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Haar null sets in non-locally compact groups

Master's Thesis

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1 Introduction

In this thesis I will be mainly concerned with the symmetric group acting on a countably infinite set (Definition 2.1.4) and its subgroups. If we equip the set these groups are acting on with the discrete topology then these groups can be equipped with the topology of pointwise convergence. In such a setting they can be equipped with a compatible complete metric and they are Polish groups.

It is an interesting question to ask how a typical element of a given group looks like. First note that it makes sense to require that if an element $g \in G$ (where G is any group) is typical then all of its conjugates are typical because conjugate elements in a group cannot be distinguished internally. Second, in order to make this question mathematically precise we should specify what we mean by a typical element. There are two common ways one can consider typicality:

1. It the sense of the Baire category theorem: typical elements shall form a comeager set,
2. In a measure theoretic sense: the set of atypical elements must have measure zero.

We can speak about typicality in the first sense without much problem because the Baire category theorem holds for Polish groups. Thus we can ask the following questions and their variants:

1. Which conjugacy classes are meager (if any)?
2. Which conjugacy classes are of second category?
3. Is there a comeager conjugacy class?

But we have a problem with the second approach: what measure shall we use? There is an obvious choice for locally compact groups, the Haar measure but most of the groups considered in this thesis are not locally compact. We will use a more general notion instead due to Christensen [2]: the notion of Haar null sets. The Haar null sets of a locally compact group in the sense of Christensen coincide with the sets of Haar measure zero but they can be defined for non-locally compact Polish groups. There is a terminology that calls Haar null sets *shy* and co-Haar null sets *prevalent*. We will not use this terminology. In this setting we can ask the following questions and their variants:

1. Which conjugacy classes are Haar null?
2. Which conjugacy classes are not Haar null?
3. Is there a co-Haar null conjugacy class?

The answers for the above questions are quite different in the two settings, for example we will see that a large class of infinite permutation groups can be divided into two sets where one set is meager and the other is Haar null.

We would like to highlight that the notion of the "group theoretic algebraic closure" turned out to be especially useful in some cases and that probably other concepts could be translated into more easily verifiable ones with its help.

2 Preliminaries and notation

In this section we will give a quick overview of the relevant definitions and theorems.

2.1 Topological groups

Definition 2.1.1 (Topological group) *If a group G has a topology defined on it such that the operations of multiplication*

$$G \times G \to G : (x, y) \mapsto xy$$

and inverse

$$G \to G : x \mapsto x^{-1}$$

*are continuous then G is called a **topological group** (with respective to that topology).*

In this thesis the topology will always be clear from the context so it will not be indicated.

Definition 2.1.2 (Polish group) *If a topological group G as a topological space is a Polish space: it is separable and admits a compatible complete metric then G is called a **Polish group**.*

Definition 2.1.3 (Group action) *If G is any group and X is a set, then a **group action** φ of G on X is a function*

$$\varphi : G \times X \to X$$

that satisfies the following two axioms. Usually we denote $\varphi(g, x)$ by $g(x)$ if the group action is obvious from the context.

1. *$1(x) = x$ for all $x \in X$ (where 1 denotes the neutral element of the group G),*
2. *$(gh)(x) = g(h(x))$ for all $g, h \in G$ and all $x \in X$.*

We will mostly be concerned with the properties of one particular Polish group and its closed subgroups, namely with the symmetric group acting on a countably infinite set. Without loss of generality we can assume that the countable set being acted on is ω.

Definition 2.1.4 *We will denote the group of all permutations acting on the set* ω *by* $\mathrm{Sym}(\omega)$.

One can find an overview of the properties of $\mathrm{Sym}(\omega)$ as a topological group in [1].

Definition 2.1.5 *The group* $\mathrm{Sym}(\omega)$ *can be equipped with the* ***topology of pointwise convergence****. This is the topology generated by the base consisting of the sets:*

$$U^{y_1,y_2\dots y_n}_{x_1,x_2\dots x_n} = \{p \in G : p(x_1) = y_1, p(x_2) = y_2 \dots p(x_n) = y_n\}.$$

It is really a topology of pointwise convergence if we consider the set ω that the permutations act on with the discrete topology. A sequence of permutations $p_1, p_2 \dots \in \mathrm{Sym}(\omega)$ converges to a permutation p if and only if for all $x \in \omega$ there is an index N such that $p_M(x) = p(x)$ for every $M > N$. From now on if we refer to a subset of $\mathrm{Sym}(\omega)$ as closed we mean it as closed in this topology. This topology is metrizable, for example by the following metric:

$$\delta(p, q) = \frac{1}{2^N} \text{ where } N = \min\{n \in \omega : p(n) \neq q(n)\}.$$

However, this metric is not complete. This can be fixed by the metric:

$$d(p, q) = \max\{\delta(p, q), \delta(p^{-1}, q^{-1})\} = \frac{1}{2^N} \text{ where}$$
$$N = \min\{n \in \omega : p(n) \neq q(n) \text{ or } p^{-1}(n) \neq q^{-1}(n)\}.$$

The group $\mathrm{Sym}(\omega)$ is a complete metric space when equipped with the metric $d(p, q)$.

Definition 2.1.6 *If* G *is a subgroup of* $\mathrm{Sym}(\omega)$ *then:*

1. *The* ***orbit of an element*** $x \in \omega$ *is the set of all elements* $y \in \omega$ *that are of the form* $g(x) = y$ *for some* $g \in G$.

2. *The* ***stabilizer of an element*** $x \in \omega$ *is the set of permutations* $g \in G$ *such that* $g(x) = x$*. The stabilizer of* x *will be denoted by* $G_{(x)}$.

3. *The* ***orbit of a tuple*** $(x_1, x_2 \dots x_n)$ *is the set of all tuples* $(y_1, y_2 \dots y_n)$ *that are of the form* $g(x_1) = y_1, g(x_2) = y_2 \dots g(x_n) = y_n$ *for some* $g \in G$.

4. *The* ***stabilizer of the tuple*** $(x_1, x_2 \dots x_n)$ *is the set of permutations* $g \in G$ *such that* $g(x_1) = x_1, g(x_2) = x_2 \dots g(x_n) = x_n$*. The stabilizer of a tuple will be denoted by* $G_{(x_1,x_2\dots x_n)}$.

5. *For a set $S \subset \omega$ the* ***pointwise stabilizer of the set****, S consisting of the permutations $g \in G$ such that $g(s) = s$ for every $s \in S$, will be denoted by $G_{(S)}$.*

Definition 2.1.7 (Topologically transitive group action) *A group action where a group G is acting on a topological space S is said to be* ***topologically transitive*** *if for every pair of non-empty open sets $U, V \subset S$ there is an element of the group $g \in G$ such that $g(U) \cap V \neq \emptyset$.*

In the above definition we can restrict ourselves to a given base of S without loss of generality. If $(U_i)_{i\in\omega}$ is a base of S and for every U_j and U_k there is an element of the group $g \in G$ such that $g(U_j) \cap U_k \neq \emptyset$ then the action is topologically transitive. Note that every group acts on itself by conjugation.

Definition 2.1.8 (σ-ideal) *Let $(X, \mathcal{S})$ be a measurable space. An subset $\mathcal{I}$ of the σ-algebra $\mathcal{S}$ is called a* ***σ-ideal*** *if it satisfies the following:*

1. *$\emptyset \in \mathcal{I}$,*

2. *If $A \in \mathcal{I}$, $B \in \mathcal{S}$ and $B \subset A$ then $B \in \mathcal{I}$,*

3. *The set $\mathcal{I}$ is closed under taking countable unions: $\{A_n\}_{n\in\omega} \subset \mathcal{I} \Rightarrow \bigcup_{n\in\omega} A_n \in \mathcal{I}$.*

If one tries to capture smallness or negligibility in some way then the resulting definition often determines a σ-ideal, as in the following examples:

1. The countable subsets of a given base set,

2. The sets of measure 0 regarding an arbitrary measure,

3. The sets of first category in the sense of Baire in an arbitrary topological space,

4. The Haar null sets in the sense of Christensen [2] in an arbitrary Polish group,

5. The openly Haar null sets in an arbitrary Polish group [13],

6. The Haar-meager sets in an arbitrary Polish group [4].

If some subset is negligible in one way that does not imply that it is negligible in some other way:

Theorem 2.1.9 ([11, Special case of Theorem 16.5]) *If we consider the set $\mathbb{R}$ with the Euclidean topology and the Lebesgue measure then there exists a decomposition $\mathbb{R} = A \cup B$ where A is meager and B has measure 0.*

Section 4 is about similar decompositions.

Definition 2.1.10 (Haar null set) *Let G be a Polish group. A subset $A \subset G$ is called* ***Haar null*** *if there exists a Borel set $U \supset A$ and a Borel probability measure μ on G such that $\mu(xUy) = 0$ for every $x, y \in G$.*

Sets that are not Haar null will be called ***Haar positive****. The complements of Haar null sets will be called* ***co-Haar null****.*

Note that for a Borel set A we can omit U from the definition and require $\mu(xAy) = 0$ instead of $\mu(xUy) = 0$. We mention that in some literature the set U in the definition of Haar null sets is required to be universally measurable instead of Borel. In most practical applications this makes little difference but the resulting notions are not the same. This is examined in [6].

Definition 2.1.11 (Openly Haar null set) *Let G be a Polish group. A subset $A \subset G$ is called* ***openly Haar null*** *if there exists a probability measure μ on G such that for all $\varepsilon > 0$ exists a $U \supset A$ open set such that for all pairs of elements $x, y \in G$ the inequality $\mu(xUy) < \varepsilon$ holds* [13]*.*

From Definition 2.1.10 and Definition 2.1.11 clearly follows that every openly Haar null set is Haar null. On the other hand there are Haar null sets that are not openly Haar null: every openly Haar null set has a Haar null G_δ hull (take a series of ε-s converging to zero then take the intersection of the corresponding open sets) but in [6] some Haar null sets are constructed without any Haar null G_δ hull. Note that although some Haar null sets do not have a Haar null G_δ hull all Haar null sets have a Haar null Borel hull.

If in the definition of Haar null sets we only require the existence of a Borel measure μ such that $\mu(xU) = 0$ for every element $x \in G$ then we get a different notion. These sets are called **left Haar null**, and they coincide with Haar null sets (and thus sets with Haar measure zero) in locally compact second countable Polish groups, coincide with Haar null sets on abelian Polish groups but are different in a general setting. For example, they do not always form a σ-ideal ([14, Theorem 3]). Their properties are studied for example in [14].

However, for conjugacy invariant Borel subsets the two definitions will yield the same result:

Lemma 2.1.12 *If a Borel set $A \subset G$ is conjugacy invariant: $gAg^{-1} = A$ for every $g \in G$ then A is Haar null if and only if there exists a Borel measure μ such that $\mu(gA) = 0$ for every element $g \in G$. Similarly A is Haar null if and only if there exists a Borel measure ν such that $\nu(Ag) = 0$ for every element $g \in G$.*

Proof. This μ works for the original definition. Let $x, y \in G$ be two arbitrary elements. Then

$$\mu(xAy) = \mu(x(yAy^{-1})y) = \mu(xyA) = 0.$$

The same works for the right translates as well. □

Lemma 2.1.13 *Let G be a Polish group. Let $S \subset G$ be a Borel subset with the property that for any compact set $F \subset G$ there is an open set $V \subset G$ and an element $g \in G$ such that $\emptyset \neq F \cap V \subset gS$. Then S is Haar positive.*

Proof. First note that changing the definition of Haar null sets in a way to require the Borel probability measure μ to have compact support yields the same notion. In order to prove this we show that if some set $X \subset G$ has a Borel probability measure μ with non-compact support satisfying the requirements of Definition 2.1.10 then there is another Borel probability measure ν with compact support that satisfies the requirements with the same U.

According to [9, 17.10] every finite Borel measure is regular. This means that the measure of every set is the supremum of the measure of all of its compact subsets. Thus for every Borel measure there is a compact set with positive measure.

Let $Y \subset G$ be a set with $\mu(Y) > 0$. Let ν_0 be the restriction of μ to ν and let ν be ν_0 normalized to $\nu(G) = 1$. This ν satisfy our requirements as a replacement for μ in the definition.

Indirectly assume that S is Haar null, let μ be a Borel probability measure with compact support as in Definition 2.1.10. Let F denote the compact support of μ. This implies that every open neighborhood N_x of every point $x \in F$ has $\mu(N_x) > 0$. There is an open set V and an element $g \in G$ such that $\emptyset \neq F \cap V \subset gS$. So $0 < \mu(F \cap V) \leq \mu(gS)$ which contradicts that $\mu(gS)$ is assumed to be zero. □

Later we will need a somewhat general of a theorem proved by Christensen [2], here we reiterate Rosendal's proof (see [12]).

Theorem 2.1.14 *(Christensen) Let $A \subset G$ be a conjugacy invariant Borel set and suppose that there exists a cover of A by Borel sets $A = \bigcup_{n\in\omega} A_n$ and a conjugacy invariant Borel set B so that $1 \in \overline{B}$ and $B \cap \bigcup_{n\in\omega} A_n^{-1} A_n = \emptyset$. Then A is Haar null.*

Proof. We claim that there exists a sequence $\{g_i : i \in \omega\} \subset B$ with $g_i \to 1$ and the following properties:

- for every $(\varepsilon_i)_{i\in\omega} \in 2^\omega$ we have that the sequence $(g_0^{\varepsilon_0} g_1^{\varepsilon_1} \dots g_n^{\varepsilon_n})_{n\in\omega}$ converges,
- the map $\varphi : 2^\omega \to G$ defined by $(\varepsilon_n)_{n\in\omega} \mapsto g_0^{\varepsilon_0} g_1^{\varepsilon_1} g_2^{\varepsilon_2} \dots$ is continuous (the right hand side expression makes sense because of the convergence).

We can choose such a sequence by induction: fix a compatible complete metric and suppose that we have already selected $g_0, g_1, \dots, g_n$. Now notice that for every $(\varepsilon_0, \dots, \varepsilon_n) \in 2^{n+1}$ the set $\{x \in G : d(g_0^{\varepsilon_0} g_1^{\varepsilon_1} \dots g_n^{\varepsilon_n} x, g_0^{\varepsilon_0} g_1^{\varepsilon_1} \dots g_n^{\varepsilon_n}) < 2^{-n-1}\}$ contains a neighbourhood of the identity. Therefore we can choose a

$$g_{n+1} \in B \cap \bigcap_{(\varepsilon_0,\dots,\varepsilon_n)\in 2^{n+1}} \{x \in G : d(g_0^{\varepsilon_0} g_1^{\varepsilon_1} \dots g_n^{\varepsilon_n} x, g_0^{\varepsilon_0} g_1^{\varepsilon_1} \dots g_n^{\varepsilon_n}) < 2^{-n-1}\}.$$

One can easily show that for every $(\varepsilon_n)_{n\in\omega} \in 2^\omega$ the sequence $(g_0^{\varepsilon_0} g_1^{\varepsilon_1} \dots g_n^{\varepsilon_n})_{n\in\omega}$ is Cauchy and the function φ is continuous.

Let λ be the usual measure on 2^ω and let $\lambda_* = \varphi_*\lambda$, its push forward. We claim that λ_* witnesses that A is left-Haar null which is equivalent to its Haar nullness, by the fact that A is conjugacy invariant using Lemma 2.1.12.

Suppose not, then there exists an $f \in G$ so that $\lambda_*(fA) > 0$, therefore $\lambda_*(fA_k) > 0$ for some $k \in \omega$. This is equivalent to $\lambda(\varphi^{-1}(fA_k)) > 0$ and if we regard 2^ω as $\mathbb{Z}_2^\omega$ by Weil's theorem (see e.g. [12]) we have that $\varphi^{-1}(fA_k) - \varphi^{-1}(fA_k)$ contains a neighbourhood of $(0, 0, \dots)$, the identity in $\mathbb{Z}_2^\omega$. Then there exists an element in $\varphi^{-1}(fA_k) - \varphi^{-1}(fA_k)$ that is zero at every coordinate except for one. Thus, $\varphi^{-1}(fA_k)$ contains two elements of the form $(\varepsilon_0, \dots, \varepsilon_{n-1}, 0, \varepsilon_{n+1}, \dots)$ and $(\varepsilon_0, \dots, \varepsilon_{n-1}, 1, \varepsilon_{n+1}, \dots)$, i. e. differing at exactly one place. Then taking the φ images of these elements we obtain that there exist $h_1, h_2 \in G$ so that $h_1 h_2 \in fA_k$ and $h_1 g_n h_2 \in fA_k$. This implies

$$h_2^{-1} h_1^{-1} h_1 g_n h_2 \in A_k^{-1} A_k$$

thus

$$h_2^{-1} g_n h_2 \in A_k^{-1} A_k$$

but by the conjugacy invariance of B we get

$$h_2^{-1} g_n h_2 \in B \cap A_k^{-1} A_k,$$

contradicting the initial assumptions of the theorem. □

2.2 Infinite permutation groups

This subsection is mostly concerned with results specific to $\mathrm{Sym}(\omega)$ and its subgroups. The interested reader can read more about the basics of model theory in [8].

Definition 2.2.1 (Structure and its automorphism group) *We will call a base set A equipped with a set of relations, a set of functions, and a set of constants a* ***structure****: $\mathcal{A}$. The* ***automorphism group*** *of $\mathcal{A}$ denoted by $\mathrm{Aut}(\mathcal{A})$ is the set of all one-to-one functions (permutations) $p : A \to A$ that preserve all of the relations, functions and constants of $\mathcal{A}$.*

All countably infinite structures mentioned in this thesis will have ω as their base set. This will simplify our arguments. Thus the automorphism groups of the structures will be subgroups of $\mathrm{Sym}(\omega)$. All automorphism groups are closed, and all closed subgroups G of $\mathrm{Sym}(\omega)$ can be obtained as an automorphism group of some structure $\mathcal{A}$ on ω. If we define an n-ary relation for each $n \in \omega$ and for each finite tuple $(x_1, x_2 \dots x_n)$ as

$$R_{(x_1,x_2\dots x_n)} = \{(y_1, y_2 \dots y_n) : (\exists g \in G) g(x_1) = y_1, g(x_2) = y_2 \dots g(x_n) = y_n\}$$

then the automorphism group of the resulting relational structure will be exactly G.

We briefly mention the following two properties:

1. A countable structure $\mathcal{A}$ is ω-categorical if it is isomorphic to all of the countable models of its theory.
2. A permutation group acting on ω is oligomorphic if for every finite set $S \subset \omega$ the number of orbits under the stabilizer $G_{(S)}$ is finite. The Engeler–Ryll-Nardzewski–Svenonius theorem states that a structure $\mathcal{A}$ is ω-categorical if and only if $\mathrm{Aut}(\mathcal{A})$ is oligomorphic ([8, Theorem 7.3.1]).

Definition 2.2.2 *Let $\mathcal{A}$ be a structure on ω, let G denote its automorphism group and let $S \subset \omega$ be a finite subset. The* ***group theoretic algebraic closure*** *of S is:*

$$\mathrm{ACL}(S) = \{x \in \omega : \text{ the orbit of } x \text{ under the stabilizer } G_{(S)} \text{ is finite}\}$$

We follow the notation of [8, notation introduced before Lemma 4.1.1]. It is known that if $\mathcal{A}$ is ω-categorical then for finite subsets the above definition coincides with the usual model-theoretic algebraic closure [8]. In particular in ω-categorical structures the $\mathrm{ACL}(S)$ is a finite set for every finite S because of the Engeler–Ryll-Nardzewski–Svenonius theorem and $\mathrm{ACL}(\mathrm{ACL}(S)) = \mathrm{ACL}(S)$ because the model-theoretic algebraic closure is a proper closure operator. We will now give easier to check characterizations of general topological properties for subgroups of $\mathrm{Sym}(\omega)$:

Theorem 2.2.3 ([1, Theorem 2.1]) *Let G be a subgroup of $\mathrm{Sym}(\omega)$. Then:*

1. *G is open if and only if for some finite set $S \subset \omega$ the stabilizer $\mathrm{Sym}(\omega)_{(S)}$ is contained in G,*
2. *G is closed if and only if it is the automorphism group of some first order structure on ω,*
3. *G is discrete if and only if for some finite $S \subset \omega$ its pointwise stabilizer $G_{(S)}$ is trivial,*
4. *G is compact if and only if it is closed and all its orbits are finite,*
5. *G is locally compact if and only if it is closed and all orbits of the stabilizer of some finite set are finite.*

Note that the last two items can be reworded as G is compact if and only if it is closed and $\mathrm{ACL}(\emptyset) = G$ and G is locally compact if and only if it is closed and there is a finite set $S \subset \omega$ such that $\mathrm{ACL}(S) = G$.

Definition 2.2.4 *We will say that a group G has a **nice algebraic closure** if for every finite set $S \subset \omega$ its group-theoretic algebraic closure $\mathrm{ACL}(S)$ is finite.*

It is easy to check that oligomorphic groups have a nice algebraic closure, but the class of closed groups having a nice algebraic closure is strictly greater.

Lemma 2.2.5 *If a group G has a nice algebraic closure (Definition 2.2.4) then the corresponding operator* ACL *is idempotent.*

Proof. We shall show that for every finite set $S \subset \omega$ the identity $\mathrm{ACL}(\mathrm{ACL}(S)) = \mathrm{ACL}(S)$ holds. Let $S \subset \omega$ be an arbitrary finite set and let $x \in \mathrm{ACL}(\mathrm{ACL}(S))$ be an arbitrary element. We will show that x has a finite orbit under $G_{(S)}$ which implies $x \in \mathrm{ACL}(S)$ because this is the definition of the group theoretic algebraic closure.

Let $X_{\mathrm{ACL}(S)}$ denote the orbit of x under $G_{(\mathrm{ACL}(S))}$ and X_S denote the orbit of x under $G_{(S)}$. Enumerate the elements of $\mathrm{ACL}(S)$ as $(x_1, x_2 \dots x_k)$. The group $G_{(S)}$ is acting on $\mathrm{ACL}(S)^k$ coordinatewise. Under this group action the stabilizer of the tuple $(x_1, x_2 \dots x_k)$ is $G_{(\mathrm{ACL}(S))}$. The Orbit-Stabilizer Theorem states that for any group action the index of the stabilizer of an element in the whole group is the same as the cardinality of its orbit. In our settings this yields that the index $[G_{(S)} : G_{(\mathrm{ACL}(S))}]$ is the same as the cardinality of the orbit of $(x_1, x_2 \dots x_k)$. This orbit is finite because the whole space $\mathrm{ACL}(S)^k$ is finite. So $G_{(\mathrm{ACL}(S))}$ has a finite index in $G_{(S)}$.

Let $g_1, g_2 \dots g_n \in G_{(S)}$ be a left transversal for $G_{(\mathrm{ACL}(S))}$ in $G_{(S)}$. Since $X_S = g_1 X_{\mathrm{ACL}(S)} \cup g_2 X_{\mathrm{ACL}(S)} \cup \dots g_n X_{\mathrm{ACL}(S)}$ is a finite union of finite sets it must be finite. □

We will formalize the following technical statement for future use:

Lemma 2.2.6 *Let G be a group that has a nice algebraic closure. Let $S \subset \omega$ be a finite subset such that $\mathrm{ACL}(S) = S$ and $x \in (\omega \setminus S)$ be an arbitrary element. Then we can order the finite set $(\mathrm{ACL}(S \cup \{x\}) \setminus S)$ as $(x_1, x_2 \dots x_k)$ such that if $1 \le a < b \le k$ then $\mathrm{ACL}(S \cup \{x_1, \dots x_{a-1}\} \cup \{x_b\}) \subsetneq\!\!\!\!/\ \mathrm{ACL}(S \cup \{x_1, \dots x_{a-1}\} \cup \{x_a\})$ where $\not\subsetneq$ denotes (not a proper subset of).*

Proof. We can always pick the next x_a by choosing it as one of the elements such that $\mathrm{ACL}(S \cup \{x_1, \dots x_a\})$ is minimal with respect to inclusion. □

Definition 2.2.7 (Partial permutation) *A **partial permutation** is an injection from a finite set $S \subset \omega$ to ω. A partial permutation p **extends** another partial permutation q if $\mathrm{Dom}(q) \subset \mathrm{Dom}(p)$ and $p|_{\mathrm{Dom}(q)} = q$. A permutation p **extends** a partial permutation q if $p|_{\mathrm{Dom}(q)} = q$.*

Definition 2.2.8 (Possible images and preimages) *Let p be a partial permutation and let $x \in \omega$ be an arbitrary element. We define the **possible images of** x **under** p as the set*

$$\{y \in \omega : (\exists g \in G) g \text{ extends } p \text{ and } g(x) = y\}.$$

*Similarly we define the **possible preimages of** x **under** p as the set*

$$\{y \in \omega : (\exists g \in G) g \text{ extends } p \text{ and } g^{-1}(x) = y\}.$$

Lemma 2.2.9 *Let G be a group that has a nice algebraic closure.. We define the numbers $\Theta_{G,S,(x_1,x_2 \dots x_k)}(a, b) \in \omega \cup \{\omega\}$ where the parameters are:*

(a) A finite subset $S \subset \omega$ such that $\mathrm{ACL}(S) = S$,

(b) *An ordering of* $(\mathrm{ACL}(S \cup \{x\}) \setminus S)$ *for some arbitrary element* x *as described in Lemma 2.2.6:* $(x_1, x_2 \ldots x_k)$,

(c) *Two indices* $1 \leq a < b \leq k$

Let q *be a partial permutation defined on the domain* $\mathrm{Dom}(q) = S \cup \{x_1, \ldots x_{a-1}\}$ *such that the set of permutations* g *extending* q *is non-empty. Then exactly one of the following two possibilities hold:*

(1) *For every possible extension* $\tilde{q}$ *of* q *to* $(\mathrm{Dom}(q) \cup \{x_a\})$ *the set of still possible images (Definition 2.2.8) of* x_b *is infinite:*

$$|\{y_b \in \omega : (\exists g \in G) g \text{ extends } \tilde{q} \text{ and } g(x_b) = y_b\}| = \omega.$$

In this case we define $\Theta_{G,S,(x_1,x_2\ldots x_k)}(a,b)$ *as* ∞.

(2) *For every element* $y \in \omega$ *the set*

$$W_y = \{z \in \omega : (\exists g \in G) g \text{ extends } q; g(x_a) = z \text{ and } g(x_b) = y\}$$

is finite. For those y *where* W_y *is non-empty, it always has the same size, define* $\Theta_{G,S,(x_1,x_2\ldots x_k)}(a,b)$ *as this size.*

Proof. All of the sets

$$S_y = \{y_b \in \omega : (\exists g \in G) g \text{ extends } q; g(x_a) = y \text{ and } g(x_b) = y_b\}$$

are translates of the orbit of x_b under $G_{(\mathrm{Dom}(q) \cup \{x_a\})}$. Thus for every element $y \in \omega$ that is a possible image for x_a under q the cardinality of the set S_y is the same. When these sets are infinite then the (1) holds. Assume that the sets S_y are finite, this means that $x_b \in \mathrm{ACL}(\mathrm{Dom}(q) \cup \{x_a\})$. Then $x_a \in \mathrm{ACL}(\mathrm{Dom}(q) \cup \{x_b\})$ also holds because the two algebraic closures must coincide since we enumerated the set $(\mathrm{ACL}(S \cup \{x\}) \setminus S)$ as described in Lemma 2.2.6. Thus for every element $y \in \omega$ the set

$$W_y = \{z \in \omega : (\exists g \in G) g \text{ extends } q; g(x_a) = z \text{ and } g(x_b) = y\}$$

is finite because these sets are either empty or translates of the orbit of x_a under $G_{(\mathrm{Dom}(q) \cup \{x_b\})}$. For those y where the set W_y is non-empty it will always have the same size because the translates of any set have the same cardinality. □

Lemma 2.2.10 *Let G be a group that has a nice algebraic closure and $S \subset \omega$ be a finite subset such that $\mathrm{ACL}(S) = S$ and $x \in (\omega \setminus S)$ be an arbitrary element. Let $(x_1, x_2 \dots x_k)$ be an ordering of $(\mathrm{ACL}(S \cup \{x\}) \setminus S)$ as described in Lemma 2.2.6.*

For every $1 \le d \le k$ there is a $1 \le c \le d$ such that if h_1 is any partial permutation defined on the domain $\mathrm{Dom}(h_1) = S \cup \{x_1, \dots x_{c-1}\}$ then the set of the possible images of x_d under h_1 is infinite and if h_2 is any partial permutation defined on the domain $\mathrm{Dom}(h_2) = S \cup \{x_1, \dots x_c\}$ then the set of the possible images of x_d under h_2 is finite.

Proof. Let x_e be an arbitrary element from $(x_1, x_2 \dots x_k)$ and h' be a partial permutation with $\mathrm{Dom}(h') = S \cup \{x_1, \dots x_e\}$.

Then the set of possible images of x_d under h' is the translate of the orbit of x_d under the stabilizer $G_{(S \cup \{x_1, \dots x_e\})}$. The element x_c is the first element such that $x_d \in \mathrm{ACL}(S \cup \{x_1, \dots x_c\})$. □

Lemma 2.2.11 *The operator* ACL *is translation invariant in the following sense: if $S \subset \omega$ is a finite set and $g \in G$ is an arbitrary permutation then*

$$\mathrm{ACL}(gS) = g\,\mathrm{ACL}(S).$$

Proof. Let $x \in \omega$ be an arbitrary element, then

$$\begin{aligned} &x \text{ and } y \text{ are in the same orbit under } G_{(S)} \Leftrightarrow \\ &\qquad \exists h \in G_{(S)} : h(y) = x \Leftrightarrow \exists h \in G_{(S)} : gh(y) = g(x) \Leftrightarrow \\ &\qquad\qquad g(x) \text{ and } g(y) \text{ are in the same orbit under } G_{(gS)} \end{aligned}$$

So an element x has a finite orbit under $G_{(S)}$ if and only if $g(x)$ has a finite orbit under $G_{(gS)}$. □

3 A random construction

In our proofs we will use the following probability measure generated by a random process. Our process makes sense only for closed permutation groups that have a nice algebraic closure (Definition 2.2.4).

Our random process will define a permutation $p \in G$ in stages. We will need a random integer sequence $(k_{i,j})_{i\in\omega, j\in\omega}$ whose values will be determined later to be appropriate for the given proof. It is indexed by a pair instead of a single index only for the simplicity of notation in some later proofs. By random integer sequence we mean that the value of $k_{i,j}$ is only needed in the ith stage of the construction, and although it is not fixed before the proof it can be determined at stage i based on the outcome of the random events during previous stages.

Let G be a closed permutation group with a nice algebraic closure. At stage i we will choose the least element from ω that either has no preimage or image, let this element be denoted by a_i. We will denote the set of those elements that have their image defined before stage i by I_i and the set of those elements that have their preimage defined before stage i by P_i. We also build a sequence of partially defined permutations (Definition 2.2.7) p_i During the process we maintain:

1. Both I_i and P_i will be finite sets such that $\mathrm{ACL}(I_i) = I_i$, $\mathrm{ACL}(P_i) = P_i$,
2. The domain of p_i is I_i and the range of p_i is P_i,
3. There is a permutation $g \in G$ that extends p_i.

In some stages we define the sets K_{i+1}, L_{i+1} and a partial permutation p'_{i+1} in such a way that $P_i \subset K_{i+1} \subset P_{i+1}$, $I_i \subset L_{i+1} \subset I_{i+1}$, $\mathrm{Dom}\left(p'_{i+1}\right) = K_{i+1}$ and $\mathrm{Im}(p'_{i+1}) = L_{i+1}$. The partial permutation p'_{i+1} will be an extension of p_i and will be extended by p_{i+1}.

At stage i we proceed as follows:

1. We will refer to this part of the construction as the **first step** of stage i. If the element $a_i \notin P_i$ then define $K_{i+1} = \mathrm{ACL}(P_i \cup \{a_i\})$. We will extend p_i to a partial permutation p'_{i+1} such that $\mathrm{Im}(p'_{i+1}) = K_{i+1}$. This includes finding an appropriate preimage for a_i as the partial permutation p_i^{-1} is defined only for P_i. Enumerate the elements of $K_{i+1} \setminus P_i$ as in Lemma 2.2.6:

   $$K_{i+1} \setminus P_i = (x_1, x_2 \dots x_k).$$

   We will determine the preimages of $(x_1, x_2 \dots x_k)$ in this order. Note that a_i is amongst these elements so this procedure will define the preimage for a_i. Denote the partial permutations defined in these sub-steps by $p_{i,j}$. So $\mathrm{Im}(p_{i,0}) = P_i, \mathrm{Im}(p_{i,1}) = P_i \cup \{x_1\}, \mathrm{Im}(p_{i,2}) = P_i \cup \{x_1, x_2\} \dots$. If the first j preimages are determined then there are two possibilities for x_{j+1}:

(a) The set of possible preimages of x_{j+1} under $p_{i,j}$ (Definition 2.2.8) is finite. Then choose one from them randomly with uniform distribution.

(b) The set of possible preimages of x_{j+1} under $p_{i,j}$ (Definition 2.2.8) is infinite. Then choose one from the smallest $k_{i,j+1}$ many possible values uniformly. The value of $k_{i,j+1}$ might depend on the choices made in previous stages, steps and sub-steps. Note that the exact values will be specified in each of the proofs.

We note that for all x_j its orbit under the stabilizer $G_{(P_i)}$ is infinite because $x_j \notin P_i = \mathrm{ACL}\,(P_i)$ so the possibility (b) must occur for at least x_1 in every stage.

Now we have K_{i+1} as the set of elements with their preimage currently defined. Let p'_{i+1} be $p_{i,k}$ and let L_{i+1} be $\mathrm{Dom}\left(p'_{i+1}\right)$. Then L_{i+1} is a strict superset of I_i containing the elements with an already defined image. The elements of K_{i+1} and I_{i+1} are in a bijection that can be extended to a permutation $g \in G$. $K_{i+1} = \mathrm{ACL}\,(K_{i+1})$ because Lemma 2.2.5 states that ACL is idempotent. $L_{i+1} = \mathrm{ACL}\,(L_{i+1})$ because of Lemma 2.2.11.

2. We will refer to this part of the construction as the **second step** of stage i. If the element $a_i \notin L_{i+1}$ then we will choose an image $p(a_i)$ similarly, starting with defining
$$I_{i+1} = \mathrm{ACL}\,(L_{i+1} \cup \{a_i\})\,.$$
Let p_{i+1} be the extension of p'_{i+1} to the whole I_{i+1} (the extension can be carried out in the same manner as the extension of p_i above). Let P_{i+1} be the set $\mathrm{Im}(p_{i+1})$. These will satisfy $\mathrm{ACL}\,(I_{i+1}) = I_{i+1}$, $\mathrm{ACL}\,(P_{i+1}) = P_{i+1}$ and that p_{i+1} is a partial permutation between them that can be extended to a permutation $g \in G$.

Let p be the union of the increasing chain of partial permutations $p_1, p_2 \ldots$. This p will be a permutation because every element $x \in \omega$ has both its preimage and image defined. The resulting p will be in G because G is closed.

Lemma 3.0.1 *Let p be a permutation generated by the previous random construction. Then the sets P_i, K_i, I_i, L_i that appeared in the different stages of the construction can be obtained as*

(A) $K_{i+1} = \mathrm{ACL}\,(a_0, p(a_0), a_1, p(a_1) \ldots a_i)$,

(B) $L_{i+1} = \mathrm{ACL}\,(p^{-1}(a_0), a_0, p^{-1}(a_1), a_1 \ldots p^{-1}(a_i))$,

(C) $I_{i+1} = \mathrm{ACL}\,(p^{-1}(a_0), a_0, p^{-1}(a_1), a_1 \ldots p^{-1}(a_i), a_i)$,

(D) $P_{i+1} = \mathrm{ACL}\,(a_0, p(a_0), a_1, p(a_1) \ldots a_i, p(a_i))$.

Proof. The set K_1 is $\mathrm{ACL}(a_0)$ so (A) holds for the index $i = 0$. Proceed by induction:

If (A) holds for the index i then (B) also holds for the index i because of Lemma 2.2.11 using that L_{i+1} is the translate of K_{i+1} by p^{-1},

If (B) holds for the index i then (C) also holds for the index i because I_{i+1} is defined as $I_{i+1} = \mathrm{ACL}(L_{i+1} \cup \{a_i\})$,

If (C) holds for the index i then (D) also holds for the index i because of Lemma 2.2.11 using that P_{i+1} is the translate of I_{i+1} by p,

If (D) holds for the index i then (A) holds for the index $(i+1)$ because K_{i+2} is defined as $K_{i+2} = \mathrm{ACL}(P_{i+1} \cup \{a_{i+1}\})$. □

4 Haar null–meager decompositions

In this section we are examining possible generalizations of Theorem 2.1.9. The statement of Theorem 2.1.9 was that $\mathbb{R}$ equipped with the Euclidean topology and the Lebesgue measure can be decomposed as $\mathbb{R} = A \cup B$ where A is meager and B has measure zero.

A proof of Theorem 2.1.9. Enumerate all rational numbers: $(q_i)_{i\in\omega}$. Let $U_{i,n}$ denote the open interval with center q_i and length $\frac{1}{n2^i}$. Let $V_n = \bigcup_{i\in\omega} U_{i,n}$ then the measure of V_n is at most $\frac{2}{n}$. Then the set $B = \bigcap_{n\in\omega} V_n$ has measure zero. Moreover the set $A = \mathbb{R} \setminus B$ is meager because $\mathbb{R} \setminus B = \bigcup_{n\in\omega} \mathbb{R} \setminus V_n$ where all of the sets $(\mathbb{R} \setminus V_n)$ are nowhere dense. □

For locally compact groups the following generalization is known:

Theorem 4.0.1 ([11, Theorem 16.5]) *Every uncountable locally compact Polish group G may be written as a disjoint union $\mathbb{R} = A \cup B$ where A is meager and B has Haar measure zero.*

There are generalizations for non-locally compact groups as well. We will state the following theorem without proof:

Theorem 4.0.2 ([3, Proposition 2]) *The following Polish groups can be decomposed as $G = A \cup B$ where A is meager and B is Haar null:*

1. *Any uncountable G that has a compatible two-sided invariant metric (TSI),*

2. $G = \mathrm{Sym}\,(\omega)$,

3. $G = \mathrm{Aut}(\mathbb{Q})$ *if only the ordering $<$ of $\mathbb{Q}$ is considered,*

4. $G = \mathcal{U}(l^2)$ *or more generally $G = \mathcal{U}(\mathcal{R})$ the unitary group of a von Neumann algebra $\mathcal{R}$ on a separable infinite dimensional complex Hilbert space $\mathcal{H}$ with the strong operator topology,*

5. *If there is a continuous surjective homomorphism from G to any of the above groups.*

Our goal is to prove the existence of a similar decomposition for a class of groups as general as possible. Considering the diversity of groups in Theorem 4.0.2 it is not a big surprise that the proof given in [3] is elaborate and uses tools from topological dynamics. Although the proof of the $G = \mathrm{Aut}\,(\mathbb{Q})$ case could be modified with little additional efforts to achieve our goals we will rather go with a new shorter proof which uses some parts of the original.

Theorem 4.0.3 *Let G be a closed subgroup of* $\mathrm{Sym}(\omega)$ *that has a nice algebraic closure and satisfies the following: for every $n \in \omega$ there is a finite set $S_n \subset \omega$ such that there is no n-element subset $X \subset \omega$ such that $S_n \subset \mathrm{ACL}(X)$.*

Then G can be decomposed as $G = A \cup B$ where A is meager and B is Haar null.

Proof. Let H be an arbitrary Polish group, then its openly Haar null subsets form a translation invariant σ-ideal [13]. If the singleton set $\{1\}$ is openly Haar null then every countable subset of H is openly Haar null. The group H is separable as a topological space so there is a countable dense set $X \subset H$ in it. This X is openly Haar null so X has a Haar null G_δ hull B. This hull can be obtained as an intersection $B = \bigcap U_i$ where the sets U are dense open sets containing X. The set $A = G \setminus B = \bigcup (G \setminus U_i)$ is meager because the sets $G \setminus U$ are nowhere dense.

We will prove that the singleton set $\{1\}$ is openly Haar null. Let p be a random permutation obtained using the construction described in Section 3. We will show that for every $\varepsilon > 0$ there is an open set $U \ni 1$ such that for every pair of permutations $g, h \in G$ the inequality $\mathbb{P}(gUh) < \varepsilon$ holds.

Let $n \in \omega$ be an arbitrary integer. Let $S_n \subset \omega$ be a finite subset such that there is no n-element subset $X \subset \omega$ such that $S_n \subset \mathrm{ACL}(X)$. The pointwise stabilizer $G_{(S_n)}$ is an open set containing $\{1\}$. We will restrict our search for appropriate open sets to this kind of stabilizers. Let $g, h \in G$ be an arbitrary pair of permutations. The two-sided translate of a stabilizer $G_{(S_n)}$ by g from the left and by h from the right consists of the following permutations:

$$gG_{(S_n)}h = \{q \in G : (\forall x \in h^{-1}S_n)\, q(x) = gh(x)\}.$$

From Lemma 2.2.11 follows that all translates of S_n, in particular $h^{-1}S_n$ also have the property that they are not contained in the algebraic closure of any n-element set.

We will use the construction described in Section 3 choosing the numbers $(k_{i,j})_{i \in \omega, j \in \omega}$ as described below. If we are at stage i and $K_{i+1} \setminus P_i = (x_1, x_2 \dots x_k)$ and we already defined the preimages for $x_1, x_2 \dots x_{j-1}$ and let $p_{i,j-1}$ denote the last partial permutation defined already. We choose $k_{i,j}$ as follows. Let M denote following maximum using Lemma 2.2.9 (the function Θ is also defined there):

$$M = \max\{\Theta_{G, P_i, (x_1, x_2 \dots x_k)}(j, b) : j < b \le k \text{ where } \Theta \text{ is finite}\}.$$

Let $k_{i,j}$ be $2^i M$. We proceed similarly for the elements of $L_{i+1} \setminus I_i$ when defining their images.

Now let i denote the stage where the image of all elements of $h^{-1}S_n$ are finally determined. Then i is at least $\frac{n-1}{2}$ because after any stage j the set $I_{j+1} = \mathrm{ACL}\left(p^{-1}(a_0), a_0, p^{-1}(a_1), a_1 \dots p^{-1}(a_j), a_j\right)$ as stated in Lemma 3.0.1, and the set $h^{-1}S_n$ cannot be covered by the algebraic closure of any set with size less than n. Let x denote an element from $h^{-1}S_n$ that has its image defined only at stage i and let y denote $p(x)$. Then at stage i exactly one of the following two possibilities happened:

(A) The element y was amongst $K_{i+1} \setminus P_i = (x_1, x_2 \dots x_k)$, let the index of y be denoted by m,

(B) The element x was amongst $I_{i+1} \setminus L_{i+1}$.

If possibility (A) occurred then there was an $x_l \in (x_1, x_2 \dots x_k)$ because of Lemma 2.2.10 such that the number of possible preimages of y under $p_{i,l-1}$ was infinite and the number of possible preimages of y under $p_{i,l}$ was finite. The number $k_{i,l}$ used when defining the preimage of x_l was chosen to be greater than $2^i \Theta_{G,P_i,(x_1,x_2\dots x_k)}(l,m)$. Define the set

$$W_x = \{z \in \omega : (\exists g \in G) g \text{ extends } p^{-1}_{i,l-1}; g(x_l) = z \text{ and } g(x_m) = x\}$$

as in Lemma 2.2.9 (note that the names of the variables changed). So $|W_x|$ is the number of elements that can be chosen as the preimage of x_l in a way that x remains a possible preimage of y under the resulting $p_{i,l}$ partial permutation. Using the definition of $\Theta_{G,P_i,(x_1,x_2\dots x_k)}(l,m) = |W_x|$ we obtain that the probability of choosing a preimage for x_l without making $p(x) = y$ impossible was at most $\frac{|W_x|}{2^i M} \le \frac{1}{2^i}$.

A similar argument works for possibility (B).

The probability of $p(x) = y$ is at most $\frac{1}{2^i}$. The permutation p cannot be in $gG_{(S_n)}h$ if $p(x) \neq gh(x)$ so

$$\mathbb{P}\left(p \in gG_{(S_n)}h\right) \le \frac{1}{2^i}.$$

This proves the statement of the theorem: for every $\varepsilon > 0$ one can choose the stabilizer of any set S that cannot be covered by the algebraic closure of any finite set with at most $2\log_2 \frac{1}{\varepsilon} + 1$ elements as a suitable U. □

5 The generalization of a theorem of Dougherty and Mycielski

In the article [5] Dougherty and Mycielski examined the group Sym (ω), the group of permutations acting on a countably infinite base set. The description of the typical permutations in the sense of Baire category had been known before [10], Dougherty and Mycielski gave the description of the typical permutations in a measure theoretic sense. The descriptions in the measure and in the category case are quite different:

Theorem 5.0.1 ([5, Theorem 1]) *The set of permutations with infinitely many infinite cycles and only finitely many finite cycles is co-Haar null.*

Theorem 5.0.2 (Trivial consequence of the theorems in [10]) *There is a comeager conjugacy class in* Sym (ω). *The permutations in this conjugacy class have infinitely many cycles of any given finite length and no infinite cycles.*

In this section our goal is to generalize Theorem 5.0.1

Lemma 5.0.3 *Assume we are constructing a permutation p using the process described in Section 3 and we are just entering stage i. Let $w \in \omega$ be an integer and let $\varepsilon > 0$ be any constant.*

Then one can choose the numbers $k_{i,j}$ in such a way that for any w-element set $S \subset \omega$ the probability of defining a preimage at the first step of stage i as some element from S or defining an image at the second step of stage i as some element from S is at most ε.

Proof. We use Lemma 2.2.10 Let $(x_1, x_2 \dots x_j)$ denote the elements of $K_{i+1} \setminus P_i$ enumerated in the same order as they appear during the construction. Let $x_d \in (x_1, x_2 \dots x_j)$ be an arbitrary element. There is an x_c such that the set of possible preimages of x_d is infinite before defining the preimage of x_c and finite after defining the preimage of x_c (x_c may be equal to x_d). We will denote this kind of relationship by $\Phi(x_d) = x_c$.

The numbers $k_{i,j}$ used to determine the number of elements out of the preimage of x_c will be chosen in a way that the inequality

$$\mathbb{P}(\text{After the preimage of } x_c \text{ is chosen there is still a possible preimage for } x_d \text{ from } S) \leq \frac{\varepsilon}{2j} \quad (5.0.1)$$

holds. The second possibility of Lemma 2.2.9 holds for the pair (x_c, x_d) so the number $\Theta_{G,P_i,(x_1,x_2\ldots x_k)}(c,d)$ (as defined in Lemma 2.2.9) is finite. Denote this $\Theta_{G,P_i,(x_1,x_2\ldots x_k)}(c,d)$ by $N(x_d)$ for better readability. This means that for every element $v \in \omega$ the set

$$W_v = \{z \in \omega : (\exists g \in G) g \text{ extends the partial permutation } p^{-1} \\ \text{already defined; } g(x_c) = z \text{ and } g(x_d) = v\}$$

is finite and is always empty or has the same non-zero size regardless of v: $N(x_d)$. The union of the W_v-s for all $v \in S$ is finite because S is finite and it has at most $wN(x_d)$ elements. If we choose the number $k_{i,c}$ to be greater than $\frac{2j}{\varepsilon} wN(x_d)$ then Inequality 5.0.1 holds for x_d.

For an $x \in (x_1, \ldots x_j)$ let us denote the maximum of the numbers $wN(x_l)$ where $\Phi(x_l) = x$ by $M(x)$. If we choose the integers $k_{i,j}$ in such a way that

$\frac{2j}{\varepsilon} M(x_l) \leq k_{i,l}$ for all $x_l \in (x_1, \ldots x_j)$ then all of the inequalities above will hold. We can choose the values $k_{i,j}$ similarly for the second step of the construction when images are determined.

In this way the probability of having any preimage or image defined from S is at most $j\frac{\varepsilon}{2j} + j'\frac{\varepsilon}{2j'} = \varepsilon$. □

Definition 5.0.4 *Let $G \leq \mathrm{Sym}(\omega)$ be a closed group. We will denote the set of permutations $p \in G$ with finitely many finite cycles by $\mathfrak{X}_{FFC}(G)$.*

Lemma 5.0.5 *The set of permutations $\mathfrak{X}_{FFC}(G)$ is Borel and conjugacy invariant.*

Proof. The set of permutations containing a given cycle is open for every cycle (it is an element of the base described in Definition 2.1.5)

$$U^{x_2,x_3\ldots x_1}_{x_1,x_2\ldots x_n} = \{p \in G : p(x_1) = x_2, p(x_2) = x_3 \ldots p(x_n) = x_1\}.$$

Thus for any finite set of finite cycles the set of permutations containing those finite cycles in their cycle decompositions is open: it can be obtained as the intersection of finitely many open sets. Thus for every $n \in \omega$ the set of permutations containing at least n finite cycles is open: it can be obtained as the union of open sets (one open set for each possible set of n cycles). Thus $\mathrm{Sym}(\omega) \setminus \mathfrak{X}_{FFC}(G)$ is G_δ: it is the intersection of the above open sets. Thus $\mathfrak{X}_{FFC}(G)$ is Borel.

The set $\mathcal{X}_{FFC}(G)$ is conjugacy invariant because the number of cycles of given length is conjugacy invariant in any symmetric group. □

Theorem 5.0.6 *Let $G \leq \mathrm{Sym}(\omega)$ be a closed group. Then G has a nice algebraic closure if and only if $\mathcal{X}_{FFC}(G)$ is co-Haar null.*

Proof. First we will prove that if G has a nice algebraic closure then $\mathcal{X}_{FFC}(G)$ is co-Haar null. We will generate a permutation p randomly using the construction described in Section 3 Let $g \in G$ be an arbitrary permutation. We will show that

$$\mathbb{P}(pg \text{ has infinitely many finite cycles}) = 0.$$

In this case it is sufficient to examine only translations by multiplication from the right because of Lemma 2.1.12 and Lemma 5.0.5 It is sufficient to prove that the expected value of the number of finite cycles is finite:

$$\mathbb{E}(\text{Number of finite cycles in } pg) = \sum_{i\in\omega} \mathbb{E}(\text{Number of finite cycles completed at stage } i \text{ in } pg) < \infty.$$

At stage i we define the image of a finite number of elements $x_1, x_2 \dots x_j$ and the preimage of another finite number of elements $y_1, y_2 \dots y_j$ under p if not already defined. We are interested what this means for the permutation pg.

At stage i we define the image of $g^{-1}(x_k)$ under pg as $p(x_k)$ and the preimage of y_l under pg as $g^{-1}p^{-1}(y_l)$. So at stage i a finite cycle is completed if $p(x_k)$ happens to be chosen as $(pg)^{-h}(g^{-1}(x_k))$ where h is the largest integer such that $(pg)^{-h}(g^{-1}(x_k))$ is already defined, or $g^{-1}p^{-1}(y_l)$ is $(pg)^h(y_l)$ where h is the largest integer such that $(pg)^h(y_l)$ is already defined.

So there is a finite number of problematic elements (the $[(pg)^{-h}(g^{-1}(x_k))]$-s and the $[(pg)^h(y_l)]$-s above) that can cause problems (finite cycles) if appear as newly defined images or preimages. Denote the set of these elements by D. We will choose the sequence $(k_{i,j})_{i\in\omega, j\in\omega}$ such that the following estimate will hold:

$$\mathbb{E}(\text{Number of finite cycles completed at stage } i \text{ in } pg) \leq \frac{1}{2^i}.$$

We will use that if for all $\alpha \in A$ where A is a partition of the sample space the inequality $\mathbb{E}(X|\alpha) \leq c$ holds for some fixed c then $\mathbb{E}(X) \leq c$. In Equation 5.0.2 we will take the conditional expectation of some value regarding P_i, I_i, and p_i. We can do this because P_i, I_i and p_i can be considered as set-valued and partial permutation-valued random variables. Note that including P_i and I_i is redundant. We are using this conditional expectation to formulate that we are now working in a setting where we consider the results of the first few stages until stage $(i-1)$ given. Since

$$\begin{aligned} &\mathbb{E}\left(\text{Number of finite cycles completed at stage } i \text{ in } pg | P_i, I_i, p_i\right) = \\ &\qquad \sum_{x_k \in (x_1, \ldots x_j)} \mathbb{P}\left(x_k \text{ is assigned an image from } D\right) + \\ &\qquad\qquad \sum_{y_k \in (y_1, \ldots y_j)} \mathbb{P}\left(y_k \text{ is assigned a preimage from } D\right) \end{aligned} \tag{5.0.2}$$

it would suffice to choose the random sequence $k_{i,j}$ in such a way that

$$\mathbb{P}\left(\text{some element is assigned an image or preimage from } D\right) \leq \frac{1}{2^i}.$$

This is exactly the setting of Lemma 5.0.3 with $w = |D|$ and $\varepsilon = \frac{1}{2^i}$ so the random sequence $k_{i,j}$ can be chosen as required. We can conclude that

$$\mathbb{E}\left(\text{Number of finite cycles in } pg\right) \leq \sum_{i \in \omega} \frac{1}{2^i}$$

which is finite.

We will now prove the other direction that can be restated as: if G does not have a nice algebraic closure then the set of permutations with infinitely many finite cycles is not Haar null. If G does not have a nice algebraic closure then there is a finite set $S \subset \omega$ such that $\mathrm{ACL}(S)$ is infinite. This means that all of the permutations in $G_{(S)}$ have infinitely many finite cycles. The stabilizer $G_{(S)}$ is a non-empty open set. Countably many translates of $G_{(S)}$ can cover the whole G. Thus $G_{(S)}$ cannot be Haar null. □

Definition 5.0.7 *Let $G \leq \mathrm{Sym}(\omega)$ be a closed group. We will denote the set of permutations $p \in G$ with infinitely many infinite cycles by $\mathfrak{X}_{IIC}(G)$.*

Lemma 5.0.8 *Let G be a closed subgroup of $\mathrm{Sym}(\omega)$ that has a nice algebraic closure. Let $S \subset \omega$ be an infinite set. Then the set of permutations that have infinitely many cycles in their cycle decomposition, each containing at least one element from S is co-Haar null. We will denote the set of these permutations by P_S.*

Proof. Let p be a permutation generated by a random construction that is a slightly modified version of the construction described in Section 3. We modify the rule to choose a_i at stage i. The original rule was to choose the smallest element from ω that has undefined preimage **or** undefined image. We keep this rule for odd i, but for even i we will choose the smallest element from S that has undefined preimage **and** undefined image.

Let $g, h \in G$ be two arbitrary permutations.

During the stages of the construction of p every element is contained in either an already defined finite cycle or a *partially defined cycle*: a tuple of elements $(x_1, x_2 \dots x_k)$ such that it is already determined that $p(x_1) = x_2, p(x_2) = x_3 \dots p(x_{k-1}) = x_k$ and the element x_1 has no preimage defined yet and the element x_k has no image defined yet. We define the length of the partially defined cycle as k. We will refer to x_1 as the tail and x_k as the head of the partially defined cycle. We will say that two partially defined cycles $(x_1, x_2 \dots x_k)$ and $(y_1, y_2 \dots y_l)$ merge at the ith stage of the construction if either $p(x_k) = y_1, p(y_l) = x_1, p^{-1}(y_1) = x_k$ or $p^{-1}(x_1) = y_l$ is defined at stage i. We will say that a partially defined cycle $(x_1, x_2 \dots x_k)$ closes at the ith stage of the construction if $p(x_k) = x_1$ or $p^{-1}(x_1) = x_k$ is defined at stage i.

Let X and X_i denote the following random variables:

$$X = (\text{the number of occasions during the construction when two partially defined cycles with length at least two merge or a partially defined cycle with length at least two closes}),$$

$$X_i = (\text{the number of occasions during stage } i \text{ when two partially defined cycles with length at least two merge or a partially defined cycle with length at least two closes}),$$

$$Y_i = (\text{a variable indicating that at stage } 2i \text{ the element } a_{2i} \text{ is assigned its preimage or image defined as the head or tail of a partially defined cycle already of length at least two}).$$

In the definition of Y_i "is assigned its preimage" means that $a_{2i} \notin P_{2i}$ so the definition of the preimage of a_{2i} happened at the first step of stage $2i$ and "is assigned its image" means that $a_{2i} \notin L_{2i+1}$ so the definition of the image of a_{2i} happened at the second step of stage $2i$.

Note that $X = \sum_{i\in\omega} X_i$. We will show that the sequence $k_{i,j}$ can be chosen in such a way that $\mathbb{E}(X)$ is finite and Y_i is always small enough. More precisely, in a way that $\mathbb{E}(X_i) \leq \frac{1}{2^i}$ and $\mathbb{P}(Y_i) \leq \frac{1}{2^i}$ for every $i \in \omega$. Then $\sum_{i\in\omega} \mathbb{P}(Y_i) < \infty$ and the Borel-Cantelli Lemma implies that it only happens for finitely many i that we define the preimage or image of a_{2i} as the head or tail of some longer than one partially defined cycle with probability 1. This will prove that P_S is co-Haar null because there are only a finite number of merges between longer than one partially defined cycles with probability 1. We will use a similar argument to that in the proof of Theorem 5.0.6.

At stage i we define the image of a finite number of elements $x_1, x_2 \dots x_j$ and the preimage of another finite number of elements $y_1, y_2 \dots y_j$ under p. For the permutation hpg this means that the image of $g^{-1}(x_k)$ under hpg is defined as $hp(x_k)$ and the preimage of $h(y_l)$ under hpg is defined as $g^{-1}p^{-1}(y_l)$.

During the construction we have partially defined cycles with two free ends. These partially defined cycles grow during the construction either when a preimage is defined for their tail or an image is defined for their head. At these events a partially defined cycle can close or two of them can merge. We will denote the set of endpoints of those partially defined cycles that contain at least two elements by E. The set E is finite at every stage. Using Lemma 5.0.3 with $w = |E|$ and $\varepsilon = \frac{1}{2^i}$ yields that the numbers $k_{i,j}$ can be chosen in an appropriate way. □

Theorem 5.0.9 *Let $G \leq \mathrm{Sym}(\omega)$ be a closed group that has a nice algebraic closure. The set $\mathcal{X}_{IIC}(G)$ is co-Haar null.*

Proof. Apply Lemma 5.0.8 with arbitrary infinite $S \subset \omega$. □

We hope that the following Theorem will be useful in concrete classifications of conjugacy classes of automorphism groups (similar to Section 6).

Theorem 5.0.10 *Let $G \leq \mathrm{Sym}(\omega)$ be a closed group that has a nice algebraic closure. Let $H \subset G$ denote the set of permutations p with the following property:*

1. *For every finite set $X \subset \omega$ and for every infinite orbit O under the stabilizer $G_{(X)}$ the orbit O is not covered by finitely many cycles of p.*

Then H is co-Haar null.

Proof. Let $H_{X,O}$ where $X \subset \omega$ is a finite set and O is an infinite orbit under $G_{(X)}$ denote the set of permutations p such that O is not covered by finitely many cycles of p. Then

$$H = \bigcap_{(X \subset \omega \text{ finite})} \bigcap_{(O \text{ is an infinite orbit under } G_{(X)})} H_{X,O}.$$

It is enough to show that all $H_{X,O}$ are co-Haar null since the countable intersection of co-Haar null sets is co-Haar null. Apply Lemma 5.0.8 with $S = O$. □

6 A concrete example

Let $\mathrm{Aut}(\mathbb{Q})$ denote the set of automorphisms of $(\mathbb{Q}, <)$, that is, the set of order preserving bijections $f : \mathbb{Q} \to \mathbb{Q}$. With the topology pointwise convergence, $\mathrm{Aut}(\mathbb{Q})$ is a Polish group. In the current section, our world is restricted to the rational numbers, hence the interval (p, q) now denotes the set $\{r \in \mathbb{Q} : p < r < q\}$. For an automorphism $f \in \mathrm{Aut}(\mathbb{Q})$, we denote the set of fixed points of f by $\mathrm{Fix}(f)$. The set of *orbitals* of f, $\mathcal{O}_f$, consists of the convex hull (relative to $\mathbb{Q}$) of the orbits of the rational numbers, that is

$$\mathcal{O}_f = \{\mathrm{conv}(\{f^n(r) : n \in \mathbb{Z}\}) : r \in \mathbb{Q}\}.$$

It is easy to see that the orbitals of f form a partition of $\mathbb{Q}$, with the fixed points determining one element orbitals, hence "being in the same orbital" is an equivalence relation. Using this fact, we define the relation $<$ on the set of orbitals by letting $O_1 < O_2$ for distinct $O_1, O_2 \in \mathcal{O}_f$ if $p_1 < p_2$ for some (and hence for all) $p_1 \in O_1$ and $p_2 \in O_2$. Note that $<$ is a linear order on the set of orbitals.

It is also easy to see that if $p, q \in \mathbb{Q}$ are in the same orbital of f then $f(p) > p \Leftrightarrow f(q) > q$, $f(p) < p \Leftrightarrow f(q) < q$ and $f(p) = p \Leftrightarrow f(q) = q \Rightarrow p = q$. This observation makes it possible to define the *parity function*, $s_f : \mathcal{O}_f \to \{-1, 0, 1\}$. Let $s_f(O) = 0$ if O consists of a fixed point of f, $s_f(O) = 1$ if $f(p) > p$ for some (and hence, for all) $p \in O$ and $s_f(O) = -1$ if $f(p) < p$ for some (and hence, for all) $p \in O$.

The main theorem of this section is the following.

Theorem 6.0.1 *The conjugacy class of $f \in \mathrm{Aut}(\mathbb{Q})$ is Haar positive if and only if $\mathrm{Fix}(f)$ is finite, and for distinct orbitals $O_1, O_2 \in \mathcal{O}_f$ with $O_1 < O_2$ such that $s_f(O_1) = s_f(O_2) = 1$ or $s_f(O_1) = s_f(O_2) = -1$, there exists an orbital $O_3 \in \mathcal{O}_f$ with $O_1 < O_3 < O_2$ and $s_f(O_3) \neq s_f(O_1)$.*

Remark 6.0.2 We actually prove that every Haar positive conjugacy class $\mathcal{C}$ is *compact biter*, that is, a portion of every compact set can be translated into $\mathcal{C}$.

Together with the next theorem this yields a complete description of the random element of $\mathrm{Aut}(\mathbb{Q})$.

Theorem 6.0.3 *The union of the Haar null conjugacy classes of $\mathrm{Aut}(\mathbb{Q})$ is Haar null.*

We say that an automorphism is *good* if it satisfies the conditions of the theorem. In the proof of the theorem, we use the following lemma to check conjugacy between good automorphisms.

Lemma 6.0.4 *Let f and g be good automorphisms. Suppose that there exists a function $\varphi : \mathbb{Q} \to \mathcal{O}_f$ with the following properties: it is monotonically increasing* *(not necessarily strictly), surjective,* $|\varphi^{-1}(p)| = 1$ *for every* $p \in \mathrm{Fix}(f)$*, and for each* $q \in \mathbb{Q}$*,*

(1) $g(q) = q \Leftrightarrow s_f(\varphi(q)) = 0$*;*

(2) $g(q) > q \Leftrightarrow s_f(\varphi(q)) = 1$*;*

(3) $g(q) < q \Leftrightarrow s_f(\varphi(q)) = -1$*.*

Then f *and* g *are conjugate automorphisms.*

Proof. We use the characterization in [7] to check the conjugacy of automorphisms: f and g are conjugate if and only if there exists an order preserving bijection $\psi : \mathcal{O}_g \to \mathcal{O}_f$ such that $s_g(O) = s_f(\psi(O))$ for every $O \in \mathcal{O}_g$.

We now show that it is legal to define the appropriate bijection ψ as $\psi(O) = O'$ where $O' = \varphi(p)$ for some $p \in O$. To show that it is a well-defined map, we need to prove that given $O \in \mathcal{O}_g$ and $p, q \in O$, $\varphi(p) = \varphi(q)$. Suppose the contrary, then $p \neq q$, hence $s_g(O) = 1$ or $s_g(O) = -1$. We now suppose that $s_g(O) = 1$, the case where $s_g(O) = -1$ is analogous. Then $g(p) > p$ and $g(q) > q$, hence $s_f(\varphi(p)) = s_f(\varphi(q)) = 1$. Since f is good, and $\varphi(p) \neq \varphi(q)$ by our assumption, there is an orbital $O' \in \mathcal{O}_f$ such that $\varphi(p) < O' < \varphi(q)$ and $s_f(O') \neq 1$. Using that φ is surjective and monotone increasing, there exists an $r \in (p, q)$ such that $\varphi(r) = O'$. Then $s_f(\varphi(r)) \neq 1$, but r is in the same orbital as p and q, since orbitals are convex, hence $g(r) > r$. This contradicts (2).

The map ψ is increasing and surjective, since φ is increasing and surjective. One can easily check that conditions (1), (2) and (3) imply that for every $O \in \mathcal{O}_g$, $s_g(O) = s_f(\psi(O))$. Hence it remains to show that ψ is injective.

Let $O, O' \in \mathcal{O}_g$ be distinct orbitals with $\psi(O) = \psi(O')$. Then using conditions (1), (2) and (3), we have $s_g(O) = s_g(O')$. If $s_g(O) = s_g(O') = 0$ then $s_f(\psi(O)) = 0$, hence $\psi(O)$ is a set consisting of a fixed point, let $\{q\} = \psi(O)$. Then $|\varphi^{-1}(q)| = 1$ using the assumption of the lemma, contradicting the fact that $O, O' \subset \varphi^{-1}(q)$. If $s_g(O) = s_g(O') = 1$ then using that g is good, there exists an orbital $O'' \in \mathcal{O}_g$ between O and O' such that $s_g(O'') \neq 1$. Then using the monotonicity of ψ one obtains $\psi(O'') = \psi(O')$, hence $1 \neq s_f(\psi(O'')) = s_f(\psi(O)) = 1$, a contradiction. An analogous argument shows that $s_g(O) = s_g(O') = -1$ also leads to a contradiction, hence the proof of the lemma is complete. □

Now we turn to the proof of the theorems.

Proof of Theorem 6.0.1. First we show the "only if" part. Using Theorem 5.0.6, for the co-Haar null $f \in \mathrm{Aut}(\mathbb{Q})$, $\mathrm{Fix}(f)$ is finite. Since the cardinality of fixed points is the same for conjugate automorphisms, it is clear that the conjugacy class of f can only be Haar positive if $\mathrm{Fix}(f)$ is finite.

The property that between any two distinct orbitals $O_1, O_2 \in \mathcal{O}_f$ with either $s_f(O_1) = s_f(O_2) = 1$ or $s_f(O_1) = s_f(O_2) = -1$, there exists an orbital $O_3 \in \mathcal{O}_f$ with $s_f(O_3) \neq s_f(O_1)$, is also conjugacy invariant. Hence it is enough to prove the following lemma to finish the "only if" part of the theorem.

Lemma 6.0.5 *For the co-Haar null $f \in \mathrm{Aut}(\mathbb{Q})$, for distinct orbitals $O_1, O_2 \in \mathcal{O}_f$ with $O_1 < O_2$ such that either $s_f(O_1) = s_f(O_2) = 1$ or $s_f(O_1) = s_f(O_2) = -1$, there exists an orbital $O_3 \in \mathcal{O}_f$ with $O_1 < O_3 < O_2$ and $s_f(O_3) \neq s_f(O_1)$.*

Proof. Let $\mathcal{H}$ be the set of those automorphisms that satisfy the property in the lemma, and let $\mathcal{A}$ denote the set of functions $f \in \mathcal{H}^c$ such that we can choose distinct orbitals $O_1^f, O_2^f \in \mathcal{O}_f$ such that $O_1^f < O_2^f$, $s_f(O_1^f) = s_f(O_2^f) = 1$, and between O_1^f and O_2^f, there is no orbital $O_3 \in \mathcal{O}_f$ with $s_f(O_3) \neq 1$. Also, let us denote by $\mathcal{A}'$ the set of functions $f \in \mathcal{H}^c$, such that we can choose distinct orbitals $O_1^f, O_2^f \in \mathcal{O}_f$ such that $O_1^f < O_2^f$, $s_f(O_1^f) = s_f(O_2^f) = -1$, and between O_1^f and O_2^f, there is no orbital $O_3 \in \mathcal{O}_f$ with $s_f(O_3) \neq -1$. We show that $\mathcal{A}$ is Haar null and the same can be proved similarly for $\mathcal{A}'$. Since it is easy to see that $\mathcal{H}^c = \mathcal{A} \cup \mathcal{A}'$, proving this will finish the proof of the lemma.

We use Theorem 2.1.14 to show that the conjugacy invariant set $\mathcal{A}$ is Haar null. Let $(p_0, q_0), (p_1, q_1), \ldots$ be an enumeration of all pairs (p, q) with $p < q$, and for all $n \in \mathbb{N}$, let

$$\begin{aligned}\mathcal{A}_n =&\{f \in \mathrm{Aut}(\mathbb{Q}) : p_n \text{ and } q_n \text{ are in distinct orbitals with respect to } f \\ &\text{ and } f(r) > r \text{ for every } r \in [p_n, q_n]\} \\ =&\bigcap_{k \in \mathbb{Z}} \bigcap_{r \in [p_n, q_n]} \{f \in \mathrm{Aut}(\mathbb{Q}) : f^k(p_n) < q_n \text{ and } f(r) > r\}.\end{aligned}$$

Note that $\mathcal{A} = \bigcup_{n \in \mathbb{N}} \mathcal{A}_n$ and $\mathcal{A}_n$ is Borel for every $n \in \mathbb{N}$. Using Theorem 2.1.14, it is enough to show that there is a conjugacy invariant set $\mathcal{B}$ with $1 = \mathrm{id}_{\mathbb{Q}} \in \overline{\mathcal{B}}$ and

$\mathcal{B} \cap \bigcup_{n \in \mathbb{N}} \mathcal{A}_n^{-1} \mathcal{A}_n = \emptyset$. Let

$$\mathcal{B} = \{f \in \operatorname{Aut}(\mathbb{Q}) : \\ \operatorname{Fix}(f) \text{ is finite, } |\mathcal{O}_f| = 2|\operatorname{Fix}(f)| + 1 \text{ and } f(r) \geq r \text{ for every } r \in \mathbb{Q}\}.$$

Note that the condition $|\mathcal{O}_f| = 2|\operatorname{Fix}(f)| + 1$ essentially states that between neighboring fixed points, every point is in the same orbital (roughly speaking, this means that there are no “irrational fixed points”).

It is easy to see that $\mathcal{B}$ is a conjugacy invariant set, and also that $\mathrm{id}_{\mathbb{Q}} \in \overline{\mathcal{B}}$. Let $n \in \mathbb{N}$ be arbitrary, it remains to show that if f, $g \in \mathcal{A}_n$ then $f^{-1}g \notin \mathcal{B}$. Let O be the orbit of p_n with respect to g and let $I = \{r \in \mathbb{Q} : r \leq r' \text{ for some } r' \in O\}$. Note that I is convex, and since $g(r) > r$ for every $r \in (p_n, q_n)$ but p_n and q_n are in different orbitals (with respect to both f and g), $q_n \notin I$.

There are two cases with respect to the relationship of I and the orbitals of f. Suppose first that I does not split orbitals of f, that is, there is no $r \in I$ and $k \in \mathbb{Z}$ such that $f^k(r) \notin I$. Then the sets I and $\mathbb{Q} \setminus I$ are invariant under both f and g (and f^{-1} and g^{-1}), thus I does not split any orbitals of $f^{-1}g$. Moreover, I has no greatest element, nor $\mathbb{Q} \setminus I$ has a least element, since any such element would need to be a fixed point of g, but g does not have a fixed point in the interval (p_n, q_n). Now suppose that $f^{-1}g \in \mathcal{B}$. Then it has a greatest fixed point (if any) that belongs to I and a least fixed point (if any) that belongs to $\mathbb{Q} \setminus I$, hence between the two, every point is in the same orbital. This contradicts the fact that I does not split the orbitals of $f^{-1}g$.

Now suppose that I splits any orbital of f, thus there exist $r \in I$ and $k \in \mathbb{Z}$ such that $f^k(r) \notin I$. Since $f(r) > r$ for every $r \in (p_n, q_n)$, it follows that there is an $r \in (p_n, \infty) \cap I$ such that $f(r) \notin I$. Then $g^{-1}(f(r)) \notin I$, since I does not split orbitals of g. By setting $r' = g^{-1}(f(r))$, we see that $f^{-1}g(r') = r \in I$, thus $f^{-1}g(r') < r'$, hence $f^{-1}g \notin \mathcal{B}$ also in this case, finishing the proof of the lemma. □

Now we prove the “if” part of the theorem. Let f be a good automorphism, we prove that $\mathcal{C}$, the conjugacy class of f is Haar positive. We use the notation $\mathcal{O} = \mathcal{O}_f$ and $s = s_f$.

To show this, using Lemma 2.1.13, it is enough to prove that for any compact set $\mathcal{F} \subset \operatorname{Aut}(\mathbb{Q})$ there is an open set $\mathcal{U} \subset \operatorname{Aut}(\mathbb{Q})$ and an element $g \in \operatorname{Aut}(\mathbb{Q})$ such that $\emptyset \neq \mathcal{F} \cap \mathcal{U} \subset g\mathcal{C}$. So let $\mathcal{F} \subset \operatorname{Aut}(\mathbb{Q})$ be compact. In our proof, we partition $\mathbb{Q}$

into finitely many intervals bounded by the fixed points of f, and on each interval, we define a suitable part of g.

Let $\{p_1, p_2, \dots, p_{k-1}\}$ be the set of fixed points of f (which is necessarily finite) with $p_1 < p_2 < \dots < p_{k-1}$. We now choose $q_1, q_2, \dots, q_{k-1} \in \mathbb{Q}$ such that if we set $\mathcal{U} = \{h \in \mathrm{Aut}(\mathbb{Q}) : \forall i(h(p_i) = q_i)\}$ then $\mathcal{U} \cap \mathcal{F} \neq \emptyset$. Let $\mathcal{K} = \mathcal{U} \cap \mathcal{F}$, we will construct an automorphism g with $\mathcal{K} \subset g\mathcal{C}$ to finish the proof of the theorem. Note that $\mathcal{U}$ is clopen, hence $\mathcal{K}$ is compact. Also note that the sets $\{h(p) : h \in \mathcal{K}\}$ and $\{h^{-1}(p) : h \in \mathcal{K}\}$ are finite, since $\mathcal{K}$ is compact.

Let us use the notation $p_0 = q_0 = -\infty$ and $p_k = q_k = +\infty$. We construct g and a function $\varphi : \mathbb{Q} \times \mathcal{K} \to \mathcal{O}$ separately on each interval (p_i, p_{i+1}), recursively. So let $i < k$ be fixed for now and let $r_1, r_2, \dots$ be an enumeration of (p_i, p_{i+1}) and $t_1, t_2, \dots$ be an enumeration of (q_i, q_{i+1}). Let $O_1, O_2, \dots$ be an infinite sequence of elements of $\mathcal{O}$ that are subsets of (p_i, p_{i+1}), containing every such element at least once. Note that there may be only finitely many such intervals, hence the sequence may contain the same element more than once. We let $\mathcal{O}' = \{O_1, O_2, \dots\}$. At the nth step of the recursive construction, we have a finite set $H_n \subset (p_i, p_{i+1})$ and functions g_n and φ_n. We preserve the following properties of these sets and functions:

For every $n \in \mathbb{N}$, $h, h_1, h_2 \in \mathcal{K}$ and $p, p', p'' \in H_n$, where $p' < p''$ and $(p', p'') \cap H_n = \emptyset$,

(i) $H_0 \subset H_1 \subset \dots$, $g_0 \subset g_1 \subset \dots$ and $\varphi_0 \subset \varphi_1 \subset \dots$;

(ii) $H_n \subset (p_i, p_{i+1})$ is finite;

(iii) $g_n : H_n \to (q_i, q_{i+1})$ is strictly increasing;

(iv) $\varphi_n : H_n \times \mathcal{K} \to \mathcal{O}'$, and $\varphi_n(., h)$ is increasing;

(v) $r_1, \dots, r_{n+1} \in H_{3n+1}$, $t_1, \dots, t_{n+1} \in g_{3n+2}(H_{3n+2})$ and $O_1, \dots, O_{n+1} \in \varphi_{3n+3}(H_{3n+3}, h)$;

(vi) it cannot happen that $h_1(p') < g_n(p') < h_2(p')$, $h_1(p'') > g_n(p'') > h_2(p'')$;

(vii) if $h(p') > g_n(p')$ and $h(p'') > g_n(p'')$ then $h(r) \geq g_n(p'')$ for every $r \in [p', p'']$; similarly, if $h(p') < g_n(p')$ and $h(p'') < g_n(p'')$ then $h(r) \leq g_n(p')$ for every $r \in [p', p'']$ (thus extending g_n in any way to a strictly increasing function on $[p', p'']$, there is no $r \in [p', p'']$ where the value of the extension can be equal to $h(r)$);

(viii) $s(\varphi_n(p,h)) = 1 \Leftrightarrow g_n(p) < h(p)$ and $s(\varphi_n(p,h)) = -1 \Leftrightarrow g_n(p) > h(p)$;

(ix) $\varphi_n(H_n, h_1) = \varphi_n(H_n, h_2)$;

(x) the value of s is alternating on the image $\varphi_n(H_n, h)$, that is, either $\varphi_n(p', h) = \varphi_n(p'', h)$ or $s(\varphi_n(p', h)) \neq s(\varphi_n(p'', h))$;

(xi) $h_i(p') > g_n(p')$ and $h_i(p'') < g_n(p'')$ $(i = 1, 2)$ (or similarly, $h_i(p') < g_n(p')$ and $h_i(p'') > g_n(p'')$ $(i = 1, 2)$) implies that $\varphi_n(p', h_1) = \varphi_n(p', h_2)$ and $\varphi_n(p'', h_1) = \varphi_n(p'', h_2)$.

Remark 6.0.6 Conditions (vi) and (vii) are equivalent to the following fact: the rectangle $\text{conv}((p', g_n(p')), (p'', g_n(p')), (p'', g_n(p'')), (p', g_n(p'')))$ has two sides that are opposite such that no $h \in \mathcal{K}$ intersects the interior of any of those sides.

First we prove the following.

Claim 6.0.7 *On each interval (p_i, p_{i+1}), the sets and functions H_n, g_n and φ_n can be constructed with the above properties.*

Proof. We prove the claim by induction on n. For $n = 0$, let $H_0 = g_0 = \varphi_0 = \emptyset$. Now suppose that H_n, g_n and φ_n are given with the above properties, using them, we construct the suitable H_{n+1}, g_{n+1} and φ_{n+1}. There are three cases according to the remainder of n mod 3.

Case 1: $n = 3m$. At this step, we make sure that $r_{m+1} \in H_{n+1}$. If already $r_{m+1} \in H_n$ then let $H_{n+1} = H_n$, $g_{n+1} = g_n$ and $\varphi_{n+1} = \varphi_n$. Otherwise, there are multiple cases according to the existence of $p' \in H_n$ with $p' < r_{m+1}$, $p'' \in H_n$ with $r_{m+1} < p''$, and whether $g_n(p') < h(p')$ or $g_n(p') > h(p')$, $g_n(p'') < h(p'')$ or $g_n(p'') > h(p'')$.

Case 1a: there are neither $p' \in H_n$ with $p' < r_{m+1}$ nor $p'' \in H_n$ with $r_{m+1} < p''$ (that is, $H_n = \emptyset$, $n = 0$). If $s(O_1) = 1$ then we find $q \in (q_i, q_{i+1})$ with $q < h(r_{m+1})$ for every $h \in \mathcal{K}$, otherwise, we find $q \in (q_i, q_{i+1})$ with $q > h(r_{m+1})$ for every $h \in \mathcal{K}$. Such a q exists, since $\mathcal{K}$ is compact, thus $\{h(r_{m+1}) : h \in \mathcal{K}\}$ is finite. Now we set $H_{n+1} = \{r_{m+1}\}$, $g_{n+1}(r_{m+1}) = q$, $\varphi_{n+1}(r_{m+1}, h) = O_1$ for every $h \in \mathcal{K}$.

Case 1b: there is a $p' \in H_n$ with $p' < r_{m+1}$ but there is no $p'' \in H_n$ with $r_{m+1} < p''$. Let p' be the largest element in H_n, clearly $p' < r_{m+1}$. Let $q' = g_n(p')$. Using (ix), $\varphi_n(p', h)$ is the same for every $h \in \mathcal{K}$, since it is the largest element in the common image $\varphi_n(H_n, h)$. Let $O = \varphi_n(p', h)$ for some $h \in \mathcal{K}$. Depending on $s(O)$, $g_n(p') < h(p')$ for every $h \in \mathcal{K}$ or $g_n(p') > h(p')$ for every $h \in \mathcal{K}$ using (viii). In the first case, choose $t \in (q_i, q_{i+1})$ such that $q' < t < h(p')$ for every $h \in \mathcal{K}$. Then set $H_{n+1} = H_n \cup \{r_{m+1}\}$ and let g_{n+1} extend g_n with $g_{n+1}(r_{m+1}) = t$, and let φ_{n+1} extend φ_n with $\varphi_{n+1}(r_{m+1}, h) = O$ for every $h \in \mathcal{K}$.

In the second case, let $h(r_{m+1}) < t$ for every $h \in \mathcal{K}$, also satisfying $t \in (q', q_{i+1})$. Choose $q \in (q', t)$ such that $q > h(r_{m+1})$ for every $h \in \mathcal{K}$. As $h^{-1}(q') > p'$ for every $h \in \mathcal{K}$, there exists $p \in (p', r_{m+1})$ such that $p < h^{-1}(q')$ for every $h \in \mathcal{K}$. Now set $H_{n+1} = H_n \cup \{r_{m+1}, p\}$, and let g_{n+1} and φ_{n+1} extend the appropriate functions with $g_{n+1}(r_{m+1}) = t$, $g_{n+1}(p) = q$ and $\varphi_{n+1}(r_{m+1}, h) = \varphi_{n+1}(p, h) = O$ for every $h \in \mathcal{K}$.

Case 1c: there is no $p' \in H_n$ with $p' < r_{m+1}$ but there is a $p'' \in H_n$ with $r_{m+1} < p''$. This case can be handled similarly as Case 1b.

Case 1d: there is a $p' \in H_n$ with $p' < r_{m+1}$, there is a $p'' \in H_n$ with $r_{m+1} < p''$, and for the largest such p' and the smallest such p'', there is no $h \in \mathcal{K}$ with $g_n(p') < h(p')$ and $g_n(p'') > h(p'')$. In this case, let $\mathcal{K}' = \{h \in \mathcal{K} : g_n(p') > h(p') \text{ and } g_n(p'') < h(p'')\}$, where $p' \in H_n$ is the largest with $p' < r_{m+1}$ and $p'' \in H_n$ is the smallest with $p'' > r_{m+1}$. Note that $\mathcal{K}'$ may be the empty set. Let $q' = g_n(p')$ and $q'' = g_n(p'')$. Choose $t \in (q', q'')$ such that $t > h(r_{m+1})$ for each $h \in \mathcal{K}'$ with $h(r_{m+1}) < q''$. Such a t exists, since the compactness of $\mathcal{K}'$ implies that $\{h(r_{m+1}) : h \in \mathcal{K}', h(r_{m+1}) < q''\}$ is finite. We will set $g_{n+1}(r_{m+1}) = t$, but we need to define the value of g_{n+1} at one more place. Choose $q \in (q', t)$ with $q > h(r_{m+1})$ for each $h \in \mathcal{K}'$ with $h(r_{m+1}) < q''$. For every $h \in \mathcal{K}'$ we have $h(p') < q'$, hence also $p' < h^{-1}(q')$. Therefore there is a $p \in (p', r_{m+1})$ for which $p < h^{-1}(q')$ for every $h \in \mathcal{K}'$.

Now let $H_{n+1} = H_n \cup \{p, r_{m+1}\}$, g_{n+1} extend g_n with $g_{n+1}(p) = q$, $g_{n+1}(r_{m+1}) = t$. For $h \in \mathcal{K}'$, either $h(r_{m+1}) < t$ or $h(r_{m+1}) > t$. If $h(r_{m+1}) < t$ then let $\varphi_{n+1}(r_{m+1}, h) = \varphi_n(p', h)$, if $h(r_{m+1}) > t$ then let $\varphi_{n+1}(r_{m+1}, h) = \varphi_n(p'', h)$. In both cases, let $\varphi_{n+1}(p, h) = \varphi_n(p', h)$. If $h \in \mathcal{K} \setminus \mathcal{K}'$ then let $\varphi_{n+1}(p, h) = \varphi_{n+1}(r_{m+1}, h) = \varphi_n(p', h)$. Note that using (viii), $s(\varphi_n(p', h)) = s(\varphi_n(p'', h))$, thus (x) implies that $\varphi_n(p', h) = \varphi_n(p'', h)$. All of the properties can be checked easily.

Case 1e: there is a $p' \in H_n$ with $p' < r_{m+1}$, there is a $p'' \in H_n$ with $r_{m+1} < p''$, and for the largest such p' and the smallest such p'', there is no $h \in \mathcal{K}$ with $g_n(p') > h(p')$ and $g_n(p'') < h(p'')$. Now let $\mathcal{K}' = \{h \in \mathcal{K} : g_n(p') < h(p') \text{ and } g_n(p'') > h(p'')\}$, where again, $p' \in H_n$ is the largest with $p' < r_{m+1}$ and $p'' \in H_n$ is the smallest with $p'' > r_{m+1}$. Let $q' = g_n(p')$ and $q'' = g_n(p'')$. The set $\{h(p'') : h \in \mathcal{K}'\}$ is finite, hence there is a $t \in (q', q'')$ with $t > h(p'')$ for every $h \in \mathcal{K}'$. Let $H_{n+1} = H_n \cup \{r_{m+1}\}$, $g_{n+1}(r_{m+1}) = t$ and $\varphi_{n+1}(r_{m+1}, h) = \varphi_n(p'', h)$ for every $h \in \mathcal{K}$. Using the fact that for no $h \in \mathcal{K}$ can h and any strictly increasing extension of g_{n+1} have the same values on $[r_{m+1}, p'']$, one can easily check that every property is satisfied.

Using (vi), these cover all sub-cases of Case 1. Now we turn to the second case.

Case 2: $n = 3m+1$. At this step, we make sure that $t_{m+1} \in g_{n+1}(H_{n+1})$. If already $t_{m+1} \in g_n(H_n)$ then let $H_{n+1} = H_n$, $g_{n+1} = g_n$ and $\varphi_{n+1} = \varphi_n$. Otherwise, similarly as in Case 1, there are multiple sub-cases according to the existence of $q' \in g_n(H_n)$ with $q' < t_{m+1}$, $q'' \in g_n(H_n)$ with $t_{m+1} < q''$, and whether there exists an $h \in \mathcal{K}$ such that $g_n(p') < h(p')$ or $g_n(p') > h(p')$, $g_n(p'') < h(p'')$ or $g_n(p'') > h(p'')$, where $p' = g_n^{-1}(q')$ and $p'' = g_n^{-1}(q'')$. These sub-cases can be handled similarly as in Case 1, but we quickly go though them. Since $r_{m+1} \in H_n$ we do not have to deal with the case $H_n = \emptyset$.

Case 2a: there is a $q' \in g_n(H_n)$ with $q' < t_{m+1}$ but there is no $q'' \in g_n(H_n)$ with $t_{m+1} < q''$. Let q' be the largest element in $g_n(H_n)$, clearly $q' < t_{m+1}$. As before, $g_n(p') < h(p')$ for every $h \in \mathcal{K}$ or $g_n(p') > h(p')$ for every $h \in \mathcal{K}$, where $p' = g_n^{-1}(q')$. In the first case, choose $r \in (p', p_{i+1})$ and $r > h^{-1}(t_{m+1})$ for every $h \in \mathcal{K}$. Such an r exists, since $h(p_{i+1}) = q_{i+1}$ for every $h \in \mathcal{K}$, and $\{h^{-1}(t_{m+1}) : h \in \mathcal{K}\}$ is finite. Let $q \in (q', t_{m+1})$ with $q < h(p')$ for every $h \in \mathcal{K}$, and choose $p \in (p', r)$ with $p > h^{-1}(t_{m+1})$ for every $h \in \mathcal{K}$. Then let $H_{n+1} = H_n \cup \{r, p\}$, and let g_{n+1} and φ_{n+1} extend g_n and φ_n, respectively, with $g_{n+1}(r) = t_{m+1}$, $g_{n+1}(p) = q$ and $\varphi_{n+1}(r, h) = \varphi_{n+1}(p, h) = \varphi_n(p', h)$ for every $h \in \mathcal{K}$.

In the second case, choose $r \in (p', p_{i+1})$ with $r < h^{-1}(q')$ for every $h \in \mathcal{K}$. Such an r exists, since for every $h \in \mathcal{K}$, $h(p') < q'$ implies $p' < h^{-1}(q')$ and $\{h^{-1}(q') : h \in \mathcal{K}\}$ is finite. Then set $H_{n+1} = H_n \cup \{r\}$, and let $g_{n+1}(r) = t_{m+1}$ and $\varphi_{n+1}(p, h) = \varphi_n(p', h)$ for every $h \in \mathcal{K}$.

Case 2b: there is no $q' \in g_n(H_n)$ with $q' < t_{m+1}$ but there is a $q'' \in g_n(H_n)$ with $t_{m+1} < q''$. This case can be handled similarly to Case 2a.

Case 2c: there is a $q' \in g_n(H_n)$ with $q' < t_{m+1}$, there is a $q'' \in H_n$ with $t_{m+1} < q''$, and for the largest such q' and the smallest such q'', there is no $h \in \mathcal{K}$ with $g_n(p') < h(p')$ and $g_n(p'') > h(p'')$, where $p' = g_n^{-1}(q')$ and $p'' = g_n^{-1}(q'')$. This is analogous to Case 1e. There exists $r \in (p', p'')$ with $h^{-1}(q') > r$ for every $h \in \mathcal{K}$ such that $g_n(p') > h(p')$ and $g_n(p') < h(p')$. As before, set $H_{n+1} = H_n \cup \{r\}$ and let g_{n+1}

extend g_n with $g_{n+1}(r) = t_{m+1}$, and φ_{n+1} extend φ_n with $\varphi_{n+1}(r, h) = \varphi_n(p', h)$ for every $h \in \mathcal{K}$.

Case 2d: there is a $q' \in g_n(H_n)$ with $q' < t_{m+1}$, there is a $q'' \in H_n$ with $t_{m+1} < q''$, and for the largest such q' and the smallest such q'', there is no $h \in \mathcal{K}$ with $g_n(p') > h(p')$ and $g_n(p'') < h(p'')$, where $p' = g_n^{-1}(q')$ and $p'' = g_n^{-1}(q'')$. This is analogous to Case 1d. Let $\mathcal{K}' = \{h \in \mathcal{K} : g_n(p') < h(p') \text{ and } g_n(p'') > h(p'')\}$, this may again be the empty set. Choose $r \in (p', p'')$ such that $r < h^{-1}(t_{m+1})$ for each $h \in \mathcal{K}'$ with $h^{-1}(t_{m+1}) > p'$. There is a $q \in (t_{m+1}, q'')$ with $q > h(p'')$ for every $h \in \mathcal{K}'$. Choose $p \in (r, p'')$ with $p < h^{-1}(t_{m+1})$ for each $h \in \mathcal{K}'$ with $h^{-1}(t_{m+1}) > p'$.

Now let $H_{n+1} = H_n \cup \{p, r\}$, g_{n+1} extend g_n with $g_{n+1}(r) = t_{m+1}$, $g_{n+1}(p) = q$. For $h \in \mathcal{K}'$, either $h^{-1}(t_{m+1}) \leq p'$ or $h^{-1}(t_{m+1}) > p$. If $h^{-1}(t_{m+1}) \leq p'$ then let $\varphi_{n+1}(r, h) = \varphi_n(p', h)$, if $h^{-1}(t_{m+1}) > p$ then let $\varphi_{n+1}(r, h) = \varphi_n(p'', h)$. In both cases, let $\varphi_{n+1}(p, h) = \varphi_n(p'', h)$. If $h \in \mathcal{K} \setminus \mathcal{K}'$ then let $\varphi_{n+1}(p, h) = \varphi_{n+1}(r, h) = \varphi_n(p', h)$.

Again using (vi), these cover all sub-cases of Case 2. Now we turn to the third case.

Case 3: $n = 3m + 2$. At this step, we make sure that $O_{m+1} \in \varphi_{n+1}(H_{n+1}, h)$ for every $h \in \mathcal{K}$. Note throughout that there is no $O \in \mathcal{O}'$ with $s(O) = 0$. If $O_{m+1} \in \varphi_n(H_n, h)$ for any (hence, by (ix) for every) $h \in \mathcal{K}$ then let $H_{n+1} = H_n$, $g_{n+1} = g_n$ and $\varphi_{n+1} = \varphi_n$. If this is not the case, we consider the sub-cases according to $\varphi_n(H_n, h_0)$ for a fixed $h_0 \in \mathcal{K}$. We suppose throughout that $s(O_{m+1}) = 1$. The case $s(O_{m+1}) = -1$ is similar. Also, note that $H_n \neq \emptyset$, as, for example, $r_1 \in H_n$.

Case 3a: $O_{m+1} > O$ for every $O \in \varphi_n(H_n, h_0)$, and for the largest $O \in \varphi_n(H_n, h_0)$ (with respect to $<$), $s(O) = -1$. Let p be the largest element in H_n, $q = g_n(p)$, then $O = \varphi_n(p, h_0)$. This means, using (ix) and (viii) that $\varphi_n(p, h) = O$ and $g_n(p) > h(p)$ for every $h \in \mathcal{K}$. As $h(p_{i+1}) = q_{i+1}$ for every $h \in \mathcal{K}$ and $g_n : H_n \to (q_i, q_{i+1})$, we can choose $t \in (q, q_{i+1})$ and as $\{h^{-1}(t) : h \in \mathcal{K}\}$ is finite, there exists $r \in (p, p_{i+1})$ with $r > h^{-1}(t)$ for every $h \in \mathcal{K}$. Now let $H_{n+1} = H_n \cup \{r\}$, let g_{n+1} extend g_n with $g_{n+1}(r) = t$ and let φ_{n+1} extend φ_n with $\varphi_{n+1}(r, h) = O_{m+1}$ for every $h \in \mathcal{K}$. One can easily check that the necessary conditions still hold.

Case 3b: $O_{m+1} > O$ for every $O \in \varphi_n(H_n, h_0)$, and for the largest $O \in \varphi_n(H_n, h_0)$ (with respect to $<$), $s(O) = 1$. Let p be the largest element in H_n, $q = g_n(p)$, then $O = \varphi_n(p, h_0)$. Using that f is good, there exists $O' \in \mathcal{O}'$ with $O < O' < O_{m+1}$ and $s(O') = -1$. Now choose $r' \in (p, p_{i+1}$ and choose $t' \in (q, q_{i+1})$ with $t' > h(r')$ for every $h \in \mathcal{K}$. Then choose $t'' \in (t', q_{i+1})$ and choose $r'' \in (r', p_{i+1})$ with $r'' >$

$h^{-1}(t'')$ for every $h \in \mathcal{K}$. Now let $H_{n+1} = H_n \cup \{r', r''\}$, and let g_{n+1} extend g_n with $g_{n+1}(r') = t'$ and $g_{n+1}(r'') = t''$, and let φ_{n+1} extend φ_n with $\varphi_{n+1}(r', h) = O'$ and $\varphi_{n+1}(r'', h) = O_{m+1}$ for every $h \in \mathcal{K}$.

The cases where $O_{m+1} < O$ for every $O \in \varphi_n(H_n, h_0)$ are similar to the ones above.

Case 3c: O_{m+1} is between elements of $\varphi_n(H_n, h_0)$, and if O' is the largest element of $\varphi_n(H_n, h_0)$ with $O' < O_{m+1}$ and O'' is the smallest element of $\varphi_n(H_n, h_0)$ with $O_{m+1} < O''$ then $s(O') = -1$ and $s(O'') = 1$. In this case, choose $O \in \mathcal{O}'$ with $O_{m+1} < O < O''$ and $s(O) = -1$, again, such an O exists because f is good. The orbitals O' and O'' are neighboring ones in $\varphi_n(H_n, h)$ for every $h \in \mathcal{K}$.

Notice that for every $h \in \mathcal{K}$ there exists a unique pair of neighboring points $p', p'' \in H_n$ with $\varphi_n(p', h) = O'$ and $\varphi_n(p'', h) = O''$. Therefore, we can partition $\mathcal{K}$ into finitely many compact sets according to this pair. We define g_{n+1} separately on each such interval (p', p''), that is, where p' and p'' are neighboring points in H_n and $\varphi_n(p', h) = O'$, $\varphi_n(p'', h) = O''$ for some $h \in \mathcal{K}$.

So let p', p'' be such elements of H_n and let $\mathcal{K}' = \{h \in \mathcal{K} : \varphi_n(p', h) = O' \text{ and } \varphi_n(p'', h) = O''\}$. Using the facts that $s(O') = -1$, $s(O'') = 1$ and (viii), we have $g_n(p') > h(p')$ and $g_n(p'') < h(p'')$ for every $h \in \mathcal{K}'$. Let $q' = g_n(p')$ and $q'' = g_n(p'')$, and choose $q \in (q', q'')$. Let $\{r^1, r^2, \ldots, r^c\} = \{h^{-1}(q) : h \in \mathcal{K}'\}$, where $r^1 < r^2 < \cdots < r^c$. Note that $h(p') < g_n(p') = q' < q < q'' = g_n(p'') < h(p'')$ for every $h \in \mathcal{K}'$, hence $p' < r^1$ and $r^c < p''$. For $1 \leq j \leq c$, let $\mathcal{K}^j = \{h \in \mathcal{K}' : h^{-1}(q) = r^j\}$.

Choose $t \in (q', q)$ with $t > h(r^j)$ for every $1 \leq j \leq c$ and every $h \in \mathcal{K}'$ such that $h(r^j) < q$. From now on, the values of $g_{n+1}|_{(p',p'')}$ on newly defined points will always be at least t. This will achieve that if we add new points to take care of the functions in $\mathcal{K}^j$ for some j, then our choices will not interfere with the functions in $\mathcal{K}' \setminus \mathcal{K}^j$.

Choose $r \in (p', p'')$ with $r < h^{-1}(q')$ for every $h \in \mathcal{K}'$. By setting $g_{n+1}(r) = t$ and extending it to a strictly increasing function, it can be easily seen that the extension cannot have a common value with any $h \in \mathcal{K}'$ on the interval (p', r). Let $t_1^1 \in (t, q)$ be arbitrary and choose $r_1^1 \in (r, r^1)$ with $t_1^1 < h(r_1^1)$ for every $h \in \mathcal{K}^1$. Then choose $r_2^1 \in (r_1^1, r^1)$ and choose $t_2^1 \in (t_1^1, q)$ such that $t_2^1 > h(r_2^1)$ for every $h \in \mathcal{K}^1$. Then let $r_3^1 = r^1$ and choose $t_3^1 \in (t_2^1, q)$.

We handle the families $\mathcal{K}^j$ for $j \geq 2$ similarly. Choose $t_1^j \in (t_3^{j-1}, q)$ and then choose $r_1^j \in (r_3^{j-1}, r^j)$ such that $h(r_1^j) > t_1^j$ for every $h \in \mathcal{K}^j$. Then let $r_2^j \in (r_1^j, r^j)$ and choose $t_2^j \in (t_1^j, q)$ with $t_2^j > h(r_2^j)$ for every $h \in \mathcal{K}^j$. Then let $r_3^j = r^j$ and choose $t_3^j \in (t_2^j, q)$.

After recursively choosing the rational numbers above for every $j \leq c$, we choose $p \in (r_3^c, p'')$ such that $p > h^{-1}(q'')$ for every $h \in \mathcal{K}'$. Now we will set $H_{n+1} \cap (p', p'') = (H_n \cap (p', p'')) \cup \{r, p, r_\ell^j : 1 \leq j \leq c, 1 \leq \ell \leq 3\}$. Let g_{n+1} extend g_n with $g_{n+1}(r) = t$, $g_{n+1}(p) = q$ and $g_{n+1}(r_\ell^j) = t_\ell^j$ for every $1 \leq j \leq c$ and $1 \leq \ell \leq 3$. Let φ_{n+1} extend φ_n with $\varphi_{n+1}(r, h) = O'$, $\varphi_{n+1}(p, h) = O''$ for every $h \in \mathcal{K}'$. Also, let $\varphi_{n+1}(r_\ell^j, h) = O'$ for every ℓ if $h \in \mathcal{K}^{j'}$ with $j' > j$, and $\varphi_{n+1}(r_\ell^j, h) = O''$ if $j' < j$. If $j' = j$ then let $\varphi_{n+1}(r_1^j, h) = O_{m+1}$, $\varphi_{n+1}(r_2^j, h) = O$ and $\varphi_{n+1}(r_3^j, h) = O''$. For every $h \in \mathcal{K} \setminus \mathcal{K}'$, let $\varphi_{n+1}(x, h) = \varphi_n(p', h) = \varphi_n(p'', h)$, for every $x \in \{r, p, r_\ell^j : 1 \leq j \leq c, 1 \leq \ell \leq 3\}$, where we used (x) for the last equality.

We do the same in every interval of the form (p', p''), where p' and p'' are neighbors in H_n, and $\varphi_n(h, p') = O'$ and $\varphi_n(h, p'') = O''$ for some $h \in \mathcal{K}$. Extending g_n and φ_n appropriately, one obtains H_{n+1}, g_{n+1} and φ_{n+1} with the necessary conditions. We note that the choice of t ensures that condition (vii) is satisfied.

Case 3d: O_{m+1} is between elements of $\varphi_n(H_n, h_0)$, and if O' is the largest element of $\varphi_n(H_n, h_0)$ with $O' < O_{m+1}$ and O'' is the smallest element of $\varphi_n(H_n, h_0)$ with $O_{m+1} < O''$ then $s(O') = 1$ and $s(O'') = -1$. This case can be handled quite similarly as Case 3c. Choose $O \in \mathcal{O}'$ with $O' < O < O_{m+1}$ and $s(O) = -1$. Again the unique pairs of neighboring points $p', p'' \in H_n$ with $\varphi_n(p', h) = O'$ and $\varphi_n(p'', h) = O''$ define a partition of $\mathcal{K}'$. So let $p', p'' \in H_n$ be such a pair, we set $q' = g_n(p')$ and $q'' = g_n(p'')$.

Let $p \in (p', p'')$ be arbitrary and let $\{t^1, \ldots, t^c\} = \{h(p) : h \in \mathcal{K}'\}$, where $\mathcal{K}' = \{h \in \mathcal{K} : h(p') > g_n(p') \text{ and } h(p'') < g_n(p'')\}$, such that $t^1 < \cdots < t^c$. We set $\mathcal{K}^j = \{h \in \mathcal{K}' : h(p) = t^j\}$. Now one can choose $r \in (p', p)$ with $h^{-1}(t^j) < r$ for every $h \in \mathcal{K}'$ and $1 \leq j \leq c$ if $h^{-1}(t^j) < p$. Let $t \in (q', t^1)$ be such that $t < h(p')$ for every $h \in \mathcal{K}'$. Now suppose that for $j' < j$ and $1 \leq \ell \leq 3$ the points $r_\ell^{j'}$ and $t_\ell^{j'}$ are given. Then choose r_1^j arbitrarily for the set (r, p) if $j = 1$ and from (r_3^{j-1}, p) if $j > 1$. Then choose t_1^j from (t, t^j) if $j = 1$ and from (t_3^{j-1}, t^j) if $j > 1$ such that $h(r_1^j) < t_1^j$ for every $h \in \mathcal{K}^j$. Then choose $t_2^j \in (t_1^j, t^j)$ and choose $r_2^j \in (r_1^j, p)$ such that $h(r_2^j) > t_2^j$ for every $h \in \mathcal{K}^j$. Finally, choose $r_3^j \in (r_2^j, p)$ and set $t_3^j = t^j$.

After recursively choosing the points r_ℓ^j and t_ℓ^j, choose $q \in (t^c, q'')$ such that

$q > h(p'')$ for every $h \in \mathcal{K}'$. As before, let $H_{n+1} \cap (p', p'') = (H_n \cap (p', p'')) \cup \{r, p, r_\ell^j : 1 \leq j \leq c, 1 \leq \ell \leq 3\}$, and define $g_{n+1}(r) = t$, $g_{n+1}(p) = q$ and $g_{n+1}(r_\ell^j) = t_\ell^j$ for every $1 \leq j \leq c$ and $1 \leq \ell \leq 3$. For $h \in \mathcal{K}^j$ let $\varphi_{n+1}(r, h) = O'$, $\varphi_{n+1}(r_\ell^{j'}, h) = O'$ for every $j' < j$ and $1 \leq \ell \leq 3$, $\varphi_{n+1}(r_1^j, h) = O$, $\varphi_{n+1}(r_2^j, h) = O_{m+1}$, $\varphi_{n+1}(r_3^j, h) = O''$, and $\varphi_{n+1}(r_\ell^{j''}, h) = \varphi_{n+1}(p, h) = O''$ for every $j'' > j$, $1 \leq \ell \leq 3$. For every $h \in \mathcal{K} \setminus \mathcal{K}'$ we set $\varphi_{n+1}(x, h) = \varphi_n(p', h)$ for every $x \in (H_{n+1} \cap (p', p'')) \setminus H_n$.

It is straightforward to check that H_{n+1}, g_{n+1} and φ_{n+1} obtained in this way satisfy the conditions. □

Now we show the following to complete the proof of the theorem.

Claim 6.0.8 *There is an automorphism $g \in \mathrm{Aut}(\mathbb{Q})$ such that $g^{-1}\mathcal{K} \subset \mathcal{C}$.*

Proof. Suppose that H_n^i, g_n^i and φ_n^i are the corresponding sets and functions on the interval (p_i, p_{i+1}) for $i < k$. Let $g(p_i) = q_i$ for every $1 \leq i < k$, and let $g|_{(p_i, p_{i+1})} = \bigcup_n g_n^i$ for every $i < k$. This makes sense, since $\bigcup_n g_n^i$ is an increasing bijection between (p_i, p_{i+1}) and (q_i, q_{i+1}) using (i), (iii) and (v). Also, let $\varphi(p_i, h) = \{p_i\}$ for every $1 \leq i < k$ and $h \in \mathcal{K}$, and let $\varphi(., h)|_{(p_i, p_{i+1})} = \bigcup_n \varphi_n^i(., h)$ for every $0 \leq i < k$ and $h \in \mathcal{K}$. This also makes sense, since using (i), (iv) and (v), $\bigcup_n \varphi_n^i(., h)$ is an increasing surjective function from (p_i, p_{i+1}) to those elements of $\mathcal{O}_f$ that are subsets of (p_i, p_{i+1}).

We now show that f, $g^{-1}h$ and $\varphi(., h)$ satisfy the conditions of Lemma 6.0.4 to prove that f and $g^{-1}h$ are conjugate automorphisms for every $h \in \mathcal{K}$. We start by showing that $g^{-1}h$ is good for every $h \in \mathcal{K}$. First of all, it only has the finitely many fixed points that f has, since if $p \in \mathbb{Q}$ is not among the fixed points of f, then p is in some interval of the form (p_i, p_{i+1}), and as (viii) covers all cases, $g_n(p) \neq h(p)$, hence $g^{-1}h(p) \neq p$. Now suppose towards a contradiction that there are distinct orbitals $O_1, O_2 \in \mathcal{O}_{g^{-1}h}$ such that either $s_{g^{-1}h}(O_1) = s_{g^{-1}h}(O_2) = 1$ or $s_{g^{-1}h}(O_1) = s_{g^{-1}h}(O_2) = -1$ and there is no orbital $O_3 \in \mathcal{O}_{g^{-1}h}$ with $s_{g^{-1}h}(O_3) \neq s_{g^{-1}h}(O_1)$ between them. We suppose for the rest of the proof that $s_{g^{-1}h}(O_1) = s_{g^{-1}h}(O_2) = 1$, the case when they equal -1 is analogous. Note that in this case,

$$g(p) < h(p) \text{ for every } p \in (O_1 \cup O_2). \tag{6.0.1}$$

There is no fixed point of $g^{-1}h$, or equivalently, there is no fixed point of f between O_1 and O_2, thus $O_1, O_2 \subset (p_i, p_{i+1})$ for some $i < k$. Let $p' \in O_1$ and

$p'' \in O_2$ be arbitrary. Then $\varphi(p', h), \varphi(p'', h) \in \mathcal{O}_f$. We consider the following two cases separately.

Case 1: $\varphi(p', h) = \varphi(p'', h)$. Let $O = \varphi(p', h)$. Then, using the fact that $\varphi(., h)$ is increasing provided by (iv), $\varphi(p, h) = O$ for every $p \in (p', p'')$. Using (6.0.1) and (viii), $s_f(\varphi(p', h)) = s_f(O) = 1$. Hence if $p \in (p', p'')$ then $g(p) < h(p)$ using (viii) and the fact that $\varphi(p, h) = O$. Let n be large enough such that $p', p'' \in H_n^i$ and let $\{r^1, \ldots, r^m\} = H_n^i \cap [p', p'']$ where $p' = r^1 < \cdots < r^m = p''$. Then applying (vii) to each of the intervals $[r^j, r^{j+1}]$, the facts that $h(r^j) > g_n^i(r^j)$ and $h(r^{j+1}) > g_n^i(r^{j+1})$ imply $h(r) \geq g_n^i(r^{j+1})$ for every $r \in [r^j, r^{j+1}]$. It follows (since g is an increasing extension of g_n^i) that $g^{-1}h(r^1) \geq g^{-1}(g(r^2)) = r^2$. Using induction, one can show with the same argument that $(g^{-1}h)^{m-1}(r^1) \geq r^m$, hence $(g^{-1}h)^{m-1}(p') \geq p''$. This fact implies that p' and p'' are in the same orbital with respect to $g^{-1}h$, contradicting our assumption.

Case 2: $\varphi(p', h) \neq \varphi(p'', h)$. Again using (6.0.1) and (viii) twice, $g(p') < h(p')$ and $g(p'') < h(p'')$, hence $s_f(\varphi(p', h)) = s_f(\varphi(p'', h)) = 1$. Using the fact that f is good, there is $O \in \mathcal{O}_f$ between $\varphi(p', h)$ and $\varphi(p'', h)$ with $s_f(O) = -1$, since there is no fixed point between O_1 and O_2. Using that $\varphi(., h)$ is increasing and surjective provided by (iv) and (v), there is $p \in (p', p'')$ with $\varphi(p, h) = O$. Then (viii) ensures that $g(p) > h(p)$, hence $g^{-1}h(p) < p$, therefore there exists $O' \in \mathcal{O}_{g^{-1}h}$ with $O_1 < O' < O_2$ and $s_{g^{-1}h}(O') = -1$, contradicting our assumptions. This completes the proof of the fact that $g^{-1}h$ is good.

The function $\varphi(., h) : \mathbb{Q} \to \mathcal{O}_f$ is increasing and surjective using its construction and (iv), (v). The fact that $|\varphi(., h)^{-1}(p)| = 1$ for every $p \in \mathrm{Fix}(f)$ readily follows from the construction of φ. Condition (1) of Lemma 6.0.4 follows from the fact that (viii) covers all cases, hence there is no fixed points of $g^{-1}h$ on any interval of the form (p_i, p_{i+1}). Now we check condition (2). Let $q \in \mathbb{Q}$ be fixed. For both direction, both the facts that $g^{-1}h(q) > q$ and $s_f(\varphi(q, h)) = 1$ imply separately that $q \neq p_i$ for any i, hence $q \in (p_i, p_{i+1})$ for some i. If n is large enough such that $q \in H_n^i$ then (viii) implies both direction in (2). The proof is analogous for (3).

Therefore the conditions of Lemma 6.0.4 are satisfied for f, $g^{-1}h$ and φ, hence f and $g^{-1}h$ are conjugate automorphisms for every $h \in \mathcal{K}$. This completes the proof of the lemma. □

And thus the proof of the theorem is also complete. □

Proof of Theorem 6.0.3. Using Theorem 6.0.1, the union of the Haar null conjugacy classes is exactly the union of the automorphisms with infinitely many fixed points and those that violate the condition of Lemma 6.0.5. The former set is Haar null using Theorem 5.0.6, and the latter is Haar null by Lemma 6.0.5. Hence the union of the two is also Haar null. □

7 Problems and questions

In this section we mention some problems and questions formulated while working on this thesis.

Theorem 2.2.3 characterizes compact and locally compact subgroups of $\mathrm{Sym}(\omega)$. One might think that having a nice algebraic closure as defined in Definition 2.2.4 is almost the exact opposite of being locally compact:

1. Being locally compact is equivalent of having a finite set $S \in \omega$ that has the largest group theoretic algebraic closure possible: $\mathrm{ACL}(S) = G$,
2. while having a nice group theoretic algebraic closure means that for every finite S its algebraic closure is small: $\mathrm{ACL}(S)$ must be finite.

In fact there is an even stricter notion that is more like the exact opposite of being locally compact:

Definition 7.0.1 (Having no algebraicity) *A closed subgroup G of* $\mathrm{Sym}(\omega)$ ***has no algebraicity*** *if for every finite set $S \subset \omega$ its group theoretic algebraic closure is itself:* $\mathrm{ACL}(S) = S$.

Note that being compact or locally compact is an intrinsic property of a Polish group G as an abstract group (i.e., we are not considering actions of G on underlying spaces). Similarly, having a dense conjugacy class, having a comeager conjugacy class, having a co-Haar null conjugacy class or having a decomposition into a meager and a Haar null set are all intrinsic properties.

On the other hand, having a nice algebraic closure or having no algebraicity also depends on the action of G on ω. These are not intrinsic properties. For example, if G is any subgroup of $\mathrm{Sym}(\omega)$ that has no algebraicity and thus has nice algebraic closure, then one can define an action of G on $(\omega \times \omega)$ as

$$g \in G : g((x, y)) = (g(x), y).$$

This group action has

$$\mathrm{ACL}(\{(x, y)\}) \supset \{(x, z) : z \in \omega\}$$

so it has algebraicity and does not have nice algebraic closure.

Question 7.0.2 (Generalization of Theorem 5.0.9) *For a closed subgroup G of* $\mathrm{Sym}(\omega)$ *is the set of permutations with infinitely many infinite cycles co-Haar null if and only if G is non-locally compact?*

Still it is possible that being locally compact is the exact opposite of some of the two definitions in the following sense:

Question 7.0.3 (Asked by Juris Steprans[1]) *Is every non-locally compact, closed subgroup of* $\mathrm{Sym}(\omega)$ *isomorphic as an abstract group to another non-locally compact, closed subgroup of* $\mathrm{Sym}(\omega)$ *that has no algebraicity?*

We ask the following similar question:

Question 7.0.4 *Is every non-locally compact, closed subgroup of* $\mathrm{Sym}(\omega)$ *homeomorphic as a topological group to another non-locally compact, closed subgroup of* $\mathrm{Sym}(\omega)$ *that has nice algebraic closure?*

In the statement of Theorem 4.0.3 there is an additional criterion on G besides having a nice algebraic closure, namely that for every $n \in \omega$ there is a finite set $S_n \subset \omega$ such that there is no n-element subset $X \subset \omega$ such that $S_n \subset \mathrm{ACL}(X)$.

Question 7.0.5 *Does the additional criterion in Theorem 4.0.3 follow from having a nice algebraic closure?*

Note that if both Question 7.0.4 and Question 7.0.5 have affirmative answers then combining them with Theorem 4.0.1 and Theorem 4.0.3 yields the following statement:

Conjecture 7.0.6 *Every closed uncountable subgroup of* $\mathrm{Sym}(\omega)$ *can be written as a disjoint union* $\mathbb{R} = A \cup B$ *where* A *is meager and* B *has Haar measure zero.*

[1] On mathoverflow: http://mathoverflow.net/questions/239285/automorphism-group-of-a-structure-without-the-sap, we are not aware of any printed appearance of this question.

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