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The Denjoy-Young-Saks theorem in higher dimensions

Master's thesis

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Introduction

This thesis examines what can be said about the behavior a.e. of Dini derivatives of one or several real variable functions.[1] For one variable functions it is mostly clear what should be meant by these notions, and the fundamental result describing the behaviour is the Denjoy-Young-Saks theorem. However, in higher dimensions the generalization of the notions involved is not self-evident. What should be meant by Dini derivatives? By taking Dini derivatives of linear sections of the function one can talk about directional or linear derivatives. Derivatives can also be obtained by taking limits of the difference quotient in decreasing angular sectors near a point which are called directed derivatives. Also, when one tries to state an analogue of the Denjoy-Young-Saks theorem using these notions, it is not clear what should be meant by almost everywhere. Should such a theorem be formulated for almost every point, in every direction or for almost every point, in almost every direction? This paper recapitulates the results that are known about such questions from both the measure and category point of view, for both ordinary and approximate derivatives.

The structure of the thesis is the following. The first chapter introduces basic definitions and gives a historical introduction to the results (most results are stated without proof). The chapter is divided into two sections; first the results are discussed from the measure point of view, then from the category point of view. In both sections Denjoy-Young-Saks type results for one and several variable functions are discussed, first for classical, second for approximate derivatives. In each case measurability properties of the different variants of Dini derivatives are examined.

The rest of the thesis presents proofs, and is only concerned with the two variable case and linear Dini derivatives. Theorems that are stated without a citation are the product of this thesis.

The second chapter presents Denjoy-Young-Saks type theorems that hold for linear Dini derivatives of two variable functions at a typical point in a typical direction. Typical here is meant both from the measure and category point of view.

First we prove Ward's theorem, which states that a Borel measurable function at almost every point in almost every direction has the Denjoy property in that direction. Then we present Davies' example of a Lebesgue measurable function that doesn't have the linear Denjoy property on a set of positive measure in a set of directions of full outer measure. We then present Ward's theorem that states that a Lebesgue measurable function at a.e. point, a.e. direction is an approximate linear Denjoy direction. We construct

[1] In the discussion that follows we state the definitions and theorems for two-variable functions. Most of the theorems can be generalized to higher dimensions.

a counterexample for arbitrary functions. The chapter is concluded by adapting Ward's theorem to category.

The third chapter is concerned with the stronger form of the previous question; what can be said about such relations at a typical point in all directions? (The answer seems to be: not much. This chapter is mainly concerned with counterexamples.)

We first present Besicovitch's example of a continuous function, such that to each point of a set of positive measure there is associated a direction, in which the function has three zero and one infinite linear Dini derivatives. Subsequently we construct a continuous function, such that to each point of a set of positive measure there is associated a direction, in which the function has finite and distinct linear Dini derivatives. We conclude the chapter by constructing a continuous function that on a set of positive measure in c many directions has finite and different one-sided approximate derivatives.

In the appendix we present some by-products that resulted from the previous examinations, in particular the following question is examined: How big can a set of disjoint (linear) segments be? We show that if the endpoints are measurable, then this set must be of measure zero. We also prove Stepanoff's theorem on differentiability a.e. by slightly extending the proof of Federer's theorem on measurability of partial derivatives. The thesis is concluded with some tables recapitulating the results discussed herein.

1 The Denjoy-Young-Saks theorem and its analogues

"If only I had the theorems!
Then I should find the proofs
easily enough."

Bernhard Riemann

This chapter presents the results that are known concerning the relations that hold between Dini derivatives of one and several variable functions. The majority of the results presented in this section are stated without proof. Most of the results presented for one variable functions can also be found in [12], [72]. For discussion of both the one and several variable case the reader is referred to Saks' *Theory of the integral*[2] [63], [13], similar and other aspects are also discussed in [22]. Many related results and also historical aspects can be found in [15].

1.1 Measure

1.1.1 Classical derivatives

One variable real functions The Denjoy-Young-Saks theorem in its final form expresses the connection between the Dini derivatives of an arbitrary function $f : \mathbb{R} \to \mathbb{R}$ almost everywhere. The *Dini derivatives* (sometimes also called derivate numbers or derivates) of a function $f : \mathbb{R} \to \mathbb{R}$ are defined as follows:

$$D^+f(x) := \limsup_{t \to x+} \frac{f(t) - f(x)}{t - x}$$

$$D^-f(x) := \limsup_{t \to x-} \frac{f(t) - f(x)}{t - x}$$

$$D_+f(x) := \liminf_{t \to x+} \frac{f(t) - f(x)}{t - x}$$

$$D_-f(x) := \liminf_{t \to x-} \frac{f(t) - f(x)}{t - x}$$

Both (D^+f, D_-f) and (D_+f, D^-f) are called *opposite derivatives* of each other. (D^+f, D_+f) and (D^-f, D_-f) are called f's *one-sided derivatives*. Finally, (D^+f, D^-f) are called *upper derivatives* and (D_+f, D_-f) are called *lower derivatives* of f.

[2]Many results that were original at the time are stated in the first edition of the book. However, unless stated, we always cite from the second (English) edition of the book.

Theorem 1 (Denjoy-Young-Saks [23], [77], [59]) *For an arbitrary function $f : E \to \mathbb{R}$ defined on an arbitrary set $E \subseteq \mathbb{R}$, at λ_1-a.e. point, one of the following three cases holds:*

- *f is differentiable;*
- *two of f's opposite derivatives (D^+f, D_-f) [(D_+f, D^-f)] are finite and equal, the two other opposite derivatives (D_+f, D^-f) [(D^+f, D_-f)] are infinite with the appropriate sign;*
- *all four derivate numbers of f are $\pm\infty$ with the appropriate sign.*

A function f with one of the previous properties at x is said to have the Denjoy property *at x.*

The theorem in its previous form was found by Arnaud Denjoy (1884-1974) in 1915, for continuous functions [23]. In 1916 Grace Chisholm Young (1868-1944) [77] weakened the condition of continuity to measurability. The theorem in its final form, for arbitrary real functions was stated by Stanisław Saks (1897-1942) [59] in 1924. For a proof based on Vitali's theorem see Eugene Harold Hanson's proof [28], for a proof based on the contingence theorem (to be discussed later), see Saks' proof [63] Chapter IX, Theorem 4.1, and for a direct proof see Riesz–Szőkefalvi-Nagy [53], pages 18-19.

Several variable real functions In generalizing the Denjoy-Young-Saks theorem to higher dimensions two questions arise naturally:

- How does one generalize the Dini derivatives? (Directional (linear) derivatives, or directed derivatives)
- What should be meant by a.e.? (for λ_2-a.e. point in every $\vartheta \in [0, 2\pi)$ direction, or λ_3-a.e. in $\mathbb{R}^2 \times [0, 2\pi)$)

Let us define the *directional* or *linear Dini derivatives* of an arbitrary function $f : E \to \mathbb{R}$ defined on $E \subseteq \mathbb{R}^2$ at a point $x \in E$ in a direction $0 \le \vartheta < 2\pi$:

$$\partial^\vartheta f(x) := \limsup_{E\cap l\ni y\to x} \frac{f(y) - f(x)}{|y - x|},$$

$$\partial_\vartheta f(x) := \liminf_{E\cap l\ni y\to x} \frac{f(y) - f(x)}{|y - x|},$$

where l denotes the half-line $l(x, \vartheta)$ extending from the point x in direction ϑ. By this we mean that the extension of the half-line has angle ϑ with the positive x-axis, and $\vartheta = 0$ if they are parallel. If the function f restricted to $l(x, \vartheta)$ has the Denjoy property as a one variable function, we say that f has the *directional* (or linear) *Denjoy property* at the point x in the direction ϑ.

Conditions for differentiability a.e. were first given by Hans Rademacher (1892-1969) [49], who examined differentiability a.e. for Lipschitz continuous functions. Vyacheslav Vassilievich Stepanoff (1889-1950) [71] gave the following conditions for Lebesgue measurable functions[3]: if $E \subseteq \mathbb{R}^2$ is Lebesgue measurable, a necessary and sufficient condition for a Lebesgue measurable function $f : E \to \mathbb{R}$ to be (totally) differentiable a.e., is that for a.e. point

$$L_f(x) := \limsup_{y \to x} \frac{|f(y) - f(x)|}{|y - x|} < \infty.$$

In Stepanoff's proof John Charles Burkill (1900-1993) and Ughtred Shuttleworth Haslam-Jones (1903-1962) [17] discover two imprecise statements: Stepanoff, in order to avoid questions of non-measurability, supposes that the Lebesgue measurable function $f : E \to \mathbb{R}$ is defined on an F_σ-set $M \subseteq E$ of equal measure. However, by defining f on an additional null-set, the differentiability properties can change on a set of positive measure as simple examples show.

Thus his theorem is only proved for Lebesgue measurable functions defined on F_σ-sets. The other shortcoming of the proof is that Stepanoff obtains a function f defined on a set P, such that

$$\frac{f(x+h, y) - f(x, y)}{h} \to \frac{\partial f}{\partial x}$$

uniformly as $h \to 0$, for $x \in P$. However $f(x+h, y)$ need not be defined ($x + h \notin P$), and even if so, it is not clear that it is continuous. The continuity of $\partial f / \partial x$ is needed in the proof.

It is interesting to note that in the Burkill–Haslam-Jones article there is also an imprecise Lemma 2, which was discovered by Clarence Raymond Adams (1898-1965) and James Andrew Clarkson (1906-1970) in a correction (1939, [2]) given for their own 1936 article [1] on functions of bounded variation. This lemma states that if $f : E \to \mathbb{R}$ is a two variable Lebesgue measurable function, defined on a Lebesgue measurable $E \subseteq \mathbb{R}^2$ set, then the partial derivatives, taken where they exist, are also Lebesgue measurable. Miloš Neubauer (1898-1959) in 1931, [45], had already published the following counterexample, originally given by Hans Hahn (1879-1934)[4]:

[3] This is what we refer to as Stepanoff's theorem, unless it is explicitly stated otherwise.

[4] It is strange that Burkill and Haslam-Jones stated the lemma, since Haslam-Jones already knew of this counterexample of Neubauer, see [29], p. 121.

Example Take a non-measurable subset of the line $N \subseteq \mathbb{R}$. Define $f(x,y) := \chi_{\mathbb{Q}}(x)\chi_N(y)$, where χ_M denotes the characteristic function of the set M. The function f is zero outside of a null-set, thus it is λ_2-measurable. By the previous notation $\partial^0 f$ $[\partial_0 f]$ denotes the upper [lower] partial derivative in the positive direction of the x-axis, and we have

$$\partial^0 f|_{\mathbb{R}\times N^c} \equiv 0, \qquad \partial_0 f|_{\mathbb{R}\times N^c} \equiv 0, \qquad \partial^\pi f|_{\mathbb{R}\times N^c} \equiv 0, \qquad \partial_\pi f|_{\mathbb{R}\times N^c} \equiv 0,$$

$$\partial^0 f|_{\mathbb{Q}\times N} \equiv 0, \qquad \partial_0 f|_{\mathbb{Q}\times N} \equiv -\infty, \qquad \partial^\pi f|_{\mathbb{Q}\times N} \equiv 0, \qquad \partial_\pi f|_{\mathbb{Q}\times N} \equiv -\infty,$$

since $\mathbb{Q}$ is dense in $\mathbb{R}$ and $f|_{\mathbb{Q}\times N} \equiv 1$, and

$$\partial^0 f|_{\mathbb{Q}^c\times N} \equiv \infty, \qquad \partial_0 f|_{\mathbb{Q}^c\times N} \equiv 0, \qquad \partial^\pi f|_{\mathbb{Q}^c\times N} \equiv \infty, \qquad \partial_\pi f|_{\mathbb{Q}^c\times N} \equiv 0.$$

Since $\mathbb{Q}^c \times N$ is non-measurable, $\partial^0 f$ is a non-measurable function. The set, where the partial derivatives in the direction of the x-axis exist, is $\mathbb{R} \times N^c$, which is not Lebesgue measurable.

As we shall see later, the theorem of Stepanoff is sound (and can be even extended to non-measurable functions); e.g. it follows both from Theorem 6 of Haslam-Jones and Theorem 7 of Saks. We give a proof of the theorem for arbitrary functions in Appendix B.

Measurability (one variable functions) The previous examples show that measurability conditions are delicate points of the investigations. For one variable functions Stefan Banach (1892-1945) showed in 1922 [5] that the Dini derivatives of a Lebesgue-measurable function are also Lebesgue-measurable. Wacław Sierpiński (1882-1969), in the same volume of *Fund. Math.* [67] obtained the same result for Borel measurable functions, by showing that if f is a real function of class Baire-α, then its Dini derivates are of class $\alpha + 2$.[5] Stronger relations do not hold, as simple examples show (i.e. there exist Lebesgue measurable functions whose derivate numbers are not Borel measurable. Also, there are arbitrary functions whose derivate numbers are not Lebesgue measurable). Herman Auerbach gave a simple proof for the Lebesgue measurable case in 1925 [4].

Remark 1 (Further results) We also mention some related further results: In [27] it is shown that the extreme bilateral derivatives of arbitrary one variable functions are of Baire class two and in [69] it is shown that this holds strictly. Surprisingly in [40] a function of Baire class two is given such that the upper symmetric derivatives are not even Borel measurable.

[5] In fact in Banach's paper it is also shown that the Dini derivatives of a *bounded* function of Borel class α are of class $\alpha + 2$.

Measurability (two variable functions) For functions of two real variables one can speak of the measurability of the $(x, \vartheta) \mapsto \partial^{\vartheta} f(x)$ directional Dini derivatives, or measurability of Dini derivatives in a fixed direction ϑ_0: $x \mapsto \partial^{\vartheta_0} f(x)$ (which are the sections of the previous function).

For a **fixed ϑ_0 direction** if f is continuous, then $\partial^{\vartheta_0} f$ is Borel measurable, however Borel measurability of f does not imply that $\partial^{\vartheta_0} f$ is Borel measurable. If f is Borel measurable, then $\partial^{\vartheta_0} f$ is Lebesgue measurable (see Neubauer [45] and also [39], pp. 512-514[6]), but if f is Lebesgue measurable, $\partial^{\vartheta_0} f$ is not necessarily Lebesgue measurable, as we have seen.

However, restricted to the set $M := \{x : L_f(x) < \infty\}$ an argument due to Herbert Federer (1920-2010) [70] p. 268 proves the following:

Theorem 2 (Federer [70]) *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a Lebesgue measurable function. Then the set*

$$M := \{x \in \mathbb{R}^2 : L_f(x) < \infty\}$$

is Lebesgue measurable and the set of points where the function is differentiable is also Lebesgue measurable.

Extending Federer's argument we obtain an extension of Stepanoff's result:

Theorem 3 *Let $f : E \to \mathbb{R}$ be an arbitrary function defined on an arbitrary set $E \subseteq \mathbb{R}^2$. Define*

$$M := \{x \in E : L_f(x) < \infty\}.$$

Then the function is differentiable a.e. on M. Moreover M is relative Lebesgue measurable (moreover relative F_σ), and the partial derivatives are relative Lebesgue measurable functions on M.

For a proof, see Appendix B.

Moshe Marcus and Victor Julius Mizel (1931-2005) have given sufficient conditions for measurability of ordinary partial derivatives in 1977 [41]. Define for an arbitrary real variable function $f : \mathbb{R} \to \mathbb{R}$ the *right cluster set* of f at x as

$$C^+ f(x) := \{y \in \mathbb{R} \cup \{\pm\infty\} : \exists x_n \to x + 0,\ f(x_n) \to y\},$$

and define the *left cluster set* $C^- f(x)$ similarly. Define $L^\pm f(x) := \sup C^\pm f(x)$ and $L_\pm f(x) := \inf C^\pm f(x)$.

[6] Actually, it is shown that the upper [lower] partial Dini derivatives are analytic [co-analytic] functions.

Theorem 4 (Marcus-Mizel [41]) *If $f : \mathbb{R}^2 \to \mathbb{R}$ is a Lebesgue measurable function, and for every $c \in \mathbb{R}$ constant the c-section $f_c(x) := f(x, c)$ satisfies*

$$\min(L^+ f_c(x), L^- f_c(x)) \le f(x, c) \le \max(L_+ f_c(x), L_- f_c(x)),$$

then the set of points Ω_x where the partial derivative $\partial_x f$ in the x-axis' direction exists is Lebesgue measurable, and $\partial_x f$ is also Lebesgue measurable on Ω_x.

Directional derivatives, λ_3-a.e. on $\mathbb{R}^2 \times [0, 2\pi)$ Augustus John Ward (1911-1984) showed in 1936 [76] that a Borel measurable function $f : \mathbb{R}^2 \to \mathbb{R}$ has Borel measurable $(x, \vartheta) \mapsto \partial^\vartheta f(x)$ directional Dini derivatives. From this, he shows that the two-variable case can be reduced to the one-variable Denjoy-Young-Saks theorem; he obtains that every Borel measurable function f has the Denjoy property at the point x on the line $l(x, \vartheta)$ for λ_3-a. e. (x, ϑ).[7] However if f is only Lebesgue measurable, Ward remarks that he cannot prove Lebesgue measurability of the directional Dini derivatives. Luckily, since Roy Osborne Davies in 1956 [21] disproves this. In fact, he constructs a function using transfinite methods, whose $(x, \vartheta) \mapsto \partial^\vartheta f(x)$ Dini derivatives are not Lebesgue measurable. He remarks that his construction can be modified in such a way that one obtains a Lebesgue measurable function $f : \mathbb{R}^2 \to \mathbb{R}$ with the following property. At every point of a subset $H \subseteq \mathbb{R}^2$ of positive measure, the set of directions in which the lower directional Dini derivatives have values a and c, the upper directional Dini derivatives have values b and d has full outer measure for every $a \le 0 \le b$, $c \le 0 \le d$. We present the proofs of these results in detail in Chapter 2.

Directional derivatives, at λ_2-a.e. point, in all directions The question whether a certain stronger form of Ward's theorem holds, was answered negatively by Abram Samuel Besicovitch (1891-1970) in a remarkable construction in his 1936 paper [8]. By this stronger form we mean the following question: Is it true that an $f : E \to \mathbb{R}$ function of the appropriate class (e.g. continuous) defined on an appropriate set E, at λ_2-a.e. point of the set E has the Denjoy property in all directions? Besicovitch constructed a continuous function $f : \mathbb{R}^2 \to \mathbb{R}$, a set $G \subseteq \mathbb{R}^2$ of positive measure such that for each point of G there exists a direction (even c many directions), where the section of the function doesn't have the Denjoy property. In these directions the function has one upper derivative that is $+\infty$ on one side, and all three other derivate numbers are 0. The construction can be sketched as follows:

Besicovitch constructs pairwise disjoint systems of circles $\mathcal{C}_n$ in a square such that for each point x of a set G of positive measure, there exist c many lines through x with

[7] More is obtained from his results on approximate derivatives; at λ_2-a.e. point the directional derivatives are not independent of each other: there exists a derivate plane, such that λ_1-a.e. directional derivative belongs to this plane. This is discussed in Chapter 2.

the following property: each line on one side of x locally intersects only finitely many circles, and on the other side of x it intersects one from each system $\mathcal{C}_n$ for $n \geq N$ for a certain N. Then by taking these $\mathcal{C}_n$ sets as the support of the function f, and placing right cones of appropriate height on each circle of $\mathcal{C}_n$, his result follows.

For a function of one variable it is impossible that any two derivate numbers be finite and distinct on a set of positive measure by the Denjoy-Young-Saks theorem. Besicovitch's example doesn't answer the seemingly natural question whether anything can be said in this sense for two variable functions, for λ_2-a.e. point of E, in all directions. In Chapter 3 we answer this question by the negative, in which a continuous function is given, and for each point of a set of positive measure there exists a direction in which the derivate numbers on one side are finite and distinct.

...

Directed derivatives In order to avoid the previously encountered problems posed by non-measurability, U. S. Haslam-Jones in [29] introduced the notion of *directed derivatives*. This notion is slightly less natural, however the stronger type of relations hold for these derivatives: Haslam-Jones already obtained positive results concerning the directed derivatives for λ_2-a.e. points in all directions.

The definition is the following. For $\vartheta \in [0, 2\pi)$ denote by e_ϑ the unit vector at an angle ϑ with the $\{(x,0) : x \geq 0\}$ half-line. Denote by $S_\alpha(\rho, \eta)$ the open angular sector originating from 0, with radius ρ, direction α and central angle 2η:

$$S_\alpha(\rho, \eta) := \{re_\vartheta : \vartheta \in [\alpha - \eta, \alpha + \eta], r \in [0, \rho]\}.$$

When α, ρ and η are of no importance, we simply write $S\angle$. Putting

$$B^\alpha(\rho, \eta, x) := \sup_{r \in S_\alpha(\rho,\eta)} \frac{f(x+r) - f(x)}{|r|},$$

the *directed upper derivative* is defined as

$$D^\alpha f(x) := \lim_{\eta \to 0} \lim_{\rho \to 0} B^\alpha(\rho, \eta, x).$$

The *directed lower derivative* is defined similarly and is denoted $D_\alpha f$. Haslam-Jones also gives the following definitions:

The function $f : \mathbb{R}^2 \to \mathbb{R}$ is said to have an *upper differential* at the point x if there exists a $d^+ f(x) \in \mathbb{R}^2$ such that for every α and η:

$$\limsup_{\rho \to 0,\, r \in S_\alpha(\rho,\eta)} \frac{f(x+r) - f(x) - < d^+ f(x), r >}{|r|} = 0.$$

We define analogously the *lower differential.* The following definition of Saks is equivalent:

The function $f : \mathbb{R}^2 \to \mathbb{R}$ is said to have an *upper differential* at the point x_0 if there exists a $d^+f(x_0) \in \mathbb{R}^2$ such that :

$$\limsup_{r\to 0} \frac{f(x_0+r) - f(x_0) - < d^+f(x_0), r >}{|r|} = 0,$$

and the contingent of the graph of f at the point $(x_0, f(x_0)) \in \mathbb{R}^3$ contains the plane[8]

$$\left\{(x,z) \in \mathbb{R}^2 \times \mathbb{R} : < d^+f(x_0), x - x_0 >= z - f(x_0)\right\}.$$

We define analogously the *lower differential.* If both exist at x, then they are necessarily equal, and the function is totally differentiable at the point x.

Concerning the measurability of directed derivatives, Haslam-Jones obtained that two variable Lebesgue measurable functions have Lebesgue measurable directed $D^\vartheta f$ Dini derivatives in every fixed direction ϑ. This obviously doesn't hold for non-measurable functions: decomposing the plane into two sets M, N of full outer measure, the directed Dini derivates of the characteristic function of N are not measurable for any fix directions.

It was already proven by G. C. Young in 1914 that for an arbitrary function, everywhere, except a countable set of points

$$D^+f(x) \geq D_-f(x) \qquad \text{and} \qquad D^-f(x) \geq D_+f(x)$$

holds. Ward in [74] by examining the structure of plane sets, obtained a two dimensional analogue of the theorem (using directed derivatives):

Theorem 5 (Ward, [74]) *For an arbitrary function $f : E \to \mathbb{R}$, defined on an arbitrary set $E \subseteq \mathbb{R}^2$, at every point except maybe a countable set of points there exists a direction α, such that*

$$D^\alpha f + D^{\alpha+\pi} \geq 0.$$

The following stronger form of the theorem does not hold: The set of points in which there exists an α such that the statement doesn't hold ($D^\alpha f + D^{\alpha+\pi} f < 0$) is not necessarily countable. To see this, simply take $f(x,y) := -|x|$. At each point of the y-axis in the direction 0 (the positive direction of the x-axis) $D^\alpha f + D^{\alpha+\pi} f < 0$ holds. However, that this set is of measure 0, follows from the following theorems.

[8] See the following paragraph for the definitions.

Haslam-Jones [29] obtained what can be considered as the analogue of the Denjoy-Young-Saks theorem, for directed derivatives of measurable functions. Subsequently Saks already removed some measurability conditions in the first edition of [63] and Ward extended the result to arbitrary functions in [75]:

Theorem 6 (Haslam-Jones–Ward, [29], [75]) *An arbitrary function $f : \mathbb{R}^2 \to \mathbb{R}$ at λ_2-a.e. point x satisfies one of the following relations:*

- *The function is totally differentiable at x,*
- *There exists an upper [lower] derivate plane and $D_\alpha f(x) = -\infty$ [$D^\alpha f(x) = +\infty$] in all directions α at the point x,*
- *$D^\alpha f(x) = \infty$ and $D_\alpha f(x) = -\infty$ in all directions α at x.*

For an arbitrary function $f : \mathbb{R}^2 \to \mathbb{R}$ define

$$H := \left\{ x \in \mathbb{R}^2 : \exists S\angle : \limsup_{S \ni h \to 0} \frac{f(x+h) - f(x)}{|h|} < \infty \right\}$$

and

$$K := \left\{ x \in \mathbb{R}^2 : \exists S\angle : \liminf_{S \ni h \to 0} \frac{f(x+h) - f(x)}{|h|} > -\infty \right\}.$$

Saks' following theorem (second edition of [63] p. 311), slightly extends the previous theorem:

Theorem 7 (Saks, [63]) *For an arbitrary function $f : \mathbb{R}^2 \to \mathbb{R}$ the following hold:*

- *In λ_2-a.e. point of $H \cap K$, f is totally differentiable,*
- *In λ_2-a.e. point of H [K] there exists an upper [lower] differential,*
- *The set*

  $$M := \left\{ x \in \mathbb{R}^2 : \exists S\angle : \lim_{S \ni h \to 0} \frac{|f(x+h) - f(x)|}{|h|} = \infty \right\}$$

  has measure zero.

Note that by definition, in every point of $H^c \cap K^c$ the directed derivatives of Haslam-Jones ($D^\alpha f$ and $D_\alpha f$) in every α direction are infinite with the appropriate sign, so this corresponds to the third case of Haslam-Jones' theorem. The proof of Saks' theorem relies on the contingence theorem in higher dimensions.

Contingent of a set The first definition of the contingent of a set is apparently due to Georges Bouligand (1889-1979) in 1932 [11]. Further examination of the contingent can be found in the thesis of Jean Mirguet [42] (and also [43]). Besicovitch in 1934 [7] obtains part of the contingence theorem, Andrei Nikolayevich Kolmogorov (1903-1987) and Ivan Yakovlevich Verčenko in 1934 [36], [37] obtain the more complete form of the theorem for plane sets. Both Saks [61] and Fréderic Roger (a student of Fréchet and Denjoy) rediscover the theorem in 1936, the latter author generalizing it to higher dimensions (presented by Émile Borel (1871-1956) at the French Academy of Sciences in 1935 [54], [56], [55], 1936 [58]). Roger called Saks' attention to his results, who stated and proved Roger's theorem in the supplement of [61]. In the subsequent paper [57], Roger remarks that this supplement states only his "simplest results", and that the article "stays silent" on his less simple, and more important results.[9] In response Saks proves Roger's theorem on one page [62].

We state the definitions and the contingence theorem for the case $\mathbb{R}^3$.

Define the *contingent* of a set $E \subseteq \mathbb{R}^3$ at a point x as the set $C(x)$ of all half-lines issuing from x, with the property that for each $l \in C(x)$ there exist l_n half-lines issuing from x, converging[10] to l such that there exist $x_n \in l_n \cap E$ such that $x_n \to x$. We say that the contingent of the set E at the point x is a half-space if $\cup C(x)$ is congruent to a half-space $\{(x, y, z) \in \mathbb{R}^3 : x \geq 0\}$. We say that the contingent of the set E at x is the whole space if $\cup C(x)$ is $\mathbb{R}^3$. We say that the contingent of the set E at x is a plane, if $\cup C(x)$ is an affine plane.

Theorem 8 (Contingence theorem) *Any set $E \subseteq \mathbb{R}^3$ can be decomposed into two sets P and Q such that*

- *at each point of P the contingent of E is the whole space,*
- *at $\mathcal{H}^2$-a.e. point of Q the contingent is either a plane or a half-space*
- *Q is $\mathcal{H}^2$ σ-finite.*

A proof of the theorem can be found in [63], p. 307.

Remark 2 (Further questions) Ward's theorem states that a Borel measurable function $f : \mathbb{R}^2 \to \mathbb{R}$ at λ_2-a.e. point, in λ_1-a.e. direction has the Denjoy-property. However it is not clear whether the domain can be partitioned to sets of positive measure, such that in λ_2-a.e.

[9] *"Aussi n'est-il pas étonnant qu'elles conduisent, comme le fait remarquer l'Auteur dans un supplément á son article, aux plus simples de mes résultats de l'espace. Cependant la tournure plus géométrique des méthodes que j'ai employées permet peut-être une extensions plus facile, notamment aux résultats d'oú je vais tirer un critère d'analyticité et sur lesquels la remarque de M. Saks reste muette."* [57]

[10] By convergence of half-lines we mean the convergence of the direction vectors of these half-lines.

point of these sets in λ_1-a.e. direction the Denjoy-behaviour of f is the same in that direction. Gholam-Hossein Mossaheb (1910-1979) answered this question in the negative (1950, [44]), by constructing a continuous function, such that at every point of a set of positive measure there are sets of directions of positive measure, on which the Denjoy-behaviour of the function is different.

For one variable functions Jerome Raymond Ravetz in [50] examined the Hausdorff dimension of the exceptional set, where the Denjoy relations do not hold. He constructed a continuous function $f : \mathbb{R} \to \mathbb{R}$, such that there exists a set $H \subseteq \mathbb{R}$ of Hausdorff dimension one, on which three of the Dini derivatives are 0, and one is $+\infty$.

1.1.2 Approximate derivatives

One-variable case [Definitions] Denjoy [24] and Aleksandr Yakovlevich Khintchine (1894-1959) [32] in 1916 both introduce the notion of approximate (or in Khintchine's words asymptotic) derivatives. These derivatives are defined by ignoring a set of outer density zero around each point.

A point $x \in E$ is called a *point of dispersion* of the set $E \subseteq \mathbb{R}$, if E has outer density 0 at the point x. Let us define the *approximate limits* of an arbitrary function $f : E \to \mathbb{R}$ defined on a set $E \subseteq \mathbb{R}$:

$$A^+ \lim_{y \to x} f(y) := \inf \{K \in \mathbb{R} \cup \{\pm\infty\} : x \text{ is a point of dispersion of } \{E \ni y > x : f(y) > K\}\},$$

$$A_+ \lim_{y \to x} f(y) := \sup \{K \in \mathbb{R} \cup \{\pm\infty\} : x \text{ is a point of dispersion of } \{E \ni y > x : f(y) < K\}\},$$

$$A^- \lim_{y \to x} f(y) := \inf \{K \in \mathbb{R} \cup \{\pm\infty\} : x \text{ is a point of dispersion of } \{E \ni y < x : f(y) > K\}\},$$

$$A_- \lim_{y \to x} f(y) := \sup \{K \in \mathbb{R} \cup \{\pm\infty\} : x \text{ is a point of dispersion of } \{E \ni y < x : f(y) < K\}\}.$$

Defining derivatives using approximate limits one obtains the *approximate Dini derivatives* of f:

$$AD^+ f(x) := A^+ \lim_{y \to x} \frac{f(y) - f(x)}{y - x},$$

$$AD_+ f(x) := A_+ \lim_{y \to x} \frac{f(y) - f(x)}{y - x},$$

$$AD^- f(x) := A^- \lim_{y \to x} \frac{f(y) - f(x)}{y - x},$$

$$AD_- f(x) := A_- \lim_{y \to x} \frac{f(y) - f(x)}{y - x}.$$

Several-variable case [Definitions] Analogously to directional classical Dini derivatives, *directional approximate Dini derivatives* can also be defined for an arbitrary two variable function $f : E \to \mathbb{R}$ defined on a set $E \subseteq \mathbb{R}^2$. Denote

$$E^+(x, \vartheta, K) := \{y \in E \cap l(x, \vartheta) : f(y) > K\},$$

and

$$E^-(x, \vartheta, K) := \{y \in E \cap l(x, \vartheta) : f(y) < K\},$$

where $l(x, \vartheta)$ denotes the half-line extending from the point x in the direction ϑ. First define the *directional* (or linear) *approximate limits* of such a function in the direction $0 \leq \vartheta < 2\pi$:

$$A^{\vartheta} \lim_{y \to x} f(y) := \sup \left\{ K \in \mathbb{R} \cup \{\pm\infty\} : x \text{ is a point of (linear) dispersion of } E^+(x, \vartheta, K) \right\},$$

$$A_{\vartheta} \lim_{y \to x} f(y) := \inf \left\{ K \in \mathbb{R} \cup \{\pm\infty\} : x \text{ is a point of (linear) dispersion of } E^-(x, \vartheta, K) \right\}.$$

Using directional approximate limits, the definition of *directional approximate Dini derivatives* follows:

$$A\partial^{\vartheta} f(x) := A^{\vartheta} \lim_{y \to x} \frac{f(y) - f(x)}{y - x},$$
$$A\partial_{\vartheta} f(x) := A_{\vartheta} \lim_{y \to x} \frac{f(y) - f(x)}{y - x}.$$

Measurability The distinction of the one and several variable case for questions of measurability of approximate derivatives is not necessary here, as the following result shows:

Theorem 9 (Khintchine-Saks, [33], [63]) *Let $f : E \to \mathbb{R}$ be a Lebesgue measurable function defined on a Lebesgue measurable set $E \subseteq \mathbb{R}^2$. For any fixed direction ϑ, the approximate partial Dini derivatives $x \mapsto A\partial^{\vartheta} f(x)$ and $x \mapsto A\partial_{\vartheta} f(x)$ are also Lebesgue measurable.*

The result for one variable functions has been obtained by Khintchine [33]. Besicovitch [9] obtained the theorem for continuous one variable functions, which has been subsequently extended to Lebesgue measurable functions by Burkill and Haslam-Jones [16] independently of Khintchine's result. The first mention of the multivariable case stating measurability of all sections was found in Saks [63], p. 299, Theorem 11.2. The following theorem of Ward [76] examines the measurability of $(x, \vartheta) \mapsto A\partial_{\vartheta} f(x)$:

Theorem 10 (Ward, [76]) *Let $f : E \to \mathbb{R}$ be a Lebesgue measurable function defined on a Lebesgue measurable $E \subseteq \mathbb{R}^2$ set. Then f has Lebesgue measurable $(x, \vartheta) \mapsto A\partial^{\vartheta} f(x)$ and $(x, \vartheta) \mapsto A\partial_{\vartheta} f(x)$ approximate directional Dini derivatives.*

Remark 3 (Further questions) Further results on measurability of ordinary approximate derivatives (as opposed to approximate Dini derivates) of one variable functions can be found in [26], [38]. In [26] it is shown that if f is an approximately differentiable function on an interval, then ADf is of Baire class one. In [38] it is shown that for any $f : \mathbb{R} \to \mathbb{R}$, if $R \subseteq \mathbb{R}$ denotes the points where the function is approximately differentiable and all points of R are points of outer density, then ADf is of Baire class two with respect to R.

One-variable functions Denjoy in 1916 [24], pp. 208–209 and Khintchine in 1924 [33] and [34] p. 212 independently discover the following theorem (which Besicovitch also discovers, however only for continuous functions [9]):

Theorem 11 (Denjoy-Khintchine [24], [33]) *Let $f : E \to \mathbb{R}$ be a Lebesgue measurable function defined on a set $E \subseteq \mathbb{R}$; then one of the following two Denjoy-properties for approximate derivatives holds λ_1-a.e.:*

- *f is approximately differentiable*
- *all four approximate derivate numbers of f are $\pm\infty$ with the appropriate sign.*

Shu-Er Chow in a 1948 paper [19] notices that Saks remarks in *Theory of the Integral* [63] [11] that the Denjoy-Khintchine theorem can be extended to arbitrary functions by a "slight modification" of the proof. However, Chow constructs a simple example, showing that the Denjoy-Khintchine theorem stated in the previous form does not generalize to arbitrary functions:

Example Take $(0,1) = I \cup J$ where I and J are disjoint and non-measurable sets of outer measure 1. Then by taking $f(x) := \chi_I(x)$ as the characteristic function of I, we obtain that at each point x of I: $AD^+f(x) = AD_-f(x) = 0$, $AD_+f(x) = -AD^-f = -\infty$, and on J: $AD^+f(x) = -AD_-f(x) = +\infty$, $AD_+f(x) = AD^-f(x) = 0$, contradicting the conclusion of the theorem for arbitrary functions.

However, upon closer inspection Saks states that the "slight modification" should be made in such a way that for arbitrary functions the definition of approximate differentiability should be modified. According to the new definition, the point x is said to be an point of approximate differentiability in the modified sense, if there *exists a set*, of which x is a point of outer density, and in which the function is differentiable. If interpreted in this light, Chow's counterexample fails since all points are approximately differentiable in the modified sense by taking I as this set for points of $x \in I$, and by taking J for points of $x \in J$.

Let us now return to the discussion of the original definition. Burkill and Haslam-Jones in 1931 [16] knowing of Besicovitch's result, first extend the result from continuous

[11] See page 297, discussion following Theorem (10.1) in Chapter IX.

functions to Lebesgue measurable functions. In their subsequent 1933 paper [18], p. 238, Theorem 10, they obtain a result for non-measurable functions. Ward in the same year obtains a partial result in [73], and completes the result of Burkill of Haslam-Jones in 1934 [75] p. 344, Theorem II. All these results are obtained using the notion of λ-*approximate derivates*, that is instead of sets of density 0, sets of density λ are ignored. The following theorem is not stated in terms of λ-approximate derivates, but it follows directly from their result.

Theorem 12 (Burkill–Haslam-Jones–Ward [18], [75]) *If $f : E \to \mathbb{R}$ is an arbitrary function defined on an arbitrary set $E \subseteq \mathbb{R}$, then λ_1-a.e. one of the following Denjoy relations holds for approximate derivates:*

- *f is approximately differentiable;*
- *two of f's opposite approximate derivatives (AD^+f, AD_-f) [(AD_+f, AD^-f)] are finite and equal, the two other opposite derivatives (AD_+f, AD^-f) [(AD^+f, AD_-f)] are infinite with the appropriate sign;*
- *all four approximate derivate numbers of f are $\pm\infty$ with the appropriate sign.*

Ralph Lent Jeffery (1889-1975) in 1935 [30], (cf. also [31] pp. 198-199) also obtained the previous result for arbitrary functions, however his results are based on the notion of metric separability[12] and the paper has been criticized for the unusual definition of the approximate derivatives, cf. [60]. Chow in his paper [19] also obtained the same result, using the metrical upper and lower boundaries $u(x), l(x)$ originally introduced by Henry Blumberg (1886-1950) in [10].

Remark 4 (Further questions) Alberti-Csörnyei-Laczkovich-Preiss in 2000 [3] examined the valid relations that hold for approximate derivates at $\mathcal{H}^1$-a.e. point of the graph of an arbitrary f real function.

. . .

Several-variable case Stepanoff in [71] obtains the following theorem (cf. also Saks [63] p. 300):

Theorem 13 (Stepanoff [71]) *If $f : \mathbb{R}^2 \to \mathbb{R}$ is a Lebesgue measurable function, then it is approximately differentiable a.e. if and only if f possesses approximate partial derivatives a.e..*

[12] Two sets of finite outer measure are said to be *metrically separable* if for every $\varepsilon > 0$ there exist neighborhoods of the sets, such that the intersection of these neighborhoods has measure less than ε. Or, the same: if the sum of their outer measures is the outer measure of their union. A function $f : E \to \mathbb{R}$ is said to be metrically separable if for every c the sets $E(f < c)$ and $E(f \geq c)$ are metrically separable.

Saks in [63], p. 312 also obtained further conditions on approximate differentiability of two variable Lebesgue measurable functions. Ward in 1937 [76] extended the result:

Theorem 14 (Saks-Ward, [63], [76]) *If $f : E \to \mathbb{R}$ is a Lebesgue measurable function defined on an $E \subseteq \mathbb{R}^2$ Lebesgue measurable set, then for λ_2-a.e. x in E, either*

- *In λ_1-a.e. ϑ direction $A\partial^\vartheta f(x) = \infty$, and there exists no approximate derivative plane, or*
- *There exists an approximate derivative plane $(a(x)\cos\vartheta + b(x)\sin\vartheta)$ and in λ_1-a.e. ϑ direction:*
$$A\partial^\vartheta f(x) = A\partial_\vartheta f(x) = a\cos\vartheta + b\sin\vartheta.$$

Andrew Michael Bruckner and Melvin Rosenfeld in 1968 [14] obtain that if a Lebesgue measurable function on the plane has approximate partial derivatives almost everywhere, then it has approximate directional derivatives at λ_2-a.e. point, in λ_1-a.e. direction. Note that this result follows from the previous two theorems.

The question arises, whether anything could be said about the approximate directional behaviour of an arbitrary function. By a transfinite construction we show in Theorem 29 that the answer is negative. In fact, we construct a function, which at no point $(x, \vartheta) \in M$ of a set $M \subseteq \mathbb{R}^2 \times [0, 2\pi)$ of positive outer measure possesses the approximate linear Denjoy property at x in the direction ϑ.

Approximate directed derivatives The approximate version of the directed derivatives AD^ϑ introduced by Haslam-Jones have been examined by Ward in [75], who obtained the following theorem:

Theorem 15 (Ward [75]) *If $f : E \to \mathbb{R}$ is an arbitrary real function defined on an arbitrary set $E \subseteq \mathbb{R}^2$, then for λ_2-a.e. point $x \in E$ we have either*

- *$AD^\vartheta f(x) = \infty$ for all $\vartheta \in [0, 2\pi]$, or*
- *there exists an upper approximate derivate plane at x.*

Note that this theorem is of the utmost generality, the strongest type of statement holds for arbitrary functions.

Remark 5 (Further questions) Fedor Isaakovich Shmidov using notions of approximate contingents also examined the problem of approximate differentiability for two variable functions in several papers, cf. [64], [65], [66], [68].

1.2 Category

1.2.1 Classical derivatives

One variable real functions The relation between the Dini derivatives from the category point of view was already examined by William Henry Young (1863-1942) [78] who proved that for continuous functions $D^+f = D^-f$ and $D_+f = D_-f$, except on a set of first category (cf. also Christoph J. Neugebauer (1927-2012) [46]).

Ludek Zajíček [80] and simultaneously Belna-Cargo-Evans-Humke [6] found the following theorem, which may be regarded as the category version of the Denjoy-Young-Saks theorem (cf. also [72] p. 176):

Theorem 16 (Zajíček, [80]) *For an arbitrary function $f : \mathbb{R} \to \mathbb{R}$ one of the following relations must hold, except on a first category set:*

- *The upper derivatives are equal ($D^+f = D^-f$) and the lower derivatives are equal ($D_+f = D_-f$);*
- *The opposite derivatives are infinite with the appropriate sign $D_-f = -\infty$, $D^+f = \infty$ [$D_+f = -\infty$, $D^-f = \infty$] and $D_+f \le D^-f$ [$D_-f \le D^+f$].*

If a function satisfies one of the previous relations at the point x, we say that it has the Zajíček property *at the point x.*

In fact it is shown that the exceptional set is σ-porous.

Directional derivatives in a generic point, generic direction If a two variable function f restricted to the line $l(x, \vartheta)$ satisfies the Zajíček property at the point x, we say that ϑ is a *Zajíček direction* of f at the point x. Adapting Ward's proof to category we obtain the following theorem:

Theorem 17 *If $f : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function (Baire measurable) then for a generic (x, ϑ), ϑ is a* Zajíček-direction *in x, that is one of the following hold must hold on a residual set $R \subseteq \mathbb{R}^2 \times [0, 2\pi)$:*

- *The upper directional derivatives are equal and the lower directional derivatives are equal;*
- *$\partial_{\vartheta+\pi} f = -\infty$, $\partial^{\vartheta} f = \infty$ and $\partial_{\vartheta} f \le \partial^{\vartheta+\pi} f$.*

For the proof, see Chapter 2.

For sevaral variables Ravetz in two articles (1955 [51], 1956 [52]) examines from the category point of view the properties of directed derivatives introduced by Haslam-Jones. He calls a point a *tangential singularity*, when there exists a direction μ, in which one of the following holds:

- $\vartheta \mapsto D^{\vartheta} f(x)$ or $\vartheta \mapsto D_{\vartheta} f(x)$ is not continuous in $\vartheta = \mu$
- $D^{\mu} f(x) > \partial^{\mu} f(x)$ or $D_{\mu} f(x) < \partial_{\mu} f(x)$
- $D^{\mu} f(x) = \infty$ or $D_{\mu} f(x) = -\infty$

Theorem 18 (Ravetz [51]) *For a continuous function $f : E \to \mathbb{R}$ on an arbitrary set $E \subseteq \mathbb{R}^2$, there exists a set $H \subseteq E$ residual in E, and which is the disjoint union of two relative open sets U and V, such that*

- *Neither point of V is a tangential singularity (in these points the directed derivatives $\vartheta \mapsto D^{\vartheta} f$ and $\vartheta \mapsto D_{\vartheta} f$ are continuous, finite and equal to the directional (linear) derivatives), and*
- *In every point x of U there is a direction μ such that in every direction $\vartheta \in [\mu, \mu + \pi]$:*
$$D^{\vartheta} f(x) = -D_{\vartheta+\pi} f(x) = \infty$$
holds.

From this theorem he obtains an analogue of W. H. Young's theorem stated at the beginning of this section:

Theorem 19 (Ravetz [51]) *If $f : E \to \mathbb{R}$ is a continuous function defined on an arbitrary set $E \subseteq \mathbb{R}^2$ and ϑ_0 is a fixed direction, then the set of points x where $\partial^{\vartheta_0} f(x) \neq \partial_{\vartheta_0+\pi} f(x)$ is a set of first category.*

1.2.2 Approximate derivatives

For approximate derivatives of one variable functions Zajíček [79] obtained a result similar to the Denjoy-Young-Saks theorem in the category sense. This was later strengthened in a joint paper with David Preiss in [48] to the following form of the theorem:

Theorem 20 (Preiss-Zajíček [48]) *For an arbitrary function $f : \mathbb{R} \to \mathbb{R}$ for all x except a set of first category at least one of the following must hold:*

- *Both upper and lower approximate derivatives are equal ($AD^+ f(x) = AD^- f(x)$ and $AD_+ f(x) = AD_- f(x)$)*
- *Two opposite derivatives are infinite with the appropriate sign (i.e. at least one of the relations $AD^+ f(x) = -AD_- f(x) = \infty$ or $AD^- f(x) = -AD_+ f(x) = \infty$ holds)*

In their paper it is also shown that for any given four (extended real) numbers satisfying any of the theorem's relations, there exists a function such that the relation is satisfied on a residual subset of the line. In this sense, this theorem is the strongest possible.

Remark 6 (Further questions) It should be mentioned that Michael J. Evans and Lee Larson in [25] examined what is in some sense the category analogue of approximate derivatives, that is this time instead of ignoring a set of outer density zero around each point, a set of first category is ignored. These are also known as qualitative derivatives, and are originally due to Solomon Marcus. They obtain a Denjoy type theorem for qualitative derivatives of arbitrary functions that hold on a residual set, similar to the theorem of Zajíček.

2 DYS for linear derivatives at a typical point and direction

"In the old days when people invented a new function they had something useful in mind. Now, they invent them deliberately just to invalidate our ancestors' reasoning, and that is all they are ever going to get out of them."

Henri Poincaré

This section deals in detail with the following question: What can be said about the linear derivatives of a two variable function at a typical point in a typical direction? Typical here is meant both from the measure and category point of view. Both ordinary linear derivates and approximate linear derivates are discussed.

2.1 Measure

2.1.1 Classical derivatives

Simple examples show that without restrictions nothing can be said about the linear Denjoy behaviour of an arbitrary two variable function.

Positive results Ward's following theorem can be regarded as the proper 2 dimensional analogue of the classical Denjoy-Young-Saks theorem for linear derivatives:

Theorem 21 (Ward, [76]) *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a two variable Borel measurable function. Then in λ_2-a.e. point, λ_1-a.e. direction is a Denjoy direction.*

Proof. First suppose that f is continuous. Then the sets

$$E(K,h) := \left\{(x,\vartheta) : \frac{f(x+re_\vartheta)-f(x)}{r} \le K, \forall r \in [0,h]\right\}$$

are closed for all (K,h). Since

$$\left\{(x,\vartheta) : \partial^\vartheta f(x) < K\right\} = \bigcup_n E\left(K-\frac{1}{n},\frac{1}{n}\right),$$

these sets are measurable (they are F_σ sets), $(x,\vartheta) \mapsto \partial^\vartheta f(x)$ is Lebesgue measurable. Denote by H the set of points (x,ϑ), for which ϑ is not a Denjoy-direction of f at x. H

is also measurable, since it can be written as a countable union and intersection of some $E(K,h)$ sets. Moreover H is of measure zero. Since, by Fubini's theorem, if H wasn't of measure zero, then there would be a direction ϑ_0, for which the section H^{ϑ_0} would be measurable and also be of positive measure (a.e. direction has this property). This would contradict the Denjoy-Young-Saks theorem, since (again using Fubini's theorem), in an appropriate point x_0 in the direction ϑ_0 the $r \mapsto f(x_0 + re_{\vartheta_0})$ one variable function would not satisfy the Denjoy-Young-Saks theorem on a set of positive measure. □

Negative results Davies' following example shows that this theorem cannot be strengthened to Lebesgue measurable functions:

Theorem 22 (Davies, [21]) *There exists a Lebesgue measurable function $f : \mathbb{R}^2 \to \mathbb{R}$, such that at λ_2-almost every point x, the set of directions ϑ, in which*

$$\partial^\vartheta f(x) = a, \qquad \partial_\vartheta f(x) = b, \qquad \partial^{\vartheta+\pi} f(x) = c, \qquad \partial_{\vartheta+\pi} f(x) = d$$

hold, is of full outer measure in $[0, 2\pi)$, for every $a \geq 0 \geq b$, $c \geq 0 \geq d$.

Proof. Denote by M the set of points with at least one rational coordinate

$$M := (\mathbb{Q} \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{Q}).$$

Using transfinite recursion we construct the function in such a way that each point of M^c will have the property of the statement. The set M of plane measure null contains the support of the resulting function f, hence the function is Lebesgue measurable.

Denote by $\mathcal{F}$ the set of all closed sets of $(0, 2\pi)$ of positive measure. Now enumerate the elements of the following set:

$$M^c \times \mathcal{F} \times \mathbb{R}^2_+ \times \mathbb{R}^2_- = (p_\alpha, F_\alpha, (a_\alpha, c_\alpha), (b_\alpha, d_\alpha))_{\alpha < c},$$

where c is the cardinal continuum.

Choose any $\vartheta_0 \in F_0$ and denote the line of angle ϑ_0 passing through p_0 by l_0. Define $f_0 : M^c \cup l_0 \to \mathbb{R}$ as 0 on M^c and such that at p_0 on the line l_0 the linear Dini derivatives are a_0, b_0, c_0, d_0 in the prescribed order.

Let $\alpha < c$, and suppose that for each $\beta < \alpha$ a line l_β of angle $\vartheta_\beta \in F_\beta$ has been selected, such that $M \cap (\cup_{\gamma<\beta} l_\gamma) \cap l_\beta = \emptyset$ and that a function

$$f_\beta : M^c \cup (\cup_{\gamma<\beta} l_\beta) \to \mathbb{R}$$

has been defined with the desired non-Denjoy behaviour at each point p_γ on the line l_γ prescribed by $(a_\gamma, b_\gamma, c_\gamma, d_\gamma)$ for all $\gamma < \beta$. Choose a line l_α passing through p_α that is disjoint from $M \cap (\cup_{\beta<\alpha} l_\beta)$. This is possible, since so far only $\beta < \alpha$ lines intersect M, each in ω points; hence these define less than c many "forbidden" directions, whereas F_α is of cardinality c. Since p_α is a limit point of $l_\alpha \cap M$ and $l_\alpha \cap M$ is disjoint from the domain of all f_β for all $\beta < \alpha$, the desired non-Denjoy behaviour can be prescribed on the line $l_\alpha \cap M$ giving a function g_α. Then we define $f_\alpha := \bigcup_{\beta<\alpha} f_\beta \cup g_\alpha$.

This gives a function

$$f_c : M^c \cup (\cup_{\alpha<c} l_\alpha) \to \mathbb{R}.$$

We define the function on the rest of the points as 0. Thus we obtained a function with the following property: in each point p of a set M^c of full measure in the plane, for any $a \geq 0 \geq b$, $c \geq 0 \geq d$, the set of directions in which

$$\partial^\vartheta f(x) = a, \qquad \partial_\vartheta f(x) = b, \qquad \partial^{\vartheta+\pi} f(x) = c, \qquad \partial_{\vartheta+\pi} f(x) = d$$

simultaneously hold intersects each closed set of positive measure. □

2.1.2 Approximate derivatives

Positive results The main result of Ward's article [76] is the result on approximate derivatives. The following theorem is obtained:

Theorem 23 (Ward, [76]) *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a two variable Lebesgue measurable function. Then at λ_2-a.e. point x, the following two cases are possible:*

- *In λ_1-a.e. direction ϑ, $A\partial^\vartheta f(x) = \infty$.*
- *The function is approximately differentiable, $ADf(x) = (a, b)$, and for λ_1-a.e. direction ϑ:*

$$A\partial^\vartheta f(x) = A\partial_\vartheta f(x) = a\cos\vartheta + b\sin\vartheta.$$

Denote by $d_\vartheta(x, E)$ the linear outer density in direction ϑ of the set $E \subseteq \mathbb{R}^2$. The proof is based on the following two lemmas:

Lemma 1 *If $E \subseteq \mathbb{R}^2$ is a Lebesgue measurable set, then for λ_3-a.e. $(x, \vartheta) \in E \times [0, 2\pi)$ the set $\{r : x + re_\vartheta \in E\}$ is Lebesgue measurable and $d_\vartheta(x, E) = 1$.*

Lemma 2 *If $f : E \to \mathbb{R}$ is a Lebesgue measurable function defined on a Lebesgue measurable set $E \subseteq \mathbb{R}^2$, then $(x, \vartheta) \mapsto A\partial^\vartheta f(x)$ is Lebesgue measurable.*

Proof. (Ward's theorem) Denote by $F := \{(x, \vartheta) \in E \times [0, 2\pi) : A\partial^\vartheta f(x) < \infty\}$. By Lemma 2 this set is measurable. We want to show that for λ_3-a.e. $(x, \vartheta) \in F$ there exists an approximate derivate plane at x. Take those points $x \in E$ in which there exist distinct (or infinite) partial approximate Dini derivatives; this set is measurable by the Khintchine-Saks theorem. Furthermore, by Stepanoff's theorem (Theorem 13) this set differs from the set $\hat{H} \subseteq E$ consisting of the points in which there is no approximate derivate plane by a set of measure 0, so $\hat{H}$ is also measurable. Denote $H := \hat{H} \times [0, 2\pi) \cap F$, that is those points $(x, \vartheta) \in F$ in which there is no approximate derivate plane of f at x. Now suppose to the contrary that H has positive measure.

By Lebesgue's density theorem, we can find a cube C of side length $\delta < \pi$ such that if $Q := H \cap C$ then $\lambda_3(Q) > \delta^3/2$. By Fubini's theorem a.e. section of the measurable set Q is measurable. It follows that there exist two directions ϑ_1, ϑ_2, such that $|\vartheta_1 - \vartheta_2| \leq \delta$ and the projections of the sections $Q_1 := \pi_x(Q^{\vartheta_1})$ and $Q_2 := \pi_x(Q^{\vartheta_2})$, are measurable and have measure greater than $\delta^2/2$; here π_x denotes the projection on the x-plane.

Since Q_1 and Q_2 are subsets of $\pi_x(F)$, in none of their points does there exist an approximate derivate plane. Also, since Q_1 and Q_2 both have plane measure greater than $\delta^2/2$, their intersection Q' has positive measure. Thus we obtained a plane set of positive measure, at each point x of which there exists no approximate derivate plane, and the linear approximate derivatives in two fix directions are finite: $A\partial^{\vartheta_1} f(x) < \infty$, and $A\partial^{\vartheta_2} f(x) < \infty$. By Stepanoff's theorem (and by a transformation of variables) it follows that

$$A\partial^{\vartheta_i} f(x) = A\partial_{\vartheta_i} f(x) = -A\partial^{\vartheta_i+\pi} f(x) = -A\partial_{\vartheta_i+\pi} f(x), \qquad i = 1, 2$$

cannot hold at any point $x \in Q'$ simultaneously, since then the approximate derivate plane would exist. Suppose that this doesn't hold in the direction ϑ_1 on a set Q'' of positive measure.

By Fubini's theorem again, since almost every ϑ_1 directional section of the Lebesgue measurable set Q'' is measurable and has positive measure, there exists a section on which on a set of positive linear measure $A\partial^{\vartheta} f$ is finite, but not all four approximate Dini derivatives of f are equal. This is impossible. $\square$

Negative results Since the Denjoy-relations hold for arbitrary one variable functions concerning approximate Dini derivatives, the question arises whether anything can be said in the case of arbitrary two variable functions for linear approximate derivatives. The following example shows that the answer is negative if one assumes the continuum hypothesis. We show the following:

Theorem 24 *Assuming the continuum hypothesis, there exists a function $f : \mathbb{R}^2 \to \mathbb{R}$, such that for a set $M \subseteq \mathbb{R}^2 \times [0, \pi)$ of positive outer measure, in each $(x, \vartheta) \in M$ the direction ϑ is not an approximate Denjoy direction for f at x.*

Proof. Following Sierpiński, we first construct the (non-measurable) set $M \subseteq \mathbb{R}^2 \times [0, \pi)$ of full outer measure with the property that each ϑ-section and each x-section contains at most one point of M. We assume the continuum hypothesis.

Let $\mathcal{F}_3$ denote the closed sets of $\mathbb{R}^2 \times [0, 2\pi)$ of positive measure. Take a well-ordering of

$$\mathcal{F}_3 = (F_\alpha)_{\alpha<c}.$$

Suppose that for $\beta < \alpha$ the $(p_\beta, \vartheta_\beta) \in F_\beta$ points have been chosen with the desired properties. Since F_α is a closed set of positive measure, its projection $\pi(F_\alpha)$ on $\mathbb{R}^2$ also has positive measure. Since less than $\alpha < c$ points of S have been chosen so far, and the projection $\pi(F_\alpha)$ is of cardinality c, there exists a $p_\alpha \in \pi(F_\alpha)$, such that the p_α-section of F_α doesn't contain any ϑ_β ($\beta < \alpha$). Take any ϑ_α such that $(p_\alpha, \vartheta_\alpha) \in F_\alpha$. The set $M = (p_\alpha, \vartheta_\alpha)_{\alpha<c}$ thus obtained has the desired properties.

Take through each point $p \in \pi_x(M)$ the line $l(p)$ of angle $\vartheta(p)$. By the construction of M, the lines $l(p)$ cannot coincide, or be parallel. Now for each such p we will construct a set $E(p) \subseteq l(p)$, with the property that p is a point of linear outer density 1 of $E(p)$ and such that the sets $E(p)$ are disjoint.

Let $\mathcal{F}_1 := (F^1_\alpha)_{\alpha<c}$ denote the closed sets of $\mathbb{R}$ of positive measure. Fix in advance linear isomorphisms between the lines $l(p)$ and $\mathbb{R}$. When we talk about the sets F of $\mathcal{F}_1$ as subsets of these lines $l(p)$, we mean it by these fix isomorphisms. Take a well-ordering of

$$M \times \mathcal{F}_1 = (p_\alpha, \vartheta_\alpha, F^1_\alpha)_{\alpha<c}.$$

For simplicity denote $l_\beta := l(p_\beta)$. For each $\alpha < c$ we choose a point $q_\alpha \in F^1_\alpha$ with the additional property that $q_\alpha \notin l_\beta$ for any $\beta < \alpha$, with $l_\beta \neq l_\alpha$. This is possible; suppose the $q_\beta \in l_\beta$ points have been selected for $\beta < \alpha$. Since less than c many points q_β have been selected so far and the set F_α being of cardinality continuum, there exists a point

$$q_\alpha \in F^1_\alpha \setminus \bigcup_{\substack{\beta<\alpha \\ l_\beta \neq l_\alpha}} (l_\beta \cap l_\alpha).$$

Now define the sets $E(p)$ as the union of all those q_β, where q_β was chosen for $p_\beta = p$. By construction, we obtain sets $E(p)$ that are disjoint and of full linear outer measure on l_α.

We have yet to check that p_α is not a linear density point of $\bigcup_{q\neq p} E(q)$ on the line l_α. Since, for each α, in at most $\alpha < c = \omega_1$ many instances can there be a q_β on the line l_α, there are only countably many $q_\beta \in l_\alpha$ (assuming the continuum hypothesis).

Now the approximate Dini derivatives can be simultaneously set, almost arbitrarily on these $E(p)$ sets, with the following restriction: on these lines the complement of $E(p)$ might also have even full outer measure, so the values on $l(p)\backslash E(p)$ might influence the approximate linear derivative behaviour. However, by defining f as zero on the complement of $\bigcup_{p\in M} E(p)$ any $a(p) \geq 0 \geq b(p)$, $c(p) \geq 0 \geq d(p)$ values (allowing $\pm\infty$) can be assigned in any points to the four linear approximate Dini derivatives. □

Remark 7 The previous proof works under the weaker condition non $\mathcal{N} = c$, where non $\mathcal{N}$ is the least cardinal of any set which has positive outer measure.

2.2 Category

2.2.1 Classical derivatives

Applying Ward's proof verbatim, only changing the wording to category equivalents, we obtain the following theorem:

Theorem 25 *If $f : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function (Baire measurable) then for a typical (x, ϑ), ϑ is a* Zajíček-direction *in x, that is one of the following hold must hold on a residual set $R \subseteq \mathbb{R}^2 \times [0, 2\pi)$:*

- *The upper directional derivatives are equal and the lower directional derivatives are equal;*
- *A $\partial_{\vartheta+\pi} f = -\infty$, $\partial^{\vartheta} f = \infty$ and $\partial_{\vartheta} f \leq \partial^{\vartheta+\pi} f$.*

Proof. First suppose that f is continuous. Then the

$$E(K, h) := \left\{(x, \vartheta) : \frac{f(x + re_\vartheta) - f(x)}{r} \leq K, \forall r \in [0, h]\right\}$$

sets are closed for all (K, h). Also, since the sets

$$\left\{(x, \vartheta) : \partial^{\vartheta} f(x) < K\right\} = \bigcup_n E\left(K - \frac{1}{n}, \frac{1}{n}\right)$$

have the Baire property (they are F_σ sets), $(x, \vartheta) \mapsto \partial^{\vartheta} f(x)$ is Baire measurable. Denote by H the set of (x, ϑ) points, for which ϑ is not a Zajíček-direction of f at x. H also has the Baire property, since it can be written as a countable union and intersection of some of the sets $E(K, h)$. Moreover if H wasn't a set of the first category. Since using the Kuratowski-Ulam theorem, if H wouldn't be of the first category, then there would be a direction ϑ_0, for which the section H^{ϑ_0} would have the Baire property and also be of the second category (a typical direction has this property). This would contradict Zajíček's theorem, since (using the Kuratowski-Ulam theorem), in an appropriate point x_0 in the direction ϑ_0 the $r \mapsto f(x_0 + re_{\vartheta_0})$ one variable function would not satisfy Zajíček's theorem on a second category set. □

3 DYS for linear derivatives (λ_2-a.e. in all directions)

"I recoil in fear and loathing from that deplorable evil: continuous functions with no derivatives."

Charles Hermite
in a letter to Stieltjes, 1893

In this chapter we only examine the question from the measure point of view.

3.1 Measure

3.1.1 Classical derivatives

The first result of this sort is due to Besicovitch, who constructed the following counterexample:

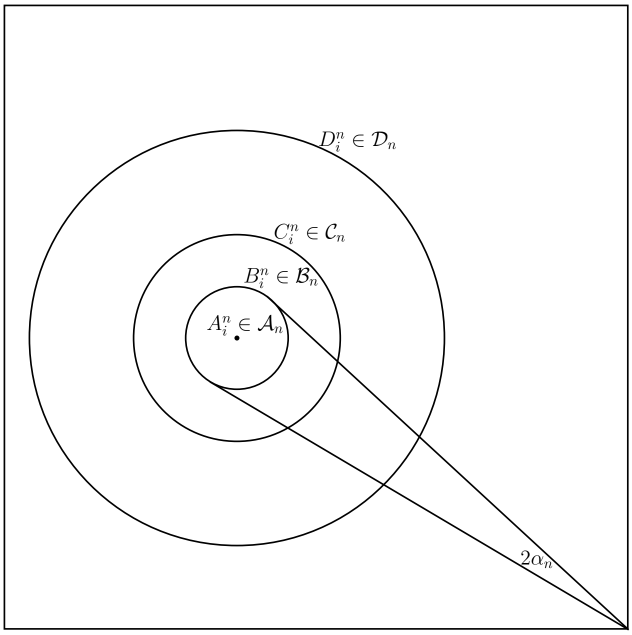
Theorem 26 (Besicovitch, [8]) *There exists a continuous function $f : [0,1]^2 \to \mathbb{R}$ such that, at every point of a set $H \subseteq [0,1]^2$ of positive measure, there exist c many directions in which the function f does not have the Denjoy property.*

Proof. We construct pairwise disjoint systems $\mathcal{C}_n$, each consisting of disjoint circles, each circle in $\mathcal{C}_n$ having radius r_n to be defined later. The aim is the following: each line on one side of x locally intersects only finitely many circles, and on the other side of x it intersects one from each system $\mathcal{C}_n$ for $n \geq N$ for a certain N. Then by taking as the support of the function f these $\mathcal{C}_n$, and placing right cylinders of appropriate height on each circle of $\mathcal{C}_n$, the function can be made continuous in such a way that three of the derivates are 0 and one is $+\infty$.

Take disjoint finite systems of points $\mathcal{A}_n$, which are the centers of the circles of $\mathcal{C}_n$. Denote by $\mathcal{B}_n$ and $\mathcal{D}_n$ the systems of circles with the same centers $\mathcal{A}_n$ with radii $r_n/2$ and $2r_n$ respectively. Denote by E_n the finite union of strips whose points through which there is a line intersecting more than one circle from $\mathcal{C}_n$. Denote by $2\alpha_n$ the smallest angle under which a circle of $\mathcal{B}_n$ can be seen from a point in $[0,1]^2$.

Let us choose the r_n radii and the centers $\mathcal{A}_n$ recursively: choose r_1 and $\mathcal{A}_1$ arbitrarily (ensuring that the entire $[0,1]^2$ square is not covered). By our previous definitions, this defines the angle α_1 and the systems $\mathcal{B}_1$ and $\mathcal{D}_1$.

Now suppose $\mathcal{A}_n$ and r_n have been defined. Choose the radius r_{n+1} and centers $\mathcal{A}_{n+1}$ in such a way that every angular sector originating in a point outside $\cup_{i=1}^{n}\mathcal{D}_i$ with central angle α_n and radius r_n contains a circle of $\mathcal{D}_{n+1}$. This is always possible, since the $\mathcal{D}_n$ systems of circles are not required to be disjoint (as opposed to the $\mathcal{C}_n$). It is also clear that for fixed $\mathcal{A}_n$ centers, by choosing r_n sufficiently small, $\lambda^2(E_n) < \varepsilon/2^n$ can simultaneously be achieved, since decreasing the radius does not destroy the previous property. Also assume $\sum_n \lambda^2(\mathcal{D}_n) < 1$.



Now we define the set G that is going to satisfy the requirements of the theorem. First, define:

$$G_n := \bigcup_{i=n}^{\infty} E_i \cup \bigcup_{i=1}^{\infty} \mathcal{D}_i.$$

Since

$$\lambda^2(G_n) \to \lambda^2 \left(\bigcup_{n=1}^{\infty} \mathcal{D}_n \right) < 1,$$

it follows that

$$G := [0,1]^2 \setminus \bigcap_{n=1}^{\infty} G_n$$

has positive measure.

Any point x not belonging to G_n has the following property: any line through x meets at most one circle from $\mathcal{D}_i$ for all $i \geq n$. Take an arbitrary angle γ and choose an N large enough, such that $2\alpha_N \leq \gamma$. An angular sector with vertex x and angle γ

contains at least two disjoint sectors with radius r_N, angle $\gamma_N < \alpha_N$ circumscribing a circle of $\mathcal{B}_{N+1}$.

By definition, $2\alpha_{N+1}$ is the smallest angle in which a circle of $\mathcal{B}_{N+1}$ can be seen, thus $2\alpha_{N+1} \leq \gamma_N$.

Repeating this argument for the sectors of angle α_{N+1} an infinity of nested angular sectors are obtained, each such nested sequence defining a common half-line L, resulting in c many lines. Every such half-line L meets a circle of $\mathcal{B}_{N+i}$ at a distance $\leq r_{N+i-1}$ for every i. By construction the opposite half-line does not meet any circle of these $\mathcal{B}_{N+i}$ since any line through $x \notin G_n$ can meet at most one circle of $\mathcal{D}_i$ for all $i \geq n$.

Now it remains to define the function: take a sequence of h_n such that $h_n \to 0$ and $h_n/r_{n-1} \to \infty$. We construct the graph of the function by placing right cylinders on each circle of $\mathcal{C}_n$ of height $2h_n$. This function is easily seen to be continuous. On the other hand the half-lines L meet each $\mathcal{B}_n$ from an n on at a distance less than r_{n-1}. Denote such a point of intersection by y_n:

$$\frac{f(y_n) - f(x)}{|y_n - x|} \geq \frac{h_n}{r_{n-1}} \to \infty$$

as $n \to \infty$ since at each $\mathcal{B}_n$ the value of the function is at least h_n. The other side of such a line meets a finite number of circles, so there is a neighborhood on this line disjoint from the support of the function. It follows that at any point belonging to the set G of positive measure in c many directions three of the linear Dini derivates are zero and one is infinite. $\square$

Remark 8 By Ward's theorem it follows that at a.e. point the set of such directions forms a set of measure zero.

What can be said about the directional one-sided Denjoy property[13]? Is it true that for every continuous function, at a.e. point, every direction is a one-sided Denjoy direction? The following counterexample shows that even when considering continuous functions, such a stronger type relation doesn't hold:

Theorem 27 *There exists a continuous function $f : [0,1]^2 \to \mathbb{R}$ such that, in every point of a set $H \subseteq [0,1]^2$ of positive measure, there exists a direction in which the function f has two finite and distinct linear Dini derivatives.*

[13] By the directional one-sided Denjoy property we mean that $-\infty < \partial_\vartheta f(x) < \partial^\vartheta f(x) < \infty$ doesn't hold at x in direction ϑ.

Proof. The construction of such a function follows the lines of Davies' construction of a positive measure accessible set.

In the rest of the proof *all* parallelograms have two sides parallel to $[0,1] \times \{0\}$ and $[0,1] \times \{1\}$; these are called the bases of the parallelograms, the other two sides are referred to as sides of the parallelogram, and the direction determined by the sides are called the direction of the parallelogram.

We are going to use the following notations: B^n $[B^{<n}]$ is the set of binary sequences (consisting of $\{0,1\}$) of length n [less than n]. (So $\varepsilon \in B^n$ if $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$, where $\varepsilon_i \in \{0,1\}$.) Denote by $B^{\mathbb{N}}$ $[B^{<\mathbb{N}}]$ the set of infinite [finite] binary sequences. The first n digits of a sequence $\varepsilon \in B^{\mathbb{N}}$ are denoted $\varepsilon|_n$; similarly for elements of B^m, where $m > n$. Denote by $\varepsilon_1 * \varepsilon_2$ the concatenation of the finite sequences ε_1 and ε_2. By a slight abuse of notation we write $B^{<n} * 2$ for the set of binary sequences of length less than n with a 2 concatenated at the end of each sequence (thus not being a binary sequence any more).

Take the square $S := [0,1] \times [0,1)$. We are going to recursively define the function and the set with the required properties simultaneously. The result of each nth step are the following:

- A system $\mathcal{S}_n$ consisting of disjoint horizontal strips $S_\varepsilon \subseteq S_{\varepsilon|_{n-1}}$ where
$$\varepsilon \in B^n \cup B^{<n} * 2.$$
For $\varepsilon \in B^n$ we call the sets S_I *nth stage binary strips*. The points that only belong to binary strips are called *binary points*. The rest of the points are called *triadic points*.

- For every $\varepsilon \in B^n$, a *principal direction* ϑ_ε.

- For every $I \in B^n$ a finite collection of parallelograms $\mathcal{P}_I$ contained in $\mathcal{S}_\varepsilon$, each of direction ϑ_ε such that every $P \in \mathcal{P}_\varepsilon$ is contained in a previous parallelogram $P' \in \mathcal{P}_{\varepsilon|_{n-1}}$, and the union of all parallelograms contained in P' almost cover $P' \backslash \mathcal{S}_{\varepsilon*2}$.

- For each $\varepsilon \in B^{n-1}$ a finite collection of linear segments $\mathcal{R}_\varepsilon \subseteq S_{\varepsilon*2}$, each in the principal direction $\vartheta_{\varepsilon*1}$ with the following property: the projection of $\mathcal{R}_\varepsilon$ in the direction $\vartheta_{\varepsilon*0}$ is almost the entire base of S, meanwhile the projection in the direction $\vartheta_{\varepsilon*1}$ is a finite number of points. We will call such a collection of segments a *venetian blind* (VB).

- A function
$$f_n : \bigcup_{\varepsilon \in B^{<n}} S_{\varepsilon*2} \to \mathbb{R},$$

always extending the previous function: $f_{n-1} \subseteq f_n$. The support of the functions is contained in a small neighborhood of the VBs $\mathcal{R}_\varepsilon$, $\varepsilon \in B^n$.

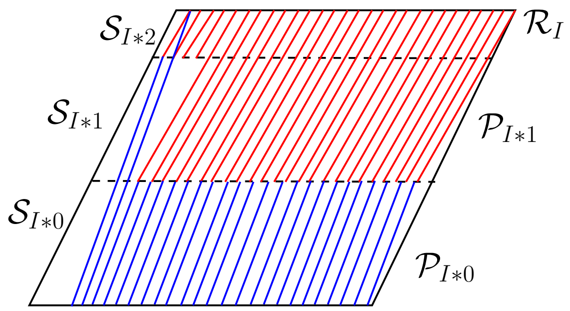


Figure 1: A venetian blind type construction

Take a point p in a $P \in \mathcal{P}_\varepsilon$ parallelogram where $\varepsilon \in B^n$. We say that p *sees the VB* $\mathcal{R}_{\varepsilon'}$ where $\varepsilon' \in B^{\leq n}$, if any line intersecting the bases of P also intersects the VB. If no such line intersects the VB, then the VB is *invisible* from that point. In order for these definitions to be well-defined we have to achieve that in each step of the recursion the previous visibility properties must not change. In other words: any point that has seen a VB in the previous stages still has to see the VB in future stages and any VB that was invisible from a point must remain invisible.

We construct the parallelograms in such a way that the intersection of any nested sequence of parallelograms $P_1 \supseteq P_2 \supseteq \dots$ containing a point p defines a line l, intersecting the bases of each P_i. This is the line through p on which the linear Dini derivatives are going to be finite and distinct.

The linear Dini derivatives should be finite. This is achieved in such a way that the line l intersects only those VBs $\mathcal{R}_\varepsilon$ where $\varepsilon \in B^n$ ends in a 0, thus the points of $\mathcal{S}_{\varepsilon*1}$, "close" to the VB cannot see it. Hence any binary point that sees a VB $\mathcal{R}_\varepsilon$ has a lower bound on the distance from it.

The linear Dini derivatives should be distinct. This can be obtained by setting the values of the function on the VBs $\mathcal{R}_\varepsilon$ such that for points of $\mathcal{S}_{\varepsilon*0}$ the difference quotient is between two fix positive numbers, $u > l > 0$.

Now we give the explicit construction. First, we define the horizontal strips S_ε. In the first step of the construction divide S into

$$S_0 := [0,1]\times[0,(1-\eta_1)/2), \qquad S_1 := [0,1]\times[(1-\eta_1)/2, 1-\eta_1), \qquad S_2 := [0,1]\times[1-\eta_1, 1).$$

If $\varepsilon \in B^n$ and

$$S_\varepsilon = [0,1] \times [a_\varepsilon, b_\varepsilon),$$

denote by $h_n := b_\varepsilon - a_\varepsilon$ the height of S_ε (depending only on n) and $m_\varepsilon := (a_\varepsilon + b_\varepsilon - \eta_n h_\varepsilon)/2$ the midpoint of a_ε and $b_\varepsilon - \eta_n h_n$. In each subsequent step we subdivide the S_ε sets into

$$S_{\varepsilon*0} := [0,1]\times[a_\varepsilon, m_\varepsilon), \qquad S_{\varepsilon*1} := [0,1]\times[m_\varepsilon, b_\varepsilon - \eta_n h_n), \qquad S_{\varepsilon*2} := [0,1]\times[b_\varepsilon - \eta_n h_n, b_\varepsilon),$$

where ε_n are to be defined later. (In the initial step $h_0 = 1$, $m_{[]} = (1-\varepsilon_1)/2$.) Notice that any point $(x,y) \in S$ has a finite triadic or an infinite binary representation

$$\varepsilon_y \in (B^{<\mathbb{N}} * 2) \cup B^{\mathbb{N}}$$

associated to it (depending on y). By an appropriate choice of the η_n (say $\eta/4^n$), the set

$$\bigcup_{\varepsilon \in B^{<\mathbb{N}}} S_{\varepsilon*2}$$

can be made to have arbitrarily small measure ($\sum_n \eta 2^n/4^n = \eta$).

We make the following obvious geometric remark: Let $ABCD$ be a parallelogram, l a fixed line parallel to AB. Denote by E and F the intersection of l and the extension of the side AD and the diagonal AC respectively. The remark is the following: As the width of the parallelogram tends to zero (keeping A and D fixed) EF also tends to zero.

Now we give the construction of the VB, the parallelograms and the function. Suppose we have constructed everything until the stage $n-1$. Take an arbitrary $\varepsilon \in B^{n-1}$ and let us do the recursion step in a single $P \in \mathcal{P}_\varepsilon$ (and then applying the same construction in every other $P' \in \mathcal{P}_\varepsilon$). We want to achieve the following:

- The area of P not covered by the parallelograms of the following stage is a δ_n (to be precised) fraction of the area of P.
- Every line, through every point of the new parallelograms $P' \in \mathcal{P}_{\varepsilon*0}$ intersecting both bases of P' intersect the new VB $\mathcal{R}_\varepsilon$.
- No line, through no point of the new parallelograms $P' \in \mathcal{P}_{\varepsilon*1}$ that intersect both bases of P' intersects the new VB $\mathcal{R}_\varepsilon$.
- Every new parallelogram $P' \subseteq P$ has the property that if a line intersects both bases of P', then it intersects both bases of P.

Take two strips $L_1, L_2 \subseteq P$ both of width δ_n in the direction ϑ_ε at both ends of the parallelogram $P \in \mathcal{P}_\varepsilon$; this is the area that is not going to be entirely covered by the new parallelograms. By choosing $\delta_n \to 0$ fast enough, in the end the loss can be achieved to be negligible.

The principal direction $\vartheta_{\varepsilon*0}$ is chosen to be ϑ_ε. Note that in this case, any line intersecting both bases of a parallelogram of $\mathcal{P}_{\varepsilon*0}$ has a direction in $(\vartheta_\varepsilon - \psi, \vartheta_\varepsilon + \psi)$ for some ψ. As the width w_0 of the parallelograms of $\mathcal{P}_{\varepsilon*0}$ tends to zero, ψ also tends to zero. By choosing any width less than δ_n, the parallelograms of $\mathcal{P}_{\varepsilon*0}$ have the property that any line intersecting both bases of the parallelogram also intersects the bases of the previous parallelogram.

When choosing the direction $\vartheta_{\varepsilon*1}$ two constraints have to be taken into account: the VB of this direction has to be always seen from points of $\mathcal{S}_{\varepsilon*0}\backslash(L_1 \cup L_2)$, and any line intersecting both bases of a parallelogram of $\mathcal{P}_{\varepsilon*1}$ must intersect the bases of the previous parallelogram. The VB is chosen as the extension of the sides of the parallelograms of $\mathcal{P}_{\varepsilon*1}$.

The first property can be achieved by choosing an angle outside the interval $(\vartheta_\varepsilon - \psi, \vartheta + \psi)$ (since the VB cannot be parallel to any line intersecting both bases of a parallelogram of $\mathcal{P}_{\varepsilon*0}$) and putting a VB densely enough (e.g. by taking the width w_1 less than the width of the parallelogram of height ε_n, direction $\vartheta_{\varepsilon*1}$ and diagonal of direction $\vartheta_\varepsilon + \psi$).

Extend the two sides of the first parallelogram of $\mathcal{P}_{\varepsilon*1}$ to the right of L_1 and denote the intersection of the extensions with the base by A and B. Since decreasing w_0 decreases ψ, $\vartheta_{\varepsilon*1}$ can be brought close enough to ϑ_ε, such that both A and B fall in the interior of L_1. Decreasing w_1 decreases the angle of the diagonal, hence by the geometric remark made in the beginning the second property can also be obtained: any line intersecting both bases intersects the bases of the previous parallelogram.

Finally take an ρ_n neighborhood of the VBs $\mathcal{R}_n$, and define a continuous function, such that it takes the value z_n on the points of the VB and is zero outside the neighborhood. The numbers z_n should be chosen such that $z_n \to 0$ and $u > z_n/d_n \geq z_n/D_n \geq l$ where $D_n = h_n$ denotes the upper and $d_n = h_n(1-\varepsilon_n)/2$ denotes the lower bound on the distance of a point of $\mathcal{S}_{\varepsilon*0}$ from $\mathcal{S}_{\varepsilon*2}$. Note that by defining the function on a neighborhood of the VB, in order for the visibility properties to remain intact, the area of the parallelograms $\mathcal{P}_{\varepsilon*1}$ must be decreased by a ratio depending on ρ_n, which can be chosen to be arbitrarily small by decreasing ρ_n. Since the f_n functions are each continuous on the $\mathcal{S}_{\varepsilon*2}$ strips, by extending the function $f := \cup_n f_n$ as zero outside the triadic strips $\cup_{\varepsilon\in B^{<\mathbb{N}}} S_{\varepsilon*2}$, since $z_n \to 0$ we obtain a continuous function. $\square$

. . .

3.1.2 Approximate derivatives

We wish to construct a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that, at each point of a subset $E \subseteq \mathbb{R}^2$ of positive measure, there exists a direction that is not a Denjoy direction.

Davies in [20], p. 231 obtained the following result:

Theorem 28 (Davies [20]) *For every $\varepsilon > 0$ there exists a set $M \subseteq [0,1]^2$ of measure greater than $1 - \varepsilon$, such that to each point p of M there is associated a set of lines of accessibility $L(p)$ with the following properties:*

- *Each angle contains c many lines of accessibility $l(p)$ from $L(p)$;*
- *To each p in M there is associated a subset $F(p) \subseteq L(p)$, such that for each $l(p)$ in $L(p)$, if we define $E(p) = l(p) \cap F(p)$, then the point p is a linear density point of $E(p)$;*
- *There exists a function $g : \mathbb{R}_+ \to \mathbb{R}_+$ such that for any choices of $l(p) \in L(p)$, if $d(p,p') > \delta$, then $d(E(p), E(p')) > g(\delta)$, where $d(\cdot,\cdot)$ denotes the Euclidean distance of point sets.*

Remark 9 The property of g can also be stated in the following way: if $d(q,q') < \varepsilon$ for some $q \in F(q)$, $q' \in F(q')$, then $d(p,p') < g^{-1}(\varepsilon) = \delta$. Notice that g can be supposed to be a nondecreasing function, and that $\lim_{t\to 0+} g(t) = 0$.

Davies originally used this theorem for proving the following result: for any given continuous function f defined on a unit square, for any $\varepsilon > 0$, one can give a function g equal to f on a set H of measure greater than $1 - \varepsilon$, with the property that through each point of H there pass c many lines, each on which g is approximately constant.

However our aims are different, and based on this result we obtain the following counterexample:

Theorem 29 *There exists a continuous function that at each point of a set of positive measure in c many directions has finite and different one-sided approximate derivatives:*

$$\pm\infty \neq A\partial^{\vartheta} f = A\partial_{\vartheta} f \neq A\partial^{\vartheta+\pi} f = A\partial_{\vartheta+\pi} f \neq \pm\infty.$$

Proof. Take the set of the previous theorem. Define f to be 0 on M. Define f on the points q of $F(p)$ as the distance from p: $f(q) := d(p,q)$. By the properties of the sets $F(p)$, the function f is approximately differentiable on both sides of all the lines $l(p) \in L(p)$, these derivatives are ± 1, thus do not possess the Denjoy property. It remains to show that f is uniformly continuous. Fix an $\varepsilon > 0$. We need that there exists a δ, such that if $d(q,q') < \delta$ ($q \in F(p)$, $q' \in F(p')$), then $|f(q) - f(q')| < \varepsilon$. Suppose $d(p,q) > d(p',q')$, then we have

$$\begin{aligned} |f(q) - f(q')| &= |d(p,q) - d(p',q')| \leq |d(p,p') + d(p',q') + d(q,q') - d(p',q')| \\ &= |d(p,p') + d(q,q')| < g^{-1}(\delta) + \delta \end{aligned}$$

by taking the limit as $\delta \to 0$, we obtain a δ, such that $g^{-1}(\delta) + \delta < \varepsilon$ holds.

If for a point $p \in M$ the relation $d(p, q') < \delta$ holds, then a point $q \in F(p)$ close enough (say $d(p, q) = \eta$) to p can be found such that $d(q, q') < \delta$ also holds, implying $|f(q) - f(q')| < \varepsilon$. We obtain that

$$|f(p) - f(q')| \leq |f(q') - f(q)| + |f(p) - f(q)| < \varepsilon + \eta,$$

for arbitrarily small η. Now it only remains to extend this uniformly continuous function to the whole domain. $\square$

Remark 10 Note that when considering approximate derivatives we can ignore the behaviour of the function on the lines $l'(p') \in L(p')$ associated to the other points $p' \neq p$. However the point p can still be a point of accumulation of such $l' \cap l$ intersections, thus changing the value of the classical Dini derivatives. It is not even clear how a transfinite construction could be executed in this case. This is what Theorem 27 achieves.

A Disjoint segments in the plane

The construction of Otto Marcin Nikodym̃ (1887-1974) [47] and later Davies [20] of a plane set of positive measure, each point of which is linearly accessible from the outside suggests that linear segments in the plane can be placed in a complicated fashion, to obtain rather unexpected behaviour. A first related question that can be posed is the following:

Problem 1
Is it possible, that each point p of a plane disk D of unit radius is the endpoint of a linear open segment $l(p) \subseteq \mathbb{R}^2$ of length 1, such that these open segments are disjoint?

Remark 11 By a linear open segment $l(p)$ of length 1 and direction $\vartheta(p)$ we mean the set

$$\{p + \lambda e_{\vartheta(p)} : \lambda \in (0, 1)\}.$$

In this section ϑ as usual denotes the positive angle with the positive x-axis.

Proof. [14] The answer to this question is negative as the following simple argument shows. Suppose that such segments exist. Take an open linear interval I in the disk D. By a theorem of Blumberg, any function $f : \mathbb{R} \to \mathbb{R}$ can be restricted to a dense set, where it is continuous. Applying this to $\vartheta|_I$, we obtain a dense set $H \subseteq I$, where $\vartheta|_H$ is continuous. Take any point p of H. Take a point $q \in l(p) \cap D$. Since there are points $p_n \in H$ converging to p from both directions, by continuity of $\vartheta|_H$ there is no direction in which an interval l can be originating from q. However, we supposed q to be in D, thus it should be the endpoint of a linear segment; contradiction. □

A next natural question that can be asked is the following:

Problem 2
Is it true, that every plane set $E \subseteq \mathbb{R}^2$, each point of which is the endpoint of pairwise disjoint linear segments, has measure zero?

We show that the answer is negative. In fact, it is possible to give a Sierpiński like construction for such a set, if all the segments are parallel. (Enumerate in the least ordinal c the closed sets of the plane of positive measure, and place a unit segment parallel to the y axis in a point $(x_\alpha, y_\alpha)_{\alpha<c}$, such that x_α is disjoint from all the previous $(x_\beta)_{\beta<\alpha}$.)

[14] I would like to thank Balázs Keszegh for this argument.

By defining a (non-measurable) function on each y-section we can obtain arbitrary Denjoy behaviour in the fixed direction of the y-axis on a set of positive outer measure (namely the endpoints of these segments). It is difficult to see how a continuous or even measurable function using such a transfinite construction can be obtained. To this end let us ask the following:

Problem 3
Is it true, that every plane *measurable* set $E \subseteq \mathbb{R}^2$, each point of which is the endpoint of pairwise disjoint linear segments, has measure zero?

The answer to this question is positive. We briefly sketch a proof for a special case. Suppose that a plane measurable set E of positive measure with the required properties exists. We can suppose that E is contained in a disk D of radius $1/10$. (Take one such that $D \cap E$ has positive measure and omit the rest of the segments.) Denote by M the midpoints of the segments. For any segment S denote by $e_S \in E$ its endpoint and $m_S \in M$ its midpoint. Take the $f : M \to E$ function that assigns to a midpoint of a segment S its endpoint (by assumption contained in D)

$$f(m_S) := e_S.$$

(This function is well-defined since the segments are disjoint.) This function is Lipschitz with constant 2:

$$|f(m_S) - f(m_{S'})| = |e_S - e_{S'}| \leq 2|m_S - m_{S'}|,$$

otherwise the segments would intersect. Similarly, if instead of midpoints we define M_λ as the points $m_S^\lambda \in S$ at distance λ from e_S, then each $f_\lambda : M_\lambda \to E$ is also Lipschitz with constant $1/\lambda$.

If the sets M_λ were measurable, we would be done, since Lipschitz functions do not increase measure and if E is a set of positive measure, this would result in c many pairwise disjoint plane sets of positive measure which would finish the proof.

However if already we allow the segments to have length other than one, using a Sierpiński like construction it is possible to take $E := S^1$ and the segments in radial direction, such that M is not measurable. It is not clear whether measurability of M can be concluded with the restriction that all segments have length one.

B Measurability of partial derivatives of arbitrary functions

The following theorem extends Federer's argument to arbitrary functions, and a result similar to Stepanoff's theorem is obtained. Although the theorems of Haslam-Jones and Saks are stronger versions of this theorem, this proof does not rely nor on directed derivatives, nor on the contingence theorem.

Theorem 30 *Let $f : E \to \mathbb{R}$ be an arbitrary function defined on an arbitrary set $E \subseteq \mathbb{R}^2$. Define*

$$M := \{x \in E : L_f(x) < \infty\}.$$

Then the function is differentiable a.e. on M. Moreover M is relative Lebesgue measurable (moreover relative F_σ), and the partial derivatives are relative Lebesgue measurable functions on M.

Proof. For each n, take the points x in which f restricted to the $1/n$ neighbourhood of x is Lipschitz with constant n:

$$M_n := \left\{ x \in E : |f(x+h) - f(x)| \le n|h|, \quad \forall |h| < \frac{1}{n}, \; x + h \in E \right\}.$$

Since $L_f(x) = K < \infty$ means that for a small enough neighborhood U_x of x, at every $y \in U_x$ $|f(y) - f(x)| \le K|y - x|$, it follows that $M = \cup_n M_n$.

The sets M_n are relative closed. It is enough to show that if $x_k \in M_n$, and $x_k \to x$, then $x \in M_n$. Take a y, such that $|y - x| < 1/n$. We want to show that

$$|f(y) - f(x)| \le n|y - x|.$$

By taking an x_k close enough to x, one has $|x_k - y| < 1/n$ for all $k > N$ for a certain N. Since $x_k \in M_n$, it follows that $|f(x_k) - f(y)| \le n|x_k - y|$. Using the triangle inequality:

$$|f(y) - f(x)| \le |f(y) - f(x_k)| + |f(x_k) - f(x)| \le n|x_k - y| + n|x_k - x|,$$

and since $|x_k - x| \to 0$ and $|x_k - y| \to |x - y|$, it follows that $x \in M_n$.

The function f restricted to M_n is continuous for each n, since it is locally Lipschitz on M_n.

In the following we show that the function is differentiable a.e. on M. Divide M_n into $M_{n,i}$, each of diameter $1/n$. It is enough to prove that f is differentiable a.e. on $M_{n,i}$ for each i and n. The function $f|_{M_{n,i}}$ is (globally) Lipschitz on $M_{n,i}$ with constant n. By known results (see Mojżesz David Kirszbraun (1904-1942) [35]) $f|_{M_{n,i}}$ can be extended to $\mathbb{R}^2$ with the same Lipschitz constant; denote this function $\tilde{f}$.

By Rademacher's theorem $\tilde{f}$ is a.e. differentiable. It is enough to show the statement for density points of $M_{n,i}$, where $\tilde{f}$ is differentiable. Let $a \in M_{n,i}$ be such a point and let $d = (\tilde{f})'(a)$. Then

$$\lim_{x \to a} \frac{\tilde{f}(x) - \tilde{f}(a) - \langle d, x - a \rangle}{|x - a|} = 0. \tag{B.1}$$

Our aim is to show that

$$\lim_{x \to a,\, x \in E} \frac{f(x) - f(a) - \langle d, x - a \rangle}{|x - a|} = 0. \tag{B.2}$$

Let $\varepsilon > 0$ is given. Since a is a density point of $M_{n,i}$, there is a $0 < \delta_1 < 1/(2n)$ such that $\lambda_2(M_{n,i} \cap B(a,h)) > (1 - \varepsilon^2/4)\lambda_2(B(a,h))$ for every $0 < h < \delta_1$. It is easy to check that if $x \in B(a, \delta_1/2)$, $x \neq a$ and $|x - a| = r$, then $B(x, \varepsilon r) \cap M_{n,i} \neq \emptyset$.

By (B.1), there is a $\delta_2 > 0$ such that

$$|\tilde{f}(x) - \tilde{f}(a) - \langle d, x - a \rangle| \leq \varepsilon |x - a|$$

for every $x \in B(a, \delta_2)$. Let $\delta = \min(\delta_1, \delta_2)/2$.

Let $x \in B(a, \delta) \cap E$, $x \neq a$ be arbitrary. Choose an element $y \in B(x, \varepsilon r) \cap M_{n,i}$. Then $y \in M_n$ and $|y - x| < 1/n$, and thus

$$|f(y) - f(x)| \leq n \cdot |y - x| \leq n\varepsilon |x - a|.$$

Therefore,

$$\begin{aligned}
|f(x) - f(a) - \langle d, x - a \rangle| &= |f(x) - \tilde{f}(a) - \langle d, x - a \rangle| \leq \\
&\leq |f(y) - \tilde{f}(a) - \langle d, y - a \rangle| + |f(x) - f(y)| + |\langle d, y - x \rangle| \leq \\
&\leq |\tilde{f}(y) - \tilde{f}(a) - \langle d, y - a \rangle| + n\varepsilon |x - a| + |d| \cdot |y - x| \leq \\
&\leq \varepsilon |x - a| + n\varepsilon |x - a| + |d|\varepsilon \cdot |x - a| = \\
&= (1 + n + |d|)\varepsilon \cdot |x - a|.
\end{aligned}$$

This proves (B.2).

To prove measurability of the partial derivatives of f, it is enough to show that the partial derivatives of $\tilde{f}$ are measurable. For proving the measurability of the $\partial^0 \tilde{f}$ functions, define $g_k : \mathbb{R}^2 \to \mathbb{R}$ as

$$g_k(x,y) := \frac{\tilde{f}(x+1/k,y) - \tilde{f}(x,y)}{1/k}$$

Since the $\tilde{f}$ functions are Lipschitz continuous and defined everywhere, it follows that the $g_k|_{M_n}$ functions are also Lipschitz continuous. Since the g_k functions are Lipschitz, they are differentiable a.e., thus $\partial^0 \tilde{f} = \lim_{k\to\infty} g_k$ a.e., thus the $\partial^0 \tilde{f}$ functions are measurable. $\square$

C Tables

In order to create some order to all the results discussed in the introduction, the following tables recapitulate the most relevant results and articles known in each case. A plus sign (+) denotes positive results for Denjoy type relations, whereas a minus sign (-) denotes the cases where quite strong counterexamples have been given. On some occasions depending on the conditions (Borel/Lebesgue/non-measurable) both a + and a - sign can be found in the same entry (Ward/Davies; Ward/Theorem 24).

1-dim	Classical	Approximate
Measure	Denjoy-Young-Saks	Denjoy-Khintchine, Ward [75]
Category	Zajíček [80], Belna-Cargo-Evans-Humke [6]	Zajíček [79], Preiss-Zajíček [48]

2-dim directed (typical point all dir)	Classical	Approximate
Measure	+: Haslam-Jones [29], Saks	+: Ward [75]
Category	Ravetz [51], [52]	

2-dim linear (typical point+dir)	Classical	Approximate
Measure	+: Ward [76], -: Davies [21]	+: Ward [76], -: Theorem 24
Category	+: Theorem 17	

2-dim linear (typical point all dir)	Classical	Approximate
Measure	-: Besicovitch [8], Theorem 27	-: Theorem 29
Category		

The following table illustrates known results about the Lebesgue measurability of derivate numbers of Lebesgue measurable functions:

$f : \mathbb{R}^2 \to \mathbb{R}$ λ_2-meas. [measurability]	Classical	Approximate
$x \mapsto \partial^\vartheta f(x)$ sections, ϑ fix	No [Neubauer]	Yes [Khintchine-Saks]
$(x, \vartheta) \mapsto \partial^\vartheta f(x)$	No [Davies]	Yes [Ward]
$x \mapsto D^\vartheta f(x)$ sections, ϑ fix	Yes [Haslam-Jones]	
$(x, \vartheta) \mapsto D^\vartheta f(x)$		

...

The following questions still remain open at the end of this thesis:

1. Does there exist a continuous function with the following properties: for each point of a set of positive measure there exists a direction such that the linear derivative

exists (upper/lower equal), in the opposite direction the linear derivative exists, but these two are finite and distinct?

2. Can the example given in Theorem 27 be improved such that all four linear derivate numbers are distinct on a set of positive measure?

3. What can be said about functions $f : \mathbb{R}^2 \to \mathbb{R}$ having the following property: for each point of a set of positive measure there exists a direction, such that the function is continuous in that direction. (With some appropriate condition on the directions.)

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