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Haar null and Haar meager sets
Small sets in non-locally-compact Polish groups

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Contents

1 Introduction and history

Many results in various branches of mathematics state that certain properties hold for *almost every* element of a space. In the continuous, large structures which are frequently studied in analysis it is common to encounter a property that is true for most points, but false on a negligibly small part of the structure. For example, for a Lebesgue measurable set $A \subseteq \mathbb{R}^n$ the set of points where the density of A is neither 0 nor 1 always has Lebesgue measure zero by Lebesgue's density theorem, but it is known that there are always such exceptional points, unless either A or $\mathbb{R}^n \setminus A$ has measure zero. These situations mean that there are facts which can be grasped only by defining a suitable notion of smallness and stating that the exceptional elements form a small set.

In the Euclidean space $\mathbb{R}^n$ there is a generally accepted, natural notion of smallness: a set is considered to be small if it has Lebesgue measure zero. As this notion is defined by a measure, which is by definition countably additive, these small sets are closed under countable unions and hence form a σ-ideal. The Lebesgue measure is essentially defined by the fact that it is translation invariant (and satisfies some technical properties). This allows us to generalize it to topological groups (we need a group structure for the translations and a topological structure for the technical properties).

This generalized notion (which was introduced by Alfréd Haar in 1933) is called the *Haar measure* (when the group is not commutative, either left multiplication or right multiplication can be the generalized notion corresponding to translation and hence we get left and right Haar measures). (We summarize the definition and the basic properties of the Haar measures in subsection 3.3, a deeper analysis can be found e.g. in [13, §15].) It is possible to show that (e.g. left) Haar measures exist on a topological group if and only if it is locally compact, and in locally compact groups the left Haar measures are unique up to multiplication by a constant (see Theorem 3.3.3 and Theorem 3.3.11). This means that in locally compact groups one can define a natural notion of smallness by saying that a set is small if it has (e.g. left) Haar measure zero, but this method says nothing about non-locally-compact groups.

In the paper [3] (which was published in 1972) Christensen introduced *Haar null sets*, which are defined in all abelian Polish groups and coincide with the sets of Haar measure zero in the locally compact case. Twenty years later Hunt, Sauer and Yorke independently introduced this notion under the name of *shy sets* in the paper [15]. Since then lots of papers were published which either study some property of Haar null sets or use this notion of smallness to state facts which are true for almost every element of some structure. It was relatively easy to generalize this notion to non-abelian groups, on the other hand, the assumption that the topology is Polish (that is, separable and completely metrizable) is still almost always assumed, because it turned out to be convenient and useful.

Haar null sets are translation invariant in the strong sense that if X is Haar null, then $gXh = \{gxh : x \in X\}$ is Haar null for every pair of elements g, h from the group. The definition of Haar null sets is chosen in a way that makes this fact trivial: a (Borel) set is Haar null if there is a (Borel probability) measure that assigns measure zero to every such translate of a set. It is possible to prove that Haar null sets form a σ-ideal (see Theorem 3.2.5).

There is another widely used notion of smallness, the notion of *meager sets* (also known as sets of the first category). Meager sets can be defined in any topological space; a set is said to be meager if it is the countable union of nowhere dense sets. It is trivial that meager sets form a σ-ideal, and it is also clear that this notion is translation invariant in topological groups. A topological space is called a Baire space if the nonempty open sets are non-meager; this basically means that one can consider the meager sets small in these spaces. The Baire category theorem states that all completely metrizable spaces and all locally compact Hausdorff spaces are Baire spaces (see [17, Theorem 8.4] for the proofs).

In locally compact groups the system of meager sets and the system of sets of Haar measure zero share many properties; for example the classical Erdős-Sierpiński duality theorem states that it is consistent that there is a bijection $f : \mathbb{R} \to \mathbb{R}$ such that $f(A)$ is meager if and only if $A \subseteq \mathbb{R}$ has Lebesgue measure zero and $f(A)$ has Lebesgue measure zero if and only if $A \subseteq \mathbb{R}$ is meager. Despite this, there are sets that are small in one sense and far from being small in the other sense, for example every abelian locally compact group can be written as the union of a meager set and a set of Haar measure zero.

In 2013, Darji defined the notion of *Haar meager sets* in the paper [5] to provide a better analog of Haar null sets in the non-locally-compact case. Darji only considered abelian Polish groups, but [7] generalized this notion to non-abelian Polish groups. Haar meager sets coincide with meager sets in locally compact Polish groups, and Haar meagerness is a strictly weaker notion than meagerness in non-locally-compact abelian groups (see Theorem 3.3.13 and Theorem 3.3.14). The difference between the definition of Haar null and Haar meager sets is that for Haar meager sets we require the existence of a continuous map from a compact metric space to the group that assigns meager preimages to the translates of our set (instead of the existence of a measure that assigns measure zero to the translates). The analogy between the definitions mean that most of the results for Haar null sets are also true for Haar meager sets and often can be proved using similar methods.

The goal of this thesis is to introduce these two notions and collect those basic results about them that are useful for proving new results. Of course the boundary between applicable and not applicable results is blurry, but we tried to include the most frequently used lemmas and the counterexamples showing that some usual property cannot be generalized and must be avoided in the proofs. This focus means that we do not include the applications of this theory in concrete cases except as illustrations for proof techniques. The majority of the results are included with proofs to illustrate the ideas and methods of this area, but especially in the later sections we frequently omit proofs that are too technical, only distantly related to this area or simply too long.

At the beginning, in section 2, we introduce the notions, definitions and conventions which are not related to our area, but used repeatedly in this thesis. Then section 3 defines the core notions and investigates their most important properties.

After these, section 4 considers the modified variants of the definitions. This section starts with a large collection of equivalent definitions for our core notions, then lists and briefly describes most of the versions which appear in the literature and are not (yet proved to be) equivalent to the "plain" versions.

The next section, section 5, considers the feasibility of generalizing three well-known results (Fubini's theorem, the Steinhaus theorem and the countable chain condition) for non-locally-compact Polish groups. Unfortunately, most of the answers are given in form of counterexamples, but weakened variants of the first two results can be salvaged and these are useful as lemmas.

Finally, in section 6 we discuss some proof techniques for questions from this area. Some of these are essentially useful lemmas, the others are just ideas and ways of thinking which can be helpful in certain cases.

2 Notation and terminology

This section is the collection of the miscellaneous notations, definitions and conventions that are used repeatedly in this thesis.

The symbols $\mathbb{N}$ and ω both refer to the set of nonnegative integers. We write $\mathbb{N}$ if we consider this set as a topological space (with the discrete topology) and ω if we use it only as a cardinal, ordinal or index set. (For example we write the Polish space of the countably infinite sequences of natural numbers as $\mathbb{N}^\omega$.) We consider the nonnegative integers as von Neumann ordinals, i.e. we identify the nonnegative integer n with the set $\{0, 1, 2, \ldots, n-1\}$.

$\mathcal{P}(S)$ denotes the power set of a set S. For a set $S \subseteq X \times Y$, $x \in X$ and $y \in Y$, S_x is the x-section $S_x = \{y : (x, y) \in S\}$ and S^y is the y-section $S^y = \{x : (x, y) \in S\}$.

If S is a subset of a topological space, $\operatorname{int}(S)$ is the interior of S and $\overline{S}$ is the closure of S. We consider $\mathbb{N}$, $\mathbb{Z}$ and all finite sets to be topological spaces with the discrete topology. (Note that this convention allows us to simply write the Cantor set as $2^\omega (= \{0, 1\}^\omega)$.) If X is a topological space, then

$\mathcal{B}(X)$ denotes its Borel subsets ($\mathcal{B}(X)$ is the σ-algebra generated by the open sets, see [17, Chapter II]),

$\mathcal{M}(X)$ denotes its meager subsets (a set is *meager* if it is the union of countably many nowhere dense sets and a set is *nowhere dense* if the interior of its closure is empty, see [17, §8.A]).

If the space X is *Polish* (that is, separable and completely metrizable), then

$\mathbf{\Sigma}^1_1(X)$ denotes its analytic subsets (a set is *analytic* if it is the continuous image of a Borel set, see [17, Chapter III]),

$\mathbf{\Pi}^1_1(X)$ denotes its coanalytic subsets (a set is *coanalytic* if its complement is analytic, see [17, Chapter IV]).

If the topological space X is clear from the context, we simply write $\mathcal{B}$, $\mathcal{M}$, $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$.

In a metric space (X, d), $\operatorname{diam}(S) = \sup\{d(x, y) : x, y \in S\}$ denotes the diameter of the subset S. If $x \in X$ and $r > 0$, then $B(x, r) = \{x' \in X : d(x, x') < r\}$ and $\overline{B}(x, r) = \{x' \in X : d(x, x') \leq r\}$ denotes respectively the open and the closed ball ball with center x and radius r in X.

If μ is an outer measure on a set X, we say that $A \subseteq X$ is *μ-measurable* if $\mu(B) = \mu(B \cap A) + \mu(B \setminus A)$ for every $B \subseteq X$. Unless otherwise stated, we identify an outer measure μ with its restriction to the μ-measurable sets and "measure" means an outer measure or the complete measure that is identified by it this way. A measure μ is said to be *Borel* if all Borel sets are μ-measurable. The support of the measure μ is denoted by $\operatorname{supp} \mu$.

Almost all of our results will be about topological groups. A set G is called a *topological group* if is equipped with both a group structure and a Hausdorff topology and these structures are *compatible*, that is, the multiplication map $G \times G \to G$, $(g, h) \mapsto gh$ and the inversion map $G \to G$, $g \mapsto g^{-1}$ are continuous functions. We make the convention that whenever we require a group to have some topological property (for example a "compact group", a "Polish group", ...), then it means that the group must be a topological group and have that property (as a topological space). The identity element of a group G will be denoted by 1_G.

Most of the results in this thesis are about certain subsets of Polish groups. Unless otherwise noted, $(G, \cdot)$ denotes an arbitrary Polish group. We denote the group operation by multiplication even when we assume that the (abstract) group under consideration is abelian, but we write the group operation of well-known concrete abelian groups like $(\mathbb{R}, +)$ or $(\mathbb{Z}^\omega, +)$ as addition.

Some techniques only work in Polish groups that *admit a two-sided invariant metric*. (A metric d on G is called *two-sided invariant* (or simply *invariant*) if $d(g_1hg_2, g_1kg_2) = d(h, k)$ for any $g_1, g_2, h, k \in G$.) Groups with this property are also called TSI groups. This class of groups properly contains all Polish, abelian groups, since each metric group G admits a left-invariant metric which, obviously, is invariant when G is abelian. Any invariant metric on a Polish group is automatically complete. For proofs of these facts and more results about TSI groups see for example [13, §8.].

Some basic results can be generalized for non-separable groups, but we will only deal with the separable case. On the other hand, many papers about this topic only consider abelian groups or some class of vector spaces. When the proof of a positive result can be generalized for arbitrary Polish group, we will usually do so, but we will usually provide counterexamples only in the special case where their construction is the simplest. If we assume that G is locally compact, our notions will coincide with simpler notions (see subsection 3.3) and the majority of the results in this thesis become significantly easier to prove, so the interesting case is when G is not locally compact.

3 Basic properties

3.1 Core definitions

This subsection introduces the core notions of this thesis. Both notions have several slightly different formalizations in the literature. The terminology used in this thesis is based on the terminology of [8]. The various equivalent forms of these definitions and some variants which are similar, but lack some important properties are discussed in section 4.

Haar null sets were first introduced by Christensen in [3] in 1972 as a generalization of the null sets of the Haar measure. (The Haar measure itself cannot be generalized for groups that are non-locally-compact, see Theorem 3.3.11.) Twenty years later in [15] Hunt, Sauer and Yorke independently introduced Haar null sets under the name "shy sets".

Definition 3.1.1. A set $A \subseteq G$ is said to be *Haar null* if there are a Borel set $B \supseteq A$ and a Borel probability measure μ on G such that $\mu(gBh) = 0$ for every $g, h \in G$. A measure μ satisfying this is called a *witness measure* for A. The system of Haar null subsets of G is denoted by $\mathcal{HN} = \mathcal{HN}(G)$.

Remark 3.1.2. Using the terminology introduced in [15], a set $A \subseteq G$ is called *shy* if it is Haar null, and *prevalent* if $G \setminus A$ is Haar null.

Some authors (including Christensen) write "universally measurable set" instead of "Borel set" when they define Haar null sets. This version is not equivalent to the original, but most results can be proved for both notions in the same way. When a paper uses both notions, sets satisfying this alternative definition are called "generalized Haar null sets".

Definition 3.1.3. If X is a Polish space, a set $A \subseteq X$ is called *universally measurable* if it is μ-measurable for any σ-finite Borel measure μ on X.

Definition 3.1.4. A set $A \subseteq G$ is said to be a *generalized Haar null* if there are a universally measurable set $B \supseteq A$ and a Borel probability measure μ on G such that $\mu(gBh) = 0$ for every $g, h \in G$. A measure μ satisfying this is called a *witness measure* for A. The system of generalized Haar null subsets of G is denoted by $\mathcal{GHN} = \mathcal{GHN}(G)$

Remark 3.1.5. As every Borel set is universally measurable, every Haar null set is generalized Haar null.

Haar meager sets were first introduced by Darji in [5] in 2013 as a topological counterpart to the Haar null sets. (Meagerness remains meaningful in non-locally-compact groups, but Haar meager sets are a better analogue for Haar null sets.)

Definition 3.1.6. A set $A \subseteq G$ is said to be *Haar meager* if there are a Borel set $B \supseteq A$, a (nonempty) compact metric space K and a continuous function $f : K \to G$ such that $f^{-1}(gBh)$ is meager in K for every $g, h \in G$. A function f satisfying this is called a *witness function* for A. The system of Haar meager subsets of G is denoted by $\mathcal{HM} = \mathcal{HM}(G)$.

3.2 Notions of smallness

Both "Haar null" and "Haar meager" are notions of smallness (i.e. we usually think of Haar null and Haar meager sets as small or negligible). This point of view is justified by the fact that both the system of Haar null sets and the system of Haar meager sets are σ-ideals.

Definition 3.2.1. A system $\mathcal{I}$ (of subsets of some set) is called a *σ-ideal* if

(I) $\emptyset \in \mathcal{I}$,
(II) $A \in \mathcal{I}, B \subseteq A \Rightarrow B \in \mathcal{I}$ and
(III) if $A_n \in \mathcal{I}$ for all $n \in \omega$, then $\bigcup_n A_n \in \mathcal{I}$.

To prove that these systems are indeed σ-ideals we will need some technical lemmas.

Lemma 3.2.2. *If μ is a Borel probability measure on G and U is a neighborhood of 1_G, then there are a compact set $C \subseteq G$ and $c \in G$ with $\mu(C) > 0$ and $C \subseteq cU$.*

Proof. Applying [17, Theorem 17.11], there exists a compact set $\tilde{C} \subseteq X$ with $\mu(\tilde{C}) \geq \frac{1}{2}$. Fix an open set V with $1_G \in V \subset \overline{V} \subset U$. The collection of open sets $\{cV : k \in \tilde{C}\}$ covers $\tilde{C}$ and $\tilde{C}$ is compact, so $\tilde{C} = \bigcup_{c \in F}(cV \cap \tilde{C})$ for some finite set $F \subseteq \tilde{C}$. It is clear that $\mu(cV \cap \tilde{C})$ must be positive for at least one $c \in F$. Choosing $C = \overline{cV} \cap \tilde{C}$ clearly satisfies our requirements. □

Corollary 3.2.3. *If μ is a Borel probability measure on G, U is a neighborhood of 1_G, $B \subseteq G$ is universally measurable and satisfies $\mu(gBh) = 0$ for every $g, h \in G$, then there exists a Borel probability measure μ' that satisfies $\mu'(gBh) = 0$ for every $g, h \in G$ and has a compact support that is contained in U.*

Proof. It is easy to check that $\mu'(X) := \frac{\mu(cX \cap C)}{\mu(C)}$ satisfies our requirements. □

Lemma 3.2.4. *Let d be a metric on G that is compatible with the topology of G. If $L \subseteq G$ is compact and $\varepsilon > 0$ is arbitrary, then there exists a neighborhood U of 1_G such that $d(x \cdot u, x) < \varepsilon$ for every $x \in L$ and $u \in U$.*

Proof. (Reproduced from [7, Lemma 2].) By the continuity of the function $(x, u) \mapsto d(x \cdot u, x)$, for every $x \in L$ there are neighborhoods V_x of x and U_x of 1_G such that the image of $V_x \times U_x$ is a subset of $[0, \varepsilon)$. Let $F \subseteq L$ be a finite set such that $L \subseteq \bigcup_{x \in F} V_x$. It is easy to check that $U = \bigcap_{x \in F} U_x$ satisfies our conditions. □

Theorem 3.2.5.

(1) *The system $\mathcal{HN}$ of Haar null sets is a σ-ideal.*
(2) *The system $\mathcal{GHN}$ of generalized Haar null sets is a σ-ideal.*

Proof. It is trivial that both $\mathcal{HN}$ and $\mathcal{GHN}$ satisfy (I) and (II) in Definition 3.2.1. The proof of (III) that is reproduced in this thesis is from the appendix of [4], where a corrected version of the proof in [20] is given. Proving this fact is easier in abelian Polish groups (see [3, Theorem 1]) and when the group is metrizable with a complete left invariant metric (this would allow the proof of [20, Theorem 3] to work without modifications). The appendix of [4] mentions the other approaches and discusses the differences between them.

The proof of (III) for Haar null and for generalized Haar null sets is very similar. The following proof will be for Haar null sets, but if "Borel set" is replaced with "universally measurable set" and "Haar null" is replaced with "generalized Haar null", it becomes the proof for generalized Haar null sets.

Let A_n be Haar null for all $n \in \omega$. By definition there are Borel sets $B_n \subseteq G$ and Borel probability measures μ_n on G such that $A_n \subseteq B_n$ and $\mu_n(gB_nh) = 0$ for every $g, h \in G$. Let d be a complete metric on G that is compatible with the topology of G (as G is Polish, it is completely metrizable).

We construct for all $n \in \omega$ a compact set $C_n \subseteq G$ and a Borel probability measure $\tilde{\mu}_n$ such that the support of $\tilde{\mu}_n$ is C_n, $\tilde{\mu}_n(gB_nh) = 0$ for every $g, h \in G$ (i.e. $\tilde{\mu}_n$ is a witness measure) and the "size" of the sets C_n decreases "quickly".

The construction will be recursive. For the initial step use Corollary 3.2.3 to find a Borel probability measure $\tilde{\mu}_0$ that satisfies $\tilde{\mu}_0(gB_0h) = 0$ for every $g, h \in G$ and that has compact support $C_0 \subseteq G$. Assume that $\tilde{\mu}_{n'}$ and $C_{n'}$ are already defined for all $n' < n$. By Lemma 3.2.4 there exists a neighborhood U_n of 1_G such that if $u \in U_n$, then $d(k \cdot u, k) < 2^{-n}$ for every k in the compact set $C_0C_1C_2 \cdots C_{n-1}$. Applying Corollary 3.2.3 again we can find a Borel probability measure $\tilde{\mu}_n$ that satisfies $\tilde{\mu}_n(gB_nh) = 0$ for every $g, h \in G$ and that has a compact support $C_n \subseteq U_n$.

If $c_n \in C_n$ for all $n \in \omega$, then it is clear that the sequence $(c_0c_1c_2 \cdots c_n)_{n \in \omega}$ is a Cauchy sequence. As (G, d) is complete, this Cauchy sequence is convergent; we write its limit as the infinite product $c_0c_1c_2 \cdots$. The map $\varphi : \prod_{n \in \omega} C_n \to G$, $\varphi((c_0, c_1, c_2, \ldots)) = c_0c_1c_2 \cdots$ is the uniform limit of continuous functions, hence it is continuous.

Let μ^Π be the product of the measures $\tilde{\mu}_n$ on the product space $C^\Pi := \prod_{n\in\omega} C_n$. Let $\mu = \varphi_*(\mu^\Pi)$ be the push-forward of μ^Π along φ onto G, i.e.

$$\mu(X) = \mu^\Pi(\varphi^{-1}(X)) = \mu^\Pi\left(\left\{(c_0, c_1, c_2, \ldots) \in C^\Pi : c_0c_1c_2\cdots \in X\right\}\right).$$

We claim that μ witnesses that $A = \bigcup_{n\in\omega} A_n$ is Haar null. Note that A is contained in the Borel set $B = \bigcup_{n\in\omega} B_n$, so it is enough to show that $\mu(gBh) = 0$ for every $g, h \in G$. As μ is σ-additive, it is enough to show that $\mu(gB_nh) = 0$ for every $g, h \in G$ and $n \in \omega$.

Fix $g, h \in G$ and $n \in \omega$. Notice that if $c_j \in C_j$ for every $j \neq n$, $j \in \omega$, then

$$\begin{aligned}&\tilde{\mu}_n\left(\{c_n \in C_n : c_0c_1c_2\cdots c_n\cdots \in gB_nh\}\right) = \\ &\quad = \tilde{\mu}_n\left((c_0c_1\cdots c_{n-1})^{-1}\cdot gB_nh\cdot (c_{n+1}c_{n+2}\cdots)^{-1}\right) = 0\end{aligned}$$

because $\tilde{\mu}_n(g'B_nh') = 0$ for all $g', h' \in G$. Applying Fubini's theorem in the product space $\left(\prod_{j\neq n} C_j\right)\times C_n$ to the product measure $\left(\prod_{j\neq n}\tilde{\mu}_j\right)\times\tilde{\mu}_n$ yields that

$$0 = \mu^\Pi\left(\left\{(c_0, c_1, \ldots, c_n, \ldots) \in C^\Pi : c_0c_1\cdots c_n\cdots \in gB_nh\right\}\right).$$

By the definition of μ this means that $\mu(gB_nh) = 0$. $\square$

Theorem 3.2.6. *The system $\mathcal{HM}$ of Haar meager sets is a σ-ideal.*

Proof. The proof of (I) and (II) in Definition 3.2.1 is trivial again. The proof of (III) is reproduced from [7, Theorem 3]. This proof will be very similar to the proof of Theorem 3.2.5, but restricting the witnesses to a smaller "part" of G is simpler in this case (we do not need an analogue of Corollary 3.2.3).

Let A_n be Haar meager for all $n \in \omega$. By definition there are Borel sets $B_n \subseteq G$, compact metric spaces $K_n \neq \emptyset$ and continuous functions $f_n : K_n \to G$ such that $f_n^{-1}(gB_nh)$ is meager in K_n for every $g, h \in G$. Let d be a complete metric on G that is compatible with the topology of G.

We construct for all $n \in \omega$ a compact metric space $\tilde{K}_n$ and a continuous function $\tilde{f}_n : \tilde{K}_n \to G$ satisfying that $\tilde{f}_n^{-1}(gB_nh)$ is meager in $\tilde{K}_n$ for every $g, h \in G$ (i.e. $\tilde{f}_n$ is a witness function) and the "size" of the images $\tilde{f}_n(\tilde{K}_n) \subseteq G$ decreases "quickly".

Unlike the Haar null case, we do not have to apply recursion in this construction. By Lemma 3.2.4 there exists a neighborhood U_n of 1_G such that if $u \in U_n$, then $d(k\cdot u, k) < 2^{-n}$ for every k in the compact set $f_0(K_0)f_1(K_1)\cdots f_{n-1}(K_{n-1})$. Let $x_n \in f_n(K_n)$ be an arbitrary element and $\tilde{K}_n = \overline{f_n^{-1}(x_nU_n)}$. The set $\tilde{K}_n$ is compact (because it is a closed subset of a compact set) and nonempty. Let $\tilde{f}_n : \tilde{K}_n \to G$, $\tilde{f}_n(k) = x_n^{-1}f_n(k)$, this is clearly continuous.

Claim 3.2.7. *For every $n \in \omega$ and $g, h \in G$, $\tilde{f}_n^{-1}(gB_nh)$ is meager in $\tilde{K}_n$.*

Proof. Fix $n \in \omega$ and $g, h \in G$. The set $\tilde{f}_n^{-1}(U_n)$ is open in K_n and because f is a witness function, the set $\tilde{f}_n^{-1}(gB_nh) = f_n^{-1}(x_ngB_nh)$ is meager in K_n. This means that $\tilde{f}_n^{-1}(U_n) \cap \tilde{f}_n^{-1}(gB_nh)$ is meager in $\tilde{f}_n^{-1}(U_n)$. Since each open subset of K_n is comeager in its closure and the closure of $\tilde{f}_n^{-1}(U_n) = f_n^{-1}(x_nU_n)$ is $\overline{f_n^{-1}(x_nU_n)} = \tilde{K}_n$, simple formal calculations yield that $\tilde{f}_n^{-1}(gB_nh) \cap \tilde{K}_n$ is meager in $\tilde{K}_n$. □

Let K be the compact set $\prod_{n\in\omega} \tilde{K}_n$ and for $n \in \omega$ let ψ_n be the continuous function $\psi_n : K \to G$,

$$\psi_n(k) = \tilde{f}_0(k_0) \cdot \tilde{f}_1(k_1) \cdot \ldots \cdot \tilde{f}_{n-1}(k_{n-1}).$$

By the choice of U_n we obtain $d(\psi_{n-1}(k), \psi_n(k)) \leq 2^{-n}$ for every $k \in K$. Using the completeness of d this means that the sequence of functions $(\psi_n)_{n\in\omega}$ is uniformly convergent. Let $f : K \to G$ be the limit of this sequence. f is continuous, because it is the uniform limit of continuous functions.

We claim that f witnesses that $A = \bigcup_{n\in\omega} A_n$ is Haar meager. Note that A is contained in the Borel set $B = \bigcup_{n\in\omega} B_n$, so it is enough to show that $f^{-1}(gBh)$ is meager in K for every $g, h \in G$. As meager subsets of K form a σ-ideal, it is enough to show that $f^{-1}(gB_nh)$ is meager in K for every $g, h \in G$ and $n \in \omega$.

Fix $g, h \in G$ and $n \in \omega$. Notice that if $k_j \in \tilde{K}_j$ for every $j \neq n$, $j \in \omega$, then Claim 3.2.7 means that

$$\begin{aligned}
&\{k_n \in \tilde{K}_n : f(k_0, k_1, \ldots, k_n, \ldots) \in gB_nh\} = \\
&\quad = \{k_n \in \tilde{K}_n : \tilde{f}_0(k_0) \cdot \tilde{f}_1(k_1) \cdot \ldots \cdot \tilde{f}_n(k_n) \cdot \ldots \in gB_nh\} \\
&\quad = \tilde{f}_n^{-1}\left(\left(\tilde{f}_0(k_0) \cdot \ldots \tilde{f}_{n-1}(k_{n-1})\right)^{-1} \cdot gB_nh \cdot \left(\tilde{f}_{n+1}(k_{n+1}) \cdot \tilde{f}_{n+2}(k_{n+2}) \cdot \ldots\right)^{-1}\right)
\end{aligned}$$

is meager in $\tilde{K}_n$. Applying the Kuratowski-Ulam theorem (see e.g. [17, Theorem 8.41]) in the product space $\left(\prod_{j\neq n} \tilde{K}_j\right) \times \tilde{K}_n$, the Borel set $f^{-1}(gB_nh)$ is meager. □

As the group G acts on itself via multiplication, it is useful if this action does not convert "small" sets into "large" ones. This means that a "nice" notion of smallness must be a translation invariant system.

Definition 3.2.8. A $\mathcal{I} \subseteq \mathcal{P}(G)$ system is called *translation invariant* if $A \in \mathcal{I} \Leftrightarrow gAh \in \mathcal{I}$ for every $A \subseteq G$ and $g, h \in G$.

Proposition 3.2.9. *The σ-ideals $\mathcal{HN}$, $\mathcal{GHN}$ and $\mathcal{HM}$ are all translation invariant.*

Proof. This is clear from Definition 3.1.1, Definition 3.1.4 and Definition 3.1.6. □

If a nontrivial notion of smallness has these "nice" properties, then the following lemma states that countable sets are small and nonempty open sets are not small. Applying this simple fact for our σ-ideals is often useful in simple cases.

Lemma 3.2.10. *Let $\mathcal{I}$ be a translation invariant σ-ideal that contains a nonempty set but does not contain all subsets of G. If $A \subseteq G$ is countable, then $A \in \mathcal{I}$, and if $U \subseteq G$ is nonempty open, then $U \notin \mathcal{I}$.*

Proof. If $x \in G$, then any nonempty set in $\mathcal{I}$ has a translate that contains $\{x\}$ as a subset, hence $\{x\} \in \mathcal{I}$. Using that $\mathcal{I}$ is closed under countable unions, this yields that if $A \subseteq G$ is countable, then $A \in \mathcal{I}$. To prove the other claim, suppose for a contradiction that $U \subseteq G$ is a nonempty open set that is in $\mathcal{I}$. It is clear that $G = \bigcup_{g \in G} gU$, and as G is Lindelöf, $G = \bigcup_{n \in \omega} g_n U$ for some countable subset $\{g_n : n \in \omega\} \subseteq G$. But here $g_n U \in \mathcal{I}$ (because $\mathcal{I}$ is translation invariant) and thus $G \in \mathcal{I}$ (because $\mathcal{I}$ is closed under countable unions), and this means that $\mathcal{I}$ contains all subsets of G, and this is a contradiction. □

Remark 3.2.11. Let $\mathcal{I}$ be one of the σ-ideals $\mathcal{HN}$, $\mathcal{GHN}$ and $\mathcal{HM}$. If G is countable, then $\mathcal{I} = \{\emptyset\}$, otherwise $\mathcal{I}$ contains a nonempty set and does not contain all subsets of G.

3.3 Connections to Haar measure and meagerness

This section discusses the connection between sets with Haar measure zero and Haar null sets and the connection between meager sets and Haar meager sets. In the simple case when G is locally compact we will find that equivalence holds for both pairs, justifying the names "Haar null" and "Haar meager". When G is non-locally-compact, we will see that the first connection is broken by the fact that there is no Haar measure on the group. For the other pair we will see that Haar meager sets are always meager, but there are Polish groups where the converse is not true.

First we recall some well-known facts about Haar measures. For proofs and more detailed discussion see for example [13, §15].

Definition 3.3.1. If (X, Σ) is a measurable space with $\mathcal{B}(X) \subseteq \Sigma$, then a measure $\mu : \Sigma \to [0, \infty]$ is called *regular* if $\mu(U) = \sup\{\mu(K) : K \subseteq U, K \text{ is compact}\}$ for every U open set and $\mu(A) = \inf\{\mu(U) : A \subseteq U, U \text{ is open}\}$ for every set A in the domain of μ.

Definition 3.3.2. If G is a topological group (not necessarily Polish), a measure $\lambda : \mathcal{B}(G) \to [0, \infty]$ is called a *left Haar measure* if it satisfies the following properties:

(I) $\lambda(F) < \infty$ if F is compact,

(II) $\lambda(U) > 0$ if U is a nonempty open set,

(III) $\lambda(gB) = \lambda(B)$ for all $B \in \mathcal{B}(\mathcal{G})$ and $g \in G$ (*left invariance*),
(IV) λ is regular.

If left invariance is replaced by the property $\lambda(Bg) = \lambda(B)$ for all $B \in \mathcal{B}(G)$ and $g \in G$ (*right invariance*), the measure is called a *right Haar measure.*

Theorem 3.3.3. *(existence of the Haar measure) If G is a locally compact group (not necessarily Polish), then there exists a left (right) Haar measure on G and if λ_1, λ_2 are two left (right) Haar measures, then $\lambda_1 = c \cdot \lambda_2$ for a positive real constant c.*

If G is compact, (I) means that the left and right Haar measures are finite measures, and this fact can be used to prove the following result:

Theorem 3.3.4. *If G is a compact group, then all left Haar measures are right Haar measures and vice versa.*

This result is also trivially true in abelian locally compact groups, but not true in all locally compact groups. However, the following result remains true:

Theorem 3.3.5. *If G is a locally compact group, then the left Haar measures and the right Haar measures are absolutely continuous relatively to each other, that is, for every Borel set $B \subseteq G$, either every left Haar measure and every right Haar measure assigns measure zero to B or no left Haar measure and no right Haar measure assigns measure zero to B.*

This allows us to define the following notion:

Definition 3.3.6. Suppose that G is a locally compact group and fix an arbitrary left (or right) Haar measure λ. We say that a set $N \subseteq G$ has *Haar measure zero* if $N \subseteq B$ for some Borel set B with $\lambda(B) = 0$. The collection of these sets is denoted by $\mathcal{N} = \mathcal{N}(G)$.

Definition 3.3.2 defines the Haar measures only on the Borel sets. If λ is an arbitrary left (or right) Haar measure, we can complete it using the standard techniques. The domain of the completion will be $\sigma(\mathcal{B}(G) \cup \mathcal{N})$ (the σ-algebra generated by $\mathcal{N}$ and the Borel sets). For every set A in this σ-algebra, let

$$\lambda(A) = \sup \left\{ \sum_{j \in \omega} \lambda(B_j) : B_j \in \mathcal{B}(G), A \subseteq \bigcup_{j \in \omega} (B_j) \right\}.$$

This completion will be a complete measure that agrees with the original λ on Borel sets and satisfies properties (I) – (IV) from Definition 3.3.2 (or right invariance instead of left invariance if λ was a right Haar measure). We will identify a left (or right) Haar measure with its completion and we will also call this extension (slightly imprecisely) a left (or right) Haar measure.

Theorem 3.3.7. *If G is a locally compact Polish group, then system of sets with Haar measure zero is the same as the system of Haar null sets and is the same as the system of generalized Haar null sets, that is, $\mathcal{N}(G) = \mathcal{HN}(G) = \mathcal{GHN}(G)$.*

Proof. $\mathcal{N}(G) \subseteq \mathcal{HN}(G)$:
Let λ be a left Haar measure and λ' be a right Haar measure. If $N \in \mathcal{N}(G)$ is arbitrary, then by definition there is a Borel set B satisfying $N \subseteq B$ and $\lambda(B) = 0$. The left invariance of G means that $\lambda(gB) = 0$ for every $g \in G$. Applying Theorem 3.3.5 this means that $\lambda'(gB) = 0$ for every $g \in G$, and applying the right invariance of λ' we get that $\lambda'(gBh) = 0$ for every $g, h \in G$. Using the regularity of λ', it is easy to see that there is a compact set K with $0 < \lambda'(K) < \infty$. The measure $\mu(X) = \frac{\lambda'(K \cap X)}{\lambda'(K)}$ is clearly a Borel probability measure. $\mu \ll \lambda'$ means that $\mu(gBh) = 0$ for every $g, h \in G$, so B and μ satisfy the requirements of Definition 3.1.1.

$\mathcal{HN}(G) \subseteq \mathcal{GHN}(G)$:
This is trivial in all Polish groups, see Remark 3.1.5.

$\mathcal{GHN}(G) \subseteq \mathcal{N}(G)$:
Suppose that $A \in \mathcal{GHN}(G)$. By definition there exists an universally measurable $B \subseteq G$ and a Borel probability measure μ such that $\mu(gBh) = 0$ for every $g, h \in G$. Notice that we will only use that $\mu(Bh) = 0$ for every $h \in G$, so we will also prove that (using the terminology of subsection 4.4) all generalized right Haar null sets have Haar measure zero. Let λ be a left Haar measure on G. Let m be the multiplication map $m : G \times G \to G$, $(x, y) \mapsto x \cdot y$.

Notice that the set $m^{-1}(B) = \{(x, y) \in G \times G : x \cdot y \in B\}$ is universally measurable in $G \times G$, because it is the preimage of a universally measurable set under the continuous map m. (This follows from that that the preimage of a Borel set under m is Borel, and for every σ-finite measure ν on $G \times G$, the preimage of a set of $m_*(\nu)$-measure zero under m must be of ν-measure zero. Here $m_*(\nu)$ is the push-forward measure: $m_*(\nu)(X) = \nu(\{(x, y) : x \cdot y \in X\})$.)

Applying Fubini's theorem in the product space $G \times G$ to the product measure $\mu \times \lambda$ (which is a σ-finite Borel measure) we get that

$$(\mu \times \lambda)(m^{-1}(B)) = \int_G \lambda(\{y : x \cdot y \in B\}) \,\mathrm{d}\mu(x) = \int_G \mu(\{x : x \cdot y \in B\} \,\mathrm{d}\lambda(y)$$

$$\int_G \lambda(x^{-1}B) \,\mathrm{d}\mu(x) = \int_G \mu(By^{-1}) \,\mathrm{d}\lambda(y)$$

As μ is a witness measure, the right hand side is the integral of the constant 0 function. On the left hand side $\lambda(x^{-1}B) = \lambda(B)$, as λ is left invariant (note that B is λ-measurable, because B is universally measurable and λ is σ-finite). Thus $0 = \int_G \lambda(B) \,\mathrm{d}\mu(x) = \lambda(B)$. As $A \subseteq B$, this means that $\lambda(A) = 0$, $A \in \mathcal{N}(G)$. $\square$

We reproduce the classical results which show that (left and right) Haar measures do not exist on topological groups that are not locally compact. We will need the following generalized version of the Steinhaus theorem.

Theorem 3.3.8. *If G is a topological group, λ is a left Haar measure on G and $C \subseteq G$ is compact with $\lambda(C) > 0$, then $1_G \in \operatorname{int}(C \cdot C^{-1})$.*

Proof. As λ is a Haar measure and C is compact, $\lambda(C) < \infty$. Using the regularity of λ, there is an open set $U \supseteq C$ that satisfies $\lambda(U) < 2\lambda(C)$.

Claim 3.3.9. *There exists an open neighborhood V of 1_G such that $V \cdot C \subseteq U$.*

Proof. For every $c \in C$ the multiplication map $m : G \times G \to G$ is continuous at $(1_G, c)$, so $c \in V_c \cdot W_c \subseteq U$ for some open neighborhood V_c of 1_G and some open neighborhood W_c of c. As C is compact and $\bigcup_{c \in C} W_c \supseteq C$, there is a finite set F with $\bigcup_{c \in F} W_c \supseteq C$. Then $V = \bigcap_{c \in F} V_c$ satisfies $V \cdot W_c \subseteq U$ for every $c \in F$, so $V \cdot C \subseteq U$. □

Now it is enough to prove that $V \subseteq C \cdot C^{-1}$. Choose an arbitrary $v \in V$. Then $v \cdot C$ and C are subsets of U and $\lambda(v \cdot C) = \lambda(C) > \frac{\lambda(U)}{2}$ (we used the left invariance of λ). This means that $v \cdot C \cap C \neq \emptyset$, so there exists $c_1, c_2 \in C$ with $vc_1 = c_2$, but this means that $v = c_2 c_1^{-1} \in C \cdot C^{-1}$. □

We note that in Polish groups it is possible to find a compact subset with positive Haar measure in every set with positive Haar measure. Hence the following version of the previous theorem is also true:

Corollary 3.3.10. *If G is a locally compact Polish group, λ is a left Haar measure on G and $A \subseteq G$ is λ-measurable with $\lambda(A) > 0$, then $1_G \in \operatorname{int}(A \cdot A^{-1})$.*

Other, more general variants of this result are examined in subsection 5.2.

Theorem 3.3.11. *If G is a topological group and λ is a left Haar measure on G, then G is locally compact.*

Proof. $\lambda(G) > 0$ as G is open. Using the regularity of G, there exists a compact set C with $\lambda(C) > 0$. The set $C \cdot C^{-1}$ is compact (it is the image of the compact set $C \times C$ under the continuous map $(x, y) \mapsto xy^{-1}$). Applying Theorem 3.3.8 yields that $C \cdot C^{-1}$ is a neighborhood of 1_G, but then for every $g \in G$ the set $g \cdot C \cdot C^{-1}$ is a compact neighborhood of g, and this shows that G is locally compact. □

The connection between meager sets and Haar meager sets is simpler. The following results are from [5] (the first paper about Haar meager sets, which only considers abelian Polish groups) and [7] (where the concept of Haar meager sets is extended to all Polish groups).

Theorem 3.3.12. *Every Haar meager set is meager, $\mathcal{HM}(G) \subseteq \mathcal{M}(G)$.*

Proof. Let A be a Haar meager subset of G. By definition there exists a Borel set $B \supseteq A$, a (nonempty) compact metric space K and a continuous function $f : K \to G$ such that $f^{-1}(gBh)$ is meager in K for every $g, h \in G$.

Consider the Borel set

$$S = \{(g, k) : f(k) \in gB\} \subseteq G \times K.$$

For every $g \in G$, the g-section of this set is $S_g = \{k \in K : f(k) \in gB\} = f^{-1}(gB)$, and this is a meager set in K. Hence, by the Kuratowski-Ulam theorem, S is meager in $G \times K$. Using the Kuratowski-Ulam theorem again, for comeager many $k \in K$, the section $S^k = \{g \in G : f(k) \in gB\} = f(k) \cdot B^{-1}$ is meager in G. Since K is compact, there is at least one such k. Then the inverse of the homeomorphism $b \mapsto f(k) \cdot b^{-1}$ maps the meager set S^k to B, and this shows that B is meager. □

Theorem 3.3.13. *In a locally compact Polish group G meagerness is equivalent to Haar meagerness, that is, $\mathcal{HM}(G) = \mathcal{M}(G)$.*

Proof. We only need to prove the inclusion $\mathcal{M}(G) \subseteq \mathcal{HM}(G)$. As G is locally compact, there is an open set $U \subseteq G$ such that $\overline{U}$ is compact. Let $f : \overline{U} \to G$ be the identity map restricted to $\overline{U}$. If M is meager in G, then there exists a meager Borel set $B \supseteq M$. The set gBh is meager in G for every $g, h \in G$ (as $x \mapsto gxh$ is a homeomorphism), so $f^{-1}(gBh) = gBh \cap \overline{U}$ is meager in $\overline{U}$ for every $g, h \in G$. □

Theorem 3.3.14. *In a non-locally-compact Polish group G that admits a two-sided invariant metric meagerness is a strictly stronger notion than Haar meagerness, that is, $\mathcal{HM}(G) \subsetneqq \mathcal{M}(G)$.*

Proof. We know that $\mathcal{HM}(G) \subseteq \mathcal{M}(G)$. To construct a meager but not Haar meager set, we will use a theorem of Solecki from [23]. As the proof of this purely topological theorem is relatively long, we do not reproduce it here.

Theorem 3.3.15. *Assume that G is a non-locally-compact Polish group that admits a two-sided invariant metric. Then there exists a closed set $F \subseteq G$ and a continuous function $\varphi : F \to 2^\omega$ such that for any $x \in 2^\omega$ and any compact set $C \subseteq G$ there is a $g \in G$ with $gC \subseteq \varphi^{-1}(\{x\})$.*

Using this we construct a closed nowhere dense set M that is not Haar meager. The system $\{f^{-1}(\{x\}) : x \in 2^\omega\}$ contains continuum many pairwise disjoint closed sets. If we fix a countable basis in G, only countably many of these sets contain an open set from that basis. If for $x_0 \in 2^\omega$ the set $M := f^{-1}(\{x_0\})$ does not contain a basic open set, then it is nowhere dense (as it is closed with empty interior). On the other hand, it is clear that M is not Haar meager, as for every compact metric space

K and continuous function $f : K \to G$ there exists a $g \in G$ such that $gf(K) \subseteq M$, thus $f^{-1}(g^{-1}M) = K$. $\square$

4 Alternative definitions

In this section first we discuss various alternative definitions which are equivalent to the "normal" definitions, but may be easier to prove or easier to use in some situations. After this, we will briefly describe some other versions which appeared in papers about this topic.

4.1 Equivalent versions

In this subsection we mention some alternative definitions which are equivalent to Definition 3.1.1, Definition 3.1.4 or Definition 3.1.6. Most of these equivalences are trivial, but even these trivial equivalences can be frequently used as lemmas. First we list some versions of the definition of Haar null sets.

Theorem 4.1.1. *For a set $A \subseteq G$ the following are equivalent:*

(1) *there exists a Borel set $B \supseteq A$ and a Borel probability measure μ on G such that $\mu(gBh) = 0$ for every $g, h \in G$ (i.e. A is Haar null),*
(2) *there exists a Borel Haar null set $B \supseteq A$,*
(3) *there exists a Borel generalized Haar null set $B \supseteq A$,*
(4) *there exists an analytic set $B \supseteq A$ and a Borel probability measure μ on G such that $\mu(gBh) = 0$ for every $g, h \in G$,*
(5) *there exists an analytic generalized Haar null set $B \supseteq A$.*

Proof. First note that Lusin's theorem (see [17, 28.7]) states that all analytic sets are universally measurable, hence gBh is μ-measurable in condition (4).

(1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) is trivial from the definitions. (3) $\Rightarrow$ (1) follows from the fact that if (3) is true, then there exists a Borel probability measure μ such that $\mu(gB'h) = 0$ for some (universally measurable) $B' \supseteq B$ and every $g, h \in G$, but this means that $\mu(gBh) = 0$ for every $g, h \in G$.

The implication (1) $\Rightarrow$ (4) follows from the fact that all Borel sets are analytic. The implication (4) $\Rightarrow$ (5) is trivial again, considering that all analytic sets are universally measurable.

Finally we prove (5) $\Rightarrow$ (1) to conclude the proof of the theorem. This proof is reproduced from [23]. Without loss of generality we may assume that the set A itself is analytic generalized Haar null. We have to prove that there exists a Borel set $B \supseteq A$ and a Borel probability measure μ on G such that $\mu(gBh) = 0$ for every $g, h \in G$.

By definition there exists a Borel probability measure μ such that $\mu(g\tilde{B}h) = 0$ for some (universally measurable) $\tilde{B} \supseteq A$ and every $g, h \in G$, but this means that $\mu(gAh) = 0$ for every $g, h \in G$.

Claim 4.1.2. *The family of sets*

$$\Phi = \{X \subseteq G : X \text{ is analytic and } \mu(gXh) = 0 \text{ for every } g, h \in G\}$$

is coanalytic on analytic*, that is, for every Polish space Y and $P \in \mathbf{\Sigma}^1_1(Y \times G)$, the set $\{y \in Y : P_y \in \Phi\}$ is $\mathbf{\Pi}^1_1$.*

Proof. Let Y be Polish space and $P \in \mathbf{\Sigma}^1_1(Y \times G)$ and let

$$\tilde{P} = \{(g, h, y, \gamma) \in G \times G \times Y \times G : \gamma \in gP_y h\}.$$

Then $\tilde{P}$ is analytic, as it is the preimage of P under $(g, h, y, \gamma) \mapsto (y, g^{-1}\gamma h^{-1})$. We will use the fact that if U and V are Polish spaces, ϱ is a Borel probability measure on V and $A \subseteq U \times V$ is analytic, then $\{u \in U : \varrho(A_u) = 0\}$ is coanalytic (this is a corollary of [17, Theorem 29.26]). Using this fact yields that $\{(g, h, y) : \mu(\tilde{P}_{(g,h,y)}) = 0\}$ is coanalytic, but then

$$\{y \in Y : \mu(\tilde{P}_{(g,h,y)}) = 0 \text{ for every } g, h \in G\} = \{y \in Y : P_y \in \Phi\}$$

is also coanalytic. □

Now, since $A \in \Phi$, by the dual form of the First Reflection Theorem (see [17, Theorem 35.10 and the remarks following it]) there exists a Borel set B with $B \supseteq A$ and $B \in \Phi$, and this B (together with μ) satisfies our requirements. □

The following proposition states the analog of the trivial equivalence (1) ⇔ (2) for generalized Haar null sets. (The other parts of Theorem 4.1.1 have no natural analogs for generalized Haar null sets.)

Proposition 4.1.3. *For a set $A \subseteq G$ the following are equivalent:*

(1) *there exists a universally measurable set $B \supseteq A$ and a Borel probability measure μ on G such that $\mu(gBh) = 0$ for every $g, h \in G$ (i.e. A is generalized Haar null),*
(2) *there exists a universally measurable generalized Haar null set $B \supseteq A$.*

In Definition 3.1.1 and Definition 3.1.4 the witness measure is required to be a Borel probability measure, but some alternative conditions yield equivalent definitions. A set $A \subseteq G$ is Haar null (or generalized Haar null) if and only if there is a Borel (or universally measurable) set $B \supseteq A$ that satisfies the equivalent conditions listed in the following theorem.

Theorem 4.1.4. *For a universally measurable set $B \subseteq G$ the following are equivalent:*

(1) *there exists a Borel probability measure* μ *on* G *such that* $\mu(gBh) = 0$ *for every* $g, h \in G$*,*
(2) *there exists a Borel probability measure* μ *on* G *such that* μ *has compact support and* $\mu(gBh) = 0$ *for every* $g, h \in G$*,*
(3) *there exists a Borel measure* μ *on* G *such that* $0 < \mu(X) < \infty$ *for some* μ*-measurable set* $X \subseteq G$ *and* $\mu(gBh) = 0$ *for every* $g, h \in G$*.*
(4) *there exists a Borel measure* μ *on* G *such that* $0 < \mu(C) < \infty$ *for some compact set* $C \subseteq G$ *and* $\mu(gBh) = 0$ *for every* $g, h \in G$ *(the paper [15] calls a Borel set* shy *if it has this property).*

Proof. The implications (2) $\Rightarrow$ (4) $\Rightarrow$ (3) are trivial. (3) $\Rightarrow$ (1) is true, because if μ and X satisfies the requirements of (3), then $\tilde{\mu}(Y) = \frac{\mu(Y \cap X)}{\mu(X)}$ is a Borel probability measure and $\tilde{\mu} \ll \mu$ means that $\tilde{\mu}(gBh) = 0$ for every $g, h \in G$. Finally (1) $\Rightarrow$ (2) follows from Corollary 3.2.3. $\square$

The following result gives an equivalent characterization of Haar null sets which allows proving that a Borel set is Haar null by constructing measures that assign small, but not necessarily zero measures to the translates of that set. In [19, Theorem 1.1] Matoušková proves this theorem for separable Banach spaces, but her proof can be generalized to work in arbitrary Polish groups.

Theorem 4.1.5. *A Borel set* B *is Haar null if and only if for every* $\delta > 0$ *and neighborhood* U *of* 1_G*, there exists a Borel probability measure* μ *on* G *such that the support of* μ *is contained in* U *and* $\mu(gBh) < \delta$ *for every* $g, h \in G$*.*

Proof. Let $P(G)$ be the set of Borel probability measures on G. As G is Polish, [17, Theorem 17.23] states that $P(G)$ (endowed with the weak topology) is also a Polish space. In particular this means that it is possible to fix a metric d such that $(P(G), d)$ is a complete metric space. If $\mu, \nu \in P(G)$, let $(\mu * \nu)(X) = (\mu \times \nu)(\{(x, y) : xy \in X\})$ be their convolution. It is straightforward to see that $*$ is associative (but not commutative in general, as we did not assume that G is commutative). The map $* : P(G) \times P(G) \to P(G)$ is continuous, for a proof of this see e.g. [14, Proposition 2.3]. Let $\delta(X) = 1$ if $1_G \in X$, and $\delta(X) = 0$ if $1_G \notin X$, then it is clear that $\delta \in P(G)$ is the identity element for $*$.

First we prove the "only if" part. Let $(U_n)_{n \in \omega}$ be open sets with $\bigcap_n U_n = \{1_G\}$. For every $n \in \omega$ fix a Borel probability measure μ_n such that

(I) $\operatorname{supp} \mu_n \subseteq U_n$ and
(II) $\mu_n(gBh) < \frac{1}{n+1}$ for every $g, h \in G$.

It is easy to see from property (I) that the sequence $(\mu_n)_{n \in \omega}$ (weakly) converges to δ, and this and the continuity of $*$ means that for any $\nu \in P(G)$ the sequence

$d(\nu, \nu * \mu_n)$ converges to zero. This allows us to replace $(\mu_n)_{n\in\omega}$ with a subsequence which also satisfies that $d(\nu, \nu * \mu_n) < 2^{-n}$ for every measure ν from the finite set

$$\{\mu_{j_0} * \mu_{j_1} * \ldots * \mu_{j_r} : r < n \text{ and } 0 \leq j_0 < j_1 < \ldots < j_r < n\}.$$

(Notice that property (II) clearly remains true for any subsequence.) Using this assumption and the completeness of $(P(G), d)$ we can define (for every $n \in \omega$) the "infinite convolution" $\mu_n * \mu_{n+1} * \ldots$ as the limit of the Cauchy sequence $(\mu_n * \mu_{n+1} * \mu_{n+2} * \ldots * \mu_{n+j})_{j\in\omega}$. We will show that the choice $\mu = \mu_0 * \mu_1 * \ldots$ witnesses that B is Haar null.

We have to prove that $\mu(gBh) = 0$ for every $g, h \in G$. To show this fix arbitrary $g, h \in G$ and $n \in \omega$; we will show that $\mu(gBh) \leq \frac{1}{n+1}$. Let $\alpha_n = \mu_0 * \mu_1 * \ldots * \mu_{n-1}$ and $\beta_n = \mu_{n+1} * \mu_{n+2} * \ldots$ and notice the continuity of $*$ yields that

$$\mu = \lim_{j\to\infty} (\alpha_n * \mu_n * (\mu_{n+1} * \mu_{n+2} * \ldots * \mu_{n+j})) =$$
$$= (\alpha_n * \mu_n * \lim_{j\to\infty} (\mu_{n+1} * \mu_{n+2} * \ldots * \mu_{n+j})) = \alpha_n * \mu_n * \beta_n.$$

This means that

$$\mu(gBh) = (\alpha_n * \mu_n * \beta_n)(gBh) = (\alpha_n \times \mu_n \times \beta_n)(\{(x, y, z) \in G^3 : xyz \in gBh\}) =$$
$$= ((\alpha_n \times \beta_n) \times \mu_n)(\{((x, z), y) \in G^2 \times G : y \in x^{-1}gBhz^{-1}\}).$$

Notice that for every $x, z \in G$ property (II) yields that $\mu_n(x^{-1}gBhz^{-1}) < \frac{1}{n+1}$. Applying Fubini's theorem in the product space $G^2 \times G$ to the product measure $(\alpha_n \times \beta_n) \times \mu_n$ yields that $\mu(gBh) \leq \frac{1}{n+1}$.

To prove the "if" part of the theorem, suppose that there exists a $\delta > 0$ and a neighborhood U of 1_G such that for every Borel probability measure μ on G if $\operatorname{supp} \mu \subseteq U$, then $\mu(gBh) \geq \delta$ for some $g, h \in G$. Let μ be an arbitrary Borel probability measure. Applying Lemma 3.2.2 yields that there are a compact set $C \subseteq G$ and $c \in G$ with $\mu(C) > 0$ and $C \subseteq cU$. Define $\mu'(X) = \frac{\mu(cX\cap C)}{\mu(C)}$, then μ' is a Borel probability measure with $\operatorname{supp} \mu' \subseteq U$, hence $\mu'(gBh) \geq \delta$ for some $g, h \in G$. This means that $\mu(gBh) \neq 0$, so μ is not a witness measure for B, and because μ was arbitrary, B is not Haar null.

□

For Haar meagerness the following analogue of Theorem 4.1.1 holds:

Theorem 4.1.6. *For a set $A \subseteq G$ the following are equivalent:*

(1) *there exists a Borel set $B \supseteq A$, a (nonempty) compact metric space K and a continuous function $f : K \to G$ such that $f^{-1}(gBh)$ is meager in K for every $g, h \in G$ (i.e. A is Haar meager),*

(2) *there exists a Borel Haar meager set $B \supseteq A$,*
(3) *there exists an analytic set $B \supseteq A$, a (nonempty) compact metric space K and a continuous function $f : K \to G$ such that $f^{-1}(gBh)$ is meager in K for every $g, h \in G$.*

Proof. (1) $\Leftrightarrow$ (2) is trivial from Definition 3.1.6, (1) $\Rightarrow$ (3) follows from the fact that all Borel sets are analytic. Finally, the implication (3) $\Rightarrow$ (1) can be found as [7, Proposition 8], and the proof is a straightforward analogue of the proof of (5) $\Rightarrow$ (1) in Theorem 4.1.1.

Without loss of generality we may assume that the set A itself is analytic and satisfies that $f^{-1}(gAh)$ is meager in K for every $g, h \in G$ for some (nonempty) compact metric space K and continuous function $f : K \to G$. We will prove that (for this K and f) there exists a Borel set $B \supseteq A$ such that $f^{-1}(gBh)$ is meager in K for every $g, h \in G$.

Claim 4.1.7. *The family of sets*

$$\Phi = \{X \subseteq G : X \text{ is analytic and } f^{-1}(gXh) \text{ is meager in } K \text{ for every } g, h \in G\}$$

is coanalytic on analytic*, that is, for every Polish space Y and $P \in \mathbf{\Sigma}^1_1(Y \times G)$, the set $\{y \in Y : P_y \in \Phi\}$ is $\mathbf{\Pi}^1_1$.*

Proof. Let Y be a Polish space and $P \in \mathbf{\Sigma}^1_1(Y \times G)$ and let

$$\tilde{P} = \{(g, h, y, k) \in G \times G \times Y \times K : f(k) \in gP_y h\}.$$

Then $\tilde{P}$ is analytic, as it is the preimage of P under $(g, h, y, k) \mapsto (y, g^{-1}f(k)h^{-1})$. Novikov's theorem (see e.g [17, Theorem 29.22]) states that if U and V are Polish spaces and $A \subseteq U \times V$ is analytic, then $\{u \in U : A_u \text{ is not meager in } V\}$ is analytic. This yields that $\{(g, h, y) : \tilde{P}_{(g,h,y)} \text{ is meager in } K\}$ is coanalytic, but then

$$\{y \in Y : \tilde{P}_{(g,h,y)} \text{ is meager in } K \text{ for every } g, h \in G\} = \{y \in Y : P_y \in \Phi\}$$

is also coanalytic. □

Now, since $A \in \Phi$, by the dual form of the First Reflection Theorem (see [17, Theorem 35.10 and the remarks following it]) there exists a Borel set B with $B \supseteq A$ and $B \in \Phi$, and this B satisfies our requirements. □

To prove our next result we will need a technical lemma. This is a modified version of the well-known result that for every (nonempty) compact metric space K, there exists a continuous surjective map $\varphi : 2^\omega \to K$.

Lemma 4.1.8. *If (K, d) is a (nonempty) compact metric space, then there exists a continuous function $\varphi : 2^\omega \to K$ such that if M is meager in K, then $\varphi^{-1}(M)$ is meager in 2^ω.*

Proof. We will use a modified version of the usual construction. Note that if $\mathrm{diam}(K) < 1$, then the φ constructed this way will be surjective (we will not need this).

Let $2^{<\omega} = \bigcup_{n\in\omega} 2^n$ be the set of finite 0-1 sequences. The length of a sequence s is denoted by $|s|$ and the sequence of length zero is denoted by $\emptyset$. If s and t are sequences (where t may be infinite), $s \preceq t$ means that s is an initial segment of t (i.e. the first $|s|$ elements of t form the sequence s) and $s \prec t$ means that $s \preceq t$ and $s \neq t$. For sequences s, t where $|s|$ is finite, $s\hat{\ }t$ denotes the concatenation of s and t. For a finite sequence s, let $[s] = \{x \in 2^\omega : s \preceq x\}$ be the set of infinite sequences starting with s. Note that $[s]$ is clopen in 2^ω for every $s \in 2^{<\omega}$.

We will choose a set of finite 0-1 sequences $S \subseteq 2^{<\omega}$, and for every $s \in S$ we will choose a point $k_s \in K$. The construction of S will be recursive: we recursively define for every $n \in \omega$ a set S_n and let $S = \dot{\bigcup}_{n\in\omega} S_n$. Our choices will satisfy the following properties:

(I) for every $n \in \omega$ and $x \in 2^\omega$ there exists a unique $s = s(x, n) \in S_n$ with $s \preceq x$, and $s(x, n) \prec s(x, n')$ if $n < n'$.

(II) for every $n \in \omega$ and $x \in 2^\omega$ the set $C_{x,n} = \bigcap_{0\le j<n} \overline{B}\left(k_{s(x,j)}, \frac{1}{j+1}\right)$ is nonempty.

First we let $S_0 = \{\emptyset\}$ and choose an arbitrary $k_\emptyset \in K$, then these trivially satisfy (I) and (II).

Suppose that we already defined $S_0, S_1, \ldots, S_{n-1}$ and let $s \in S_{n-1}$ be arbitrary. Notice that for $x, x' \in [s]$, $C_{x,n-1} = C_{x',n-1}$ and denote this common set with C_s. The set C_s is (nonempty) compact and it is covered by the open sets $\{B\left(c, \frac{1}{n}\right) : c \in C_s\}$, hence we can select a collection $(c_j^{(s)})_{j\in I_s}$ where I_s is a finite index set such that $\bigcup_{j\in I_s} B\left(c_j^{(s)}, \frac{1}{n}\right) \supseteq C_s$. We may increase the cardinality of this collection by repeating one element several times if necessary, and thus we can assume that the index set I_s is of the form 2^{ℓ_s} for some integer $\ell_s \ge 1$ (i.e. it consists of the 0-1 sequences with length ℓ_s). Now we can define $S_n = \{s\hat{\ }t : s \in S_{n-1}, t \in 2^{\ell_s}\}$. If $s' \in S_n$, then there is a unique $s \in S_{n-1}$ such that $s \preceq s'$, if t satisfies that $s' = s\hat{\ }t$ (i.e. t is the final segment of s'), then let $k_{s'} = c_t^{(s)}$.

It is straightforward to check that these choices satisfy (I). Property (II) is satisfied because if $x \in 2^\omega$, and s, s' and t are the sequences that satisfy $s = s(x, n-1)$ and $s\hat{\ }t = s' = s(x, n)$, then $k_{s'} = c_t^{(s)} \in C_{x,n}$, because it is contained in both $C_{x,n-1} = C_s$ and $\overline{B}\left(k_{s'}, \frac{1}{n}\right)$, and this shows that $C_{x,n}$ is not empty.

We use this construction to define the function φ: let $\{\varphi(x)\} = \bigcap_{n\in\omega} C_{x,n}$ (as the system of nonempty compact sets $(C_{x,n})_{n\in\omega}$ is descending and $\mathrm{diam}(C_{x,n}) \le$

diam $\left(B\left(k_{s(x,n-1)}, \frac{1}{n}\right)\right) \leq 2 \cdot \frac{1}{n} \to 0$, the intersection of this system is indeed a singleton). This function is continuous, because if $\varepsilon > 0$ and $\varphi(x) = k \in K$, then $2 \cdot \frac{1}{n_0+1} < \varepsilon$ for some n_0, and then for every x' in the clopen set $[s(x, n_0)]$ the set $\overline{B}\left(k_{s(x,n_0)}, \frac{1}{n_0+1}\right) = \overline{B}\left(k_{s(x',n_0)}, \frac{1}{n_0+1}\right)$ has diameter $< \varepsilon$ and contains both $\varphi(x)$ and $\varphi(x')$.

Now we prove that if $U \subseteq 2^\omega$ is open, then $\varphi(U)$ contains an open set V. It is clear from (I) that $\{[s] : s \in S\}$ is a base of the topology of 2^ω. This means that there exist a $n \in \omega \setminus \{0\}$ and $s \in S_{n-1}$ satisfying $[s] \subseteq U$. Let $V = \bigcap_{0 \leq j < n} B\left(k_{t_j}, \frac{1}{j+1}\right)$ where $t_j = s(x, j)$ for an arbitrary $x \in [s]$ (this is well-defined as $t_j \in S_j$ is the only element of S_j with $s(x, j) \preceq s(x, n-1) = s$). It is clear that V is open and $V \subseteq C_{x,n-1} = C_s$ (where $x \in [s]$ is arbitrary, this is again well-defined), to show that $V \subseteq \varphi(U)$ let $v \in V$ be arbitrary. Define $u_0 \in 2^{\ell_s}$ such that $v \in B\left(k_{s^\frown u_0}, \frac{1}{n}\right)$ (this is possible as $\{B\left(k_{s^\frown u_0}, \frac{1}{n}\right) : u_0 \in 2^{\ell_s}\}$ was a cover of $C_{x,n-1} = C_s$). Repeat this to define $u_1 \in 2^{\ell_{s^\frown u_0}}$ such that $v \in B\left(k_{s^\frown u_0{}^\frown u_1}, \frac{1}{n+1}\right)$, then $u_2 \in 2^{\ell_{s^\frown u_0{}^\frown u_1}}$ such that $v \in B\left(k_{s^\frown u_0{}^\frown u_1{}^\frown u_2}, \frac{1}{n+2}\right)$ etc. and let y be the infinite sequence $y = s^\frown u_0{}^\frown u_1{}^\frown u_2{}^\frown \ldots$. It is easy to see that $\varphi(y) = v$, as $d(\varphi(y), v) \leq 2 \cdot \frac{1}{n+1}$ for every $n \in \omega$.

Finally we show that if $M \in \mathcal{M}(K)$ is arbitrary, then $\varphi^{-1}(M) \in \mathcal{M}(2^\omega)$. As M is meager $M \subseteq \bigcup_{n\in\omega} F_n$ for a system $(F_n)_{n\in\omega}$ of nowhere dense closed sets. If $\varphi^{-1}(M)$ is not meager, then $\varphi^{-1}(F_n)$ contains an open set for some $n \in \omega$. But then F_n contains an open set, which is a contradiction. □

We use this lemma to show that in Definition 3.1.6 we can also restrict the choice of the compact metric space K. In the following theorem the equivalence (1) ⇔ (2) is [8, Proposition 3], the equivalence (1) ⇔ (3) is [7, Theorem 2.11].

Theorem 4.1.9. *For a Borel set $B \subseteq G$ the following are equivalent:*

1. *there exists a (nonempty) compact metric space K and a continuous function $f : K \to G$ such that $f^{-1}(gBh)$ is meager in K for every $g, h \in G$ (i.e. B is Haar meager),*
2. *there exists a continuous function $f : 2^\omega \to G$ such that $f^{-1}(gBh)$ is meager in 2^ω for every $g, h \in G$,*
3. *there exists a (nonempty) compact set $C \subseteq G$, a continuous function $f : C \to G$ such that $f^{-1}(gBh)$ is meager in C for every $g, h \in G$,*

Proof. (1) ⇒ (2):
This implication is an easy consequence of Lemma 4.1.8. If K and f satisfies the requirements of (1) and φ is the function granted by Lemma 4.1.8, then $\tilde{f} = f \circ \varphi : 2^\omega \to G$ will satisfy the requirements of (2), because it is continuous and for every $g, h \in G$ the set $f^{-1}(gBh)$ is meager in K, hence $\varphi^{-1}(f^{-1}(gBh)) = \tilde{f}^{-1}(gBh)$ is meager in 2^ω.

(2) ⇒ (3):
If G is countable, the only Haar meager subset of G is the empty set. In this case, any nonempty $C \subseteq G$ and continuous function $f : C \to G$ is sufficient. If G is not countable, then it is well known that there is a (compact) set $C \subseteq G$ that is homeomorphic to 2^ω. Composing the witness function $f : 2^\omega \to G$ granted by (2) with this homeomorphism yields a function that satisfies our requirements (together with C).

(3) ⇒ (1):
This implication is trivial. □

4.2 Coanalytic hulls

In Theorem 4.1.1 and Theorem 4.1.6 we proved that the Borel hull in the definition of Haar null sets and Haar meager sets can be replaced by an analytic hull. The following theorems show that it cannot be replaced by a coanalytic hull. As the proofs of these theorems are relatively long, we do not reproduce them here. Note that these theorems were proved in the abelian case, but they can be generalized to the case when G is TSI.

Theorem 4.2.1. *(Elekes-Vidnyánszky, see [11]) If G is a non-locally-compact abelian Polish group, then there exists a coanalytic set $A \subseteq G$ that is not Haar null, but there is a Borel probability measure μ on G such that $\mu(gAh) = 0$ for every $g, h \in G$.*

Corollary 4.2.2. *If G is non-locally-compact and abelian, then $\mathcal{GHN}(G) \supsetneq \mathcal{HN}(G)$.*

This (and its generalization to TSI groups) is the best known result for the following question.

Question 4.2.3. *(Elekes-Vidnyánszky, see [11, Question 5.4]) Is $\mathcal{GHN}(G) \supsetneq \mathcal{HN}(G)$ in all non-locally-compact Polish groups?*

Theorem 4.2.4. *(Doležal-Vlasák, see [8]) If G is a non-locally-compact abelian Polish group, then there exists a coanalytic set $A \subseteq G$ that is not Haar meager, but there is a (nonempty) compact metric space K and a continuous function $f : K \to G$ such that $f^{-1}(gAh)$ is meager in K for every $g, h \in G$.*

4.3 Naive versions

It is possible to eliminate the Borel/universally measurable hull from our definitions completely. Unfortunately, the resulting “naive” notions will not share the nice properties of a notion of smallness. The results in this subsection will show some of these problems. Most of these counterexamples are provided only in special groups or as a corollary of the Continuum Hypothesis, as even these "weak" results are enough to show that these notions are not very useful.

Definition 4.3.1. A set $A \subseteq G$ is called *naively Haar null* if there is a Borel probability measure μ on G such that $\mu(gAh) = 0$ for every $g, h \in G$.

Definition 4.3.2. A set $A \subseteq G$ is called *naively Haar meager* if there is a (nonempty) compact metric space K and a continuous function $f : K \to G$ such that $f^{-1}(gAh)$ is meager in K for every $g, h \in G$.

The following two examples show that in certain groups the whole group is the union of countably many sets which are both naively Haar null and naively Haar meager, and hence neither the system of naively Haar null sets, nor the system of naively Haar meager sets is a σ-ideal. The first example is from [9], it uses the Continuum Hypothesis to partition a group into two naively Haar null sets (hence it also proves that naively Haar null sets do not form an ideal in this case). The second example is a certain partition of $\mathbb{R}^2$ into countably many sets by Davies (using only ZFC), where the sets were proved to be naively Haar null in [12, Example 5.4]. In these papers the sets forming the partitions were only proved to be naively Haar null, but similar proofs show that they are also naively Haar meager.

Example 4.3.3. *Let G be an uncountable Polish group. Assuming the Continuum Hypothesis, there exists a subset W in the product group $G \times G$ such that both W and $(G \times G) \setminus W$ are naively Haar null and naively Haar meager.*

Proof. Let $<_W$ be a well-ordering of G in order type ω_1, and let $W = \{(g, h) \in G \times G : g <_W h\}$ be this relation considered as a subset of $G \times G$. Let μ_1 be an non-atomic measure on G and μ_2 be a measure on G that is concentrated on a single point. It is clear that $(\mu_1 \times \mu_2)(gWh) = 0$ for every $g, h \in G$ (as $gWh \cap \operatorname{supp}(\mu_1 \times \mu_2)$ is countable). Similarly $(\mu_2 \times \mu_1)(g((G \times G) \setminus W)h) = 0$ for every $g, h \in G$, but these mean that W and $(G \times G) \setminus W$ are both naively Haar null. Let $K \subseteq G$ be a nonempty perfect compact set (it is well known that such set exists) and let $f_1, f_2 : K \to G$ be the continuous functions $f_1(k) = (k, 1_G)$, $f_2(k) = (1_G, k)$ (here 1_G could be replaced by any fixed element of G). Then $f_1^{-1}(gWh)$ is countable (hence meager) for every $g, h \in G$, so W is naively Haar meager, similarly f_2 shows that $(G \times G) \setminus W$ is also naively Haar meager. □

Example 4.3.4. *The Polish group $(\mathbb{R}^2, +)$ is the union of countably many sets that are both naively Haar null and naively Haar meager.*

Proof. We use the following result of Davies, its proof can be found in [6].

Theorem 4.3.5. *Suppose that $(\theta_i)_{i\in\omega}$ is a countably infinite system of directions, such that θ_i and θ_j are not parallel if $i \neq j$. Then the plane can be decomposed as $\mathbb{R}^2 = \dot{\bigcup}_{i\in\omega} S_i$ such that each line in the direction θ_i intersects the set S_i in at most one point.*

To prove that in this construction S_i is naively Haar null for every $i \in \omega$ simply let μ_i be the 1-dimensional Lebesgue measure on an arbitrary line with direction θ_i. Then $\mu_i(g + S_i + h) = 0$ for every $g, h \in \mathbb{R}^2$, because at most one point of $g + S_i + h$ is contained in the support of μ_i. This means that S_i is indeed naively Haar null.

Similarly let K_i be a nonempty perfect compact subset of an arbitrary line with direction θ_i, and let $f_i : K_i \to \mathbb{R}^2$ be the restriction of the identity function. Then $f_i^{-1}(g + S_i + h)$ contains at most one point for every $g, h \in \mathbb{R}^2$, hence it is meager. This means that S_i is indeed naively Haar meager. □

The following theorem shows that in a relatively general class of groups the naive notions are strictly weaker than the corresponding "canonical" notions.

Theorem 4.3.6. *Let G be an uncountable abelian Polish group.*

(1) *There exists a subset of G that is naively Haar null but not Haar null.*
(2) *There exists a subset of G that is naively Haar meager but not Haar meager.*

We do not reproduce the relatively long proof of these results. The proof of (1) can be found in [12, Theorem 1.3], the proof of (2) can be found in [7, Theorem 16]. Note that when G is non-locally-compact, these results are corollaries of Theorem 4.2.1 and Theorem 4.2.4.

The following example from [7, Proposition 17] yields the results of Example 4.3.3 in a different class of groups. The cited paper only proves this for the naively Haar meager case, but states that it can be proved analogously in the naively Haar null case. We do not reproduce this proof, as it is significantly longer than the proof of Example 4.3.3.

Example 4.3.7. *Let G be an uncountable abelian Polish group. Assuming the Continuum Hypothesis, there exists a subset $X \subseteq G$ such that both X and $G \setminus X$ are naively Haar null and naively Haar meager.*

4.4 Left and right Haar null sets

When we defined Haar null sets in Definition 3.1.1, we used multiplication from both sides by arbitrary elements of G. If we replace this by multiplication from one side, we get the following notions:

Definition 4.4.1. A set $A \subseteq G$ is said to be *left Haar null* (or *right Haar null*) if there are a Borel set $B \supseteq A$ and a Borel probability measure μ on G such that $\mu(gB) = 0$ for every $g \in G$ ($\mu(Bg) = 0$ for every $g \in G$).

Definition 4.4.2. A set $A \subseteq G$ is said to be *left-and-right Haar null* if there are a Borel set $B \supseteq A$ and a Borel probability measure μ on G such that $\mu(gB) = \mu(Bg) = 0$ for every $g \in G$.

If "Borel set" is replaced by "universally measurable set", we can naturally obtain the generalized versions of these notions. As the papers about this topic happen to follow Christensen in defining "Haar null set" to mean generalized Haar null set in the terminology of this thesis, most results in this subsection were originally stated for these generalized versions.

Notice that even these "one-sided" notions form systems that are translation invariant, because if e.g. B is Borel left Haar null and a Borel probability measure μ satisfies $\mu(gB) = 0$ for every $g \in G$ and we consider a right translate Bh (this is the interesting case, invariance under left translation is trivial), then $\mu'(X) = \mu(Xh^{-1})$ is a Borel probability measure which satisfies $\mu'(g \cdot Bh) = 0$ for every $g \in G$. An analogous argument shows that the system of right Haar null sets is translation invariant; using Proposition 4.4.7 it follows from these that the system of left-and-right Haar null sets is also translation invariant. It is clear that this reasoning works for the generalized versions, too.

Unfortunately, these notions are not "good" notions of smallness in general, because they fail to form σ-ideals in some groups. In [26] Solecki gives a sufficient condition which guarantees that the generalized left Haar null sets form a σ-ideal and gives another sufficient condition which guarantees that the left Haar null sets do not form a σ-ideal. We state these results and show examples of groups satisfying these conditions without proofs:

Definition 4.4.3. A Polish group G is called *amenable at 1* if for any sequence $(\mu_n)_{n\in\omega}$ of Borel probability measures on G with $1_G \in \operatorname{supp}\mu_n$, there are Borel probability measures ν_n and ν such that

(I) $\nu_n \ll \mu_n$,
(II) if $K \subseteq G$ is compact, then $\lim_n(\nu * \nu_n)(K) = \nu(K)$.

This class is closed under taking closed subgroups and continuous homomorphic images. A relatively short proof also shows that (II) is equivalent to

(II') if $K \subseteq G$ is compact, then $\lim_n(\nu_n * \nu)(K) = \nu(K)$.

This means that if G is amenable at 1, then so is the opposite group G^{opp} (G^{opp} is the set G considered as a group with $(x, y) \mapsto y \cdot x$ as the multiplication).

Examples of groups which are amenable at 1 include:

(1) abelian Polish groups,
(2) locally compact Polish groups,
(3) countable direct products of locally compact Polish groups such that all but finitely many factors are amenable,
(4) inverse limits of sequences of amenable, locally compact Polish groups with continuous homomorphisms as bonding maps.

Theorem 4.4.4. *If G is amenable at 1, then the generalized left Haar null sets form a σ-ideal.*

Our note about the opposite group (and the fact that the intersection of two σ-ideals is also a σ-ideal) means that generalized right Haar null sets and generalized left-and-right Haar null sets also form σ-ideals.

The following definition is the sufficient condition for the "bad" case, note that this condition is also symmetric (in the sense that if G satisfies it, then G^{opp} also satisfies it).

Definition 4.4.5. A Polish group G is said to *have a free subgroup at 1* if it has a non-discrete free subgroup whose all finitely generated subgroups are discrete.

The paper [26] mentions several groups which all have a free subgroup at 1, we mention some of these:

(1) countably infinite products of Polish groups containing discrete free non-Abelian subgroups,
(2) S_∞, the group of all permutations of a countably infinite set,
(3) the group of all permutations of $\mathbb{Q}$ which preserve the standard linear order.

Theorem 4.4.6. *If G has a free subgroup at 1, then there are a Borel left Haar null set $B \subseteq G$ and $g \in G$ such that $B \cup Bg \in G$. As the left Haar null sets are translation invariant and G is not left Haar null, this means that they do not form an ideal.*

As the notion of having a free subgroup at 1 is symmetric, the same is true for right Haar null sets.

The paper [26] uses these notions for problems related to automatic continuity and the generalizations of the Steinhaus theorem, in subsection 5.2 we mention a few of these results.

We finish the examination of these notions with some results about the connections among these notions and the connections between these notions and the "plain", two-sided notions defined in subsection 3.1.

It is clear that all Haar null sets are left-and-right Haar null and all left-and-right Haar null sets are both left Haar null and right Haar null. In abelian groups these notions all trivially coincide with Haar nullness (and the generalized versions coincide with generalized Haar nullness).

Also note that if $A \subseteq G$ is a conjugacy invariant set (that is, $gAg^{-1} = A$ for every $g \in G$), then $gAh = gh \cdot h^{-1}Ah = ghA$ (and similarly $gAh = Agh$) for every $g, h \in G$, hence if A is left (or right) Haar null, then A is Haar null.

For locally compact groups Theorem 3.3.7 and the remark in its proof shows that all generalized right Haar null sets are Haar null. An analogous proof works for generalized left Haar null sets and thus in locally compact groups eight versions of Haar null (generalized or not, left or right or left-and-right or "plain") are all equivalent to each other and equivalent to having Haar measure zero.

The following simple result mentioned in [22] can be proved by modifying the proof of [20, Theorem 2]:

Proposition 4.4.7. *A set $A \subseteq G$ is left-and-right Haar null if and only if it is both left Haar null and right Haar null.*

Proof. We only have to show that if A is both left Haar null and right Haar null, then it is left-and-right Haar null, as the other direction is trivial. By definition there exist a Borel set $B \supseteq A$ and Borel probability measures μ_1, μ_2 on G such that $\mu_1(gB) = \mu_2(Bg) = 0$ for every $g \in G$.

Define

$$\mu(X) = (\mu_1 \times \mu_2)\left(\{(x, y) \in G^2 : yx \in X\}\right),$$

then μ is clearly a Borel probability measure and if the characteristic function of a set S is denoted by χ_S, then using Fubini's theorem we have

$$\mu(gB) = \int_G \int_G \chi_{gB}(yx)\ \mathrm{d}\mu_1(x)\ \mathrm{d}\mu_2(y) = \int_G \mu_1(y^{-1}gB)\ \mathrm{d}\mu_2(y) = 0$$

and

$$\mu(Bg) = \int_G \int_G \chi_{Bg}(yx)\ \mathrm{d}\mu_2(y)\ \mathrm{d}\mu_1(x) = \int_G \mu_2(Bgx^{-1})\ \mathrm{d}\mu_1(x) = 0$$

and these show that μ satisfies the requirements of Definition 4.4.2. □

There are results that show that (aside from this proposition) the weaker ones among these notions do not imply the stronger ones.

If G has a free subgroup at 1, Theorem 4.4.6 provides a left Haar null set B and $g \in G$ such that $B \cup Bg \in G$; this set B is not generalized Haar null as the generalized Haar null sets form a σ-ideal that does not contain G. Notice that there are groups that admit a two-sided invariant metric and have a free subgroup at 1 (for example F_2^{ω} where F_2 is the free group of rank 2 with discrete topology).

In the earlier article [22] Shi and Thompson used more elementary techniques to find examples in the group $\mathcal{H}[0,1] = \{f : f \text{ is continuous, strictly increasing}, f(0) = 0, f(1) = 1\}$ (this is the automorphism group of $[0,1]$; the group operation is composition, the topology is the compact-open topology). We state these results without proofs:

Example 4.4.8. *There exists a Borel set $B \subseteq \mathcal{H}[0,1]$ and a Borel probability measure μ such that $\mu(Bg) = 0$ for every $g \in \mathcal{H}[0,1]$ (this implies that B is right Haar null), but $\mu(gB) \neq 0$ for some $g \in \mathcal{H}[0,1]$ (i.e. μ does not witness that B is left Haar null).*

Example 4.4.9. *The group $\mathcal{H}[0,1]$ has a Borel subset that is left-and-right Haar null but not Haar null.*

The following result is [25, Theorem 6.1], a result from another paper of Solecki, which provides a necessary and sufficient condition for the equivalence of the notions generalized left Haar null and generalized Haar null in a special class of groups.

Theorem 4.4.10. *Let H_n $(n \in \omega)$ be countable groups and consider the group $G = \prod_n H_n$. The following conditions are equivalent:*

(1) *In G the system of generalized left Haar null sets is the same as the system of generalized Haar null sets.*
(2) *For each universally measurable set $A \subseteq G$ that is not generalized Haar null, $1_G \in \operatorname{int}(AA^{-1})$.*
(3) *For each closed set $F \subseteq G$ that is not (generalized) Haar null, FF^{-1} is dense in some non-empty open set.*
(4) *For all but finitely many $n \in \omega$ all elements of H_n have finite conjugacy classes in H_n, that is, for all but finitely many $n \in \omega$ and for all $x \in H_n$ the set $\{yxy^{-1} : y \in H_n\}$ is finite.*

4.5 Openly Haar null sets

The notion of openly Haar null sets was introduced in [24], and more thoroughly examined in [4].

Definition 4.5.1. A set $A \subseteq G$ is said to be *openly Haar null* if there is a Borel probability measure μ on G such that for every $\varepsilon > 0$ there is an open set $U \supseteq A$ such that $\mu(gUh) < \varepsilon$ for every $g, h \in G$. If μ has these properties, we say that μ witnesses that A is openly Haar null.

The following simple proposition shows that this is a stronger property than Haar nullness:

Proposition 4.5.2. *Every openly Haar null set is contained in a G_δ Haar null set.*

Proof. For every $n \in \omega$ there is an open set $U_n \supseteq A$ such that $\mu(gU_nh) < \frac{1}{n+1}$ for every $g, h \in G$. Then $B = \bigcap_{n\in\omega} U_n$ is a G_δ set that is Haar null because it satisfies $\mu(gBh) \leq \mu(gU_nh) < \frac{1}{n+1}$ for every $g, h \in G$, hence $\mu(gBh) = 0$ for every $g, h \in G$. □

To prove that the system of openly Haar null sets forms a σ-ideal, we will need some lemmas.

Lemma 4.5.3. *If μ witnesses that $A \subseteq G$ is openly Haar null and V is a neighborhood of 1_G, then there exists a measure μ' which also witnesses that A is openly Haar null and has a compact support that is contained in V.*

Proof. Lemma 3.2.2 states that there are a compact set $C \subseteq G$ and $c \in G$ such that $\mu(C) > 0$ and $C \subseteq cV$. Let $\mu'(X) := \frac{\mu(cX\cap C)}{\mu(C)}$, this is clearly a Borel probability measure and has a compact support that is contained in V. Fix an arbitrary $\varepsilon > 0$. We will find an open set $U \supseteq A$ such that $\mu'(gUh) < \varepsilon$ for every $g, h \in G$. Notice that

$$\mu'(gUh) < \varepsilon \quad \Leftrightarrow \quad \mu(cgUh \cap C) < \mu(C) \cdot \varepsilon \quad \Leftarrow \quad \mu(cgUh) < \mu(C) \cdot \varepsilon.$$

There exists an open set $U \supseteq A$ with $\mu(gUh) < \mu(C) \cdot \varepsilon$ for every $g, h \in G$, and this satisfies $\mu(cgUh) < \mu(C) \cdot \varepsilon$ for every $g, h \in G$, hence μ' has the required properties. □

Theorem 4.5.4. *The system of openly Haar null sets is a translation invariant σ-ideal.*

Proof. It is trivial that the system of openly Haar null sets satisfy (I) and (II) in Definition 3.2.1. The following proof of (III) is described in the appendix of [4] and is similar to the proof of Theorem 3.2.5.

Let A_n be openly Haar null for all $n \in \omega$, we prove that $A = \bigcup_{n\in\omega} A_n$ is also openly Haar null. For every set A_n fix a measure μ_n which witnesses that A_n is openly Haar null. Let d be a complete metric on G that is compatible with the topology of G.

We construct for all $n \in \omega$ a compact set $C_n \subseteq G$ and a Borel probability measure $\tilde{\mu}_n$ such that the support of $\tilde{\mu}_n$ is C_n, $\tilde{\mu}_n(gU_nh) < \varepsilon$ for every $g, h \in G$ and the "size" of the sets C_n decreases "quickly".

The construction will be recursive. For the initial step use Lemma 4.5.3 to find a measure $\tilde{\mu}_0$ witnessing that A_0 is openly Haar null and has compact support $C_0 \subseteq G$. Assume that $\tilde{\mu}_{n'}$ and $C_{n'}$ are already defined for all $n' < n$. By Lemma 3.2.4 there exists a neighborhood V_n of 1_G such that if $v \in V_n$, then $d(k \cdot v, k) < 2^{-n}$ for every k in the compact set $C_0 C_1 C_2 \cdots C_{n-1}$. Applying Lemma 4.5.3 again we can find a measure $\tilde{\mu}_n$ with compact support $C_n \subseteq V_n$ which is witnessing that A_n is openly Haar null.

If $c_n \in C_n$ for all $n \in \omega$, then it is clear that the sequence $(c_0 c_1 c_2 \cdots c_n)_{n \in \omega}$ is a Cauchy sequence. As (G, d) is complete, this Cauchy sequence is convergent; we write its limit as the infinite product $c_0 c_1 c_2 \cdots$. The map $\varphi : \prod_{n \in \omega} C_n \to G$, $\varphi((c_0, c_1, c_2, \ldots)) = c_0 c_1 c_2 \cdots$ is the pointwise limit of continuous functions, hence it is Borel.

Let μ^Π be the product of the measures $\tilde{\mu}_n$ on the product space $C^\Pi := \prod_{n \in \omega} C_n$. Let $\mu = \varphi_*(\mu^\Pi)$ be the push-forward of μ^Π along φ onto G, i.e.

$$\mu(X) = \mu^\Pi(\varphi^{-1}(X)) = \mu^\Pi\left(\left\{(c_0, c_1, c_2, \ldots) \in C^\Pi : c_0 c_1 c_2 \cdots \in X\right\}\right).$$

We claim that μ witnesses that $A = \bigcup_{n \in \omega} A_n$ is openly Haar null. Fix an arbitrary $\varepsilon > 0$, we will show that there is an open set $U \supseteq A$ such that $\mu(gUh) < \varepsilon$ for every $g, h \in G$. It is enough to find open sets $U_n \supseteq A_n$ such that $\mu(gU_nh) < \varepsilon \cdot 2^{-(n+2)}$ for every $g, h \in G$ and $n \in \omega$, because then $U = \bigcup_{n \in \omega} U_n$ satisfies that for every $g, h \in G$

$$\mu(gUh) \leq \sum_{n \in \omega} \mu(gU_nh) \leq \frac{\varepsilon}{2} < \varepsilon.$$

Fix $g, h \in G$ and $n \in \omega$. Choose an $U_n \supseteq A_n$ open set satisfying that $\tilde{\mu}_n(gU_nh) < \varepsilon \cdot 2^{-(n+2)}$ for every $g, h \in G$. Notice that if $c_j \in C_j$ for every $j \neq n$, $j \in \omega$, then

$$\begin{aligned} &\tilde{\mu}_n\left(\{c_n \in C_n : c_0 c_1 c_2 \cdots c_n \cdots \in gU_nh\}\right) = \\ &\quad = \tilde{\mu}_n\left((c_0 c_1 \cdots c_{n-1})^{-1} \cdot gU_nh \cdot (c_{n+1} c_{n+2} \cdots)^{-1}\right) < \varepsilon \cdot 2^{-(n+2)} \end{aligned}$$

because $\tilde{\mu}_n(g'U_nh') < \varepsilon \cdot 2^{-(n+2)}$ for all $g', h' \in G$. Applying Fubini's theorem in the product space $\left(\prod_{j \neq n} C_j\right) \times C_n$ to the product measure $\left(\prod_{j \neq n} \tilde{\mu}_j\right) \times \tilde{\mu}_n$ yields that

$$\varepsilon \cdot 2^{-(n+2)} > \mu^\Pi\left(\left\{(c_0, c_1, \ldots, c_n, \ldots) \in C^\Pi : c_0 c_1 \cdots c_n \cdots \in gU_nh\right\}\right).$$

By the definition of μ this means that $\mu(gU_nh) < \varepsilon \cdot 2^{-(n+2)}$. □

The surprisingly simple proof of the following result from [4] demonstrates that this is a useful notion:

Theorem 4.5.5. *If there is a nonempty openly Haar null set in G, then every countable subset $C \subseteq G$ is contained in a comeager Haar null set. In particular, G may be written as the disjoint union $G = A \dot{\cup} B$ where A is a Haar null set and B is meager in G.*

Proof. If there is a nonempty openly Haar null set in G, then applying Lemma 3.2.10 yields that every countable set in G is openly Haar null. In particular a dense countable set $C' \supseteq C$ is openly Haar null. Then Proposition 4.5.2 yields that $C' \subseteq A$ for a G_δ Haar null set A. The dense G_δ set A is a countable intersection of dense open sets, hence $B := G \setminus A$ is a countable union of nowhere dense sets, i.e. B is (F_σ and) meager. $\square$

This reasoning shows that a large class of Polish groups can be written as the union of a Haar null set and a meager set, but unfortunately there are groups where only the empty set is openly Haar null. We list some results about this in the following propositions, the proofs of these results can be found in [4].

Proposition 4.5.6. *In the Polish group G there is a nonempty openly Haar null subset if at least one of the following conditions holds:*

(1) *G is uncountable and admits a two-sided invariant metric,*
(2) *$G = S_\infty$ is the group of permutations of $\mathbb{N}$ with the topology of pointwise convergence,*
(3) *$G = \mathrm{Aut}(\mathbb{Q}, \leq)$ is the group of order-preserving self-bijections of the rationals with the topology of pointwise convergence on $\mathbb{Q}$ viewed as discrete (i.e. a sequence $(f_n)_{n\in\omega} \in G^\omega$ is said to be convergent if for every $q \in \mathbb{Q}$ there is a $n_0 \in \omega$ such that the sequence $(f_n(q))_{n\geq n_0}$ is constant),*
(4) *$G = \mathcal{U}(\ell^2)$ is the unitary group on the separable infinite-dimensional complex Hilbert space with the strong operator topology,*
(5) *G admits a continuous surjective homomorphism onto a group listed above.*

Proposition 4.5.7. *In the Polish group G the empty set is the only openly Haar null subset if for every compact subset $C \subseteq G$ and every nonempty open subset $U \subseteq G$ there are $g, h \in G$ with $gCh \subseteq U$. In particular $G = \mathcal{H}[0,1]$, the group of order-preserving self-homeomorphisms of $[0,1]$ (endowed with the compact-open topology) has this property, hence in $\mathcal{H}[0,1]$ only the empty set is openly Haar null.*

4.6 Strongly Haar meager sets

Definition 3.1.6 has another interesting version, where we require the witness function to be the identity function of G restricted to a compact subset $C \subseteq G$. This is motivated by the fact that when we prove that some set is Haar meager, we frequently use witness functions of this kind.

Definition 4.6.1. A set $A \subseteq G$ is said to be *strongly Haar meager* if there are a Borel set $B \supseteq A$ and a (nonempty) compact set $C \subseteq G$ such that $gBh \cap C$ is meager in C for every $g, h \in G$.

Unfortunately, we know almost nothing about this notion. In [5] Darji asks the following basic questions for the case when G is abelian (as [5] only considers abelian groups), but these questions are also interesting in the general case.

Question 4.6.2. *Is every Haar meager set strongly Haar meager?*

Question 4.6.3. *Is the system of strongly Haar meager sets a σ-ideal?*

(Of course if the answer for Question 4.6.2 is yes, then the answer for Question 4.6.3 is also trivially yes.)

A natural idea for showing that all strongly Haar meager sets are Haar meager would be showing that if $f : K \to G$ is a witness function for a Haar meager set, then $C := f(K)$ satisfies the requirements of Definition 4.6.1. Unfortunately, this kind of proof will not work: we reproduce without proof the statement of [5, Example 11], which shows a Haar meager set where the image of one particular witness function does not satisfy the requirements of Definition 4.6.1.

Example 4.6.4. *There exists a G_δ Haar meager set $A \subseteq \mathbb{R}$, a compact metric space K and a witness function $f : K \to \mathbb{R}$ such that $A \cap f(K)$ is comeager in $f(K)$.*

5 Analogs of the results from the locally compact case

In this section we discuss generalizations and analogs of a few theorems that are well-known for locally compact groups. Unfortunately, although these are true in locally compact groups for the sets of Haar measure zero and the meager sets, neither of them remains completely valid for Haar null sets and Haar meager sets. However, in some cases weakened versions remain true, and these often prove to be useful.

5.1 Fubini's theorem and the Kuratowski-Ulam theorem

Fubini's theorem, and its topological analog, the Kuratowski-Ulam theorem (see e.g. [17, 8.41]) describes small sets in product spaces. They basically state that a set (which is measurable in the appropriate sense) in the product of two spaces is small if and only if co-small many sections of it are small. Notice that because the product of left (or right) Haar measures of two locally compact groups of is trivially a left (or right) Haar measure of the product group, (a special case of) Fubini's theorem connects the sets of Haar measure zero in the two groups and the sets of Haar measure zero in the product group.

Unfortunately, analogs of these theorems are proved only in very special cases, and there are counterexamples known in otherwise "nice" groups. We provide a simple counterexample (which can be found as [7, Example 20]) that works in both the Haar null and Haar meager case.

Example 5.1.1. *There exists a closed set $A \subseteq \mathbb{Z}^\omega \times \mathbb{Z}^\omega$ that is neither Haar null nor Haar meager, but in one direction all its sections are Haar null and Haar meager. (In the other direction, non-Haar-null and non-Haar-meager many sections are non-Haar-null and non-Haar-meager.)*

Proof. The group operation of $\mathbb{Z}^\omega \times \mathbb{Z}^\omega$ is denoted by $+$.

The set with these properties will be

$$A = \{(s,t) \in \mathbb{Z}^\omega \times \mathbb{Z}^\omega : t_n \geq s_n \geq 0 \text{ for every } n \in \omega\}.$$

It is clear from the definition that A is closed.

Note that for $t \in \mathbb{Z}^\omega$, the section $A^t = \{s \in \mathbb{Z}^\omega : t_n \geq s_n \geq 0 \text{ for every } n \in \omega\}$ is compact (as it is the product of finite sets with the discrete topology), and it follows from Theorem 6.3.1 and Corollary 6.3.6 that all compact sets are Haar null and Haar meager in $\mathbb{Z}^\omega$.

To show that A is not Haar null and not Haar meager, we will use the technique described in subsection 6.5 and show that for every compact set $C \subseteq \mathbb{Z}^\omega \times \mathbb{Z}^\omega$ the set A contains a translate of C. As $C = \emptyset$ satisfies this, we may assume that $C \neq \emptyset$. Let $\pi_n^1(s,t) = s_n$ and $\pi_n^2(s,t) = t_n$, then $\pi_n^1, \pi_n^2 : \mathbb{Z}^\omega \times \mathbb{Z}^\omega \to \mathbb{Z}$ are continuous functions. Let a and b be the sequences satisfying $a_n = -\min \pi_n^1(C)$ and $b_n = -\min \pi_n^2(C) + \max \pi_n^1(C) - \min \pi_n^1(C)$. It is straightforward to check that this choice guarantees that if $(s,t) \in C + (a,b)$, then $t_n \geq s_n \geq 0$ for every $n \in \omega$.

Finally, if $s \in \mathbb{Z}^\omega$ satisfies $s_n \geq 0$ for every $n \in \omega$, then similar, but simpler arguments show that the section $A_s = \{t \in \mathbb{Z}^\omega : t_n \geq s_n \text{ for every } n \in \omega\}$ contains a translate of every compact set $C \subseteq \mathbb{Z}^\omega$, hence it is neither Haar null nor Haar meager. Similarly, the set $\{s \in \mathbb{Z}^\omega : s_n \geq 0 \text{ for every } n \in \omega\}$ is also neither Haar null nor Haar meager, so we proved the statement about the sections in the other direction. □

In [3, Theorem 6] Christensen gives a counterexample for Haar nullness where one of the two groups in the product is locally compact. We state the properties of this example without proving them.

Example 5.1.2. *Let H be a separable infinite dimensional Hilbert space (with addition as the group operation) and let $\mathbb{S}^1$ be the unit circle in the complex plane (with complex multiplication as the group operation). There exists in the product group $H \times \mathbb{S}^1$ a Borel set A such that*

(I) *For every $h \in H$, the section A_h has Haar measure one in $\mathbb{S}^1$.*
(II) *For every $s \in \mathbb{S}^1$, the section A^s is Haar null in H.*
(III) *The complement of A is Haar null in the product group $H \times \mathbb{S}^1$.*

The connection between (I) and (III) in this example is not accidental; Christensen notes that if one of the two groups in a product is locally compact (and both are abelian), then one direction of Fubini's theorem holds, that is, a Borel set is Haar null in the product group if and only if for co-Haar-null many h in the "large" group, the h-section of the set has Haar measure zero in the locally compact group. The proof of this remark can be found in [1, Theorem 2.3], we include the proof of the following generalized version.

Theorem 5.1.3. *Suppose that G and H are Polish groups and $B \subseteq G \times H$ is a Borel subset. Then if G is locally compact, then the following conditions are all equivalent. Moreover, the implication (1) $\Rightarrow$ (2) remains valid even if G is not necessarily locally compact.*

(1) *there exists a Haar null set $E \subseteq H$ and a Borel probability measure μ on G such that every $h \in H \setminus E$ and $g_1, g_2 \in G$ satisfies $\mu(g_1 B^h g_2) = 0$ (i.e. these sections are Haar null and μ is witness measure for each of them),*

(2) *B is Haar null in $G \times H$,*
(3) *there exists a Haar null set $E \subseteq H$ such that for every $h \in H \setminus E$ the section B^h has Haar measure zero in G.*

In the theorem equivalence (2) $\Leftrightarrow$ (3) (which holds when G is locally compact) is the statement of the remark (generalized for not necessarily abelian groups), condition (1) is a trivial corollary of (3) that remains meaningful when G is non-locally-compact and (1) $\Rightarrow$ (2) is the direction of the remark that remains true when G is not necessarily locally compact.

Proof. (1) $\Rightarrow$ (2):
As the set E is Haar null, there exists a Borel set $E' \supseteq E$ and a Borel probability measure ν on H such that $\nu(h_1 E' h_2) = 0$ for every $h_1, h_2 \in H$. We show that the measure $\mu \times \nu$ witnesses that B is Haar null in $G \times H$. Fix arbitrary $g_1, g_2 \in G$ and $h_1, h_2 \in H$, we have to prove that $(\mu \times \nu)((g_1, h_1) \cdot B \cdot (g_2, h_2)) = 0$.

Notice that

$$(g_1, h_1) \cdot B \cdot (g_2, h_2) \subseteq (G \times (h_1 \cdot E' \cdot h_2)) \cup ((g_1, h_1) \cdot (B \setminus (G \times E')) \cdot (g_2, h_2))$$

and in this union the $(\mu \times \nu)$-measure of the first term is zero (by the choice of μ and E'). For every $h \in H$, the h-section of the Borel set $(g_1, h_1) \cdot (B \setminus (G \times E')) \cdot (g_2, h_2)$ is $g_1 \cdot (B \setminus (G \times E'))^{h_1^{-1} h h_2^{-1}} \cdot g_2$, and this is either the empty set (if $h_1^{-1} h h_2^{-1} \in E'$) or a set of μ-measure zero (if $h_1^{-1} h h_2^{-1} \in H \setminus E'$). Applying Fubini's theorem in the product space $G \times H$ to the product measure $\mu \times \nu$ yields that $(\mu \times \nu)((g_1, h_1) \cdot B \cdot (g_2, h_2)) = 0$, and this is what we had to prove.

(2) $\Rightarrow$ (3) (when G is locally compact):
Let $B \subseteq G \times H$ be Haar null and suppose that μ is a witness measure, i.e. $\mu((g_1, h_1) \cdot B \cdot (g_2, h_2)) = 0$ for every $g_1, g_2 \in G$ and $h_1, h_2 \in H$. Fix a left Haar measure λ on G, let δ denote the Dirac measure at 1_H (i.e. for $X \subseteq H$, $\delta(X) = 1$ if $1_H \in X$ and 0 otherwise) and let $\tilde{\lambda} = \lambda \times \delta$. Let $E = \{h \in H : \lambda(B^h) \neq 0\}$, it is clearly enough to prove that $E \subseteq H$ is Haar null. First we use a standard argument to show that E is a Borel set.

If $(X, \mathcal{S})$ is a measurable space, Y is a separable metrizable space, $P(Y)$ is the space of Borel probability measures on Y and $A \subseteq X \times Y$ is measurable (i.e. $A \in \mathcal{S} \times \mathcal{B}(Y)$), then [17, Theorem 17.25] states that the map $X \times P(Y) \to \mathbb{R}_+, (x, \varrho) \mapsto \varrho(A_x)$ is measurable (for $\mathcal{S} \times \mathcal{B}(P(Y))$). Applying this for $(X, \mathcal{S}) := (H, \mathcal{B}(H))$, $Y := G$, and $A := B$ yields that $\tilde{E} = \{(h, \varrho) \in H \times P(G) : \varrho(B^h) \neq 0\}$ is Borel (the Borel preimage of an open set in $\mathbb{R}_+$). If ϱ is a Borel probability measure on G that is equivalent to λ in the sense that they have the same zero sets, then $E = \tilde{E}^\varrho$ is Borel, because it is the section of a Borel set.

We will show that the measure $\nu(X) = \mu(G \times X)$ witnesses that E is Haar null. Fix arbitrary $h_1, h_2 \in H$, we have to prove that $0 = \nu(h_1 E h_2) = \mu(G \times (h_1 E h_2))$.

Consider the set

$$S = \{((u_G, u_H), (v_G, v_H)) \in ((G \times H) \times (G \times H)) : \\ (u_G, u_H) \cdot (v_G, v_H) \in (1_G, h_1) \cdot B \cdot (1_G, h_2)\},$$

it is easy to see that this is a Borel set. Applying Fubini's theorem in the product space $(G \times H) \times (G \times H)$ to the product measure $\mu \times \tilde{\lambda}$ yields that

$$(\mu \times \tilde{\lambda})(S) = \int_{G \times H} \mu((1_G, h_1) \cdot B \cdot (v_G^{-1}, h_2 v_H^{-1}))\, \mathrm{d}\tilde{\lambda}((v_G, v_H)) = 0$$

because μ witnesses that B is Haar null. Applying Fubini's theorem again for the other direction yields that

$$0 = (\mu \times \tilde{\lambda})(S) = \int_{G \times H} \tilde{\lambda}\left((u_G^{-1}, u_H^{-1} h_1) \cdot B \cdot (1_G, h_2)\right)\, \mathrm{d}\mu((u_G, u_H)) =$$
$$= \int_{G \times H} \lambda\left(\{g \in G : (g, 1_H) \in (u_G^{-1}, u_H^{-1} h_1) \cdot B \cdot (1_G, h_2)\}\right)\, \mathrm{d}\mu((u_G, u_H)) =$$
$$= \int_{G \times H} \lambda\left(\{g \in G : (u_G g, h_1^{-1} u_H h_2^{-1}) \in B\}\right)\, \mathrm{d}\mu((u_G, u_H)) =$$
$$= \int_{G \times H} \lambda\left(u_G^{-1} \cdot B^{h_1^{-1} u_H h_2^{-1}}\right)\, \mathrm{d}\mu((u_G, u_H)) = \int_{G \times H} \lambda\left(B^{h_1^{-1} u_H h_2^{-1}}\right)\, \mathrm{d}\mu((u_G, u_H)).$$

It is clear from the definition of E that the function

$$\varphi : G \times H \to \mathbb{R}_+, \qquad (u_G, u_H) \mapsto \lambda\left(B^{h_1^{-1} u_H h_2^{-1}}\right)$$

takes strictly positive values on the set $G \times (h_1 E h_2)$. However, our calculations showed that $\int \varphi \, \mathrm{d}\mu$ is zero, hence the μ-measure of the set $G \times (h_1 E h_2)$ must be zero.

(3) $\Rightarrow$ (1) (when G is locally compact):
Fix a left Haar measure λ on G. (3) states that $\lambda(B^h) = 0$ for every $h \in H \setminus E$. Using Theorem 3.3.5 it is easy to see that $\lambda(g_1 B^h g_2) = 0$ for every $g_1, g_2 \in G$. Let $C \subseteq G$ be a Borel set such that $0 < \lambda(C) < \infty$ (the regularity of the Haar measure guarantees a compact C satisfying this) and let μ be the Borel probability measure $\mu(X) = \frac{\lambda(X \cap C)}{\lambda(C)}$. Then $\mu \ll \lambda$ guarantees that μ satisfies condition (1). □

We also prove the analog of this for Haar meager sets:

Theorem 5.1.4. *Suppose that G and H are Polish groups and $B \subseteq G \times H$ is a Borel subset. Then if G is locally compact, then the following conditions are all equivalent.*

Moreover, the implication (1) ⇒ (2) *remains valid even if* G *is not necessarily locally compact.*

(1) *there exists a Haar meager set* $E \subseteq H$*, a (nonempty) compact metric space* K *and a continuous function* $f : K \to G$ *such that every* $h \in H \setminus E$ *and* $g_1, g_2 \in G$ *satisfies that* $f^{-1}(g_1 B^h g_2)$ *is meager in* K *(i.e. these sections are Haar meager and* f *is a witness function for each of them),*
(2) B *is Haar meager in* $G \times H$*,*
(3) *there exists a Haar meager set* $E \subseteq H$ *such that for every* $h \in H \setminus E$ *the section* B^h *is meager in the locally compact group* G*.*

Proof. (1) ⇒ (2):
As the set E is Haar meager, there exist a Borel set $E' \supseteq E$, a (nonempty) compact metric space K' and a continuous function $\varphi : K' \to H$ such that $\varphi^{-1}(h_1 E' h_2)$ is meager in K' for every $h_1, h_2 \in H$.

Let $f \times \varphi : K \times K' \to G \times H$ be the function $(f \times \varphi)(k, k') = (f(k), \varphi(k'))$ (we use this notation because this is the analog of the product measure). We show that the function $f \times \varphi$ witnesses that B is Haar meager in $G \times H$. Fix arbitrary $g_1, g_2 \in G$ and $h_1, h_2 \in H$, we have to prove that $(f \times \varphi)^{-1}((g_1, h_1) \cdot B \cdot (g_2, h_2))$ is meager in $K \times K'$.

Notice that

$$(g_1, h_1) \cdot B \cdot (g_2, h_2) \subseteq (G \times (h_1 \cdot E' \cdot h_2)) \cup ((g_1, h_1) \cdot (B \setminus (G \times E')) \cdot (g_2, h_2))$$

and in this union

$$(f \times \varphi)^{-1} (G \times (h_1 \cdot E' \cdot h_2)) = K \times \varphi^{-1}(h_1 \cdot E' \cdot h_2)$$

and using the choice of φ and the Kuratowski-Ulam theorem it is clear that this is a meager subset of $K \times K'$. For every $h \in H$, the h-section of the Borel set $(g_1, h_1) \cdot (B \setminus (G \times E')) \cdot (g_2, h_2)$ is $g_1 \cdot (B \setminus (G \times E'))^{h_1^{-1} h h_2^{-1}} \cdot g_2$, and this is either the empty set (if $h_1^{-1} h h_2^{-1} \in E'$) or a set whose preimage under f is meager in K (if $h_1^{-1} h h_2^{-1} \in H \setminus E'$). This means that for every $k' \in K'$, the k'-section of the set

$$(f \times \varphi)^{-1} ((g_1, h_1) \cdot (B \setminus (G \times E')) \cdot (g_2, h_2)) \subseteq K \times K'$$

is meager in K. Applying the Kuratowski-Ulam theorem in the product space $K \times K'$ yields that the set $(f \times \varphi)^{-1}((g_1, h_1) \cdot B \cdot (g_2, h_2))$ is indeed meager in $K \times K'$, and this is what we had to prove.

(2) ⇒ (3) (when G is locally compact):
Let $B \subseteq G \times H$ be Haar meager and suppose that $f : K \to G \times H$ is a witness function (where K is a nonempty compact metric space), i.e. $f^{-1}((g_1, h_1) \cdot B \cdot (g_2, h_2))$

is meager in K for every $g_1, g_2 \in G$ and $h_1, h_2 \in H$. Let $f(k) = (f_G(k), f_H(k))$ for every $k \in K$, then $f_G : K \to G$ and $f_H : K \to H$ are continuous functions. Let $E = \{h \in H : B^h$ is not meager in $G\}$, then [17, Theorem 16.1] states that E is Borel. It is enough to prove that $E \subseteq H$ is Haar meager. We show that this is witnessed by the function $f_H : K \to H$. Fix arbitrary $h_1, h_2 \in H$, we have to prove that $f_H^{-1}(h_1 E h_2)$ is meager in K.

As in the case of measure, define the Borel set

$$S = \{((u_G, u_H), (v_G, v_H)) \in ((G \times H) \times (G \times H)) : \\ (u_G, u_H) \cdot (v_G, v_H) \in (1_G, h_1) \cdot B \cdot (1_G, h_2)\},$$

and the function $\psi : G \to G \times H$, $\psi(g) = (g, 1_H)$ (ψ is the analog of $\tilde{\lambda}$ from the case of measure). Let $f \times \psi : K \times G \to (G \times H) \times (G \times H)$ be the function $(f \times \psi)(k, g) = (f(k), \psi(g))$. First notice that for every $g \in G$, the g-section of the set $(f \times \psi)^{-1}(S) \subseteq K \times G$ is

$$\begin{aligned}
&((f \times \psi)^{-1}(S))^g = \{k \in K : (f \times \psi)((k, g)) \in S\} = \\
&\quad = \{k \in K : (f(k), \psi(g)) \in S\} = \{k \in K : (f(k), (g, 1_H)) \in S\} = \\
&\quad = \{k \in K : f(k) \in (1_G, h_1) \cdot B \cdot (1_G, h_2) \cdot (g, 1_H)^{-1}\} = \\
&\quad = f^{-1}((1_G, h_1) \cdot B \cdot (g^{-1}, h_2))
\end{aligned}$$

and this is meager in K (as f witnesses that B is Haar meager), hence the Kuratowski-Ulam theorem yields that $(f \times \psi)^{-1}(S)$ is meager in $K \times G$.

Applying the Kuratowski-Ulam theorem in the other direction yields that the set

$$\{k \in K : ((f \times \psi)^{-1}(S))_k \text{ is not meager in } G\}$$

is meager in K. Here if we let $u_G = f_G(k)$ and $u_H = f_H(k)$, then

$$\begin{aligned}
&((f \times \psi)^{-1}(S))_k = \{g \in G : (f \times \psi)((k, g)) \in S\} = \\
&\quad = \{g \in G : (f(k), \psi(g)) \in S\} = \{g \in G : ((u_G, u_H), (g, 1_H)) \in S\} = \\
&\quad = \{g \in G : (g, 1_H) \in (u_G, u_H)^{-1} \cdot (1_G, h_1) \cdot B \cdot (1_G, h_2)\} = \\
&\quad = \{g \in G : (g, 1_H) \in (u_G^{-1}, u_H^{-1} h_1) \cdot B \cdot (1_G, h_2)\} = \\
&\quad = \{g \in G : (u_G g, h_1^{-1} u_H h_2^{-1}) \in B\} = \\
&\quad = u_G^{-1} \cdot B^{h_1^{-1} u_H h_2^{-1}} = (f_G(k))^{-1} \cdot B^{h_1^{-1} f_H(k) h_2^{-1}}.
\end{aligned}$$

Thus we know that

$$\{k \in K : (f_G(k))^{-1} \cdot B^{h_1^{-1} f_H(k) h_2^{-1}} \text{ is not meager in } G\} \text{ is meager in } K,$$

and because meagerness is translation invariant

$$\{k \in K : B^{h_1^{-1} f_H(k) h_2^{-1}} \text{ is not meager in } G\} \text{ is meager in } K.$$

But notice that

$$f_H^{-1}(h_1 E h_2) = \{k \in K : f_H(k) \in h_1 E h_2\} = \{k \in K : h_1^{-1} f_H(k) h_2^{-1} \in E\} =$$
$$= \{k \in K : B^{h_1^{-1} f_H(k) h_2^{-1}} \text{ is not meager in } G\}$$

so we proved that f_H is indeed a witness measure for E.

(3) ⇒ (1) (when G is locally compact):
Notice that the proof of Theorem 3.3.13 shows that in a locally compact Polish group every meager set is Haar meager and there is a function which is a witness function for each of them. The implication that we have to prove is clearly a special case of this observation. □

Finally, we state another special case when the analog of the Kuratowski-Ulam theorem is valid. The proof of this result can be found in [7, Theorem 18].

Theorem 5.1.5. *Suppose that G and H are Polish groups and $A \subseteq G$ and $B \subseteq H$ are analytic sets. Then $A \times B$ is Haar meager in $G \times H$ if and only if at least one of A and B is Haar meager in the respective group.*

5.2 The Steinhaus theorem

The Steinhaus theorem and its generalizations state that if a set $A \subseteq G$ is not small (and satisfies some measurability condition, e.g. it is in the σ-ideal generated by the Borel sets and the small sets), then $AA^{-1} = \{xy^{-1} : x, y \in A\}$ contains a neighborhood of 1_G.

The original result of Steinhaus stated this in the group $(\mathbb{R}, +)$ and used the sets of Lebesgue measure zero as the small sets. Weil extended this for an arbitrary locally compact group, using the sets of Haar measure zero as the small sets. We already stated this result as Corollary 3.3.10 and used a special case of it to prove that there are no (left or right) Haar measures in non-locally-compact groups.

The proof of Weil's version also works for non-locally-compact groups, but in those groups the result is vacuously true, as the statement starts with "If λ is a left Haar measure on G…". Using some variant of Haar null sets as the small sets is a natural idea; this subsection will list some results about these versions.

We already stated the result [25, Theorem 6.1] as Theorem 4.4.10 in the section about left and right Haar null sets. This theorem shows examples of groups where the straightforward generalization of the Steinhaus theorem using generalized Haar null sets as small sets is false.

The following positive result can be found as [21, Theorem 2.8] and is a slight variation of a result in [2]. This result does not claim that AA^{-1} will be a neighborhood of 1_G, only that finitely many conjugates of it will cover a neighborhood of 1_G. Also, it uses generalized right Haar null sets as small sets (see subsection 4.4 for the definition), which is a weaker notion than generalized Haar null sets (hence this result does *not* imply the variant where "right" is omitted from the text).

Theorem 5.2.1. *Suppose that $A \subseteq G$ is a universally measurable subset which is not generalized right Haar null. Then for any neighborhood W of 1_G there are $n \in \omega$ and $h_0, h_1, h_2, \dots, h_{n-1} \in W$ such that*

$$h_0 AA^{-1} h_0^{-1} \cup h_1 AA^{-1} h_1^{-1} \cup \ldots \cup h_{n-1} AA^{-1} h_{n-1}^{-1}$$

is a neighborhood of 1_G.

Proof. This theorem can be found as [21, Theorem 2.8] and it is a slight variation of a result in [2].

Suppose that the conclusion fails for A and W, that is, for every $n \in \omega$ and $h_0, h_1, \dots h_{n-1} \in W$ and any neighborhood $V \ni 1_G$, there is some

$$g \in V \setminus \left(h_0 AA^{-1} h_0^{-1} \cup h_1 AA^{-1} h_1^{-1} \cup \ldots \cup h_{n-1} AA^{-1} h_{n-1}^{-1}\right).$$

Then we can inductively choose a sequence $(g_j)_{j\in\omega}$ such that $g_j \to 1_G$ and for every $j_0 < j_1 < j_2 < \ldots$ index sequence and $r \in \omega$

(I) the infinite product $g_{j_0} g_{j_1} g_{j_2} \cdots$ converges (this can be achieved e.g. by requiring $d(g_{j_0} \cdots g_{j_{r-1}}, g_{j_0} \cdots g_{j_{r-1}} \cdot g_{j_r}) < 2^{-j_r}$ where d is a fixed complete metric on G).

(II) $g_{j_r} \notin (g_{j_0} \cdots g_{j_{r-1}})^{-1} AA^{-1} (g_{j_0} \cdots g_{j_{r-1}})$

Using (I) we can define a continuous map $\varphi : 2^\omega \to G$ by

$$\varphi(\alpha) = g_0^{\alpha(0)} g_1^{\alpha(1)} g_2^{\alpha(2)} \cdots,$$

where $g^0 = 1_G$ and $g^1 = g$. Let λ be the Haar measure on the Cantor group $(\mathbb{Z}_2)^\omega = 2^\omega$ and notice that as A is not generalized right Haar null, there is some $g \in G$ such that

$$\lambda(\varphi^{-1}(Ag)) = \varphi_*(\lambda)(Ag) > 0$$

So by Corollary 3.3.10,

$$\varphi^{-1}(Ag)(\varphi^{-1}(Ag))^{-1}$$

contains a neighborhood of the identity $(0,0,\dots)$ in 2^ω. This neighborhood must contain an element with exactly one "1" coordinate, so there are $\alpha,\beta \in \varphi^{-1}(Ag)$ and $m \in \omega$ such that $\alpha(m)=1$, $\beta(m)=0$ and $\alpha(j)=\beta(j)$ for every $j\in\omega, j\neq m$. This means that there are $h = g_{j_0}g_{j_1}\cdots g_{j_{r-1}}$, $j_0<j_1<\ldots<j_{r-1}<m$ and $k\in G$ such that $\varphi(\alpha)=hg_mk$ and $\varphi(\beta)=hk$. It follows that

$$hg_mh^{-1} = hg_mk\cdot k^{-1}h^{-1} \in Agg^{-1}A = AA^{-1}$$

and so $g_m \in h^{-1}AA^{-1}h = (g_{j_0}\cdots g_{j_{r-1}})^{-1}AA^{-1}(g_{j_0}\cdots g_{j_{r-1}})$, contradicting the choice of g_m. □

The following special case is often useful:

Corollary 5.2.2. *Suppose that $A\subseteq G$ is a universally measurable subset which is* conjugacy invariant *(that is, $gAg^{-1}=A$ for every $g\in G$; in abelian groups every subset has this property) and not Haar null. Then AA^{-1} is a neighborhood of the identity.*

Proof. If A is conjugacy invariant, then as we noted, $gAh = gAg^{-1}\cdot gh = Agh$ and thus A is generalized right Haar null if and only if A is generalized Haar null. Moreover, if A is conjugacy invariant, then $gAA^{-1}g^{-1} = (gAg^{-1})\cdot(gAg^{-1})^{-1} = AA^{-1}$, hence AA^{-1} is also conjugacy invariant and thus in this case Theorem 5.2.1 states that AA^{-1} is a neighborhood of identity. □

Variants of the Steinhaus theorem can be used to prove results about automatic continuity (results stating that all homomorphisms $\pi: G\to H$ which satisfy certain properties are continuous). For example Corollary 3.3.10 yields that any universally measurable homomorphism from a locally compact Polish group into another Polish group is continuous (see e.g. [21, Corollary 2.4]; a map is said to be *universally measurable* if the preimages of open sets are universally measurable). We prove the following automatic continuity result as a corollary of Theorem 5.2.1.

Corollary 5.2.3. *If G and H are Polish groups, H admits a two-sided invariant metric and $\pi: G\to H$ is a universally measurable homomorphism, then π is continuous.*

Proof. It is enough to prove that π is continuous at 1_G (a homomorphism is continuous if and only if it is continuous at the identity element). Let $V\subseteq H$ be an arbitrary neighborhood of 1_H.

Using the continuity of the map $(x,y)\mapsto xy^{-1}$ at $(1_H,1_H)\in H\times H$, there is an open set U with $1_H\in U$ satisfying that $UU^{-1}\subseteq V$. If d is a two-sided invariant metric on H, then the open balls $B(1_H,r)$ are conjugacy invariant sets and form a neighborhood base of 1_H, hence we may also assume that U is conjugacy invariant.

G can be covered by countably many right translates of $\pi^{-1}(U)$, because there is a countable set $S \subseteq \pi(G)$ that is dense in $\pi(G)$ and thus the system $\{U \cdot s : s \in S\}$ covers $\pi(G)$. This means that the universally measurable set $\pi^{-1}(U)$ is not right Haar null, hence we may apply Theorem 5.2.1 to see that for some $n \in \omega$ and $h_0, h_1, \ldots, h_{n-1} \in G$ the set

$$W = \bigcup_{0 \le j < n} h_j \pi^{-1}(U)(\pi^{-1}(U))^{-1} h_j^{-1}$$

is a neighborhood of 1_G. For every $g \in W$ there is a $0 \le j < n$ such that $g \in h_j \pi^{-1}(U)(\pi^{-1}(U))^{-1} h_j^{-1}$, but then

$$\pi(g) \in \pi(h_j) U U^{-1} \pi(h_j)^{-1} = \pi(h_j) U \pi(h_j)^{-1} \cdot (\pi(h_j) U \pi(h_j)^{-1})^{-1} = UU^{-1},$$

thus $\pi(W) \subseteq UU^{-1} \subseteq V$ and this shows that π is continuous at 1_G. □

In [26] Solecki proved that if G is amenable at 1 (see Definition 4.4.3), then a simpler variant of the Steinhaus theorem is true, but if G has a free subgroup at 1 (see Definition 4.4.5) and satisfies some technical condition, then this variant is false. The following theorems state these results.

Theorem 5.2.4. *If G is amenable at 1 and $A \subseteq G$ is universally measurable and not generalized left Haar null, then $A^{-1}A$ contains a neighborhood of 1_G.*

Definition 5.2.5. A Polish group G is called *strongly non-locally-compact* if for any neighborhood U of 1_G there exists a neighborhood V of 1_G such that U cannot be covered by finitely many sets of the form gVh with $g, h \in G$.

Note that the examples we mentioned after Definition 4.4.5 are all strongly non-locally-compact.

Theorem 5.2.6. *If G has a free subgroup at 1 and is strongly non-locally-compact, then there is a Borel set $A \subseteq G$ which is not left Haar null and satisfies $1_G \notin \operatorname{int}(A^{-1}A)$.*

It is possible to prove an analog of the Steinhaus theorem for the case of category, using the meager sets as the small sets. This result is known as Piccard's theorem or as Pettis' theorem (see e.g. [18, Theorem 2.9.6] and [17, Theorem 9.9]). The paper [16] generalizes this result to abelian, not necessarily locally compact Polish groups by proving the analog of the Steinhaus theorem that uses the Haar meager sets as small sets. We state this theorem without proof; the proof is similar to that of Theorem 5.2.1 and can be found in [16].

Theorem 5.2.7. *Let G be an abelian Polish group. If $A \subseteq G$ is a Borel set that is not Haar meager, then AA^{-1} is a neighborhood of 1_G.*

5.3 The countable chain condition

The countable chain condition (often abbreviated as kc) is a well-known property of partially ordered sets or structures with associated partially ordered sets. A partially ordered set $(P, \leq)$ is said to satisfy the countable chain condition if every strong antichain in P is countable. (A set $A \subseteq P$ is a strong antichain if $\forall x, y \in A : (x \neq y \Rightarrow \nexists z \in P : (z \leq x \text{ and } z \leq y))$.) This is called countable "chain" condition because in some particular cases this condition happens to be equivalent to a condition about lengths of certain chains.

We will apply the countable chain condition to notions of smallness in the following sense:

Definition 5.3.1. Suppose that X is a set and $\mathcal{S} \subseteq \mathcal{A} \subseteq \mathcal{P}(X)$. We say that $\mathcal{S}$ *has the countable chain condition in* $\mathcal{A}$ if there is no uncountable system $\mathcal{U} \subseteq \mathcal{A} \setminus \mathcal{S}$ such that $U \cap V \in \mathcal{S}$ for any two distinct $U, V \in \mathcal{U}$.

In our cases $\mathcal{A}$ will be a σ-algebra and $\mathcal{S}$ will be the σ-ideal of "small" sets. Notice that $\mathcal{S}$ has the countable chain condition in $\mathcal{A}$ if and only if the partially ordered set $(\mathcal{A} \setminus \mathcal{S}, \subseteq)$ satisfies the countable chain condition. Also notice that if $\tilde{\mathcal{S}} \subseteq \mathcal{S}$ and $\tilde{\mathcal{A}} \supseteq \mathcal{A}$, then "$\tilde{\mathcal{S}}$ has the countable chain condition in $\tilde{\mathcal{A}}$" is a stronger statement than "$\mathcal{S}$ has the countable chain condition in $\mathcal{A}$".

We will generalize the following two classical results (these are stated as [17, Exercise 17.2] and [17, Exercise 8.31]).

Proposition 5.3.2. *If μ is a σ-finite measure, the σ-ideal of sets with μ-measure zero has the countable chain condition in the σ-algebra of μ-measurable sets.*

Proposition 5.3.3. *In a second countable Baire space, the σ-ideal of meager sets has the countable chain condition in the σ-algebra of sets with the Baire property.*

The theorems in subsection 3.3 state that if G is locally compact, then $\mathcal{N} = \mathcal{HN} = \mathcal{GHN}$ (i.e. of Haar measure zero $\Leftrightarrow$ Haar null $\Leftrightarrow$ generalized Haar null) and $\mathcal{M} = \mathcal{HM}$ (i.e. meager $\Leftrightarrow$ Haar meager). The special case of Proposition 5.3.2 where $\mu = \lambda$ for a left Haar measure λ on G means that the σ-ideal $\mathcal{HN} = \mathcal{GNH}$ has the countable chain condition in the σ-algebra of λ-measurable sets. As every universally measurable set is λ-measurable, this clearly implies that $\mathcal{HN} = \mathcal{GHN}$ has the countable chain condition in the σ-algebra of universally measurable sets. Analogously, Proposition 5.3.3 means that in a locally compact Polish group G the σ-ideal $\mathcal{HM}$ has the countable chain condition in the σ-algebra of sets with the Baire property.

For the case of measure Christensen asked in [3, Problem 2] whether is this true in the non-locally-compact case (the paper [3] considers only abelian groups, but the problem is interesting in general).

The following simple example shows that the answer for this is negative in the group $\mathbb{Z}^\omega$ (a variant of this is stated in [9, Proposition 1]). It also answers the analogous question in the case of category.

Example 5.3.4. *For $A \subseteq \omega$ let*

$$S(A) = \{s \in \mathbb{Z}^\omega : s_n \geq 0 \text{ if } n \in A \text{ and } s_n < 0 \text{ if } n \notin A\}.$$

Then the system $\{S(A) : A \in \mathcal{P}(\omega)\}$ consists of continuum many pairwise disjoint Borel (in fact, closed) subsets of $\mathbb{Z}^\omega$ which are neither generalized Haar null nor Haar meager.

Proof. It is clear that $S(A)$ is closed for every $A \subseteq \omega$. If A and B are two different subsets of ω, then some $n \in \omega$ satisfies for example $n \in A \setminus B$ and thus $\forall s \in S(A) : s_n \geq 0$, but $\forall s \in S(B) : s_n < 0$.

Finally, for every $A \subseteq \omega$ the set $S(A)$ contains a translate of every compact subset $C \subseteq \mathbb{Z}^\omega$, because if we define $t^{(C)} \in \mathbb{Z}^\omega$ by

$$t_n^{(C)} = \begin{cases} \min\{c_n : c \in C\} & \text{if } n \in A, \\ -1 - \max\{c_n : c \in C\} & \text{if } n \notin A, \end{cases}$$

then clearly $C + t^{(C)} \subseteq A$. Applying Lemma 6.5.1 concludes our proof. $\square$

In [23] Solecki showed that the situation is the same in all non-locally-compact groups that admit a two-sided invariant metric. This is a corollary of Theorem 3.3.15, which we already stated without proof. As we will use Lemma 6.5.1 again, this also answers the question in the case of category.

Example 5.3.5. *Suppose that G is non-locally-compact and admits a two-sided invariant metric. Then none of $\mathcal{HN}$, $\mathcal{GHN}$ and $\mathcal{HM}$ has the countable chain condition in $\mathcal{B}(G)$.*

Proof. By Theorem 3.3.15 there exists a closed set $F \subseteq G$ and a continuous function $\varphi : F \to 2^\omega$ such that for any $x \in 2^\omega$ and any compact set $C \subseteq G$ there is a $g \in G$ with $gC \subseteq \varphi^{-1}(\{x\})$. Then the system $\{\varphi^{-1}(\{x\}) : x \in 2^\omega\}$ consists of continuum many pairwise disjoint closed sets and they all satisfy the requirements of Lemma 6.5.1, hence they are neither generalized Haar null nor Haar meager. $\square$

6 Common techniques

In this section we introduce five techniques, which are frequently useful in practice. The first four of these can be used to show that a set is small, and the last one can be used to show that a set is not small. Note that some of the results from the earlier sections (for example, the basic properties in subsection 3.2 or the equivalent definitions in subsection 4.1) are also very useful in practice.

6.1 Probes

Probes are a very basic technique for constructing witness measures. The core of this idea is fairly straightforward and the only surprising thing about probes is the fact that despite their simplicity they are often useful.

Probes were introduced with the following definition in [15] (this paper examines the Haar null sets in completely metrizable linear space).

Definition 6.1.1. Suppose that V is an (infinite-dimensional) completely metrizable linear space. A finite dimensional subspace $P \subseteq V$ is called a *probe for* a set $A \subseteq V$ if the Lebesgue measure on P witnesses that A is Haar null.

There is nothing "magical" about this definition, but it is easy to handle these simple witness measures in the calculations and if there is a probe for a set $A \supseteq V$, then by definition A is Haar null. In arbitrary Polish groups it is easy to generalize this idea and consider a witness measure which is the "natural" measure supported on a small and well-understood subgroup or subset. If the considered set and the candidate for the probe are not too contrived, then it is often easy to see that it is indeed a probe.

For Haar meager sets the analogue of this is basically proving that the set is strongly Haar meager (see subsection 4.6) and this is witnessed by a "naturally chosen" set.

The proof of the following example illustrates the usage of probes. Several more examples which demonstrate the usage of probes are collected in the paper [15].

Example 6.1.2. *In the Polish group* $(\mathcal{C}[0,1],+)$ *of continuous real-valued functions on* $[0,1]$*, the set* $M = \{f \in \mathcal{C}[0,1] : f \text{ is monotone on some interval}\}$ *is Haar null.*

Proof. For a proper interval $I \subseteq [0,1]$ let

$$M(I) = \{f \in \mathcal{C}[0,1] : f \text{ is monotone on } I\}.$$

As the Haar null sets form a σ-ideal and

$$M = \bigcup \{M([q,r]) : 0 \leq q < r \leq 1 \text{ and } q, r \in \mathbb{Q}\},$$

it is enough to show that $M(I)$ is Haar null for every proper interval $I \subseteq [0,1]$. It is straightforward to check that $M(I)$ is Borel (in fact, closed).

Fix a function $\varphi \in \mathcal{C}[0,1]$ such that its restriction to I is not of bounded variation. We show that the one-dimensional subspace $\mathbb{R}\varphi = \{c \cdot \varphi : c \in \mathbb{R}\}$ is a probe for $M(I)$, that is, the measure μ on $\mathcal{C}[0,1]$ that is defined by

$$\mu(X) = \lambda(\{c \in \mathbb{R} : c \cdot \varphi \in X\})$$

(where λ is the Lebesgue measure on $\mathbb{R}$) is a witness measure for $M(I)$.

We have to prove that $\mu(M(I) + f) = 0$ for every $f \in \mathcal{C}[0,1]$. By definition

$$\mu(M(I) + f) = \lambda(\{c \in \mathbb{R} : c \cdot \varphi \in M(I) + f\})$$

and here the set $S_f = \{c \in \mathbb{R} : c \cdot \varphi \in M(I) + f\}$ has at most one element, because if $c_1, c_2 \in S_f$, then $c_1 \cdot \varphi = m_1 + f$ and $c_2 \cdot \varphi = m_2 + f$ for some functions $m_1, m_2 \in M(I)$ that are monotone on I and hence $(c_1 - c_2) \cdot \varphi = m_1 - m_2$ is of bounded variation when restricted to I, but this is only possible if $c_1 = c_2$. Thus $\lambda(S_f) = \mu(M(I) + f) = 0$ and this shows that $M(I)$ is Haar null. □

6.2 Application of the Steinhaus theorem

Sometimes the application of one of the results in subsection 5.2 can yield very short proofs for the Haar nullness and Haar meagerness of certain sets. Unfortunately, this technique is restricted in the sense that "good" analogs of the Steinhaus theorem are known only in special groups.

We illustrate this technique by proving Example 6.1.2 again. This new proof is not as elementary as the one using probes, but also proves that the set under consideration is Haar meager.

Example 6.2.1. *In the Polish group* $(\mathcal{C}[0,1], +)$ *of continuous real-valued functions on* $[0,1]$*, the set* $M = \{f \in \mathcal{C}[0,1] : f \text{ is monotone on some interval}\}$ *is Haar null and Haar meager.*

Proof. As we noted in the proof using probes, if $I \subseteq [0,1]$ is a proper interval, then the set

$$M(I) = \{f \in \mathcal{C}[0,1] : f \text{ is monotone on } I\}$$

is Borel and it is enough to see that this set is Haar null and Haar meager for every proper interval $I \subseteq [0, 1]$ (we use the fact that the Haar meager sets also form a σ-ideal).

Assume for contradiction that there exists a proper interval $I \subseteq [0, 1]$ such that $M(I)$ is either not Haar null or not Haar meager. If $M(I)$ is not Haar null, then Corollary 5.2.2 implies that $M(I) - M(I)$ is a neighborhood of the constant 0 function. Similarly, if $M(I)$ is not Haar meager, then Theorem 5.2.7 implies that $M(I) - M(I)$ is a neighborhood of the constant 0 function.

It is well-known and easy to prove that the difference of two monotone functions is a function of bounded variation. But $M(I)-M(I)$ is a neighborhood of the constant 0 function and every $f \in M(I)-M(I)$ satisfies that the restriction $f|_I$ is of bounded variation. This is clearly a contradiction, as it is easy to construct a continuous function f such that $\|f\|_\infty$ is small and $f|_I$ is not of bounded variation. □

The proof of Proposition 6.3.2 is another example of this technique.

6.3 Compact sets are small

This technique is based on the idea that in non-locally-compact groups the compact sets are "small" in the sense that they have empty interior, and in some groups we can use this to prove other kinds of smallness. The question whether all compact sets are Haar null and/or Haar meager in some Polish group is interesting on its own right, and is not yet answered in general.

Several known results state that in groups satisfying certain properties the compact sets are Haar null or Haar meager, and these positive answers are also useful as lemmas. As both the system of Haar null sets and the system of Haar meager sets are σ-ideals, these results will also mean that K_σ sets (countable unions of compact sets) are Haar null and/or Haar meager in these groups.

The following theorem is [9, Proposition 8], one of the earliest results in this topic.

Theorem 6.3.1. *Let G be a non-locally-compact Polish group admitting a two-sided invariant metric. Then every compact subset of G is Haar null.*

Proof. We will use Theorem 4.1.5 to prove this result; this proof is not essentially different from the proof in [9], but separates the ideas specific to compact sets (this proof) and the construction of a limit measure (the proof of Theorem 4.1.5).

Fix a two-sided invariant metric d on G and let $C \subseteq G$ be an arbitrary compact subset. We need to prove that for every $\delta > 0$ and neighborhood U of 1_G there exists a Borel probability measure μ on G such that the support of μ is contained in U and $\mu(gCh) < \delta$ for every $g, h \in G$.

Fix $\delta > 0$ and a neighborhood U of 1_G. We may assume that U is open. As G is non-locally-compact, the open set U is not totally bounded, hence there exists an $\varepsilon > 0$ such that U cannot be covered by finitely many open balls of radius 2ε.

As C is compact, hence totally bounded, there exists a $N \in \omega$ such that C can be covered by N open balls of radius ε. This means that if $X \subseteq C$ and every $x, x' \in X$ satisfies $x \neq x' \Rightarrow d(x, x') \geq 2\varepsilon$, then $|X| \leq N$ (because each of the N open balls of radius ε covering C may contain at most one element of X). Using the invariance of d this yields that for every $g, h \in G$ if $X \subseteq gCh$ and every $x, x' \in X$ satisfies $x \neq x' \Rightarrow d(x, x') \geq 2\varepsilon$, then $|X| \leq N$.

As U cannot be covered by finitely many open balls of radius 2ε, it is possible to choose a sequence $(u_n)_{n\in\omega}$ such that $u_n \in U$ and $u_n \notin \bigcup_{i=0}^{n-1} B(u_i, 2\varepsilon)$ for every $n \in \omega$. Choose an integer M that is larger than $\frac{N}{\delta}$ and let $Y = \{u_n : 0 \leq n < M\}$. Let μ be the measure on Y which assigns measure $\frac{1}{M}$ to every point in Y. If $g, h \in G$ are arbitrary, then $\mu(gCh) = \frac{|gCh \cap Y|}{M}$ and here every $y, y' \in gCh \cap Y$ satisfies $y \neq y' \Rightarrow d(y, y') \geq 2\varepsilon$, and hence $|gCh \cap Y| \leq N$, and thus $\mu(gCh) \leq \frac{N}{M} < \delta$. □

The following result works in all non-locally-compact Polish groups, but only proves that the compact sets are right Haar null (this is a weaker notion than Haar nullness, see subsection 4.4 for the definition and properties).

Proposition 6.3.2. *Let G be a non-locally-compact Polish group. Then every compact subset of G is right Haar null.*

Proof. Suppose that $C \subseteq G$ is compact but not right Haar null. Applying Theorem 5.2.1 yields that there exist a $n \in \omega$ and $h_0, h_1, \dots, h_{n-1} \in G$ such that

$$h_0 C C^{-1} h_0^{-1} \cup h_1 C C^{-1} h_1^{-1} \cup \ldots \cup h_{n-1} C C^{-1} h_{n-1}^{-1}$$

is a neighborhood of 1_G. But CC^{-1} is compact (as it is the image of $C \times C$ under the continuous map $(x, y) \mapsto xy^{-1}$), thus its conjugates are also compact, and the union of finitely many compact sets is also compact, and this is a contradiction, because a neighborhood cannot be compact in G. □

The paper [8] investigates the question in the case of Haar meager sets, we state the main results without proofs. This article introduces the *finite translation property* with the following definition:

Definition 6.3.3. A set $A \subseteq G$ is said to have the *finite translation property* if for every open set $\emptyset \neq U \subseteq G$ there exists a finite set $M \subseteq U$ such that for every $g, h \in G$ we have $gMh \nsubseteq A$.

The first part of the proof is the following result which allows using this property to prove that a set is strongly Haar meager (this is a stronger notion than Haar meagerness, see subsection 4.6 for the definition and properties). The role of this result is roughly similar to the role of Theorem 4.1.5 in the case of measure; its proof involves a relatively complex recursive construction.

Theorem 6.3.4. *If an F_σ set $A \subseteq G$ has the finite translation property, then A is strongly Haar meager.*

The second part is showing that the compact sets have the finite translation property when there is a two-sided invariant metric; the proof is relatively simple and very similar to the one used in Theorem 6.3.1.

Theorem 6.3.5. *Let G be a non-locally-compact Polish group admitting a two-sided invariant metric. Then every compact subset of G has the finite translation property.*

These results yield the analogue of Theorem 6.3.1. Note that in the case when G is abelian, it is also possible to prove this as a corollary of Theorem 5.2.7, using the method of the proof of Proposition 6.3.2.

Corollary 6.3.6. *Let G be a non-locally-compact Polish group admitting a two-sided invariant metric. Then every compact subset of G is (strongly) Haar meager.*

In addition to these, [8] also shows that compact sets have the finite translation property in the group S_∞ of all permutations of a countably infinite set.

We illustrate the usage of this technique with a simple example.

Example 6.3.7. *In the non-locally-compact Polish group $(\mathbb{Z}^\omega, +)$ there are subsets $A, B \subseteq \mathbb{Z}^\omega$ such that they are neither Haar null nor Haar meager, but for every $x \in \mathbb{Z}^\omega$ the intersection $(A + x) \cap B$ is both Haar null and Haar meager.*

Proof. It is well-known that in $\mathbb{Z}^\omega$ a closed set C is compact if and only if

$$C \subseteq \prod_{n\in\omega} \{u_n, u_n + 1, u_n + 2, \ldots, v_n - 1, v_n\} \text{ for some } u, v \in \mathbb{Z}^\omega,$$

as sets of this kind are closed subsets in a product of compact sets and in the other direction if $C \subseteq \mathbb{Z}^\omega$ is compact, then the projections map it into compact subsets of $\mathbb{Z}$.

Let $A = \{a \in \mathbb{Z}^\omega : a_n \leq 0 \text{ for every } n \in \omega\}$ and $B = \{b \in \mathbb{Z}^\omega : b_n \geq 0 \text{ for every } n \in \omega\}$. It is easy to check that these sets satisfy the condition of Lemma 6.5.1 (the result used in the last technique) and this implies that A and B are neither Haar null nor Haar meager.

On the other hand, the set $(A + x) \cap B = \{z \in \mathbb{Z}^\omega : 0 \leq z_n \leq x_n\}$ is compact, hence Theorem 6.3.1 and Corollary 6.3.6 shows that it is Haar null and Haar meager. □

Note that this phenomenon is impossible in the locally compact case, where non-small sets have density points and if we translate a density point of one set onto a density point of the other, then the intersection will be non-small.

6.4 Random construction

This is a technique that is useful when one wants to prove that a not very small set is Haar null. (For example this does not prove Haar meagerness, hence this can work for sets that are not Haar meager.) The main idea of this technique is that a witness measure for a Haar null set $A \subseteq G$ is a Borel *probability* measure and one can use the language of probability theory (e.g random variables, conditional probabilities, stochastic processes) to construct it and prove that it is indeed a witness measure. For example, the paper [10] applies this to prove a result in S_∞, the group of all permutations of the natural numbers (endowed with the topology of pointwise convergence). We illustrate this technique by reproducing the core ideas of this proof. (The proof also contains relatively long calculations which we omit.)

Theorem 6.4.1. *In the Polish group S_∞ let X be the set of permutations that have infinitely many infinite cycles and finitely many finite cycles. Then the complement of X is Haar null.*

Proof (sketch). We will find a Borel probability measure μ on S_∞ such that $\mu(gXh) = 1$ for every $g, h \in S_\infty$. Using that X is conjugacy invariant $\mu(gXh) = \mu(gh(h^{-1}Xh)) = \mu(ghX)$ and here $\{gh : g, h \in S_\infty\} = S_\infty = \{g^{-1} : g \in S_\infty\}$, thus it is enough to show that $\mu(g^{-1}X) = 1$ for every $g \in S_\infty$.

We define the probability measure μ by describing a procedure which chooses a random permutation p with distribution μ. (This way we can describe a relatively complex measure in a way that keeps the calculations manageable.)

Fix a sequence $k_0 < k_1 < k_2 < \ldots$ of natural numbers which are large enough (the actual growth rate is used by the omitted parts of the proof). The procedure will choose values for $p(0)$, $p^{-1}(0)$, $p(1)$, $p^{-1}(1)$, $p(2)$, $p^{-1}(2)$, $\ldots$ in this order, skipping those which are already defined (e.g. if we choose $p(0) = 1$ in the first step, then the step for $p^{-1}(1)$ is omitted, as we already know that $p^{-1}(1) = 0$). When we have to choose a value for $p(n)$, we choose randomly a natural number $n' < k_n$ which is still available as an image (that is, $n' \geq n$ and n' is not among the already determined values $p(0), p(1), \ldots, p(n-1)$); we assign equal probabilities to each of these choices. Similarly, when we have to choose a value for $p^{-1}(n)$, we choose randomly a natural number $n' \leq k_n$ which is still available as a preimage (that is, $n' > n$ and n' is not among the already determined values $p^{-1}(0)$, $p^{-1}(1)$, ..., $p^{-1}(n-1)$); we assign equal probabilities to each of these choices again. When we are finished with these steps, the resulting object p is clearly a well-defined permutation, as every $n \in \omega$ has exactly one image and exactly one preimage assigned to it. We can assume that k_n is large enough to satisfy $k_n > 2n + 1$ and this guarantees that we never "run out" of choices.

We say that p_0 is a *possible partial result*, if it can arise after finitely many steps of this process. Relatively long combinatorial arguments show that the following claim is true:

Claim 6.4.2. *Assume that p_0 is a possible partial result, $g \in S_\infty$ is an arbitrary element and M is a natural number. Then there is a natural number N such that the conditional probability with respect to μ, under the condition of extending p_0, of the event that the permutation p chosen by our process will be such that gp has no finite cycles including a number greater than N and no two of the numbers $N+1, \ldots, N+M$ are in the same cycle of gp is at least $\frac{1}{2}$.*

Using this claim, it is possible to show the following claim by induction on i:

Claim 6.4.3. *Assume that (as in the previous claim) p_0 is a possible partial result, $g \in S_\infty$ is an arbitrary element and M is a natural number. Then the conditional probability (with respect to μ, under the condition of extending p_0) of the event that the permutation p chosen by our process satisfies that gp has only finitely many finite cycles and at least M infinite cycles is at least $1 - 2^{-i}$.*

Applying this second claim for every $i \in \omega$ in the special case when p_0 is the empty partial permutation yields that the (unconditional) probability (with respect to μ) of the event that the permutation p chosen by our process satisfies that gp has only finitely many finite cycles and at least M infinite cycles is 1. Since this is true for every $M \in \omega$, the permutation gp has infinitely many infinite cycles with μ-probability 1. This shows that $\mu(g^{-1}X) = 1$ for the arbitrary permutation g, so we are done. □

This set X is the union of countably many conjugacy classes of permutations, one for each finite list of sizes for the finite cycles in the permutation; the paper [10] also shows that none of these conjugacy classes are Haar null.

6.5 Sets containing translates of all compact sets

Proving that a set is not Haar null from the definitions requires showing that *all* Borel probability measures fail to witness that it is Haar null, which is frequently harder than just showing *one* measure witnesses that the set is Haar null. The situation is similar for Haar meager sets, where even the choice of the domain of the witness function is not straightforward, although the equivalence (1) $\Leftrightarrow$ (2) in Theorem 4.1.9 can be used to eliminate this extra choice. Fortunately, in many cases the following simple sufficient condition is enough to show that a set is not Haar null (in fact, not even generalized Haar null) and not Haar meager.

Lemma 6.5.1. *Suppose that a set $A \subseteq G$ satisfies that for every compact set $C \subseteq G$ there are $g, h \in G$ such that $gCh \subseteq A$. Then A is neither generalized Haar null nor Haar meager.*

Proof. This lemma is stated e.g. as [22, Lemma 2.1], but this reasoning is also frequently used without being stated separately.

If A were generalized Haar null, then by Theorem 4.1.4 there would be an universally measurable set $B \supseteq A$ and a Borel probability measure μ with compact support $C \subseteq G$ such that $\mu(g'Bh') = 0$ for every $g', h' \in G$, but there are $g, h \in G$ such that $gCh \subseteq A \subseteq B$ and thus $\mu(g^{-1}Bh^{-1}) \geq \mu(C) = 1$, a contradiction.

Similarly, if A would be Haar meager, then there would be a Borel set $B \supseteq A$, a (nonempty) compact metric space K and a continuous function $f : K \to G$ such that $f^{-1}(g'Bh')$ is meager in K for every $g', h' \in G$, but there are $g, h \in G$ such that $gf(K)h \subseteq A \subseteq B$ and thus $f^{-1}(g^{-1}Bh^{-1}) \supseteq f^{-1}(f(K)) = K$ is not meager in K, a contradiction. $\square$

References

[1] J. M. Borwein, W. B. Moors, Null sets and essentially smooth Lipschitz functions. *SIAM J. Optim.* 8 (1998), no. 2, 309–323.

[2] J. P. R. Christensen, Borel Structures in Groups and Semigroups. *Math. Scand.* 28 (1971), 124–128.

[3] J. P. R. Christensen, On sets of Haar measure zero in abelian Polish groups. *Israel J. Math.* 13 (1972) 255–260 (1973).

[4] M. P. Cohen, R. R. Kallman, Openly haar null sets and conjugacy in Polish groups. To appear in *Israel J. Math.*

[5] U. B. Darji, On Haar meager sets, *Topology Appl.* 160 (18) (2013) 2396–2400.

[6] R. O. Davies, Covering the plane with denumerably many curves, *J. London Math. Soc.* 38 (1963), 433–438.

[7] M. Doležal, M. Rmoutil, B. Vejnar, V. Vlasák, Haar meager sets revisited. *J. Math. Anal. Appl.* 440 (2016), no. 2, 922–939.

[8] M. Doležal, V. Vlasák, Haar meager sets, their hulls, and relationship to compact sets. Available at `arXiv:1603.09510`.

[9] R. Dougherty, Examples of non-shy sets. *Fund. Math.* 144 (1994), 73–88.

[10] R. Dougherty, J. Mycielski, The prevalence of permutations with infinite cycles. *Fund. Math.* 144 (1994), no. 1, 89–94.

[11] M. Elekes, Z. Vidnyánszky, Haar null sets without G_δ hulls. *Israel J. Math.* 209 (1) (2015) 199–214.

[12] M. Elekes, Z. Vidnyánszky, Naively Haar null sets in Polish groups. To appear in *J. Math. Anal. Appl.*

[13] E. Hewitt, K. A. Ross, Abstract harmonic analysis. Vol. I. Structure of topological groups, integration theory, group representations. Second edition. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 115. *Springer-Verlag, Berlin-New York*, 1979.

[14] G. Högnäs, A. Mukherjea, Probability measures on semigroups. Convolution products, random walks, and random matrices. Second edition. Probability and its Applications (New York). *Springer, New York*, 2011.

[15] B. R. Hunt, T. Sauer, J. A. Yorke, Prevalence: A translation-invariant "almost every" on infinite-dimensional spaces. *Bull. Amer. Math. Soc. (N.S.)* 27 (1992), 217–238.

[16] E. Jabłońska, Some analogies between Haar meager sets and Haar null sets in abelian Polish groups. *J. Math. Anal. Appl.* 421 (2015), no. 2, 1479–1486.

[17] A. S. Kechris, Classical descriptive set theory. Graduate Texts in Mathematics, 156. *Springer-Verlag, New York*, 1995

[18] M. Kuczma, An Introduction to the Theory of Functional Equations and Inequalities. Cauchy's Equation and Jensen's Inequality. 2nd edition. *Birkhäuser Verlag, Basel*, 2009, edited by A. Gilányi.

[19] E. Matoušková, The Banach-Saks property and Haar null sets. *Comment. Math. Univ. Carolin.* 39 (1998), no. 1, 71–80.

[20] J. Mycielski, Some unsolved problems on the prevalence of ergodicity, instability, and algebraic independence. *Ulam Quarterly*, 1 (1992), no. 3, 30–37.

[21] C. Rosendal, Automatic continuity of group homomorphisms. *Bull. Symbolic Logic* 15 (2009), no. 2, 184–214.

[22] H. Shi, B. S. Thomson, Haar null sets in the space of automorphisms of $[0,1]$. *Real Anal. Exchange* 24 (1998/99), 337–350.

[23] S. Solecki, On Haar null sets. *Fund. Math.* 149 (1996), no. 3, 205–210.

[24] S. Solecki, Haar null and non-dominating sets. *Fund. Math.* 170 (2001), 197–217.

[25] S. Solecki, Size of subsets of groups and Haar null sets. *Geom. Funct. Anal.* 15 (2005), no. 1, 246–273.

[26] S. Solecki, Amenability, free subgroups, and Haar null sets in non-locally compact groups. *Proc. London Math. Soc. (3)* 93 (2006), no. 3, 693–722.