# RATIONAL BLOW-DOWN ALONG A FOUR BRANCHED PLUMBING TREE AND AN EXOTIC $\mathbb{CP}^2\sharp 9\overline{\mathbb{CP}^2}$

Master's thesis


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# Introduction

Classification of manifolds is one of the main problems in topology. For example, the Poincare conjecture and its generalisations can be seen as the problem of classifying homotopy spheres. By the works of Smale [29], Freedman [8] and Perelman [27], homotopy spheres in each dimension greater than or equal to 3 only have one topological type. It is a classical result that the same statement holds true in dimensions 1 and 2. It was very surprising when Milnor [19] showed that there are more than one different smooth structures on the 7-sphere. In what follows, for a given topological manifold $M$ together with a standard smooth structure on it, an exotic $M$ is a manifold which is homeomorphic to $M$, but carries a smooth structure which is non-diffeomorphic to the standard one. Exotic manifolds do not exist in dimensions less than 4 (see [21]). Exotic spheres were classified in dimensions greater than 4 by the work of Milnor and Kervaire [20]. The existence of exotic 4-spheres is a major open problem.

4-dimensional topology experienced a revolution in early 1980s by the works of Freedman and Donaldson. On the one hand, Freedman's work [8] provides a complete classification for a large class of topological 4-manifolds, as well as examples of many topological 4-manifolds which do not carry any smooth structure. On the other hand, Donaldson's result [6] provides smooth invariants which paved the way for the discovery of many surprising results concerning smooth structures in dimension 4. For instance, many exotic $\mathbb{R}^4$'s were found by Freedman and Taylor [9] while it is known that the smooth structures on $\mathbb{R}^n$'s are unique for $n \neq 4$ (see [21], [30]). It was later shown by Taubes [33] that there are continuum many pairwise non-diffeomorphic smooth structures on $\mathbb{R}^4$. In [6] , Donaldson found an example of an exotic $\mathbb{CP}^2\sharp 9\overline{\mathbb{CP}^2}$ as a complex surface. Later, Friedman and Morgan [10] showed that there are infinitely many pairwise non-diffeomorphic smooth structures on $\mathbb{CP}^2\sharp 9\overline{\mathbb{CP}^2}$. Also by using Donaldson's theory, Kotschick [16] showed that a certain complex surface called the Barlow surface is an exotic $\mathbb{CP}^2\sharp 8\overline{\mathbb{CP}^2}$.

In 1994, Witten [34] introduced new smooth invariants for smooth 4-manifolds called the Seiberg-Witten invariants (see Section 2 for the definition of Seiberg-Witten invariants) which is easier to work with. In [7], Fintushel and Stern introduced a topological procedure called the rational blow-down (see Section 1.5 for an explanation of the rational blow-down procedure) and investigated the change of Seiberg-Witten invariants of smooth 4-manifolds under this procedure. Jongil Park later extended the rational blow-down technique, and constructed the first example of an exotic $\mathbb{CP}^2\sharp 7\overline{\mathbb{CP}^2}$ in [25]. Following this line, Stipsicz and Szabó [31], and later Jongil Park, Stipsicz and Szabó [26] constructed exotic $\mathbb{CP}^2\sharp 6\overline{\mathbb{CP}^2}$'s and exotic $\mathbb{CP}^2\sharp 5\overline{\mathbb{CP}^2}$'s respectively. Note that other topological methods also have been applied to construct exotic 4-manifolds of the form $\mathbb{CP}^2\sharp k\overline{\mathbb{CP}^2}, 2 \leq k \leq 4$ by Akhmedov and Doug Park [1], [2]. The central remaining problem in this direction is to construct exotic structures on $\mathbb{CP}^2\sharp\overline{\mathbb{CP}^2}, \mathbb{CP}^2$ and $S^2 \times S^2$, if they exist. Also one might hope to find an approach to the smooth Poincare conjecture, i.e constructing exotic structures on $S^4$.

In this thesis, we only concern about the rational blow-down technique. In [7], [25], [26] and [31], the blowing-down configurations are all linear plumbing trees (see Section 1.3 for the definition of plumbing trees). The question of which plumbing trees can be used in the rational blow-down procedure was investigated extensively by Stipsicz, Szabó and Wahl in [32], and by Bhupal and Stipsicz in [4]. A list of linear, three branched and four branched plumbing trees is provided. The problem then is to realise plumbing manifolds associated to these plumbing trees as embedded submanifolds, and perform the rational blow-down procedure to construct new exotic smooth 4-manifolds. In [18], some three branched plumbing trees were used by Michalogiorgaki to construct new exotic 4-manifolds $\mathbb{CP}^2 \sharp k \overline{\mathbb{CP}^2}, 6 \leq k \leq 9$. No previously known example of exotic 4-manifolds obtained by rational blowing-down along four branched plumbing trees has been constructed. In this thesis, we will construct the first example of this kind.

In general, constructing an exotic structure on a given 4-manifold with a standard smooth structure requires two steps. First, one needs to construct a new smooth 4-manifold in such a way that it can be shown to be homeomorphic to the given 4-manifold. This usually can be done by applying Freedman's classification theorem (see Theorem 1.6). Secondly, one must compute values of smooth invariants (Donaldson's invariants, Seiberg-Witten invariants) for the given 4-manifold, as well as for the new manifold in order to show that their values are different. This is enough to conclude that the new 4-manifold carries a smooth structure which is non-diffeomorphic to the standard structure on the given manifold. The Seiberg-Witten invariants are usually used in this step as it is easier to work with, and many computational results about Seiberg-Witten invariants are known (see for example Theorem 2.50, Theorem 2.58). When $b_2^+ > 1$ (see Section 1.1 for the definition of $b_2^+, b_2^-$), the Seiberg-Witten invariants are well defined (see Section 2.6). However, when $b_2^+ = 1$, the Seiberg-Witten invariants may depend on the choice of metrics, and one has to modify its definition more carefully. If in addition to $b_2^+ = 1$, one also has $b_2^- \leq 9$, then the Seiberg-Witten invariants are again well defined.

All blowing-down configurations used in [25], [31], [26] and [18] are constructed from singular fibers of some elliptic fibrations obtained by blowing-up the base points of pencil of two certain complex projective cubic curves (see the Appendices of [31] and [18]). Stipsicz suggests that one might look at pencil of two complex projective quartic curves to find new blowing-down configurations. Following his suggestion, we construct a plumbing manifold associated to a four branched plumbing tree as an embedded submanifold in $\mathbb{CP}^2 \sharp 17 \overline{\mathbb{CP}^2}$, and perform rational blow-down procedure along it to construct a new exotic structure on the 4-manifold $\mathbb{CP}^2 \sharp 9 \overline{\mathbb{CP}^2}$. Our construction is new in the sense that this is the first example of rational blow-down along a four branched plumbing tree.

## Acknowledgements

First and foremost, I would like to thank my advisor, professor András Stipsicz for his guidance and support, for generously sharing with me his ideas, for spending an enormous amount of time on teaching the subject of this thesis to me, and for carefully reading my manuscripts. I also would like to thank professors Péter Komjáth and Katalin Gyarmati for their help on my application process to the Master program at Eötvös Loránd University. Finally, I would like to thank professor Ánh Pham Ngoc for personal help and encouragement.

# 1. Topological constructions

1.1. **Topological 4-manifolds.** Let $X$ be a closed, oriented, simply connected 4-manifold. The cohomology groups of $X$ are

$$H^0(X;\mathbb{Z}) \cong \mathbb{Z}, H^4(X;\mathbb{Z}) \cong \mathbb{Z},$$
$$H^1(X;\mathbb{Z}) = 0, H^3(X;\mathbb{Z}) = 0,$$
$$H^2(X;\mathbb{Z}) \cong \mathbb{Z}^{b_2(X)},$$

where $b_2(X)$ is the second Betti number of $X$.

The orientation on $X$ fixes the fundamental class $[X]$ in $H_4(X;\mathbb{Z})$, the ring structure of $H^*(X;\mathbb{Z})$ gives an intersection form on $H^2(X;\mathbb{Z})$.

**Definition 1.1.** *The intersection form of $X$ is*

$$Q_X : H^2(X;\mathbb{Z}) \times H^2(X;\mathbb{Z}) \longrightarrow \mathbb{Z},$$

*defined by $Q_X(a, b) = \langle a \smile b, [X]\rangle$.*

$Q_X$ is symmetric and bilinear. Poincare duality implies that $Q_X$ is also unimodular which means that the matrix of $Q_X$ has determinant $+1$ or $-1$ in any basis of $H^2(X;\mathbb{Z})$, i.e it is invertible over $\mathbb{Z}$.

Denote by $b_2^+(X)$ and $b_2^-(X)$ the numbers of positive and negative eigenvalues of $Q_X$ respectively.

**Definition 1.2.** *The signature of $X$ is $\sigma(X) = b_2^+(X) - b_2^-(X)$.*

In general, a form $Q$ over $\mathbb{Z}$ is a map $Q : \mathbb{Z}^n \times \mathbb{Z}^n \longrightarrow \mathbb{Z}$ for some positive integer $n$. We define types of forms as follows.

**Definition 1.3.** *A symmetric, bilinear, unimodular form $Q$ over $\mathbb{Z}$ is called even if $Q(v, v)$ is always even. Otherwise, $Q$ is called odd.*

**Example 1.4.** *a) The complex projective plane $\mathbb{CP}^2$ with its usual orientation has the second homology group $H_2(\mathbb{CP}^2;\mathbb{Z}) \cong \mathbb{Z}$ whose generator is usually denoted by $h$. The intersection form $Q_{\mathbb{CP}^2}$ is represented by the $1\times 1$ identity matrix: $Q_{\mathbb{CP}^2} = \langle 1\rangle$.*

*b) The complex projective plane $\overline{\mathbb{CP}^2}$ with the opposite orientation has the second homology class $H_2(\overline{\mathbb{CP}^2};\mathbb{Z}) \cong \mathbb{Z}$ whose generator is $-h$. The intersection form $Q_{\overline{\mathbb{CP}^2}}$ is represented by the $1 \times 1$ matrix: $Q_{\overline{\mathbb{CP}^2}} = \langle -1\rangle$.*

*c) The intersection form of the manifold $S^2 \times S^2$ is represented by the following matrix:*

$$Q_{S^2\times S^2} = H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

*d) We denote by $E_8$ the intersection form which is represented by the following matrix:*

$$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 \end{bmatrix}.$$

**Theorem 1.5** ([11], Theorem 1.2.1)**.** *Suppose that $Q$ is a symmetric, bilinear, unimodular form. If $Q$ is indefinite, then $Q$ is equivalent over $\mathbb{Z}$ to $b_2^+\langle 1\rangle \oplus b_2^-\langle -1\rangle$ when $Q$ is odd; and to $nH \oplus mE_8$ for some non-negative integers $n, m$, when $Q$ is even.*

We recall the following fundamental theorem about homeomorphic classification of topological 4-manifolds due to Michael Freedman.

**Theorem 1.6** (Freedman, [8])**.** *For every symmetric, bilinear, unimodular form $Q$ over $\mathbb{Z}$, there exist a closed, simply connected topological 4-manifold $X$ such that $Q_X$ and $Q$ are equivalent over $\mathbb{Z}$.*

*If $Q$ is even, then such manifold is unique up to homeomorphims. If $Q$ is odd, then there are exactly two such manifolds up to homeomorphisms, and at least one of them does not admit differentiable structure.*

If $X$ carries smooth structures, then its intersection form has more special properties.

**Theorem 1.7** (Rokhlin, [28])**.** *If a simply connected 4-manifold $X$ admits a smooth structure and $Q_X$ is even, then the signature of $X$ is divisible by 16.*

**Theorem 1.8** (Donaldson, [6])**.** *If a simply connected 4-manifold $X$ admits a smooth structure and $Q_X$ is definite, then $Q_X$ is diagonalisable over $\mathbb{Z}$.*

Combining Theorem 1.5, Theorem 1.6 and Theorem 1.8 leads to the following criterion for smooth 4-manifolds being homeomorphic.

**Theorem 1.9** (Donaldson, [6])**.** *If two closed, simply connected differentiable 4-manifolds have the same Euler characteristic, signature and parity, then they are homeomorphic.*

1.2. **The blow-up process.** Blow-up is a type of geometric transformations originated from algebraic geometry. The idea is to replace a point of an algebraic variety with the projectivised tangent cone at that point. It is usually used for resolving singular points. Here, we are only interested in the differential topology of blow-up at a point in smooth 4-manifolds. We give a brief review of blow-up process at a point in a smooth 4-manifold. A complete explanation can be found in Chapter 2 of [11].

**Definition 1.10.** *For a oriented smooth 4-manifold $X$, the connected sum $X' = X\sharp\overline{\mathbb{CP}^2}$ is called the blow-up of $X$.*

There is a map $\pi : X' \longrightarrow X$ called the projection map with the property that $\pi$ is a diffeomorphism between $X' - \mathbb{CP}^1$ and $X - P$, where $P$ is a point of $X$, and $\pi^{-1}(P) = \mathbb{CP}^1$. The sphere $\mathbb{CP}^1$ lies in the $\overline{\mathbb{CP}^2}$ summand of $X'$, and is called the exceptional sphere.

The homology class of the exceptional sphere is denoted by $e \in H_2(X';\mathbb{Z}) = H_2(X;\mathbb{Z}) \oplus H_2(\overline{\mathbb{CP}^2};\mathbb{Z})$. Note that $Q_X(e,e) = e^2 = -1$. Since $X' = X\sharp\overline{\mathbb{CP}^2}$, we have that $b_2^+(X') = b_2^+(X)$ and $b_2^-(X') = b_2^-(X) + 1$, $\chi(X') = \chi(X) + 1$ and $\sigma(X') = \sigma(X) - 1$.

**Definition 1.11.** *If we perform a blow-up at a point $P$ on a smooth surface $\Sigma \subset X$ in a smooth 4-manifold $X$, and denote the projection by $\pi : X' \longrightarrow X$, then the inverse image $\Sigma' = \pi^{-1}(\Sigma) \subset X'$ is called the total transform of $\Sigma$, and the closure $\tilde{\Sigma} = \overline{\pi^{-1}(\Sigma - \{P\})}$ is called the proper transform.*

The homology class of the proper transform is $[\tilde{\Sigma}] = [\Sigma] - e \in H_2(X';\mathbb{Z})$. If two smooth surfaces $\Sigma_1, \Sigma_2$ in $X$ intersect each other transversally at one point $P$ with the sign of intersection is positive, then after blowing-up $X$ at $P$, the proper transforms $\tilde{\Sigma}_1, \tilde{\Sigma}_2$ are disjoint in $X'$.

**Example 1.12.** *a) Blow-up at the intersection of two complex projective curves is illustrated below.*

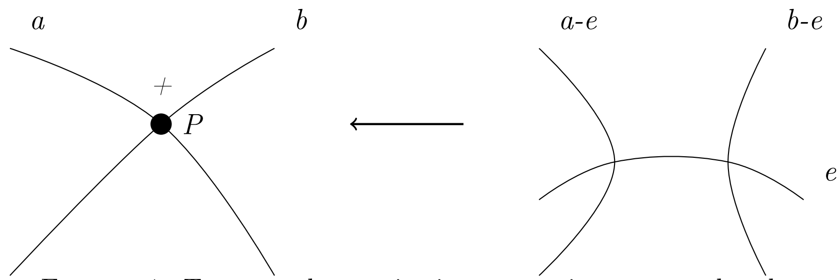


FIGURE 1. Two complex projective curves intersect each other transversally at $P$ in a smooth 4-manifold $X$. The sign of the intersection point $P$ is positive. Homology classes of the curves are $a, b \in H_2(X;\mathbb{Z})$. After blowing-up at $P$, homology classes of the two proper transforms are $a-e, b-e \in H_2(X\sharp\overline{\mathbb{CP}^2};\mathbb{Z})$, homology class of the exceptional sphere is $e \in H_2(X\sharp\overline{\mathbb{CP}^2};\mathbb{Z})$.

*b) Blow-up at the intersection of three complex projective curves is illustrated below.*

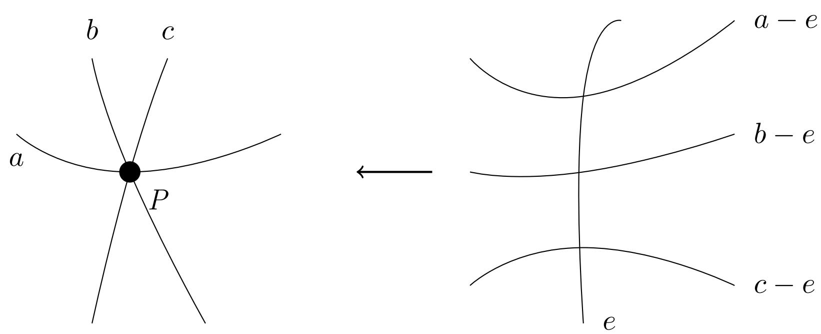


FIGURE 2. Three complex projective curves intersect each other transversally at $P$. The sign of the intersection point $P$ is positive. Homology classes of the curves are $a, b, c \in H_2(X;\mathbb{Z})$.After blowing-up at $P$, homology classes of the proper transforms are $a-e, b-e, c-e \in H_2(X\sharp\overline{\mathbb{CP}^2};\mathbb{Z})$, homology class of the exceptional sphere is $e \in H_2(X\sharp\overline{\mathbb{CP}^2};\mathbb{Z})$.

*c) Blow-up at the tangent point of a complex projective quadratic curve and a complex projective line is illustrated below.*

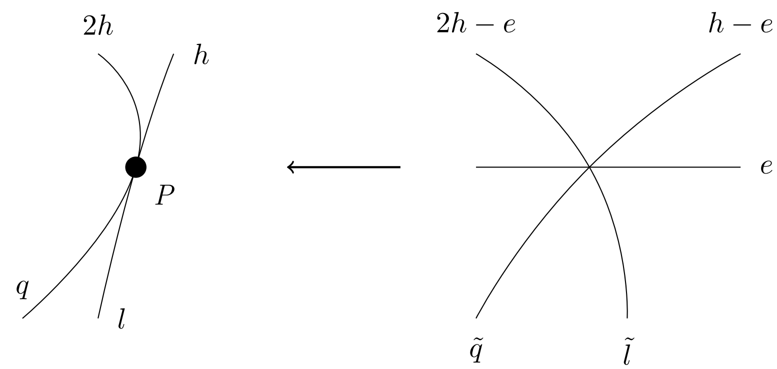


FIGURE 3. A projective quadratic curve $q$ and a projective line $l$ in $\mathbb{CP}^2$ are tangent to each other at $P$. The homology class of $q$ is $2h \in H_2(\mathbb{CP}^2;\mathbb{Z})$.The homology class of $l$ is $h \in H_2(\mathbb{CP}^2;\mathbb{Z})$. After blowing-up at $P$, the proper transforms $\tilde{q}, \tilde{l}$ and the exceptional sphere intersect each other transversally at one point. Homology classes of the proper transforms are $[\tilde{q}] = 2h - e \in H_2(\mathbb{CP}^2\sharp\overline{\mathbb{CP}^2};\mathbb{Z})$, $[\tilde{l}] = h - e \in H_2(\mathbb{CP}^2\sharp\overline{\mathbb{CP}^2};\mathbb{Z})$, homology class of the exceptional sphere is $e \in H_2(\mathbb{CP}^2\sharp\overline{\mathbb{CP}^2};\mathbb{Z})$.

1.3. **The plumbing construction.** Plumbing is a way to associate 4-manifolds to graphs with weighted vertices. Let $E_1, E_2$ be two $D^2$-disk bundles over the 2-spheres $S_1, S_2$ with Euler numbers $k_1, k_2$ respectively. On each $S_i$, we pick a disk $B_i$ and the trivialisation of $E_i$ on $B_i$ as $B_i \times D^2$. Now gluing two disk products over the $B_i$'s by the switching diffeomorphism $B_1 \times D^2 \cong D^2 \times B_2$ gives us the plumbing of $E_1$ and $E_2$. This manifold is presented by a graph with one edge and two vertices weighted by $k_1, k_2$ (see Figure 4).

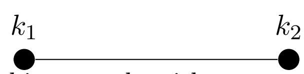


FIGURE 4. A plumbing graph with one egde and two vertices weighted by $k_1, k_2$.

By iterating this process, we can construct a 4-manifold denoted by $P\Gamma$ from any tree $\Gamma$ of finite number of vertices and edges with vertices weighted by intergers.

**Lemma 1.13.** *The 4-manifold $P\Gamma$ is simply connected for any tree $\Gamma$ of finite number of vertices and edges.*

*Proof.* As it is explaned in [11, Example 4.6.2], $P\Gamma$ admits a handlebody decomposition consisting only a 0-handle and a set of 2-handles. The fundamental group of $P\Gamma$ admits a presentation whose generators are 1-handles. Thus, this gives a presentation of fundamental group of $P\Gamma$ with empty set of generators, i.e the fundamental group of $P\Gamma$ is trivial. □

Consider the case when $\Gamma$ is a star-shaped graph with $s$ branches $l_1, l_2, \cdots, l_s$ and with the central vertex of weight $b$. Weights along the branch $l_i$ are given by the negatives of the continued fraction coefficients of $\frac{n_i}{m_i} > 1$.

By Theorem 5.1 in [24], $\partial P\Gamma$ is a Seifert manifold with Seifert invariant $\{0; (n_1, m_1), \cdots (n_s, m_s)\}$, that is $\partial P\Gamma$ can be described as an $S^1$-fibration over a 2-sphere with a finite number of singular fibers corresponding to the branches of $\Gamma$ (see [24],[15]).

The fundamental group of $\partial P\Gamma$ admits an explicit presentation as follows.

**Theorem 1.14** ([15], Theorem 6.1)**.**

$$\pi_1(\partial P\Gamma) = \langle q_1, \cdots q_s, h | [h, q_i] = 1, q_i^{n_i} h^{m_i} = 1, i = 1, \cdots, s\rangle. \tag{1.1}$$

*Moreover, $q_i$ can be represented by normal circle of the singular fiber corresponding to branch $l_i$ for each $i$.*

The Seifert manifold $\partial P\Gamma$ can be obtained by performing Dehn surgeries along $s+1$ knots $K_1, \cdots, K_{s+1}$ in $S^3$ with coefficients $n_i/m_i$. Recall that for a knot $K$ in $S^3$, a Dehn surgery on $K$ with coefficient $p/q$ is an operation done by removing an open neighbourhood of $K$ from $S^3$, and then gluing back a solid torus $S^1 \times D^2$ by a diffeomorphism of the boundary tori in such a way that the meridian curve on the solid torus goes to the $(p, q)$-curve on the boundary of the knot exterior in $M$. This is explaned in [15, Section 1].

The first homology group of $\partial P\Gamma$ has the following presentation.

**Proposition 1.15** ([11], Proposition 5.3.11)**.**

$$H_1(\partial P\Gamma; \mathbb{Z}) = \{\mu_1, \cdots, \mu_s | n_i\mu_i + m_i \sum_{j \neq i} lk(K_i, K_j)\mu_j = 0\}, \tag{1.2}$$

*where $\mu_i$'s are meridian curves and $lk(K_i, K_j)$ is the linking number between $K_i, K_j$.*

For each star-shaped plumbing graph $\Gamma$, its dual graph is defined as follows.

**Definition 1.16.** *The dual graph $\Gamma'$ of a star-shaped plumbing graphd $\Gamma$ is defined to be the star-shaped graph with $s$ branches $l'_1, l'_2, \cdots, l'_s$ and with the central vertex of weight $b-s$. Weights along the branch $l'_i$ are given by the negatives of the continued fraction coefficients of $\frac{n_i}{n_i - m_i}$.*

**Lemma 1.17** ([32],Section 8.1)**.** *The boundary of $P\Gamma$ is orientation reserving diffeomorphic to the boundary of $P\Gamma'$: $\partial P\Gamma \cong -\partial P\Gamma'$. In addition, the union $P\Gamma \cup P\Gamma'$ is diffeomorphic to $\mathbb{CP}^2 \sharp n\overline{\mathbb{CP}^2}$ for some positive integer $n$.*

A proof of Lemma 1.17 can be found in [4, Lemma 2.8].

**Example 1.18.** *Of particular interest for us is the plumbing graph $\mathcal{A}_2$ defined in* [32, Example 8.12]. *The plumbing tree $\mathcal{A}_2$ and its dual graph $\mathcal{A}'_2$ are given below.*

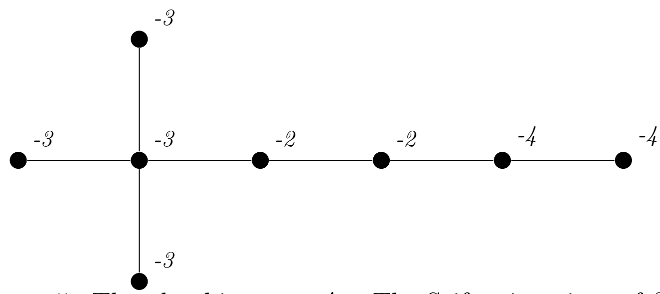


Figure 5. The plumbing tree $\mathcal{A}_2$ : The Seifert invariant of $\partial P\mathcal{A}_2$ is $\{(0;(3,1),(3,1),(3,1),(37,26)\}$. Here, we calculate the continued fraction $[2;2,4,4]$ to obtain $(37,26)$.

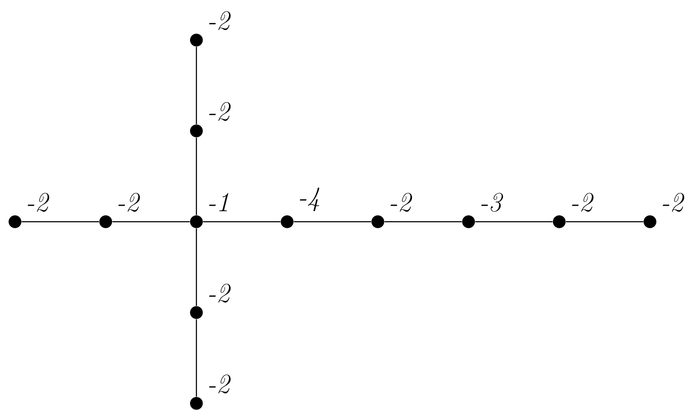


Figure 6. The dual graph $\mathcal{A}_2'$ of the plumbing tree $\mathcal{A}_2$.

1.4. **A rational homology ball.** We give a brief discussion about the fact that the 3-manifold $\partial P\mathcal{A}_2$ bounds a four dimensional rational homology ball $B$.

**Theorem 1.19** ([32], Theorem 1.8, Example 8.12)**.** *The 3-manifold $\partial P\mathcal{A}_2$ bounds a four dimensional rational homology ball $B$, that is, there exists a smooth 4-manifold $B$ such that $\partial B$ is diffeomorphic to $\partial P\mathcal{A}_2$, and the rational homology groups of $B$ are the same as the rational homology groups of the 4-dimensional disk $D^4$: $H_i(B;\mathbb{Q}) \cong H_i(D^4;\mathbb{Q})$ for any non-negative integer $i$.*

In [4, Section 4.3, Figure 13], Bhupal and Stipsicz give an explicit construction of a rational homology ball $B$ as follows. By blowing-up intersection points of a special configuration of projective curves in $\mathbb{CP}^2$ as in [4, Figure 13], one can obtain an embedding of $P\mathcal{A}_2'$ into $\mathbb{CP}^2\sharp 11\overline{\mathbb{CP}^2}$. Now take $B$ to be the complement $B = \mathbb{CP}^2\sharp 11\overline{\mathbb{CP}^2} - \mathrm{int}(P\mathcal{A}_2')$. Notice that the second homology groups of $\mathbb{CP}^2\sharp 11\overline{\mathbb{CP}^2}$ and $P\mathcal{A}_2'$ both have rank 12. Their first and third homology groups have rank 0. Both of them are simply connected oriented 4-manifolds. Thus, a homological calculation by Mayer-Vietoris sequence shows that $B$ is a rational homology ball. From this construction we can see that $\partial B \cong -\partial P\mathcal{A}_2'$. Since $\partial P\mathcal{A}_2 \cong -\partial P\mathcal{A}_2'$, we conclude that $\partial B \cong \partial P\mathcal{A}_2$, i.e $\partial P\mathcal{A}_2$ bounds the rational homology ball $B$.

Moreover, the generators $h, e_1, \cdots, e_{11}$ of $H_2(\mathbb{CP}^2 \sharp 11 \overline{\mathbb{CP}^2}; \mathbb{Z})$ can be arranged so that the homology classes of embedding spheres at vertices of $\mathcal{A}'_2$ are given as follows.

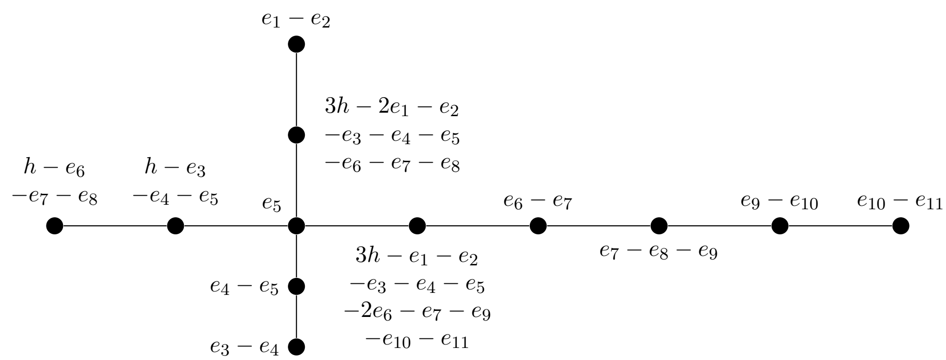


FIGURE 7. Homology classes of embedding spheres of $P\mathcal{A}'_2$.

For the sake of completeness, we give detailed explanation of blow-up processes and explicit homological computations of the above construction of Bhupal and Stipsicz in the Appendix.

The embedding $j : \partial P\mathcal{A}'_2 \hookrightarrow B$ induces a map $j_* : \pi_1(\partial P\mathcal{A}'_2) \longrightarrow \pi_1(B)$.

**Proposition 1.20.** *The map $j_*$ is surjective.*

*Proof.* Applying the van Kampen theorem to the decomposition

$$\mathbb{CP}^2 \sharp 11 \overline{\mathbb{CP}^2} = P\mathcal{A}'_2 \bigcup_{\partial P\mathcal{A}'_2} B$$

yields

$$\pi_1(P\mathcal{A}'_2) *_N \pi_1(B) = \pi_1(\mathbb{CP}^2 \sharp 11 \overline{\mathbb{CP}^2}) = 1,$$

where $N = \langle i_*(x).j_*(x)^{-1} | x \in \pi_1(\partial P\mathcal{A}'_2) \rangle$ with $i : \partial P\mathcal{A}'_2 \hookrightarrow P\mathcal{A}'_2$ and $j : \partial P\mathcal{A}'_2 \hookrightarrow B$ are two embeddings of the boundary. Since $P\mathcal{A}'_2$ is simply connected by Theorem 1.13, we obtain that $\pi_1(B)/\langle j_*(x)^{-1} | x \in \pi_1(\partial P\mathcal{A}'_2) \rangle = 1$, or equivalently $j_*$ is surjective. □

1.5. **A topological construction.** Suppose that we have an embedding of a plumbing manifold $P\Gamma$ into a smooth 4-manifold $X$. When the boundary of $P\Gamma$ bounds a rational homology ball $B$, we can construct a new smooth 4-manifold as follows.

**Definition 1.21.** *The rational blow-down of $X$ along the plumbing manifold $P\Gamma$ is defined to be*

$$X_1 = (X - int(P\Gamma)) \bigcup_{\partial P\Gamma} B.$$

The rational blow-down procedure was first introduced by Fintushel and Stern in [7] for certain cases of linear plumbing trees. It has been extended and applied to construct new exotic 4-manifolds by Park in [25], Stipsicz and Szabó in [31], Park, Stipsicz and Szabó in [26] and Michalogiorgaki in [18].

We will apply the rational blow-down procedure along $P\mathcal{A}_2$ to construct a new exotic manifold. In order to do that, we first give a construction of the plumbing manifold $P\mathcal{A}_2$ by embedding as a submanifold in $\mathbb{CP}^2\sharp 17\overline{\mathbb{CP}^2}$.

**Proposition 1.22.** *The 4-manifold $P\mathcal{A}_2$ embeds into $\mathbb{CP}^2\sharp 17\overline{\mathbb{CP}^2}$.*

*Proof.* We consider two complex projective quartic curves $C_1$ and $C_2$ in the projective plane $\mathbb{CP}^2$ given as follows. Let $q_1$ be a complex projective quadratic curve in $\mathbb{CP}^2$ and $l_1$ be a complex projective line which is tangent to $q_1$. Choose another complex projective quadratic curve $q_2$ which intersects $q_1$ and $l_1$ transversely at their intersection point. The curve $q_2$ meets $q_1$ at three further points, and $l_1$ at one other point. Now choose three complex projective lines $l_2, l_3, l_4$ such that their intersection points with each other all lie on either $q_1$ or $q_2$, while their intersection points with $l_1$ do not, and also each of $l_3, l_4$ goes through each of two intersection points of $q_1$ and $q_2$. Finally, let

$$C_1 = \{[x : y : z] \in \mathbb{CP}^2 | q_1(x, y, z)q_2(x, y, z) = 0\},$$

$$C_2 = \{[x : y : z] \in \mathbb{CP}^2 | l_1(x, y, z)l_2(x, y, z)l_3(x, y, z)l_4(x, y, z) = 0\}.$$

The configurations of $C_1$ and $C_2$ are indicated as below.

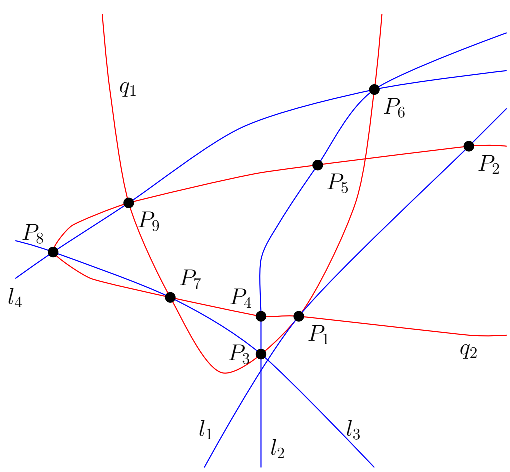


FIGURE 8. Depiction of the configurations of $C_1$ and $C_2$ in $\mathbb{CP}^2$ : The curve $C_1$ is the union of the complex projective quadratic curves $q_1, q_2$. The curve $C_2$ is the union of the complex projective lines $l_1, l_2, l_3, l_4$. $C_1$ and $C_2$ intersect each other at nine point $P_1, P_2, \cdots, P_9$.

For an explicit example, one can take

$$q_1(x, y, z) = xy + z^2,$$

$$q_2(x, y, z) = xz + y^2,$$

$$l_1(x, y, z) = z,$$

$$l_2(x, y, z) = (-\zeta_3) \cdot x + (1 - \zeta_3) \cdot y + z,$$

$$l_3(x, y, z) = x + y,$$

$$l_4(x, y, z) = x + (\zeta_3 - 1) \cdot y + \zeta_3 \cdot z,$$

where $\zeta_3 = \frac{-1+i\sqrt{3}}{2}$ is the primitive cube root of unity.

One can verify that $l_1$ intersects $q_1$ at a single point $P_1 = [1 : 0 : 0]$, and $l_1$ intersects $q_2$ at two points $P_1$ and $P_2 = [0 : 1 : 0]$. $l_2$ intersects $q_1$ at two points

$$P_4 = [-(\frac{i+\sqrt{3}+i\sqrt{2(7-9i\sqrt{3})}}{2i+2\sqrt{3}})^2 : \frac{i+\sqrt{3}+i\sqrt{2(7-9i\sqrt{3})}}{2i+2\sqrt{3}} : 1] \text{ and}$$
$$P_5 = [-(\frac{i+\sqrt{3}-i\sqrt{2(7-9i\sqrt{3})}}{2i+2\sqrt{3}})^2 : \frac{i+\sqrt{3}-i\sqrt{2(7-9i\sqrt{3})}}{2i+2\sqrt{3}} : 1],$$

and $l_2$ intersects $q_2$ at two points $P_3 = [-1, 1, -1]$ and $P_6 = [1 : -1 : 1 - 2\zeta_3^2]$. $l_3$ intersects $q_1$ and $q_2$ transversally at an intersection point $P_7 = [-1, 1, 1]$ of $q_1$ and $q_2$. $l_3$ intersects $q_1$ at another point $P_3$ and $l_3$ intersects $q_2$ at another point $P_8 = [1 : -1 : 2 - \zeta_3^2]$. $l_4$ intersects $l_3$ and $q_2$ transversally at $P_8$. $l_4$ intersects $q_1, q_2$ transversally at an intersection point $P_9 = [(1 - \zeta_3)^2 : -1 : 1 - \zeta_3]$ of $q_1$ and $q_2$. Finally, $l_4$, $l_3$ and $q_1$ intersect each other transversally at $P_6$.

Let us consider the pencil of quartic curves defined by $C_1$ and $C_2$:

$$C_t = \{z \in \mathbb{CP}^2 | p_t(z) = t_1.q_1(z)q_2(z) + t_2.l_1(z)l_2(z)l_3(z)l_4(z) = 0, t = [t_1 : t_2] \in \mathbb{CP}^1\}.$$

The pencil $C_t$ gives us a map from $\mathbb{CP}^2$ to $\mathbb{CP}^1$ which is well-defined away from nine intersection points of $C_1$ and $C_2$ by sending $z \in \mathbb{CP}^2$ to $t \in \mathbb{CP}^1$ such that $p_t(z) = 0$. By performing blow-ups at the intersection points of $C_1$ and $C_2$ totally 16 times, we get a fibration over $\mathbb{CP}^1$. More precisely, we need to perform

*i*) three infinitely close blow-ups at $P_1$,
*ii*) one blow-up at $P_2$,
*iii*) two infinitely close blow-ups at $P_3$,
*iv*) one blow-up at $P_4$,
*v*) one blow-up at $P_5$,
*vi*) two infinitely close blow-ups at $P_6$,
*vii*) two infinitely close blow-ups at $P_7$,
*viii*) two infinitely close blow-ups at $P_8$,
*ix*) two infinitely close blow-ups at $P_9$.

Here, by infinitely close blow-ups we mean that the blow-ups are performed at different points on exceptional curves.

The blow-up processes are illustrated below. We also calculate the homology classes of proper tranforms in each step so that at the end we will obtain a fibration: After each blow-up, the proper transforms of two curves will be choosen together with possibly a certain multiple of exceptional curves (this depends on whether the blowing-up point is a multiple point of the curves on which we perform blow-ups or not) so that they define a pencil on the blown-up manifold, i.e they must represent the same homology classes. We will explain the first blow-up at $P_1$, the remaining 15 blow-ups can be done in the same way.

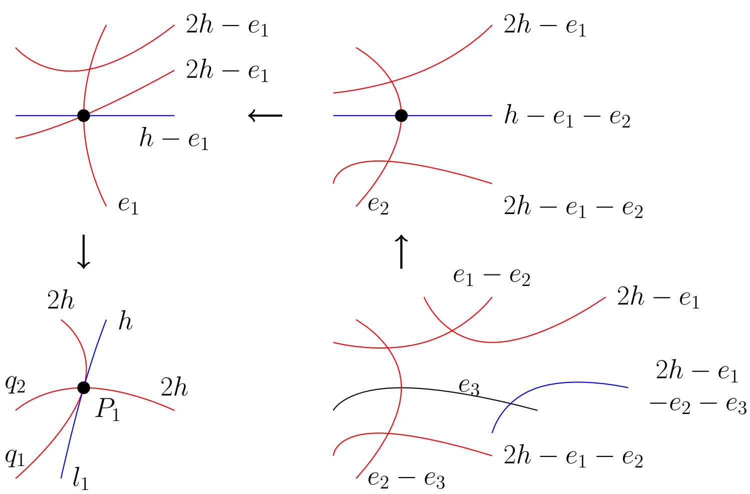


FIGURE 9. Three infinitely close blow-ups at $P_1$: The arrows can be regarded as the projection maps. Counter-clockwise: *i*) At the start, the intersection point $P_1$ is a double point of $C_1$ and a single point of $C_2$; $q_2$ intersect $q_1, l_1$ transversally; and $P_1$ is also the tangent point of $q_1$ and $l_1$. *ii*) After the first blow-up, we get the second configuration: in the pencil, we need to take the curve given by the proper transform of $C_1$ together with the exceptional curve of multiplicity one whose homology class is $e_1$; the proper transform of $q_2$ is disjoint from the proper transforms of $q_1, l_1$; the proper transform of $q_1$ and $l_1$ and the exceptional curve intersect each other transversally at one point. *iii*) The first blow-up can be done in a similar way as in Example 1.12 a) and c). The second and the third blow-ups can be done in a similar way as in Example 1.12 b) and a) respectively.

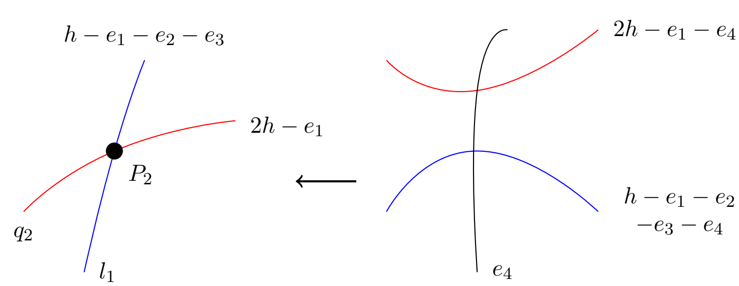


FIGURE 10. One blow-up at $P_2$: This can be done in a similar way as in Example 1.12 a).

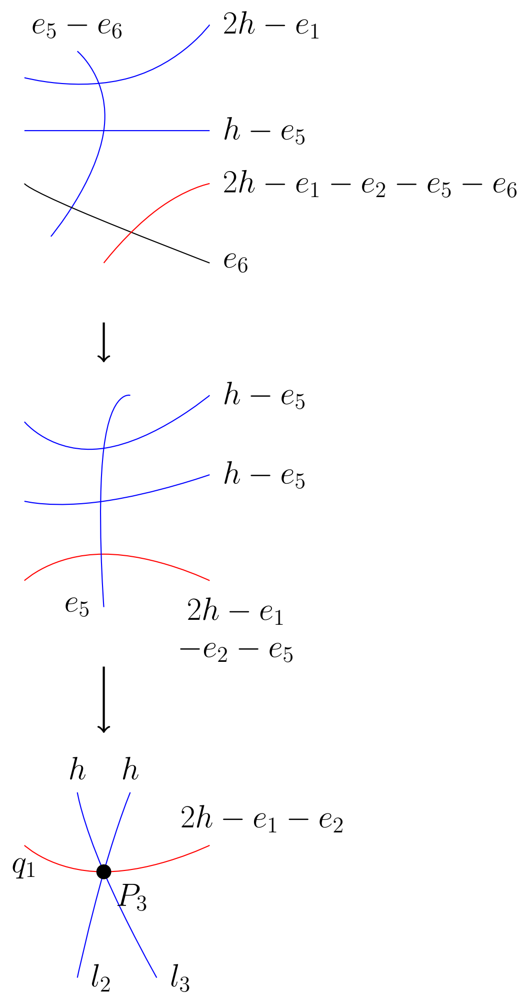


FIGURE 11. Two infinitely close blow-ups at $P_3$: The first blow-up can be done in a similar way as in Example 1.12 b). The second blow-up can be done in a similar way as in Example 1.12 a).

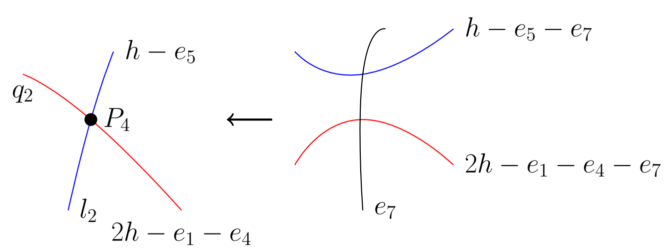


FIGURE 12. One blow-up at $P_4$: This can be done in a similar way as in Example 1.12 a)

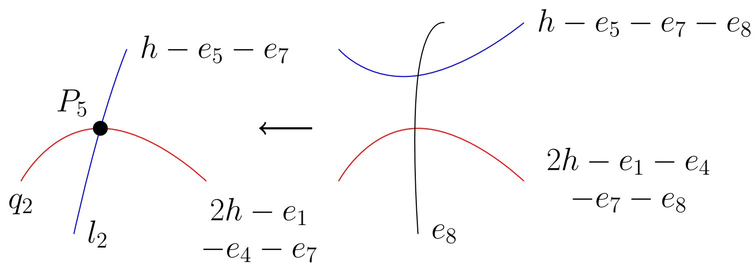


FIGURE 13. One blow-up at $P_5$: This can be done in a similar way as in Example 1.12 a)

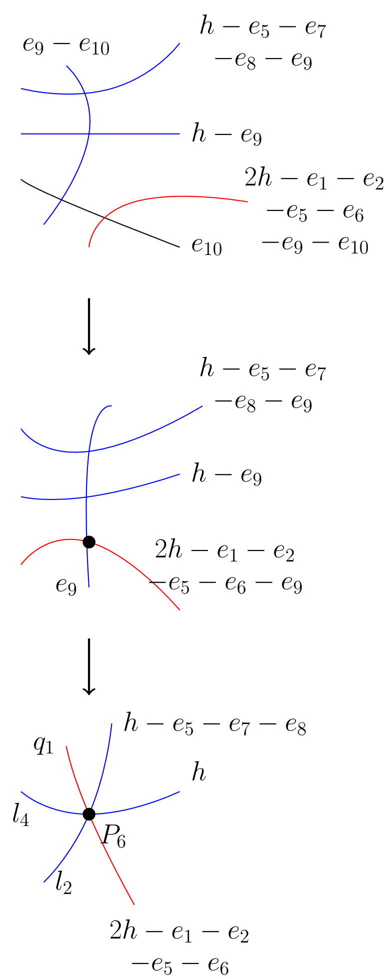


FIGURE 14. Two infinitely close blow-ups at $P_6$: The first blow-up can be done in a similar way as in Example 1.12 b). The second blow-up can be done in a similar way as in Example 1.12 a)

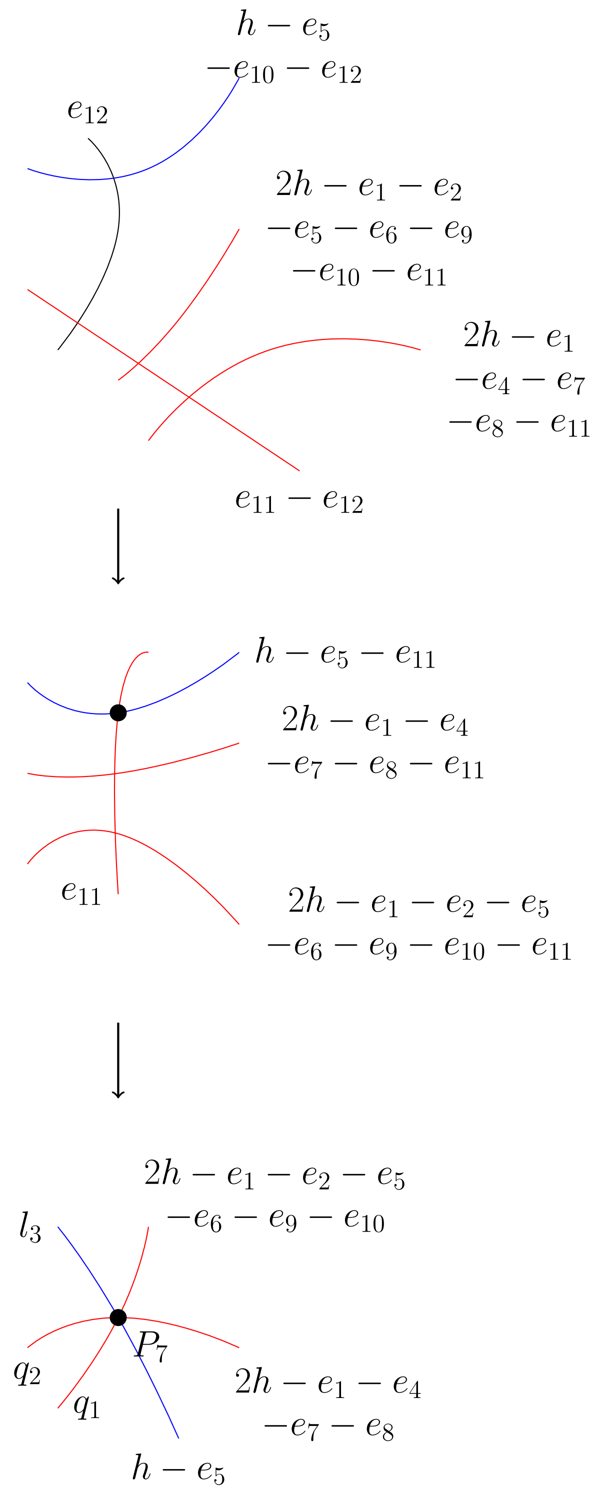


FIGURE 15. Two infinitely close blow-ups at $P_7$: The first blow-up can be done in a similar way as in Example 1.12 b). The second blow-up can be done in a similar way as in Example 1.12 a).

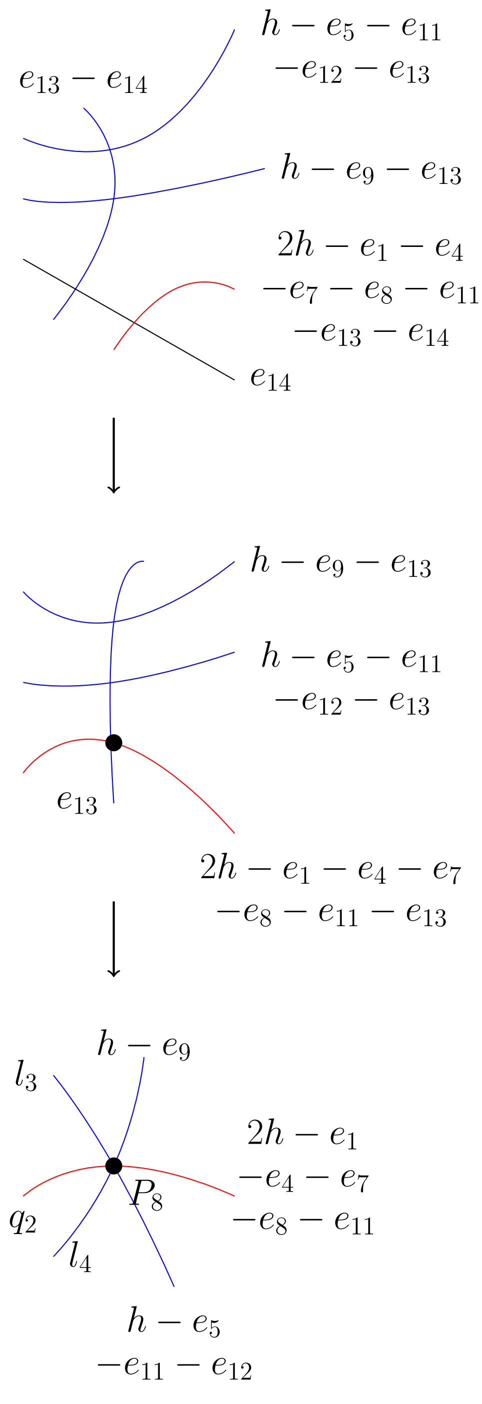


FIGURE 16. Two infinitely close blow-ups at $P_8$: The first blow-up can be done in a similar way as in Example 1.12 b). The second blow-up can be done in a similar way as in Example 1.12 a).

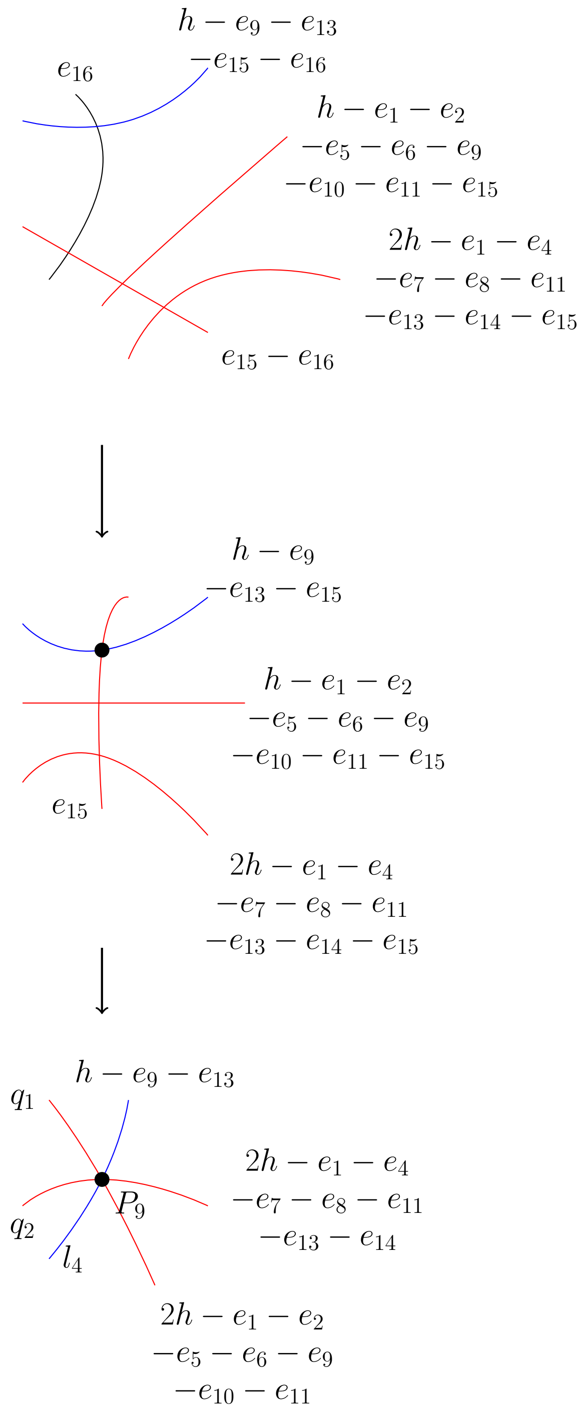


FIGURE 17. Two infinitely close blow-ups at $P_9$: The first blow-up can be done in a similar way as in Example 1.12 b). The second blow-up can be done in a similar way as in Example 1.12 a).

Finally, blowing-up another point on $e_3$ we have the desired configuration of embedding curves in $\mathbb{CP}^2\sharp 17\overline{\mathbb{CP}^2}$ as follows.

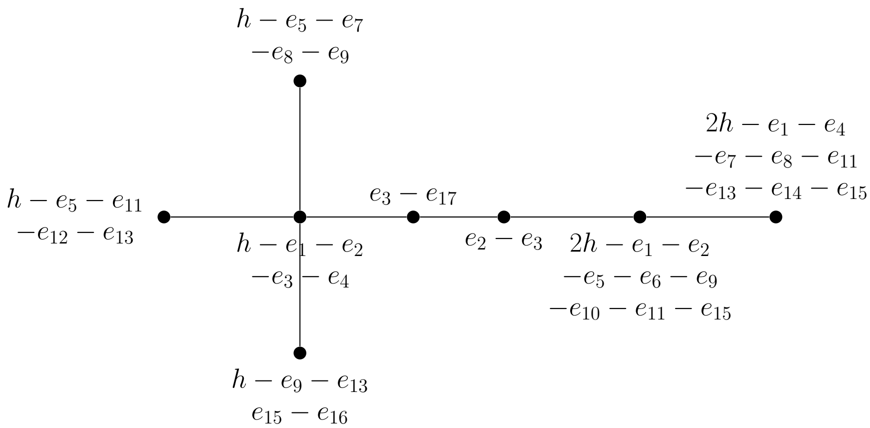


FIGURE 18. Embedded curves in $\mathbb{CP}^2\sharp 17\overline{\mathbb{CP}^2}$ obtained from the above blow-up processes: By taking the square of the corresponding homology class at each vertex, we obtain weight at the corresponding vertex of the plumbing graph $\mathcal{A}_2$.

Thus, we obtain an embedding of $P\mathcal{A}_2$ into $\mathbb{CP}^2\sharp 17\overline{\mathbb{CP}^2}$. □

**Definition 1.23.** *We define $X_1$ as the rational blow-down of $\mathbb{CP}^2\sharp 17\overline{\mathbb{CP}^2}$ along $P\mathcal{A}_2$:*

$$X_1 = (\mathbb{CP}^2\sharp 17\overline{\mathbb{CP}^2} - \ int(P\mathcal{A}_2)) \bigcup_{\partial P\mathcal{A}_2} B.$$

**Proposition 1.24.** *$X_1$ is simply connected.*

*Proof.* We write

$$U = \mathbb{CP}^2\sharp 17\overline{\mathbb{CP}^2}, V = \mathbb{CP}^2\sharp 17\overline{\mathbb{CP}^2} - \text{ int}(P\mathcal{A}_2) \text{ and } W = \partial P\mathcal{A}_2.$$

Thus, $X_1 = V\bigcup_W B$. First, we show that $V = \mathbb{CP}^2\sharp 17\overline{\mathbb{CP}^2} - \text{ int}(P\mathcal{A}_2)$ is simply connected, i.e $\pi_1(V) = 1$. Applying the van Kampen theorem to the following decomposition

$$U = V\bigcup_W P\mathcal{A}_2,$$

we have

$$\pi_1(U) = \pi_1(V) *_N \pi_1(P\mathcal{A}_2),$$

where $N = \langle i_*(x).j_*(x)^{-1} | x \in \pi_1(W)\rangle$ with $i : W \hookrightarrow V$, $j : W \hookrightarrow P\mathcal{A}_2$ are two embeddings of the boundary $W$.

Since $\mathbb{CP}^2$ and $\overline{\mathbb{CP}^2}$ are simply connected, we have $\pi_1(U) = 1$. By Theorem 1.13, we also have $\pi_1(P\mathcal{A}_2) = 1$. We will show that the map $i_* : \pi_1(W) \longrightarrow \pi_1(V)$ has trivial image. Notice that $W$ is a Seifert fibered 3-manifold over $S^2$ with 4 exceptional fibers corresponding to 4 branches of the plumbing graph $\mathcal{A}_2$. By applying Theorem 1.14, the fundamental group of $W = \partial P\mathcal{A}_2$ admits a presentation with 5 generators $a_1, a_2, a_3, a_4, h$ and a set of relations :

$$a_1a_2a_3a_4 = 1;$$
$$[h, a_1] = [h, a_2] = [h, a_3] = [h, a_4] = 1;$$
$$a_i^3 h = 1, \text{ for } i = 1, 2, 3; \text{ and } a_4^{37} h^{26} = 1.$$

Furthermore, the homotopy classes $a_1, a_2, a_3$ can be represented by normal circles of $\widetilde{L_2}, \widetilde{L_3}, \widetilde{L_4}$, the proper transforms of $l_2, l_3, l_4$, respectively, after 17 blow-ups at the intersections of $C_1, C_2$. This is because the branches of $\mathcal{A}_2$ corresponding to $\widetilde{L_2}, \widetilde{L_3}, \widetilde{L_4}$ only contain one vertex, and therefore, the normal circles of these branches are exactly the normal circles of $\widetilde{L_2}, \widetilde{L_3}, \widetilde{L_4}$.

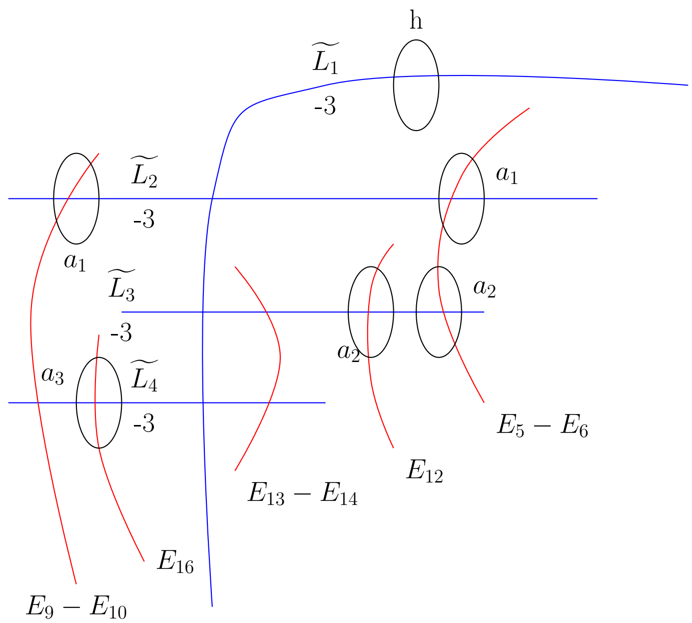


Figure 19. Homotopy classes $a_1, a_2, a_3$ are presented by normal circles of $\widetilde{L_2}, \widetilde{L_3}, \widetilde{L_4}$.

Now we observe that the circles $i(a_1), i(a_2), i(a_3)$ in $V$ have the following properties:

*i*) The circle $i(a_2)$ can be homotopically moved to the sphere $E_{12}$ in such a way that it bounds a disk which completely lies in $V$. Thus, we have $i_*(a_2) = 1$.

*ii*) The circle $i(a_3)$ can be homotopically moved to the sphere $E_{16}$ in such a way that it bounds a disk which completely lies in $V$. Thus, we have $i_*(a_3) = 1$.

*iii*) The circles $i(a_1)$ and $i(a_2)$ can be homotopically moved to the sphere $E_5 - E_6$ in such a way that together they bound an annulus which completely lies in $V$. Thus, $i(a_1)$ and $i(a_2)$ represent the same homotopy class in $\pi_1(V) : i_*(a_1) = i_*(a_2) = 1$.

Combining these with the relations of $\pi_1(W)$ gives us $i_*(h) = 1$ and $i_*(a_4) = 1$. Since all generators of $\pi_1(W)$ map to 1, we conclude that $i_*(x) = 1$ for any $x \in \pi_1(W)$.

Now to prove that $V = \mathbb{CP}^2\sharp 17\overline{\mathbb{CP}^2} - \text{ int}(P\mathcal{A}_2)$ is simply connected, or in other words $\pi_1(V) = 1$, it is enough to show that the map $j_* : \pi_1(W) \longrightarrow \pi_1(P\mathcal{A}_2)$ is surjective. Since $P\mathcal{A}_2$ is simply connected, this follows immediately.

Applying van Kampen theorem once again to the decomposition

$$X_1 = V \bigcup_W B$$

yields

$$\pi_1(X_1) = \pi_1(V) *_{N'} \pi(B),$$

where $N^{'} = \langle i^{'}_*(x).j^{'}_*(x)^{-1} | x \in \pi_1(W)\rangle$ with $i^{'} : W \hookrightarrow V$, $j^{'} : W \hookrightarrow B$ are two embeddings of the boundary $W$. Since $\pi_1(V) = 1$ and $j^{'}_*$ is surjective by Proposition 1.20, we conclude that $\pi_1(X_1) = 1$. □

**Remark 1.25.** *i) In the construction of $X_1$, we glue together two smooth manifold $\mathbb{CP}^2\sharp 17\overline{\mathbb{CP}^2} -$ int$(P\mathcal{A}_2)$ and $B$ along their boundaries. In order to do that, we need to specify an orientation reserving self-diffeomorphism of $\partial P\mathcal{A}_2$. For each such self-diffeomorphism of $\partial P\mathcal{A}_2$, we can construct a smooth 4-manifold $X_1$. Thus, we possibly obtain many smooth 4-manifold $X_1$'s. In what follows, we fix a choice of $X_1$.*

*ii) In* [25], [31], [26], *the blowing-down configurations are linear plumbing trees whose boundaries are lens spaces of the form $L(p^2, pq-1)$, where $p, q$ are relatively prime integers. By a theorem of Bonahon* [3], *any lens space of the form $L(p^2, pq-1)$, where $p, q$ are relatively prime integers, bounds a rational homology ball in such a way that any self-diffeomorphism of the boundary can be extended to a self-diffeomorphism of the rational homology ball. It can be shown that in this case, the rational blow-down 4-manifolds obtained from different self-diffeomorphisms of the boundary are all diffeomorphic to each other. In* [18], *as well as in our case, it is not known whether every self-diffeomorphism of the boundary can be extended to a self-diffeomorphism of the rational homology ball. Nevertheless, any rational blow-down 4-manifold obtained in* [18] *and in our case is still exotic.*

We are now able to apply Theorem 1.9 to $X_1$.

**Theorem 1.26.** *The 4-manifold $X_1$ is homeomorphic to $\mathbb{CP}^2\sharp 9\overline{\mathbb{CP}^2}$.*

*Proof.* Both $X_1$ and $\mathbb{CP}^2\sharp 9\overline{\mathbb{CP}^2}$ are simply conneted. Since $B$ is a rational homology ball, we have $b_2^+(X_1) = b_2^+(\mathbb{CP}^2\sharp 17\overline{\mathbb{CP}^2}) - b_2^+(P\mathcal{A}_2) = 1 - 0 = 1$ and $b_2^-(X_1) = b_2^-(\mathbb{CP}^2\sharp 17\overline{\mathbb{CP}^2}) - b_2^-(P\mathcal{A}_2) = 17 - 8 = 9$. Thus $\sigma(X_1) = -8$ and $\chi(X_1) = 12$. Theorem 1.9 implies that $X_1$ and $\mathbb{CP}^2\sharp 9\overline{\mathbb{CP}^2}$ are homeomorphic. □

In conclusion, we have constructed a smooth 4-manifold $X_1$ which is homeomorphic to $\mathbb{CP}^2\sharp 9\overline{\mathbb{CP}^2}$. The rest of this thesis is devoted to show that there exists a smooth structure on $X_1$ which is non-diffeomorphic to the standard smooth structure on $\mathbb{CP}^2\sharp 9\overline{\mathbb{CP}^2}$.

1.6. **Homological calculations.** In order to show that two smooth structures on a 4-manifold are different, one can use the Seiberg-Witten invariants which are defined on the set of characteristic elements. Recall that for a closed, oriented, smooth 4-manifold $X$, a second cohomology class $K \in H^2(X;\mathbb{Z})$ is called a characteristic element if $K \equiv w_2(X)$ $(mod\ 2)$, i.e $\langle K, \alpha \rangle \equiv \alpha^2 = Q_X(\alpha, \alpha)$ $(mod\ 2)$ for every homology class $\alpha \in H_2(X; \mathbb{Z})$.

Let us denote the standard basis of $H_2(\mathbb{CP}^2 \sharp 17\overline{\mathbb{CP}^2}; \mathbb{Z})$ by $h, e_1, \cdots, e_{17}$. We write $K = PD(3h - \sum_{i=1}^{17} e_i)$.

**Lemma 1.27.** *$K$ is a characteristic element of $H^2(\mathbb{CP}^2 \sharp 17\overline{\mathbb{CP}^2}; \mathbb{Z})$.*

*Proof.* For a second homology class $\alpha = x_0 h + \sum_{i=1}^{17} x_i e_i$, where $x_i$'s are integers, we have

$$\langle K, \alpha \rangle = 3x_0 + \sum_{i=1}^{17} x_i \equiv x_0^2 - \sum_{i=1}^{17} x_i^2 = \alpha^2 \ (mod\ 2)$$

as $x_0^2 \equiv 3x_0$ $(mod\ 2)$ and $x_i \equiv -x_i^2$ $(mod\ 2)$ for any integer $x_i, i = 0, \cdots, 17$. □

The cohomology class $K$ restricts to a cohomology class $K_{|\mathbb{CP}^2 \sharp 17\overline{\mathbb{CP}^2} -\ \mathrm{int}(P\mathcal{A}_2)}$ of $\mathbb{CP}^2 \sharp 17\overline{\mathbb{CP}^2} -\ \mathrm{int}(P\mathcal{A}_2)$, and a cohomology class $K_{|P\mathcal{A}_2}$ of $P\mathcal{A}_2$. Our goal is to show that the cohomology class $K_{|\mathbb{CP}^2 \sharp 17\overline{\mathbb{CP}^2} -\ \mathrm{int}(P\mathcal{A}_2)}$ extends to a characteristic element $\widetilde{K}$ of $X_1$ which will be used later for computing the Seiberg-Witten invariants in Theorem 2.60. In order to do that, we need to find a cohomology class $\bar{K}$ of the manifold $\mathbb{CP}^2 \sharp 11\overline{\mathbb{CP}^2} = P\mathcal{A}'_2 \bigcup B$ such that $\bar{K}_{|\partial B}$ and $K_{|\partial P\mathcal{A}_2}$ can be identified under the isomorphisms $H_1(\partial B; \mathbb{Z}) \cong H_1(\partial P\mathcal{A}'_2; \mathbb{Z}) \cong H_1(\partial P\mathcal{A}_2; \mathbb{Z}) \cong H_1(\mathbb{CP}^2 \sharp 17\overline{\mathbb{CP}^2} -\ \mathrm{int}(P\mathcal{A}_2); \mathbb{Z})$, and then glue the cohomology classes $\bar{K}_{|B}$ and $K_{|\mathbb{CP}^2 \sharp 17\overline{\mathbb{CP}^2} -\ \mathrm{int}(P\mathcal{A}_2)}$ together to get the desired cohomology class $\widetilde{K}$.

**Lemma 1.28.** *The first homology group of $\partial P\mathcal{A}_2$ admits the following presentation*

$$H_1(\partial P\mathcal{A}_2; \mathbb{Z}) = \{\mu_1, \mu_2, \mu_3, \mu_8 | \sum_{i=1}^{3} \mu_i = 85\mu_8, 3\mu_i = 37\mu_8, i = 1, 2, 3\} \tag{1.3}$$

*where the homology classes $\mu_i$ can be represented by normal circles to the disks bundles $u_i$'s, as illustrated in Figure 20.*

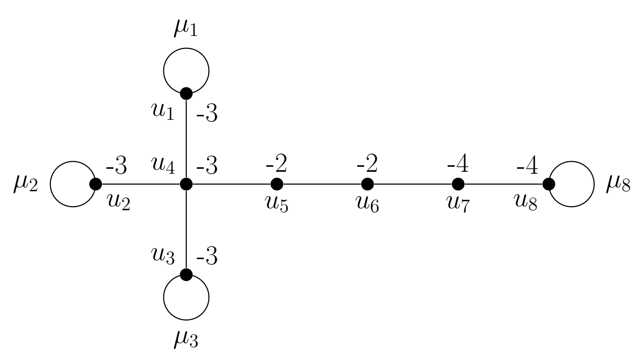


FIGURE 20. Generators of $H_1(\partial P\mathcal{A}_2; \mathbb{Z})$ as normal circles to the far most disk bundles of $P\mathcal{A}_2$: The disk bundles are denoted by $u_i$'s. The homology classes of the embedding spheres at the corresponding vertices of $\mathcal{A}_2$ are also denoted by $u_i$'s.

*Proof.* Recall from Figure 18 that the homology classes $u_i$'s are

$$\begin{aligned}
u_1 &= h - e_5 - e_7 - e_8 - e_9, \\
u_2 &= h - e_5 - e_{11} - e_{12} - e_{13}, \\
u_3 &= h - e_9 - e_{13} - e_{15} - e_{16}, \\
u_4 &= h - e_1 - e_2 - e_3 - e_4, \\
u_5 &= e_3 - e_{17}, \\
u_6 &= e_2 - e_3, \\
u_7 &= 2h - e_1 - e_2 - e_5 - e_6 - e_9 - e_{10} - e_{11} - e_{15}, \\
u_8 &= 2h - e_1 - e_4 - e_7 - e_8 - e_{11} - e_{13} - e_{14} - e_{15}.
\end{aligned}$$

By the Equation (1.2), $H_1(\partial P\mathcal{A}_2)$ admits a presentation with generators $\mu_1, \cdots, \mu_8$ which can be represented by the normal circles to the disk bundles $u_1, \cdots, u_8$ respectively, and the following relations:

$$\begin{cases}
-4\mu_8 + \mu_7 = 0, \\
-4\mu_7 + \mu_6 + \mu_8 = 0, \\
-2\mu_6 + \mu_5 + \mu_7 = 0, \\
-2\mu_5 + \mu_4 + \mu_6 = 0, \\
-3\mu_4 + \mu_1 + \mu_2 + \mu_3 + \mu_5 = 0, \\
-3\mu_1 + \mu_4 = 0, \\
-3\mu_2 + \mu_4 = 0, \\
-3\mu_3 + \mu_4 = 0.
\end{cases}
\iff
\begin{cases}
\mu_7 = 4\mu_8, \\
\mu_6 = 15\mu_8, \\
\mu_5 = 26\mu_8, \\
\mu_4 = 37\mu_8, \\
\mu_1 + \mu_2 + \mu_3 = 85\mu_8, \\
3\mu_1 = 3\mu_2 = 3\mu_3 = \mu_4.
\end{cases}$$

Thus, we obtain Equation (1.3). □

**Proposition 1.29.** *We have* $PD(K_{|\partial P\mathcal{A}_2}) = -132\mu_8$.

*Proof.* For the pair $(P\mathcal{A}_2, \partial P\mathcal{A}_2)$, the homological long exact sequence reads as

$$\cdots \longrightarrow H_2(P\mathcal{A}_2;\mathbb{Z}) \xrightarrow{j} H_2(P\mathcal{A}_2, \partial P\mathcal{A}_2;\mathbb{Z}) \xrightarrow{\delta} H_1(\partial P\mathcal{A}_2;\mathbb{Z}) \longrightarrow \cdots$$

Since $K_{|P\mathcal{A}_2}$ is an element of $H^2(P\mathcal{A}_2;\mathbb{Z})$, its Poincare dual $PD(K_{|P\mathcal{A}_2})$ is an element of $H_2(P\mathcal{A}_2, \partial P\mathcal{A}_2;\mathbb{Z})$ by Poincare duality. We have $\delta(PD(K_{|P\mathcal{A}_2})) = PD(K_{|\partial P\mathcal{A}_2}) \in H_1(\partial P\mathcal{A}_2;\mathbb{Z})$.

Since the homology classes $\{u_1, \cdots, u_8\}$ form a basis of $H_2(P\mathcal{A}_2;\mathbb{Z})$, there is a basis $\{\gamma_1, \cdots, \gamma_8\}$ of $H^2(P\mathcal{A}_2;\mathbb{Z})$ called dual basis such that

$$\langle \gamma_i, u_j \rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

By Poincare duality, $\{PD(\gamma_1), \cdots, PD(\gamma_8)\}$ form a basis of $H_2(P\mathcal{A}_2, \partial P\mathcal{A}_2;\mathbb{Z})$. Since $PD(\gamma_i)$ can be represented by a generic fiber of the disk bundle $u_i$, the boundary map $\delta$ sends $PD(\gamma_i)$ to an element of $H_1(\partial P\mathcal{A}_2;\mathbb{Z})$ which can be represented by the boundary of a generic fiber of the disk bundle $u_i$, i.e the normal circle to the disk bundle $u_i$. Thus, we obtain that $\delta(PD(\gamma_i)) = \mu_i$.

Notice that

$$\begin{aligned}\langle K, u_1\rangle &= (3h - \sum_{i=1}^{17} e_i).(h - e_5 - e_7 - e_8 - e_9)\\ &= 3h^2 + e_5^2 + e_7^2 + e_8^2 + e_9^2\\ &= 3.1 + (-1) + (-1) + (-1) + (-1) = -1.\end{aligned}$$

The remaining $\langle K, u_i\rangle$'s can be calculated in the same way:

$$\langle K, u_2\rangle = -1, \langle K, u_3\rangle = -1, \langle K, u_4\rangle = -1,$$
$$\langle K, u_5\rangle = 0, \langle K, u_6\rangle = 0, \langle K, u_7\rangle = -2, \langle K, u_8\rangle = -2.$$

Thus, we have

$$\begin{aligned}PD(K_{|P\mathcal{A}_2}) &= \sum_{i=1}^{8} \langle K, u_i\rangle PD(\gamma_i)\\ &= -(PD(\gamma_1) + PD(\gamma_2) + PD(\gamma_3) + PD(\gamma_4)) - 2(PD(\gamma_7) + PD(\gamma_8)).\end{aligned}$$

It follows that

$$\begin{aligned}PD(K_{|\partial P\mathcal{A}_2}) &= \delta(PD(K_{|P\mathcal{A}_2}))\\ &= \delta(-(PD(\gamma_1) + PD(\gamma_2) + PD(\gamma_3) + PD(\gamma_4)) - 2(PD(\gamma_7) + PD(\gamma_8)))\\ &= -(\mu_1 + \mu_2 + \mu_3 + \mu_4) - 2(\mu_7 + \mu_8).\end{aligned}$$

Since $\mu_7 = 4\mu_8, \mu_4 = 37\mu_8$ and $\sum_{i=1}^{3} \mu_i = 85\mu_8$, we have

$$PD(K_{|\partial P\mathcal{A}_2}) = -(85\mu_8 + 37\mu_8) - 2(4\mu_8 + \mu_8) = -132\mu_8.$$

□

We need to find a charateristic element $\bar{K} \in H^2(P\mathcal{A}'_2 \bigcup B; \mathbb{Z})$ such that $PD(\bar{K}_{|\partial P\mathcal{A}'_2})$ and $PD(K_{|\partial P\mathcal{A}_2}) = -132\mu_8$ are identified under the isomorphism $H_1(\partial P\mathcal{A}'_2; \mathbb{Z}) \cong H_1(\partial P\mathcal{A}_2; \mathbb{Z})$. Without any confusion, in the rest of this section, we denote the standard basis of $H_2(\mathbb{CP}^2 \sharp 11\overline{\mathbb{CP}^2}; \mathbb{Z}) = H_2(P\mathcal{A}'_2 \bigcup B; \mathbb{Z})$ by $h, e_1, \cdots, e_{11}$. We choose $\bar{K} = PD(-h - \sum_{i=1}^{8} e_i + 11(e_9 + e_{10} + e_{11}))$.

**Lemma 1.30.** *$\bar{K}$ is a characteristic element of $H^2(\mathbb{CP}^2 \sharp 11\overline{\mathbb{CP}^2}; \mathbb{Z})$.*

*Proof.* For a second homology class $\alpha = x_0 h + \sum_{i=1}^{11} x_i e_i \in H_2(\mathbb{CP}^2 \sharp 11\overline{\mathbb{CP}^2}; \mathbb{Z})$, where $x_i$'s are integers, we have

$$\langle \bar{K}, \alpha\rangle = -x_0 + \sum_{i=1}^{8} x_i - 11(x_9 + x_{10} + x_{11}) \equiv x_0^2 - \sum_{i=1}^{11} x_i^2 = \alpha^2 \ (mod\ 2)$$

as $(2k + 1)x \equiv x^2 \ (mod\ 2)$ for any integers $x$ and $k$. □

Now we can do calculations for the case of $P\mathcal{A}'_2, \partial P\mathcal{A}'_2$ and $\bar{K}$ in a similar way as we did above for the case of $P\mathcal{A}_2, \partial P\mathcal{A}_2$ and $K$. Recall that $\partial P\mathcal{A}'_2 \cong -\partial P\mathcal{A}_2$, and the generators of $H_1(\partial P\mathcal{A}'_2)$ can be represented by the normal circles to the disk bundles of the plumbing manifold $P\mathcal{A}'_2$ (see Figure 21). Denote by $\tau_i$ the homology class in $H_1(\partial P\mathcal{A}'_2)$ represented by the normal circle to the disk bundle $v_i$ in the plumbing manifold $P\mathcal{A}'_2$, $1 \le i \le 12$.

From equation (1.2), we also obtain the relations between $\tau_i$'s as follows:

$$\begin{cases} -2\tau_{12}+\tau_{11}=0, \\ -2\tau_{11}+\tau_{10}+\tau_{12}=0, \\ -3\tau_{10}+\tau_9+\tau_{11}=0, \\ -2\tau_9+\tau_8+\tau_{10}=0, \\ -4\tau_8+\tau_7+\tau_9=0, \\ -2\tau_1+\tau_2=0, -2\tau_2+\tau_1+\tau_7=0, \\ -2\tau_3+\tau_4=0, -2\tau_4+\tau_3+\tau_7=0, \\ -2\tau_5+\tau_6=0, -2\tau_6+\tau_5+\tau_7=0, \\ -\tau_7+\tau_2+\tau_4+\tau_6+\tau_8=0. \end{cases} \iff \begin{cases} \tau_{11}=2\tau_{12}, \\ \tau_{10}=3\tau_{12}, \\ \tau_9=7\tau_{12}, \\ \tau_8=11\tau_{12}, \\ \tau_7=37\tau_{12}, \\ \tau_2=2\tau_1, \tau_4=2\tau_3, \tau_6=2\tau_5, \\ 3\tau_1=3\tau_3=3\tau_5=\tau_7, \\ 2(\tau_1+\tau_3+\tau_5)=26\tau_{12}. \end{cases}$$

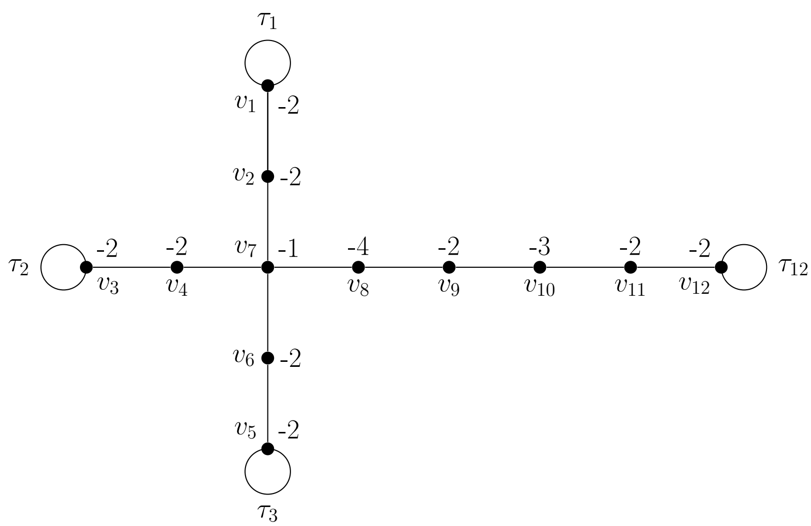


FIGURE 21. Generators of $H_1(\partial P\mathcal{A}'_2;\mathbb{Z})$ as normal circles to the far most disk bundles of $P\mathcal{A}'_2$: The disk bundles are denoted by $v_i$'s. The homology classes of the embedding spheres at the corresponding vertices of $\mathcal{A}_2$ are also denoted by $v_i$'s. The homology classes in $H_1(\partial P\mathcal{A}'_2;\mathbb{Z})$ represented by $\tau_1, \tau_2, \tau_3, \tau_{12}$ are identified under the isomorphism $H_1(\partial P\mathcal{A}'_2) \cong H_1(\partial P\mathcal{A}_2)$to the homology classes in $H_1(\partial P\mathcal{A}_2;\mathbb{Z})$ represented by $\mu_1, \mu_2, \mu_3, \mu_8$ respectively.

**Proposition 1.31.** *We have* $PD(\bar{K}_{|\partial P\mathcal{A}'_2}) = -132\tau_{12}$.

*Proof.* Calculations can be carried out in a similar way as in the proof of Proposion 1.31. We refer to Figure 7 for the values of homology classes $v_i$ in the basis $h, e_1, \cdots, e_{11}$ of $H_2(\mathbb{CP}^2\sharp 11\overline{\mathbb{CP}^2};\mathbb{Z})$. We have

$$\langle \bar{K}, v_1\rangle = (-h-\sum_{i=1}^{8} e_i+11(e_9+e_{10}+e_{11}))(e_1-e_2) = -e_1^2+e_2^2 = -(-1)+(-1) = 0.$$

We can calculate the remaining $\langle \bar{K}, v_i\rangle$'s in the same way to obtain:

$$\langle \bar{K}, v_2\rangle = -12, \langle \bar{K}, v_3\rangle = -4, \langle \bar{K}, v_4\rangle = -4, \langle \bar{K}, v_5\rangle = 0, \langle \bar{K}, v_6\rangle = 0,$$

$\langle \bar{K}, v_7\rangle = 1, \langle \bar{K}, v_8\rangle = 22, \langle \bar{K}, v_9\rangle = 0, \langle \bar{K}, v_{10}\rangle = 11, \langle \bar{K}, v_{11}\rangle = 0, \langle \bar{K}, v_{12}\rangle = 0.$
It follows that

$$\begin{aligned} PD(\bar{K}_{|\partial P\mathcal{A}'_2}) &= \sum_{i=1}^{12} \langle \bar{K}, v_i\rangle \tau_i = -12\tau_2 - 4\tau_3 - 4\tau_4 + \tau_7 + 22\tau_8 + 11\tau_{10} \\ &= -8(3\tau_1) - 4\tau_3 - 4(2\tau_3) + \tau_7 - 22\tau_8 + 11\tau_{10} \\ &= -8(37\tau_{12}) - 4(37\tau_{12}) + 37\tau_{12} - 22(11\tau_{12}) + 11(3\tau_{12}) \\ &= -11(37\tau_{12}) + 25.11\tau_{12} = -11.12\tau_{12} = -132\tau_{12}. \end{aligned}$$

□

Since $\tau_{12}$ and $\mu_8$ are identified under the isomorphism $H_1(\partial P\mathcal{A}'_2;\mathbb{Z}) \cong H_1(\partial P\mathcal{A}_2;\mathbb{Z})$, the cohomology classes $K$ and $\bar{K}$ restrict to the same cohomology class of the boundaries $\partial P\mathcal{A}_2 \cong -\partial P\mathcal{A}'_2$.

**Definition 1.32.** *The cohomology class $\widetilde{K} \in H^2(X_1;\mathbb{Z})$ is defined as*

$$\widetilde{K}_{|\mathbb{CP}^2\sharp 17\overline{\mathbb{CP}^2} -\ int(P\mathcal{A}_2)} = K_{|\mathbb{CP}^2\sharp 17\overline{\mathbb{CP}^2} -\ int(P\mathcal{A}_2)}, \widetilde{K}_{|B} = \bar{K}_{|B}$$

*and*

$$\widetilde{K}_{|\partial P\mathcal{A}_2} = K_{|\partial P\mathcal{A}_2} = \bar{K}_{\partial P\mathcal{A}'_2}.$$

**Proposition 1.33.** *$\widetilde{K}$ is a characteristic element.*

*Proof.* Since $\partial P\mathcal{A}_2$ is a rational homology ball, for an arbitrary homology class $\alpha \in H_2(X_1;\mathbb{Z})$, we have

$$\begin{aligned} \langle \widetilde{K}, \alpha\rangle &= \langle K_{|\mathbb{CP}^2\sharp 17\overline{\mathbb{CP}^2} -\ \mathrm{int}(P\mathcal{A}_2)}, \alpha_{|\mathbb{CP}^2\sharp 17\overline{\mathbb{CP}^2} -\ \mathrm{int}(P\mathcal{A}_2)}\rangle + \langle \bar{K}_{|B}, \alpha_{|B}\rangle \\ &\cong \langle \alpha_{|\mathbb{CP}^2\sharp 17\overline{\mathbb{CP}^2} -\ \mathrm{int}(P\mathcal{A}_2)}, \alpha_{|\mathbb{CP}^2\sharp 17\overline{\mathbb{CP}^2} -\ \mathrm{int}(P\mathcal{A}_2)}\rangle + \langle \alpha_{|B}, \alpha_{|B}\rangle = \alpha^2 \ (mod\ 2). \end{aligned}$$

□

To summerise, we have constructed a smooth 4-manifold $X_1$ which is homeomorphic to $\mathbb{CP}^2\sharp 9\overline{\mathbb{CP}^2}$, together with a characteristic element $\widetilde{K} \in H^2(X_1;\mathbb{Z})$ such that $\widetilde{K}_{|\mathbb{CP}^2\sharp 17\overline{\mathbb{CP}^2} -\ \mathrm{int}(P\mathcal{A}_2)} = K_{|\mathbb{CP}^2\sharp 17\overline{\mathbb{CP}^2} -\ \mathrm{int}(P\mathcal{A}_2)}$.

## 2. Seiberg-Witten invariants of smooth 4-manifolds

Seiberg-Witten invariants are differential topological invariants of smooth 4-manifolds introduced by Witten in [34]. In this section, we will define these invariants and discuss some of their properties which will be used to show that $X_1$ carries a smooth structure which is non-diffeomorphic to the standard smooth structure on $\mathbb{CP}^2\sharp 9\overline{\mathbb{CP}^2}$. Our main references are [23], [22] and [14].

### 2.1. *Spin$^c$*-structures.

The Seiberg-Witten invariants of a smooth 4-manifold are defined on the set of its *Spin$^c$*-structures which we will define in this section. We also discuss equivalent descriptions of *Spin$^c$*-structures as they will be used later.

For $n \geq 3$, the Lie group $SO(n)$ has fundamental group $\mathbb{Z}_2$. Therefore, its double covering manifold exists and admits a unique Lie group structure which makes the covering map countinuous.

**Definition 2.1.** *The group $Spin(n)$ is defined to be the double covering Lie group of $SO(n)$ for $n \geq 3$.*

Write $\pi : Spin(n) \longrightarrow SO(n)$ for the double covering map. $Spin(n)$ contains $\mathbb{Z}_2$ as a subgroup in such a way that $\mathbb{Z}_2 \cong \ker(\pi)$. The group $U(1)$ also contains $\mathbb{Z}_2$ as a subgroup of the identity element and the rotation by $180^0$.

**Definition 2.2.** *The group $Spin^c(n)$ is defined to be $Spin(n) \times_{\mathbb{Z}_2} U(1)$.*

Concretely, $Spin^c(n) = (Spin(n) \times U(1))/\{(I,1) \sim (-I,-1)\}$ where $I$ and 1 are the identity elements of $Spin(n)$ and $U(1)$ respectively. The map $\tilde{\pi} : Spin^c(n) \longrightarrow SO(n)$ which sends $(x, \lambda)$ to $\pi(x)$ is an $S^1$-fibering map.

For an orientable smooth $n$-dimensional manifold $M$, fixing a Riemannian metric on $M$ reduces the structure group of the tangent bundle $TM \longrightarrow M$ from $GL(n)$ to $O(n)$. An orientation on $M$ reduces the structure group from $O(n)$ to $SO(n)$. Thus, we obtain a principal $SO(n)$-bundle $\phi : P_{SO(n)} \longrightarrow M$ on $M$.

**Definition 2.3.** *A Spin-structure on an oriented Riemannian manifold $M$ is a principal $Spin(n)$-bundle $\psi : P_{Spin(n)} \longrightarrow M$ such that there exists a map $\alpha : P_{Spin(n)} \longrightarrow P_{SO(n)}$ which has the properties that $\psi = \phi \circ \alpha$ and $\alpha$ is the the covering map $Spin(n) \longrightarrow SO(n)$ fiberwise.*

In particular, $\alpha : P_{Spin(n)} \longrightarrow P_{SO(n)}$ is a double covering map.

**Definition 2.4.** *A $Spin^c$-structure on an oriented Riemannian manifold $M$ is a principal $Spin^c(n)$-bundle $\psi : P_{Spin^c(n)} \longrightarrow M$ such that there exists a map $\alpha : P_{Spin^c(n)} \longrightarrow P_{SO(n)}$ which has the properties that $\psi = \phi \circ \alpha$ and $\alpha$ is the $S^1$-fibering map $Spin^c(n) \longrightarrow SO(n)$ fiberwise.*

In particular, $\alpha : P_{Spin^c(n)} \longrightarrow P_{SO(n)}$ is an $S^1$-fibering map.

The set $\mathcal{S}_M$ of $Spin^c$-structures on $M$ is parametrized by $H^2(M;\mathbb{Z})$ as it will be explained below.

If we consider a good covering $\{U_\alpha\}$ of $M$, which means that multiple intersections of $U_\alpha$-s are contractible or empty, then the corresponding transition functions $\{g_{\alpha\beta}\}$ of the tangent bundle take values in $SO(n)$. A $Spin^c$-structure $\mathfrak{s}$ on $M$ is given by a collection of transition functions $\{\tilde{g}_{\alpha\beta}\}$:

$$\tilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow Spin^c(n)$$

which satisfies the cocycle conditions $\tilde{g}_{\alpha\beta}\tilde{g}_{\beta\gamma}\tilde{g}_{\gamma\alpha} \equiv 1$ and $\tilde{\pi} \circ \tilde{g}_{\alpha\beta} = g_{\alpha\beta}$.

If we write $\tilde{g}_{\alpha\beta} = (\pm h_{\alpha\beta}, \pm z_{\alpha\beta})$, then we have two maps

$$h_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow Spin(n)$$

and

$$z_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow U(1)$$

satisfying $\tilde{\pi} \circ h_{\alpha\beta} = g_{\alpha\beta}$ and $(h_{\alpha\beta}h_{\beta\gamma}h_{\gamma\alpha}, z_{\alpha\beta}z_{\beta\gamma}z_{\gamma\alpha}) \in \{(I,1),(-I,-1)\}$.

The collection of transition functions $\{\epsilon_{\alpha\beta\gamma} = h_{\alpha\beta}h_{\beta\gamma}h_{\gamma\alpha}\}$ satisfying the 2-cocycle conditions represents the second Stiefel-Whitney class $w_2(M)$ of $M$.

The collection of $U(1)$-valued transition functions $\{\lambda_{\alpha\beta} = z^2_{\alpha\beta}\}$ satisfying the cocycle conditions determines a principal $U(1)$-bundle on $M$, and equivalently, a complex line bundle $L$ on $M$.

**Definition 2.5.** *The complex line bundle $L$ is called the determinant line bundle of the $Spin^c$-structure, and we write $det(\mathfrak{s}) = L$.*

We write $c_1(\mathfrak{s}) = c_1(L)$ as the first Chern class of the complex line bundle $L$. The cohomology class $c_1(L)$ in $H^2(M;\mathbb{Z})$ is represented by the collection of transition functions $\{\eta_{\alpha\beta\gamma} = \frac{1}{2\pi i}\log(\lambda_{\alpha\beta}\lambda_{\beta\gamma}\lambda_{\gamma\alpha})\}$ which gives a $\mathbb{Z}$-valued Cech 2-cocyle.

The condition $(h_{\alpha\beta}h_{\beta\gamma}h_{\gamma\alpha}, z_{\alpha\beta}z_{\beta\gamma}z_{\gamma\alpha}) \in \{(I,1),(-I,-1)\}$ implies that

$$c_1(L) \equiv w_2(M)(\text{ mod } 2).$$

In conclusion, a $Spin^c$-structure on $M$ gives rise to a complex line bundle on $M$ for which $c_1$ reduces (mod 2) to the second Stiefel-Whitney class of $M$.

Denote by $\mathcal{C}_X$ the set of all characteristic elements of $X$. For a given $Spin^c$-structure $\mathfrak{s}$ on $X$, the first Chern class of the determinant line bundle $c_1(det(\mathfrak{s})) \in H^2(X;\mathbb{Z})$ is a characteristic element. Conversely, for every characteristic element $K \in C_X$, there is a $Spin^c$-structure $\mathfrak{s}$ on $X$ such that $c_1(det(\mathfrak{s})) = K$. If $X$ is simply connected, then $H^2(X;\mathbb{Z})$ does not have 2-torsions and the map

$$\begin{aligned} \mathcal{S}_M &\longrightarrow C_X \\ \mathfrak{s} &\mapsto c_1(det(\mathfrak{s})) \end{aligned}$$

is bijective.

In dimension 4, $Spin^c$-structures always exist by a theorem of Hopf and Hirzebruch [13].

**Proposition 2.6** ([11], Proposition 2.4.16)**.** *$Spin^c$-structures on a simply connected, oriented 4-manifold $X$ always exist, and the set of $Spin^c$-structures $\mathcal{S}_X$ can be identified with the set of characteristic elements $C_X$.*

*Spin*-structures on the other hand do not always exist. If a smooth 4-manifold admits a *Spin*-structure, then its intersection form is even. When the 4-manifold is simply connected, then the converse is also true.

**Theorem 2.7** (Rokhlin,[28])**.** *If a compact, smooth 4-manifold admits a Spin-structure, then its signature is divisible by 16.*

2.2. $Spin^c(3)$ **and** $Spin^c(4)$**.** We now return to the case of dimensions 3 and 4.

Viewing $\mathbb{R}^3$ as the set of imaginary quaternions $Im(\mathbb{H})$. The group $S^3$ of unit quaternions in $\mathbb{H}$ comes with the adjoint action on its Lie algebra $Im(\mathbb{H})$ defined by $q \cdot h = qhq^{-1}$, where $q \in S^3$ and $h \in Im(\mathbb{H})$. This action induces a map $S^3 \longrightarrow SO(3)$ which is a continuous double covering. Thus, we have

$$Spin(3) \cong S^3.$$

Every element of $SU(2)$, the special unitary group of degree 2, is of the form $x = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}$, where $a, b \in \mathbb{C}$ such that $|a|^2 + |b|^2 = 1$. The map $x \mapsto (a,b)$ is an isomorphism between $SU(2)$ and $S^3$.

Notice also that $U(2) \cong (SU(2) \times U(1))/\{(I,1) \sim (-I,-1)\}$. In conclusion, we have

$$\begin{aligned} Spin^c(3) &\cong (Spin(3) \times U(1))/(I,1) \sim (-I,-1) \\ &\cong (SU(2) \times U(1))/(I,1) \sim (-I,-1) \\ &\cong U(2). \end{aligned}$$

For the descriptions of $Spin^c(4)$, we choose two copies $SU_+(2)$ and $SU_-(2)$ of the group

$$SU(2) = \{\begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} : a, b \in \mathbb{C} \text{ such that } |a|^2 + |b|^2 = 1\}.$$

Using the fibration $SO(3) \longrightarrow SO(4) \longrightarrow S^3$, we see that $SO(4)$ is homeomorphic to $S^3 \times SO(3)$. Considering $S^3$ as the group of unit quaternions and $\mathbb{R}^4$ as the set of all quaternions $\mathbb{H}$, we have a double covering map

$$\begin{aligned} S^3 \times S^3 &\longrightarrow SO(4) \\ (q_1, q_2) &\mapsto \phi \end{aligned}$$

where the rotation $\phi$ is defined by $\phi(h) = q_1 h q_2^{-1}$ for $h \in \mathbb{H}$. Thus, we have

$$Spin(4) \cong S^3 \times S^3 \cong SU_+(2) \times SU_-(2)$$

and

$$\begin{aligned} Spin^c(4) &\cong \left(S^3 \times S^3 \times S^1\right) / \{(1,1,1) \sim (-1,-1,-1)\} \\ &\cong \left(SU_+(2) \times SU_-(2) \times U(1)\right) / \{(I, I, 1) \sim (-I, -I, -1)\}. \end{aligned}$$

The bijective map from $(SU_+(2) \times SU_-(2) \times U(1)) / \{(I, I, 1) \sim (-I, -I, -1)\}$ to $\{(A, B) \in U(2) \times U(2) : \det A = \det B\}$ which sends $(x, y, \lambda)$ to $(\lambda x, \lambda y)$ allows us to write

$$Spin^c(4) = \{(A, B) \in U(2) \times U(2) : \det A = \det B\}.$$

Also, we can define two representations of $Spin^c(4)$ into $U(2)$ as

$$s_\pm([h_+, h_-, \lambda]) = [h_\pm, \lambda]. \tag{2.1}$$

2.3. **Clifford multiplication.** In general, for a principal $G$-bundle $P \longrightarrow M$ and a given representation $\rho : G \longrightarrow \mathrm{GL}(V)$, where $V$ is a complex vector space, we can define an associated vector bundle by considering

$$E = P \times_\rho V = (P \times V) / \{(p.g, v) \sim (p, \rho(g)v)\}.$$

The maps $s_\pm$ given in (2.1) give two representations of $Spin^c(4)$ into $\mathrm{GL}(\mathbb{C}^2)$. Thus, we can define two complex vector bundles from $P_{Spin^c(4)} \longrightarrow X$ as

**Definition 2.8.** *Two spinor bundles associated to $P_{Spin^c(4)} \longrightarrow X$ are defined by*

$$W^\pm = P_{Spin^c(4)} \times_{s_\pm} \mathbb{C}^2.$$

Note that we have $\det(W^+) = \det(W^-) = \det(\mathfrak{s})$.

Again, regarding $\mathbb{R}^4$ as the set $\{\begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} : a, b \in \mathbb{C}\}$, we can define a map from $\mathbb{R}^4 \times \mathbb{C}^2$ to $\mathbb{C}^2$ by matrix multiplication $(x, v) \mapsto xv$. Since this map commutes with the representations of $Spin^c(4)$ by

$$(h_+ x (h_-)^{-1})(h_- \lambda v) = h_+ \lambda x v$$

for all $[h_+, h_-, \lambda] \in Spin^c(4)$, it gives rise to a global map

$$T^*X \otimes W^- \longrightarrow W^+.$$

There is also a map $T^*X \otimes W^+ \longrightarrow W^-$ if we replace the matrix multiplication by the map $(x, v) \mapsto -\bar{x}^T v$.

These two bundle maps together give rise to a bundle map called the Clifford multiplication

$$\gamma : T^*X \longrightarrow Hom_{\mathbb{C}}(W^+, W^-).$$

Note that $\gamma(v)^2 = -|v|^2$, and if $|v| = 1$, then $\gamma(v)$ is a unitary tranformation fiberwise.

From now on, for a 1-form $\alpha$ and a spinor $\psi$, we write $\alpha \cdot \psi$ for the Clifford multiplication $\gamma(\alpha)(\psi)$.

In conclusion, we have another description of a $Spin^c$-structure on the 4-manifold $X$ as a pair of spinor bundles $W^{\pm}$ together with a Clifford multiplication

$$\gamma : T^*X \longrightarrow Hom_{\mathbb{C}}(W^+, W^-).$$

2.4. **Connections and curvatures.** For a principal $G$-bundle $\pi : P \longrightarrow M$, at any point $p \in P$ there is an identification of the subspace of vertical vectors $\mathrm{Ker}(d\pi)_p$ in $T_pP$ and the Lie algebra Lie$(G)$ of $G$: since an element of $\mathrm{Ker}(d\pi)_p$ can be represented by $\{p.\gamma_t\}$ where $\{\gamma_t\}$ is an 1-parameter subgroups in $G$, the identification $\phi_p$ simply takes $\{p.\gamma_t\}$ to $\{\gamma_t\}$ which is an element of Lie$(G)$ viewed as an 1-parameter subgroups in $G$.

**Definition 2.9.** *A Lie$(G)$-valued 1-form $\omega : TP \longrightarrow$ Lie$(G)$ is a connection if it satisfies the following two conditions:*
*i) $\omega|_{Ker(d\pi)_p} = \phi_p$ for all $p$ in $P$,*
*ii) $\omega_{gp}(dg(v)) = g^{-1}.\omega_p(v).g$ for all $g$ in $G$.*

**Definition 2.10.** *Given a Lie$(G)$-valued 1-form $\omega$ on $P$, the horizontal distribution $\{H_p\}$ given by $\omega$ is defined at every point $p$ in $P$ to be the subspace $H_p = Ker\ \omega_p$ of $T_pP$.*

For every point $p$ in $P$, the map $(d\pi)_p$ gives an isomorphism between $H_p$ and $T_{\pi(p)}M$, thus a connection $\omega$ allows us to lift curves uniquely from $M$ to $P$ by its horizontal distribution, and therefore makes sense of taking directional differentiation on $P$.

For a given Riemannian manifold $(M, g)$, there is a unique connection $\nabla$ on the tangent bundle $TM$ which is compatible with the metric $g$ and torsion-free:

$$Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \text{ and } \nabla_X Y - \nabla_Y X = [X, Y].$$

**Definition 2.11.** *The connection $\nabla$ is called the Levi-Civita connection on $M$.*

**Definition 2.12.** *For a Riemannian 4-manifold $X$, there exists a canonical connection $\omega$ called Levi-Civita connection on the principal $SO(4)$-bundle $P_{SO(4)}$:*

$$\omega : TP_{SO(4)} \longrightarrow \ Lie(SO(4)).$$

Since $Spin(4) \longrightarrow SO(4)$ is a double covering map, we have

$$\mathrm{Lie}(Spin(4)) \cong \ \mathrm{Lie}(SO(4)).$$

The Lie algebra Lie$(Spin^c(4))$ splits as Lie$(Spin^c(4)) \cong$ Lie$(Spin(4)) \oplus$ Lie$(U(1))$.

Therefore, the generalization of Levi-Civita connection on principal $Spin^c$-bundle $P_{Spin^c(4)} \longrightarrow X$ needs a complement factor to the pull back connection $\pi^*\omega : P_{Spin^c(4)} \longrightarrow$ Lie$(Spin(4))$ which comes from a connection $A$ on the determinant line bundle $p : L \longrightarrow X$.

**Definition 2.13.** *Fixing a connection $A$ on the determinant line bundle, we define a generalization of Levi-Civita connection on the principal $Spin^c$-bundle $P_{Spin^c(4)}$ to be $\tilde{\omega} = \pi^*\omega \oplus p^*A$.*

The connection $\tilde{\omega}$ allows us to take the covariant derivative

$$\nabla_A : \Gamma(TX) \times \Gamma(W^+) \longrightarrow \Gamma(W^+).$$

**Definition 2.14.** *For a given $Spin^c$-structure $(W^{\pm}, \gamma)$, the Dirac-operator*

$$\partial\!\!\!/_A : \Gamma(W^+) \longrightarrow \Gamma(W^-)$$

*is defined to be the composition of the following two maps:*

$$\nabla_A : \Gamma(W^+) \longrightarrow \Gamma(T^*X \otimes W^+)$$

*and*

$$\gamma : \Gamma(T^*X \otimes W^+) \longrightarrow \Gamma(W^-).$$

For a Riemannian manifold $(X, g)$ with Levi-Civita connection $\nabla$, we have various notions of curvature: Riemannian curvature tensor, Ricci curvature, scalar curvature.

**Definition 2.15.** *The Riemannian curvature tensor is a 2-form with values in the skew adjoint endomorphisms of the tangent bundle*

$$R : \Gamma(TX) \longrightarrow \Gamma(TX \otimes \wedge^2(X))$$

*defined by*

$$R(U, V)W = \nabla_U \nabla_V W - \nabla_V \nabla_U W - \nabla_{[U,V]} W,$$

*where $U, V, W \in \Gamma(TX)$.*

We write $R(U, V, W, Z) = g(R(U, V)W, Z)$.

**Definition 2.16.** *The Ricci curvature of $X$ is a bilinear form on $TX$ defined by*

$$Ric(U, V) = g(-\sum_{i=1}^{n} R(e_i, U)(e_i), V),$$

*where $\{e_1, \cdots, e_n\}$ is an othonormal basis of $T_xX$ and $U, V \in \Gamma(TX)$.*

Note that the Ricci curvature is a symmetric bilinear form.

**Definition 2.17.** *The scalar curvature of $X$ is a real valued function*

$$\kappa : X \longrightarrow \mathbb{R}$$

*defined by*

$$\kappa(x) = Tr(Ric) = \sum_{i=1}^{n} R(e_i, e_i).$$

Suppose that $(X, g)$ is an oriented Riemannian 4-manifold with a $Spin^c$-structure $\mathfrak{s} = (W^+, W^-, \gamma)$. For a $U(1)-$connection $A$ on the determinant line bundle $L$, the formal adjoint $\nabla_A^*$ of the covariant derivative $\nabla_A$ is well-defined by

$$\int_X \langle \nabla_A \psi, \xi \rangle = \int_X \langle \psi, \nabla_A^* \xi \rangle$$

for $\psi \in \Gamma(W^+), \xi \in \Gamma(T^*X \otimes W^+)$.

We recall the following important formula due to Bochner and Lichnerowicz.

**Theorem 2.18.** [17]*(Bochner-Lichnerowicz) For any spinor $\psi \in \Gamma(W^+)$, we have*

$$\partial\!\!\!/_A \partial\!\!\!/_A \psi = \nabla_A^* \nabla_A \psi + \frac{\kappa}{4}\psi + \frac{F_A^+}{4} \cdot \psi, \tag{2.2}$$

*where the last term is the Clifford multiplication of 2-form on a spinor.*

Recall that the Lie algebra of $SO(n)$ consists of $n \times n$ matrices $A$ such that $A + A^T = 0$. Fixing a symmetric, bilinear, positive definite form $<,>$ on the Euclidean space $\mathbb{R}^n$, there is an isomorphism between Lie$(SO(n))$ and $\wedge^2(\mathbb{R}^n)$ given by $w_A(x, y) = \langle Ax, y\rangle$ for each $A \in$ Lie$(SO(n))$.

**Definition 2.19.** *The linear map $* : \wedge^i(\mathbb{R}^n) \longrightarrow \wedge^{n-i}(\mathbb{R}^n)$ is defined in an othonormal oriented basis $\{e_1, \cdots, e_n\}$ of $\mathbb{R}^n$ by*

$$*(e_{j_1} \wedge \cdots \wedge e_{j_i}) = e_{k_1} \wedge \cdots \wedge e_{k_{n-i}}$$

*where $\{j_1, \cdots, j_i, k_1, \cdots, k_{n-i}\}$ is an even permutation of $\{1, 2, \cdots, n\}$.*

When $n = 4$ and $i = 2$, we have $*^2 = Id$ which induces a splitting of vector spaces

$$\wedge^2(\mathbb{R}^4) \cong \wedge_+(\mathbb{R}^4) \oplus \wedge_-(\mathbb{R}^4),$$

where $*|_{\wedge_+(\mathbb{R}^4)} = Id$ and $*|_{\wedge_-(\mathbb{R}^4)} = -Id$.

We can globalise this construction to get a splitting of the vector bundle of differential 2-forms:

$$\wedge^2(T^*X) \cong \wedge_+(T^*X) \oplus \wedge_-(T^*X).$$

We write $\Omega_+(X) = \Gamma(\wedge_+(T^*X))$ and $\Gamma(\Omega_-(X) = \wedge_-(T^*X))$.

**Definition 2.20.** *An element of $\Omega_+(X)$ is called a self-dual 2-form, and an element of $\Omega_-(X)$ is called an anti-self-dual 2-form.*

The splitting of $\wedge^2(\mathbb{R}^4)$ is compatible through the isomorphism

$$w : \text{Lie}(SO(4)) \longrightarrow \wedge^2(\mathbb{R}^4)$$

with the splitting of the Lie algebra Lie$(SO(4))$:

$$\text{Lie}(SO(4)) \cong \text{Lie}(SO(3)) \oplus \text{Lie}(SO(3)) \cong Im(\mathbb{H}) \oplus Im(\mathbb{H}),$$

and we have

$$Im(\mathbb{H}) \cong \wedge_+(\mathbb{R}^4) \cong \wedge_-(\mathbb{R}^4).$$

There is also a splitting of the Lie algebra Lie$(Spin^c(4))$:

$$\text{Lie}(Spin^c(4)) \cong \text{Lie}(SO(3)) \oplus \text{Lie}(SO(3)) \oplus \text{Lie}(S^1) \cong Im(\mathbb{H}) \oplus Im(\mathbb{H}) \oplus i\mathbb{R}.$$

The adjoint representation of $Spin^c(4)$ on Lie$(Spin^c(4))$ gives rise to two representations $r_\pm$ of $Spin^c(4)$ on $Im(\mathbb{H})$: for $[q_+, q_-, \lambda] \in Spin^c(4)$ with two unit quaternions $q_\pm$ and a complex unit $\lambda$, and an imaginary quaternion $h \in Im(\mathbb{H})$, these two maps are defined explicitely by

$$r_\pm([q_+, q_-, \lambda])(h) = q_\pm h (q_\pm)^{-1}.$$

**Definition 2.21.** *The map $\sigma : \mathbb{H} \longrightarrow Im(\mathbb{H})$ is given by $\sigma(q) = -q.i.\bar{q}$ for $q \in \mathbb{H}$.*

**Lemma 2.22.** *The map $\sigma$ is $Spin^c(4)$-equivariant with respect to the representations $s_\pm$ and $r_\pm$ of $Spin^c(4)$ into $\mathbb{C}^2 \cong \mathbb{H}$ and $Im(\mathbb{H})$ respectively.*

*Proof.* For an element $[q_+, q_-, \lambda] \in Spin^c(4)$ and $h \in \mathbb{H}$, we have

$$r_+(\sigma(h)) = r_+(-h.i.\bar{h}) = -q_+.h.i.\bar{h}.(q_+)^{-1},$$

and

$$\sigma(s_+(h)) = \sigma(q_+ h\lambda) = -q_+.h.\lambda.i.\bar{\lambda}.\bar{h}.(q_+)^{-1}.$$

Since $\lambda \in U(1)$, we have $r_+(\sigma(h)) = \sigma(s_+(h))$.
Similarly, we have $r_-(\sigma(h)) = \sigma(s_-(h))$. □

The map $\sigma$ therefore can be globalised to a map

$$\sigma : \Gamma(W^+) \longrightarrow i\Omega_+(X)$$

by noticing that a fiber of $W^+$ is $\mathbb{C}^2 \cong \mathbb{H}$, and a fiber of $\wedge_+(T^*X)$ is $\wedge_+(\mathbb{R}^4) \cong Im(\mathbb{H})$.

**Lemma 2.23.** [23, Lemma 4.1.1] *For $\psi \in \Gamma(W^+)$, the map $\sigma$ reads*

$$\sigma(\psi) = \psi \otimes \psi^* - \frac{1}{2}|\psi|^2 Id, \tag{2.3}$$

*where $\psi \otimes \psi^*$ acts as an endomorphism on $W^+$ which sends $\phi$ to $\langle \phi, \psi \rangle w$.*

For a principal $G$-bundle $\pi : P \longrightarrow X$, the set of connections on $P$ is an affine space. It means that if $A_1, A_2$ are two connections on $P$, then $A_1 - A_2$ is the pull-back by $\pi$ of a Lie$(G)$-valued 1-form on $X$:

$$A_1 - A_2 = \pi^*\alpha, \text{ where } \alpha \in \Omega^1(X, \text{ Lie}(G)).$$

Recall that if $A_1$ is a Lie$(G)$-valued $k$-form, and $A_2$ is a Lie$(G)$-valued $l$-form, then the wedge product $A_1 \wedge A_2$ of $A_1$ and $A_2$ is defined as a Lie$(G)$-valued $(k+l)$-form by matrix multiplication on Lie$(G)$.

**Definition 2.24.** *For a connection $A$ on principal $G$-bundle $\pi : P \longrightarrow X$, the curvature of $A$ is a Lie$(G)$-valued 2-form defined by $F_A = dA + A \wedge A$.*

That is, $F_A$ is the pull-back by $\pi$ of a 2-form on $X$

$$F_A = \pi^*\omega, \text{ where } \omega \in \Omega^2(X, \text{ Lie}(G)).$$

When the bundle is a principal $U(1)$-bundle $L \longrightarrow X$, i.e $L$ is a complex line bundle over $X$, we denote by $\mathcal{A}_L$ the set of $U(1)$-connections on $L$.

Notice that

$$\Omega^2(X, \text{ Lie}(U(1))) = i\Omega^2(X, \mathbb{R}) = i\Gamma(\wedge^2(T^*X)) \cong i\Gamma(\wedge_+(T^*X)) \oplus i\Gamma(\wedge_-(T^*X)).$$

Any differential 2-form $\omega \in \Omega^2(X)$ can be decomposed as a direct sum

$$\omega = \omega_+ + \omega_-,$$

where

$$\omega_+ = \frac{1}{2}(\omega + *\omega) \in \Omega_+(X), \text{ and } \omega_- = \frac{1}{2}(\omega - *\omega) \in \Omega_-(X).$$

Thus, for a $U(1)$-connection $A \in \mathcal{A}_L$, the curvature of $A$ splits as

$$F_A = F_A^+ + F_A^-, \text{ where } F_A^{\pm} \in i\Omega_{\pm}(X).$$

When $X$ is a compact, oriented Riemannian manifold, we can define a symmetric, bilinear, positive definite form on $\Omega^k(X)$ by

$$(\omega_1, \omega_2) = \int_X \omega_1 \wedge *\omega_2.$$

There is an adjoint operator of the exterior derivative $d : \Omega^{k-1}(X) \longrightarrow \Omega^k(X)$ defined by

$$d^* = - * d* : \Omega^k(X) \longrightarrow \Omega^{k-1}(X)$$

which satisfies $(d\alpha, \beta) = (\alpha, d^*\beta)$ for $\alpha \in \Omega^{k-1}(X)$ and $\beta \in \Omega^k(X)$.

The Hodge-Laplacian then is defined by

$$\Delta = dd^* + d^*d : \Omega^k(X) \longrightarrow \Omega^k(X).$$

**Definition 2.25.** *The space of harmonic $k$-forms on $X$ is defined to be*

$$\mathcal{H}^k(X) = \{\omega \in \Omega^k(X) : \Delta(\omega) = 0\}.$$

We recall the following fundamental theorem of Hodge.

**Theorem 2.26.** *For a compact, oriented Riemannian manifold $X$, we have*

$$H^k_{dR}(X; \mathbb{R}) \cong \mathcal{H}^k(X).$$

Since $*$ interchanges kernels of $d$ and $\delta$, we also have the decomposition

$$\Omega^2(X) \cong \mathcal{H}^2(X) \oplus d(\Omega^1(X)) \oplus \delta(\Omega^3(X)).$$

Thus, self-dual and anti-self-dual parts of a harmonic 2-form are again harmonic 2-forms, which means that

$$\mathcal{H}^2(X) \cong \mathcal{H}^+(X) \oplus \mathcal{H}^-(X),$$

where $\mathcal{H}^+(X)$ and $\mathcal{H}^-(X)$ are spaces of self-dual and anti-self-dual harmonic 2-forms respectively.

2.5. **The Seiberg-Witten equations.** We are now able to define the Seiberg-Witten equations.

Let $X$ be a closed, oriented, simply connected smooth 4-manifold, $g$ be a Riemannian metric, and $\mathfrak{s}$ be a $Spin^c$-structure on $X$. The $Spin^c$-structure $\mathfrak{s}$ is given by a triple $(W^+, W^-, \gamma)$ of two spinor bundles $W^\pm$ and a Clifford multiplication $\gamma : T^*X \longrightarrow Hom_{\mathbb{C}}(W^+, W^-)$. The associated determinant line bundle is $L = \det W^+ = \det W^-$.

**Definition 2.27.** *The Seiberg-Witten equations for $(X, g, \mathfrak{s})$ are*

$$\begin{cases} \partial\!\!\!/_A \psi = 0 \\ F_A^+ = \sigma(\psi), \end{cases}$$

*where $A \in \mathcal{A}_L$ and $\psi \in \Gamma(W^+)$.*

Applying the Bochner-Lichnerowicz formula (2.2) to solutions of the Seiberg-Witten equations, we obtain the following:

**Proposition 2.28.** *For a solution $(A, \psi)$ of the Seiberg-Witten equations, we have*

$$\|\nabla_A \psi\|^2_{L^2} + \frac{1}{4}\langle \kappa\psi, \psi\rangle + \frac{1}{4}\|\psi\|^4_{L^2} = 0. \tag{2.4}$$

*In particular, we have*

$$\kappa^-_X . \|\psi\|^2_{L^2} \geq \|\psi\|^4_{L^2} \tag{2.5}$$

*where $\kappa^-_X = \max\{0, -\kappa(x) : x \in X\}$.*

*We also have the following pointwise bound for any $x \in X$*

$$|\psi(x)|^2 \leq \kappa^-_X. \tag{2.6}$$

*Proof.* Notice that $\not{\partial}_A\psi = 0$ and $F_A^+ = \psi\otimes\psi^* - \frac{|\psi|^2}{2}Id$, hence taking the integral $\int_X\langle .,\psi\rangle$ with two sides of (2.2) gives us the desired formula (2.4).

For (2.5), we observe that $\|\nabla_A\psi\|^2_{L^2}\geq 0$ implies $\langle\kappa\psi,\psi\rangle + \|\psi\|^4_{L^2}\leq 0$, or equivalently

$$\|\psi\|^4_{L^2}\leq -\langle\kappa\psi,\psi\rangle\leq \kappa_X^-.\|\psi\|^2_{L^2}.$$

For (2.6), it is enough to show that if $x_0\in X$ is a point where the function $|\psi(x)|^2$ with $x\in X$ obtains its maximal value, then $|\psi(x_0)|^2\leq\max(0,-\kappa(x_0))$. Notice that for any $x\in X$, taking the product $\langle .,\psi(x)\rangle$ with two sides of (2.2) implies that

$$0 = \langle\nabla_A^*\nabla_A\psi(x),\psi(x)\rangle + \frac{\kappa(x)}{4}|\psi(x)|^2 + \frac{1}{4}|\psi(x)|^4.$$

For the given metric $g$ on $X$ and an orthonormal basis $\{e_1,e_2,e_3,e_4\}$ of $TX$, the Laplacian $\Delta_g$ is defined by $\Delta_g = -\sum_{i=1}^{4}\frac{\partial^2}{\partial e_i^2}$. We have

$$\Delta_g|\psi(x)|^2 = -2|\nabla_A\psi(x)|^2 + 2Re\langle\nabla_A^*\nabla_A\psi(x),\psi(x)\rangle.$$

Since the maximal value of the function $|\psi(x)|^2$ attained at $x_0$, we have $\sum_{i=1}^{4}\frac{\partial^2|\psi(x_0)|^2}{\partial e_i^2}\leq 0$, i.e $-2|\nabla_A\psi(x)|^2 + 2Re\langle\nabla_A^*\nabla_A\psi(x),\psi(x)\rangle = \Delta_g|\psi(x_0)|^2\geq 0$.

Since $0\geq -2|\nabla_A\psi(x_0)|^2$ and $2\langle\nabla_A^*\nabla_A\psi(x_0),\psi(x_0)\rangle = 2Re\langle\nabla_A^*\nabla_A\psi(x_0),\psi(x_0)\rangle$, we obtain

$$-\frac{\kappa(x_0)}{4}|\psi(x_0)|^2 - \frac{1}{4}|\psi(x_0)|^4 = 2\langle\nabla_A^*\nabla_A\psi(x_0),\psi(x_0)\rangle\geq 0$$

which implies that either $|\psi(x_0)|^2 = 0$ or $|\psi(x_0)|^2\leq -\kappa(x_0)$. It follows that $|\psi(x_0)|^2\leq\max(0,-\kappa(x_0))$.

If $|\psi(x_0)|^2 = 0$, then we also have $0\leq|\psi(x)|^2\leq|\psi(x_0)|^2 = 0$, i.e $\psi(x)\equiv 0$. □

**Corollary 2.29.** *For any solution $(A,\psi)$ of the Seiberg-Witten equations with moduli space of non-negative dimension, we have*

$$\|F_A^+\|^2_{L^2}\leq\frac{(\kappa_X^-)^2}{4}vol(X), \tag{2.7}$$

$$\|F_A^-\|^2_{L^2}\leq\frac{(\kappa_X^-)^2}{4}vol(X) - 8\pi^2\chi(X) - 12\pi^2\sigma(X). \tag{2.8}$$

*Proof.* Since $F_A^+ = \psi\otimes\psi^* - \frac{1}{2}|\psi|^2Id$, we have $|F_A^+(x)| = \frac{1}{2}|\psi(x)|^2\leq\frac{1}{2}\kappa_X^-$ for any $x\in X$ by the inequality (2.6). By integrating over $X$, we obtain the inequality (2.7). For the inequality (2.8), notice that we have $c_1(L) = \frac{1}{2\pi i}[F_A]$, which implies

$$\begin{aligned} c_1^2(L) &= \frac{1}{4\pi^2}\int_X F_A\wedge F_A = \frac{1}{4\pi^2}\int_X (F_A^+\wedge F_A^+ + F_A^-\wedge F_A^-) \\ &= \frac{1}{4\pi^2}\int_X (F_A^+\wedge *F_A^+ - F_A^-\wedge *F_A^-) = \frac{1}{4\pi^2}(\|F_A^+\|^2 - \|F_A^-\|^2). \end{aligned}$$

Here we use the fact that $F_A^+\wedge F_A^- = 0$, $*F_A^+ = F_A^+$ and $*F_A^- = -F_A^-$.

Since $d = \frac{1}{4}(c_1^2(L) - 2\chi(X) - 3\sigma(X))\geq 0$, we have

$$c_1^2(L)\geq 2\chi(X) + 3\sigma(X).$$

Thus, we obtain the inequality $\frac{1}{4\pi^2}(\|F_A^+\|^2 - \|F_A^-\|^2) \geq 2\chi(X) + 3\sigma(X)$ which together with (2.7) imply the inequality (2.8). □

Consider the map

$$\mathcal{SW} : \Gamma(W^+) \times \mathcal{A}_L \to \Gamma(W^-) \times i\Omega_+(X)$$
$$(\psi, A) \mapsto (\partial\!\!\!/_A \psi, F_A^+ - \sigma(\psi)).$$

The solution space $\mathcal{SW}^{-1}(0,0)$ is usually an infinite dimensional space. It admits an action of an infinite dimensional Lie group $\mathcal{G}$ of symmetries of the $Spin^c$-structure $\mathfrak{s}$ which is isomorphic to the group $Map(X, S^1)$ with the group structure given by pointwise multiplication. The group $\mathcal{G}$ is called the gauge group.

The group $\mathcal{G}$ acts on $\Gamma(W^+) \times \mathcal{A}_L$ coordinate-wise. On $\Gamma(W^+)$, $\mathcal{G}$ acts by pointwise multiplication on $X$

$$\mathcal{G} \times \Gamma(W^+) \to \Gamma(W^+)$$
$$(u, \psi) \mapsto u^*\psi = u.\psi$$

which means that $u^*\psi(x) = u(x)\psi(x)$ for all $x \in X$.

On $\mathcal{A}_L$, $\mathcal{G}$ acts by

$$\mathcal{G} \times \mathcal{A}_L \to \mathcal{A}_L$$
$$(u, A) \mapsto u^*A = A + udu^{-1}.$$

In term of covariant derivative $\nabla_A$, $\mathcal{G}$ acts by

$$u^*\nabla_A(\psi) = u(\nabla_A(u^{-1}\psi)).$$

**Lemma 2.30.** *The solution space* $\mathcal{SW}^{-1}(0,0)$ *is invariant under the action of* $\mathcal{G}$.

*Proof.* Since $u^*\nabla_A(u^*\psi) = u(\nabla_A(u^{-1}(u\psi))) = u\nabla_A(\psi)$, we have

$$\partial\!\!\!/_{u^*A}(u^*\psi) = u\partial\!\!\!/_A(\psi)$$

which means that $\mathcal{G}$ sends a solution of the equation $\partial\!\!\!/_A(\psi) = 0$ to another solution of it.

Since the manifold $X$ is simply connected, a map $u \in Map(X, S^1)$ can be written in the form $u = e^{if}$ for some real valued function $f$ on $X$. We have

$$u^*A = A + udu^{-1} = A - idf.$$

The curvature is left invariant under the action of $\mathcal{G}$ as

$$\begin{aligned} F_{u^*A} &= d(u^*A) + u^*A \wedge u^*A \\ &= d(A - idf) + (A - idf) \wedge (A - idf) \\ &= dA + A \wedge A = F_A. \end{aligned}$$

Notice here that $df$ is an exact 1-form which makes $d(df) = 0$ and $df \wedge df = 0$.

For any $x \in X$ we have

$$\begin{aligned} \sigma(u^*\psi(x)) = \sigma(u(x)\psi(x)) &= -\psi(x).u(x).i.\overline{u(x)}.\overline{\psi(x)} \\ &= -\psi(x).i.\overline{\psi(x)} = \sigma(\psi(x)), \end{aligned}$$

as $u(x)$ is a complex unit.

Both sides of the equation $F_A^+ = \sigma(\psi)$ are invariant under the action of $\mathcal{G}$.

In conclusion, the action of $\mathcal{G}$ sends a given solution to another solution of the Seiberg-Witten equations. □

We now define the Seiberg-Witten moduli spaces as

$$\mathcal{M} = \mathcal{SW}^{-1}(0,0)/\mathcal{G},$$
$$\mathcal{M}^0 = \mathcal{SW}^{-1}(0,0)/\mathcal{G}_0.$$

Here $\mathcal{G}_0 = Map((X, x_0), (S^1, 1)) = \{u : M \longrightarrow S^1 | u(x_0) = 1\}$ is the based gauge group for a fixed point $x_0$ in $X$.

**Lemma 2.31.** *The action of $\mathcal{G}_0$ on $\mathcal{SW}^{-1}(0,0)$ is always free.*

*Proof.* Note that for $u \in \mathcal{G}_0$ and $(A, \psi) \in \mathcal{SW}^{-1}(0,0)$ such that $(u^*A, u^*\psi) = (A, \psi)$, we have $u^*A = A + udu^{-1} = A$. Thus $udu^{-1} = 0$, which means that $u$ is a constant map. Since $u(x_0) = 1$, we have $u \equiv 1$ is the identity element in $\mathcal{G}_0$. □

However, the action of $\mathcal{G}$ might have non-trivial stabilizers which make the moduli space $\mathcal{M}$ to be singular.

**Lemma 2.32.** *Under the action of $\mathcal{G}$, the stabilizer of $(A, \psi) \in \mathcal{SW}^{-1}(0,0)$ is trivial when $\psi$ is not identically zero. When $\psi \equiv 0$, the stabilizer contains all constant maps from $X$ to $S^1$.*

*Proof.* For $u \in \mathcal{G}$ such that $(u^*A, u^*\psi) = (A, \psi)$, the equation $u^*A = A + udu^{-1} = A$ implies that $u$ is a constant map. From the second equation $u^*\psi = u.\psi = \psi$, we see that when $u$ is not a constant map whose value is not 1, then $\psi \equiv 0$. When $\psi$ is not identically equal 0, then $u \equiv 1$ is the identity element of $\mathcal{G}$. □

Since $\mathcal{G}/\mathcal{G}_0 \cong S^1$, we have a map $\mathcal{M}^0 \longrightarrow \mathcal{M}$ which is an $S^1$-bundle away form those points where $\psi \equiv 0$.

The topologies on $\mathcal{M}$ and $\mathcal{M}^0$ are obtained from the topology of $\Gamma(W^+) \times \mathcal{A}_L$ considered as a space of smooth functions.

Sometimes, $\mathcal{M}$ is not a smooth manifold and we need to modify the Seiberg-Witten equations.

**Definition 2.33.** *For $(X, g, \mathfrak{s})$ and $g$-self-dual 2-form $\delta$, the perturbed Seiberg-Witten equations are*

$$\begin{cases} \partial\!\!\!/_A \psi = 0 \\ F_A^+ = \sigma(\psi) + i\delta, \end{cases}$$

*where $A \in \mathcal{A}_L$ and $\psi \in \Gamma(W^+)$.*

In a similar fashion as in Proposion 2.28, we can obtain the following bounds for solutions of the perturbed Seiberg-Witten equations.

**Proposition 2.34.** *If $(A, \psi)$ is a solution of the perturbed Seiberg-Witten equations, then we have the following pointwise bound for any $x \in X$*

$$|\psi(x)|^2 \leq \max(0, 4|\delta(y)| - \kappa(y) : y \in X). \tag{2.9}$$

*Proof.* The proof can be done in the same way as in the proof of Proposition 2.28, now with the additional term $i\delta$ in the second equation of the Seiberg-Witten equations. Applying formula 2.2 gives us

$$0 = \nabla_A^* \nabla_A \psi + \frac{\kappa}{4}\psi + \frac{|\psi|^2}{4} \cdot \psi + i\delta \cdot \psi.$$

If $|\psi(x)|, x \in X$ attains its maximal value at $x_0$, then

$$-\frac{\kappa(x_0)}{4}|\psi(x_0)|^2-\frac{1}{4}|\psi(x_0)|^4-Re(\langle i\delta(x_0)\cdot\psi(x_0),\psi(x_0)\rangle)=2\langle\nabla_A^*\nabla_A\psi(x_0),\psi(x_0)\rangle\geq 0.$$

Now (2.9) is obtained by noticing that $|\delta(x_0)||\psi(x_0)|^2 \geq |Re(\langle i\delta(x_0)\cdot\psi(x_0),\psi(x_0)\rangle)|$. □

In the same way, we define the perturbed moduli spaces as

$$\mathcal{M}_\delta = \mathcal{SW}^{-1}(0, i\delta)/\mathcal{G},$$
$$\mathcal{M}_\delta^0 = \mathcal{SW}^{-1}(0, i\delta)/\mathcal{G}_0.$$

Now, if the action of the gauge group on $\mathcal{M}_\delta$ has a non-trivial stabilizer at the solution $(A,\psi)$ with $\psi \equiv 0$, then the second equation is of the form $F_A^+ = i\delta$.

**Definition 2.35.** *Reducible solutions of the perturbed Seiberg-Witten equations are those of the form* $(A, 0)$.

Fixing a metric on $X$, the Hodge theorem reads as

$$H^2(X;\mathbb{R}) \cong \mathcal{H}^2(X) = \mathcal{H}^+(X) \oplus \mathcal{H}^-(X),$$

with $b_2^+(X) = \dim \mathcal{H}^+(X)$.

For a given $Spin^c$-structure $\mathfrak{s}$ with the determinant line bundle $L = det(\mathfrak{s})$, the curvature $F_A$ represents $c_1(L)$ as a 2-form: $c_1(L) = \frac{1}{2\pi i}[F_A]$, and therefore, $F_A^+$ is the self-dual part of a fixed harmonic 2-form. Thus, when $b_2^+(X) > 0$, the equation $F_A^+ = 0$ does not have any solution for generic choice of the metric $g$. If $\delta$ is choosen to have norm $\|\delta\| = \int_X \delta\wedge\delta$ small enough, then $F_A^+ = i\delta$ also does not admit any solution, which means that no solution with non-trivial stabilizer exists, and therefore, the moduli space $\mathcal{M}_\delta$ does not have any singular point for generic choice of metric $g$. Because of this reason, we will consider the case when $b_2^+ > 0$.

Fixing the $Spin^c$-structure $\mathfrak{s}$, the perturbed moduli spaces have following properties:

**Theorem 2.36.** [23, Proposition 6.4.1] *The moduli space* $\mathcal{M}_\delta$ *is always compact.*

*Proof.* We only sketch the proof. Detailed treatments can be found in [23, Section 5.3] and [22, Section 3.3].

We need to show that if $\{(A_n,\psi_n)\}$ is a sequence of solutions to the perturbed Seiberg-Witten equations, then there exists a sequence of gauge transformations $\{u_n\}$ such that $\{u_n^*A_n, u_n^*\psi_n\}$ contains a subsequence which converges smoothly to a solution of the perturbed Seiberg-Witten equations.

By fixing a base connection $A_0$ on the determinant line bundle $L$, any $U(1)$-connection on $L$ can be written as $A = A_0 + \alpha_0$ for $\alpha_0 \in \Omega^1(X, i\mathbb{R})$. Moreover, for any connection $A$ there is a gauge transformation $u \in Map(X, S^1)$ such that $u^*(A) = A_0 + \alpha_1$, where $\alpha_1$ has the properties that $d^*\alpha_1 = 0$ and $\|\alpha_1\|^2 \leq c_1.\|F_A^+\|^2 + c_2$ for some constants $c_1, c_2$.

Recall that in this case the adjoint operator $d^*$ of derivative $d : \Omega^0(X, i\mathbb{R}) \longrightarrow \Omega^1(X, i\mathbb{R})$ is defined by $d^* = -*d*$. Since $(df, \alpha_0) = (f, d^*\alpha_0) = 0$ for any constant function $f$ on $X$, $d^*(\alpha_0)$ is orthogonal to all constant functions on $X$. Denote the orthogonal complement of the constant functions on $X$ by $I$. The restriction of the Laplace operator to $I$ has an inverse $\Delta^{-1} : I \longrightarrow I$. Define an $i\mathbb{R}$-valued function $s_0 = \frac{-1}{2}\Delta^{-1}(d^*\alpha_0)$ on $X$, and a gauge transformation $u = exp(s_0)$. Consider $\alpha_1 = \alpha_0 + 2du$, we have $u^*(A) = A_0 + \alpha_1$ and $d^*(\alpha_1) = 0$.

For the determinant line bundle $L$ with a given connection $A_0$, we can define the following norms on $\Gamma(L)$:

For $p > 1, k \geq 1$ and $f \in \Gamma(L)$, let

$$\|f\|_{L^p_k} = \Big(\int_X (|f|^p + |d_{A_0} f|^p + \cdots + |d^k_{A_0} f|^p) dx\Big)^{1/p}.$$

The completion $L^p_k$ of $\Gamma(L)$ with respect to the norm $\|.\|_{L^p_k}$ are called the Sobolev spaces. For any $p > 1$, $L^p_k$ is a Banach space. In our case of the 4-dimensional manifold $X$, the Sobolev embedding theorem says that there is a countinous embedding of $L^p_k$ into the space of $C^r$ sections of the line bundle $L$ when $k - 4/p > r$. Moreover, by the Rellich-Kondrachov theorem, the inclusion $L^p_{k+1} \longrightarrow L^p_k$ is compact for all $p, k$, which means that any bounded sequence in $L^p_{k+1}$ admits a subsequence which converges in $L^p_k$.

For two Sobolev spaces $L^p_k, L^q_l$ and a positive integer $n$ such that $k/n - 1/p < 0$ and $l/n - 1/q < 0$, the pointwise multiplication of functions extends to a continous multiplication, called the Sobolev multiplication of Sobolev spaces

$$L^p_k \times L^q_l \longrightarrow L^r_m$$

when $0 \leq m \leq \min(k, l)$ and $0 < m/n + (-k/n + 1/p) + (-l/n + 1/q) \leq 1/r \leq 1$.

Now for a sequence $\{(A_n, \psi_n)\} = \{(A_0 + \alpha_n, \psi_n)\}, \alpha_n \in \Omega^1(X, i\mathbb{R})$ of solutions to the perturbed Seiberg-Witten equations, we can suppose $d^*\alpha_n = 0$ by using gauge transformations. The perturbed Seiberg-Witten equations read as

$$\begin{cases} \not{\partial}_{A_0} \psi_n + \alpha_n \cdot \psi_n = 0, \\ F^+_{A_0} + (d\alpha_n)^+ = \sigma(\psi_n) + i\delta, \\ d^*\alpha_n = 0, \end{cases}$$

which can be rewritten as

$$\begin{cases} \not{\partial}_{A_0} \psi_n = -\alpha_n \cdot \psi_n, \\ (d^* + d^+)\alpha_n = \sigma(\psi_n) + i\delta - F^+_{A_0}. \end{cases} \tag{2.10}$$

We have the pointwise bound (2.9) for $\psi_n$ and an $\|.\|_{L^2}$ upper-bound for $\nabla_{A_n} \psi_n$. Since $\alpha_n$ is bounded in $L^2_2$, we obtain that $\|\psi_n\|_{L^2_1} = \big(\int_X (|\psi_n|^2 + |d_{A_0} \psi_n|^2) dx\big)^{1/2}$ is also bounded from the first equation of (2.10).

Using the Sobolev multiplications and the first equation of (2.10), one can showed that $\psi_n$ is bounded in $L^2_k, k = 1, 2, 3$ (See [23, Theorem 5.3.6]). From the second equation of (2.10), it follows that $\|F^+_A\|$ is bounded in $L^2_3$, and therefore, $\alpha_n$ is bounded in $L^2_4$.

Suppose that for $k \geq 3$, $\psi_n$ and $\alpha_n$ are bounded in $L^2_k$. From the first equation of (2.10) and the Sobolev multiplication $L^2_k \times L^2_k \longrightarrow L^2_k$, we obtain that $\not{\partial}_{A_0} \psi_n$ is bounded in $L^2_k$, and therefore $\psi_n$ is bounded in $L^2_{k+1}$. From the second equation of (2.10), we have an $L^2_k$ bound for $F^+_{A_0}$, which implies that $\alpha_n$ is bounded in $L^2_{k+1}$.

We conclude that $\psi_n$ and $\alpha_n$ are bounded in $L^2_k$ for all $k$. Therefore, $(A_0 + \alpha_n, \psi_n)$ contains a subsequence $(A_0 + \alpha_{n_i}, \psi_{n_i})$ which converges in $L^2_k$ for all $k$ by the Rellich-Kondrachov theorem. The Sobolev embedding theorem then implies that $(A_0 + \alpha_{n_i}, \psi_{n_i})$ also converges in $C^r$ for all $r$, i.e $(A_0 + \alpha_{n_i}, \psi_{n_i})$ converges smoothly to a solution of the perturbed Seiberg-Witten equations. □

**Theorem 2.37.** [23, Theorem 6.1.1] *$\mathcal{M}_\delta^0$ is a smooth manifold of finite dimension for generic choices of $g$ and $\delta$. When $b_2^+ > 0$, $\mathcal{M}_\delta$ is also a smooth manifold for generic choice of $\delta$ and we have an principal $S^1$-bundle $\mathcal{M}_\delta^0 \longrightarrow \mathcal{M}_\delta$.*

**Theorem 2.38.** [23, Corollary 6.6.3] *For a generic choice of $\delta$, the manifold $\mathcal{M}_\delta$ is orientable. Orientations on $\mathcal{M}_\delta$ are in one-to-one correspondence with orientations of the vector space $H^0(X;\mathbb{R}) \oplus H^2_+(X;\mathbb{R})$.*

Here $H^2_+(X;\mathbb{R})$ is the maximal positive definite subspace of $H^2(X;\mathbb{R})$ with respect to the quadratic form $Q$ on $H^2(X;\mathbb{R})$ defined by $Q(\alpha,\beta) = \int_X \alpha \wedge \beta$.

**Theorem 2.39.** [23, Proposition 6.2.2] *Suppose that $b_2^+(X) > 0$. Fix a Riemannian metric $g$ on $X$ and an orientation of the vector space $H^0(X;\mathbb{R}) \oplus H^2_+(X;\mathbb{R})$. For a generic choice of $g$-self-dual 2-form $\delta$, the moduli space $\mathcal{M}_\delta$ is a smooth, compacted, oriented manifold of dimension $d(\mathfrak{s}) = \frac{1}{4}(c_1^2(L) - 3\sigma(X) - 2\chi(X))$ for any $Spin^c$-structure $\mathfrak{s}$ on $X$ with determinant line bundle $L = det(\mathfrak{s})$.*

2.6. **The Seiberg-Witten invariants.** Assuming that $b_2^+(X) > 0$, we have an principal $S^1$-bundle $\mathcal{M}_\delta^0 \longrightarrow \mathcal{M}_\delta$. Denote the first Chern class of this bundle by $\mu \in H^2(\mathcal{M}_\delta;\mathbb{Z})$. We fix an orientation on the vector space $H^0(X;\mathbb{R}) \oplus H^2_+(X;\mathbb{R})$. This orientation induces an orientation on $\mathcal{M}_\delta$, and we have a homology class $[\mathcal{M}_\delta] \in H_d(\mathcal{M}_\delta;\mathbb{Z})$.

**Definition 2.40.** *For a closed, oriented, simply connected smooth 4-manifold $X$ with $b_2^+(X) > 1$, the Seiberg-Witten invariant of $X$*

$$SW_X : \mathcal{S}_X \longrightarrow \mathbb{Z}$$

*is defined for a generic metric $g$ and a generic choice of a $g$-self-dual 2-form $\delta$ as follows. Denote the dimension $\mathcal{M}_\delta$ by $d$.*

*i) If $d < 0$ for a $Spin^c$-structure $\mathfrak{s}$, then $SW_X(\mathfrak{s}) = 0$.*

*ii) If $d = 0$ for a $Spin^c$-structure $\mathfrak{s}$, then $\mathcal{M}_\delta$ is a finite set of points. The orientation on $\mathcal{M}_\delta$ assigns $+1$ or $-1$ to each point. We define $SW_X(\mathfrak{s})$ to be the sum of these $\pm 1$ numbers over $\mathcal{M}_\delta$.*

*iii) If $d$ is positive and odd, then $SW_X(\mathfrak{s}) = 0$.*

*iv) If $d$ is positive and even, then $SW_X(\mathfrak{s}) = \langle \mu^{\frac{d}{2}}, [\mathcal{M}_\delta] \rangle$.*

The fundamental property of Seiberg-Witten invariants is the following:

**Lemma 2.41.** [23, Lemma 6.7.1] *If $b_2^+ > 1$, then the definition of $SW_X(\mathfrak{s})$ does not depend on the choice of pertubation $\delta$ and the choice of metric $g$.*

Thus, we obtain that

**Theorem 2.42.** [23, Theorem 6.7.3] *For a closed, oriented, simply connected smooth 4-manifold $X$ with $b_2^+(X) > 1$, $SW_X(\mathfrak{s})$ is a differential topological invariant which depends only on $Spin^c$-structure $\mathfrak{s}$. Moreover, $SW_X$ vanishes for all but finitely many $Spin^c$-structures.*

Since orienting moduli spaces is rather complicated, and in fact, we will only need the cases when $d(\mathfrak{s}) = 0$, i.e when the moduli space is a finite set of points, the following alternative definition of the Seiberg-Witten invariant allows us to avoid orientation issues.

**Definition 2.43.** *The (mod 2) Seiberg-Witten invariant of X*

$$SW_X^{(2)} : \mathcal{S}_X \longrightarrow \mathbb{Z}$$

*is defined by*

*i) If* $d(\mathfrak{s}) = 0$, *then* $SW_X^{(2)}(\mathfrak{s}) = \#\mathcal{M}_\delta$ *(mod 2).*

*ii) If* $d(\mathfrak{s})$ *is positive and even, then* $SW_X^{(2)}(\mathfrak{s}) = \langle \mu_2^{\frac{d}{2}}, [\mathcal{M}_\delta] \rangle$. *Now without orientations, we take the homology class* $[\mathcal{M}_\delta] \in H_d(\mathcal{M}_\delta; \mathbb{Z}_2)$, *and* $\mu_2 \in H^2(\mathcal{M}_\delta; \mathbb{Z}_2)$ *is the reduction (mod 2) of the first Chern class of the* $S^1$*-bundle* $\mathcal{M}_\delta^0 \longrightarrow \mathcal{M}_\delta$ .

*iii)* $SW_X^{(2)}(\mathfrak{s}) = 0$ *for all other values of* $d(\mathfrak{s})$.

**Remark 2.44.** *Although the (mod 2) Seiberg-Witten invariants are enough for us to prove the exoticness of* $X_1$*, the integral invariants are indispensable for other purposes. For instance, Park, Stipsicz and Szabó showed in [26] that there are infinitely many pairwise non-diffeomorphic structures on* $\mathbb{CP}^2 \sharp 5\overline{\mathbb{CP}^2}$. *This kind of result clearly cannot obtained by using the (mod 2) invariants.*

Since we are dealing with $\mathbb{CP}^2 \sharp 9\overline{\mathbb{CP}^2}$, we need to define the invariant for the case $b_2^+(X) = 1$. Note that for each Riemannian metric $g$ of $X$, there exist a unique $g$-self-dual 2-form $\omega_g$ in $\Omega_+(X)$ such that $[\omega_g]^2 = 1$ and $\langle \omega, h \rangle > 0$, where $h$ is the generator of $H_2(\mathbb{CP}^2; \mathbb{Z})$ and $h$ is considered as an element of $H_2(\mathbb{CP}^2 \sharp 9\overline{\mathbb{CP}^2}; \mathbb{Z})$ . Since we also have $\dim H_+^2(X; \mathbb{R}) = b_2^+(X) = 1$, the 2-form $\omega_g$ is determined by the metric $g$ up to sign. As we already fixed an orientation on $H_+^2(X; \mathbb{R})$, we can choose the sign of $\omega_g$ to be positive.

**Lemma 2.45.** *For a metric* $g$ *on* $X$ *and a* $g$*-self-dual 2-form* $\delta$*, the perturbed Seiberg-Witten equations have a reducible solution if and only if*

$$(2\pi c_1(L) + [\delta]).[\omega_g] = 0.$$

*Proof.* The Seiberg-Witten equations have a reducible solution if $F_A^+ = i\delta$. Since $c_1(L) = \frac{1}{2\pi i}[F_A]$, we have $(2\pi c_1(L) + [\delta])^+ = 0$ which means that $2\pi c_1(L) + [\delta]$ is an anti-self-dual 2-form. Because $[\omega_g]$ is a self-dual 2-form, this is equivalent to

$$(2\pi c_1(L) + [\delta]).[\omega_g] = 0.$$

□

**Proposition 2.46.** *If a smooth 4-manifold* $X$ *is simply connected with* $b_2^+(X) = 1$ *and* $b_2^-(X) \leq 9$*, then the Seiberg-Witten invariant of* $X$ *does not depend on the choice of generic metrics.*

*Proof.* Let $L$ be a determinant line bundle of some $Spin^c$-structure on $X$ such that the formal dimension $d(L) = c_1^2(L) - 3\sigma(X) - 2\chi(X)$ is non negative and even. Notice that we have

$\sigma(X) = b_2^+(X) - b_2^-(X) = 1 - b_2^-(X)$ and $\chi(X) = 2 + b_2^+(X) + b_2^-(X) = 3 + b_2^-(X)$.

Thus, $c_1^2(L) = d(L) + 3\sigma(X) + 2\chi(X) = d(L) + 9 - b_2^-(X)$. When $b_2^-(X) \leq 9$, we have $c_1^2(L) \geq 0$.

If $c_1(L) = 0$, then $c_1^2 = 0$, i.e $d(L) + 9 - b_2^-(X) = 0$. Since $d(L) \geq 0$ and $b_2^-(X) \leq 9$, we have $d(L) = 0$ and $b_2^-(X) = 9$. Since $L$ is a characteristic element, we also have $w_2(X) \equiv c_1(L) = 0$ ( mod 2), which means that $X$ admit a *Spin*-structure. Rokhlin's theorem (Theorem 2.7) implies that the signature of $X$ is divisible by 16. But in this case, the signature $\sigma(X) = 1 - 9 = -8$ is not divisible by 16. This contradiction shows that $c_1(L) \neq 0$.

Thus, we obtain that $c_1(L).[\omega_g]$ is not equal to zero and therefore has a fixed sign for all generic metrics.

The harmonic 2-form representing $c_1(L)$ is non-zero and lies in $H^2_+(X;\mathbb{R})$. Therefore, we have $2\pi c_1(L).[\omega_g] \neq 0$ for generic metric $g$. In conclusion, the sign of $(2\pi c_1(L) + [\delta]).[\omega_g] = 0$ is independent of the choice of metric $g$ for generic perturbation $\delta$ with norm small enough. □

Now we can see that in the case of $b_2^+ = 1$ and $b_2^-$ arbitrary, the space of generic metrics and generic perturbations with small norm for which the Seiberg-Witten equations do not admit reducible solutions is divided into two disjoined parts (exactly one of them is empty when $b_2^+ \leq 9$). We call them plus and minus chamber according to the sign of $(2\pi c_1(L)+[\delta]).[\omega_g]$. The Seiberg-Witten invariants are now well defined in the following sense.

**Theorem 2.47.** [23, Theorem 6.9.2] *Let $X$ be a closed, oriented, simply connected smooth 4-manifold with $b_2^+(X) = 1$. Then for any Riemannian metric $g$ and any $Spin^c$-tructure on $X$ with determinant line bundle $L$ for which $c_1(L).[\omega_g] \neq 0$, Seiberg-Witten invariants are well defined and constant on each of two chambers.*

We write $SW_{X,g,\delta}(\mathfrak{s})$ and $SW^{(2)}_{X,g,\delta}(\mathfrak{s})$ to indicate that the invariant depending on metric, perturbation and $Spin^c-$structure.

When $b_2^+(X) = 1$ and $b_2^-(X) \leq 9$, one of two chambers is empty according to Proposition 2.46.

When $b_2^+(X) = 1$ and $b_2^-(X) > 9$, there are two chambers and we fix cohomology classes $\eta_+, \eta_-$ with non-negative squares which represent the positively oriented harmonic self-dual 2-forms for some metrics on plus and minus chambers respectively, then we have the so called wall-crossing formula.

**Theorem 2.48.** [23, Proposition 6.9.4] *For a $Spin^c$-structure $\mathfrak{s}$ of even formal dimension $d$ on $X$ with $b_2^+(X) = 1$ and $b_2^-(X) > 9$, we have*

$$SW_{X,g_+,\delta_+}(\mathfrak{s}) = SW_{X,g_-,\delta_-}(\mathfrak{s}) - (-1)^{\frac{d}{2}},$$

*where the pairs $(g_+, \delta_+)$ and $(g_-, \delta_-)$ are choosen arbitraryly from the two chambers $\eta_+$ and $\eta_-$ respectively.*

In particular, when $d = 0$, crossing a wall once will change the Seiberg-Witten invariants by 1.

**Theorem 2.49.** *For a $Spin^c$-structure $\mathfrak{s}$ of formal dimension $d = 0$ on $X$ with $b_2^+(X) = 1$ and $b_2^-(X) > 9$, we have*

$$SW^{(2)}_{X,g_+,\delta_+}(\mathfrak{s}) = SW^{(2)}_{X,g_-,\delta_-}(\mathfrak{s}) + 1.$$

2.7. **The Seiberg-Witten invariants in the case of $b_2^+ = 1$.** Inequality (2.5) means that if the metric $g$ on $X$ has positive scalar curvature, then every solution of the Seiberg-Witten equations must satisfy $\psi \equiv 0$. This already implies the following vanishing result of Witten.

**Theorem 2.50** (Witten, [34])**.** *All Seiberg-Witten invariants of an oriented Riemannian 4-manifold with metric of positive scalar curvature vanish.*

Of particular important for us, the 4-manifolds $\mathbb{CP}^2$ and $\overline{\mathbb{CP}^2}$ admit metrics of positive curvature. On $\mathbb{CP}^2$, we consider the $(1,1)-$form

$$\omega_{FS}([z_0 : z_1 : z_2]) = \frac{i}{2\pi}\sum_{i,j=0}^{2}\frac{\partial^2}{\partial z_i \partial \bar{z}_j}\log(|z_0|^2 + |z_1|^2 + |z_2|^2)dz_i d\bar{z}_j.$$

The complex structure on $\mathbb{CP}^2$ induces a vector bundle isomorphism $J : T\mathbb{CP}^2 \longrightarrow T\mathbb{CP}^2$ which satisfies $J^2 = -1$. $J$ is called an almost complex structure on $\mathbb{CP}^2$. We can associate a Hermitian metric to $\omega_{FS}$ as follows:

$$g_{FS}(U,V) = \omega_{FS}(U, JV), \text{ where } U, V \in T\mathbb{CP}^2.$$

**Definition 2.51.** *The Hermitian metric $g_{FS}$ is called the Fubini-Study metric on $\mathbb{CP}^2$.*

The Fubini-Study metric on $\overline{\mathbb{CP}^2}$ is the same as that on $\mathbb{CP}^2$. It is known that the Fubini-Study metric have positive Ricci curvature, and therefore its scalar curvatures are positive (see [5]).

**Theorem 2.52.** [12, Gromov-Lawson] *If $X_1$ and $X_2$ are compact $n$-manifolds, $n \geq 3$, with positive scalar curvature, then their connected sum also carries positive scalar curvature.*

Thus, the connected sums of $\mathbb{CP}^2$'s and $\overline{\mathbb{CP}^2}$'s also admit metrics with positive scalar curvature.

**Proposition 2.53.** *The smooth 4-manifold $\mathbb{CP}^2\sharp n\overline{\mathbb{CP}^2}$ admits a Riemannian metric $g_0$ with positive scalar curvature for any non-negative integer $n$.*

Since $b_2^-(\mathbb{CP}^2\sharp 17\overline{\mathbb{CP}^2}) = 17 > 9$, the Seiberg-Witten invariants of $\mathbb{CP}^2\sharp 17\overline{\mathbb{CP}^2}$ also depend on the choosen metric. Combining Theorem 2.50 and Proposition 2.53 implies that

**Proposition 2.54.** *For all parameter $\delta$ with norm small enough or $\delta = 0$, we have*

$$SW^{(2)}_{\mathbb{CP}^2\sharp 17\overline{\mathbb{CP}^2}, g_0, \delta}(\mathfrak{s}) = 0$$

*for every $Spin^c-$structure $\mathfrak{s}$ such that $d(\mathfrak{s}) = 0$.*

Let us choose a second homology class $h \in H_2(\mathbb{CP}^2\sharp 17\overline{\mathbb{CP}^2}; \mathbb{Z})$ such that $PD(h) = [\omega_{g_0}]$. We fix a basis of $H_2(\mathbb{CP}^2\sharp 17\overline{\mathbb{CP}^2}; \mathbb{Z})$ by $h, e_1, e_2, \cdots, e_{17}$, and write $K = PD(3h - \sum_{i=1}^{17} e_i)$. We will find a homology class representing the chamber which does not contain $PD(h)$.

**Lemma 2.55.** *There is a homology class $\alpha \in H_2(\mathbb{CP}^2\sharp 17\overline{\mathbb{CP}^2}; \mathbb{Z})$ such that $\alpha.h > 0$, $\alpha.v = 0$ for all homology classes $v$ at vertices of the plumbing diagram given in Figure 18, and $K(\alpha) < 0$.*

*Proof.* Suppose that $\alpha = a.h + \sum_{i=1}^{17} b_i.e_i$ is a homology class to be determined later with unknown integers $a > 0, b_1, \cdots, b_{17}$. The condition $\alpha.v = 0$ for all homology classes $v$ at vertices of the plumbing diagram given in Figure 18 implies the following system of linear equations

$$\begin{cases} a + b_1 + b_2 + b_3 + b_4 = 0 \\ a + b_5 + b_7 + b_8 + b_9 = 0 \\ a + b_5 + b_{11} + b_{12} + b_{13} = 0 \\ a + b_9 + b_{13} + b_{15} + b_{16} = 0 \\ b_3 - b_{17} = 0 \\ b_2 - b_3 = 0 \\ 2a + b_1 + b_2 + b_5 + b_6 + b_9 + b_{10} + b_{11} + b_{15} = 0 \\ 2a + b_1 + b_4 + b_7 + b_8 + b_{11} + b_{13} + b_{14} + b_{15} = 0, \end{cases}$$

which is equivalent to

$$\begin{cases} b_3 = b_{17} = b_2 \\ b_4 = -a - b_1 - 2b_2 \\ b_9 = -a - b_5 - b_7 - b_8 \\ b_{13} = -a - b_5 - b_{11} - b_{12} \\ b_{14} = a + b_2 - b_4 + 2b_5 + b_6 - b_7 - b_8 + b_9 + b_{10} + b_{11} + b_{12} \\ b_{15} = -2a - b_1 - b_2 - b_5 - b_6 - b_9 - b_{10} - b_{11} \\ b_{16} = 2a + b_1 + b_2 + 2b_5 + b_6 - b_{10} + b_{12}. \end{cases}$$

The second condition $K(\alpha) < 0$ means that $3a + \sum_{i=1}^{17} b_i < 0$. We can choose a solution, for instance, $a = 1$, $b_4 = b_9 = b_{14} = -1$, $b_{13} = 1$, $b_{15} = -2$ and $b_i = 0$ for all other values of $i$, and obtain a suitable homology class

$$\alpha = h - e_4 - e_9 - 2e_{12} + e_{13} - e_{14} - 2e_{15},$$

for which $K(\alpha) = -1$ as desired. □

Now we consider the $Spin^c$−structure $\mathfrak{s}_K$ associated to the cohomology class $K = PD(3h - \sum_{i=1}^{17} e_i)$. Note that we will prove later in the proof of Theorem 2.59 that $d(\mathfrak{s}_K) = 0$.

**Lemma 2.56.** *The cohomology classes $PD(h)$ and $PD(\alpha)$ correspond to different chambers.*

*Proof.* This is obtained by noticing that $K(h) = 3 > 0, h.\alpha = 3 > 0$ and $K(\alpha) = -1 < 0$. □

The chamber corresponding to $PD(h)$ contains $g_0$. Choose a metric $g_1$ in the chamber corresponding to $PD(\alpha)$. Theorem 2.49 implies that

**Proposition 2.57.** *For all perturbed parameter $\delta$ with norm small enough, we have*

$$SW^{(2)}_{\mathbb{CP}^2\sharp 17\overline{\mathbb{CP}^2}, g_1, \delta}(\mathfrak{s}_K) = 1.$$

2.8. **Seiberg-Witten invariants of $X_1$.** Our goal is to show that the 4-manifold

$$X_1 = (\mathbb{CP}^2 \sharp 17\overline{\mathbb{CP}^2} - \text{ int}(P\mathcal{A}_2)) \bigcup_{\partial P\mathcal{A}_2} B$$

admits a metric $\tilde{g}_1$ such that $SW^{(2)}_{X_1,\tilde{g}_1,\delta}(\mathfrak{s}_{\widetilde{K}}) = 1$, where $\delta$ is a perturbation with norm small enough and $\mathfrak{s}_{\widetilde{K}}$ is the $Spin^c-$structure asscociated to the cohomology class $\widetilde{K}$ defined in Definition 1.32, and we will show that $d(\mathfrak{s}_{\widetilde{K}}) = 0$ in the proof of Theorem 2.59. To this end, we use a scheme for computing Seiberg-Witten invariants of blow-downs along Wahl-type plumbing trees introduced by Michalogiorgaki in [18].

We provide a brief review of Michalogiorgaki's argument in [18] as follows. Let $\Gamma$ be a Wahl-type plumbing tree. Suppose that we have an embedding of $P\Gamma$ into a closed, oriented, simply connected smooth 4-manifold $X$. The boundary of $P\Gamma$ bounds a rational homology ball $B$. We define the rational blow-down of $X$ along $P\Gamma$ as $X' = (X - int(P\Gamma)) \bigcup_{\partial P\Gamma} B$.

Using monopole Floer homology of Kronheimer and Mrowka, Michalogiorgaki proved that

**Theorem 2.58** (Michalogiorgaki, [18], Theorem 1)**.** *Suppose that $H_1(\partial P\Gamma)$ is finite, $P, B$ have negative definite intersection forms, and the first Betti numbers of $P$ and $B$ are zero . If $\mathfrak{s} \in \mathcal{S}_X$ and $\mathfrak{s}' \in \mathcal{S}_{X'}$ are $Spin^c$-structures such that $d(\mathfrak{s}) = d(\mathfrak{s}') = 0$ and $\mathfrak{s}_{|X-int(P\Gamma)} = \mathfrak{s}'_{|X-int(P\Gamma)}$, then $SW^{(2)}_X(\mathfrak{s}) = SW^{(2)}_{X'}(\mathfrak{s}')$.*

*In the case $b_2^+(X) = 1$, $SW^{(2)}_{X,g_1}(\mathfrak{s}) = SW^{(2)}_{X',g_2}(\mathfrak{s}')$, where $g_1$ and $g_2$ are metrics induced by $a_1 \in H_2(X;\mathbb{Z})$ and $a_2 \in H_2(X';\mathbb{Z})$ homology classes with properties that $a_{1|P} = a_{2|B} = 0$ and $a_{1|X-int(P\Gamma)} = a_{2|X-int(P\Gamma)}$.*

Note that $\partial P\Gamma$ is a so called monopole L-space as explained in [18]. Applying Theorem 2.58 to our case of the plumbing tree $\Gamma = \mathcal{A}_2$, $X = \mathbb{CP}^2\sharp 17\overline{\mathbb{CP}^2}$, $Z = \mathbb{CP}^2\sharp 17\overline{\mathbb{CP}^2} - int(P\mathcal{A}_2)$ and the rational homology ball $B$ defined in Section 1.4, we obtain the following:

**Proposition 2.59.** *We have $SW^{(2)}_{X_1,\tilde{g}_1}(\mathfrak{s}_{\widetilde{K}}) = SW^{(2)}_{\mathbb{CP}^2\sharp 17\overline{\mathbb{CP}^2},g_1}(\mathfrak{s}_K) = 1$, where $X_1$ is our rational blow-down 4-manifold, $\tilde{g}_1$ is a metric on $X_1$ induced by $PD(\widetilde{K})$, where $\widetilde{K}$ is defined in Definition 1.32, and $g_1$ is a metric on $\mathbb{CP}^2\sharp 17\overline{\mathbb{CP}^2}$ defined in Proposition 2.57.*

*Proof.* We verify the conditions of Theorem 2.58 one by one. The fact that $H_1(\partial P\mathcal{A}_2)$ is finite follows from Formula 1.3. The intersection form of $P\mathcal{A}_2$ is negative definite by the definition of $\mathcal{A}_2$. Since $P\mathcal{A}_2$ admits a handlebody decompostion with only one 0-handle and a collection of 2-handles, its first Betti number is 0. Since $B$ is a rational homology ball, the intersection form of $B$ vanishes and its first Betti number is also 0. We have $d(\mathfrak{s}_K) = \frac{1}{4}(c_1^2(K) - \sigma(\mathbb{CP}^2\sharp 17\overline{\mathbb{CP}^2}) - \chi(\mathbb{CP}^2\sharp 17\overline{\mathbb{CP}^2})) = \frac{1}{4}((9-17) - 3(1-17) - 2(2+1+17)) = 0$. We can use familiar properties of the first Chern number, signature and Euler number to compute $d(\mathfrak{s}_{\widetilde{K}})$ as following:

$$\begin{aligned} d(\mathfrak{s}_{\widetilde{K}_{|Z}}) &= \frac{1}{4}(c_1^2(\widetilde{K}_{|Z}) - 3\sigma(Z) - 2\chi(Z)) \\ &= \frac{1}{4}(c_1^2(K_{|Z}) - 3(\sigma(\mathbb{CP}^2\sharp 17\overline{\mathbb{CP}^2}) - \sigma(P\mathcal{A}_2)) \\ &- 2(\chi(\mathbb{CP}^2\sharp 17\overline{\mathbb{CP}^2}) - \chi(P\mathcal{A}_2) + \chi(\partial P\mathcal{A}_2)))) \\ &= \frac{1}{4}((c_1^2(K) - c_1^2(K_{|P\mathcal{A}_2})) - 3(1-9) - 2(20-10+1)) \\ &= \frac{1}{4}((9-17+8) - 3(1-9) - 2(20-10+1)) = \frac{1}{2}, \end{aligned}$$

$$\begin{aligned} d(\mathfrak{s}_{\bar{K}_{|B}}) &= \frac{1}{4}(c_1^2(\bar{K}_{|B}) - 3\sigma(B) - 2\chi(B)) \\ &= \frac{1}{4}(c_1^2(\bar{K}) - c_1^2(\bar{K}_{|P\mathcal{A}'_2}) - 3(\sigma(\mathbb{CP}^2\sharp 11\overline{\mathbb{CP}^2}) - \sigma(P\mathcal{A}'_2)) \\ &- 2(\chi(\mathbb{CP}^2\sharp 11\overline{\mathbb{CP}^2}) - \chi(P\mathcal{A}'_2))) \\ &= \frac{1}{4}(0 - 3.0 - 2.1) = \frac{-1}{2}, \end{aligned}$$

$$d(\mathfrak{s}_{\widetilde{K}}) = d(\mathfrak{s}_{\widetilde{K}_{|Z}}) + d(\mathfrak{s}_{\bar{K}_{|B}}) = 0.$$

Notice that we have constructed the homology classes $\widetilde{K}$ and $\alpha$ in such away that they satisfy the properties: $\alpha_{|P} = \widetilde{K}_{|B} = 0$ and $\alpha_{|\mathbb{CP}^2\sharp 17\overline{\mathbb{CP}^2} - int(P\mathcal{A}_2)} = \widetilde{K}_{|\mathbb{CP}^2\sharp 17\overline{\mathbb{CP}^2} - int(P\mathcal{A}_2)}$. Thus, from Theorem 2.58 and Proposition 2.57 we obtain $SW^{(2)}_{X_1,\tilde{g}_1}(\mathfrak{s}_{\widetilde{K}}) = SW^{(2)}_{\mathbb{CP}^2\sharp 17\overline{\mathbb{CP}^2}, g_1}(\mathfrak{s}_K) = 1$ for the metric $\tilde{g}_1$ on $X_1$ induced by $\widetilde{K}$. □

We are now in the position to prove that $X_1$ is non-diffeomorphic to $\mathbb{CP}^2\sharp 9\overline{\mathbb{CP}^2}$.

**Theorem 2.60.** *There is a smooth structure on $X_1$ which is non-diffeomorphic to the standard smooth structure on $\mathbb{CP}^2\sharp 9\overline{\mathbb{CP}^2}$.*

*Proof.* By Theorem 2.50 and Proposition 2.53, if we denote by $g_0$ the metric on $\mathbb{CP}^2\sharp 9\overline{\mathbb{CP}^2}$ induced by Fubini-Study metrics, then $SW^{(2)}_{\mathbb{CP}^2\sharp 9\overline{\mathbb{CP}^2}, g_0}(\mathfrak{s}) = 0$ for any $Spin^c$-structure $\mathfrak{s}$ on $\mathbb{CP}^2\sharp 9\overline{\mathbb{CP}^2}$.

By Proposition 2.59, there exists a metric $\tilde{g}_1$ and a $Spin^c$-structure $\mathfrak{s}_{\widetilde{K}}$ on $X_1$ such that $SW^{(2)}_{X_1,\tilde{g}_1}(\mathfrak{s}_{\widetilde{K}}) = 1$.

Thus, the fact that Seiberg-Witten invariant is a differential topological invariant for smooth 4-manifolds implies that $X_1$ with the smooth structure induced by the metric $\tilde{g}_1$ and the $Spin^c$-structure $\mathfrak{s}_{\widetilde{K}}$ is non-diffeomorphic to the canonical smooth structure on $\mathbb{CP}^2\sharp 9\overline{\mathbb{CP}^2}$. □

In conclusion, we have constructed an exotic structure on $\mathbb{CP}^2\sharp 9\overline{\mathbb{CP}^2}$ by rational blowing-down along the four branched plumbing tree $\mathcal{A}_2$.

## Appendix: An embedding of $P\mathcal{A}'_2$ into $\mathbb{CP}^2\sharp 11\overline{\mathbb{CP}^2}$

In Section 1.4, we use a topological construction of Bhupal and Stipsicz in [4] to construct a rational homology ball $B$ whose boundary is diffeomorphic to the boundary of the manifold $P\mathcal{A}_2$. More precisely, Bhupal and Stipsicz give an embedding of $P\mathcal{A}'_2$ into $\mathbb{CP}^2\sharp 11\overline{\mathbb{CP}^2}$ by blowing-up intersection points of a special configuration of complex projective curves (see [4, Section 4.3, Figure 13]). Since we also need the homology classes represented by embedding spheres of the plumbing manifold $P\mathcal{A}'_2$ (see Figure 7) for homological calculations in Section 1.6, we give here a detailed explanation of the blow-up processes in the construction of Bhupal and Stipsicz in [4], as well as compute homology classes of curves in each step.

Recall from [4, Section 4.3] that the curves used by Bhupal and Stipsicz are:

$$
\begin{aligned}
&L_1 = \{[x:y:z] \in \mathbb{CP}^2 | z = 0\},\\
&L_2 = \{[x:y:z] \in \mathbb{CP}^2 | -i\sqrt{3}(x + \frac{8}{9}z) + y = 0\},\\
&C_1 = \{[x:y:z] \in \mathbb{CP}^2 | x^3 + x^2 z - y^2 z = 0\},\\
&C_2 = \{[x:y:z] \in \mathbb{CP}^2 | \frac{-1+i\sqrt{3}}{2}x^3 + (-2+i\sqrt{3})x^2 z + (1-i\sqrt{3})xyz\\
&\quad + \frac{4(3-i\sqrt{3})}{9}xz^2 + y^2 z \frac{4(3-i\sqrt{3})}{9}yz^2 = 0\}.
\end{aligned}
$$

which have the properties that: $C_1, C_2$ are rational nodal cubics; $C_1, C_2$ intersect each other at 3 point $P_1 = [0:0:1], P_2 = [0:1:0], P_3 = [-12 : -4i\sqrt{3} : 9]$, each point with multiplicity 3; $C_1, C_2$ are triply tangent to $L_1$ at $P_2$; $C_1$ intersects $L_2$ at $P_3$ with multiplicity 3; $C_2$ is triply tangent to $L_2$ at $P_3$ (see Figure 22 for a depiction of the curves $L_1, L_2, C_1, C_2$ in $\mathbb{CP}^2$).

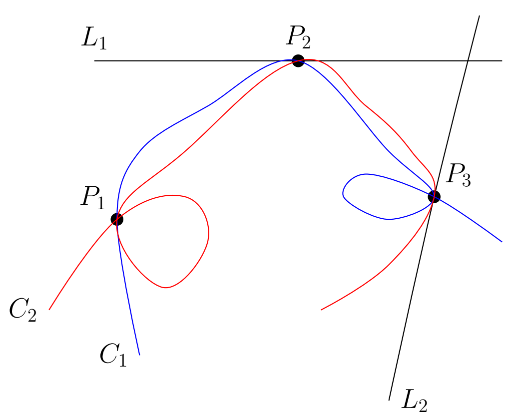


Figure 22. Depiction of the curves $L_1, L_2, C_1, C_2$. We will perform totally 11 blow-ups: 2 infinitely close blow-ups at $P_1$, 3 infinitely close blow-ups at $P_2$ and 6 infinitely close blow-ups at $P_3$ to get an embedding of $P\mathcal{A}'_2$ into $\mathbb{CP}^2\sharp 11\overline{\mathbb{CP}^2}$.

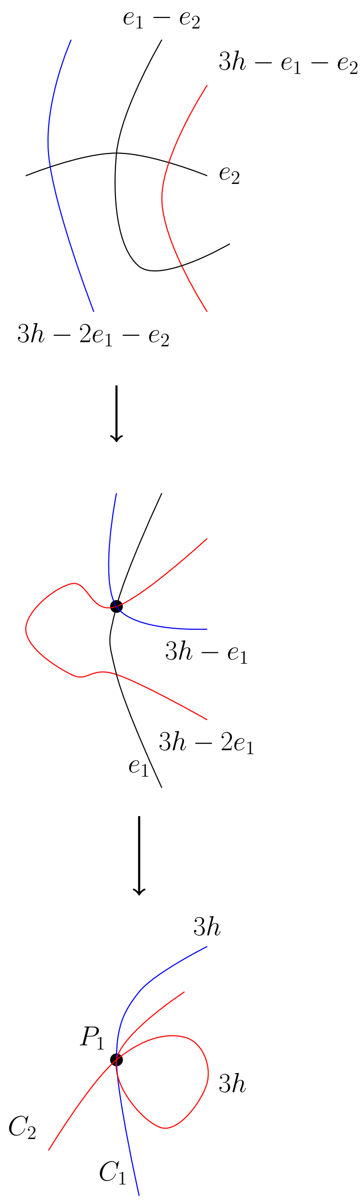


FIGURE 23. Two infinitely close blow-ups at $P_1$: At the beginning, homology classes represented by the curves $C_1, C_2$ are $3h \in H_2(\mathbb{CP}^2; \mathbb{Z})$.

After the first 2 infinitely close blow-ups at $P_1$, the proper transforms $\tilde{C_1}, \tilde{C_2}$ of $C_1, C_2$ represent homology classes $3h - e_1 - e_2, 3h - 2e_1 - e_2 \in H_2(\mathbb{CP}^2 \sharp 2\overline{\mathbb{CP}^2}; \mathbb{Z})$ respectively.

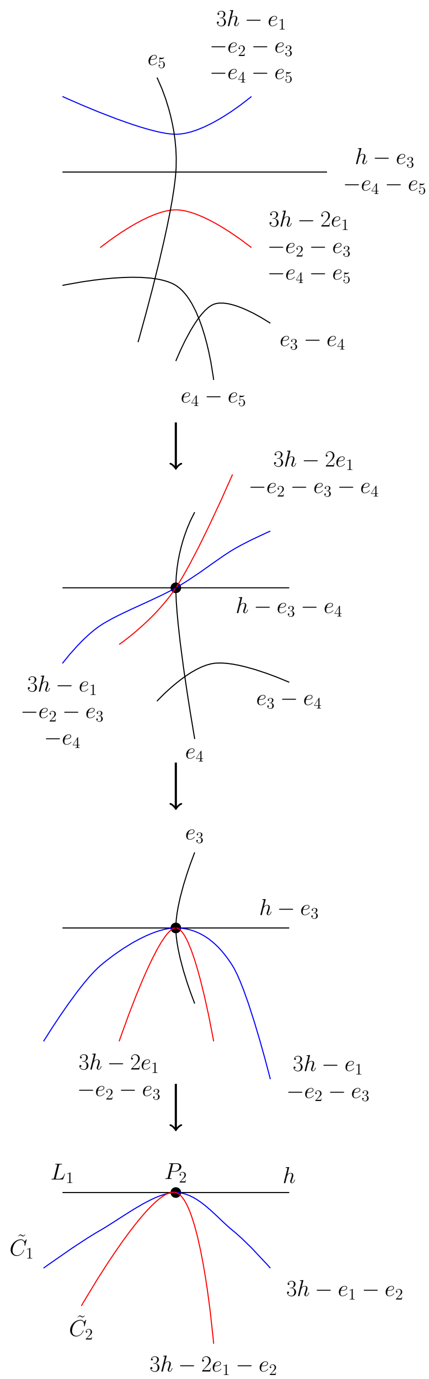


FIGURE 24. Three infinitely close blow-ups at $P_2$

Without any confusion, we also denote the proper transforms of $C_1, C_2$ by $\tilde{C}_1, \tilde{C}_2$ after the first 5 blow-ups. The homology classes represented by $\tilde{C}_1, \tilde{C}_2$ are $3h - e_1 - e_2 - e_3 - e_4 - e_5, 3h - 2e_1 - e_2 - e_3 - e_4 - e_5 \in H_2(\mathbb{CP}^2 \sharp 5\overline{\mathbb{CP}^2}; \mathbb{Z})$ respectively.

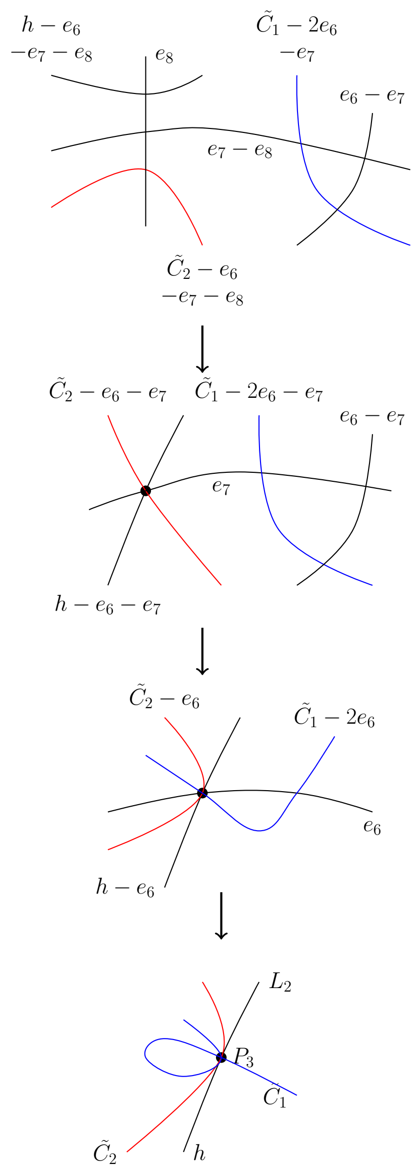


FIGURE 25. The first three infinitely close blow-ups at $P_3$

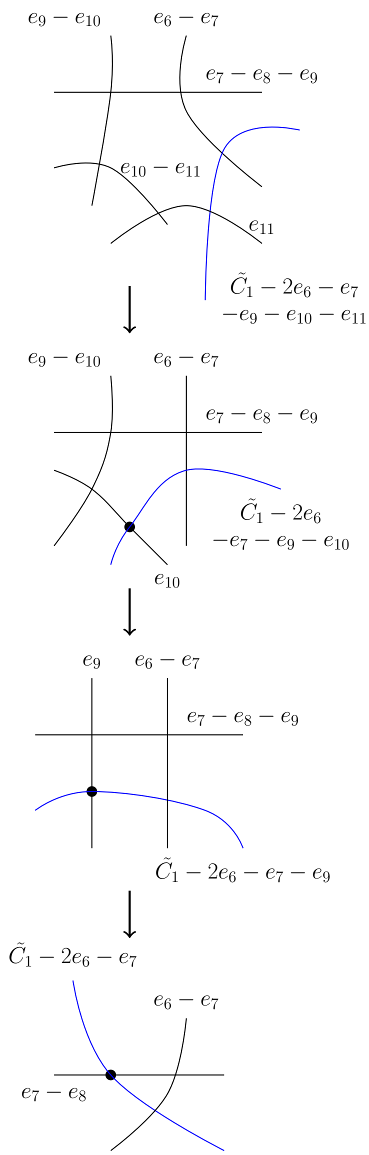


FIGURE 26. The last three infinitely close blow-ups at $P_3$

After blowing-up totally 11 times at the intersection points of $L_1, L_2, C_1, C_2$, the proper transforms of $L_1, L_2, C_1, C_2$ represent the homology classes $h - e_3 - e_4 - e_5, h - e_6 - e_7 - e_8, 3h - e_1 - e_2 - e_3 - e_4 - e_5 - 2e_6 - e_7 - e_9 - e_{10} - e_{11}, 3h - 2e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 - e_8 \in H_2(\mathbb{CP}^2 \sharp 11 \overline{\mathbb{CP}^2}; \mathbb{Z})$ respectively. Thus, we obtain an embedding of $P\mathcal{A}'_2$ into $\mathbb{CP}^2 \sharp 11 \overline{\mathbb{CP}^2}$ with the homology classes represented by embedding spheres at vertices of $P\mathcal{A}'_2$ are exactly those in Figure 7.

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