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A Study of Forcings

Master's Thesis

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Abstract

In this report, we propose to study some forcings of two kinds: the forcings with partial functions and the Prikry-type forcings. The motivation is to construct different models of the Zermelo-Fraenkel with Choice axiom system with the help of forcings and see how this way of construction may implie some properties to the models. We will follow the work of Kenneth Kunen and Moti Gitic. Thus no original result or proof will be presented here.

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1 Introduction

Forcing is a mathematical method used in set theory and developped in 1963 by the American mathematician Paul Cohen. It is a strong tool to prove consistency and independence results. Indeed, in his publication of 1963 *The independence of the continuum hypothesis*, Paul Cohen shows that the continuum hypothesis is independent from the Zermelo-Fraenkel with Choice axiom system. Meaning that, it cannot be decided whether the continuum hypothesis is true or not in the Zermelo-Fraenkel with Choice axiom system. The continuum hypothesis is probably one of the most famous hypothesis of mathematics, it has been formulated by the German mathematician Georg Cantor in 1878 and states that:

There is no set whose cardinality is strictly between that of the set of the integers and of the real numbers.

Showing it was the first of Hilbert's problems presented in 1900. But, by forcing, Paul Cohen gave the proof that this hypothesis was neither true nor false.

In this report, we propose to prove some results with the help of different forcings. The philosophy will be, for each forcing, to create an extension of a model of the Zermelo-Fraenkel with Choice axiom system in which the desired property is satisfied. This is an interesting way to proceed because the extension model we construct by forcing is also a model of the Zermelo-Fraenkel with Choice axiom system (Theorem 2.18). In this way, we will work with five different forcings of two kinds: the forcings with partial functions and the Prikry-type forcings. They will compose the third and the fourth chapter.

But before working with forcings, we have to define what is a forcing. That will be the motivation of the second chapter. This chapter will define some notions and give some results that will be necessary for the good understanding of the following chapters. It will be composed by two sections, the first one will be focused on the notion of forcing and the second one on the notion of complete embeddings. It is clear to see why the first section will be useful, it is not necessary the case for the second section. In fact, this section will be useful for only one forcing, the Easton forcing that will be studied in the third chapter. Hence the next chapter will be a preparation of the others.

The third chapter will be the one of the forcings with partial functions. The motivation of this chapter will be to go directly further than the 1963's publication of Paul Cohen. It will be composed by two sections that will use two different forcings. In the first one, we want to construct a model in which $2^{\aleph_n} = \aleph_{n+2}$ for every $n < \omega$ is satisfied. This property reminds the generalized continuum hypothesis which is, as its name says, a generalization of the continuum hypothesis and states that $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for every ordinal α. Thus we propose to do a slight shifting of the generalized continuum hypothesis by forcing. The second section will go deeper in this way by building a model in which we can choose, respecting some constraints, the cardinality of the power sets of the cardinals. It will be done by forcing, called the Easton forcing.

The fourth and last chapter of this report will introduce a new type of forcings, the Prikry-type forcings. The name comes from the American mathematician Karel Prikry who developped these forcings. They differ from the previous forcings we studied by the fact that they are based on ultrafilters. Hence, a good understanding of ultrafilters is necessary. That is why the first section will be a small recap of the definitions and results important about ultrafilters for the following. There will be three other sections composing this chapter where each of them will introduce a Prikry-type forcing. The first two will have exactly the same goal, the only difference between them is the assumption we will need. For the first one, the basic Prikry forcing, we will need a κ-complete, normal ultrafilter, instead of the second one, the tree Prikry forcing, for which we will only need a κ-complete ultrafilter. Thus the second section will give a stronger result by using a weaker assumption. Their common goal will be to construct a model in which cardinalities are preserved but not cofinalites. This is an interesting result because we will show in the second chapter that if a model constructed by forcing preserves cofinalities then it preserves also cardinalities (Lemma 2.20). Finally, we will conclude this report by a last forcing that will add to our basic model a sequence with a dominating property. It will use twice the construction made with the two previous Prikry-type forcings, making remarkable the common construction done by Prikry-type forcing.

2 Forcing Tools

In this chapter, we will introduce the notion of forcing and some important results for the constructions done in the two following chapters. This chapter will be splitted into two sections. In the first one, we will show the construction of a forcing starting from a partial order with largest element and some properties kept in the new model created from the original model. This section will give a necessary knowledge for the good understanding of the whole report. In the second section, we will focus, as its name says, on the complete embeddings. Contrary to the first section, this section will only be useful to understand the Easton forcing that will be introduced in the second chapter. That is why it can easily be omitted by the reader if he is not interested by the Easton forcing construction. Or, in the opposite case, it may be clever to skip, at first, this section and come back before starting to read about the Easton forcing.

The goal of this chapter is to give a general background of the forcing. It is a kind of tool box. Thus there will be a lot of definitions and some of the results will not be proved. None of the results reached in this chapter is a goal in itself, it will be used later in the report. That is why, some of them may appear a bit randomly and without consistency. If a result is not proved, we will always give a reference where the proof can be found. More details can be found in [2] and [5].

As a matter of simplicity and readability, we will directly introduce a notation for the whole report, M will denote a countable transitive model for the Zermelo-Fraenkel with Choice axiom system (also written ZFC).

2.1 Forcing

As said previously, in this section, a lot of definitions will be given in the perspective of introducing the notion of forcing and some necessary results for the report. The main result of this section is the Theorem 2.18. It is because of this theorem that the construction of a model extension using a forcing is relevent. But let us start this report by defining partial orders.

Definition 2.1. Let $\mathbb{P}$ be a non-empty set and $\leq$ be a relation on $\mathbb{P}$.
$\leq$ is a *transitive* relation if $p \leq q$ and $q \leq r$ implies that $p \leq r$ for every $p, q, r \in \mathbb{P}$.
$\leq$ is a *reflexive* relation if $p \leq p$ for every $p \in \mathbb{P}$.

Definition 2.2. Let $\mathbb{P}$ be a non-empty set, $\leq_{\mathbb{P}}$ be a transitive and reflexive relation on $\mathbb{P}$ and $\mathbf{1}_{\mathbb{P}}$ be an element of $\mathbb{P}$ such that $p \leq_{\mathbb{P}} \mathbf{1}_{\mathbb{P}}$ for every $p \in \mathbb{P}$. Then the pair $\langle \mathbb{P}, \leq_{\mathbb{P}} \rangle$ is a *partial order* and the triple $\langle \mathbb{P}, \leq_{\mathbb{P}}, \mathbf{1}_{\mathbb{P}} \rangle$ is a *partial order with largest element.* If no confusion is possible, we will usually denote them by $\langle \mathbb{P}, \leq \rangle$ and $\langle \mathbb{P}, \leq, \mathbf{1} \rangle$.
$\mathbf{1}$ is called a *largest element* of $\mathbb{P}$ with respect to $\leq$.
Elements of $\mathbb{P}$ are called *conditions* and we say that p *extends* q when $p \leq q$ for some conditions p and q.

The distinction between partial order and partial order with largest element may look irrelevant at the first sight but it will be essential when we will work with Prikry-type forcings (in the fourth section) and, especially, to understand the so-called Prikry condition. The next proposition is an obvious relation between partial orders and partial orders with largest elements.

Proposition 2.3. *Any partial order with largest element is a partial order.*

Proof. Let $\langle \mathbb{P}, \leq, \mathbf{1} \rangle$ be a partial order with largest element. Then $\leq$ is a transitive and reflexive relation on $\mathbb{P}$. Thus $\langle \mathbb{P}, \leq, \mathbf{1} \rangle$ is a partial order.

□

In the same way as we could define, in algebra, the product of two algebraic structures, we define, in a natural way, the partial order that we get by taking the product of two of them.

Definition 2.4. Let $\langle \mathbb{P}, \leq_{\mathbb{P}}, \mathbf{1}_{\mathbb{P}} \rangle$ and $\langle \mathbb{Q}, \leq_{\mathbb{Q}}, \mathbf{1}_{\mathbb{Q}} \rangle$ be partial orders with largest element. Then the *product partial order with largest element* $\langle \mathbb{P}, \leq_{\mathbb{P}}, \mathbf{1}_{\mathbb{P}} \rangle \oplus \langle \mathbb{Q}, \leq_{\mathbb{Q}}, \mathbf{1}_{\mathbb{Q}} \rangle$ is defined in such a way:

$$\langle \mathbb{P}, \leq_{\mathbb{P}}, \mathbf{1}_{\mathbb{P}} \rangle \oplus \langle \mathbb{Q}, \leq_{\mathbb{Q}}, \mathbf{1}_{\mathbb{Q}} \rangle = \langle \mathbb{P} \oplus \mathbb{Q}, \leq, \mathbf{1} \rangle$$

where

$$\langle p_0, q_0 \rangle \leq \langle p_1, q_1 \rangle \iff p_0 \leq_{\mathbb{P}} p_1 \text{ and } q_0 \leq_{\mathbb{Q}} q_1$$

and

$$\mathbf{1} = \langle \mathbf{1}_{\mathbb{P}}, \mathbf{1}_{\mathbb{Q}} \rangle$$

The *product partial order* is defined the same way without largest element.

The following definition introduce some vocabulary and notations that will be used during the whole report.

Definition 2.5. Let $\langle \mathbb{P}, \leq \rangle$ be a partial order. A *chain* in $\mathbb{P}$ is a set $C \subseteq \mathbb{P}$ such that:

$$\forall p, q \in C,\ p \leq q \text{ or } q \leq p$$

Let $p, q \in \mathbb{P}$, then p and q are *compatible* if:

$$\exists r \in \mathbb{P} \text{ such that } r \leq p \text{ and } r \leq q$$

p and q are *incompatible* (we write $p \perp q$) if:

$$\neg \exists r \in \mathbb{P} \text{ such that } r \leq p \text{ and } r \leq q$$

An *antichain* in $\mathbb{P}$ is a subset $A \subseteq \mathbb{P}$ such that $\forall p, q \in A$:

$$p \neq q \Rightarrow p \perp q$$

The two following definitions (2.6 and 2.9) are essential properties for a partial order in the forcing constructions we want to do later. The lemmas in between will only be used during the Easton forcing construction.

Definition 2.6. Let $\langle \mathbb{P}, \leq \rangle$ be a partial order and θ be an ordinal in M. Then $\langle \mathbb{P}, \leq \rangle$ satisfies the *θ-chain condition* (we write also θ-c.c.) if and only if every antichain in $\mathbb{P}$ has cardinality smaller than θ.

Lemma 2.7. *Let $\langle \mathbb{P}, \leq \rangle$ be a partial order such that $\mathbb{P} \in M$, A, B be two sets in M and θ be a cardinal in M. Moreover, let G be a $\mathbb{P}$-generic over M and $f \in M$ be a maps from A to B. If $\langle \mathbb{P}, \leq \rangle$ satisfies the θ-c.c., then there is a map $F : A \to P(B)$ with $F \in M$ such that $f(a) \in F(a)$ and $(|F(a)| < \theta)^M$ $\forall a \in A$.*

Lemma 2.8. (König). *If κ is infinite and cf$(\kappa) \leq \lambda$, then $\kappa < \kappa^\lambda$.*

The proofs are not given but can be found, respectively, in [5][Chapter VII, Lemma 5.5] and [5][Chapter I, Lemma 10.40].

Definition 2.9. Let $\langle \mathbb{P}, \leq \rangle$ be a partial order and λ be an ordinal in M. Then $\langle \mathbb{P}, \leq \rangle$ is λ-*closed* if and only if whenever $\gamma < \lambda$ and $\langle p_\xi \mid \xi < \gamma \rangle$ is a $\leq$-decreasing sequence of elements of $\mathbb{P}$, then

$$\exists q \in \mathbb{P} \text{ such that } \forall \xi < \gamma,\ q \leq p_\xi$$

Now that partial orders have been defined and some of their properties given, we introduce partial order generic, which we will central elements of the model extension we will create with forcing. The following lemmas are useful results concerning these partial order generic.

Definition 2.10. Let $\langle \mathbb{P}, \leq \rangle$ be a partial order. $D \subseteq \mathbb{P}$ is *dense* in $\mathbb{P}$ if $\forall p \in \mathbb{P}\ \exists q \in D$ such that $q \leq p$.
$F \subseteq \mathbb{P}$ is a *filter* in $\mathbb{P}$ if:
(a) $\forall p, q \in F\ \exists r \in F$ such that $r \leq p$ and $r \leq q$
(b) $\forall p \in F$ and $\forall q \in \mathbb{P}$ such that $p \leq q$ implies that $q \in F$

Definition 2.11. Let $\langle \mathbb{P}, \leq, \mathbf{1} \rangle$ be a partial order with largest element. G is $\langle \mathbb{P}, \leq, \mathbf{1} \rangle$-*generic* (we will write $\mathbb{P}$-*generic* if no confusion is possible) over M if G is a filter on $\mathbb{P}$ and, for all dense set D of $\mathbb{P}$ such that $D \in M$, $G \cap D \neq \emptyset$.

Lemma 2.12. *Let $\langle \mathbb{P}, \leq, \mathbf{1} \rangle$ be a partial order with largest element in M, $E \subseteq \mathbb{P}$ be a set in M and G be $\mathbb{P}$-generic over M. Then either $G \cap E \neq \emptyset$ or $\exists q \in G$ such that $\forall r \in E$, $r \perp q$.*

Proof. Let us consider the set

$$D = \{p \in \mathbb{P} \mid \exists r \in E \text{ such that } p \leq r\} \cup \{q \in \mathbb{P} \mid \forall r \in E, r \perp q\}$$

We want to show that D is dense. Let $p \in \mathbb{P} \setminus D$ and $r \in E$ such that $\exists q \in \mathbb{P}$ such that $q \leq p$ and $q \leq r$. But, then $q \in D$ and $q \leq p$. Hence D is dense. The first set composing D shows the first part of the lemma and the second set shows the second part.

□

Lemma 2.13. *Let $\langle \mathbb{P}, \leq, \mathbf{1} \rangle$ be a partial order with largest element in M and $G \subseteq \mathbb{P}$. Then G is $\mathbb{P}$-generic over M if:*
(a) $\forall p, q \in G, \exists r \in \mathbb{P}$ *such that* $r \leq p$ *and* $r \leq q$
(b) $\forall p \in G$ *and* $\forall q \in \mathbb{P}$ *such that* $p \leq q$, $q \in G$
(c) $\forall D \subseteq \mathbb{P}$ *such that* $D \in M$ *and* D *is dense in* $\mathbb{P}$, $G \cap D \neq \emptyset$

Proof. (b) and (c) are the same properties as the ones of the Definition 2.11, thus we only need to show that the property (a) of the Definition 2.10 is satisfied when we assume the three properties of the lemma.
Let $p, q \in G$ and consider the set:

$$D = \{r \in \mathbb{P} \mid r \perp p \text{ or } r \perp q \text{ or } (r \leq p \text{ and } r \leq q)\}$$

We want to show that D is dense. Let $\bar{p} \in \mathbb{P}$, then there are three possibilities:

- If $\forall r \leq \bar{p}$, $\neg(r \leq p)$:
$$\bar{p} \perp p \Rightarrow \bar{p} \in D$$
- If $\forall r \leq \bar{p}$, $\neg(r \leq q)$:
$$\bar{p} \perp q \Rightarrow \bar{p} \in D$$

- If $\exists r \leq \bar{p}$ such that $r \leq p$ and $r \leq q$:

$$r \in D$$

This shows that D is dense. Thus $D \cap G \neq \emptyset$ so $\exists r \in G$ such that $r \perp p$ or $r \perp q$ or $r \leq p$ and $r \leq q$. But, by (a), $r \perp p$ and $r \perp q$ are impossible.

□

Lemma 2.14. *If $\langle \mathbb{P}, \leq, \mathbf{1} \rangle$ is a partial order with largest element and $p \in \mathbb{P}$, then there is a $\mathbb{P}$-generic G over M such that $p \in G$.*

The proof is not given but can be found in [5][Chapter VII, Lemma 2.3].

We can, now, define the model extension given by forcing with a partial order with largest element and a partial order generic. But, to do that, we need, first, to introduce the notion of name.

Definition 2.15. Let $\langle \mathbb{P}, \leq, \mathbf{1} \rangle$ be a partial order with largest element, then τ is called a $\mathbb{P}$-*name* if it is a relation and $\forall \langle \sigma, p \rangle \in \tau$, σ is a $\mathbb{P}$-name and $p \in \mathbb{P}$. Any element $x \in M$ is represented, in a canonical way, by a $\mathbb{P}$-name, called $\check{x}$.

We denote by $\mathbf{V}^{\mathbb{P}}$ the class of $\mathbb{P}$-names and by $M^{\mathbb{P}}$ the class of $\mathbb{P}$-names in M. Thus $M^{\mathbb{P}} = \mathbf{V}^{\mathbb{P}} \cap M$.

Definition 2.16. Let $\langle \mathbb{P}, \leq, \mathbf{1} \rangle \in M$ be a partial order with largest element and G be a $\mathbb{P}$-generic over M. Then $M[G] = \{\tau_G : \tau \in M^{\mathbb{P}}\}$, where $\tau_G = \{\sigma_G : \exists p \in G(\langle \sigma, p \rangle \in \tau)\}$.

If φ is a formula, we will write φ^M to say that φ is a formula in M and $\varphi^{M[G]}$ to say that φ is a formula in $M[G]$.

To reduce the study of the set theory of $M[G]$ to that of M, one works with the *forcing language*, which is built up like ordinary first-order logic, with membership as the binary relation and all the $\mathbb{P}$-names as constants.

Definition 2.17. Let $\langle \mathbb{P}, \leq, \mathbf{1} \rangle \in M$ be a partial order with largest element. If ψ is a sentence of the forcing language, we say that p *forces* ψ (and write $p \Vdash \psi$) to mean that for all G that are $\mathbb{P}$-generic over M, if $p \in G$, then ψ is true in $M[G]$.

It is because of the two facts that are statted below and the following theorem that the study of forcings is relevent. The two facts say that we can understand what happens in $M[G]$ by working in M. A good analogy would be to think of a field extension in algebra. For example, the study of the field extension $\mathbb{Z}[\sqrt{2}]$ is essentially a study of $\mathbb{Z}$. Then the Theorem 2.18 shows that a model extension of a model of ZFC given by a forcing remains a model of ZFC. Hence, the goal of constructing forcings is to create a model of ZFC in which some properties are satisfied.

Fact 1: It may be decided within M whether or not $p \Vdash \psi$.
Fact 2: If G is $\mathbb{P}$-generic over M and ψ is true in $M[G]$, then for some $p \in G$, $p \Vdash \psi$.

Theorem 2.18. *Let $\langle \mathbb{P}, \leq, \mathbf{1} \rangle \in M$ be a partial order with largest element and G be a $\mathbb{P}$-generic over M. Then $M[G]$ satisfies* ZFC.

The proof is not given but can be found in [5][Chapter VII, Theorem 4.2].

Finally, we are interested in how cardinals and cofinalities act in a model extension given by forcing. The last lemma shows that preservation of cofinalities implies preservation of cardinals. But the opposite is not true and we will create a model extension showing it thanks to Prikry-type forcings.

Definition 2.19. Let $\langle \mathbb{P}, \leq, \mathbf{1} \rangle \in M$ be a partial order with largest element.

$\mathbb{P}$ *preserves cardinals* if whenever G is $\mathbb{P}$-generic over M,

$$(\beta \text{ is a cardinal in } M) \iff (\beta \text{ is a cardinal in } M[G])$$

$\mathbb{P}$ *preserves cofinalities* if whenever G is $\mathbb{P}$-generic over M and γ is a limit ordinal in M,

$$cf(\gamma)^M = cf(\gamma)^{M[G]}$$

Lemma 2.20. *Let $\langle \mathbb{P}, \leq, \mathbf{1} \rangle \in M$ be a partial order with largest element. If $\mathbb{P}$ preserves cofinalities, then it preserves cardinals.*

Proof. We assume that $\mathbb{P}$ preserves cofinalities and we want to show that

$$(\beta \text{ is a cardinal})^M \iff (\beta \text{ is a cardinal})^{M[G]}.$$

$\Leftarrow$: If $(\beta \neq |\beta|)^M$, then $(\beta \neq |\beta|)^{M[G]}$. Thus a cardinal in $M[G]$ is a cardinal in M.
$\Rightarrow$: If $\beta < \omega$, then it is clear. So let us suppose that β is an infinite cardinal in M. Then β is a regular cardinal, a limit cardinal or both.

If β is a regular cardinal in M, then $cf(\beta)^M = \beta$. Thus, by assumption,

$$cf(\beta)^{M[G]} = cf(\beta)^M = \beta$$

Hence β is a regular cardinal in $M[G]$.

If β is a limit cardinal in M, then the regular cardinals less than β of M are unbounded in β. By the previous argument, they remain regular in $M[G]$ and unbounded in β. So β is a limit cardinal in $M[G]$.

Thus, in any case, β remains a cardinal in $M[G]$.

□

2.2 Complete Embeddings

This section is a preparation for the Easton forcing that will be constructed later and has no other meaning. We will define complete embeddings and use them on product of partial orders in the perspective of studying extensions with product of partial order generics and extensions of model extensions. Every lemma, theorem and corollary of this section will be used to show the Theorem 2.26.

Definition 2.21. Let $\langle \mathbb{P}, \leq_{\mathbb{P}}, \mathbf{1}_{\mathbb{P}} \rangle$ and $\langle \mathbb{Q}, \leq_{\mathbb{Q}}, \mathbf{1}_{\mathbb{Q}} \rangle$ be two partial orders with largest element. Then $i : \mathbb{P} \to \mathbb{Q}$ is a *complete embedding* if:

(a) $\forall p, p' \in \mathbb{P}$:

$$p' \leq_{\mathbb{P}} p \Rightarrow i(p') \leq_{\mathbb{Q}} i(p)$$

(b) $\forall p, p' \in \mathbb{P}$:

$$p \perp p' \iff i(p) \perp i(p')$$

(c) $\forall q \in \mathbb{Q}, \exists p \in \mathbb{P}$ such that $\forall p' \in \mathbb{P}$:

$$p' \leq_{\mathbb{P}} p \Rightarrow \exists q' \in \mathbb{Q} \text{ such that } q' \leq_{\mathbb{Q}} q \text{ and } q' \leq_{\mathbb{Q}} i(p')$$

In this case, p is called a *reduction* of q to $\mathbb{P}$.

Lemma 2.22. *Let $\langle \mathbb{P}_0, \leq_{\mathbb{P}_0}, \mathbf{1}_{\mathbb{P}_0}\rangle$ and $\langle \mathbb{P}_1, \leq_{\mathbb{P}_1}, \mathbf{1}_{\mathbb{P}_1}\rangle$ be two partial orders with largest elements. Then*

$$i_0 : \mathbb{P}_0 \to \mathbb{P}_0 \times \mathbb{P}_1 \qquad\qquad i_1 : \mathbb{P}_1 \to \mathbb{P}_0 \times \mathbb{P}_1$$
$$p_0 \mapsto \langle p_0, \mathbf{1}_{\mathbb{P}_1}\rangle \qquad\qquad p_1 \mapsto \langle \mathbf{1}_{\mathbb{P}_0}, p_1\rangle$$

are complete embeddings.

Proof. The proof is the same for i_0 and i_1 so we will only show that i_0 is a complete embedding. To show that, we just need to check if i_0 satisfy the properties of the Definition 2.21.

(a) Let $p, p' \in \mathbb{P}_0$ such that $p' \leq_{\mathbb{P}_0} p$. Then:

$$i_0(p') = \langle p', \mathbf{1}_{\mathbb{P}_1}\rangle \leq \langle p, \mathbf{1}_{\mathbb{P}_1}\rangle = i_0(p)$$

(b) $\Rightarrow$: Let $p, p' \in \mathbb{P}_0$ such that $p \perp p'$ and let us suppose, for contradiction, that $i_0(p) \not\perp i_0(p')$. Hence $\exists \langle p_0, p_1\rangle \in \mathbb{P}_0 \times \mathbb{P}_1$ such that $\langle p_0, p_1\rangle \leq i_0(p) = \langle p, \mathbf{1}_{\mathbb{P}_1}\rangle$ and $\langle p_0, p_1\rangle \leq i_0(p') = \langle p', \mathbf{1}_{\mathbb{P}_1}\rangle$. But, then, $p_0 \leq_{\mathbb{P}_0} p, p'$ which is a contradiction to $p \perp p'$.

$\Leftarrow$: Let $p, p' \in \mathbb{P}_0$ such that $i_0(p) \perp i_0(p')$ and let us suppose, for contradiction, that $p \not\perp p'$. Hence $\exists p_0 \in \mathbb{P}_0$ such that $p_0 \leq_{\mathbb{P}_0} p, p'$. But, then, $i_0(p_0) = \langle p_0, \mathbf{1}_{\mathbb{P}_1}\rangle \leq \langle p, \mathbf{1}_{\mathbb{P}_1}\rangle = i_0(p)$ and $i_0(p_0) = \langle p_0, \mathbf{1}_{\mathbb{P}_1}\rangle \leq \langle p', \mathbf{1}_{\mathbb{P}_1}\rangle = i_0(p')$ which is a contradiction to $i_0(p) \perp i_0(p')$.

(c) Let $\langle p_0, p_1\rangle \in \mathbb{P}_0 \times \mathbb{P}_1$ and $p \in \mathbb{P}_0$ such that $p \leq_{\mathbb{P}_0} p_0$. Then consider $\langle p, p_1\rangle$:

$$\langle p, p_1\rangle \leq \langle p, \mathbf{1}_{\mathbb{P}_1}\rangle = i_0(p) \text{ and } \langle p, p_1\rangle \leq \langle p_0, p_1\rangle$$

Thus p_0 is a reduction of $\langle p_0, p_1\rangle$ to $\mathbb{P}_0$

□

Theorem 2.23. *Let $\langle \mathbb{P}, \leq_{\mathbb{P}}, \mathbf{1}_{\mathbb{P}}\rangle$ and $\langle \mathbb{Q}, \leq_{\mathbb{Q}}, \mathbf{1}_{\mathbb{Q}}\rangle$ be two partial orders with largest element, $i : \mathbb{P} \to \mathbb{Q}$ be a complete embedding and H be $\mathbb{Q}$-generic over M. Then $i^{-1}(H)$ is $\mathbb{P}$-generic over M and $M[i^{-1}(H)] \subseteq M[H]$.*

Proof. Let us show first that $i^{-1}(H)$ is $\mathbb{P}$-generic over M. To do that we will show that G satisfies the three properties of the Lemma 2.13

(a) Let $q_1, q_2 \in H$. Since H is a filter in $\mathbb{Q}$, $\exists \tilde{q} \in H$ such that $\tilde{q} \leq_{\mathbb{Q}} q_1$ and $\tilde{q} \leq_{\mathbb{Q}} q_2$ so $q_1 \not\perp q_2$. As i is a complete embedding, $i^{-1}(q_1) \not\perp i^{-1}(q_2)$ so

$$\exists p \in \mathbb{P} \text{ such that } p \leq_{\mathbb{P}} i^{-1}(q_1), i^{-1}(q_2)$$

(b) Let $q \in H$ and $p \in \mathbb{P}$ such that $i^{-1}(q) \leq_{\mathbb{P}} p$. Hence, since i is a complete embedding, $q \leq_{\mathbb{Q}} i(p)$. As H is a filter in $\mathbb{Q}$,

$$i(p) \in H \text{ so } p \in i^{-1}(H)$$

(c) Let D be a dense set of $\mathbb{P}$ and suppose, for contradiction, that $i^{-1}(H) \cap D = \emptyset$. Then $H \cap i(D) = \emptyset$ so, by the Lemma 2.12, $\exists q \in H$ such that $\forall q' \in i(D)$, $q \perp q'$. Hence,

$$\forall p \in D,\ q \perp i(p)$$

Since i is an complete embedding, $\exists p' \in \mathbb{P}$ which is a reduction of q to $\mathbb{P}$. Thus

$$i(\bar{p}) \not\perp q\ \forall \bar{p} \leq p'$$

As D is a dense set, $\exists \bar{p} \in D$ such that $\bar{p} \leq p'$. But, then, $i(\bar{p}) \not\perp q$ and $i(\bar{p}) \perp q$ which is impossible.

Thus $i^{-1}(H)$ is $\mathbb{P}$-generic over M.

Let us show now that $M[i^{-1}(H)] \subseteq M[H]$. Since $i \in M \subseteq M[H]$ and $H \in M[H]$, we have that $i^{-1}(H) \in M[H]$. Thus, by construction of $M[i^{-1}(H)]$, $M[i^{-1}(H)] \subseteq M[H]$.

□

Corollary 2.24. *Let $\langle \mathbb{P}, \leq_{\mathbb{P}}, \mathbf{1}_{\mathbb{P}} \rangle$ and $\langle \mathbb{Q}, \leq_{\mathbb{Q}}, \mathbf{1}_{\mathbb{Q}} \rangle$ be two partial orders with largest element, $i : \mathbb{P} \to \mathbb{Q}$ be an isomorphism and a complete embedding and $G \subseteq \mathbb{Q}$. Then G is $\mathbb{Q}$-generic over M if and only if $i^{-1}(G)$ is $\mathbb{P}$-generic over M. Moreover, $M[G] = M[i^{-1}(G)]$.*

Proof. Since i is an isomorphism, we can use the Theorem 2.23 with i and i^{-1}, which is enough to prove this corollary.

□

Lemma 2.25. *Let $\langle \mathbb{P}_0, \leq_{\mathbb{P}_0}, \mathbf{1}_{\mathbb{P}_0} \rangle$ and $\langle \mathbb{P}_1, \leq_{\mathbb{P}_1}, \mathbf{1}_{\mathbb{P}_1} \rangle$ be two partial orders with largest elements such that $\mathbb{P}_0, \mathbb{P}_1 \in M$ and G be a $\mathbb{P}_0 \times \mathbb{P}_1$-generic over M. Consider the embeddings:*

$$\begin{aligned} i_0 :\ & \mathbb{P}_0 \to \mathbb{P}_0 \times \mathbb{P}_1 \\ & p_0 \mapsto \langle p_0, \mathbf{1}_{\mathbb{P}_1} \rangle \end{aligned} \qquad \begin{aligned} i_1 :\ & \mathbb{P}_1 \to \mathbb{P}_0 \times \mathbb{P}_1 \\ & p_1 \mapsto \langle \mathbf{1}_{\mathbb{P}_0}, p_1 \rangle \end{aligned}$$

Then $i_0^{-1}(G)$ is $\mathbb{P}_0$-generic over M, $i_1^{-1}(G)$ is $\mathbb{P}_1$-generic over M and $G = i_0^{-1}(G) \times i_1^{-1}(G)$.

Proof. By the Lemma 2.22, we know that i_0 and i_1 are complete embeddings. Then, we can apply the Theorem 2.23, to get that $i_0^{-1}(G)$ is $\mathbb{P}_0$-generic over M and $i_1^{-1}(G)$ is $\mathbb{P}_1$-generic over M.

Let us show now that $G = i_0^{-1}(G) \times i_1^{-1}(G)$:

$\subseteq$: Let $\langle p_0, p_1 \rangle \in G$, then $\langle p_0, p_1 \rangle \leq \langle p_0, \mathbf{1}_{\mathbb{P}_1} \rangle = i_0(p_0)$ and $\langle p_0, p_1 \rangle \leq \langle \mathbf{1}_{\mathbb{P}_0}, p_1 \rangle = i_1(p_1)$. Hence $i_0(p_0), i_1(p_1) \in G$ so $p_0 \in i_0^{-1}(G)$ and $p_1 \in i_1^{-1}(G)$. Thus $\langle p_0, p_1 \rangle \in i_0^{-1}(G) \times i_1^{-1}(G)$.

$\supseteq$: Let $\langle p_0, p_1 \rangle \in i_0^{-1}(G) \times i_1^{-1}(G)$, then we have that $\langle p_0, \mathbf{1}_{\mathbb{P}_1} \rangle = i_0(p_0) \in G$ and $\langle \mathbf{1}_{\mathbb{P}_0}, p_1 \rangle = i_1(p_1) \in G$. But, then, $\exists \langle q_0, q_1 \rangle \in G$ such that $\langle q_0, q_1 \rangle \leq \langle p_0, \mathbf{1}_{\mathbb{P}_1} \rangle, \langle \mathbf{1}_{\mathbb{P}_0}, p_1 \rangle$. Thus $q_0 \leq_{\mathbb{P}_0} p_0$ and $q_1 \leq_{\mathbb{P}_1} p_1$ so $\langle q_0, q_1 \rangle \leq \langle p_0, p_1 \rangle$. Hence, finally, $\langle p_0, p_1 \rangle \in G$.

□

Theorem 2.26. *Let $\langle \mathbb{P}_0, \leq_{\mathbb{P}_0}, \mathbf{1}_{\mathbb{P}_0} \rangle$ and $\langle \mathbb{P}_1, \leq_{\mathbb{P}_1}, \mathbf{1}_{\mathbb{P}_1} \rangle$ be two partial orders with largest elements such that $\mathbb{P}_0, \mathbb{P}_1 \in M$, $G_0 \subseteq \mathbb{P}_0$ and $G_1 \subseteq \mathbb{P}_1$. Then the following are equivalent:*

(1) *$G_0 \times G_1$ is $\mathbb{P}_0 \times \mathbb{P}_1$-generic over M*

(2) *G_0 is $\mathbb{P}_0$-generic over M and G_1 is $\mathbb{P}_1$-generic over $M[G_0]$*

(3) *G_1 is $\mathbb{P}_1$-generic over M and G_0 is $\mathbb{P}_0$-generic over $M[G_1]$*

Furthermore, if one of these is satisfied, then $M[G_0 \times G_1] = M[G_0][G_1] = M[G_1][G_0]$.

Proof. We only need to show (1) $\iff$ (2) because (1) $\iff$ (3) is shown in the same way.

(1) $\Rightarrow$ (2): Considering the embedding:

$$\begin{aligned} i_0 :\ & \mathbb{P}_0 \to \mathbb{P}_0 \times \mathbb{P}_1 \\ & p_0 \mapsto \langle p_0, \mathbf{1}_{\mathbb{P}_1} \rangle \end{aligned}$$

and, by the Lemma 2.25, we have that $G_0 = i_0^{-1}(G_0 \times G_1)$ is $\mathbb{P}_0$-generic over M. Hence, it just remains to show that G_1 is $\mathbb{P}_1$-generic over $M[G_0]$. Let $D \in M[G_0]$ be a dense set in $\mathbb{P}_1$ and we want to show that $G_1 \cap D \neq \emptyset$. Since $D \in M[G_0]$, there is a name $\tau \in M^{\mathbb{P}_0}$ such that $D = \tau_{G_0}$. Moreover, by the Fact 2 of the forcing language, there is $p_0 \in G_0$ such that $p_0 \Vdash (\tau \text{ is dense in } \check{\mathbb{P}}_1)$. Then consider the set:

$$D' = \{\langle q_0, q_1 \rangle \in \mathbb{P}_0 \times \mathbb{P}_1 \mid q_0 \leq_{\mathbb{P}_0} p_0 \text{ and } q_0 \Vdash \check{q}_1\}$$

Let $\langle r_0, r_1\rangle \leq \langle p_0, \mathbf{1}_{\mathbb{P}_1}\rangle$, then $r_0 \leq_{\mathbb{P}_0} p_0$. Thus, as $p_0 \Vdash (\tau$ is dense in $\check{\mathbb{P}}_1)$,

$$r_0 \Vdash (\exists x \in \mathbb{P}_1 \text{ such that } x \in \tau \text{ and } x \leq_{\mathbb{P}_1} \check{r}_1)$$

Hence, there is $q_1 \in \mathbb{P}_1$ and $q_0 \leq_{\mathbb{P}_0} r_0$ such that $q_0 \Vdash (\check{q}_1 \in \tau$ and $\check{q}_1 \leq_{\mathbb{P}_1} \check{r}_1)$. Then we have that $\langle q_0, q_1\rangle \leq \langle r_0, r_1\rangle$ and $\langle q_0, q_1\rangle \in D'$. Thus D' is dense below $\langle p_0, \mathbf{1}_{\mathbb{P}_1}\rangle$. But $\langle p_0, \mathbf{1}_{\mathbb{P}_1}\rangle \in G_0 \times G_1$, so $\langle q_0, q_1\rangle \in ((G_0 \times G_1) \cap D')$. Then $q_0 \Vdash \check{q}_1 \in \tau$ so $q_1 \in \tau_{G_0} = D$. Hence $q_1 \in (G_1 \cap D)$ so $G_1 \cap D \neq \emptyset$. Moreover, since $G_0 \times G_1$ is $\mathbb{P}_0 \times \mathbb{P}_1$-generic over M, G_1 is a filter in $\mathbb{P}_1$. Thus G_1 is $\mathbb{P}_1$-generic over $M[G_0]$.

(2) $\Rightarrow$ (1): Let $D \in M$ be a dense set in $\mathbb{P}_0 \times \mathbb{P}_1$ and consider the set:

$$D^* = \{p_1 \in \mathbb{P}_1 \mid \exists p_0 \in G_0 \text{ such that } \langle p_0, p_1\rangle \in D\}$$

$D^* \in M[G_0]$ and if $D^* \cap G_1 \neq \emptyset$, we would have $D \cap (G_0 \times G_1) \neq \emptyset$. Thus, since G_1 is $\mathbb{P}_1$-generic over $M[G_0]$, we only need to show that D^* is dense in $\mathbb{P}_1$. Let $r_1 \in \mathbb{P}_1$ and

$$D_0 = \{p_0 \in \mathbb{P}_0 \mid \exists p_1 \leq_{\mathbb{P}_1} r_1 \text{ such that } \langle p_0, p_1\rangle \in D\}$$

$D_0 \in M$ and, since D is dense in $\mathbb{P}_0 \times \mathbb{P}_1$, D_0 is dense in $\mathbb{P}_0$. Moreover, G_0 is $\mathbb{P}_0$-generic over M, so there is $p_0 \in D_0 \cap G_0$. Thus there is $p_1 \leq_{\mathbb{P}_1} r_1$ such that $\langle p_0, p_1\rangle \in D$. But, then, $p_1 \leq_{\mathbb{P}_1} r_1$ and $p_1 \in D^*$ so D^* is dense in $\mathbb{P}_1$.

Let us show now that $M[G_0 \times G_1] = M[G_0][G_1] = M[G_1][G_0]$. We will only show $M[G_0 \times G_1] = M[G_0][G_1]$ because $M[G_0 \times G_1] = M[G_1][G_0]$ is shown in the same way.

$\subseteq$: $M \subseteq M[G_0][G_1]$ and $G_0 \times G_1 \in M[G_0][G_1]$ so $M[G_0 \times G_1] \subseteq M[G_0][G_1]$.

$\supseteq$: $M \subseteq M[G_0][G_1]$ and $G_0 \in M[G_0 \times G_1]$ so $M[G_0] \subseteq M[G_0 \times G_1]$. Moreover $G_1 \in M[G_0 \times G_1]$ so $M[G_0][G_1] \subseteq M[G_0 \times G_1]$.

$\square$

3 Forcing with partial functions

This chapter is a the real start of the report. Indeed, we will construct two model extensions by forcing. The two constructions will compose the two sections of this chapter. Both of the forcings will be based on partial functions. The goal is to create models in which every cardinals satisfy some rules about the size of their power set. In the first construction, we start with a model, in which the Generalized Continuum Hypothesis (also written GCH) is assumed, and create an extension of it, in which $2^{\aleph_n} = \aleph_{n+2}$ for every $n \in \omega$. In the second construction, given by the so-called Easton forcing, we will create a model, in which the size of the power set of every cardinal can be, respecting some constraints, more or less chosen. This second forcing, as you could think, will be more complicated to construct and will use a stronger machinery. A part of the results used for the Easton forcing is due to the study of complete embeddings done in the Chapter II.

These two sections are constructed in a more consistent way than the previous chapter. Indeed, the two sections will be organised in such a way: first, we will introduce partial orders with largest element, with which we will extend M by forcing and then, we will show that the model constructed in this way satisfies the required properties. That is why the last theorem of the two sections is the most important result of the section. In both forcings, M is assumed to be a model of ZFC in which GCH is assumed. Further details and results concernings these forcings can be found in [2] and [5].

3.1 Consistency of $2^{\aleph_n} = \aleph_{n+2}$

In this section, we will proceed by recursion to extend M with ω different partial orders with largest element. These partial orders with largest element will be based on sets of functions from a cardinal of M to 2. 2 is obviously not chosen randomly and come directly from the left hand-side of the equality we have to show in the constructed model. The most important result of this section is the Theorem 3.11. It is the last of the section and will use every proposition, lemma or theorem that are showed before.

Let us, first, introduce these sets of functions and then the partial orders with largest element.

Definition 3.1. Let κ and λ be some cardinals in M, then we define:

$$\text{Add}(\kappa,\lambda) = \{p \mid p \text{ is a function, } \text{Dom}(p) \in [\kappa \times \lambda]^{<\kappa} \text{and } \text{Ran}(p) \subseteq \{0,1\}\}$$

Where $[\kappa \times \lambda]^{<\kappa}$ denote the set of the subsets of $\kappa \times \lambda$ of size smaller than κ.
We order $\text{Add}(\kappa,\lambda)$ in the following way:

$$\forall p, q \in \text{Add}(\kappa,\lambda) : p \leq q \iff q \subseteq p$$

Proposition 3.2. *Let κ, λ be some cardinals in M and* 0 *be the empty function. Then $\langle Add(\kappa,\lambda), \leq, 0\rangle$ is a partial order with largest element.*

Proof. $\subseteq$ is a transitive and reflexive relation so $\leq$ is also a transitive and reflexive relation on $\text{Add}(\kappa,\lambda)$. Moreover $\forall p \in \text{Add}(\kappa,\lambda)$, $0 \subseteq p$ so $p \leq 0$.

□

The Δ-system Lemma (Theorem 3.4) is a very general result that can be used in many cases of set theory. It will be recalled here because we will use it later but it does not depend on the context of these forcings. The next definition introduces some notions used in the Δ-system Lemma.

Definition 3.3. A family $\mathcal{A}$ of sets is called a Δ-*system*, or a *quasi-disjoint* family if there is a set R, called the *root* of the Δ-system, such that $A \cap B = R$ whenever A and B are distinct members of $\mathcal{A}$.

Theorem 3.4. (Δ-system Lemma). *Let κ be any infinite cardinal. Let $\theta > \kappa$ be regular and satisfy $\forall \alpha < \theta(|\alpha^{<\kappa}| < \theta)$. Assume $|\mathcal{A}| \geq \theta$ and $\forall x \in \mathcal{A}(|x| < \kappa)$, then there is a $\mathcal{B} \subseteq \mathcal{A}$, such that $|\mathcal{B}| = \theta$ and forms a Δ-system.*

The proof is not given but can be found in [2][Chapter 13, Theorem 13.1].

The following lemma will show the chain-condition of some partial order. It is an essential point of every forcing that we will do during this report.

Lemma 3.5. *Let τ be a cardinals in M and $\mathbb{P} = Add(\tau, \tau^{++})$. Then $\langle \mathbb{P}, \leq, 0\rangle$ satisfies the τ^+-c.c.*

Proof. Let us consider the set $\mathcal{A} = \{\text{Dom}(p_\alpha) \mid p_\alpha \in \mathbb{P}, \alpha < \tau^+\}$. Then we can apply the Δ-system Lemma (Theorem 3.4) with $\kappa = \tau$ and $\theta = \tau^+$. Thus we can assume that $\forall \alpha < \tau^+$:

$$\text{Dom}(p_\alpha) = S \cup S_\alpha$$

Here S is the root of the Δ-system so $S \cap S_\alpha = \emptyset$ and $|S| < \tau$. Since $S_\alpha \cap S_\beta = \emptyset \ \forall \alpha \neq \beta$, $p_\alpha \cup p_\beta \in \mathbb{P}$. But $p_\alpha \cup p_\beta$ is a common extension of p_α and p_β. Thus two elements of $\mathbb{P}$ are compatible if they coincide on S.

Then the number of functions from S to $\{0, 1\}$ is smaller or equal to $2^{|S|}$ which is itself smaller or equal, by GCH, to τ. Hence the number of elements of an antichain in $\mathbb{P}$ cannot be bigger than τ.

□

In a similar way to the Δ-system Lemma, the Erdős-Rado Partition Theorem (Theorem 3.7) is a very general result and does not depend on the context of these forcings. But it will be useful later so we recall it. The next definition introduces some notions that are used in the Erdős-Rado Partition Theorem.

Definition 3.6. Let κ, μ and λ be regular cardinals such that $\mu < \kappa$ and $\lambda < \kappa$. Then we write $\kappa \to (\mu, \lambda)^2$ if for every $f : [\kappa]^2 \to \{0, 1\}$, one of the following occurs:

(a) There is a homogeneous set $\mathcal{A}$ of size μ such that $f(a_1, a_2) = 0 \ \forall a_1, a_2 \in \mathcal{A}$

(b) There is a homogeneous set $\mathcal{B}$ of size λ such that $f(b_1, b_2) = 1 \ \forall b_1, b_2 \in \mathcal{B}$

Where $[\kappa]^2$ is the set of pairs of elements of κ.

Theorem 3.7. (Erdős-Rado Partition Theorem). *Let κ be a regular cardinal and λ be a regular cardinal such that $\lambda < \kappa$ and*

$$\forall \tilde{\kappa} < \kappa, \ \forall \tilde{\lambda} < \lambda \ (\tilde{\kappa}^{\tilde{\lambda}} < \kappa)$$

Then

$$\kappa \to (\kappa, \lambda)^2$$

The proof is not given but can be found in [2][Chapter 14, Theorem 14.5].

The two following lemmas will show the chain condition for the product of finitely many partial orders with largest element. We will proceed by induction on the number of partial orders, where the Lemma 3.5 will be the base step and the Lemma 3.8 will be the induction step.

Lemma 3.8. *Let τ be a cardinal in M. If $\langle \mathbb{P}, \leq, 0_{\mathbb{P}} \rangle$ satisfies the τ^+-c.c. and $\mathbb{Q} = Add(\tau^+, \tau^{+++})$, then $\langle \mathbb{P} \oplus \mathbb{Q}, \leq, \tilde{0} \rangle$ satisfies the τ^{++}-c.c. where $\tilde{0} = \langle 0_{\mathbb{P}}, 0_{\mathbb{Q}} \rangle$.*

Proof. Let $\langle p_\alpha, q_\alpha \rangle$ be some conditions of $\mathbb{P} \oplus \mathbb{Q}$ for every $\alpha < \tau^{++}$. We need to show that some two of them are compatible.

By Lemma 3.5, $\mathbb{Q}$ satisfies the τ^{++}-c.c. Thus, without loss of generality, for every $\alpha, \beta < \tau^{++}$, q_α and q_β are compatible. Then, we can apply the Erdős-Rado partition Theorem (Theorem 3.7) with $\kappa = \tau^{++}$ and $\lambda = \tau^+$. Hence, we have $\tau^{++} \to (\tau^{++}, \tau^+)^2$ so

(a) there are τ^+ p_α's such that any two are incompatible or

(b) there are τ^{++} p_α's such that any two of them are compatible

Since (a) is impossible by hypothesis ($\langle \mathbb{P}, \leq, 0_{\mathbb{P}} \rangle$ satisfies the τ^+-c.c.), (b) applies. Thus any two $\langle p_\alpha, q_\alpha \rangle$ are compatible.

□

Lemma 3.9. *Let $\mathbb{Q}_n = Add(\aleph_n, \aleph_{n+2})$, then $\mathbb{Q}_1 \oplus \mathbb{Q}_2 \oplus \ldots \oplus \mathbb{Q}_n$ satisfies the $\aleph_{n+1}$-c.c.*

Proof. We will proceed by induction on n:

- $\mathbb{Q}_1 = \text{Add}(\aleph_1, \aleph_3)$ so $\langle \mathbb{Q}_1, \leq_{\mathbb{Q}_1}, 0_{\mathbb{Q}_1} \rangle$ satisfies the $\aleph_2$-c.c. by the Lemma 3.5.
- We suppose that $\langle \mathbb{Q}_1 \oplus \mathbb{Q}_2 \oplus \ldots \oplus \mathbb{Q}_k, \leq, \tilde{0} \rangle$ satisfies the $\aleph_{k+1}$-c.c. and we need to show that $\langle \mathbb{Q}_1 \oplus \mathbb{Q}_2 \oplus \ldots \oplus \mathbb{Q}_{k+1}, \leq, \tilde{0} \rangle$ satisfies the $\aleph_{k+2}$-c.c.
  Since $\mathbb{Q}_{k+1} = \text{Add}(\aleph_{k+1}, \aleph_{k+3}) = \text{Add}(\aleph_k^+, \aleph_k^{+++})$, we can apply the Lemma 3.8. Thus $\langle \mathbb{Q}_1 \oplus \mathbb{Q}_2 \oplus \ldots \oplus \mathbb{Q}_{k+1}, \leq, \tilde{0} \rangle = \langle \mathbb{Q}_1 \oplus \mathbb{Q}_2 \oplus \ldots \mathbb{Q}_k, \leq, \tilde{0} \rangle \oplus \langle \mathbb{Q}_{k+1}, \leq_{\mathbb{Q}_{k+1}}, 0_{\mathbb{Q}_{k+1}} \rangle$ satisfies the $\aleph_{k+2}$-c.c.

□

The following lemma will show that combining the chain-condition of the first partial orders and the closeness of the other partial orders give the property that we were looking for. In fact, the proof of this lemma is the most important of the section. The last theorem will just put together all the results of the section to give the consistency of $2^{\aleph_n} = \aleph_{n+2}$.

Lemma 3.10. *Let τ be a cardinal in M, $\langle \mathbb{P}, \leq_{\mathbb{P}}, 0_{\mathbb{P}} \rangle$ and $\langle \mathbb{Q}, \leq_{\mathbb{Q}}, 0_{\mathbb{Q}} \rangle$ be partial orders with largest element. If $\langle \mathbb{P}, \leq_{\mathbb{P}}, 0_{\mathbb{P}} \rangle$ satisfies the τ^+-c.c. and $\langle \mathbb{Q}, \leq_{\mathbb{Q}}, 0_{\mathbb{Q}} \rangle$ is τ^+-closed, then $\langle \mathbb{P} \oplus \mathbb{Q}, \leq, \tilde{0} \rangle$ does not collapse τ^+.*

Proof. Suppose, for contradiction, that $\langle p, q \rangle \in \mathbb{P} \oplus \mathbb{Q}$ forces that $f : \tau \to \tau^+$ is surjective. Let us choose $\langle p_\xi, q_\xi \rangle \in \mathbb{P} \oplus \mathbb{Q}$ by recursion in such a way:

- $p_\xi \leq_{\mathbb{P}} p$ such that p_ξ and p_η are incompatible if $\xi \neq \eta$.
- $q_\xi \leq_{\mathbb{Q}} q_\eta$ if $\eta \leq \xi$.

Since $\langle \mathbb{P}, \leq_{\mathbb{P}}, 0_{\mathbb{P}} \rangle$ satisfies the τ^+-c.c., the recusion stops after ν steps where $\nu < \tau^+$. Since $\langle \mathbb{Q}, \leq_{\mathbb{Q}}, 0_{\mathbb{Q}} \rangle$ is τ^+-closed, there is $\tilde{q} \in \mathbb{Q}$ such that $\tilde{q} \leq_{\mathbb{Q}} q_\xi$ $\forall \xi < \nu$.

Claim: Let α_ξ be the forced value of $f(0)$ by $\langle p_\xi, q_\xi \rangle$, then $\langle p, \tilde{q} \rangle \Vdash f(0) \in \{\alpha_\xi \mid \xi < \nu\}$.

Suppose, for contradiction, that there exists $\langle \tilde{p}, \tilde{q} \rangle \leq \langle p, \tilde{q} \rangle$ such that $\langle \tilde{p}, \tilde{q} \rangle \Vdash f(0) = \beta$ for some $\beta \notin \{\alpha_\xi \mid \xi < \nu\}$. By construction of the p_ξ's and the fact that $\tilde{p} \leq_{\mathbb{P}} p$, there is some $\xi < \nu$ such that $\tilde{p}$ and p_ξ are compatible. Hence there is $p^* \in \mathbb{P}$ such that $p^* \leq_{\mathbb{P}} \tilde{p}$ and $p^* \leq_{\mathbb{P}} p_\xi$. Then

$$\left.\begin{array}{l} \langle p^*, \tilde{q} \rangle \leq \langle p_\xi, q_\xi \rangle \Vdash f(0) = \alpha_\xi \\ \langle p^*, \tilde{q} \rangle \leq \langle \tilde{p}, \tilde{q} \rangle \Vdash f(0) = \beta \end{array}\right\} \text{Contradiction}$$

Thus the claim is proved.

Let us now choose a $\leq_{\mathbb{Q}}$-decreasing sequence $\langle q_\xi \mid \xi \leq \tau \rangle$ by recursion in such a way:

- $q_0 = q$
- If q_ξ is already choosen, we choose $q_{\xi+1} \leq q_\xi$ such that $q_{\xi+1} \Vdash f(\xi) < \alpha_\xi$ for some $\alpha_\xi < \tau^+$.
- If q_η is choosen for every $\eta < \xi$, we choose q_ξ as a lower bound of $\{q_\eta \mid \eta < \xi\}$. It is possible because $\langle \mathbb{Q}, \leq_{\mathbb{Q}}, 0_{\mathbb{Q}} \rangle$ is τ^+-closed.

Hence $\langle p, q_\tau \rangle \Vdash f(\xi) < \alpha_\xi$ for every $\xi < \tau$. Thus

$$\mathrm{Ran}(f) \subseteq \sup\{\alpha_\xi \mid \xi < \tau\} < \tau^+$$

This is in contradiction with the surjectivity of f.

□

As we said before, this theorem is the goal of the section but its proof is just using all the results showed in the section together to show that the extension of M that we get by these forcings satisfies that $2^{\aleph_n} = \aleph_{n+2}$ for every finite ordinal n.

Theorem 3.11. $2^{\aleph_n} = \aleph_{n+2}$ *for every* $n < \omega$ *is consistent.*

Proof. Let $\mathbb{P} = \bigoplus_{n<\omega} \mathbb{Q}_n$ where $\mathbb{Q}_n = \mathrm{Add}(\aleph_n, \aleph_{n+2})$. We need to show that ω_{n+1} is not collapsed for every $n < \omega$. Let $n < \omega$ and let us split $\mathbb{P}$ in such a way:

$$\mathbb{P} = (\mathbb{Q}_0 \oplus \mathbb{Q}_1 \oplus \ldots \oplus \mathbb{Q}_n) \oplus (\mathbb{Q}_{n+1} \oplus \mathbb{Q}_{n+2} \ldots)$$

We know that:

- $\langle \mathbb{Q}_0 \oplus \mathbb{Q}_1 \oplus \ldots \oplus \mathbb{Q}_n, \leq, \tilde{0} \rangle$ satisfies the $\aleph_{n+1}$-c.c. by the Lemma 3.9.
- $\langle \mathbb{Q}_{n+1} \oplus \mathbb{Q}_{n+2} \ldots, \leq, \tilde{0} \rangle$ is $\aleph_{n+1}$-closed.

Hence we can apply the Lemma 3.10, which shows that $\langle \mathbb{P}, \leq, \tilde{0} \rangle$ does not collapse ω_{n+1}.

□

3.2 Easton Forcing

In a similar way to the previous forcing, the Easton forcing is a forcing made from partial orders based on partial functions. The goal is also to build a model of ZFC in which the cardinality of the power set of the cardinals is different than in M. In the previous section, these cardinals were modified in a regular way ($2^{\aleph_n} = \aleph_{n+2}$) but the Easton forcing gives more freedom in the modification of these cardinals. The partial orders we will work with are based on so-called Easton index functions. These sets of functions will be more complicated to handle than the sets of functions we worked with in the previous section and we will need some results about complete embeddings get through it. We studied these results in the section Complete Embeddings of the second chapter. The aim of this section is to show the last theorem (Theorem 3.20). We will, at first, construct partial orders with largest element and, then, show that these partial orders extend M into a model in which the cardinality of the power set of the cardinals are defined by an Easton index function. This section will follow, on the main lines, [5][Chapter 8, §4].

Hence, let us start by defining some sets of functions and, then, the Easton index functions.

Definition 3.12. Let λ be an infinite cardinal in M, then we define:

$$\mathrm{Add}(I, J, \lambda) = \{p \mid p \text{ is a function}, |p| < \lambda, \mathrm{Dom}(p) \subseteq I \text{ and } \mathrm{Ran}(p) \subseteq J\}$$

We order $\mathrm{Add}(I, J, \lambda)$ in the following way:

$$\forall p, q \in \mathrm{Add}(I, J, \lambda) : p \leq q \iff q \subseteq p$$

Definition 3.13. An *index function* is a function E, such that $\mathrm{Dom}(E)$ is a set of regular cardinals.
An *Easton index function* is an index function E such that:
(a) $\forall \kappa \in \mathrm{Dom}(E)$, $E(\kappa)$ is a cardinal and $cf(E(\kappa)) > \kappa$
(b) $\forall \kappa, \kappa' \in \mathrm{Dom}(E)$ such that $\kappa < \kappa'$, $E(\kappa) \leq E(\kappa')$
If E is an index function, $\mathbb{P}(E)$ is the set of functions p such that:
(a) $\mathrm{Dom}(p) = \mathrm{Dom}(E)$
(b) $\forall \kappa \in \mathrm{Dom}(p)$, $p(\kappa) \in \mathrm{Add}(E(\kappa), 2, \kappa)$
(c) For every regular λ, $|\{\kappa \in (\lambda \cap \mathrm{Dom}(E)) \mid p(\kappa) \neq 0\}| < \lambda$
We order $\mathbb{P}(E)$ is ordered in the following way:

$$\forall p, p' \in \mathbb{P}(E) : p \leq p' \iff p'(\kappa) \subseteq p(\kappa)\ \forall \kappa \in \mathrm{Dom}(E)$$

The next proposition constructs the partial orders with largest element based on Easton index functions that we will work with during this section.

Proposition 3.14. *Let $\mathbf{1} \in \mathbb{P}(E)$ be such that $\mathbf{1}(\kappa) = 0$ for every $\kappa \in Dom(E)$. Then $\langle \mathbb{P}(E), \leq, \mathbf{1} \rangle$ is a partial order with largest element.*

Proof. $\subseteq$ is a transitive and reflexive relation so $\leq$ is also a transitive and reflexive relation on $\mathbb{P}(E)$.
Let $p \in \mathbb{P}(E)$ and $\kappa \in \mathrm{Dom}(E)$, then $\mathbf{1}(\kappa) = 0 \subseteq p(\kappa)$. Thus $p \leq \mathbf{1}$ and $\mathbf{1}$ is a largest element of $\mathbb{P}(E)$.

□

As we did in the previous forcing, we will split the partial order with largest element into two parts. We will show that the first part satisfies a chain-condition (Lemma 3.17) and the second part satisfies a closeness property (Lemma 3.18). The last step, which will be the last theorem (Theorem 3.20), will use this partition of $\mathbb{P}(E)$ and combine the properties of the two parts to show that the model $M[G]$ satisfies the properties that we are looking for.

Definition 3.15. Let E be an index function, then we define $E^+_\lambda = E \upharpoonright \{\kappa \mid \kappa > \lambda\}$ and $E^-_\lambda = E \upharpoonright \{\kappa \mid \kappa \leq \lambda\}$.

The next lemma splits the partial orders into two parts and shows that it is a correct partition.

Lemma 3.16. *Let E be an index function and λ be a cardinal, then $\mathbb{P}(E)$ is isomorphic to $\mathbb{P}(E^+_\lambda) \times \mathbb{P}(E^-_\lambda)$*

Proof. Consider the natural map:

$$\begin{aligned} f:\ &\mathbb{P}(E) \to \mathbb{P}(E^+_\lambda) \times \mathbb{P}(E^-_\lambda) \\ &p\ \mapsto \langle p \upharpoonright E^+_\lambda, p \upharpoonright E^-_\lambda \rangle \end{aligned}$$

□

Lemma 3.17. *If E is an index function, λ a regular cardinal, $Dom(E) \subseteq \lambda^+$ and $2^{<\lambda} \leq \lambda$, then $\langle \mathbb{P}(E), \leq, \mathbf{1} \rangle$ satisfies the λ^+-c.c.*

Proof. Let $p \in \mathbb{P}(E)$ and $d(p) = \bigcup\{\{\kappa\} \times \mathrm{Dom}(p(\kappa)) \mid \kappa \in \mathrm{Dom}(E)\}$. Since λ is regular,

$$|d(p)| \leq |\{\kappa \in (\lambda \cap \mathrm{Dom}(E)) \mid p(\kappa) \neq 0\}| < \lambda$$

Consider now $\langle p_\alpha \in \mathbb{P}(E) \mid \alpha < \lambda^+ \rangle$, we need to show that λ^+ elements of this set are compatible.

We can apply the Δ-system Lemma (Theorem 3.4) with $\kappa = \lambda$ and $\theta = \lambda^+$. Thus there is $X \subseteq \lambda^+$ such that $|X| = \lambda^+$ and $\{d(p_\alpha) \mid \alpha \in X\}$ is a Δ-system with root R. Moreover $|R| < \lambda$ so, by assumption, $2^{|R|} \leq \lambda$. But then we can find $Y \subseteq X$ such that $|Y| = \lambda^+$ and

$$\forall \alpha, \beta \in Y, \forall \langle \kappa, i \rangle \in R,\ p_\alpha(\kappa)(i) = p_\beta(\kappa)(i)$$

Hence the p_α's are compatible for every $\alpha \in Y$.

□

Lemma 3.18. *If E is an index function and $Dom(E) \cap \lambda^+ = 0$, then $\langle \mathbb{P}(E), \leq, \mathbf{1} \rangle$ is λ^+-closed.*

Proof. Let $\nu \leq \lambda$ and $\langle p_\alpha \mid \alpha < \nu \rangle$ be a $\leq$-decreasing sequence of length ν. Let us consider $p = \bigcup_{\alpha<\nu} p_\alpha$, we claim that $p \leq p_\alpha$ for every $\alpha < \nu$ and $p \in \mathbb{P}(E)$.
Let $\beta < \nu$ and $\kappa \in \mathrm{Dom}(E)$, then:

$$p_\beta(\kappa) \subseteq (\bigcup_{\alpha<\nu} p_\alpha)(\kappa) = p(\kappa)$$

Thus $p \leq p_\alpha$ for every $\alpha < \nu$. Let us show that $p \in \mathbb{P}(E)$.

(a) $\mathrm{Dom}(p) = \mathrm{Dom}(\bigcup_{\alpha<\nu} p_\alpha) = \bigcup_{\alpha<\nu} \mathrm{Dom}(p_\alpha) = \bigcup_{\alpha<\nu} \mathrm{Dom}(E) = \mathrm{Dom}(E)$

(b) Let $\kappa \in \mathrm{Dom}(E)$, then

$$p(\kappa) = (\bigcup_{\alpha<\nu} p_\alpha)(\kappa)$$

so p is a function, $\mathrm{Dom}(p(\kappa)) \subseteq E(\kappa)$ and $\mathrm{Ran}(p(\kappa)) \subseteq \kappa$. Moreover

$$\nu \leq \lambda \Rightarrow \nu \notin \mathrm{Dom}(E)$$

Since $\kappa \in \mathrm{Dom}(E)$, $\nu < \kappa$. Thus

$$|p(\kappa)| = |(\bigcup_{\alpha<\nu} p_\alpha)(\kappa)| \leq \bigcup_{\alpha<\nu} |p_\alpha(\kappa)| < \nu \times \kappa = \kappa$$

Hence $p(\kappa) \in \mathrm{Add}(E(\kappa), 2, \kappa)$

(c) Let μ be regular and suppose, for contradiction, that

$$|\{\kappa \in (\mu \cap \mathrm{Dom}(E)) \mid p(\kappa) \neq 0\}| \geq \mu$$

But, then, by construction of p, there must be $\beta < \nu$ such that

$$|\{\kappa \in (\mu \cap \mathrm{Dom}(E)) \mid p_\beta(\kappa) \neq 0\}| \geq \mu$$

which is a contradiction.

Hence $p \in \mathbb{P}(E)$ and $\langle \mathbb{P}(E), \leq, \mathbf{1} \rangle$ is λ^+-closed.

□

The following lemma is the last detail needed before beeing able to show the Theorem 3.20. Since the Easton index function is defined in M, it is important to know that the Easton forcing preserves the cardinalities. Otherwise, the cardinalities of the power set of the cardinals in $M[G]$ would not be defined by an Easton index function.

Lemma 3.19. *Let E be an index function, then $\mathbb{P}(E)$ preserves cardinals.*

Proof. By the Lemma 2.20, we only need to show that $\mathbb{P}(E)$ preserves cofinalities. We suppose, for contradiction, that $\mathbb{P}(E)$ does not. Thus there is a $\mathbb{P}(E)$-generic G over M and $\theta > \omega$ such that θ is regular in M and singular in $M[G]$. Let $\lambda = cf(\theta)^{M[G]}$, then $\lambda < \theta$ since θ is singular in $M[G]$. Moreover λ is regular in $M[G]$ and, hence, in M.

Let us work first in M. Consider then $\mathbb{P}_0 = \mathbb{P}(E_\lambda^-)$, $\mathbb{P}_1 = \mathbb{P}(E_\lambda^+)$, the complete embeddings (by the Proposition 2.22):

$$i_0 : \mathbb{P}_0 \to \mathbb{P}_0 \times \mathbb{P}_1 \qquad i_1 : \mathbb{P}_1 \to \mathbb{P}_0 \times \mathbb{P}_1$$
$$p_0 \mapsto \langle p_0, \mathbf{1}_{\mathbb{P}_1}\rangle \qquad p_1 \mapsto \langle \mathbf{1}_{\mathbb{P}_0}, p_1\rangle$$

and the isomorphism:

$$j : \mathbb{P}_0 \times \mathbb{P}_1 \to \mathbb{P}(E)$$
$$\langle p_0, p_1\rangle \mapsto p_0 \cup p_1$$

By the Corollary 2.24, we know that $j^{-1}(G)$ is $\mathbb{P}_0 \times \mathbb{P}_1$-generic over M and $M[G] = M[j^{-1}(G)]$.

Then let $G_0 = i_0^{-1}(j^{-1}(G))$ and $G_1 = i_1^{-1}(j^{-1}(G))$. Thus, by the Lemma 2.25, G_0 is $\mathbb{P}_0$-generic over M, G_1 is $\mathbb{P}_1$-generic over M and $G = G_0 \times G_1$. Moreover, by the Theorem 2.26, G_0 is $\mathbb{P}_0$-generic over $M[G_1]$ and $M[G] = M[G_1][G_0]$.

Let us work now in $M[G]$ and let $f : \lambda \to \theta$ be a cofinal map so $\mathrm{Ran}(f)$ is unbounded in θ. We can apply the Lemma 3.18 to $\mathbb{P}_1$ to get that $(\langle \mathbb{P}_1, \leq, \mathbf{1}\rangle$ is λ^+-closed$)^M$. Moreover, since $(2^{<\lambda} = \lambda)^{M[G_1]}$ by GCH, we can apply the Lemma 3.17 and have that $(\langle \mathbb{P}_0, \leq, \mathbf{1}\rangle$ satisfies the λ^+-c.c.$)^{M[G_1]}$. Then, we can apply the Lemma 2.7 with $A = \lambda$, $B = \theta$, $M = M[G_1]$, $\theta = \lambda^+$, $G = G_0$ and $\mathbb{P} = \mathbb{P}_0$ to get that there is a map $F : \lambda \to P(\theta)$ such that $F \in M[G_1]$ and, $f(\alpha) \in F(\alpha)$ and $(|F(\alpha)| \leq \lambda)^{M[G_1]}$ $\forall \alpha < \lambda$. But $(\langle \mathbb{P}_1, \leq, \mathbf{1}\rangle$ is λ^+-closed$)^M$ so $F \in M$ and $(|F(\alpha)| \leq \lambda)^{M[G]}$ $\forall \alpha < \lambda$.

Let us finally come back to work in M. Since $(|F(\alpha)| \leq \lambda)^{M[G]}$ $\forall \alpha < \lambda$, we have that $\bigcup_{\alpha<\lambda} F(\alpha)$ has cardinality smaller or equal to λ. Moreover, as $f(\alpha) \in F(\alpha)$ $\forall \alpha < \lambda$, $\bigcup_{\alpha<\lambda} F(\alpha)$ is cofinal in θ. Then, by the two last observations,

$$cf(\theta)^M = |\bigcup_{\alpha<\lambda} F(\alpha)| \leq \lambda < \theta$$

Thus θ is not regular in M which is a contradiction.

□

The last theorem of these section is using all the results we proved to show that the extension model of M constructed by the Easton forcing satisfies that 2^θ is given by the Easton index function used for every infinite cardinal θ.

Theorem 3.20. *Let E be an Easton index function, $\mathbb{P} = \mathbb{P}(E)$ and G be a $\mathbb{P}$-generic over M. Then $(2^\kappa = E(\kappa), \forall \kappa \in Dom(E))^{M[G]}$. More generally, let θ be an infinite cardinal in $M[G]$, define*

$$E'(\theta) = \max(\theta^+, \sup\{E(\kappa) \mid \kappa \in Dom(E) \text{ and } \kappa \leq \theta\})$$

and

$$E^*(\theta) = \begin{cases} E'(\theta) & \textit{if } cf(E'(\theta)) > \theta \\ (E'(\theta))^+ & \textit{otherwise} \end{cases}$$

Then $(2^\theta = E^*(\theta))^{M[G]}$.

Proof. Using Lemma 3.19, we know that $\mathbb{P}$ preserves cardinals. Since θ and $E(\kappa)$ are cardinals in $M[G]$ for every $\kappa \in \mathrm{Dom}(E)$, $E'(\theta)$ is a cardinal in $M[G]$ and, thus, $E^*(\theta)$ is also a cardinal in $M[G]$. Hence E' and E^* are absolute for M and $M[G]$.

At first, let us show that $E^*(\kappa) = E(\kappa)$ for any $\kappa \in \mathrm{Dom}(E)$. By definition of E, we know that $E(\kappa)$ is a cardinal and $\mathrm{cf}(E(\kappa)) > \kappa$ so $E^*(\kappa) = E'(\kappa)$. Let $\theta < \kappa$, then, again by definition of E, $E(\theta) < E(\kappa)$. Thus $\sup\{E(\theta) \mid \theta \in \mathrm{Dom}(E) \text{ and } \theta \leq \kappa\} = E(\kappa)$. Moreover, since $\mathrm{cf}(E(\kappa)) > \kappa$, $E(\kappa) \geq \kappa^+$. Hence $E'(\kappa) = E(\kappa)$ and so $E^*(\kappa) = E(\kappa)$. Now, let θ be an infinite cardinal of M. We want to show that $2^\theta = E^*(\theta)$ holds in $M[G]$. We will show that $2^\theta \leq E^*(\theta)$ and then that $2^\theta \geq E^*(\theta)$.

$\leq$: To show this, we will need the following claim:

Claim: $E(\kappa) \leq 2^\kappa$ in $M[G]$ for any $\kappa \in \mathrm{Dom}(E)$.

Let us give the demonstration by showing that $\mathrm{Add}(E(\kappa), 2, \kappa)^M$ in $\mathbb{P}$ forces an $E(\kappa)$-sequence of distinct subsets of κ to be added. In M, let $f : E(\kappa) \times \kappa \to E(\kappa)$ be any injective map and in $M[G]$, for some $\alpha < E(\kappa)$, let

$$A_\alpha = \{\xi < \kappa \mid \exists p \in G \text{ such that } p(\kappa)(f(\alpha, \xi)) = 1\}$$

Thus, by injectivity of f, $A_\alpha \neq A_\beta$ if $\alpha \neq \beta$. Moreover, $\{A_\alpha \mid \alpha < E(\kappa)\} \in M[G]$. We can consider, now, the map:

$$\begin{aligned} s:\ & E(\kappa) \to \{A_\alpha \mid \alpha < E(\kappa)\} \\ & \alpha \ \mapsto A_\alpha \end{aligned}$$

Since s is an injective map from $E(\kappa)$ to $\{A_\alpha \mid \alpha < E(\kappa)\}$ and $|A_\alpha| = 2^\kappa$, we have that $E(\kappa) \leq 2^\kappa$ in $M[G]$.

Then we can continue the proof with the general case of θ:

$$E'(\theta) = \max\{\theta^+, \sup\{2^\kappa \mid \kappa \in \mathrm{Dom}(E) \text{ and } \kappa \leq \theta\}\} \leq \max\{\theta^+, 2^\theta\} = 2^\theta$$

where the last equality is given by the König's Lemma (Lemma 2.8). Thus, by GCH, $E^*(\theta) \leq 2^\theta$. Hence, we only need to show that $E(\kappa) \leq 2^\kappa$ for any $\kappa \in \mathrm{Dom}(E)$.

$\geq$: The proof of the Lemma 3.19 can be repeated to show that $\mathbb{P}$ preserves cofinalities so $\mathrm{cf}(\theta)^M = \mathrm{cf}(\theta)^{M[G]}$. Let $\lambda = \mathrm{cf}(\theta)^M = \mathrm{cf}(\theta)^{M[G]}$, $\mathbb{P}_0 = \mathbb{P}(E^-_\lambda)$ and $\mathbb{P}_1 = \mathbb{P}(E^+_\lambda)$. Then, by applying the Lemma 3.16 in M, we know that $\mathbb{P}$ is isomorphic to $\mathbb{P}_0 \times \mathbb{P}_1$. Moreover, by the Lemma 3.17, $\langle \mathbb{P}_0, \leq, \mathbf{1} \rangle$ satisfies the λ^+-c.c. and by the Lemma 3.18, $\langle \mathbb{P}_1, \leq, \mathbf{1} \rangle$ is λ^+-closed. Finally, by the Theorem 2.26, $M[G] = M[G_1][G_0]$ where G_1 is $\mathbb{P}_1$-generic over M and G_0 is $\mathbb{P}_0$-generic over $M[G_1]$.

Let us, firstly, investigate the case where $\lambda = \theta$. We can see in M:

$$|\mathrm{Add}(E(\kappa), 2, \kappa)| \leq |\mathrm{Add}(E^*(\kappa), 2, \kappa)| \leq E^*(\lambda)^\lambda \ \forall \kappa \in \mathrm{Dom}(E^-_\lambda) \text{ such that } \kappa \leq \lambda$$

Thus $|\mathbb{P}_0| \leq (E^*(\lambda)^\lambda)^\lambda$. But, since $\mathrm{cf}(E^*(\lambda)) > \lambda$ and GCH holds, $E^*(\lambda)^\lambda = E^*(\lambda)$. Hence $|\mathbb{P}_0| \leq E^*(\lambda)$. Then, since $\langle \mathbb{P}_1, \leq, \mathbf{1} \rangle$ is λ^+-closed in M, we have, in $M[G_1]$, that $E^*(\lambda)^\lambda = E^*(\lambda)$, $\langle \mathbb{P}_0, \leq, \mathbf{1} \rangle$ satisfies the λ^+-c.c. and $|\mathbb{P}_0| \leq E^*(\lambda)$. Thus there are, at most, $(E^*(\lambda)^\lambda)^\lambda = E^*(\lambda)$ nice $\mathbb{P}_0$-names for subsets of $\check{\lambda}$ in $M[G_1]$, which means that:

$$2^\lambda \leq E^*(\lambda) \text{ in } M[G_1][G_0]$$

Thus, since θ is regular ($\lambda = \theta$), $2^\theta = E^*(\theta)$ in $M[G]$.

We will now investigate the second and last case where θ is not regular. To do that, we need to show that $E^*(\theta)^\lambda = E^*(\theta)$ in $M[G]$. Let us consider the map f in $M[G]$ such that $f : \lambda \to E^*(\theta)$. Then, since $(\langle \mathbb{P}_0, \leq, \mathbf{1} \rangle$ satisfies the λ^+-c.c.$)^{M[G_1]}$ and by the Lemma 2.7, $\exists F \in M[G_1]$, $F : \lambda \times \lambda \to E^*(\theta)$ such that $f(\alpha) = F(\alpha, \beta)$ $\forall \alpha < \lambda$ and $\beta < \lambda$. But $\langle \mathbb{P}_1, \leq, \mathbf{1} \rangle$ is λ^+-closed in M, so $F \in M$. Then, as GCH holds in M, there are only $E^*(\theta)$ such map F. For each such map F, the set of $f \in M[G]$ satisfying that $f(\alpha) = F(\alpha, \beta)$ $\forall \alpha < \lambda$ and $\beta < \lambda$ has size $\lambda^\lambda = E^*(\lambda) \leq E^*(\theta)$ in $M[G]$ because λ is regular in $M[G]$. Thus

$$(|{}^\lambda E^*(\theta)| \leq E^*(\theta))^{M[G]}$$

Then, in $M[G]$, let B be the set of bounded subsets of θ and if $\delta < \theta$ is regular, we have that $|P(\delta)| = E^*(\delta) \leq E^*(\theta)$ and that the regular cardinals are cofinal in θ. Hence $|B| \leq E^*(\theta)$. Finally, considering the map:

$$\begin{aligned} H :\ & {}^\lambda B \to P(\theta) \\ & h \ \mapsto \bigcup_{\alpha < \lambda} \{h(\alpha)\} \end{aligned}$$

where $g : \lambda \to B$. Hence $2^\theta \leq E^*(\theta)^\lambda = E^*(\theta)$.

□

4 Prikry-type Forcings

This chapter, which will be the last of the report, will introduce other forcings that we have seen in the previous chapter. Instead of working with forcings based on partial functions, we will work on forcings based on ultrafilter. Using four different of these forcings, called Prikry-type forcings, we will construct different models of ZFC. Thanks to the first two, the models constructed will preserve cardinalities but not cofinalites, showing that the reverse of the Lemma 2.20 is not true. Thanks to the two other Prikry-type forcings, we will add to a model a dominating sequence to a singular cardinal. The main strategy of all of these forcings will be to provide partial orders and partial orders with largest element and to show, thanks to the Prikry-condition, that we can also force with elements of the partial order. Since the partial orders and the partial orders with largest element will have different properties, we will be able to show the required properties of the constructed model.

This chapter is composed by four sections. The first one is a short recall of some results about ultrafilters that will be useful in the following. The second section will provide, starting with a normal ultrafilter, the construction of a model of ZFC in which cardinalities are preserved but not cofinalities. The third section has the same goal as the second one but starts with a weaker assumption. Indeed, we will not start with a normal ultrafilter but with a κ-complete ultrafilter where κ is a measurable cardinal. Finally, the last section will use two different Prikry-type forcings to add to a model a dominating sequence to a singular cardinal. This chapter will follow, on the main lines, [1][Chapter 1, §1]. Further details and results concernings these forcings can be found in [1] and [2].

4.1 Ultrafilters

In this section, we will define ultrafilters and show some of their properties. The goal of this section is to have a background on ultrafilters which will be necessary to the good understanding of the following three sections.

Definition 4.1. Let X be a non-empty set, then $F \subseteq P(X)$ is a *filter* on X if it is non-empty, $\emptyset \notin F$ and:

(a) $A, B \in F \Rightarrow A \cap B \in F$

(b) $A \in F$ and $A \subseteq B \Rightarrow B \in F$

F is an *ultrafilter* on X if it is a filter on X and for every $A \in P(X)$, $A \in F$ or $X \setminus A \in F$. F is a *principal* ultrafilter on X if it is an ultrafilter on X and there is $\alpha \in X$ such that $\{\alpha\} \in F$.

Proposition 4.2. *Let X be a non-empty set and F be an ultrafilter on X. If $A, B \subseteq P(X)$ such that $A \cup B \in F$, then $A \in F$ or $B \in F$.*

Proof. Suppose, for a contradiction, that $A \notin F$ and $B \notin F$. Thus, by the ultrafilter property, $X \setminus A \in F$ and $X \setminus B \in F$.

We can then apply (a) of Definition 4.1, $(X \setminus A) \cap (X \setminus B) \in F$. Hence

$$X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B) \in F$$

Finally, by the ultrafilter property again, $A \cup B \notin F$ which is a contradiction.

□

Definition 4.3. Let X be a non-empty set and $\kappa \geq \omega$ be a cardinal, then $F \subseteq P(X)$ is a *κ-complete* ultrafilter on X if it is an ultrafilter on X and for every $F' \subseteq F$ such that $|F'| < \kappa$, $\bigcap F' \in F$.

Definition 4.4. An infinite cardinal κ is called a *measurable* cardinal if there is a κ-complete ultrafilter on κ that is not a principal filter.

Definition 4.5. Let κ be an infinite cardinal, then $U \subseteq P(\kappa)$ is a *normal* ultrafilter on κ if it is an ultrafilter on κ and for every $A \in U$ and $f : A \setminus \{0\} \to \kappa$ such that $f(\alpha) < \alpha$ $\forall \alpha \in A \setminus \{0\}$, there is $\beta \in A \setminus \{0\}$ such that $f^{-1}(\beta) \in U$.

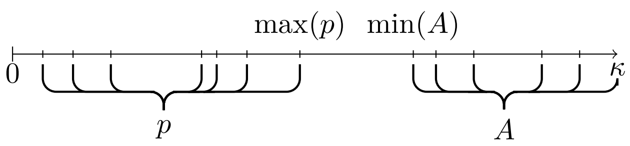
4.2 Basic Prikry Forcing

This section will build a partial order with largest element and a partial order. In the previous chapter, we splitted the the partial order with largest element between a part satisfying a chain-condition and part with a property of closeness. In this case, the partial order with largest element will satisfy the chain-condition and the partial order will satisfy the property of closeness. It is because of the Prikry condition (Lemma 4.16) that we can link the two partial orders and prove the Theorem 4.18 which is the goal of the section.

To build and show that, we must fix a measurable cardinal κ and a non-principal, κ-complete, normal ultrafilter U on κ. Let us start by constructing the partial order with largest element.

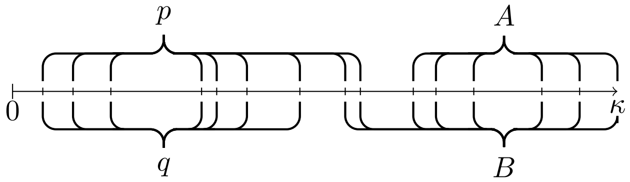
Definition 4.6. Let $\mathbb{P}$ be the set of all pairs $\langle p, A \rangle$ such that:
(a) p is a finite subset of κ
(b) $A \in U$
(c) $\min(A) > \max(p)$

The following figure should help to visualize a pair $\langle p, A \rangle \in \mathbb{P}$.



Definition 4.7. Let $\langle p, A \rangle, \langle q, B \rangle \in \mathbb{P}$. We say that $\langle p, A \rangle$ *is stronger than* $\langle q, B \rangle$ and denote this by $\langle p, A \rangle \leq \langle q, B \rangle$ if:
(a) p is an extension of q, i.e. $p \cap (\max(q) + 1) = q$
(b) $A \subseteq B$
(c) $p \setminus q \subseteq B$

The following figure should help to visualize what $\langle p, A \rangle \leq \langle q, B \rangle$ means for two pairs $\langle p, A \rangle, \langle q, B \rangle \in \mathbb{P}$.



Proposition 4.8. *$\langle \mathbb{P}, \leq, \langle 0, \kappa \rangle \rangle$ is a partial order with largest element.*

Proof. First, let us show that $\leq$ is a transitive and reflexive relation on $\mathbb{P}$.

Let $\langle p, A\rangle, \langle q, B\rangle, \langle r, C\rangle \in \mathbb{P}$ such that $\langle p, A\rangle \leq \langle q, B\rangle$ and $\langle q, B\rangle \leq \langle r, C\rangle$. We want to show that $\langle p, A\rangle \leq \langle r, C\rangle$:

(a) p is an extension of q and q is an extension of r. Hence

$$p \cap (\max(r) + 1) = p \cap (\max(q) + 1) \cap (\max(r) + 1) = q \cap (\max(r) + 1) = r$$

Thus p is an extension of r.

(b) $A \subseteq B$ and $B \subseteq C$ so $A \subseteq C$.

(c) $p \setminus q \subset B$ and $q \setminus r \subseteq C$. Let $s \in p \setminus r$, then $s \in q \setminus r$ or $s \in p \setminus q$.

- If $s \in q \setminus r$:
$$q \setminus r \subseteq C \text{ so } s \in C$$
- If $s \in p \setminus q$:
$$p \setminus q \subseteq B \text{ and } B \subseteq C \text{ so } s \in C$$

Thus, in any case, $s \in C$ so $p \setminus r \subseteq C$.

Hence $\langle p, A\rangle \leq \langle r, C\rangle$ so $\leq$ is a transitive relation on $\mathbb{P}$.
Moreover:

(a) p is an extension of itself
(b) $A \subseteq A$
(c) $p \setminus p = 0 \subseteq A$

Thus $\langle p, A\rangle \leq \langle p, A\rangle$ so $\leq$ is a reflexive relation on $\mathbb{P}$.
Let us show now that $\langle 0, \kappa\rangle$ is a largest element of $\mathbb{P}$ with respect to $\leq$:

(a) $p \cap (\max(0) + 1) = p \cap 1 = 0$
(b) $A \subseteq \kappa$
(c) $p \setminus 0 = p \subseteq \kappa$

Thus $\langle \mathbb{P}, \leq, \langle 0, \kappa\rangle\rangle$ is a partial order with largest element.

□

Let us, now, construct the partial order.

Definition 4.9. Let $\langle p, A\rangle, \langle q, B\rangle \in \mathbb{P}$. We say that $\langle p, A\rangle$ is a *direct* (or *Prikry*) *extension* of $\langle q, B\rangle$ and denote this by $\langle p, A\rangle \leq^* \langle q, B\rangle$ if:

(a) $p = q$
(b) $A \subseteq B$

Proposition 4.10. *$\langle \mathbb{P}, \leq^*\rangle$ is a partial order.*

Proof. $\subseteq$ is a transitive and reflexive relation on κ so $\leq^*$ is also a transitive and reflexive relation on $\mathbb{P}$. Hence $\langle \mathbb{P}, \leq^*\rangle$ is a partial order.

□

The next lemma just shows how the two partial orders are related to each other.

Lemma 4.11. $\leq^* \subseteq \leq$.

Proof. Let $\langle\langle p, A\rangle, \langle p, B\rangle\rangle \in \leq^*$, so $A \subseteq B$.
p is an extension of itself and $p \setminus p = 0 \subseteq B$. Thus $\langle p, A\rangle \leq \langle p, B\rangle$.
Hence $\langle\langle p, A\rangle, \langle p, B\rangle\rangle \in \leq$.

□

The following lemma is the first step of the Theorem 4.18 and, thus, of the goal of the section. Indeed, it will show that cofinalities are not preserved in the model forced by the Basic Prikry forcing.

Lemma 4.12. *Let G be a $\langle \mathbb{P}, \leq, \langle 0, \kappa \rangle \rangle$-generic. Then $\bigcup\{p \mid \exists A(\langle p, A\rangle \in G)\}$ is an ω-sequence cofinal in κ.*

Proof. We need to show that for every $\alpha < \kappa$ and $\langle q, B\rangle \in \mathbb{P}$, the set

$$D_\alpha = \{\langle p, A\rangle \in \mathbb{P} \mid \langle p, A\rangle \leq \langle q, B\rangle \text{ and } \max(p) > \alpha\}$$

is dense in $\langle \mathbb{P}, \leq, \langle 0, \kappa \rangle \rangle$ below $\langle q, B\rangle$.

Let $\langle r, C\rangle \leq \langle q, B\rangle$. Since U is κ-complete and non-principal, C is cofinal in κ. Thus $\exists c \in C$ such that $c > \alpha$. Then consider $\langle r \cup \{c\}, C \setminus (c+1)\rangle$:

(a) $r \cup \{c\}$ is an extension of r

(b) $C \setminus (c+1) \subseteq C$

(c) $r \cup \{c\} \setminus r = \{c\} \subseteq C$

Hence $\langle r \cup \{c\}, C \setminus (c+1)\rangle \leq \langle r, C\rangle$. Moreover

$$\max(r \cup \{c\}) \geq c > \alpha$$

So $\langle r \cup \{c\}, C \setminus (c+1)\rangle \in D_\alpha$. Thus D_α is dense in $\langle \mathbb{P}, \leq, \langle 0, \kappa \rangle \rangle$ below $\langle q, B\rangle$. Hence $D_\alpha \cap G \neq \emptyset \ \forall \alpha < \kappa$. Finally, $\bigcup\{p \mid \exists A(\langle p, A\rangle \in G)\}$ is an ω-sequence cofinal in κ.

□

The two next lemmas will show the chain-condition of the partial order with largest element (Lemma 4.13) and the closeness of the partial order (Lemma 4.14). Those properties are the essential properties of all the forcings we are studying in this report.

Lemma 4.13. *$\langle \mathbb{P}, \leq, \langle 0, \kappa \rangle \rangle$ satisfies the κ^+-c.c.*

Proof. Let $\langle p, A\rangle, \langle p, B\rangle \in \mathbb{P}$ and consider $\langle p, A \cap B\rangle$:

(a) p is an extension of itself

(b) $A \cap B \subseteq A$ and $A \cap B \subseteq B$

(c) $p \setminus p = 0 \subseteq A \cap B$

So $\langle p, A \cap B\rangle \leq \langle p, A\rangle, \langle p, B\rangle$.

Thus two elements of an antichain in $\langle \mathbb{P}, \leq, \langle 0, \kappa \rangle \rangle$ must have different finite subset of κ. But the number of finite subsets of κ has cardinality $\leq \kappa$. Hence the cardinality of an antichain in $\langle \mathbb{P}, \leq, \langle 0, \kappa \rangle \rangle$ is $\leq \kappa$. Thus $\langle \mathbb{P}, \leq, \langle 0, \kappa \rangle \rangle$ satisfies the κ^+-c.c.

□

Lemma 4.14. *$\langle \mathbb{P}, \leq^* \rangle$ is κ-closed.*

Proof. Let $\lambda < \kappa$ and $\langle \langle p_\alpha, A_\alpha\rangle \mid \alpha < \lambda\rangle$ be a $\leq^*$-decreasing sequence of length λ. By definition of $\leq^*$, we know that all the p_α's are the same so we recall them p. Then let $A = \bigcap\limits_{\alpha<\lambda} A_\alpha$. By κ-completeness of U, $A \in U$. Thus $\langle p, A\rangle \leq^* \langle p_\alpha, A_\alpha\rangle \ \forall \alpha < \lambda$.

□

The next theorem is a needed result for the proof of the following lemma about the Prikry condition. This Prikry condition is the reason why we are working with two different partial orders in the Prikry-type forcings. We will also have a similar condition to show in the two last sections.

Theorem 4.15. (Rowbottom). *Let $[\kappa]^{<\omega}$ be the set of the finite subsets of κ and $\gamma < \kappa$, then for every $f : [\kappa]^{<\omega} \to \gamma$ there is $A \in U$ such that $f(s) = f(s') \ \forall s, s' \in [A]^n \ \forall n < \omega$.*

The proof is not given but can be found in [3][Theorem 70] or in [4][Chapter 7, Theorem 7.70].

Lemma 4.16. (The Prikry Condition). *Let $\langle q, B\rangle \in \mathbb{P}$ and σ be a statement of the forcing language of $\langle \mathbb{P}, \leq, \langle 0, \kappa\rangle\rangle$. Then there is a $\langle p, A\rangle \leq^* \langle q, B\rangle$ such that $\langle p, A\rangle \parallel \sigma$ (i.e. $\langle p, A\rangle \Vdash \sigma$ or $\langle p, A\rangle \Vdash \neg\sigma$).*

Proof. Let $[\kappa]^{<\omega}$ be the set of the finite subsets of κ and define a partition $h : [B]^{<\omega} \to 2$ such that:

$$h(s) = \begin{cases} 1 & \text{if there is a } C \text{ such that } \langle q \cup s, C\rangle \Vdash \sigma \\ 0 & \text{otherwise} \end{cases}$$

We can apply the Rowbottom Theorem (Theorem 4.15) so there is $A \in U, A \subseteq B$ such that $\forall n < \omega$ and $\forall s_1, s_2 \in [A]^n$ we have $h(s_1) = h(s_2)$.

Consider now $\langle q, A\rangle$ and suppose, for contradiction, that $\neg(\langle q, A\rangle \parallel \sigma)$. Thus there must be $\langle q \cup s_1, B_1\rangle, \langle q \cup s_2, B_2\rangle \leq \langle q, A\rangle$ such that $\langle q \cup s_1, B_1\rangle \Vdash \sigma$ and $\langle q \cup s_2, B_2\rangle \Vdash \neg\sigma$. Without loss of generality, we can assume that $|s_1| = |s_2| = n$. But then $s_1, s_2 \in [A]^n$ and $h(s_1) \neq h(s_2)$ which is in contradiction with the Rowbottom Theorem.

Hence $\langle q, A\rangle \leq^* \langle q, B\rangle$ and $\langle q, A\rangle \parallel \sigma$.

□

The next lemma is another important step of the Theorem 4.18 and to attempt the goal of the section. Indeed, it will be necessary to show that the model constructed with the help of the Basic Prikry forcing preserves cardinalities.

Lemma 4.17. *$\langle \mathbb{P}, \leq, \langle 0, \kappa\rangle\rangle$ does not add new bounded subsets of κ.*

Proof. Let $t \in \mathbb{P}$, $\bar{a}$ be a $\mathbb{P}$-name, $\lambda < \kappa$ such that $t \Vdash \bar{a} \subseteq \check{\lambda}$. Then let

$$\sigma_\alpha \text{ be the statement "}\check{\alpha} \in \bar{a}\text{" for every } \alpha < \lambda$$

We will now define, by recursion, a $\leq^*$-decreasing sequence of conditions $\langle t_\alpha \mid \alpha < \lambda\rangle$ such that $t_\alpha \parallel \sigma_\alpha$ $\forall \alpha < \lambda$:

- $\exists t_0 \leq^* t$ such that $t_0 \parallel \sigma_0$ by Lemma 4.16.
- We suppose, then, that $\langle t_\beta \mid \beta < \alpha\rangle$ is defined: $\exists t_\gamma \leq^* t_\beta$ $\forall \beta < \alpha$ by Lemma 4.14 and, then, $\exists t_\alpha \leq^* t_\gamma \leq^* t_\beta$ $\forall \beta < \alpha$ such that $t_\alpha \parallel \sigma_\alpha$ by Lemma 4.16.

We use again the Lemma 4.14 to find $t^* \leq^* t_\alpha$ $\forall \alpha < \lambda$. Thus, by construction of t^*, we have that $t^* \leq^* t$ and so $t^* \leq t$ by Lemma 4.11. Hence $t^* \Vdash \bar{a} = \check{b}$ where $b = \{\alpha < \lambda \mid t^* \Vdash \alpha \in \bar{a}\}$. $\bar{a}$ is, then, not a new $\mathbb{P}$-name and there is no new bounded subsets of κ.

□

The last theorem will recap and assemble all the results that we showed in this section.

Theorem 4.18. *The following holds in $M[G]$ for any $\langle \mathbb{P}, \leq, \langle 0, \kappa\rangle\rangle$-generic G:*

(a) *κ has cofinality $\aleph_0$*
(b) *All the cardinals are preserved*
(c) *No new bounded subsets are added to κ*

Proof. (a) κ is an infinite cardinal so $cf(\kappa) \geq \aleph_0$. But, by Lemma 4.12, we know that there is an ω-sequence cofinal in κ so $cf(\kappa) \leq \aleph_0$. Hence $cf(\kappa) = \aleph_0$.

(c) Lemma 4.17.

(b) By (c), all cardinals $\leq \kappa$ are preserved and, by Lemma 4.13, all the cardinals $\geq \kappa^+$ are preserved. Thus all the cardinals are preserved.

□

4.3 Tree Prikry Forcing

This section will be organised in the same way as the previous one and will have analogous definitions, propositions, lemmas and theorems to the ones of the previous section. The only difference is the assumption, we start with a κ-complete ultrafilter where κ is a measurable cardinal and not a normal ultrafilter. Because of this, the partial orders that will be built will be based on trees which will make the visualization a bit more complicated but the philosophy of the construction done with the Tree Prikry forcing is exactly the same as the philosophy of the construction done with the Basic Prikry forcing.

To build and show that, we must fix a measurable cardinal κ and a non-principal, κ-complete ultrafilter U on κ. Let us start by constructing the partial order with largest element.

Definition 4.19. $\langle T, \trianglelefteq \rangle$ is a *tree* if:
(a) $T \neq \emptyset$
(b) $\trianglelefteq$ is a transitive and reflexive relation on T
(c) For every $x \in T$, the set $\{y \in T \mid y \trianglelefteq x\}$ is well-ordered by $\trianglelefteq$

Definition 4.20. A set T is called a *U-tree with a trunk t* if:
(a) T consists of finite increasing sequences of ordinals below κ
(b) $\langle T, \trianglelefteq \rangle$ is a tree, where $\eta \trianglelefteq \nu$ if $\nu \restriction \mathrm{dom}(\eta) = \eta$
(c) t is a *trunk* of T, i.e. $t \in T$ and, for every $\eta \in T$, $\eta \trianglelefteq t$ or $t \trianglelefteq \eta$
(d) For every $\eta \in T$ such that $t \trianglelefteq \eta$, $\mathrm{Suc}_T(\eta) = \{\alpha < \kappa \mid \eta^\frown\langle \alpha \rangle \in T\} \in U$
Define $\mathrm{Lev}_n(T) = \{\eta \in T \mid \mathrm{length}(\eta) = n\}$ for every $n < \omega$.

Definition 4.21. The set $\mathbb{P}$ contains all the pairs of the form $\langle t, T \rangle$ such that T is a U-tree with trunk t.

Definition 4.22. Let $\langle t, T \rangle, \langle s, S \rangle \in \mathbb{P}$. We say that $\langle t, T \rangle$ *is stronger than* $\langle s, S \rangle$ and denote this by $\langle t, T \rangle \leq \langle s, S \rangle$ if $T \subseteq S$.

Proposition 4.23. *Let $\langle t, T \rangle, \langle s, S \rangle \in \mathbb{P}$ such that $\langle t, T \rangle \leq \langle s, S \rangle$. Then $t \in S$ and $s \trianglelefteq t$.*

Proof. $\langle t, T \rangle \leq \langle s, S \rangle$ implies that $T \subseteq S$. But $t \in T$ so $t \in S$. Since s is a trunk of S, we have that $s \trianglelefteq t$ or $t \trianglelefteq s$.

Suppose, for a contradiction, that $t \trianglelefteq s$ but $t \neq s$. Hence $s \restriction \mathrm{dom}(t) = t$. We can assume, without loss of generality, that

$$\exists \alpha < \kappa \text{ such that } s = t^\frown\langle \alpha \rangle$$

We also know that $\mathrm{Suc}_T(t) \in U$ and U is not a principal filter. Thus $\mathrm{Suc}_T(t) \neq \{\alpha\}$. Hence $\exists \beta \neq \alpha$ such that $\beta \in \mathrm{Suc}_T(t)$ and so $t^\frown\langle \beta \rangle \in T$. But $\neg(s \trianglelefteq t^\frown\langle \beta \rangle)$ and $\neg(t^\frown\langle \beta \rangle \trianglelefteq s)$ which is a contradiction to the fact that s is a trunk in S.

□

Proposition 4.24. *Let $\tilde{T}$ be the set of all the finite increasing sequences of ordinals below κ. Then $\langle \mathbb{P}, \leq, \langle 0, \tilde{T} \rangle \rangle$ is a partial order with largest element.*

Proof. $\subseteq$ is a transitive and reflexive relation so $\leq$ is also a transitive and reflexive relation on $\mathbb{P}$.

Now let us show that $\langle 0, \tilde{T} \rangle$ is a largest element of $\mathbb{P}$. Let $\langle t, T \rangle \in \mathbb{P}$, then $T \subseteq \tilde{T}$ and $0 \trianglelefteq t$. Thus $\langle t, T \rangle \leq \langle 0, \tilde{T} \rangle$.

□

Let us, now, construct the partial order.

Definition 4.25. Let $\langle t, T\rangle, \langle s, S\rangle \in \mathbb{P}$. We say that $\langle t, T\rangle$ is a *direct* (or *Prikry*) *extension of* $\langle s, S\rangle$ and denote this by $\langle t, T\rangle \leq^* \langle s, S\rangle$ if:
(a) $T \subseteq S$
(b) $s = t$

Proposition 4.26. *$\langle \mathbb{P}, \leq^*\rangle$ is a partial order.*

Proof. $\subseteq$ is a transitive and reflexive relation so $\leq^*$ is also a transitive and reflexive relation on $\mathbb{P}$.

□

The two partial orders that we built are related with each other in the same way as the partial orders of the previous section are related to each other. It is shown by this lemma.

Lemma 4.27. $\leq^* \subseteq \leq$.

Proof. Let $\langle\langle t, T\rangle, \langle t, S\rangle\rangle \in \leq^*$, so $T \subseteq S$.
Thus $\langle t, T\rangle \leq \langle t, S\rangle$. Hence $\langle\langle t, T\rangle, \langle t, S\rangle\rangle \in \leq$.

□

The next lemma is a nice observation that will be useful later. Since it is an observation specific to the tree construction that we did, it has no analogous lemma in the previous section.

Lemma 4.28. *Let $\langle T_\alpha \mid \alpha < \lambda\rangle$ be a sequence of U-trees with the same trunk t and $\lambda < \kappa$. Then $T = \bigcap_{\alpha<\lambda} T_\alpha$ is a U-tree with trunk t.*

Proof. (a), (b) and (c) of Definition 4.20 are satisfied by construction of T. Concerning (d), let $\eta \in T$ such that $t \trianglelefteq \eta$. Then

$$\mathrm{Suc}_T(\eta) = \bigcap_{\alpha<\lambda} \mathrm{Suc}_{T_\alpha}(\eta)$$

By κ-completeness of U, $\mathrm{Suc}_T(\eta) \in U$.

□

The following lemma is the analogous of the Lemma 4.12.

Lemma 4.29. *Let G be a $\langle \mathbb{P}, \leq, \langle 0, \tilde{T}\rangle\rangle$-generic. Then $\bigcup\{t \mid \exists T(\langle t, T\rangle \in G)\}$ is an ω-sequence cofinal in κ.*

Proof. We need to show that for every $\alpha < \kappa$ and $\langle s, S\rangle \in \mathbb{P}$, the set

$$D_\alpha = \{\langle t, T\rangle \in \mathbb{P} \mid \langle t, T\rangle \leq \langle s, S\rangle \text{ and } \max(t) > \alpha\}$$

is dense in $\langle \mathbb{P}, \leq, \langle 0, \tilde{T}\rangle\rangle$ below $\langle s, S\rangle$. Let $\langle r, R\rangle \leq \langle s, S\rangle$.

- If $\max(r) > \alpha$:
$$\langle r, R\rangle \in D_\alpha$$
- If $\max(r) \leq \alpha$:
Since U is a non-principal, κ-complete ultrafilter, $\mathrm{Suc}_R(r) \in U$ is cofinal in κ. Hence
$$\exists \beta > \alpha \text{ such that } \beta \in \mathrm{Suc}_R(r) \text{ so } r^\frown\langle\beta\rangle \in R$$
Since $r \trianglelefteq r^\frown\langle\beta\rangle$, $\langle r^\frown\langle\beta\rangle, R\rangle \leq \langle r, R\rangle$. Moreover $\langle r^\frown\langle\beta\rangle, R\rangle \in D_\alpha$.

Thus D_α is dense in $\langle \mathbb{P}, \leq, \langle 0, \tilde{T} \rangle \rangle$ below $\langle s, S \rangle$. Hence $D_\alpha \cap G \neq \emptyset$ $\forall \alpha < \kappa$. Finally, $\bigcup \{t \mid \exists T(\langle t, T \rangle \in G)\}$ is an ω-sequence cofinal in κ.

□

This lemma is the analogous of the Lemma 4.13

Lemma 4.30. *$\langle \mathbb{P}, \leq, \langle 0, \tilde{T} \rangle \rangle$ satisfies the κ^+-c.c.*

Proof. Let $\langle t, T \rangle, \langle t, S \rangle \in \mathbb{P}$.

By Lemma 4.28, $T \cap S$ is a U-tree with trunk t so $\langle t, T \cap S \rangle \leq \langle t, T \rangle, \langle t, S \rangle$. Thus two elements of an antichain in $\langle \mathbb{P}, \leq, \langle 0, \tilde{T} \rangle \rangle$ must have different trunk. But the number of trunk has cardinality $\leq \kappa$. Hence the cardinality of an antichain in $\langle \mathbb{P}, \leq, \langle 0, \tilde{T} \rangle \rangle$ is $\leq \kappa$. Thus $\langle \mathbb{P}, \leq, \langle 0, \tilde{T} \rangle \rangle$ satisfies the κ^+-c.c.

□

The next lemma is the analogous of the Lemma 4.14

Lemma 4.31. *$\langle \mathbb{P}, \leq^* \rangle$ is κ-closed.*

Proof. Let $\lambda < \kappa$ and $\langle \langle t_\alpha, T_\alpha \rangle \mid \alpha < \lambda \rangle$ be a $\leq^*$-decreasing sequence of length λ. By definition of $\leq^*$, we know that all the t_α's are the same so we recall them t. Then let $T = \bigcap_{\alpha < \lambda} T_\alpha$ which is a U-tree with trunk t by Lemma 4.28. Thus $\langle t, T \rangle \leq^* \langle t_\alpha, T_\alpha \rangle$ $\forall \alpha < \lambda$.

□

As we did in the previous section, we will show that the Prikry condition is satisfied for these partial orders. It also the analogous lemma of the Lemma 4.16.

Lemma 4.32. (The Prikry Condition). *Let $\langle t, T \rangle \in \mathbb{P}$ and σ be a statement of the forcing language of $\langle \mathbb{P}, \leq, \langle 0, \tilde{T} \rangle \rangle$. Then there is a $\langle s, S \rangle \leq^* \langle t, T \rangle$ such that $\langle s, S \rangle \parallel \sigma$.*

Proof. Suppose, for a contradiction, that $\neg(\langle s, S \rangle \parallel \sigma)$ for every $\langle s, S \rangle \leq^* \langle t, T \rangle$. Then let us define three sets as follows:

$$X_0 = \{\alpha \in \mathrm{Suc}_T(t) \mid \exists S_\alpha \subseteq T \text{ a } U\text{-tree with trunk } t^\frown \langle \alpha \rangle \text{ such that } \langle t^\frown \langle \alpha \rangle, S_\alpha \rangle \Vdash \sigma\}$$

$$X_1 = \{\alpha \in \mathrm{Suc}_T(t) \mid \exists S_\alpha \subseteq T \text{ a } U\text{-tree with trunk } t^\frown \langle \alpha \rangle \text{ such that } \langle t^\frown \langle \alpha \rangle, S_\alpha \rangle \Vdash \neg\sigma\}$$

$$X_2 = \mathrm{Suc}_T(t) \setminus (X_0 \cup X_1)$$

By the same reasoning given in the proof of Lemma 4.30, conditions with same trunk are compatible. Thus $X_0 \cap X_1 = \emptyset$. Since $X_0 \cup X_1 \cup X_2 = \mathrm{Suc}_T(t) \in U$, we can apply the Proposition 4.2 so $X_0 \in U$, $X_1 \in U$ or $X_2 \in U$.

Let us now shrink T to a tree $T^\#$ with the same trunk t such that $\mathrm{Suc}_{T^\#}(t) = X_i$.

- If $i \in \{0, 1\}$:

$$\text{let } T^\# \text{ be } S_\alpha \text{ above } t^\frown \langle \alpha \rangle \forall \alpha \in X_i$$

- If $i = 2$:

$$\text{let } T^\# \text{ be the same as } T \text{ above } t^\frown \langle \alpha \rangle \forall \alpha \in X_2$$

We continue the shrinking of the initial tree T by recusion level by level. To do that, we define a decreasing sequence $\langle T_\alpha \mid n < \omega\rangle$ of U-trees with trunk t so that:

(1) $T_0 = T^{\#}$

(2) For every $n > 0$ and $m > n$:

$$T_m \upharpoonright (n + |t|) = T_n \upharpoonright (n + |t|)$$

i.e. after stage n, the n-th level above the trunk remains unchanged in all T_m's for $m > n$

(3) For every $n > 0$, if $i \in \{0, 1\}$:

$$\eta \in \mathrm{Lev}_{n+|t|}(T_n) \text{ and for some } U\text{-tree } S \text{ with trunk } \eta \text{ we have } \langle \eta, S\rangle \Vdash \sigma^i$$

where σ^0 denotes σ and σ^1 denotes $\neg\sigma$. Then

- $\langle \eta, (T_n)_\eta\rangle \Vdash \sigma^i$ where $(T_n)_\eta = \{\nu \in T_n \mid \eta \trianglelefteq \nu\}$
- For every $\nu \in \mathrm{Lev}_{n+|t|}(T_n)$ having the same immediate predecessor as η:

$$\langle \nu, (T_n)_\nu\rangle \Vdash \sigma^i$$

Let us define $T^* = \bigcap_{n<\omega} T_n$. By Lemma 4.28, T^* is a U-tree with trunk t so $\langle t, T^*\rangle \in \mathbb{P}$. Moreover $T^* \subseteq T$ so $\langle t, T^*\rangle \leq^* \langle t, T\rangle$. Thus, by assumption, $\neg(\langle t, T^*\rangle \Vdash \sigma)$. By the Fact 2 of the forcing language, we can find $\langle s, S\rangle \leq \langle t, T^*\rangle$ such that $\langle s, S\rangle \Vdash \sigma$. We choose such a $\langle s, S\rangle$ to have $n = |s - t|$ as small as possible. Thus

$$s \in \mathrm{Lev}_{n+|t|}(T^*) = \mathrm{Lev}_{n+|t|}(T_n) \text{ by (2)}$$

Then, by (3), we have

$$\langle s, (T_n)_s\rangle \Vdash \sigma$$

and for every $s' \in \mathrm{Lev}_{n+|t|}(T_n)$ with the same predecessor as s we have

$$\langle s', (T_n)_{s'}\rangle \Vdash \sigma$$

But $T^* \subseteq T$ so, by (2) again

$$\langle s, (T^*)_s\rangle \Vdash \sigma \text{ and } \langle s', (T^*)_{s'}\rangle \Vdash \sigma$$

for every $s' \in \mathrm{Lev}_{n+|t|}(T_n)$ with the same predecessor as s.

Let s^* be the immediate predecessor of s and consider $\langle s^*, (T^*)_{s^*}\rangle$. Since for every $\langle r, R\rangle \leq \langle s^*, (T^*)_{s^*}\rangle$, $r = s'^\frown r'$ for some $s' \in \mathrm{Lev}_{n+|t|}(T^*)$ and $s^* \trianglelefteq s'$, we have that $\langle s^*, (T^*)_{s^*}\rangle \Vdash \sigma$. But $|s^*| = |s| - 1$ so $|s^* - t| = |s - 1 - t| < n$ which is in contradiction with the minimality of n.

□

The next lemma is the analogous lemma of the Lemma 4.17.

Lemma 4.33. *$\langle \mathbb{P}, \leq, \langle 0, \tilde{T}\rangle\rangle$ does not add new bounded subsets of κ.*

Proof. The proof of Lemma 4.17 does not use intrinsic properties of the forcing it is talking about. It only uses Lemma 4.11, Lemma 4.14 and Lemma 4.16 but $\langle \mathbb{P}, \leq, \langle 0, \tilde{T}\rangle\rangle$ satisfies the same properties (respectively Lemma 4.27, Lemma 4.31 and Lemma 4.32). Thus the proof of Lemma 4.17 is also a proof for this Lemma.

□

The last theorem, like the Theorem 4.18, will recap and assemble all the results that we showed in this section. It shows the same properties as the Theorem 4.18.

Theorem 4.34. *The following holds in $M[G]$ for any $\langle \mathbb{P}, \leq, \langle 0, \tilde{T} \rangle \rangle$-generic G:*
(a) *κ has cofinality $\aleph_0$*
(b) *All the cardinals are preserved*
(c) *No new bounded subsets are added to κ*

Proof. (a) κ is an infinite cardinal so $cf(\kappa) \geq \aleph_0$. But, by Lemma 4.29, we know that there is an ω-sequence cofinal in κ so $cf(\kappa) \leq \aleph_0$. Hence $cf(\kappa) = \aleph_0$.

(c) Lemma 4.33.

(b) By (c), all cardinals $\leq \kappa$ are preserved and, by Lemma 4.30, all the cardinals $\geq \kappa^+$ are preserved. Thus all the cardinals are preserved.

□

4.4 One-Element Prikry Forcing and Adding a Prikry sequence to a Singular Cardinal

The goal of this section will be to add a dominating sequence to a singular cardinal, it will be shown that the construction that will be done satisfies it by the last theorem (Theorem 4.52). To get this result, we will construct two pairs of partial orders where the second one will be built on the top of the first one. They will be constructed in the same way as we did in the two first sections of this chapter. For both, we will construct a partial order with largest element which satisfies a chain-condition (we will not need to show it for the first one but it could be shown), a partial order which satifies a closeness property and show that these two satisfy the Prikry condition.

To build and show that, we must fix an increasing sequence of measurable cardinals $\langle \kappa_n \mid n < \omega \rangle$ with limit κ and a κ_n-complete ultrafilter U_n on κ_n for every $n < \omega$. Let us begin by defining the partial order with largest element of the first pair.

Definition 4.35. For every $n < \omega$, let $\mathbb{Q}_n = U_n \cup \kappa_n$. Then $\forall p, q \in \mathbb{Q}_n$, $p \leq_n q$ if one of these conditions is satisfied:
(a) $p, q \in U_n$ and $p \subseteq q$
(b) $p \in \kappa_n$, $q \in U_n$ and $p \in q$
(c) $p, q \in \kappa_n$ and $p = q$

Proposition 4.36. *$\langle \mathbb{Q}_n, \leq_n, \kappa_n \rangle$ is a partial order with largest element for every $n < \omega$.*

Proof. Let $n < \omega$.

We will first show that $\leq_n$ is a transitive relation on $\mathbb{Q}_n$. Let $p, q, r \in \mathbb{Q}_n$ such that $p \leq_n q$ and $q \leq_n r$, then we must show that $p \leq_n r$. There will be 8 cases depending on the reason why $p \leq_n q$ and $q \leq_n r$. But $q \leq_n r$, $p \leq_n q$ and $q \leq_n r$ are impossible cases so there remain 4 cases:

- If $p, q, r \in U_n$:

$$\subseteq \text{ is transitive so } p \subseteq r \text{ and } p \leq_n r$$

- If $q, r \in U_n$ and $p \in \kappa_n$:

$$p \in q \subseteq r \text{ so } p \in r \text{ and } p \leq_n r$$

- If $r \in U_n$ and $p, q \in \kappa_n$:

$$p = q \in r \text{ so } p \in r \text{ and } p \leq_n r$$

- If $p, q, r \in \kappa_n$:

$$= \text{ is transitive so } p = r \text{ and } p \leq_n r$$

Thus $\leq_n$ is a transitive relation on $\mathbb{Q}_n$.

Since $\subseteq$ and $=$ are reflexive relations, $\leq_n$ is a also a reflexive relation. It only remains to show that κ_n is a largest element of $\mathbb{Q}_n$. Let $p \in \mathbb{Q}_n$, then:

- If $p \in U_n$:

$$p \subseteq \kappa_n \text{ so } p \leq_n \kappa_n$$

- If $p \in \kappa_n$:

$$p \leq_n \kappa_n$$

□

We can now define the partial order.

Definition 4.37. For every $n < \omega$, and $p, q \in \mathbb{Q}_n$, we write $p \leq_n^* q$ if one of these conditions is satisfied:

(a) $p, q \in U_n$ and $p \subseteq q$
(b) $p, q \in \kappa_n$ and $p = q$

Proposition 4.38. *$\langle \mathbb{Q}_n, \leq_n^* \rangle$ is a partial order for every $n < \omega$.*

Proof. Let $n < \omega$.

$\subseteq$ and $=$ are transitive and reflexive relations so $\leq_n^*$ is also a transitive and reflexive relation on $\mathbb{Q}_n$.

□

The next property was also satisfied by the previous partial orders we built in this chapter, it will be the case for the last ones too.

Proposition 4.39. *$\leq_n^* \subseteq \leq_n$ for every $n < \omega$.*

Proof. (a) and (b) of the Definition 4.37 are the same as, respectively, (a) and (c) of the Definition 4.35. Thus every element of $\leq_n^*$ is also an element of $\leq_n$.

□

The following lemmas, which will be the last concerning the first forcing of this section, are the ones showing, respectively, the closeness of the partial orders and the Prikry condition. They are analogous to the Lemma 4.14 and the Lemma 4.16.

Lemma 4.40. *$\langle \mathbb{Q}_n, \leq_n^* \rangle$ is κ_n-closed for every $n < \omega$.*

Proof. Let $n < \omega$, $\lambda < \kappa_n$ and $\langle p_\alpha \mid \alpha < \lambda \rangle$ be a $\leq_n^*$-decreasing sequence of length λ. By definition of $\leq_n^*$, all the p_α's are in U_n or κ_n.

- If $p_\alpha \in U_n\ \forall \alpha < \lambda$:
  Consider $\bigcap\limits_{\alpha<\lambda} p_\alpha$. It is in U_n by κ_n-completeness of U_n and $\bigcap\limits_{\alpha<\lambda} p_\alpha \subseteq p_\alpha$ for every $\alpha < \lambda$.
- If $p_\alpha \in \kappa_n\ \forall \alpha < \lambda$:
  All the p_α's are the same so we can take p_α itself.

Thus, in any case, we can find an element of $\mathbb{Q}_n$ which is smaller or equal with respect to $\leq_n^*$ than all the p_α's.

□

Lemma 4.41. (The Prikry Condition). *For every $n < \omega$, let $q_n \in \mathbb{Q}_n$ and σ be a statement of the forcing language of $\langle \mathbb{Q}_n, \leq_n, \kappa_n \rangle$. Then there is a $p_n \leq_n^* q_n$ such that $p_n \parallel \sigma$.*

We can now introduce the last pair of partial orders of this report that will be constructed on the top of the one that we just defined. It will bring us to the last result of this report, the Theorem 4.52. We start, as always, to define the partial order with largest element and the partial order.

Definition 4.42. Let $\mathbb{P}$ be the set of all sequences $p = \langle p_n \mid n < \omega \rangle$ so that:

(a) For every $n < \omega$, $p_n \in \mathbb{Q}_n$

(b) There is an $l(p) < \omega$ so that for every $n < l(p)$, $p_n \in \kappa_n$ and for every $n \geq l(p)$, $p_n \in U_n$.

Definition 4.43. Let $p = \langle p_n \mid n < \omega \rangle$ and $q = \langle q_n \mid n < \omega \rangle$ be two elements of $\mathbb{P}$. Then $p \leq q$ (respectively $p \leq^* q$) if $p_n \leq_n q_n$ (respectively $p_n \leq_n^* q_n$) for every $n < \omega$.

We denote $\langle p_m \mid m < n \rangle$ by $p \restriction n$ and $\langle p_m \mid m \geq n \rangle$ by $p \setminus n$. Then let $\mathbb{P} \restriction n = \{p \restriction n \mid p \in \mathbb{P}\}$ and $\mathbb{P} \setminus n = \{p \setminus n \mid p \in \mathbb{P}\}$.

Proposition 4.44. *$\langle \mathbb{P}, \leq, \prod_{n<\omega} \kappa_n \rangle$ is a partial order with largest element and $\langle \mathbb{P}, \leq^* \rangle$ is a partial order.*

Proof. $\leq_n$ and $\leq_n^*$ are reflexive and transitive relations on $\mathbb{Q}_n$ for every $n < \omega$ so $\leq$ and $\leq^*$ are reflexive and transitive relations on $\mathbb{P}$. Moreover, κ_n are largest elements of $\mathbb{Q}_n$ with respect to $\leq_n$ for every $n < \omega$ so $\prod_{n<\omega} \kappa_n$ is a largest element of $\mathbb{P}$ with respect to $\leq$.

□

Like all the Prikry-type forcing we studied, the partial orders satisfy the next relation between each other.

Proposition 4.45. *$\leq^* \subseteq \leq$.*

Proof. $\leq_n^* \subseteq \leq_n$ for every $n < \omega$ by the Proposition 4.39 so $\leq^* \subseteq \leq$.

□

We will need, for this forcing, to split our partial order but this is more a detail for the proofs later than an important observation.

Lemma 4.46. *$\mathbb{P} \simeq \mathbb{P} \restriction n \times \mathbb{P} \setminus n$ for every $n < \omega$.*

Proof. Let $n < \omega$ and consider the natural map:

$$\begin{aligned} f : \ & \mathbb{P} \to \mathbb{P} \restriction n \times \mathbb{P} \setminus n \\ & p \mapsto \langle p \restriction n, p \setminus n \rangle \end{aligned}$$

□

We are now able to show the closeness of the partial order (Lemma 4.47) and the chain-condition of the partial order with largest element (Lemma 4.48).

Lemma 4.47. *$\langle \mathbb{P} \setminus n, \leq^* \rangle$ is κ_n-closed for every $n < \omega$.*

Proof. Let $n < \omega$, $\lambda < \kappa_n$ and $\langle p_\alpha \setminus n \mid \alpha < \lambda \rangle$ be a $\leq^*$-decreasing sequence of length λ. By the Lemma 4.40, we know that $\langle \mathbb{Q}_m, \leq^*_m \rangle$ is κ_m-closed for every $n \leq m < \omega$. Since $n \leq m$, $\langle \mathbb{Q}_m, \leq^*_m \rangle$ is κ_n-closed. Thus, for every $n \leq m < \omega$, we can find an element $\tilde{p}_m$ of $\mathbb{Q}_m$ such that $\tilde{p}_m \leq^*_m (p_\alpha \setminus n)_m$. Then $\prod_{n=m}^{\omega} \tilde{p}_m \leq^* p_\alpha \setminus n$ for every $\alpha < \lambda$.

□

Lemma 4.48. *$\langle \mathbb{P}, \leq, \prod_{n<\omega} \kappa_n \rangle$ satisfies the κ^+-c.c.*

Proof. Let $p = \langle p_n \mid n < \omega \rangle$ and $q = \langle q_n \mid n < \omega \rangle$ be elements of $\mathbb{P}$ such that $l(p) = l(q)$ and $\langle p_n \mid n < l(p) \rangle = \langle q_n \mid n < l(q) \rangle$. Then, for every $n \geq l(p)$, by the ultrafilter property of U_n, $p_n \cap q_n \in U_n$ and $p_n \cap q_n \subseteq p_n, q_n$. Thus $\langle p_n \mid n < l(p) \rangle^\frown \langle p_n \cap q_n \mid n \geq l(p) \rangle \leq p, q$ so p and q are compatible. Hence every two elements p and q of an antichain must be such that $l(p) \neq l(q)$ or if $l(p) = l(q)$, p and q must have different initial sequences of length $l(p) - 1$. But then the number of possible such elements is smaller or equal than κ.

□

The two following lemmas are again analogous to some lemmas that we showed during the chapter. The first lemma shows that the Prikry condition is satisfied in the present case and the second one is the analogous of the Lemma 4.17. After that, it will remain show that the model constructed adds a dominating sequence to a singular cardinal and the last theorem will recap everything.

Lemma 4.49. (The Prikry Condition). *Let $p = \langle p_n \mid n < \omega \rangle \in \mathbb{P}$ and σ be a statement of the forcing language of $\langle \mathbb{P}, \leq, \prod_{n<\omega} \kappa_n \rangle$. Then there is a $q \leq^* p$ such that $q \parallel \sigma$.*

Proof. We suppose, for contradiction, that $\neg(q \parallel \sigma)$ for every $q \leq^* p$. Without loss of generality, we can assume that $l(p) = 0$. Then let $p_n = A_n \in U_n$ for every $n < \omega$ and define, by recursion on $n < \omega$, a $\leq^*$-decreasing sequence $\langle q(n) \mid n < \omega \rangle$ of $\leq^*$-extensions of p satisfying:

(1) If $n \leq m$:
$$q(m) \restriction n = q(n) \restriction n$$

(2) If $q = \langle q_n \mid n < \omega \rangle \leq q(n)$, $q \parallel \sigma$ and $l(q) = n + 1$:
$$\langle q_m \mid m \leq n \rangle^\frown \langle q(n)_m \mid m > n \rangle \parallel \sigma \text{ in the same way as } q.$$

Moreover, for every $\tau_n \in q(n)_n$:
$$\langle q_m \mid m < n \rangle^\frown \langle \tau_n \rangle^\frown \langle q(n)_m \mid m > n \rangle \parallel \sigma \text{ also in the same way.}$$

Let us show that this construction is possible. Let $n < \omega$, the number of possibilities for initial sequences of length $n - 1$ below κ_n is $|\prod_{i \leq n-1} \kappa_i| = \kappa_{n-1} < \kappa_n$. But U_m is κ_n-complete for every $m \geq n$, so we can define $s = \langle s_n \mid n < \omega \rangle$ to be $\langle q(n)_n \mid n < \omega \rangle$ by taking the intersection of the possible initial sequences of length $n - 1$ below κ_n. Then, by construction, $s \in \mathbb{P}$ and $s \leq^* p$.

Let $q = \langle q_n \mid n < \omega \rangle$ be an extension of s such that $q \parallel \sigma$ with $l(q)$ as small as possible. We can find such an element by the Fact 2 of the forcing language. By assumption, $l(q) > 0$ so let $n = l(q) - 1$. Then, by (2) of the construction, for every $\tau_n \in q(n)_n = s_n$:
$$\langle q_m \mid m < n \rangle^\frown \langle \tau_n \rangle^\frown \langle s_m \mid m > n \rangle \Vdash \sigma$$
But then $\langle q_m \mid m < n \rangle^\frown \langle s_m \mid m > n \rangle \Vdash \sigma$ too, contradicting the minimality of $l(q)$.

□

Lemma 4.50. $\langle \mathbb{P}, \leq, \prod_{n<\omega} \kappa_n \rangle$ *does not add new bounded subsets to* κ.

Proof. The proof of the Lemma 4.17 does not use intrinsic properties of the forcing it is talking about. It uses the Lemma 4.11 and the Lemma 4.16 but $\langle \mathbb{P}, \leq, \prod_{n<\omega} \kappa_n \rangle$ satisfies the same properties (respectively the Lemma 4.45 and the Lemma 4.49). Concerning the Lemma 4.14, the equivalent for $\langle \mathbb{P}, \leq, \prod_{n<\omega} \kappa_n \rangle$ is given by combining the Lemma 4.46 and the Lemma 4.47. Thus the proof of Lemma 4.17 is also a proof for this Lemma.

□

More than the last theorem, this is this lemma that will show that the model extension $M[G]$ adds a dominating sequence to a singular cardinal, which was the goal of this forcing.

Lemma 4.51. *Let* G *be a* $\langle \mathbb{P}, \leq, \prod_{n<\omega} \kappa_n \rangle$*-generic. Define an* ω*-sequence* $\langle t_n \mid n < \omega \rangle \in \prod_{n<\omega} \kappa_n$ *such that:*

$$t_n = \tau \text{ if } p_n = \tau \text{ for some } p = \langle p_m \mid m < \omega \rangle \in G \text{ with } l(p) > n$$

Then for every $\langle s_n \mid n < \omega \rangle \in (\prod_{n<\omega} \kappa_n) \cap M$*, there is an* $n_0 < \omega$ *such that* $t_n > s_n$ *for every* $n \geq n_0$*.*

Proof. Let $n_0 < \omega$ and suppose, for contradiction, that there is $n \geq n_0$ such that $p_n \leq s_n$ $\forall p = \langle p_m \mid m < \omega \rangle \in G$ with $l(p) > n$. Then consider the set:

$$D = \{q = \langle q_m \mid m < \omega \rangle \in \mathbb{P} \mid l(q) > n \text{ and } q_n > s_n\}$$

This set is dense in $\mathbb{P}$ so, as G is a $\langle \mathbb{P}, \leq, \prod_{n<\omega} \kappa_n \rangle$-generic, $G \cap D \neq \emptyset$. Thus there is $q = \langle q_m \mid m < \omega \rangle \in G$ with $l(q) > n$ such that $q_n > s_n$, which is a contradiction to the assumption.

□

Finally and again, the last theorem will recap and assemble all the results that we showed in this section.

Theorem 4.52. *The following holds in* $M[G]$ *for any* $\langle \mathbb{P}, \leq, \prod_{n<\omega} \kappa_n \rangle$*-generic* G*:*

(a) *All cardinals and cofinalities are preserved*
(b) *No new bounded subsets are added to* κ
(c) *There is a sequence in* $\prod_{n<\omega} \kappa_n$ *dominating every sequence in* $(\prod_{n<\omega} \kappa_n) \cap M$

Proof. (a) and (b) follow from the Lemma 4.50 and (c) follows from the Lemma 4.51.

□

5 Further Topics

In this conclusion chapter, we will have a look, in an informal way, to some forcings that we have not studied in the report. Some of them are similar to the forcings we have seen and some others are going in a very different direction.

To continue in the same philosophy of our chapter about forcings with partial functions, we could have studied a forcing called the *Lévy collapsing order* with which it is possible to build a model extension in which every regular and uncountable cardinal collapses to ω_1. It is also an interesting forcing to study an important statement of set theory, the *Kurepa's Hypothesis* stating that:

There is an ω_1-Kurepa tree, where an ω_1-Kurepa tree is an ω_1-tree with at least ω_2 paths.

Indeed, thanks to the Lévy collapsing order, we can construct models of ZFC in which the Kurepa's Hypothesis is satisfied or not. Another interesting question we can ask ourself is how to iterate forcings. We used, to show the consistency of $2^{\aleph_n} = \aleph_{n+2}$ in the third chapter, countably many forcings but how to iterate forcings beyond limit ordinals? That can also be done with the help of supports in ideals. If we work with iterated forcings with finite supports, it possible to construct a model of ZFC satisfying another important statement of set theory, the *Martin's Axiom* which states that:

$\forall\kappa < 2^\omega$, for any partial order $\langle\mathbb{P}, \leq\rangle$ satisfying the ω-c.c. and any family $\mathcal{D}$ of $\leq_\kappa$ dense subsets of $\mathbb{P}$, then there is a filter G in $\mathbb{P}$ such that $G \cap D \neq \emptyset$ $\forall D \in \mathcal{D}$.

To continue the study of Prikry-type forcings that has been started in the fourth chapter, it may be interesting to read [1]. To go further than we did in the last chapter with the last forcing of the report, we can add multiple Prikry sequences to a singular cardinal. Another possible notion to work on is super compact and strongly compact cardinals. Indeed, since we have an ultrafilter, we can define these notions and build a Prikry-type forcing on it. Then, a bit similarily to the Lévy collapsing order, it is possible to build, by a Prikry-type forcing, a model in which every cardinal below a limit κ of measurables cardinals collapse to $\aleph_\omega$ and in which there are finitely many cardinals between $\aleph_\omega$ and 2^κ. We can also think of iterations of Prikry-type forcings. If we work with full support iteration, we construct iterations called Magidor iterations after the Israeli mathematician Menachem Magidor. We can then combine this idea of iteration with the Easton forcing theory we studied in the thrid chapter to make iterated Prikry-type forcings with Easton supports. Also considered as Prikry-type forcings, we can study Magidor forcings and Radin forcings. We already saw quickly the name of Magidor but not Radin, it comes from the American mathematician Lon Berk Radin.

As we can see, there are many different forcings. The strongeness of this tool make them very useful in set theory and, more especially, to build specific models of ZFC. But, as always in Mathematics, many results remain to be shown with the help or not of forcings. Let us look at some unknown results that could maybe be solved by forcing.

We said previously that it was possible to build a model in which every cardinal below a limit κ of measurables cardinals collapse to $\aleph_\omega$ and in which there are finitely many cardinals between $\aleph_\omega$ and 2^κ but we do not know if the same construction is possible with uncountably many cardinals between $\aleph_\omega$ and 2^κ.

We do not know neither if it is possible to have a model of ZFC in which $\aleph_{\omega_1} < 2^{\aleph_\omega}$ if $2^{\aleph_n} = \aleph_{n+1}$ for every $n < \omega$.

Finally, using stationary sets, the question of finding a model in which $2^{\aleph_{\omega_1}} = \aleph_{\omega_1+2}$ and in which we can find two stationary sets S_1 and S_2 contained in ω_1 such that $S_1 \cup S_2 = \omega_1$ and

$$(\alpha \in S_1 \Rightarrow 2^{\aleph_\alpha} = \aleph_{\alpha+2}) \text{ and } (\alpha \in S_2 \Rightarrow 2^{\aleph_\alpha} = \aleph_{\alpha+3})$$

is still unsolved.

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