E(G)| \not\equiv |T|$, or equivalently

$$e(\mathcal{P}) + \sum_{C\in\mathcal{P}} i(C) \not\equiv \sum_{C\in\mathcal{P}} |C \cap T|. \tag{2.6}$$

We rearrange the equation:

$$e(\mathcal{P}) \not\equiv \sum_{C\in\mathcal{P}} (|C \cap T| - i(C)) \equiv |\mathcal{P}| + \mathrm{bp}_T(\mathcal{P})$$ ☐

Lemma 2.2.7. *Suppose that there is a counterexample for Theorem 2.2.2, such that* (2.4) *holds but the graph does not have the desired orientation. Take a counterexample G that has a minimum number of edges. Then G has a nontrivial tight partition.*

Proof. Assume that G does not have a nontrivial tight partition. Let G' be the graph we get if we erase the edge uv from G, and let $T' = T \oplus u$. Let $\mathcal{P}$ be any nontrivial partition. This operation decreases $e(\mathcal{P})$ by at most one, and increases bp_T by at most 1, so using Lemma 2.2.6 we can conclude that (2.4) stands for G'.

If G' has a $(T' \oplus r)$-odd which r-rooted connected orientation, then putting the edge uv back and directing it towards u would give a $(T \oplus r)$-odd r-rooted connected orientation in G, which is a contradiction. So G' does not have such an orientation, hence it is a counterexample with fewer edges, which contradicts the minimality of G. ☐

Lemma 2.2.8. *Let C be a class in a tight partition $\mathcal{P}$ of G. Let $T_C = T \cap C$ if C is odd, and $T_C = (T \cap C) \oplus w$ for an arbitrary $w \in C$ if C is even. If* (2.4) *holds strictly for all refinements of $\mathcal{P}$ (the refinements of $\mathcal{P}$ are not tight), then* (2.4) *holds for T_C and $G[C]$, that is for every partition $\mathcal{Q}$ of C*

$$e(\mathcal{Q}) \geq |\mathcal{Q}| + \mathrm{bp}_{T_C}(\mathcal{Q}) - 1 \tag{2.7}$$

Proof. Since $i(C) \not\equiv |C \cap T_C|$, Lemma 2.2.6 can be used, so $e(\mathcal{Q}) \equiv |\mathcal{Q}| + \mathrm{bp}_{T_C}(\mathcal{Q}) - 1$ for any partition $\mathcal{Q}$ of C. Suppose $\mathcal{Q}$ is a counterexample, that is $e(\mathcal{Q}) \leq |\mathcal{Q}| + \mathrm{bp}_{T_C}(\mathcal{Q}) - 3$. We denote the restriction of the partition $\mathcal{P}$ to C by $\mathcal{P}|_C$. Let $\mathcal{P}'$ be the partition for which $\mathcal{P}'|_{V-C} = \mathcal{P}|_{V-C}$ and $\mathcal{P}|_C = \mathcal{Q}$.

First, $\mathcal{P}$ is tight, so $e(\mathcal{P}) = |\mathcal{P}| + \mathrm{bp}_T(\mathcal{P}) - 1$. Second, $|\mathcal{P}| + |\mathcal{Q}| - 1 = |\mathcal{P}'|$, and

$$\mathrm{bp}_T(\mathcal{P}) + \mathrm{bp}_{T_C}(\mathcal{Q}) - 2 \leq \mathrm{bp}_T(\mathcal{P}'), \tag{2.8}$$

since there are at most two classes on the left side that are not counted on the right: class C and the class in which w is in $\mathcal{Q}$.

Third, (2.4) holds for $\mathcal{P}'$. Putting these observations together we get

$$\begin{aligned} e(\mathcal{P}') = e(\mathcal{P}) + e(\mathcal{Q}) &\leq |\mathcal{P}| - 1 + \mathrm{bp}_T(\mathcal{P}) + |\mathcal{Q}| + \mathrm{bp}_{T_C}(\mathcal{Q}) - 3 \\ &\leq |\mathcal{P}'| + \mathrm{bp}_T(\mathcal{P}') - 1 \leq e(\mathcal{P}'). \end{aligned} \tag{2.9}$$

Consequently all inequalities are in fact equalities in (2.9), so $\mathcal{P}'$ is tight wich is a contradiction. □

Lemma 2.2.9. *Let C be a class in a tight partition $\mathcal{P}$ of G, and let v_C be the vertex obtained by contracting C. Suppose* (2.4) *holds for all partitions $\mathcal{P}$. Let $T' = T - C$ if C is even, and $T' = (T - C) \cup \{v_C\}$ if C is odd. Then* (2.4) *holds for $G' = G/C$ and T', that is for every partition $\mathcal{R}$ of G'*

$$e(\mathcal{R}) \geq |\mathcal{R}| + \mathrm{bp}_{\mathrm{T'}}(\mathcal{R}) - 1 \tag{2.10}$$

Proof. For a partition $\mathcal{R}$ of G' let $\mathcal{P}$ be the partition of G for which the contraction of C results in $\mathcal{R}$. Now

$$e(\mathcal{R}) = e(\mathcal{P}) \geq |\mathcal{P}| + \mathrm{bp}_T(\mathcal{P}) - 1 = |\mathcal{R}| + \mathrm{bp}_{T'}(\mathcal{R}) - 1, \tag{2.11}$$

since $\mathrm{bp}_T(\mathcal{P}) = \mathrm{bp}_{T'}(\mathcal{R})$ by the definition of T'. □

The following well-known result will be needed.

Lemma 2.2.10 (A. Frank, A. Gyárfás, [6])**.** *Let $f : V \to \mathbb{N}$ be fixed. A graph has an orientation in which $\varrho(v) = f(v)$ for all vertices if and only if*

$$\sum_{v \in V} f(v) = |E(V)| \tag{2.12}$$

and for all $X \subseteq V$

$$\sum_{v \in X} f(X) \geq i(X). \tag{2.13}$$

Lemma 2.2.11. *Let $f : V \to \mathbb{N}$ be fixed. The graph G has a r-rooted connected orientation in which $f(v) = \varrho(v)$ for all vertices $v \in V$ if and only if* (2.12) *holds and for all $X \subseteq V$*

$$\sum_{v \in X} f(X) \geq i(X) + 1 - |X \cap \{r\}|. \tag{2.14}$$

Proof. It is easy to verify that the conditions are necessary. To prove the existence of an orientation, take a graph G and a function f that satisfy the conditions. Since (2.14) implies (2.13), there is an orientation for which $f(v) = \varrho(v)$, but it is not necessarily rooted connected. Take a set $S \subseteq V - r$. We need to show that S has an incoming arc.

$$\varrho(S) = \sum_{s \in S} \varrho(s) - i(S) = \sum_{s \in S} f(s) - i(S) \geq i(S) + 1 - |S \cap \{r\}| - i(S) = 1 \quad \square$$

Now we are ready to prove the theorem.

Proof of Theorem 2.2.2. First we prove that (2.4) holds if there is an r-rooted connected orientation which is $(T \oplus r)$-odd. The orientation is rooted connected, so every class of the partition has an incoming arc, except maybe the class which contains r. Thus it would suffice to prove that all even classes have at least one more incoming arc. Let C be an even class.
If $r \in C$, then

$$\varrho(C) = \sum_{v \in C} \varrho(v) - i(C) \equiv |(T \oplus r) \cap C| - i(C) \equiv 1. \tag{2.15}$$

If $r \notin C$, then

$$\varrho(C) = \sum_{v \in C} \varrho(v) - i(C) \equiv |T \cap C| - i(C) \equiv 0. \tag{2.16}$$

Thus in both cases there is at least one more incoming arc.

Note that in a $(T \oplus r)$-odd rooted connected orientation the number of incoming edges for a class D of a tight partition is determined above:

$$\varrho(C) = \begin{cases} 0 & \text{if } C \text{ is odd and } r \in C \\ 1 & \text{if } C \text{ is even and } r \in C, \text{ or } C \text{ is odd and } r \notin C \\ 2 & \text{if } C \text{ is even and } r \notin C \end{cases} \tag{2.17}$$

To prove the existence of an orientation, we are going to use induction on the number of edges. The statement trivially holds for a graph on one vertex. By Lemma 2.2.7 we may assume that there is a nontrivial tight partition $\mathcal{P}$. We may also assume that no refinement of $\mathcal{P}$ is tight.

Case 1. There is a class $C \in \mathcal{P}$ which has at least two vertices.

We contract the edges of $G[C]$ to get the graph G'. Since $G[C]$ is a connected graph on at least two vertices, it has at least one edge, thus $|E(G')| < |E(G)|$. So we can use the induction for G' and T' as well by Lemma 2.2.9. Let $r' = r$ if $r \in V_C$ and $r' = v_C$ if $r \in C$. There is a $(T' \oplus r')$-odd r'-rooted connected orientation in G'; we are going to call this orientation $\widehat{G}'$.

Note that $\widehat{G}'$ gives an orientation on the edges of $G[V - C]$. Let $\widehat{G}$ be this partial orientation of G. Since $\mathcal{P} - C + \{v_C\}$ is a tight partition in G' for T', we can use (2.17) to determine the number of incoming edges of v_C in the orientation of G', which is the same as the number of incoming edges of C in $\widehat{G}$.

By (2.17) we see that C has either 0, 1, or 2 incoming arcs. Let P_1 be the endpoint of the incoming arc if $\varrho(C) = 1$, end let P_1 and P_2 be the enpoints if the incoming arcs if $\varrho(C) = 2$. Note that by the definiton of T' in Lemma 2.2.9, $\{v_C\}$ is odd in G' if and only if C is odd in G. We also chose the root r' of the orientation $\widehat{G}'$ so that $r = v_C$ if and only if $r \in C$.

We will use Lemma 2.2.8 to orient the undirected edges (the edges spanned by C) in $\widehat{G}$. Choose the vertices r_C and w of the lemma according to this table:

	C is even	C is odd
$r \in C$	$r_C = r$, $w = P_1$	$r_C = r$
$r \notin C$	$r_C = P_1$ and $w = P_2$	$r_C = P_1$

Lemma 2.2.8 states that (2.4) holds for $G[C]$ and T_C. Since $\mathcal{P}$ is not trivial, $i(C) < |E(G)|$, we can use the induction hyphothesis on the subgraph spanned by C, so there is $(T_C \oplus r_C)$-odd r_C-rooted connected orientation in $G[C]$. It is easy to verify that we got a $(T \oplus r)$-odd r-rooted connected orientation of G.

Case 2. All classes of $\mathcal{P}$ consist of 1 vertex. We are going to use Lemma 2.2.11 to show that there is a $(T \oplus r)$-odd orientation rooted at r. Choose the values of f according to 2.17, so

$$f(v) = \begin{cases} 0 & \text{if } v = r \in T \\ 1 & \text{if } v = r \notin T, \text{ or } v \neq r \text{ and } v \in T \\ 2 & \text{if } v \neq r \text{ and } v \notin T \end{cases} \tag{2.18}$$

If there is rooted connected orientation of G with $\varrho = f$, then it will be $(T \oplus r)$-odd. Thus we only need to verify the conditions of Lemma 2.2.11. The equation (2.12) is easy to prove:

$$\sum_{v \in V} f(v) = |V| + |V - T| - 1 = |\mathcal{P}| + \mathrm{bp}_T(\mathcal{P}) - 1 = e(\mathcal{P}) = |E(G)| \tag{2.19}$$

For the verification of (2.14), let $\mathcal{Q} = \mathcal{P}|_{V-X} \cup \{X\}$.

First,

$$\sum_{v \in X} f(v) = |X| + |X - T| - |X \cap \{r\}|. \tag{2.20}$$

Second,

$$e(\mathcal{Q}) \geq |\mathcal{Q}| + \mathrm{bp}_T(\mathcal{Q}) - 1 \geq |V| - |X| + 1 + |(V - X) - T| - 1, \tag{2.21}$$

since $\mathrm{bp}_T(\mathcal{Q}) \geq |(V - X) - T|$.

Third,

$$e(\mathcal{Q}) = e(\mathcal{P}) - i(X) = |V| + |V - T| - 1 - i(X). \tag{2.22}$$

So using the previous lower bound on $e(\mathcal{Q})$ we get

$$\begin{aligned} |V| + |V - T| - 1 - i(X) &\geq |V| - |X| + 1 + |(V - X) - T| - 1 \\ \Rightarrow i(X) + 1 &\leq |V - T| + |X| - |(V - X) - T| = |X| + |X - T|. \end{aligned} \tag{2.23}$$

Finally, equation (2.20) and (2.23) imply

$$\sum_{v \in X} f(v) \geq i(X) + 1 - |X \cap \{r\}|. \qquad \square$$

Corollary 2.2.12. *A 3-regular graph is dual-critical if and only if* (2.3) *holds for all partitions.*

Proof. A 3-regular graph is dual-critical if and only if condition (7) holds in 2.1.2. Thus we can use Theorem 2.2.1. □

2.3 Upper-embeddable graphs

In this section we will use orientable 2-dimensional surfaces. A surface S is uniquely defined by its genus $\gamma(S)$ up to homeomorphism. Graphs will be regarded as 1-complexes.

Definition 2.3.1 (Embedding)**.** An embedding of a graph into a surface S is an injective continuous mapping $f : G \to S$.

For a fixed embedding we call the components of $S - f(G)$ regions.

Definition 2.3.2 (Cellular embedding)**.** An embedding is cellular if all regions are homeomorphic to an open 2-dimensional disk. For cellular embeddings regions will be called faces.

Definition 2.3.3 (Cycle rank)**.** The cycle rank of a connected graph G is the rank of its cographic matroid. Since it is the dual of the graphic matroid, the rank is $|E(G)| - |V(G)| + 1$. We will use the notation $\beta(G)$ for cycle rank. The cycle rank is also known as the first Betti number of the graph.

Remark 2.3.4. Suppose that G has a cellular embedding into the orientable surface S. There is a simple upper bound on the genus of S. We can use Euler's polyhedral equation for the embedded graph. Let f be the number of faces in the embedding.

$$2 - 2\gamma(G) = |V(G)| - |E(G)| + f = f - \beta(G) + 1 \;\Rightarrow\; \gamma(G) \leq \left\lfloor \frac{\beta(G)}{2} \right\rfloor,$$

since $f \geq 1$.

Definition 2.3.5 (Maximum genus, upper-embeddability)**.** The maximum genus of the graph G is the maximum genus of the surface S for which there is a cellular embedding $f : G \to S$. We will use the notation $\gamma(G)$ for maximum genus. A graph is upper-embeddable if its maximum genus is $\left\lfloor \frac{\beta(G)}{2} \right\rfloor$.

Definition 2.3.6 (Deficiency)**.** The deficiency of a spanning tree T of the graph G is the number of components in $G - T$ that have an odd number of edges. The deficiency of G is the minimum deficiency of G's spanning trees. Deficiency will be denoted by $\xi(G)$.

Theorem 2.3.7 (Xuong, [8])**.** *The maximum genus of a graph G is given by the formula*

$$\gamma(G) = \frac{\beta(G) - \xi(G)}{2}.$$

Corollary 2.3.8. *A graph is upper-embeddapble if and only if its deficiency is* 0*, i.e. it has a spanning tree T for which all components of $G - T$ have an even edge count.*

Corollary 2.3.9. *A 3-regular graph that has good parity is upper-embeddable if and only if it is dual-critical.*

Proof. It is evident by Corollary 2.3.8 and the equivalence (1)⇔(5) in Proposition 2.1.2. □

Proposition 2.3.10. *All dual-critical graphs are upper-embeddable.*

Proof. By Proposition 1.2.8, there is a tree T for which there is an adjacency-matching. Since adjacent edge pairs of $G - T$ are in the same component of $G - T$, the components have an even number of edges. Hence the deficiency of T is zero, so by Xuong's theorem G is upper-embeddable. □

Remark 2.3.11*.* The contrary is not true: the graph K_5 has zero deficiency (so it is upper-embeddable), but it is not dual-critical (since it is Eulerian).

Theorem 2.3.12 (Furst, Gross, McGeoch [7])**.** *There is a polynomial algorithm that decides whether a graph is upper-embeddable or not. It runs in $O(end \log^6 n)$ time where e, n and d denote the number of edges, the number of vertices and the maximum degree respectively.*

Corollary 2.3.13. *There is an algorithm for deciding dual-criticality in the 3-regular case which runs in $O(n^2 \log^6 n)$ time.*

Chapter 3

Algorithms for deciding dual-criticality

In this chapter we analyze algorithms for deciding dual-criticality. No deterministic polynomial algorithm is known in the general case, however there is a randomized polynomial algorithm which uses the Schwartz–Zippel lemma.

3.1 The greedy algorithm

Before I started to write this thesis, I had tried to find dual-critical and non-dual-critical graphs. Bad parity graphs, Eulerian graphs and complete or almost complete graphs (graphs with at least $\binom{n}{2} - \lfloor \frac{n}{2} \rfloor$ edges) were the first obvious examples of non-dual-critical graphs. After finding the conditions described in Section 1.3 it was hard to produce non-dual-critical graphs by hand that did not fit into any of these categories.

That is why I decided to implement the greedy algorithm, with the ultimate goal of finding an interesting non-dual-critical graph. For the implementation I used the C++ language and the LEMON Graph Library. LEMON is an open source project which was launched by the Egerváry Research Group on Combinatorial Optimization (EGRES).

In the following box there is a simplified code to show how the greedy algorithm works for a fixed graph G. (The graphs tested are always simple, so they do not contain loops or parallel edges.) The algorithm searches for a good ordering that shows dual-criticality, finding the last vertex of the ordering first.

```
bool Dualcritical(G)
{
    if |V(G)|==1
        return true;
    for every v in V(G)
    {
        if v has an odd degree
        {
            erase v;
            if Dualcritical(G)
                return true;
            reattach v;
        }
    }
    return false;
}
```

Naturally, an erased vertex cannot be reattached without saving the data of its incident edges first. In my implementation I used an array of integers (called `deg_parity`) to store the current state of each vertex: it could be an erased vertex, a vertex with an odd degree, or a vertex with an even degree. This was an argument of the Dualcritical function. I used an array with the same structure to save the content of `deg_parity`. This allowed the reattaching of a vertex with a single command. The other argument was the number of non-erased vertices. I used a global variable (`adjacent`) for the adjacency matrix of the original graph for fast adjacency checking. The code:

```
bool dualcritical (vector<signed char> deg_parity,
                   const int & active_vertex_count)
//deg_parity[v]=-1......erased vertex
//deg_parity[v]= 0......even degree vertex
//deg_parity[v]= 1......odd degree vertex
{
    vector<signed char> deg_save(VERTEX_COUNT);
    if (active_vertex_count==1)
        return true;
    for (int v=0; v<VERTEX_COUNT; ++v)
    {
        if (deg_parity[v]==1)
        {
            deg_save=deg_parity;      //we save deg_parity
            deg_parity[v]=-1;         //we erase vertex v
            for (int w=0; w<VERTEX_COUNT; ++w)
                if (adjacent[v][w] && deg_parity[w]!=-1)
                    deg_parity[w]=1-deg_parity[w];
            //recursive step:
            if (dualcritical(deg_parity,active_vertex_count-1))
                return true;
            deg_parity=deg_save;      //reattach v
        }
    }
    return false;
}
```

First I tested random graphs chosen from the Erdős-Rényi model. After taking out bad parity graphs, most of the examples found were not 2-connected, or were Eulerian. Thus I decided to search among non-Eulerian 3-vertex-connected good parity graphs. (The LEMON Graph Library was very useful here: some functions for testing these properties were already implemented.)

I was able to generalize some examples. The first proposition show that one can build a non-dual-critical graph by taking any graph and attaching a clique properly.

Proposition 3.1.1. *Let G be a simple graph on vertex set V. If $V = A \cup B \cup C$, where the sets A, B, C are disjoint and B, C are non-empty, C is a clique, B is a separating set, $(B, C; E(B, C))$ is a complete bipartite graph and $|B| < |C|$ then G is non-dual-critical.*

Proof. Suppose that G has a good ordering. We use induction on $|V|$. Since B and C are non-empty and $|B| < |C|$ it follows that $|V| = |A| + |B| + |C| \geq 3$. If $|B| + |C| = 3$ then $G[B \cup C]$ is a triangle in G, thus G is non-dual-critical.

For the inductive step take a graph with $|B| + |C| = m$. If $|C| = |B| + 1$, then all vertex degrees in C are $2|B|$. Fix a good ordering of G. Let v be the last vertex in the good ordering. Erasing v results in a dual-critical graph. We show that $|B| < |C|$ stands in $G - a$, so using the induction we get that $G - a$ is non-dual-critical, which is a contradiction.

If $v \in A$, $v \in B$, or $v \in C$ and $|C| > |B| + 1$ then the statement trivially holds. Thus we only need to show that if $|C| = |B| + 1$, then $v \notin C$. If $|C| = |B| + 1$, then all vertices in C have degree $|B| + |C| - 1 = 2|B|$, which is even, thus a vertex of C cannot be the last vertex in a good ordering. □

Proposition 3.1.2. *Let G be a simple graph on vertex set V. If $V = A \cup B \cup C$, where the sets A, B, C are disjoint and B, C are non-empty, C is a clique, B is a separating set, $(B, C; E(B, C))$ is a complete bipartite graph, $|B| = |C|$, the vertices of A have an even degree and the vertices of B have an odd degree, then G is non-dual-critical.*

Proof. Suppose that G is dual-critical. Observe that $v \in C$ cannot be the last vertex of a good ordering, since $G - v$ is Eulerian. Thus the last vertex could only be an element of B. Its deletion results in a graph that satisfies the conditions of Proposition 3.1.1, so G is non-dual-critical. □

I could not find any non-dual-critical graph with the algorithm that has at least 15 vertices which falls in the "interesting" category. For 5,6,7 etc. 15 vertices each I tested millions of good parity graphs. The number of non-dual-critical graphs found by the algorithm started decreasing at 11 vertices. I checked one million graphs on 15 vertices, all of them had good parity, they were 3-connected, and they were all dual-critical. It seems that almost all good parity random graphs are dual-critical.

Conjecture. Let p_n be the probability that a random graph on n vertices (where all edges are included with $\frac{1}{2}$ probability) is dual-critical. Then

$$\lim_{n\to\infty} p_n = \frac{1}{2}$$

3.2 A randomized algorithm by Balázs Szegedy and Christian Szegedy

Definition 3.2.1 ([2]). Let M be a connected, bridgeless matroid. We denote by $\varphi(M)$ the minimal possible value of the number of even ears in an ear-decomposition of M. If M is bridgeless but not connected, we define $\varphi(M)$ to be the sum of $\varphi(K)$ over all blocks K of M. In particular $\varphi(M) = 0$ if and only if every block of M has an odd ear-decomposition.

Theorem 3.2.2 (Szegedy–Szegedy, Theorem 10.8 in [2]). *Let M be a matroid that is representable over a field of characteristic 2. There is a randomized polynomial algorithm which computes $\varphi(M)$.*

This gives a randomized polynomial algorithm for deciding dual-criticality, if we can represent the cographic matroid of graphs over a field of characteristic two: a graph is dual-critical if and only if $\varphi(M^*(G)) = 0$.

We would like to outline this algorithm for cographic matroids. Let T be the edge set of a spanning tree of our graph. We associate independent indeterminates with each edge of T: x_e for all $e \in T$. The tree edges of the fundamental cycle of $i \in E(G) - T$ will be denoted by T_i. Let $A = (a_{i,j})$ be the following matrix ($i \in E(G) - T$ and $j \in E(G) - T$):

$$a_{ij} = \sum_{e \in T_i \cap T_j} x_e.$$

If the fundamental cycles have no common edge (the sum is empty), then the matrix entry is 0. The corank of A is equal to $\varphi(M^*)$ by the theorem of Szegedy–Szegedy ([2]). So G is dual-critical if and only if $det(A)$ is not the constant zero polynomial.

This can be determined by the Schwarz–Zippel lemma ([10]), which provides a polynomial randomized algorithm. The randomized algorithm might require changing to a larger field, but this won't arise any problems: if we choose the larger field properly, the original field will be a subfield of the larger one, thus M can be represented by the same vector set over the larger field.

Notations

$\equiv$	Congruence. If the notation of the modulus is missing, then the modulus is 2.
$e(H)$	The number of edges incident with the vertex set H
$i(H)$	The number of edges spanned by the vertex set H
$G[H]$	The subgraph of G which is spanned by H
$E(G)$	Edge set of G
$V(G)$	Vertex set of G
$E(A,B)$	The set of edges with one endpoint in A and the other one in B
$e(A,B)$	The number of edges with one endpoint in A and the other one in B
$E(\mathcal{P})$	The set of edges that have their endpoints in two different classes of the partition $\mathcal{P}$
$e(\mathcal{P})$	The number of edges that have their endpoints in two different classes of the partition $\mathcal{P}$
$\oplus$	Symmetric difference.
$\varrho(.)$	Indegree of a vertex, or number of arcs entering a vertex set
$\delta(.)$	Outdegree of a vertex, or number of arcs exiting a vertex set
G/e	The graph obtained by contracting edge e in G
G/Z	The graph obtained by contracting the edge set Z in G
M/T	The matroid obtained by contracting the elements of T in M
$G-v$	The graph obtained by deleting vertex v and incident edges from G
$G-Z$	The graph obtained by deleting the edge set Z from G
$\mathrm{bp}(\mathcal{P})$	Number of classes of $\mathcal{P}$ that span a bad parity subgraph
$\mathrm{bp}_T(\mathcal{P})$	Number of classes C of $\mathcal{P}$ for which $i(C) \equiv \|C \cap T\|$
$\gamma(.)$	Genus of the orientable surface or maximum genus of a graph
$\beta(G)$	Cycle rank of the graph G. Equals $\|E(G)\| - \|V(G)\| + 1$
$\xi(G)$	Deficiency of G. See Definition 2.3.6.
$[k]$	The set $\{1, 2, \ldots k\}$

Bibliography

[1] L. Lovász: A Note on Factor-Critical Graphs, *Studia Sci. Math. Hungar.* **7** (1972), 287–290.

[2] B. Szegedy and C. Szegedy: Symplectic spaces and ear-decomposition of matroids, *Combinatorica* **26** (2006), 353–377.

[3] A. Frank, T. Jordán, Z. Szigeti: An orientation theorem with parity conditions, *Discrete Appl. Math.*, **115(1-3)** (2001) 37–47.

[4] L. Nebeský: A new characterization of the maximum genus of a graph, *Czechoslovak Mathematical J.*, **31(106)** (1981) 604–613.

[5] J. Oxley: Matroid Theory, *Oxford University Press, New York* (1992)

[6] A. Frank, A. Gyárfás: How to orient the edges of a graph, *Combinatorics (Keszthely 1976), Coll. Math. Soc. J. Bolyai* **18** (1976) 353–364.

[7] M. L. Furst, J. L. Gross, L. A. McGeoch: Finding a Maximum-Genus Graph Imbedding, *J. ACM* **35-3** (1988) 523-534.

[8] N. H. Xuong: How to determine the maximum genus of a graph, *J. Comb. Theory Ser. B* **26** (1979) 216-225.

[9] E. L. Lawler: Combinatorial Optimization: Networks and Matroids, *Holt, Rinehart and Winston, New York*, (1976)

[10] J. Schwartz: Fast probabilistic algorithms for verification of polynomial identities, *Journal of the ACM* **27** (1980), 701–717