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# Machine scheduling to minimize weighted completion time

MSc Thesis

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*...to those who ever*

*believed in me...*

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# 1 Introduction

Scheduling Theory is a branch of Mathematics, more precisely of Operations Research, which deals with finding the best allocation of scarce resources to activities over time. More generally we say that scheduling problems involve jobs and machines, and we want to find the optimal ordering of the jobs over the machines in such a way to minimize (or maximize) an objective function. Obviously there can be some constraints on the possible orderings, so we want to find a schedule specifying when and on which machine every job must be processed. For a precise definition see the Enciclopedya of Mathematics, [20].

In all our problems we have a set of jobs $N = \{j_1, j_2, \ldots, j_n\}$, and each of them has associated a processing time $p_j$. Then, following the notation in [17], we can define a scheduling problem by the three field notation: $\alpha|\beta|\gamma$, where $\alpha$ is the machine eviroinment, $\beta$ represents the set of constraints and characteristics and $\gamma$ denotes the optimization criterion. We also assume that any machine can process at most one job at any time and any job can be processed simoultaneously on at most one machine.

The machine enviroinment specifies the number of machines and, in case of more than one machines, the possible difference of speed between them. The easiest case is with a single machine, and it will be our basis case. In addition, in section 7, we will introduce the concept of machines with different speeds, so if $m_1$ has normal speed and $m_2$ has double speed we have that job $j$ takes $p_j$ times to be processed on machine $m_1$ and $\frac{p_j}{2}$ if it is processed on machine $m_2$. Then, in section 8, we will use several machines that are identical from the view of point of the jobs, so we say that these are $m$ parallel machines.

The set of constraints and characteristics can be various, so we will introduce just the ones we will use:

- *pmtn*: this means that the job can be preempted, so it can be started on one machine, stopped, and resumed in a later time, also on a different machine;

- $r_j$: this indicates the presence of some release dates, so if for job $j$ we have $r_j = x$ it means that job $j$ can be started only after $x$ units of time;

- *prec*: this means that there is a partial order between the jobs, so if, for jobs $j$ and $k$, $j \prec k$, then job $j$ must be completed before starting job $k$;

- $d_{ij}$: this is a more generic parameter and can generalize both $r_j$ and *prec*. The set of $d_{ij}$ defines a partial order on the jobs and we have that, for two jobs $j$ and $k$, $d_{jk}$ means that job $k$ can start only when $d_{jk}$ units of time passed after the completion of job $j$.

However, thruoghout the thesis these constraints will be better explained.

The optimality criterion depends on the problem. Let us define $C_j^S$ as the completion time of job $j$ in schedule $S$. Then the usually used criteria are the minimization of $\sum C_j$, so the sum of completion times, and, if we associate to each job $j$ a weight $w_j$, the minimization of the weighted sum of completion time, so $\min \sum w_j C_j$ (we can consider the former one as the latter with $w$ identically 1, $w \equiv 1$). These two criteria could be seen also as the average completion and the average weighted completion, respectively, since we have just to divide by the number of jobs.

Some scheduling problems can be solved optimally, like $1||\sum C_j$, by scheduling jobs by a shortest processing time first rule, $1||\sum w_j C_j$ and the unit processing time problem $1|r_j, p_j = 1|\sum w_j C_j$, by using the Smith's rule that is scheduling jobs by nonincreasing $\frac{w_j}{p_j}$ (see [34] and [42]), and $1|r_j, pmtn|\sum C_j$, scheduling the jobs with a shortest remaining processing time first rule (see [3] and [22]).

Unfortunately, with precedence constraints, release dates or parallel machines, scheduling problems turn out to be (strongly) NP-hard, by example $1|r_j|\sum w_j C_j$ is NP-hard, also for $w \equiv 1$, see [27]. In such cases we can just search for approximate solutions: a $\rho$-approximation algorithm is a polynomial time algorithm guaranteed to deliver a solution of cost at most $\rho$ times the optimum value, we refere to [18] for a survey.

One of the key ingredients in the design of approximation algorithms is the choice of a bound on the optimum value.

In Chapter 3 we will see some linear programming based lower bounds. Dyer and Wolsey, in [13], formulated two time-indexed relaxations, one preemptive and one nonpreemptive. They showed that the nonpreemptive one is as least as good as the preemptive, but they couldn't say that it is always better. We will see the preemptive time-indexed relaxation and we will compare it with a completion time relaxation, proposed first in [37]. By the use of the shifted parallel inequalities, firstly presented in [32], we will prove the equality between them, see [14]. In [13] the authors proved the polynomial time solvability of the preemptive time-indexed relaxation since it turn out to be a transportation problem, hence it can be solved

in polynomial time, despite its exponential size. In [45] these formulations were proved to be equal also to the combinatiorial lower bounds given in [6], that is based on job splitting. We will also prove the supermodularity of these functions, in order to see, as pointed in [34], that we can find the LP schedule from a greedy algorithm. Other important consequences following the supermodularity can be seen in [33] and [34].

In Chapter 4 we introduce the concept of $\alpha$-point of a job, defined the first time in [30]. In the same paper were described an algorithm that converts a preemptive schedule to a nonpreemptive schedule, this conversion algorithm is called $\alpha$-CONV and it plays a crucial role in the study of the algorithms, since it delays the start of the jobs in such a way to relate them with the LP formulation. In order to have a detailed discussion of the use of the $\alpha$-points it is possible to see [40].

In Chapter 5, we will make use of the $\alpha$-conversion algorithm. We will use also the concept of canonical decomposition, introduced in [14] in order to obtain better bounds. We will present first the bound given by using a fixed $\alpha$ and then an improved result for a uniformly random chosen $\alpha$ for $1|r_j|\sum w_jC_j$, proved in [15]. Then we will show some improvements, presenting the current best bounds given in [16], by using some specific random $\alpha$ distribution and the concept of conditional probabilities, see [29].

In Chapter 6 we study the problem $1|r_j|\sum C_j$, so the particular case of the problem studied in Chapter 5 with the restriction that we have, now, identical weights. Following the ideas in [26] and [30] we use the concept of $\alpha$-point in a different manner in order to present the job-by-job bound proved by Chekuri et al. in [10] that improves the results of the previous sections. We also present a family of instances for this problem for which our LP-based algorithm reaches the approximation guaranteed, so the bound, for this particular LP relaxation is tight . Hence if we want to improve this result we have to study some different relaxation as basis.

In Chapter 7 we study the problem $1|r_j, prec|\sum w_jC_j$. We will show the recent result given by Skutella in [41], that improves the previous best approximation ratios, see [36] and [37]. In order to prove the result the author has little strenghened the 2-approximation of Hall et al., see [19], for $1|r_j, prec, pmtn|\sum w_jC_j$, showing that scheduling in order of increasing LP completion times on a double speed machine yields a schedule whose cost is at most the cost of an optimal schedule on a regular machine. He modified the analysis studied by Chekuri et al. in [10] showing how to turn the preemptive schedule on a double speed machine into a nonpreemptive schedule on a regular machine with an increasing of the objective function by at most a fixed factor.

In Chapter 8 we will start by presenting an approximation algorithm for the problem $P|r_j|\sum C_j$, see [10]. This algorithm improves the previous best bound given in [8] that is not too efficient since it uses a polynomial approximation scheme for makespan due to Hochbaum and Shmoys [21]. The idea of that result is to divide all the processing times by $m$, where $m$ is the number of parallel identical machines, and then to use a general conversion algorithm developed for this case. Then we give the explanation of the DELAY-LIST algorithm, that, given an algorithm to find a $\rho$-approximation for the single machine case, produces a $m$-machine schedule, with a certain approximation, see [9]. Chekuri et al. gave an algorithm, that uses the DELAY-LIST, with a better guarantee. It is important to note that the DELAY-LIST algorithm gives some schedules which are good both for the average completion time and the makespan, the existence of such schedules was shown also in [8] and [43].

Finally in Chapter 9, we present an approximation algorithm for the problem $P|d_{ij}|\sum w_jC_j$, proved in [35]. This problem generalizes problems with precedente constraints and release dates, since they can be represented by $d_{ij}$. Even special cases of this problem, as $P2||\sum w_jC_j$, are NP-hard, see [7], [25], [27] and [28] for other examples. In order to prove this result we pass through an LP relaxation that is a direct extension of those of Hall et al. in [19] and Schulz in [37], by using stronger inequalities given in [37] and [46]. Then a general list scheduling algorithm is used and, listing the jobs by midpoints, it is possible to find an approximation ratio for the general problem with $d_{ij}$'s.

## 2 List of Main Results

In this Chapter we summarize the best known results for the problems we will study throughout this thesis. We just note that in Chapter 8 of the thesis we study the problem $P|r_j|\sum C_j$ and we present an algortihm that is a consequence of the DELAY-LIST algorithm, while the best approximation known, for the more general weighted problem, is given by Skutella in [40].

| Problem | Best known approximation | Chapter | Ref |
|---|---|---|---|
| $1\|r_j\|\sum C_j$ | $\frac{e}{e-1} \approx 1.58$ | 6 | [10] |
| $1\|r_j\|\sum w_jC_j$ | 1.6853 | 5 | [16] |
| $1\|r_j, prec\|\sum w_jC_j$ | $\frac{\sqrt{e}}{\sqrt{e}-1} \approx 2.54$ | 7 | [41] |
| $P\|r_j\|\sum w_jC_j$ | 2 | 8 | [40] |
| $P\|d_{ij}\|\sum w_jC_j$ | 4 | 9 | [35] |

# 3 LP Relaxations for the Release Dates Case

In this section we will present two different relaxed formulations for the easiest problem we will study throughout the thesis, namely $1|r_j|\sum w_jC_j$, and their relations with a preemptive schedule that can be constructed in polynomial time. The first one studied will be a time-indexed relaxation formulated by Dyer and Wolsey in [13], while the second one is a mean busy time relaxation, proposed the first time by Schulz, see [37]. Then we will show their equivalence and we will give a proof of the supermodularity of the solutions, as shown by Goemans in [14].

We first introduce a preemptive schedule that will be used as a basis for many of our next results. It works by scheduling at any point in time the avaliable job with biggest $\frac{w_j}{p_j}$ ratio, where we can assume the jobs to be indexed so that $\frac{w_1}{p_1} \geq \frac{w_2}{p_2} \geq \cdots \geq \frac{w_n}{p_n}$ and ties are broken according to this order. Since when a job $j$ is released and the job currently processed (if any) is $k$ with $j < k$, then we preempt $k$, this rule yields a preemptive schedule. In the next sections we will call this schedule LP schedule or fractional schedule.

In [24], the Labetoulle et al. proved that the problem $1|r_j, pmtn|\sum w_jC_j$ is (strongly) NP-hard, so this LP schedule does not solve the problem in the preemptive case, but we know, see [39], that the solution is always within a factor of 2 of the optimum and this bound is tight.

**Proposition 1.** *The LP schedule can be constructed in $O(n \log n)$ time.*

*Proof.* Let us create a priority queue of the avaliable jobs not yet completed using as key the ratio $\frac{w_j}{p_j}$ and using a field to express the remaining processing time (for some properties of priority queues see [11]). Then, when a job is released we add it to the priority queue, while, when a job is completed, we remove it from the queue. At any moment the element of the queue with highest priority is processed, if the queue is empty we move to the next release date. Since the number of jobs is $n$ we have at most $O(n)$ operations on the queue, each of them can be implemented in $O(\log n)$ time, hence the algorithm has running time $O(n \log n)$. □

## 3.1 Time-Indexed Relaxation

The first relaxation we study is a time-indexed relaxation. In [13] there were defined two possible time-indexed relaxations, for our aims we just need the weaker one. In this time-indexed relaxation we defines two types

of variables: $s_j$, representing the starting time of job $j$, and $y_{j,\tau}$, representing the processing of a job $j$ in the interval defined by $\tau$, so that

$$y_{j,\tau} = \begin{cases} 1 & \text{job } j \text{ is processed in } [\tau, \tau+1) \\ 0 & \text{otherwise} \end{cases}$$

For the sake of simplicity, since we consider nonpreemptive schedule, we replace $s_j$ by $C_j$ just by adding $p_j$, that gives an equivalent relaxation. Let $T$ be an upper bound of the makespan of an optimal schedule, we can assume $T = \max_j r_j + \sum p_j$. Then we can define the linear program

$$\begin{aligned} Z_D \quad = \quad & \min \sum_{j \in N} w_j C_j \\ & \text{subject to} \\ & \sum_{j: r_j \le \tau} y_{j,\tau} \le 1 && \tau = 0, 1, \ldots, T \\ \text{(D)} \qquad & \sum_{\tau = r_j}^{T} y_{j,\tau} = p_j && j \in N \\ & C_j = \frac{p_j}{2} + \frac{1}{p_j} \sum_{\tau = r_j}^{T} \left(\tau + \frac{1}{2}\right) y_{j,\tau} && j \in N \qquad (*) \\ & 0 \le y_{j,\tau} && j \in N, \tau = r_j, r_j+1, \ldots, T. \end{aligned}$$

In the LP we have that the equation for $C_j$ corresponds to the correct completion time value of job $j$ when it is nonpredempted, i.e $y_{j,s_j} = y_{j,s_j+1} = \cdots = y_{j,s_j+p_j-1} = 1$, because

$$\frac{p_j}{2} + \frac{1}{p_j} \sum_{\tau = r_j}^{T} \left(\tau + \frac{1}{2}\right) y_{j,\tau} = \frac{p_j}{2} + \frac{1}{p_j} \left(s_j + \frac{1}{2} + \cdots + \left(s_j + p_{j-1} + \frac{1}{2}\right)\right) =$$
$$= \frac{p_j}{2} + s_j + \frac{1}{p_j}\left(p_j \frac{p_j}{2}\right) = \frac{p_j}{2} + s_j + \frac{p_j}{2} = s_j + p_j = C_j.$$

The number of variables is pseudo-polynomial. Then, substituting $C_j$ by the equality in constraint $(*)$, we get a transportation problem, see [13], for which we know:

**Lemma 1.** *There is an optimal solution for the LP program D that satisfies $y_{j,\tau} \in \{0, 1\}$ for all $j$ and $\tau$.*

By the work of Posner, in [31], the time-indexed LP relaxation can be solved in $O(n \log n)$. One can derive a feasible solution $y^{LP}$ to D from the LP schedule by letting $y^{LP}_{j,\tau}$ be equal 1 if job $j$ is being processed in $[\tau, \tau+1)$, and 0 otherwise.

**Theorem 1.** *The solution $y^{LP}$ derived from the LP schedule is an optimal solution for the linear program D.*

*Proof.* We use an exchange argument to prove the theorem. So consider any optimum 0/1 solution $y^*$ of D such that we have $j < k$ and $\sigma > \tau \geq r_j$ with $y^*_{j,\tau} = y^*_{k,\tau} = 1$. By replacing $y^*_{j,\sigma}$ and $y^*_{k,\tau}$ to 0, and $y^*_{k,\sigma}$ and $y^*_{j,\tau}$ to 1, we obtain another feasible solution in which we increase the objective function by $(\sigma - \tau)\left(\frac{w_k}{p_k} - \frac{w_j}{p_j}\right) \leq 0$. Hence another optimal solution in which we decreases the wrong (for the algorithm) pairs of pieces of jobs by 1. Repeating this interchange argument we have, at the end, that there exists an optimum soution $y'$ with no $j < k$ and $\sigma > \tau \geq r_j$ such that $y'_{j,\sigma} = y'_{k,\tau} = 1$, hence the solution $y'$ corresponds to the LP schedule. □

The linear program D solves neither the problem $1|r_j|\sum w_j C_j$ nor, in general, $1|r_j, pmtn|\sum w_j C_j$, but we can find an optimum solution efficiently despite the big number of variables.

## 3.2 Mean Busy Time Relaxation

In the mean busy time relaxation we define an indicator function $I_j(t)$, that takes value 1 if job $j$ is processed at time $t$ and 0 otherwise. In order to exclude possible degenerate cases we assume that, when the machine starts processing a job, it does so for a positive amount of time. Then we can define the mean busy time

$$M_j = \frac{1}{p_j} \int_{r_j}^{T} I_j(t)\ t\ dt$$

to be the average time at which the machine is processing $j$.

**Lemma 2.** *Let $C_j$ and $M_j$ be, respectively, the completion time and the mean busy time of job $j$ for any nonpreemptive schedule. Then it follows $C_j \geq M_j + \frac{p_j}{2}$. Moreover we have equality if and only if job $j$ is not preempted.*

*Proof.* If job $j$ is processed nonpreemptively if $C_j - p_j \leq t \leq C_j$ we have $I_j(t) = 1$, otherwise 0, hence $C_j = M_j + \frac{p_j}{2}$. Suppose now job $j$ is not

processed for some interval in $[C_j - p_j, C_j]$. We know $\int_{r_j}^{T} I_j(t)\,dt = p_j$, so job $j$ is processed for some interval before $C_j - p_j$. Thus

$$M_j = \frac{1}{p_j}\int_{r_j}^{C_j} I_j(t)\ t\ dt < \frac{1}{p_j}\int_{C_j-p_j}^{C_j} t\ dt = \frac{1}{p_j}\left(\frac{C_j^2}{2} - \frac{C_j^2 + p_j^2 - 2\,C_j\,p_j}{2}\right),$$

hence $M_j < C_j - \frac{p_j}{2}$. □

In order to give another important property of $M_j$ we have to give some definitions: let $S \subseteq N$ be a set of jobs, then we have $p(S) := \sum_{j\in S} p_j$, $r_{min}(S) := \min_{j\in S} r_j$ and $I_S(t) = \sum_{j\in S} I_j(t)$. Since only one job can be processed at any time we have $I_S(t) \in \{0,1\}$, so this parameter can be seen as the indicator function for the job set $S$, hence we can define the mean busy time of the set $S$ as $M_S := \frac{1}{p(S)}\int_0^T I_S(t)\ t\ dt$. From this definition it follows that the mean busy time of $S$ is just a nonnegative combination of the mean busy times of its elements:

$$p(S)M_S = \int_0^T \left(\sum_{j\in S} I_j(t)\right) t\ dt = \sum_{j\in S}\int_0^T I_j(t)\ t\ dt = \sum_{j\in S} p_j M_j.$$

So we can, now, prove the validity of the shifted parallel inequalities:

**Lemma 3.** *Let $S$ be a set of jobs and consider any nonpreemptive schedule, then*

$$\sum_{j\in S} p_j M_j \geq p(S)\left(r_{min}(S) + \frac{p(S)}{2}\right) \tag{1}$$

*holds. Moreover we have equality if the jobs in $S$ are scheduled continuously from $r_{min}(S)$ to $r_{min}(S) + p(S)$.*

*Proof.* We know that $I_S(t) = 0$ for $t < r_{min}(S)$ and $I_S(t) \leq 1$ for each $t$, that $\int_{r_{min}(S)}^{T} I_S(t)\ dt = p(S)$ and that $\sum_{j\in S} p_j M_j = p(S)M_S = \int_{r_{min}(S)}^{T} I_S(t)\ t\ dt$. So, from these constraints, it follows that $M_S$ is minimized when $I_S(t) = 1$ for $t \in [r_{min}(S), r_{min}(S) + p(S)]$ and 0 otherwise, that is when jobs in $S$ are processed from $r_{min}(S)$, without interruption. The lower bound of $M_S$ is $r_{min}(S) + \frac{1}{2}p(S)$, so $p(S)\left(r_{min}(S) + \frac{1}{2}p(S)\right)$ minimizes $\sum_{j\in S} p_j M_j$ in every feasible preemptive schedule. □

Hence the linear program is

$$
\begin{array}{lll}
Z_R & = & \min \quad \sum_{j \in N} w_j \left( M_j + \frac{p_j}{2} \right) \\
\text{(R)} & & \text{subject to} \\
& & \sum_{j \in S} p_j M_j \geq p(S) \left( r_{min}(S) + \frac{p(S)}{2} \right) \qquad \text{for any } S \subseteq N
\end{array}
$$

and it provides a lower bound on the optimum value for $1|r_j, pmtn| \sum w_j C_j$, hence also on the optimum value of the problem $1|r_j| \sum w_j C_j$.

Let us now consider the schedule that, given a set of jobs $S$, processes these jobs as early as possible. This schedule partitions the set $S$ into $\{S_1, \ldots, S_k\}$, called canonical decomposition of $S$, and the machine processes the set $S$ in disjoint intervals $[r_{min}(S_l), r_{min}(S_l) + p(S_l)]$ for $l = 1, \ldots, k$. We call a set $S$ *canonical* if it corresponds to its canonical decomposition, that is, all jobs in $S$ are processed in $[r_{min}(S), r_{min}(S) + p(S))$. We can now reformulate Lemma 3 by defining, for the canonical decomposition $\{S_1, \ldots, S_k\}$ of $S \subseteq N$, the parameter

$$
h(S) := \sum_{l=1}^{k} p(S_l) \left( r_{min}(S_l) + \frac{p(S_l)}{2} \right). \tag{2}
$$

Thus, the relaxation R can be written

$$
min \left\{ \sum_{j \in N} w_j \left( M_j + \frac{p_j}{2} \right) : \sum_{j \in S} p_j M_j \geq h(S) \text{ for all } S \subseteq N \right\}. \tag{3}
$$

**Theorem 2.** *Let $M_j^{LP}$ be the mean busy time of job $j$ in the LP schedule, then $M^{LP}$ is an optimum solution of the linear program R.*

*Proof.* By Lemma 3 we know $M^{LP}$ is a feasible solution for the linear program R. Recall the jobs are indexed by nonincreasing $\frac{w_j}{p_j}$ ratio. Then, for the set $[i] := \{1, 2, \ldots, i\}$, let $S_1^i, \ldots, S_{k(i)}^i$ denote its canonical decomposition. Assuming $\frac{w_{n+1}}{p_{n+1}} = 0$, then for any vector $M = (M_j)_{j \in N}$ we have

$$
\sum_{j \in N} w_j M_j = \sum_{i=1}^{n} \left( \frac{w_i}{p_i} - \frac{w_{i+1}}{p_{i+1}} \right) \sum_{j \leq i} p_j M_j = \sum_{i=1}^{n} \left( \frac{w_i}{p_i} - \frac{w_{i+1}}{p_{i+1}} \right) \sum_{l=1}^{k(i)} \sum_{j \in S_l^i} p_j M_j. \tag{4}
$$

Hence we expressed $\sum w_j M_j$ as a nonnegative combination of summands over canonical sets. By construction the LP schedule continuously processes the jobs in the canonical sets $S_l^i$ in the interval $\left[r_{min}\left(S_l^i\right), r_{min}\left(S_l^i\right) + p\left(S_l^i\right)\right]$. So for any canonical set and each feasible solution $M$ of the linear program R it follows, by Lemma 3,

$$\sum_{j \in S_l^i} p_j M_j \geq h\left(S_l^i\right) = p\left(S_l^i\right)\left(r_{min}\left(S_l^i\right) + \frac{p\left(S_l^i\right)}{2}\right) = \sum_{j \in S_l^i} p_j M_j^{LP}. \tag{5}$$

So together with equation (4) we derive a lower bound on $\sum p_j M_j$ for each feasible solution $M$ of R, that it is tight for the LP schedule. □

We, now, prove the equivalence of the two relaxations:

**Corollary 1.** *For any weight $w \geq 0$ we have that the two LP relaxations D and R give the same optimal value, that is $Z_D = Z_R$.*

*Proof.* Given the solution $y^{LP}$ to R we can express the mean busy time $M_j^{LP}$ of any job as

$$M_j^{LP} = \frac{1}{p_j} \sum_{\tau = r_j}^{T} y_{j,\tau}^{LP}\left(\tau + \frac{1}{2}\right).$$

So by theorems 1 and 2 we have the desired equality. □

We recall, before studying some polyhedral consequences of the LP formulations, that the LP schedule solves optimally $1|r_j|\sum w_j M_j$ over the preemptive schedules but it does not, generally, minimize $1|r_j|\sum w_j C_j$ over the preemptive schedules.

### 3.3 Polyhedral Consequences

We study now some polyhedral consequences of the two LP relaxations. Let $P_D^\infty$ be the feasibility region for D when $T = \infty$. So

$$P_D^\infty := \left\{ y \geq 0 : \sum_{j : r_j \leq \tau} y_{j,\tau} \leq 1 \text{ for } \tau \in \mathbb{N},\ \sum y_{j,\tau} = p_j \text{ for all } j \in N \right\}.$$

Similarly the LP relaxation R, with its constraints, defines the polyhedron $P_R$.

**Theorem 3.** *The polyhedron $P_R$ is the convex hull of the mean busy time vectors $M$ of all the preemptive schedules. Moreover each vertex of $P_R$ is the mean busy time vector of an LP schedule.*

*Proof.* We already proved in Lemma 3 that $P_R$ contains the convex hull of the mean busy time vectors $M$ of all feasible preemptive schedules. In order to prove the other direction we need to show that every extreme point of $P_R$ corresponds to a preemptive schedule and each extreme ray of $P_R$ is a recession direction for the convex hull of mean busy time vectors. Since every extreme point of $P_R$ is the unique minimizer of $\sum w_j M_j$ for some $w \geq 0$ and by Theorem 2, it follows that each extreme point of $P_R$ corresponds to a preemptive schedule. In order to finish the proof we note that the extreme rays of $P_R$ are the $n$ unit vectors of $\mathbb{R}^N$. An extension of the results in [4] to preemptive schedules and mean busy times implies that the directions of recession for the convex hull of mean busy time vectors are those unit vectors of $\mathbb{R}^N$. □

**Theorem 4.** *Let be given the mapping $\mathcal{M} : y \mapsto \mathcal{M}(y) \in \mathbb{R}^N$ defined by $\mathcal{M}(y)_j = \frac{1}{p_j} \sum_{\tau \geq r_j} y_{j,\tau} \left(\tau + \frac{1}{2}\right)$ for all $j \in N$. Then we have that $P_R$ is the image of $P_D^\infty$ in the space of $M$-variables under $\mathcal{M}$.*

*Proof.* Let $y \in P_D^\infty$ and $S \subseteq N$ with canonical decomposition $\{S_1, \ldots, S_k\}$. Then, by definition of $\mathcal{M}(y)_j$ it follows:

$$\begin{aligned}\sum_{j \in S} p_j \mathcal{M}(y)_j &= \sum_{j \in S} \sum_{\tau \geq r_j} y_{j,\tau} \left(\tau + \frac{1}{2}\right) \geq \sum_{l=1}^{k} \sum_{\tau = r_{min}(S_l)}^{r_{min}(S_l)+p(S_l)} \left(\tau + \frac{1}{2}\right) \\ &= \sum_{l=1}^{k} p(S_l) \left(r_{min}(S_l) + \frac{p(S_l)}{2}\right) = h(S),\end{aligned}$$

where the inequality follows from the definition of $P_D^\infty$ and by applying an interchange argument as we already did in Theorem 1. Hence $\mathcal{M}(y) \in P_R$ and $\mathcal{M}(P_D^\infty) \subseteq P_R$. In order to prove the other direction we recall that $P_R$ can be represented as the sum of the convex hull of all mean busy time vectors of LP schedules and the nonnegative orthant. We know that the mean busy time vector $M^{LP}$ of any LP schedule is the projection of the corresponding vector $y^{LP}$, hence we only need to prove that every unit vector $e_j$ is a recession direction for $\mathcal{M}(P_D^\infty)$. In order to prove it we can fix an LP schedule where we denote the associated 0/1 vector and the mean busy time vector, respectively, by $y^{LP}$ and $M^{LP} = \mathcal{M}(y^{LP})$. Then, for any job $j \in N$ and any real $\lambda > 0$, we have to prove $M^{LP} + \lambda e_j \in \mathcal{M}(P_D^\infty)$.

Let now $\tau_{max} = \max\{\tau : y^{LP}_{k,\tau} = 1 : k \in N\}$, then we choose $\theta$ so that $y^{LP}_{j,\theta} = 1$ and $\mu > \max\{\lambda p_j, \tau_{max} - \theta\}$. Hence it is possible to define $y'_{j,\theta} = 0$, $y'_{j,\theta+\mu} = 1$; and $y'_{k,\tau} = y^{LP}_{k,\tau}$ otherwise. In the associated preemptive schedule we postpone the processing of job $j$ by $\mu$ units, so from interval $[\theta, \theta+1)$ to $[\theta+\mu, \theta+\mu+1)$, hence $M' = \mathcal{M}(y')$ equals $M'_j = M^{LP}_j + \frac{\mu}{p_j}$ and $M'_k = M^{LP}_k$ for $k \neq j$. Calling $\lambda' = \frac{\mu}{p_j} \geq \lambda$, we have $M' = M^{LP} + \lambda' e_j$, hence $M^{LP} + \lambda' e_j$ is a convex combination of $M^{LP}$ and $M'$. Let now $y$ correspond to a convex combination of $y^{LP}$ and $y'$. We conclude saying that $M^{LP} + \lambda e_j = \mathcal{M}(y) \in \mathcal{M}(P^{LP}_D)$ by the convexity of $P^{LP}_D$, so the implication $y \in P^{LP}_D$ follows. □

As last result about the LP relaxations, we note that the feasible set $P_R$ defines a particular polyhedron.

**Proposition 2.** *Given the function $h$ as defined in equation* (2)*, it defines a supermodular set function.*

*Proof.* Let $j, k \in N$ be two jobs and consider any subset $S \subseteq N \setminus \{j, k\}$. By using a job-based method, hence considering job $k$ after jobs in $S$, we construct an LP schedule minimizing $\sum_{i \in S \cup \{k\}} p_i M_i$. We know, from the definition of $h$, that considering the mean busy time $M^{LP}$ we have $\sum_{i \in S} p_i M^{LP}_i = h(S)$ and $\sum_{i \in S \cup \{k\}} p_i M^{LP}_i = h(S \cup \{k\})$. By construction we have that job $k$ is scheduled in the first $p_k$ units of idle time after $r_k$, hence we can see $M^{LP}_k$ as the mean of these time units. The next step in the proof is to construct another LP schedule, with mean busy time vector $\widetilde{M}^{LP}$, considering before set $S$, then job $j$ and at the end job $k$, in such a way to have $\widetilde{M}^{LP}_i = M^{LP}_i$ for $i \in S$, $\sum_{i \in S \cup \{j\}} p_i \widetilde{M}^{LP}_i = h(S \cup \{j\})$ and $\sum_{i \in S \cup \{j,k\}} p_i \widetilde{M}^{LP}_i = h(S \cup \{j, k\})$. For sure job $k$ cannot have processing time earlier than the former LP schedule, since, there only $S$ was considered, and now $S \cup \{j\}$, hence we have $\widetilde{M}^{LP}_k \geq M^{LP}_k$ that implies the supermodularity, because

$$h(S \cup \{j, k\}) - h(S \cup \{j\}) = \widetilde{M}^{LP}_k \geq M^{LP}_k = h(S \cup \{k\}) - h(S).$$

□

From Proposition 2 it follows that the job-based method for the construction of the LP schedule is like the greedy algorithm that minimizes $\sum w_j M_j$ over the supermodular $P_R$.

# 4 Conversion Algorithm

In this section we will explain a general algorithm that converts the preemptive LP schedule into a nonpreemptive one, by using the $\alpha$-points of the jobs, introduced for the first time by Phillips et al. in [30]. Then we will show an example in such a way to understand better how the algorithm works. After we give some easy bounds that will be improved in the next sections by Goemans et al., see [16]. Finally, we will also present a general technique to derandomize the algorithm, as shown by Goemans in [15].

## 4.1 The Algorithm

We start by defining what is the $\alpha$-point of a job $j$, denoted by $t_j(\alpha)$. Given $0 < \alpha \leq 1$ the *$\alpha$-point* of job $j$ is the first point in time when job $j$ has been processed for a total of $\alpha \cdot p_j$ time units, so an $\alpha$-fraction of job $j$ is completed. Trivially we can denote the starting and the completion time of job $j$ as, respectively, $t_j(0^+)$ and $t_j(1)$.

By definition, the mean busy time $M_j^{LP}$ of job $j$ in the $LP$-schedule is the average value of all its $\alpha$-points, so

$$M_j^{LP} = \int_0^1 t_j(\alpha)\, d\alpha. \tag{6}$$

We also define, for a given $\alpha$, $0 < \alpha \leq 1$, and a fixed job $j$, the parameter $\eta_k(j)$ to be the fraction of job $k$ that is completed in the LP schedule by $t_j(\alpha)$. It can be seen easily that $\eta_j(j) = \alpha$, since at $t_j(\alpha)$ the processed fraction of job $j$ is $\alpha$.

By the construction of the preemptive schedule there is no idle time between the start and the completion of any job $j$, so, if there is some idle time, it must be before $t_j(0^+)$. This quantity depends on the specific job, hence we can denote by $\tau_j$ the idle time that occurs between time 0 and the starting time of job $j$ in the LP schedule. By the previous observations we can write the $\alpha$-point of $j$ as

$$t_j(\alpha) = \tau_j + \sum_{k \in N} \eta_k(j) \cdot p_k. \tag{7}$$

For a given $\alpha$, $0 < \alpha \leq 1$, consider the $\alpha$-schedule which processes the jobs as early as possible in order of nondecreasing $\alpha$-point and in a nonpreemptive way. Then the completion time of job $j$ in this $\alpha$-schedule it is denoted by $C_j^\alpha$. The conversion algorithm, called $\alpha$-CONV, works as follows:

Consider the jobs $j \in N$ in order of nonincreasing $\alpha$-points and iteratively change the preemptive LP schedule to a nonpreemptive schedule by applying the following steps:

- remove the $\alpha \cdot p_j$ units of job $j$ processed before $t_j(\alpha)$ from the machine and make them idle time, we say this idle time is *caused by* job $j$;
- delay the whole processing done after the $\alpha$-point of job $j$ by $p_j$ units of time;
- remove the remaining units of job $j$ from the machine (they are $(1-\alpha) \cdot p_j$) and move earlier the processings occurring later. Hence job $j$ is processed in a nonpreemptive way exactly in the interval $[t_j(\alpha), t_j(\alpha) + p_j]$.

The feasibility of this schedule follows by the fact that the jobs are scheduled in nondecreasing order of $\alpha$-point, so no job is started before $t_j(\alpha) \geq r_j$.

## 4.2 Example

We provide an example of the conversion algorithm. Let $\alpha = \frac{1}{2}$ and let $N = \{j_1, j_2, j_3, j_4\}$ be the set of jobs, with $\frac{w_1}{p_1} \leq \cdots \leq \frac{w_4}{p_4}$, so that:

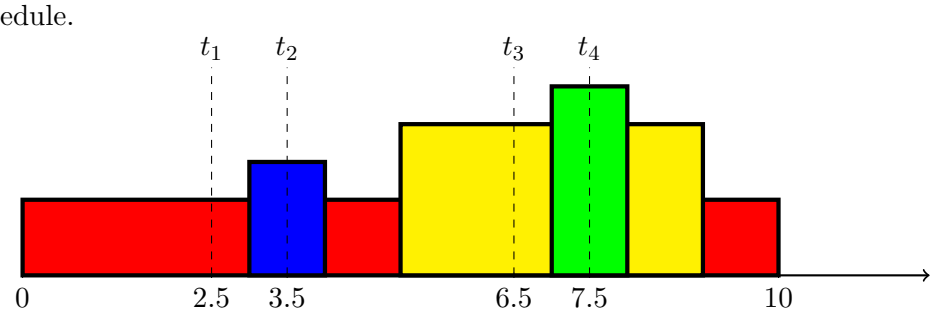
Job 1: $p_1 = 5, r_1 = 0$, then we obtain $t_1(\alpha) = t_1 = 2.5$ and $C_1 = 10$;

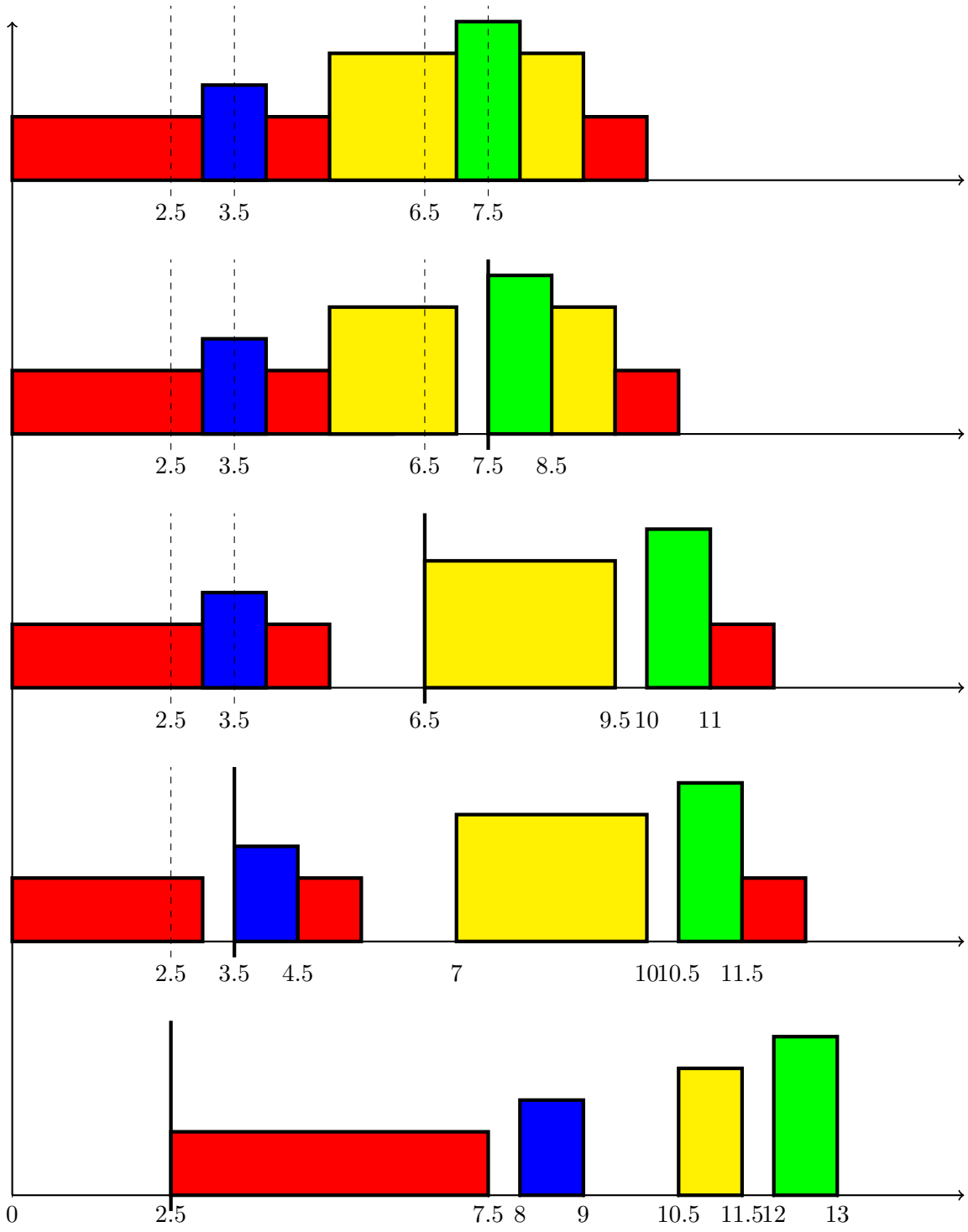
Job 2: $p_2 = 1, r_2 = 3$, then we obtain $t_2(\alpha) = t_2 = 3.5$ and $C_2 = 4$;

Job 3: $p_3 = 3, r_3 = 5$, then we obtain $t_3(\alpha) = t_3 = 6.5$ and $C_3 = 9$;

Job 4: $p_4 = 1, r_4 = 7$, then we obtain $t_4(\alpha) = t_4 = 7.5$ and $C_4 = 8$.

We give two illustration: the first is the LP schedule and the second shows the individual iterations of the conversion algorithm, starting from the LP schedule.

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13

### 4.3 Some Consequenques

Directly from the algorithm it is easy to find some bound on the completion time of a job. These bounds will be used in the next chapters to obtain improved approximation guarantees.

**Lemma 4.** *The algorithm $\alpha$-CONV set the completion time for job $j$ to be*

$$C_j^\alpha = t_j(\alpha) + \sum_{k:\eta_k(j)\geq\alpha} (1 - \eta_k(j) + \alpha) \cdot p_k. \tag{8}$$

*Proof.* Consider job $j$, after the algorithm its completion time equals the sum of the processing times of jobs scheduled no later than $j$ and the idle time before $t_j(\alpha)$. By the $\alpha$-point ordering, we have that, before the completion time of job $j$, the machine is busy a total of $\sum_{k:\eta_k(j)\geq\alpha} p_k$ time. We now have to reformulate the idle time, since now it is the sum of $\tau_j$ and the delay in the start of the other jobs in the conversion algortihm. Each job $k$ such that $\eta_k(j) \geq \alpha$ contributes $\alpha \cdot p_k$ units of idle time, while the other jobs just $\eta_k(j) \cdot \alpha$ units of idle time. Hence the total idle time before the start of job $j$ in $\alpha$-CONV is

$$\tau_j + \sum_{k:\eta_k(j)\geq\alpha} \alpha \cdot p_k + \sum_{k:\eta_k(j)<\alpha} \eta_k(j) \cdot p_k. \tag{9}$$

Then the completion time of job $j$ in the schedule constructed by the algorithm is the sum of the processing times and the idle time before the completion of job $j$, thus

$$\begin{aligned} C_j^\alpha &= \sum_{k:\eta_k(j)\geq\alpha} p_k + \tau_j + \sum_{k:\eta_k(j)\geq\alpha} \alpha \cdot p_k + \sum_{k:\eta_k(j)<\alpha} \eta_k(j) \cdot p_k = \\ &= t_j(\alpha) + \sum_{k:\eta_k(j)\geq\alpha} (1 - \eta_k(j) + \alpha) \cdot p_k, \end{aligned} \tag{10}$$

where the second equality follows from (7). □

**Corollary 2.** *In an $\alpha$-schedule we have the following bound for the completion time of job $j$:*

$$C_j^\alpha \leq t_j(\alpha) + \sum_{k:\eta_k(j)\geq\alpha} (1 - \eta_k(j) + \alpha) \cdot p_k. \tag{11}$$

*Proof.* Since the schedule constructed by $\alpha$-CONV processes the jobs in order of nondecreasing $\alpha$-point and by Lemma 4 the result follows easily. □

We will see in the next section that for a fixed $\alpha$, $0 < \alpha \leq 1$, the value of the $\alpha$-schedule is within a factor of $\max\{1 + \frac{1}{\alpha}, 1 + 2\alpha\}$ of the optimum LP value, so it is minimized for $\alpha = \frac{1}{\sqrt{2}}$ and the bound is $1 + \sqrt{2}$. We will see how to beat this bound by choosing $\alpha$ randomly from some probability distribution.

We explain here how is it possible to derandomize easily the algorithm. We prove it now since it is a general idea and it can be used for all the LP-based schedules with random $\alpha \in (0, 1]$.

**Lemma 5.** *Over all choices of $\alpha$, there are at most $n$ different orderings given by the algorithm.*

*Proof.* We have changes in the $\alpha$-schedule only when a job is preempted. Thus the total number of changes in the $\alpha$-schedule is bounded from above by the total number of preemptions. From the fact that a preemption in the LP can occur only when a new job is released, we have that the total number of preemptions is at most $n - 1$, so there are at most $n$ different orderings. □

**Proposition 3.** *There are at most $n$ different schedules and they can be computed in $O(n^2)$ time.*

*Proof.* By Lemma 5 we have that the total number of preemptions is at most $n-1$, hance there are at most $n$ different orderings, i.e. combinatorial values of $\alpha$. Since we can compute each $\alpha$-schedule in $O(n)$ time the total running time is at most $O(n^2)$. □

Obviously the schedule can be constructed by using job-dependent $\alpha_j$, but we preferred to explain for a common $\alpha$ in such a way to keep the notation easier, anyway the generalization to the job-dependent case is straightforward.

# 5 Approximations for $1|r_j|\sum w_jC_j$

In this section we will see some approximation algorithms for the scheduling problem $1|r_j|\sum w_jC_j$. Starting from the preemptive LP optimal solution, we will use a nonpreemptive feasible schedule, so we can relate the value to $Z_D = Z_R$, in such a way to obtain some specific approximation guarantee. In Chapter 5.1 we will prove the results given by Goemans in [15], namely the best possible guarantee of $1+\sqrt{2}$ for fixed $\alpha$ and a 2-approximation algorithm by choosing $\alpha$ randomly. We will present also an easy way to derandomize this algorithm. In Chapter 5.2, following the ideas of Goemans et al. in [16], we will derive another proof for the 2-approximation performance, but using job-dependent $\alpha_j$ values. In the same paper the authors proved stronger approximation algorithms, involving some specific probability distribution, we will present them in Chapters 5.3 and 5.4. These algorithms are, respectively, for a common $\alpha$ and for job-dependent $\alpha_j$, and we will prove that, with respect to those probability distributions, they are the best possible. Finally in Chapter 5.5 we give an example showing that the conversion algorithm based on the LP relaxations used can reach an approximation guarantee of at most $\frac{e}{e-1} \approx 1.5819$, see [16].

## 5.1 2-approximation algorithm

Given the LP schedule and $\alpha \in (0,1]$ the algorithm $A_\alpha$ orders the jobs in order of increasing $\alpha$-points and it schedules the jobs as early as possible following this order. For a given $\alpha$, $0 < \alpha \leq 1$, a valid $\alpha$-schedule is the schedule which processes the jobs in order of increasing $t_j(\alpha)$ and so that job $j$ is not processed before $t_j(\alpha)$, so an $\alpha$-schedule is valid if it is the output of a conversion algorithm.

For any set of jobs $S \subseteq N$ let us recall that the processing time of $S$ is the sum of processing times of the jobs in $S$, i.e. $p(S) = \sum_{j\in S} p_j$.

In order to semplify the notation, since we use a common $\alpha$, we may denote the $\alpha$-point of job $j$ just by $t_j$.

**Proposition 4.** *If the preemptive schedule processes all the jobs in $S$ in the interval $[0, p(S)]$, then there exist a valid $\alpha$-schedule which processes the jobs in $S$ in the interval $[\alpha p(S), (\alpha+1)p(S)]$.*

*Proof.* Given the set $S$, with $|S| = s$, we can reindex the jobs in such a way that $t_1 \leq t_2 \leq \cdots \leq t_s$. For any index $i$, at time $t_i$, the preemptive schedule can have processed at most all the jobs $1, 2, \ldots, i-1$ completely and the

jobs $i+1,\dots,s$ for an $\alpha$ fraction. So it follows that

$$t_i(\alpha) \le p_1+\cdots+p_{i-1}+\alpha p_i+\cdots+\alpha p_s = \sum_{k=1}^{i-1} p_k + \alpha\sum_{k=i}^{s} p_k \le \sum_{k=1}^{i-1} p_k + \alpha p(S)$$

Hence job $j_1$ can start at time $\alpha p(S)$, job $j_2$ at time $p_1+\alpha p(S)$ and so on. So the jobs can be scheduled consecutively in this order in the interval $[\alpha p(S),(\alpha+1)p(S)]$, and it is a valid $\alpha$-schedule. □

This proposition was specific for the case if the set $S$ started to be processed at time 0. For the general case we have to define before some values: given a canonical set $S\subseteq N$, let $s(S)$ be the first time when jobs in $S$ are processed and let $\mu_S(j)$ be the fraction of job $j$ that is processed in the preemptive schedule before $s(S)$. Then we have following statement.

**Proposition 5.** *Given any canonical set $S$, there is a valid $\alpha$-schedule which processes the jobs in $S$ in the interval $[t(S),t(S)+p(S)]$, where we have to define $t(S) := \max\{(1+\frac{1}{\alpha})s(S), s(S)+\alpha p(S)\}$.*

In order to have the desired approximation guarantee we prove a stronger statement.

**Proposition 6.** *Let $S$ be any canonical set, all the jobs in $S$ are processed, by a valid $\alpha$-schedule, in the interval $[t(S),t(S)+p(S)]$, where, in this case, we have $t(S) := s(S)+\max\{\sum_{j:\mu_S(j)\ge\alpha} p_j, \alpha p(S)\}$.*

*Proof.* From the definition of the algorithm $A_\alpha$, before to start the process of a job in $S$ it is required exactly $\sum_{j:\mu_S(j)\ge\alpha} p_j$ time, so the sum of the processing times of the jobs with $t_j \le t_k$, where $k$ is the first job processed in $S$. We can see, that for all those jobs $t_j \le s(S)$, otherwise, since $S$ is a canonical set the jobs $j$ would be scheduled before $k$. So these jobs can be scheduled in the interval $[s(S), s(S)+\sum_{j:t_j\le t_k} p_j]$. If $s(S)+\sum_{j:t_j\le t_k} p_j \ge s(S)+\alpha p(S)$, then we have, according to a shifted version of Proposition 4, that jobs in $S$ can follow immediately. In the other case, if $s(S)+\sum_{j:t_j\le t_k} p_j < s(S)+\alpha p(S)$, it is just needed to delay the start of the jobs preceeding $S$ by $\alpha p(S)-\sum_{j:t_j\le t_k} p_j$ in order to have the desired result. □

**Observation** Proposition 6 implies Proposition 5.

*Proof.* We know, by definition, that the $\mu_S$ fraction of every job is processed before than $s(S)$, that means $\sum \mu_S(j)p_j \le s(S)$. Thus it follows that $\sum_{j:\mu_S(j)\ge\alpha} p_j \le \sum \frac{\mu_S(j)}{\alpha} p_j \le \frac{s(S)}{\alpha}$. □

From these propositions the two main results follow easily.

**Theorem 5.** *The algorithm $A_\alpha$ is a $\rho$-approximation for $1|r_j|\sum w_jC_j$, where $\rho = \max\{1+\frac{1}{\alpha}, 1+2\alpha\}$. In particular for $\alpha = \frac{1}{\sqrt{2}}$ the performance guarantee is $1+\sqrt{2}$.*

*Proof.* For any canonical set $S$, according to Proposition 5, we have that the mean busy time given by $A_\alpha$ satisfies

$$\begin{aligned}\tilde{M}(S) &\leq \max\{(1+\frac{1}{\alpha})s(S)\ ,\ s(S)+\alpha p(S)\} + \frac{p(S)}{2} \\ &\leq (1+\frac{1}{\alpha})s(S) + (\alpha+\frac{1}{2})p(S) \leq \max\{1+\frac{1}{\alpha}\ ,\ 1+2\alpha\}\left(s(S)+\frac{p(S)}{2}\right)\end{aligned}$$

and the result follows from the mean busy time LP relaxation defined in Chapter 3.2. It is easily seen that $\max\{1+\frac{1}{\alpha}\ ,\ 1+2\alpha\}$ is minimized for $\alpha = \frac{1}{\sqrt{2}}$. Hence $A_{1/\sqrt{2}}$ is a $1+\sqrt{2}$-approximation algorithm. □

We can modify the algorithm $A_\alpha$ by choosig a random value of $\alpha$, so the algorithm $R_\alpha$ behaves in the same way of $A_\alpha$ with the difference that $\alpha$ is chosen accordingly to a uniform distribution in $(0,1]$.

**Theorem 6.** *The algorithm $R_\alpha$ outputs a schedule with expected weight at most $2\cdot OPT$.*

*Proof.* For any canonical set $S$, its expected mean busy time is

$$\begin{aligned}E\left[\tilde{M}(S)\right] &\leq E\left[\max\{s(S)+\sum_{j:\mu_S(j)\geq\alpha} p_j\ ,\ s(S)+\alpha p(S)\} + \frac{p(S)}{2}\right] \leq \\ &\leq s(S)+\frac{p(S)}{2} + E\left[\sum_{j:\mu_S(j)\geq\alpha} p_j\right] + E\left[\alpha p(S)\right] = \\ &= s(S)+\frac{p(S)}{2} + \sum Pr\left[\alpha\leq\mu_S(j)\right]\cdot p_j + E[\alpha]\cdot p(S) = \\ &= s(S)+p(S)+\sum \mu_S(j)p_j \leq 2s(S)+p(S) = 2\left(s(S)+\frac{p(S)}{2}\right)\end{aligned}$$

by definition of $\mu_S(j)$ and Propostion 6. □

In order to derandomize the algorithm we have two different ways: the first, that is general, is explained in Chapter 4, by Lemma 5 and Proposition 3, while the second applies for our case, it returns a slightly smaller guarantee, but, given a fixed integer $k$, it runs in $O(n\log n + kn)$ time, so it is faster.

**Proposition 7.** *There is a deterministic algorithm with a guarantee of* $\left(2+\frac{1}{k}\right)$ *running in* $O(n\log n+kn)$ *time.*

*Proof.* Let $k$ be a fixed integer, then for $\alpha_i = \frac{i}{k+1}$, for $i = 1,\dots,k$, we run the algorithm $A_{\alpha_i}$. The best schedule will be at least as good as the schedule returned by the randomized algorithm which selects the value of $\alpha$ uniformly between the $k$ values $\alpha_i$. Following the proof of Theorem 6 we derive that, for this randomized algorithm, we have:

$$\begin{aligned}
E[\tilde{M}(S)] &\le s(S) + \frac{p(S)}{2} + \sum \Pr(\alpha \le \alpha_j)\cdot p_j + E[\alpha p(S)] \\
&\le s(S) + p(S) + \frac{k+1}{k}\sum \alpha_j p_j \le \left(2+\frac{1}{k}\right)s(S) + p(S) \\
&\le \left(2+\frac{1}{k}\right)\left(s(S)+\frac{p(S)}{2}\right).
\end{aligned}$$

that implies the performance guarantee.
The running time is just $O(n\log n+kn)$ since $O(n\log n)$ follows by the algorithm and $O(kn)$ from the fact that we consider only $k$ schedules. □

## 5.2 Another 2-approximation algorithm

Here we will derive, by using some probability distribution, a different proof for the approximation guarantee of Section 5.1, i.e. a 2-approximation, by using job-dependent $\alpha_j$ values. Then we will refine the analysis in order to have some stricter bounds that will be used after.

**Theorem 7.** *Let* $\alpha_j$ *be random variables drawn from* $(0,1]$ *pairwise independently and uniformly. Then we have that the expected value of the resulting* $(\alpha_j)$*-schedule is within a factor of 2 of the optimum LP value. The same result holds for the case of common random* $\alpha$ *drawn uniformly from* $(0,1]$.

*Proof.* Let us define $E_U[F(\boldsymbol{\alpha})]$ as the expectation of the function $F$ in the variable $\boldsymbol{\alpha}$ when $\boldsymbol{\alpha}$ is uniformly distributed. We prove the statement by using some job-by-job bound and by the linearity of expectation.

Let us consider an arbitrary fixed job $j$. Suppose, for now, that $\alpha_j$ is fixed and we study $E_U[C_l^{\boldsymbol{\alpha}}|\alpha_j]$. By the fact that $\alpha_j$ and $\alpha_k$ are independent

for each $j \neq k$, we have, by Corollary 2, the following:

$$\begin{aligned} E_U[C_l^{\boldsymbol{\alpha}}|\alpha_j] &\leq t_j(\alpha_j) + \sum_{k\neq j} p_k \int_0^{\eta_k(j)} (1-\eta_k(j)+\alpha_k)\, d\alpha_k + p_j \\ &= t_j(\alpha_j) + \sum_{k\neq j} \left(\eta_k(j) - \frac{\eta_k(j)^2}{2}\right)\cdot p_k + p_j \\ &\leq t_j(\alpha_j) + \sum_{k\neq j} \eta_k(j)\cdot p_k + p_j \leq 2\cdot\left(t_j(\alpha_j) + \frac{p_j}{2}\right). \end{aligned}$$

Now we just have to integrate over all possible choices of $\alpha_j$ yielding:

$$E_U[C_j^{\boldsymbol{\alpha}}] = \int_0^1 E_U[C_j^{\boldsymbol{\alpha}}|\alpha_j]\, d\alpha_j \leq 2\cdot\left(\int_0^1 t_j(\alpha_j)\, d\alpha_j + \frac{p_j}{2}\right) = 2\cdot\left(M_j^{LP} + \frac{p_j}{2}\right).$$

The optimum value, $Z_D$, is given by $\sum w_j \left(M_j^{LP} + \frac{p_j}{2}\right)$, so we have, by linearity of the expectations, that $E_U[\sum w_jC_j^{\boldsymbol{\alpha}}] \leq 2\cdot Z_D$. □

We can start by observing that if a job $j$ is preempted in the LP schedule at time $s$ and it is resumed at time $t > s$ in the interval $[s,t]$ the jobs processed have greater $\frac{w_k}{p_k}$ ratio, so these jobs are completely processed in this interval. Hence, in the LP schedule, in $[s,t]$, the machine is costantly busy, alternating between complete processing of jobs $k$ with $\frac{w_k}{p_k} > \frac{w_j}{p_j}$ and portions of job $j$. At the same time we can say that any job preempted at the starting time $t_j(0^+)$ of job $j$ must wait, at least until $t_j(1)$, to be resumed. Hence, for any fixed job $j$, we partition the jobs of $N\setminus\{j\}$ into two subsets: $N_2$, the set of jobs processed between $t_j(0^+)$ and $t_j(1)$, and $N_1$, the set of all remaining jobs. We recall two parameters defined before: $\eta_k(j)$ is the fraction of job $k$ completed by time $t_j(\alpha_j)$ and $\mu_j(k)$ is the fraction of job $j$ completed by time the starting time of job $k$, i.e. by $t_k(0^+)$. If $j$ is fixed and $k \in N_1$ than the function $\eta_k(j)$ is constant, so we define $\eta_k := \eta_k(j)$, otherwise we have

$$\eta_k(j) = \begin{cases} 0 & \text{if } \alpha_j \leq \mu_k(j) \\ 1 & \text{if } \alpha_j > \mu_k(j) \end{cases} \qquad \text{for } k \in N_2. \tag{12}$$

We can now rewrite the expression (7) for the $\alpha_j$-point as

$$t_j(\alpha_j) = \tau_j + \sum_{k\in N_1} \eta_k\cdot p_k + \sum_{k\in N_2:\alpha_j>\mu_k(j)} p_k + \alpha_j\cdot p_j. \tag{13}$$

If we insert this reformulation in the formula (6) in order to calculate the LP-midpoint it yields

$$M_j^{LP} = t_j(0^+) + \sum_{k\in N_2} (1-\mu_k(j)) \cdot p_k + \frac{p_j}{2}; \tag{14}$$

at the same time we can rewrite Corollary 2 as

$$\begin{aligned} C_j^{\boldsymbol{\alpha}} \leq t_j(0^+) + &\sum_{k\in N_1:\alpha_k\leq\eta_k(j)} (1-\eta_k(j)+\alpha_k)\cdot p_k + \\ + &\sum_{k\in N_2:\alpha_j>\mu_k(j)} (1+\alpha_k)\cdot p_k + (1+\alpha_j)\cdot p_j. \end{aligned} \tag{15}$$

Here when we considered $k \in N_2$ we used the equivalence between the inequalities $\alpha_k \leq \eta_k(j)$ and $\alpha_j > \mu_k(j)$. It is easy to see that, in equation (15), small values of $\alpha$ keep the terms of the form $(1-\eta_k+\alpha_k)$ and $(1+\alpha_k)$ small, while increasing the $\alpha$ values decrease the number of summands, so we have to balance the contributes of the two effects to reduce the bound on the expected value of the completion time $E[C_j^{\alpha}]$.

### 5.3 Bounds for Common $\alpha$ value

In this section we will prove some better bounds for the common $\alpha$ case, by using a suitable probability distribution, with a truncated exponential function as density function. Then we will give the better approximation guarantee for this case.

Let $\gamma \approx 0.467$ be the unique solution of

$$1 - \frac{\gamma^2}{1+\gamma} = \gamma + \ln(1+\gamma) \tag{16}$$

in the interval $(0,1)$. We define $c := \frac{1+\gamma}{1+\gamma-e^{\gamma}} < 1.745$ and we let $\alpha$ be chosen according to the density function:

$$f(\alpha) = \begin{cases} (c-1)\cdot e^{\alpha} & \text{if } \alpha \leq \delta \\ 0 & \text{otherwise.} \end{cases} \tag{17}$$

Since it must be a density function, the integral must be 1, hence we can find the value $\delta := \delta(c) = \ln\left(\frac{c}{c-1}\right)$, so that

$$\int_0^{\delta} f(\alpha)\, d\alpha = \int_0^{\delta} (c-1)\cdot e^{\alpha}\, d\alpha = (c-1)(e^{\delta}-1) = (c-1)\left(\frac{c}{c-1}-1\right) = 1.$$

We prove now a technical lemma that is the key for the proof of the main theorem in this section. The function $f$ has two important properties bounding the delay of job $j$, inequality (18) refers to jobs in $N_1$, while inequality (19) is for the jobs in $N_2$.

**Lemma 6.** *The above defined function $f$ is a density function satisfying the following properties:*

$$\int_0^\eta f(\alpha)\cdot(1+\alpha-\eta)\,d\alpha \le (c-1)\cdot\eta \quad \textit{for all } \eta\in[0,1], \tag{18}$$

$$\int_\mu^1 f(\alpha)\cdot(1+\alpha)\,d\alpha \le c\cdot(1-\mu) \qquad \textit{for all } \mu\in[0,1]. \tag{19}$$

*Proof.* In order to prove the inequality (18) we have to split the interval of possible $\eta$ into $\eta\in[0,\delta]$, so we get

$$\begin{aligned}\int_0^\eta f(\alpha)(1+\alpha-\eta)\,d\alpha &= (c-1)\int_0^\eta e^\alpha(1+\alpha-\eta)\,d\alpha = \\ &= (c-1)\cdot[(e^\eta-1)(1-\eta)+\eta e^\eta-e^\eta+1] = (c-1)\cdot\eta\end{aligned}$$

and $\eta\in(\delta,1]$, for which we have

$$\int_0^\eta f(\alpha)(1+\alpha-\eta)\,d\alpha < \int_0^\delta f(\alpha)(1+\alpha-\eta)\,d\alpha = (c-1)\cdot\delta < (c-1)\cdot\eta.$$

For equation (19) we have to study $\mu\in(\delta,1]$, that has integral 0, and $\mu\in[0,\delta]$ for which

$$\int_\mu^1 f(\alpha)(1+\alpha)\,d\alpha = (c-1)\int_\mu^\delta e^\alpha(1+\alpha)\,d\alpha = (c-1)\cdot(\delta e^\delta-\eta e^\eta)$$

here it is easier to pass through a different variable $\gamma$, so that we can define $c := \frac{1+\gamma}{1+\gamma-e^{-\gamma}}$, with $\delta = 1-\frac{\gamma^2}{1+\gamma}$ and $\frac{c-1}{c} = \frac{e^{-\gamma}}{1+\gamma}$. The motivation will be seen later. Now, by substitution, we have

$$\begin{aligned}\int_\mu^1 f(\alpha)(1+\alpha)\,d\alpha &= c\cdot\left(1-\frac{\gamma^2}{1+\gamma}-\frac{e^{\mu-\gamma}\mu}{1+\gamma}\right) \le c\cdot\left(1-\frac{\gamma^2+(1+\mu-\gamma)\mu}{1+\gamma}\right) \\ &= c\cdot\left(1-\frac{(\gamma-\mu)^2+(1+\gamma)\mu}{1+\gamma}\right) \le c\cdot(1-\mu).\end{aligned}$$

□

**Theorem 8.** *If $\alpha$ is chosen accordingly to the truncated exponential function $f$, as defined above, then the expected value of the resulting random $\alpha$-schedule is bounded by $c \cdot Z_D$, so $c$ times the optimum value.*

*Proof.* From equation (15) and by Lemma 6 we have that

$$\begin{aligned} E_f[C_j^\alpha] &\leq t_j(0^+) + (c-1)\sum_{k\in N_1} \eta_k \cdot p_k + c\sum_{k\in N_2}(1-\mu_k)\cdot p_k + c\cdot p_j \\ &\leq c\cdot t_j(0^+) + c\sum_{k\in N_2}(1-\mu_k)p_k + c\cdot p_j = c\cdot\left(M_j^{LP} + \frac{p_j}{2}\right) \end{aligned}$$

where the last inequality follows from the definitions of $\eta_k$ in the set $N_1$, so $\sum_{k\in N_1}\eta_k\cdot p_k \leq t_j(0^+)$, and the equality by equation (14). □

This is a general theorem, so any function satisfying the hypothesis of Lemma 6 would define an upper bound of $c \cdot Z_D$.

The truncated exponential function (17) minimizes $c$ in (19), since any function $\alpha \mapsto (c-1)\cdot e^\alpha$ verifies (18) with equality.

**Theorem 9.** *Let the density function be defined, for $0 \leq a < b \leq 1$, by*

$$f(\alpha) = \begin{cases} (c-1)\cdot e^\alpha & \text{if } \alpha \in [a,b] \\ 0 & \text{otherwise.} \end{cases} \tag{20}$$

*Then the best bound satisfying Lemma 6 is reached for $a = 0$ and $b = \ln\frac{c}{c-1}$.*

*Proof.* Since it must define a density function over a probability distribution it follows $(c-1)(e^b - e^a) = 1$, hence $c - 1 = \frac{1}{e^b - e^a}$. Considering equation (18) we have equality for $a = 0$, hence

$$\int_a^\eta f(\alpha)(1+\alpha-\eta)\,d\alpha \leq \int_0^\eta f(\alpha)(1+\alpha-\eta)\,d\alpha \leq (c-1)\cdot\eta \quad \text{for } \eta \in [0,1]$$

Thus we can focus on equation (19) and we want to minimize $c$.

$$\int_{\max\{\mu,a\}}^b f(\alpha)(1+\alpha)\,d\alpha \leq \int_\mu^b f(\alpha)(1+\alpha)\,d\alpha \qquad \text{for } a = 0$$

Then we have to find the smallest value of $c$ in order to satisfy, for all $\mu \in [0,1]$ the inequality $\int_\mu^b f(\alpha)(1+\alpha)\,d\alpha \leq c\cdot(1-\mu)$. Hence

$$\frac{be^b - \mu e^\mu}{e^b - e^a} = \int_\mu^b \frac{f(\alpha)(1+\alpha)}{e^b - e^a}\,d\alpha \leq c\cdot(1-\mu) = \frac{e^b - e^a + 1}{e^b - e^a}(1-\mu), \tag{21}$$

so, in order to minimize $c$, we can derive over the variable $\mu$ and equate to 0, hence

$$\frac{(1+\mu)e^{\mu}}{e^b-e^a}=\frac{e^b-e^a+1}{e^b-e^a}=c. \tag{22}$$

Then we plug in this value for $c$ in (21), so

$$\frac{be^b-\mu e^{\mu}}{e^b-e^a}=\frac{(1+\mu)e^{\mu}}{e^b-e^a}(1-\mu) \implies \frac{be^b}{e^b-e^a}=\frac{(1+\mu-\mu^2)e^{\mu}}{e^b-e^a}.$$

Since we can multiply both sides for $e^b-e^a$ we have $be^b=(1+\mu-\mu^2)e^{\mu}$, which is not $a$-dependent. Since $c-1$ is minimized for $a=0$, plugging this value in (22) we have

$$\frac{(1+\mu)e^{\mu}}{e^b-1}=\frac{e^b-1+1}{e^b-1}=c \implies b=\ln\left(\frac{c}{c-1}\right).$$

We show now why we have equation (16). We have $e^b=(1+\mu)e^{\mu}$, that equals $b=\mu+\ln(1+\mu)$. At the same time, if we substitute $e^b$ and $e^a$ in (21) it follows

$$b(1+\mu)e^{\mu}=(1+\mu-\mu^2)e^{\mu} \qquad \text{that is} \qquad b(1+\mu)=1+\mu-\mu^2$$

$$\text{hence} \qquad b=\frac{1+\mu-\mu^2}{1+\mu}=1-\frac{\mu^2}{1+\mu} \implies 1-\frac{\mu^2}{1+\mu}=\mu+\ln(1+\mu)$$

and we have the equation for $c=\frac{e^b}{e^b-1}$, by the result found before, namely $e^b=(1+\mu)e^{\mu}$, hence it follows

$$c=\frac{e^b}{e^b-1}=\frac{(1+\mu)e^{\mu}}{(1+\mu)e^{\mu}-1}=\frac{1+\mu}{1+\mu-e^{-\mu}}.$$

□

We conlcude this section just pointing out that there is an easy way to derandomize the algorithm, see Section 4.3.

## 5.4 Bounds for Job-Dependent $\alpha_j$ values

Following the previous section we will now prove a better bound for the case of job-dependent $\alpha_j$ and we will present a different deterministic algorithm running in $O(n^2)$ time, see [15].

Let us have a probability distribution over $(0,1]$ with density function

$$g(\alpha) = \begin{cases} (c-1)\cdot e^{\alpha} & \text{if } \alpha \le \gamma + \ln(2-\gamma) = \delta \\ 0 & \text{otherwise} \end{cases} \tag{23}$$

where $\gamma \approx 0.4835$ is a solution to the equation $e^{-\gamma} + 2\gamma + \ln(2-\gamma) = 2$ and $c = 1 + \frac{1}{(2-\gamma)\cdot e^{\gamma}-1} < 1.6853$. Let the $\alpha_j$'s be chosen pairwise independently from a probability distribution over $(0,1]$ with density function (23).

We start by proving a Lemma similar to Lemma 6, where the following inequalities (24) and (25) bound the delay of job $j$ caused, respectively, by jobs in $N_1$ and in $N_2$.

**Lemma 7.** *The function $g$ defined in* (23) *is a density function with the following properties:*

$$\int_0^{\eta} g(\alpha)\cdot(1+\alpha-\eta)\,d\alpha \le (c-1)\cdot\eta \quad \textit{for all } \eta \in [0,1], \tag{24}$$

$$(1+E_g[\alpha])\int_{\mu}^{1} g(\alpha)\,d\alpha \le c\cdot(1-\mu) \quad \textit{for all } \mu \in [0,1]. \tag{25}$$

*where $E_g[\alpha]$ is the expected value of the random variable $\alpha$ distributed according to $g$.*

*Proof.* In order to have a density function we must have $\delta = \ln\left(\frac{c}{c-1}\right)$, since the calculation is identical to that in Lemma 6. We do not prove inequality (24) either because, also here, the proof is identical to the one in Lemma 6. In order to prove inequality (25) the idea is to compute first $E_g[\alpha]$, and then to show the correctness. So we have, by the definition of $\delta$, that $E_g[\alpha]$ equals

$$\int_0^1 \alpha\, g(\alpha)\,d\alpha = (c-1)\int_0^{\delta} \alpha e^{\alpha}\,d\alpha = (c-1)\cdot[\delta e^{\delta} - e^{\delta} + 1] = c\delta - 1.$$

The inequality is certainly true for $\mu \in (\delta, 1]$, while for $\mu \in [0,\delta]$ we have

$$\begin{aligned}(1+E_g[\alpha])\int_{\mu}^{1} g(\alpha)\,d\alpha &= c\delta\cdot(c-1)\int_{\mu}^{\delta} e^{\alpha}d\alpha \\ &= ce^{-\gamma}\left((2-\gamma)e^{\gamma} - e^{\mu}\right) \\ &= c\,(2-\gamma-e^{\mu-\gamma}) \\ &\le c\,(2-\gamma-(1+\mu-\gamma)) = c\,(1-\mu).\end{aligned}$$

□

We can now state and proof the following

**Theorem 10.** *If $\alpha_j$'s are chosen pairwise independently from a probability ditribution over $(0,1]$ with density function $g$ as defined in (23), then the expected value of the resulting $\alpha_j$-schedule is bounded by $c$ times the optimum LP value.*

*Proof.* We start by considering a fixed choice of $\alpha_j$ in such a way to bound the conditional expectation of $C_j^{\boldsymbol{\alpha}}$. Then we integrate over all possible choices to have the desired final bound. For a given job $j$ and a fixed $\alpha_j$ value, we have, according to equation (15) and by the property (24) stated in Lemma 7 that $E_g[C_j^{\boldsymbol{\alpha}}|\alpha_j]$ is less or equal than

$$\begin{aligned} t_j(0^+) + (1+\alpha_j)\cdot p_j + (c-1)\sum_{k\in N_1}\eta_k\cdot p_k + \sum_{k\in N_2:\alpha_j>\mu_k(j)}(1+E_g[\alpha_k])\cdot p_k \\ \leq c\cdot t_j(0^+) + (1+E_g[\alpha_1])\sum_{k\in N_2:\alpha_j>\mu_k(j)} p_k + (1+\alpha_j)\cdot p_j, \end{aligned}$$

where the second inequality follows by (13) and by the equality of the expectations, that is $E_g[\alpha_k] = E_g[\alpha_1]$ for all $k \in N$. Using now the equation (14) for the midpoints and property (25) of Lemma 7 we have

$$\begin{aligned} E_g[C_j^{\boldsymbol{\alpha}}] &\leq c\cdot t_j(0^+) + (1+E_g[\alpha_1])\sum_{k\in N_2} p_k\int_{\mu_k}^1 g(\alpha_j)\,d\alpha_j + (1+E_g[\alpha_j])\cdot p_j \\ &\leq c\cdot t_j(0^+) + \sum_{k\in N_2}(1-\mu_k)\cdot p_k + c\cdot p_j = c\cdot\left(M_j^{LP} + \frac{p_j}{2}\right). \end{aligned}$$

The final result follows easily from the linearity of the expectations. □

To describe a deterministic algorithm we need to know how many possible $(\alpha_j)$-schedules there are. We know that the number of possible orderings of $n$ jobs is $n! = 2^{O(n\log n)}$, but the number of $(\alpha_j)$-schedules is bounded.

**Lemma 8.** *The maximum number of $(\alpha_j)$-schedules is at most $2^{n-1}$ and the bound is tight.*

*Proof.* Let $q_i$ denote the number of different pieces of the job $j$ in the LP schedule, it means $q_j$ is one more than the number of preemptions of job $j$. Since in the LP schedule there are at most $n-1$ preemptions, it follows $\sum_{j=1}^n q_j \leq 2n-1$. The number of $(\alpha_j)$-schedules, denoted by $s$, is bounded by the $q_j$'s. So $s = \prod_{j=1}^n q_j = \prod_{j=2}^n q_j$ since the job with highest $\frac{w_j}{p_j}$ ratio

is never preempted. Hence, just by applying the arithmetic-geometric mean inequality, we have

$$s = \prod_{j=2}^{n} q_j \leq \left( \frac{\sum_{j=2}^{n} q_j}{n-1} \right)^{n-1} \leq 2^{n-1}.$$

The proof of tightness can be easily seen in the instance with $p_j = 2$ and $r_j = n - j$ for all $j$, then we get $q_j = 2$ for $j = 2, \ldots, n$. □

The number of possible $(\alpha_j)$-schedules is exponential, so enumerating all of them does not give a polytime algorithm. Instead, we use a different method, called of conditional probabilities.

**Proposition 8.** *There is a deterministic algorithm that is based on the 1.6853-approximation algorithm, with the same performance guarantee and running time of $O(n^2)$.*

*Proof.* Let us denote the right hand side of inequality (15) by $RHS(\boldsymbol{\alpha})$ and let $\boldsymbol{\alpha} = (\alpha_j)$. Then denoting $\sum w_j RHS_j(\boldsymbol{\alpha}) := V(\boldsymbol{\alpha})$ we already showed, for $c < 1.6853$ that $E_g[\sum_j w_j C_j^{\boldsymbol{\alpha}}] \leq E_g[V(\boldsymbol{\alpha})] \leq c \cdot Z_D$. Let us denote, for each job $j \in N$, by $\mathcal{Q}_j = \{\mathcal{Q}_{j1}, \ldots, \mathcal{Q}_{jq_j}\}$ the set of intervals for $\alpha_j$ corresponding to the $q_j$ pieces of job $j$ in the LP schedule. Let us consider now the jobs, one by one, in arbitrary order, say, without loss of generality, $j = 1, \ldots, n$. Assume that, at the step $j$ of the derandomized algorithm, we identified the intervals $Q_1^d \in \mathcal{Q}_{j1}, \ldots, Q_{j-1}^d \in \mathcal{Q}_{j-1}$ so that

$$E_g[V(\boldsymbol{\alpha})|\alpha_i \in Q_i^d \text{ for } i = 1, \ldots, j-1] \leq c \cdot Z_D.$$

By the use of the conditional expectations, we have that the left hand side of this inequality is

$$\begin{aligned} &E_g[V(\boldsymbol{\alpha})|\alpha_i \in Q_i^d \text{ for } i = 1, \ldots, j-1] \\ &\quad = \sum_{l=1}^{q_j} \Pr(\alpha_j \in Q_{jl}) \cdot E_g[V(\boldsymbol{\alpha})|\alpha_i \in Q_i^d \text{ for } i = 1, \ldots, j-1 \text{ and } \alpha_j \in Q_{jl}]. \end{aligned}$$

Since $\sum_{l=1}^{q_j} \Pr(\alpha_j \in Q_{jl}) = 1$, there exists at least one interval $Q_{jl} \in \mathcal{Q}_j$ such that

$$\begin{aligned} &E_g[V(\boldsymbol{\alpha})|\alpha_i \in Q_i^d \text{ for } i = 1, \ldots, j-1 \text{ and } \alpha_j \in Q_{jl}] \\ &\quad \leq E_g[V(\boldsymbol{\alpha})|\alpha_i \in Q_i^d \text{ for } i = 1, \ldots, j-1]. \end{aligned} \tag{26}$$

Therefore we just have to identify an interval $Q_j^d = Q_{jl}$ such that the inequality (26) is satisfied. Then we may conclude that

$$E_g\left[\sum_{h\in N} w_h C_j^{\boldsymbol{\alpha}} | \alpha_i \in Q_i^d, i = 1,\dots,j\right] \leq E_g[V(\boldsymbol{\alpha})|\alpha_i \in Q_i^d, i = 1,\dots,j] \leq c{\cdot}Z_D.$$

So we determined an interval $Q_j^d$ for every job $j = 1,\dots,n$. Then we have that the $(\alpha_j)$-schedule is the same for all $\boldsymbol{\alpha} \in Q_1^d \times \cdots \times Q_n^d$. So the deterministc objective value is now

$$\sum_{j\in N} w_j C_j^{\alpha} \leq E_g[V(\boldsymbol{\alpha})|\alpha_i \in Q_i^d, i = 1,\dots,n] \leq E_g[V(\boldsymbol{\alpha})] \leq c \cdot Z_D.$$

For every $j = 1,\dots,n$ we need to evaluate $O(n)$ terms for checking if $Q_j^d$ satisfies inequality (26), where each evaluation can be computed in constant time. By Lemma 8 there are at most $2n-1$ intervals, hence the derandomized algotihm runs in $O(n^2)$ time. □

## 5.5 Bad Instances for the LP Relaxations

In this section we will present a family of bad instances, showing that the ratio between the optimum value of the $1|r_j|\sum w_jC_j$ problem and the lower bounds for the LP is arbitrarily close to $\frac{e}{e-1} > 1.5819$.

We can define these instances $I_n$, where $n$ is the number of the jobs, such that there is one large job denoted by $n$, with processing time $p_n = n$, weight $w_n = \frac{1}{n}$ and release date $r_n = 0$. The other $n-1$ jobs, denoted $j = 1,\dots,n-1$, are small, they have 0 processing time, release date $r_j = j$ and weight $w_j = \frac{1}{n(n-1)}\left(1+\frac{1}{n-1}\right)^{n-j}$. If zero processing times are not allowed it is just sufficient to impose $1/k$ as processing time of the small jobs and multiply all the processing times and the release dates by $k$, and then let $k$ tend to infinity. But for the sake of simplicity we assume that zero processing times are possible. In the LP solution job $n$ starts at time 0 and it is preempted by each of the small jobs, hence it follows that $M_n^{LP} = \frac{n}{2}$ and $M_j^{LP} = r_j = j$ for the small jobs $j = 1,\dots,n-1$. Thus we can calculate the objective value, so $Z_R = \left(1+\frac{1}{n-1}\right)^n - \left(1+\frac{1}{n-1}\right)$.

Consider now an optimal nonpreemptive schedule $C^*$ and let $k = \lfloor C_n^* \rfloor$, it means that $k$ is the number of small jobs processed before $n$. It would be optimal to process these small jobs $j = 1,\dots,k$ at their release dates and

start processing $n$ at $r_k = k$. Then the remaining jobs would be processed at time $k+n$, hence after job $n$. Calling the obtained schedule $C^k$ we have $C_j^k = j$ for $j \leq k$ and $C_j^k = n+k$ otherwise.

The objective value of $C^k$ is $\left(1+\frac{1}{n-1}\right)^n - \frac{1}{n-1} - \frac{k}{n(n-1)}$.

The optimum schedule is $C^{n-1}$ with value $\left(1+\frac{1}{n-1}\right)^n - \frac{1}{n-1} - \frac{1}{n}$. Since $n$ grows large we have that the optimum nonpreemptive cost tends to $e$, while the LP objective value is closer to $e-1$, thus we have the desired ratio.

# 6 Approximations for $1|r_j|\sum C_j$

In this section we will study the problem $1|r_j|\sum C_j$, that can be seen as the problem of Section 5 with uniform weights. Chekuri et al., in [10], studied some $\alpha$-schedule in a different way from the previous sections, we will present their result, namely an $\frac{e}{e-1}$-approximation algorithm for $1|r_j|\sum C_j$. We will also show some examples showing the tightness of the results, see [10] and the example of Torng and Uthaisombut in [44].

## 6.1 $\frac{e}{e-1}$-approximation algorithm

We already proved in Theorem 5 that we have an $f(\alpha)$-approximation algorithm, for $f(\alpha) = \max\{1 + \frac{1}{\alpha}, 1 + 2\alpha\}$. In the following, we prove a similar statement in a different way and then we refine the analysis by the use of some random $\alpha$-points, in such a way to create the structure for the main result of this chapter.

Let $C_i^P$ and $C_i^\alpha$ be the completion times of job $i$ in a preemptive schedule $P$ and in a nonpreemptive $\alpha$-schedule $S^\alpha$ derived from $P$, respectively.

**Theorem 11.** *Let us consider the problem $1|r_j|\sum C_j$. For any $\alpha \in (0,1]$ an $\alpha$-schedule satisfies $\sum C_j^\alpha \leq \left(1 + \frac{1}{\alpha}\right) \sum C_j^P$.*

*Proof.* Given a preemptive schedule $P$, let us reindex the jobs in order of $\alpha$-point, i.e. $t_1 \leq t_2 \leq \cdots \leq t_n$, and let $r_j^{max} = \max_{1\leq k\leq j} r_k$. By maximality we have that $r_j^{max} \geq r_k$ for $k = 1, \ldots, j$, hence

$$C_j^\alpha \leq r_j^{max} + \sum_{k=1}^{j} p_k. \tag{27}$$

Since by $r_j^{max}$ only an $\alpha$ fraction of job $j$ has finished we have $C_j^P \geq r_j^{max}$. We also know, by the ordering, that jobs $k \leq j$ are processed for at least $\alpha \cdot p_k$ before $C_j^P$, hence $C_j^P \geq \alpha \sum_{k=1}^{j} p_k$. Substituting these two inequalities in (27) yields $C_j^\alpha \leq \left(1 + \frac{1}{\alpha}\right) C_j^P$. Thus we just have to sum over all the jobs to obtain the statement. □

In the following the conversion applies in general, so, in order to prove upper bounds on the performance ratio for the sum of completion times, we assume the preemptive schedule to be optimal. Hence we have a lower bound on any optimal nonpreemptive schedule.

We are now ready to define the main ideas on which the proof is based: let $S_i^P(\beta)$ denote the set of jobs that complete exactly a $\beta$ fraction of their

processing time at time $C_i^P$, or the sum of the processing times of the jobs in this set (it will be clear case by case). Clearly $i \in S_i^P(1)$. We define $\tau_i$ as the idle time before $C_i^P$.

**Lemma 9.** *The completion time of job $i$ in the preemptive schedule $P$ equals $C_i^P = \tau_i + \sum_{0<\beta\leq 1} \beta\, S_i^P(\beta)$.*

*Proof.* The completion time of job $i$ corresponds to the sum of the idle time before $C_i^P$ and the fractional processing times before $C_i^P$. □

**Lemma 10.** *We can bound the completion time in the $\alpha$-schedule by*

$$C_i^\alpha \leq \tau_i + (1+\alpha) \sum_{\beta\geq\alpha} S_i^P(\beta) + \sum_{\beta<\alpha} \beta\, S_i^P(\beta).$$

*Proof.* We give a procedure converting the preemptive schedule $P$ to a different schedule $Q$ such that:

($c_1$) jobs $1, \ldots, i-1$ run nonpreemptively in that order,

($c_2$) the remaining jobs can run preemptively

($c_3$) $C_i^Q$ satisfies the bound given in the lemma.

The proof of the lemma follows from the fact that $C_i^\alpha \leq C_i^Q$. By Lemma 9 we have

$$C_i^P = \tau_i + \sum_{\beta<\alpha} \beta\, S_i^P(\beta) + \sum_{\beta\geq\alpha} \alpha\, S_i^P(\beta) + \sum_{\beta\geq\alpha} (\beta-\alpha)\, S_i^P(\beta). \tag{28}$$

Let us partition the set on jobs into $J^B := \bigcup_{\beta\geq\alpha} S_i^P(\beta)$ and $J^A = N - J^B$. So the equation (28) can be seen as the sum of:

($p_1$) the idle time in $S^P$ before $C_i^P$,

($p_2$) the pieces of jobs in $J^A$ running before $C_i^P$,

($p_3$) the pieces of $j \in J^B$ running before $t_j$,

($p_4$) the pieces of $j \in J^B$ running in $[t_j, C_i^P]$.

Let us denote by $x_j$ the fraction of job $j$ completed before $C_i^P$, that is the $\beta$ for which $j \in S_i^P(\beta)$. So $(x_j - \alpha)\, p_j$ is just the fraction of $j$ done in the interval $[t_j, C_i^P]$, hence we can write $\sum_{j\in J^B}(x_j-\alpha)\, p_j$ instead of $\sum_{\beta\geq\alpha}(\beta-\alpha)\, S_i^P(\beta)$. Let now $J^C = \{1, \ldots, i\} =: [i]$ be a subset of $J^B$ and consider $P$ as an ordered list of pieces of jobs. Then for each $j \in J^C$ we:

- remove all pieces of the jobs running in $[t_j, C_i^P]$
- insert a piece of size $(x_j - \alpha)p_j$ at time $t_j$.

So for jobs $j = 1, \ldots, i$ we have pieces of size $(x_j - \alpha)\, p_j$ in the correct order. Then we define a schedule from this list by scheduling the pieces of the jobs in the list order, respecting release dates. Obviously job $i$ still completes at $C_i^P$ since the pieces of jobs different by $(x_j - \alpha)\, p_j$ are moved later in time, so we don't add idle time. Then the algorithm extends the pieces of jobs by adding $p_j - (x_j - \alpha)\, p_j$, in such a way to have, for each job $j$, only one piece of size $p_j$. The pieces of $j$ processed earlier, in total $\alpha\, p_j$, are replaced by idle time. The resulting schedule is the desired $Q$, in which the jobs $1, \ldots, i$ are scheduled nonpreemptively for their processing time. Then the completion time of $i$ is:

$$C_i^Q = C_i^P + \sum_{j \in J^C} (p_j - (x_j - \alpha)\, p_j) \leq C_i^P + \sum_{j \in J^B} (p_j - (x_j - \alpha)\, p_j) =$$
$$= C_i^P + \sum_{\beta \geq \alpha} (1 - \beta + \alpha)\, S_i^P(\beta) = \tau_i + (1 + \alpha) \sum_{\beta \geq \alpha} S_i^P(\beta) + \sum_{\beta < \alpha} \beta\, S_i^P(\beta),$$

where the last equation follows by substituting the value of $C_i^P$ as in Lemma 9. The remaining pieces in the schedule are all from jobs in $N - J^C$, so we are done since it satisfies $(c_1), (c_2)$ and $(c_3)$. □

We now restate the procedure used in Lemma 10 in such a way to define a conversion algorithm, called $Q_\alpha$:

- let $J^B = \cup_{\beta \geq \alpha} S_i^P(\beta)$ and, for a fixed $i$, $J^C = \{1, \ldots, i\} \subseteq J^B$,
- for each $j \in J^C$ remove all pieces of jobs running in $[t_j, C_i^P]$,
- insert a piece of size $(x_j - \alpha) \cdot p_j$ at the point corresponding to $t_j$,
- extend the piece of job $j$ of size $(x_j - \alpha) \cdot p_j$ to a piece of size $p_j$,
- replace the other pieces of job $j$ by idle time.

This algorithm has its full strength with a random choice of $\alpha$.

**Lemma 11.** *Let $\alpha$ be drawn randomly from $(0, 1]$ over a probability distribution with density function $f$. Let $\delta = \max_{0<\beta\leq 1} \int_0^\beta \frac{1+\alpha-\beta}{\beta} f(\alpha)\, d\alpha$, then, for each job $i$ we have $E[C_i^\alpha] \leq (1 + \delta)\, C_i^P$.*

*Proof.* By lemma 10 we have

$$C_i^\alpha \leq \tau_i + (1+\alpha)\sum_{\beta\geq\alpha} S_i^P(\beta) + \sum_{\beta<\alpha} \beta S_i^P(\beta).$$

Hence choosing $\alpha$ accordingly to $f$, since $\tau$ does not depend on $\alpha$ we have $E[C_i(\alpha)] =$

$$\begin{aligned}
\int_0^1 f(\alpha)\, C_i^\alpha \, d\alpha &\leq \tau_i + \int_0^1 f(\alpha) \left( (1+\alpha)\sum_{\beta\geq\alpha} S_i^P(\beta) + \sum_{\beta<\alpha} \beta\, S_i^P(\beta) \right) d\alpha \\
&= \tau_i + \sum_{0<\beta\leq 1} S_i^P(\beta) \left( \int_0^\beta (1+\alpha)\, f(\alpha)\, d\alpha + \int_\beta^1 \beta\, f(\alpha)\, d\alpha \right) \\
&= \tau_i + \sum_{0<\beta\leq 1} \beta\, S_i^P(\beta) \left( 1 + \int_0^\beta \frac{1+\alpha-\beta}{\beta} f(\alpha)\, d\alpha \right) \\
&\leq \tau_i + \left( 1 + \max_{0<\beta\leq 1} \int_0^\beta \frac{1+\alpha-\beta}{\beta} f(\alpha)\, d\alpha \right) \sum_{0<\beta\leq 1} \beta\, S_i^P(\beta) \\
&\leq \tau_i + (1+\delta) \sum_{0<\beta\leq 1} \beta\, S_i^P(\beta) \leq (1+\delta)\, C_i^P.
\end{aligned}$$

Then, by the linearity of expectations, we have the desired result. □

With the next lemma we will prove some bound for specific density function over $(0,1]$.

**Theorem 12.** *Let $\alpha$ be chosen accordingly to a specific probability distribution, then:*

1. *if $\alpha$ is chosen uniformly in $(0,1]$ the expected approximation ratio is at most 2,*
2. *if $\alpha$ is drawn from $(0,1]$ accordingly to the density function $f(\alpha) = \frac{e^\alpha}{e-1}$, then the expected approximation ratio is at most $\frac{e}{e-1} \approx 1.58$.*

*Proof.* By Lemma 11 we want to bound the value $1+\delta$, so:

1. in this case we have $f(\alpha) = 1$, from which it follows $\delta =$

   $$\delta = \max_{0<\beta\leq 1} \int_0^\beta \frac{1+\alpha-\beta}{\beta}\, d\alpha = \max_{0<\beta\leq 1} \left( 1 - \beta + \frac{\beta}{2} \right) = \max_{0<\beta\leq 1} \left( 1 - \frac{\beta}{2} \right),$$

   hence $\delta \leq 1$, and it follows $1+\delta \leq 2$.

2. in this case we have $f(\alpha) = \frac{e^\alpha}{e-1}$, hence

$$\begin{aligned}
\delta &= \max_{0<\beta\leq 1} \int_0^\beta \left(\frac{1-\beta+\alpha}{\beta}\right)\left(\frac{e^\alpha}{e-1}\right) d\alpha = \\
&= \max_{0<\beta\leq 1} \frac{1}{\beta\,(e-1)} \left[e^\alpha - \beta\, e^\alpha + \alpha\, e^\alpha - e^\alpha\right]_0^\beta = \\
&= \max_{0<\beta\leq 1} \frac{1}{\beta\,(e-1)} \left[e^\beta - 1 - \beta\, e^\beta + \beta + \beta\, e^\beta - e^\beta + 1\right] = \\
&= \max_{0<\beta\leq 1} \frac{1}{\beta\,(e-1)}\,\beta = \max_{0<\beta\leq 1} \frac{1}{e-1}
\end{aligned}$$

so $1+\delta \leq 1 + \frac{1}{e-1} = \frac{e}{e-1}$.

□

As we proved in Lemma 5, for a given preemptive schedule, there are at most $n$ combinatorially distinct values of $\alpha$. Since each of them can be computed in $O(n)$ time and the fractional schedule can be computed in $O(n \log n)$ time we have the following:

**Theorem 13.** *There is a deterministic $\frac{e}{e-1}$-approximation algorithm for $1|r_j|\sum C_j$ running in $O(n^2)$ time.*

## 6.2 Tightness

**Proposition 9.** *There are instances in which the bound given by Theorem 11 is asymptotically tight.*

*Proof.* Let $\epsilon > 0$. In the desired instance there are $n+2$ jobs: job 1 such that $r_1 = 0$ and $p_1 = 1$, job 2 such that $r_2 = \alpha - \epsilon$ and $p_2 = \epsilon$ and job $i$, $i = 3, \ldots, n+2$ such that $r_i = \alpha + \epsilon$ and $p_i = 0$. The optimal preemptive completion time is $\alpha + n\,(\alpha+\epsilon) + 1 + \epsilon$, but the completion time of the derived nonpreemptive $\alpha$-schedule is $\alpha + (\alpha+1) + n\,(1+\alpha)$. So for big $n$ and small $\epsilon$ the ratio tends to $1 + \frac{1}{\alpha}$. □

**Theorem 14.** *There are instances in which the bound given by Theorem 12 is asimptotically tight.*

*Proof.* In order to prove the statement, we just present the instances and give the results, without explaining in detail the results:

- Let us have $n+1$, jobs $j$, for $j \in [n]$, are such that $r_j = \delta < 1$ and $p_j = 0$, and job $n+1$ so that $p_{n+1} = 1$ and $r_{n+1} = 0$. The optimal nonpreemptive schedule has a completion time $C^* = n\,\delta + (1+\delta)$. There are only two combinatorially distinct values of $\alpha$, so either $\alpha \leq \delta$ or $\alpha > \delta$. If $\alpha$ is chosen uniformly random, the algorithm output $C' = \delta\,(1+n) + (1-\delta)\,(1+\delta+n\,\delta)$. For $n \gg 1 \gg \delta$ the ratio between $C'$ and $C^*$ tends to 2.

- Let us have the following class of instances: $r_1 = 0, r_j = 1+(n-j+1)\,\epsilon$ for $i = 2, \ldots, \lceil \frac{n}{e} \rceil - 1$ and $r_j = \sum_{k=j}^{n} \frac{1}{k} + (n-j)\,\epsilon$ for $j = \lceil \frac{n}{e} \rceil, \ldots, n$ and $p_1 = 1+\epsilon, p_j = \epsilon$ for $j = 2, \ldots, n$. Then it follows that the algortihm computes $C' \geq n + \left(\frac{n\,(n+1)}{2}\right)\epsilon$, while the otpimum value is

$$C^* < \left(\frac{e-1}{e}\right) n + \left(2 - \frac{1}{e}\right) + \left(\frac{n\,(n+1)}{2}\right)\epsilon + 1 + \left\lfloor \frac{n}{e} \right\rfloor \epsilon.$$

  For any integer $N > 5$ there is an integer $n \geq 6$ such that we have $N - 4 \leq n \leq N$, $0 < \epsilon < \frac{1}{n}$ and $\sum_{k=\lceil n/e \rceil}^{n} \frac{1}{k} < 1$. Hence for any $\delta > 0$ there exists some $\epsilon > 0$ that makes the ratio $\frac{C'}{C^*} \geq \frac{e}{e-1} - \delta$, so it can be made arbitrarily close to $\frac{e}{e-1}$.

$\square$

# 7 Approximation for $1|r_j, prec|\sum w_jC_j$

We now consider the more general problem $1|r_j, prec|\sum w_jC_j$, so there is a partial order defined by "$\prec$", meaning that, whenever $j \prec k$, we need to complete job $j$ before to start job $k$. It is obvious that for $j \prec k$ we can assume $r_j \leq r_k$, or more specifically $r_j + p_j \leq r_k$.

In this section we will first explain a general algorithm that is called list scheduling. Then we will present the recent result of Skutella, in [41], where he obtained a $\frac{\sqrt{e}}{\sqrt{e}-1}$-approximation algorithm. So we will give the LP relaxation used and we will show a "nice" trick used by the author.

Since the problems $1|r_j|\sum C_j$, so with $w \equiv 1$ and no precedence constraints, and $1|prec|\sum C_j$, so with $w = 1$ and $r = 0$, are strongly NP-hard also the problem $1|r_j, prec|\sum w_jC_j$ is strongly NP-hard. If we would allow preemptions we could solve optimally in polynomial time $1|r_j, pmtn|\sum C_j$, but even the problems $1|r_j, pmtn|\sum w_jC_j$ and $1|prec, pmtn|\sum C_j$, that is equivalent to $1|prec|\sum C_j$, are strongly NP-hard. We have to note, that Bansal and Khot proved in [5] a bound on the possible approximability, that is, by using a stronger version of the Unique Games Conjecture, there are no $(2-\epsilon)$-approximation algorithm for the problem $1|prec|\sum w_jC_j$. Another interesting fact is the relation between approximability of $1|prec|\sum w_jC_j$ and the vertex cover problem, see [1], [2] and [12].

## 7.1 LP Relaxation

We define a *list schedule* to be a schedule that processes the jobs as early as possible, with respect to the release dates, in a given order. This order is given by a total ordering that extends (if any) the partial order given by $\prec$. If we are in the preemptive case we can define the preemptive list scheduling such that we process at any point in the time the first available job, in this way we obtain a preemptive list schedule.

From the problem $1|r_j, prec|\sum w_jC_j$ we can obtain the preemptive relaxation $1|r_j, prec, pmtn|\sum w_jC_j$ for which it is known a 2-approximation algorithm, see [19].

Let $S \subseteq N$ be a set of jobs and recall, as we defined in the previous sections, that $p(S) = \sum_{j\in S} p_j$ and $r_{min}(S) = \min_{j\in S} r_j$. Then using the

variables $C_j$ for $j \in N$ we can define the relaxation:

$$\min \quad \sum_{j \in N} w_j C_j$$

subject to

$$C_j + p_k \leq C_k \qquad \text{for all } j \prec k \qquad \text{(s1)}$$

$$\frac{1}{p(S)} \sum_{j \in S} p_j C_j \geq r_{min}(S) + \frac{p(S)}{2} \qquad \text{for all } S \subseteq N \qquad \text{(s2)}$$

Any optimal solution $C^*$ of the LP relaxation can be found in polynomial time, because, also if there are exponentially many constraints, they can be separated in polynomial time by efficient submodular minimization algorithm, see [14]. Then we have that $\sum w_j C_j^*$ is a lower bound for the sum of weighted completion times over the preemptive schedules. Let us now reindex the jobs in order to have

$$C_1^* \leq C_2^* \leq \cdots \leq C_n^* \tag{29}$$

By the LP formulation it is possible that there are some jobs $j$ and $k$ such that $j \prec k$ but $C_j^* = C_k^*$, but, in order to avoid this degenerate case, we assume the processing time to be nonzero, i.e. $p_j > 0$ for $j \in N$.

**Lemma 12.** *Given the ordering of jobs obtained by the reindexing* (29) *and a double speed machine, if we apply the preemptive list scheduling algorithm we obtain a feasible preemptive schedule with completion times $C'$ such that $C'_j \leq C_j^*$ for all $j \in N$.*

*Proof.* To prove the theorem we define, for any fixed job $j \in N$, a specific subset of jobs $S = \{k : k \leq j\} \subseteq N$ satisfying:

1. $C'_k \leq C'_j$,
2. there is no idle time in $[C'_k, C'_j]$ by the preemptive list scheduling,
3. in $[C'_k, C'_j]$ only jobs $l \leq j$ are processed.

The idea is to prove a nice property of set $S$, namely that jobs in $S$ are processed continuously in $I := [r_{min}(S), C'_j]$, so let $h \in S$ be the job with $r_h = r_{min}(S)$. By condition (s1) and since $p_j > 0$ we can assume that $l \prec h$ implies $r_l < r_h$. So the algorithm processes only jobs $l \leq h \leq j$ and it does not leave idle time in the interval $[r_h, C'_h]$. By the third condition on the set $S$, in $[C'_h, C'_j]$ only jobs $l \leq j$ are processed and, by the second one, there is

no idle time. Hence we have that in $I$ there is no idle time and only jobs $l \leq j$ are processed. Any job $l$ processed in $I$, since $l < j$, is also completed in $I$, i.e. $C'_l \in I$. Hence $C'_l$ satisfies all the conditions and, therefore, it is contained in $S$.

Hence, since the jobs run on a double speed machine, it follows that $C'_j = r_{min}(S) + \frac{p(S)}{2}$, from which we have

$$C^*_j \geq \frac{1}{p(S)} \sum_{k \in S} p_k C^*_k \geq r_{min}(S) + \frac{p(S)}{2} = C'_j,$$

since $k \leq j$ implies $C^*_k \leq C^*_j$ and by the LP constraints (s2). □

**Theorem 15.** *Given the LP lower bound $\sum w_j C^*_j$ of a preemptive schedule on the optimal total weighted completion time. Then, using a double speed machine we can obtain, in polynomial time, a preemptive list schedule whose total weighted completion time satisfies $\sum w_j C'_j \leq \sum w_j C^*_j$.*

*Proof.* By the LP relaxation we can find in polynomial time an optimal solution $C^*$, we then apply the preemptive list scheduling algorithm on the order given by the relation (29) on a double speed machine. By Lemma 12 it follows $\sum w_j C'_j \leq \sum w_j C^*_j$. □

## 7.2 2.54-approximation algorithm

In this section we will do list scheduling in order of $\alpha$-points. The definition of $\alpha$-point for schedules on regular speed machine is known and it is easy to generalize the same parameter for a double speed machine: the $\alpha$-point $t'_j(\alpha)$, or just $t'_j$ for common $\alpha$, of job $j$ with respect to schedule $S'$ is the first time when $\alpha \cdot \frac{p_j}{2}$ units of time of job $j$ have been processed on the double speed machine. Let now $S^\alpha$ be the schedule obtained by processing jobs in order of increasing $t'_j(\alpha)$ on a regular speed machine. Since $S'$ is feasible this order respect the precedence contraints. We denote, as usual, by $C^\alpha_j$ the completion time of job $j$ in the list schedule $S^\alpha$. To be precise we redefine also the parameter $\eta$ for the double speed machine, so, for a fixed job $k$, $\eta_j(k)$ (or just $\eta_j$ when job $k$ is clear from the context) is the fraction of job $j$ done on the double speed machine by time $C'_k$. We have

$$C'_k \geq \sum_{j \in N} \eta_j \frac{p_j}{2} \tag{30}$$

**Lemma 13.**

$$C^\alpha_k \leq C'_k + \sum_{j:\eta_j \geq \alpha} \left(1 + \frac{\alpha - \eta_j}{2}\right) \cdot p_j.$$

*Proof.* Let $X$ be the subset of all jobs $j$ scheduled by $S_\alpha$ not later than $k$ for which in $[C_j^\alpha, C_k^\alpha]$ there is no idle time, clearly $k \in X$. Let us denote by $l$ the first job scheduled in $X$, hence $s_l = r_l$, where $s_l$ is the starting time of job $l$. We have, by the definition of the set $X$, that:

$$C_k^\alpha = r_l + \sum_{j \in X} p_j. \tag{31}$$

Considering now the schedule $S'$, by feasibility, for every job $j \in X$ we have that $r_l \leq t'_l \leq t'_j$ and, by definition of $\eta_j$, that the double speed machine processes $j$ in $[t'_j, C'_k]$ for exactly $\frac{1}{2}(\eta_j - \alpha)\, p_j$. Since a machine cannot schedule more than one job at any time we have

$$C'_k \geq r_l + \sum_{j \in X} \frac{\eta_j - \alpha}{2}\, p_j. \tag{32}$$

Now we just have to combine equation (31) with (32) in order to get

$$C_k^\alpha \leq C'_k + \sum_{j \in X} \left(1 + \frac{\alpha - \eta_j}{2}\right) p_j. \tag{33}$$

Since $t'_j \leq t'_k$ we have $\eta_j \geq \alpha$ for $j \in X$, then $X \subseteq \{j : \eta_j \geq \alpha\}$ and, by the nonnegativity of $\alpha$, $\left(1 + \frac{\alpha - \eta_j}{2}\right)$ is at least $\frac{1}{2}$, hence we have

$$C_k^\alpha \leq C'_k + \sum_{j \in X} \left(1 + \frac{\alpha - \eta_j}{2}\right) p_j \leq C'_k + \sum_{j : \eta_j \geq \alpha} \left(1 + \frac{\alpha - \eta_j}{2}\right) \cdot p_j.$$

□

**Theorem 16.** *Let $S'$ be a feasible preemptive schedule on a double speed machine and denote the completion time of job $j$ in $S'$, for every $j \in N$, by $C'_j$. It is possible to obtain, in polynomial time, a feasible nonpreemptive schedule on a regular speed machine with completion times $C_j$ so that*

$$\sum_{j \in N} w_j C_j \leq \frac{\sqrt{e}}{\sqrt{e} - 1} \sum_{j \in N} w_j C'_j.$$

*Proof.* In order to prove the theorem we define the value of $\alpha$ by a suitable probability distribution and then we schedule the jobs on the regular speed

machine in order of $\alpha$-points with respect to the schedule $S'$. Let $\alpha \in (0,1]$ be drawn randomly according to the density function

$$f(\alpha) := \frac{e^{\alpha/2}}{2(\sqrt{e}-1)}.$$

For $0 \leq \eta \leq 1$, we have

$$\int_0^\eta f(\alpha)\left(1+\frac{\alpha-\eta}{2}\right)d\alpha = \int_0^\eta \frac{2-\eta+\alpha}{2}\, e^{\alpha/2}\, d\alpha = \frac{\eta}{2(\sqrt{e}-1)}.$$

By Lemma 13 and by inequality (30) it follows

$$\begin{aligned} E[C_k^\alpha] &\leq C'_k + \sum_{j\in N} p_j \int_0^{\eta_j} f(\alpha)\left(1+\frac{\alpha-\eta_j}{2}\right)d\alpha \\ &= C'_k + \frac{1}{\sqrt{e}-1}\sum_{j\in N}\eta_j\,\frac{p_j}{2} \leq \frac{\sqrt{e}}{\sqrt{e}-1}\,C'_k. \end{aligned}$$

Then, by linearity of expectations, it follows that, in the nonpreemptive list schedule in order of random $\alpha$-points, the sum of weighted completion times is within a factor of $\frac{\sqrt{e}}{\sqrt{e}-1}$ times the sum of weighted completion times in $S'$. As we proved in Theorem 3 there are at most $n$ combinatorially different values of $\alpha$, each of them can be computed in $O(n)$ time, so in total the running time is $O(n^2)$. □

**Corollary 3.** *For the scheduling problem $1|r_j, prec|\sum w_j C_j$, any optimal solution of the LP relaxation defined in Chapter 7.1 is at most a factor $\frac{\sqrt{e}-1}{\sqrt{e}}$ smaller than the total weighted completion time of an optimal schedule.*

*Proof.* Let $C'$ denotes the vector of completion times in the preemptive double speed machine schedule $S'$ and let $C^*$ be an optimal LP solution. Then by Theorems 15 and 16 the algorithm constructs a feasible schedule in which the sum of weighted completion times is bounded by

$$\sum_{j\in N} w_j C_j \leq \frac{\sqrt{e}}{\sqrt{e}-1}\sum_{j\in N} w_j C'_j \leq \frac{\sqrt{e}}{\sqrt{e}-1}\sum_{j\in N} w_j C^*_j.$$

We complete the proof just by orserving that $\sum w_j C_j$ is an upper bound of the value for an optimal schedule. □

# 8 Approximations for $P|r_j|\sum C_j$

In this section we will first show an easy 3-approximation algorithm for the problem $P|r_j|\sum C_j$, given by Chekuri et al. in [10]. In the same paper the authors developed the DELAY-LIST algorithm that produces, given any $\rho$-approximation for the weighted sum of completion times on a single machine, a $((1+\beta)\rho + (1+1/\beta))$-approximation guarantee for the general $m$-machines case. Note that the DELAY-LIST algorithm works even with precedence constraints. At the end, we will present the 2.83-approximation algorithm obtained by combining the DELAY-LIST and a list scheduling algorithm, see [10].

## 8.1 3-approximation algorithm

Let $M$ be an instance for a nonpreemptive scheduling on $m$ machines, then we can define a preemptive one-machine instance $I$ on the same set of jobs with $p'_j = \frac{p_j}{m}$ and $r'_j = r_j$.

**Lemma 14.** *Let $M^*$ be an optimal value for the instance $M$ and $I^*$ be an optimal value for $I$. Then $I^* \leq M^*$.*

*Proof.* Let $S^m$ be a feasible schedule for the input $M$. Then it is possible to convert $S^m$ to a feasible schedule $S^I$ for input $I$, without increasing the average completion time. Consider a time unit $t$ sufficiently small and denote, for $S^m$, the $k \leq m$ jobs running at time $t$ by $1, \ldots, k$. For each job $1, \ldots, k$ at time $t$, in $S^I$, we can run $\frac{1}{m}$ units of each jobs. It follows that, for any job $j$ the completion time in $S^I$, $C_j^I$, is not greater than the completion time in $S^m$, denoted $C_j^m$. □

Let $S^I$ be an optimal preemptive schedule for $I$, then we can create a list schedule $S^m$ by ordering the jobs by increasing value of $C_j^I$ and scheduling them in that order, in a nonpreemptive way, respecting the release dates. Let $C_j^*$ be completion time of $j$ in an optimal schedule for $M$, then, despite $I$ could be a bad relaxation, we can derive a good approximation guarantee.

**Lemma 15.** *Given a list of jobs in order of increasing $C_j^I$, let $S^m$ be the nonpreemptive list schedule generated, then we have $\sum C_j^m \leq \left(3 - \frac{1}{m}\right) \sum C_j^*$.*

*Proof.* We can assume (by reindexing) that the jobs are ordered by completion times in $S^I$, so $C_1^I \leq C_2^I \leq \cdots \leq C_n^I$, i.e job $j$ is the $j$-th job completed

in $S^I$. Then we have two easy bounds $C_j^I \geq r'_j + p'_j$ and

$$C_j^I \geq \sum_{k=1}^{j} p'_k = \sum_{k=1}^{j} \frac{p_k}{m}. \tag{34}$$

We can derive another trivial bound after defining $r_j^{max} := \max_{k\in[j]} r_k$, namely $C_j^I \geq r_j^{max}$. By definition we know that all the jobs $k \leq j$ have been released at time $r_j^{max}$, so we can bound the completion time of job $j$ in $S^m$:

$$C_j^m \leq r_j^{max} + \sum_{k=1}^{j-1} \frac{p_k}{m} + p_j \leq C_j^I + C_j^I + p_j \left(1 - \frac{1}{m}\right), \tag{35}$$

by inequality (34) and since $C_j^I \geq r_j^{max}$. Then if we sum (35) over all jobs, by Lemma 14, we get

$$\sum_j C_j^m \leq 2 \cdot \sum_j C_j^I + \left(1 - \frac{1}{m}\right) \sum_j p_j \leq \left(3 - \frac{1}{m}\right) \sum_j C_j^* \tag{36}$$

since we know $\sum C_j^* \geq \sum p_j$. □

## 8.2 The DELAY-LIST algorithm

In this section we first sketch the DELAY-LIST algorithm and then we give a detailed explanation.

We can derive a list from the one-machine schedule by ordering the jobs in order of nondecreasing completion times. The precedence constraints prohibit the complete parallelization, so, in order to benefit from the parallelism, we may process jobs out-of-order. A job is scheduled in *order* whenever it is scheduled as first of the list, otherwise it is scheduled *out-of-order*. If the $p_j$'s are identical we could schedule jobs out-of-order without any problem, otherwise a job could delay "better" jobs that would become avaliable soon, so we have to balance this effects. We will schedule jobs out-of-order only if there is enough idle time to justify this change in the list.

We start by giving some definitions that will be used throughout this section. For any job $j$ we can define recursively the value of the *smallest feasible completion time* of $j$, namely $\kappa_j$, as $r_j + p_j$ if the job has no predecessors and $p_j + \max\{r_j, \max_{i\prec j} \kappa_i\}$ otherwise. A path $P$ from $i$ to $j$ is called *critical path to* $j$ if $p(P) = \kappa_j$. If we fix a schedule $S$ we can denote by $q_j^S$

the time at which job $j$ is ready (released and all predecessors completed) and by $s_j^S$ the starting time of $j$ in the schedule $S$.

We can now describe the algorithm where, for the sake of simplicity, we assume $p_j$ to be integers and to have discrete time. In the algorithm the idle time is *charged* on the jobs, in such a way to have a rule to decide when to schedule a job out-of-order. Let $\beta > 0$ be a fixed constant and suppose the jobs are ordered on a list, then at each time step $t$ the algorithm applies one of the following:

($a$) if $M$ is an idle machine and the first job on the list $j$ is ready at time $t$, then schedule $j$ on $M$ and charge all uncharged idle time in $(q_j, s_j)$ to $j$,

($b$) if $M$ is an idle machine and the first job on the list $j$ is not ready at time $t$, then we consider the first ready job $k$ in the list: if there is at least $\beta\, p_k$ uncharged idle time among all machine then we schedule $k$ and we charge $\beta\, p_k$ idle time on $k$,

($c$) if either there is no idle time or the previous cases do not apply we merely increment $t$.

For any job $j$ we partition $N \setminus \{j\}$ in $A_j = \{k \text{ after } j \text{ in the list}\}$ and in $B_j = \{k \text{ before } j \text{ in the list}\}$. We also denote by $O_j$ the set of jobs that are after $j$ on the list but are scheduled before job $j$, and we know $O_j \subseteq A_j$.

We can see that the algorithm considers only the first ready job in the list, even if jobs ordered after it in the list are ready.

Given a schedule $S$, for each job $j$, we can define a path $P'_j = k_1, \ldots, k_l = j$, where $k_i$ is the largest completion time predecessor of $k_{i+1}$ and ties are broken arbitrarily. Obviously job $k_1$ does not have predecessors which satisfiy the condition.

**Lemma 16.** *Given a schedule $S$, we can find the path $P'_j$. The jobs in $P'_j$ define a disjoint set of time intervals $(0, r_{k_1}], (s_{k_1}, C_{k_1}], \ldots, (s_{k_l}, C_{k_l}]$. Let $\kappa'_j$ be the sum of lengths of these intervals, then $\kappa'_j \leq \kappa_j$.*

*Proof.* It is easy to see that $s_k = \max\{r_k, C_{k-1}\}$ cannot be greater than $\max\{r_k, \max_{i \prec k} \kappa_i\}$, then, by definition of $\kappa_i$, we have the desired result. □

**Lemma 17.** *The algorithm charges on each job $j$ at most $\beta\, p_j$ time.*

*Proof.* By the algorithm if case $(b)$ applies we are done. So suppose $j$ is ready at $q_j$ and was not scheduled before according to case $(b)$, then the idle time in $(q_j, s_j)$ is not greater than $\beta\, p_j$. For this proof we assume the time to be continuous, so $j$ is scheduled at the first time when $\beta\, p_j$ units of idle time have been accumulated, otherwise the algorithm might charge some extra idle time by the integrality. □

**Lemma 18.** *For each job $j$, there is no uncharged idle time in the interval $(q_j, s_j)$. Moreover all the idle time is charged to jobs in $B_j$.*

*Proof.* By the algorithm no jobs in $A_j$ are scheduled in $(q_j, s_j)$ since $j$ was ready at $q_j$, hence the idle time is charged on $B_j \cup \{j\}$. Since $j$ is ready at $q_j$, but started only at $s_j$, by cases $(a)$ and $(b)$ of the algorithm there is no uncharged idle time to $j$. □

**Lemma 19.** *Let $j$ be any job, the idle time in $(0, s_j)$ charged to jobs in $A_j$ is at most $m\,(\kappa'_j - p_j)$. Hence $p(O_j) \leq m\,\frac{(\kappa'_j - p_j)}{\beta} \leq m\,\frac{\kappa_j - p_j}{\beta}$.*

*Proof.* Let us consider job $k \in B_j$ and let job $k'$ be ready at the completion time of job $k$, that is $q_{k'} = C_k$. Since $k$ is before $j$ in the list, then it follows $A_j \subset A_k$, then by Lemma 18 in the interval $(C_k, s'_k)$ no idle time is charged on $A_j$. Hence the idle time of jobs in $A_j$ is accumulated in the intersection between the $P'_j$ intervals and $(0, s_j)$, that is bounded by $m\,(\kappa'_j - p_j)$, since there are $m$ machines and $|(\cup P'_j) \cap (0, s_j)| \leq \kappa'_j - p_j$. Since $O_i \subseteq A_i$ the total idle time charged to $O_i$ is at most $\frac{1}{\beta}$ times the possible idle time charged on $A_i$, then by Lemma 16 we have $p(O_i) \leq \frac{m\,(\kappa'_j - p_j)}{\beta} \leq \frac{m\,(\kappa_j - p_j)}{\beta}$. □

**Theorem 17.** *Given a list the algorithm DELAY-LIST produces a schedule $S$, so that, for each job $j$, we have*

$$C_j \leq \frac{(1+\beta)p(B_j)}{m} + \left(1 + \frac{1}{\beta}\right)\kappa'_j - \frac{p_j}{\beta}.$$

*Proof.* Let us consider job $j$, then we can partition the interval $T = (0, C_j)$ in $T_1 = \{\text{intervals defined by } P'_j\}$ and $T_2 = T - T_1$, and we define $t_1$ and $t_2$ as the sum, respectively, of the intervals in $T_1$ and $T_2$. By definition we have $t_1 = \kappa'_j \leq \kappa_j$. By Lemma 18 the idle time in $T_2$ is charged on jobs in the set $B_j$ and the running jobs are from $B_j \cup O_j$, then by Lemma 17 we have

$$C_j = t_1 + t_2 \leq \kappa'_j + \frac{\beta\, p(B_j) + p(B_j) + p(O_j)}{m} \leq$$
$$\leq \kappa'_j + \frac{(\beta+1)\, p(B_j)}{m} + \frac{\kappa'_j - p_j}{\beta} = \kappa'_j\left(1 + \frac{1}{\beta}\right) + \frac{(\beta+1)\, p(B_j)}{m} - \frac{p_j}{\beta},$$

where the second inequality follows from Lemma 19. □

**Corollary 4.** *Let $S^I$ be a single machine schedule. Then, running the DELAY-LIST algorithm with the list generated by $S^I$, it outputs the schedule $S^m$ such that*

$$C_j^m \leq \frac{C_j^I}{\beta}(1+\beta) + \left(1 + \frac{1}{\beta}\kappa_j\right).$$

*Proof.* In $S^1$ all jobs in $B_j$ are scheduled before $j$, hence $p(B_j) \leq C_j^I$. If we plug this inequality in Theorem 17 and by Lemma 16 we have

$$C_j^m \leq (1+\beta)\frac{C_j^I}{m} + \left(1 + \frac{1}{\beta}\right)\kappa_j.$$

□

The DELAY-LIST algorithm needs a one-machine schedule. So we see some relations between schedules with different machines. Let $C_{OPT}^m$ be the total weighted completion time of an optimal $m$-machine schedule.

**Lemma 20.** *Given a single machine and a $m$-machine formulation we have $m \cdot C_{OPT}^m \geq C_{OPT}^I$, where $C_{OPT}^I$ is the total weighted completion time for an optimal single machine schedule.*

*Proof.* Given a schedule $S^m$ on $m$ machines with total weighted completion time $C^m$ we can construct a single machine schedule $S^I$ with $C^I \leq m\,C^m$. Let us order the jobs by $S^m$ completion time and use this order for $S^I$. By the release dates there could be some idle time. Let us denote the sum of completion times of the jobs finishing before $j$, included $j$, by $P$ and the total idle time in $S^m$ before $C_j^m$ by $ID$. Then $m\,C_j^m \geq P + ID$. Since $P$ is the sum of all jobs before $j$ in $S^I$ and the idle time in $S^I$ can be charged to idle time in $S^m$ we get $C_j^I \leq P + ID \leq m\,C_j^m$. □

**Lemma 21.** $C_{OPT}^m \geq \sum w_j C_j = C_{OPT}^\infty$.

*Proof.* By definition $C_j^m \geq \kappa_j$, hence summing over $j$ we get the inequality. If we have an unbounded number of machines any job $j$ can be scheduled at the earliest available time, so it will finish at $\kappa_j$, hence we have the equality. □

**Theorem 18.** *Suppose to have a $\rho$-approximation algorithm for minimizing the sum of completion times in a one-machine problem, then DELAY-LIST gives an $m$-machine schedule within a factor of $(1+\beta)\,\rho + \left(1 + \frac{1}{\beta}\right)$ of an optimal $m$-schedule.*

*Proof.* Let $S^I$ be a one-machine schedule within a factor of $\rho$ of the optimal one-machine schedule, then $\sum w_j C_j^I = C^I \leq \rho\, C_{OPT}^I$. By Corollary 4 the algorithm DELAY-LIST creates a schedule such that

$$\begin{aligned} C^m = \sum w_j C_j^m &\leq \sum w_j \left( (1+\beta)\, \frac{C_j^I}{m} + \left(1 + \frac{1}{\beta}\right) \kappa_j \right) \\ &= \frac{1+\beta}{m} \sum w_j C_j^I + \left(1 + \frac{1}{\beta}\right) \sum w_j \kappa_j. \end{aligned}$$

Then by Lemmas 20 and 21 we get

$$\begin{aligned} C^m &\leq \frac{(1+\beta)\, \rho\, C_{OPT}^I}{m} + \left(1 + \frac{1}{\beta}\right) C_{OPT}^{\infty} \\ &\leq \left( (1+\beta)\, \rho + \left(1 + \frac{1}{\beta}\right) \right) C_{OPT}^m. \end{aligned}$$

□

The power of the algorithm derives from the fact that the bounds are given job-by-job, hence we can apply the algorithm even for more general classes of metric.

## 8.3 2.83-approximation algorithm

We will derive a 2.83-approximation algorithm for the sum of completion times on $m$ machines with release dates but no precedence constraints. We will follow the proof of Chekuri et al., see [10], that works by combining a list scheduling and the DELAY-LIST algorithms.

Let $M$ and $I$ be defined as in Lemma 14, and let $C_j^I$ be the completion time of job $j$ in an optimal schedule $I^*$.

**Lemma 22.** *Let $S^m$ be the schedule obtained by applying the DELAY-LIST algorithm on $I^*$ with parameter $\beta$. Denoting the completion time of job $j$ in $S^m$ by $C_j^m$, then it follows*

$$\sum C_j^m \leq (2+\beta)\, C_j^* + \frac{1}{\beta} \sum r_j.$$

*Proof.* Let us consider job $j$. Trivially, since we don't have any precedence constraint, $\kappa_j = r_j + p_j$. By Lemma 16 and Theorem 17 we have that

$C_j^m \leq (1+\beta)\frac{p(B_j)}{m} + \left(1+\frac{1}{\beta}\right)\kappa_j - \frac{p_j}{\beta}$. Since the jobs are ordered by the single machine completion time we have $\frac{p(B_j)}{m} \leq C_j^I$. So combining these inequalities we get $C_j^m \leq (1+\beta)\,C_j^I + p_j + \frac{r_j}{\beta}$. Summing over all $j$ we have

$$\sum C_j^m \leq (1+\beta)\sum C_j^I + \sum p_j + \frac{1}{\beta}\sum r_j \leq (2+\beta)\sum C_j^* + \frac{1}{\beta}\sum r_j,$$

because $\sum C_j^I$ and $\sum p_j$ are lower bounds on the optimal value. □

**Theorem 19.** *For any $m$-machine instance, applying either list scheduling or DELAY-LIST for an appropriate $\beta$ we obtain a schedule $S^m$ such that $\sum C_j^m \leq 2.83\sum C_j^*$. Moreover, it runs $O(n\log n)$ time.*

*Proof.* By equation (36) we know $\sum C_j^m \leq 2\sum C_j^I + \sum p_j$.

- if $\alpha$ is such that $\sum p_j \leq \alpha\sum C_j^*$, then the list scheduling algorithm has an approximation guarantee of $(2+\alpha)$,
- if $\sum p_j > \alpha\sum C_j^*$, since we know that $\sum C_j^* \geq \sum r_j + p_j$, then we obtain $\sum r_j \leq (1-\alpha)\sum C_j^*$. So, by Lemma 22, we get

$$\sum C_j^m \leq \left(2+\beta+\frac{1-\alpha}{\beta}\right)\sum C_j^*.$$

Given $\alpha$ we can choose $\beta$ so that we minimize $\min\{2+\alpha, 2+\beta+\frac{1-\alpha}{\beta}\}$. Choosing $\beta = \sqrt{1-\alpha}$, we have that $\min\{2+\alpha, 2+2\sqrt{1-\alpha}\}$ is minimized by $\alpha = 2\sqrt{2}-2$ and it gives an approximation guarantee of $2\sqrt{2} \approx 2.83$. So the algorithm chooses $\beta = \sqrt{3-2\sqrt{2}}$ and runs both the algorithms, keeping, at the end, the best schedule. The running time of both algorithms is $O(n\log n)$, so the final schedule can be computed in $O(n\log n)$ time. □

# 9 Approximation for $P|d_{ij}|\sum w_jC_j$

In this Chapter we will explain the list scheduling algorithm for parallel machines, defining the job-driven algorithm that will be used. Then we will show the proof for the 4-approximation algorithm for the problem $P|d_{ij}|\sum w_jC_j$ given by Queyranne and Schulz in [35], for which we will present the LP relaxation used. At the end we will give an example that proves the tightness of this algorithm, see [35].

## 9.1 List-scheduling algorithms

One of the easiest methods to find approximate solutions for parallel machines scheduling problems consists of list scheduling algorithms. Given a list satisfying the precedence constraints, these algorithms schedule, when one of the $m$ machines becomes idle, the first avaliable job. This approach can cause some troubles because we may delay the schedule of jobs with large weight that may be avaliable soon, by scheduling less important jobs. Hence sometimes adding some idle time would be useful to obtain a better approximation. We provide here an easy example to show this problem:

**Example.** *Let us have a single machine instance, with jobs $j$ and $k$ for the problem $1|r_j|\sum w_jC_j$. For $q \geq 2$ assume: $p_j = q, r_j = 0$ and $w_j = 1$ for job $j$ and $p_k = 1, r_k = 1$ and $w_k = q^2$ for job $k$. The optimal schedule leaves the machine idle in $[0, 1)$, then processes $k$ and after $j$, so to have optimum value of $2q^2 + q + 2$. The list scheduling algorithm would process before $j$ and after $k$, in such a way the objective value is $q^3 + q^2 + q$. Hence, for $q$ big, we have unbounded permormance ratio.*

This example could be generalized also with precedence constraints. So the idea is to find a different way of list scheduling the jobs.

A simple strategy is to consider the jobs one by one in the given list order without altering this order, so we have a specific job-driven list scheduling. Given a list, that extends the partial order defined by precedence constraints, the algorithm works as follows:

1. assume the list to be $L = l_1, \ldots, l_n$, the machines to be empty and define the machine completion times $\Gamma_h := 0$ for $h = 1, \ldots, m$.

2. for $j$ from $l_1$ to $l_n$ define the starting and the completion times of $j$ and assign $j$ to a machine:

   (a) $s_j := \max\left(\max\{C_i + d_{ij} : (i, j) \in A\}, \min\{\Gamma_h : h = 1, \ldots, m\}\right)$ is the starting time of job $j$ and $C_j := s_j + p_j$ is its completion time,

(b) take a machine $h$ with $\Gamma_h \leq s_j$ and assign job $j$ to it. Then update the completion time of machine $h$, hence $\Gamma_h := C_j$.

We could use many rules for the choice of the machine to which assign a job, but for the purposes of the analysis we use a general one. Also the definition of the list $L$ could be done in many ways, but we consider the list generated by sorting the jobs in order of nondecreasing completion times on a relaxation of the problem.

## 9.2 4-approximation algorithm

The precedence constraints define a partial order, hence a directed acyclic graph $D = (N, A)$ where $N$ is the set of jobs and $A$ represents the precedence constraints, i.e. we have $(i, j) \in A$ if $i \prec j$. In order to study the problem we need to define what is the parameter $d_{ij}$: it is a nonnegative precedence delay so that job $j$ can start only $d_{ij}$ units of time after the completion time of job $i$. This parameter is very general because it can represent the normal precedence constraints, by replacing $i \prec j$ with $d_{ij} = 0$, release dates , by adding a dummy 0-processing time vertex $v$ and by setting $d_{vj} = r_j$, and also delivery times.

For the problem of minimizing the weighted sum of completion times $\sum w_jC_j$ subject to precedence delay constraints we define an LP relaxation:

$$
\begin{array}{lll}
\min & \displaystyle\sum_{j\in N} w_jC_j & \\
\text{subject to} & & \\
(a) & C_j \geq p_j & \text{for all } j \in N \\
(b) & C_j \geq r_j + d_{ij} + p_j & \text{for all } (i,j) \in A \\
(c) & \displaystyle\sum_{j\in S} p_jC_j \geq \frac{1}{2m}\left(\sum_{j\in S} p_j\right)^2 + \frac{1}{2}\sum_{j\in S} p_j^2 & \text{for all } S \subseteq N.
\end{array}
$$

Constraints $(a)$ impose trivial lower bounds on the completion times of the jobs, constraints $(b)$ insert the ordinary precedence constraints, while $(c)$ were introduced in [38], for the $m$ machines case. We show now the feasibility.

**Lemma 23.** *Given the completion times of jobs in any single machine feasible schedule we have*

$$\sum_{j\in S} p_jC_j \geq \frac{1}{2}\left(\sum_{j\in S} p_j^2 + \left(\sum_{j\in S} p_j\right)^2\right) \qquad \text{for all } S \subseteq N.$$

*Proof.* We can suppose, in case by reindexing, that the jobs are ordered by completion time, so $C_1 \leq \cdots \leq C_n$. Let $S = N$, clearly we have that $C_j \geq \sum_{k\leq j} p_k$ and multiplying by $p_j$ and summing over $j$ it follows

$$\sum_{j=1}^{n} p_jC_j \geq \sum_{j=1}^{n} p_j \sum_{k=1}^{j} p_k = \frac{1}{2}\left(\sum_{j\in N} p_j^2 + \left(\sum_{j\in N} p_j\right)^2\right).$$

For any other set $S$, the jobs in $S$ are feasibly scheduled by the schedule $\{1,\ldots,n\}$ just by ignoring the other jobs. So we may see $S$ as the new entire set of jobs and then we apply the previous argument. □

**Proposition 10.** *Given a feasible schedule on $m$ parallel machines, the completion times satisfy constraints* $(c)$.

*Proof.* Let $S_i$ be the set of jobs processed on machine $i$, for $i = 1,\ldots,m$. From Lemma 23 we have

$$\sum_{j\in S} p_jC_j = \sum_{i=1}^{m}\sum_{j\in S_i} p_jC_j \geq \sum_{i=1}^{m}\frac{1}{2}\left(\sum_{j\in S_i} p_j^2 + \left(\sum_{j\in S_i} p_j\right)^2\right) =$$

$$= \frac{1}{2}\sum_{j\in S} p_j^2 + \frac{1}{2}\sum_{i=1}^{m}\left(\sum_{j\in S_i} p_j\right)^2 \geq \frac{1}{2}\sum_{j\in S} p_j^2 + \frac{1}{2m}\left(\sum_{j\in S} p_j\right)^2.$$

□

So given any feasible solution $C^{LP}$ of this linear program we can, applying the list scheduling algorithm, define the schedule $H$, with completion time $C^H$. We will study the relation between the completion time of jobs, so between $C_j^H$ and $C_j^{LP}$. Let us define the LP midpoints as $M_j^{LP} := C_j^{LP} - \frac{p_j}{2}$ and assume they define the list $L$ that will be used in the job-driven list scheduling.

**Theorem 20.** *Let $M_j^{LP}$ be the midpoint of job $j$ in the LP relaxation. Let $H$ be the schedule obtained by running the job-driven list scheduling algorithm in order of LP midpoints, and denote by $s_j^H$ the starting time of job $j$ in the schedule $H$. Then it follows*

$$s_j^H \leq 4\, M_j^{LP} \text{ for all } j \in N. \tag{37}$$

*Proof.* Let the jobs be indexed, in case by reindexing, in order of LP midpoint, i.e. $M_1^{LP} \leq M_2^{LP} \leq \cdots \leq M_n^{LP}$. Let $j$ be a fixed job and suppose the algorithm runs until job $j$ is considered. We can denote by $\mu$ the total time in $[0, s_j^H)$ where all the machines are busy. It follows, since only jobs $i = 1, \ldots, j-1$ have been processed, that $\mu \leq \sum_{i=1}^{j-1} \frac{p_i}{m}$. Then, in order to prove equation (37), we need to show:

$$(i)\ \frac{1}{m}\sum_{i=1}^{j-1} p_i \leq 2M_j^{LP} \qquad \text{and} \qquad (ii)\ s_j^H - \mu \leq 2M_j^{LP},$$

where we can denote $s_j^H - \mu =: \lambda$.

$(i)$ : first we need to reformulate $(c)$: for each $S \subseteq N$ this constraint is equivalent to

$$\sum_{i\in S} p_i M_i \geq \frac{1}{2m}\left(\sum_{i\in S} p_i\right)^2.$$

By the reindexing, for each $i < j$, we have $M_i^{LP} \leq M_j^{LP}$, hence

$$\left(\sum_{i=1}^{j-1} p_i\right) M_j^{LP} \geq \sum_{i=1}^{j-1} p_i M_i^{LP} \geq \frac{1}{2m}\left(\sum_{i=1}^{j-1} p_i\right)^2.$$

So, after dividing by $\sum_{i=1}^{j-1} p_i$ it follows $\sum_{i=1}^{j-1} p_i \leq 2m \cdot M_j^{LP}$.

$(ii)$ : we want to prove $\lambda \leq 2M_j^{LP}$. Let $q$ be the number of maximal intervals in the schedule $H$ such that at least a machine is idle in $\left[0, s_j^H\right)$. If we denote this idle intervals by $[b_h, e_h)$ for $h = 1, 2, \ldots, q$, we have $0 \leq b_1 < e_1 < b_2 < \cdots < b_q < e_q \leq s_j^H$. So it follows $\lambda = \sum_{h=1}^{q}(e_h - b_h)$ and all machines are busy in the complementary intervals. We know that $[j]$ is the set of jobs with LP midpoint not greater than $M_j^{LP}$. We consider now the directed graph $G^j$ with vertices the jobs $[j]$ and with tight precedence constraints, so

$$G^j = \left([j], A^j\right) := \left([j], \{(k,i) \in A : k, i \in [j] \text{ and } C_i^H = C_k^H + d_{ki} + p_i\}\right).$$

If $e_q > 0$ then there is a job, namely $x(q)$, starting at time $e_q$ ($x(q)$ can be either $j$). So there is a job $x(q)$ such that $s^H_{x(q)} = e_q$ and, since $x(q) \in [j]$, we have $M^{LP}_{x(q)} \leq M^{LP}_j$. Let $v_1, \ldots, v_z = x(h)$ be a maximal path in $A^j$ ending at $x(h)$, with $h = q$. We repeat this process for decreasing value of $h$. By the maximality of the path and by $e_q > 0$, job $v_1$ must have started in a busy interval $[e_g, b_{g+1})$, hence it follows $e_g < s^H_{v_1} < b_{g+1}$ for some busy interval $[e_g, b_{g+1})$ with $b_{g+1} < e_q$, otherwise job $v_1$ would have started before. So we have

$$e_h - b_{g+1} \leq s^H_{v_z} - s^H_{v_1} = \sum_{i=1}^{z-1} \left( s^H_{v_{i+1}} - s^H_{v_i} \right) = \sum_{i=1}^{z-1} \left( p_{v_i} + d_{v_i v_{i+1}} \right). \tag{38}$$

The constraints $(b)$ imply, for all $i = 1, 2, \ldots, z-1$, that

$$M^{LP}_{v_{i+1}} \geq M^{LP}_{v_i} + \frac{p_{v_i}}{2} + d_{v_i v_{i+1}} + \frac{p_{v_{i+1}}}{2}. \tag{39}$$

Hence, equations (38) and (39) together give

$$M^{LP}_{x(h)} - M^{LP}_{v_1} \geq \frac{1}{2} \sum_{i=1}^{z-1} \left( p_{v_i} + d_{v_i v_{i+1}} \right) \geq \frac{1}{2}(e_h - b_{g+1})$$

If $e_g > 0$, then there exists a job $x(g)$ such that $s^H_{x(g)} = e_g$. By the indices of the jobs in the list we have that $s^H_{x(g)} < s^H_{v_1}$ implies $M^{LP}_{x(g)} \leq M^{LP}_{v_1}$, hence

$$M^{LP}_{x(h)} - M^{LP}_{x(g)} \geq \frac{1}{2}(e_h - b_{g+1}). \tag{40}$$

Since $s_{x(g)} = e_g$ (for the case $e_g > 0$) it follows that there is a $k \in [j]$ such that $(k, x(g)) \in A^j$ and $s^H_k < e_g = s^H_{x(g)}$. We can repeat the above procedure with $h = g$, and $x(h) = x(g)$. The whole process will terminate by the fact that at each step we have $g < h$. So we generated a decreasing sequence of indices $q = y(1) > \cdots > y(q') = 0$, in such a way that the intervals $[b_{y(i+1)+1}, e_{y(i)}]$ contain every idle interval. Then, if we add the inequalities (40) it follows:

$$\lambda \leq \sum_{i=1}^{q'-1} (e_{y(i)} - b_{y(i+1)+1}) \leq 2\,(M^{LP}_{x(y(1))} - M^{LP}_{x(y(q'))}) \leq 2\,(M^{LP}_j - 0) = 2\,M^{LP}_j. \tag{41}$$

□

**Corollary 5.** *Let us denote by $C^{LP}$ the optimal solution of the LP relaxation defined by the constraints $(a),(b),(c)$ for the scheduling problem $P|d_{ij}|\sum w_jC_j$, by $C^H$ the solution of the job-driven list scheduling algorithm using the LP midpoints list and by $C^*$ an optimum schedule, then*

$$\sum_{j\in N} w_jC_j^H \leq 4\sum_{j\in N} w_jC_j^*, \tag{42}$$

*and the bound is tight.*

*Proof.* The first part of the theorem, namely equation (42), is obvious by Theorem 20. For the tightness we show now an example.

Suppose we have $m \geq 2$ parallel machines and $m$ sets of jobs $J_h$, for $h = 1,\dots,m$ with $m+1$ jobs each: one medium job called $a(h)$ and $m$ small jobs called $b_g(h)$, $g = 1,\dots,m$. We have $p_{a(h)} = 2^{h-1-1}$, $p_{b_g(h)} = 0$ and $a(h) \prec b_g(h)$. We also have $m$ jobs $u(i)$, for $i = 1,\dots,m$, with $p_{u(i)} = 1$, a job $j_{n-1}$ with processing time $p_{n-1} = \frac{2}{m}$ and a job $j_n$ with $p_n = 0$, such that $a(m) \prec j_{n-1} \prec j_n$. We can see that in total $N = (m+1)^2 + 1$ jobs.

If $m$ is big enough the constraints $(a),(b),(c)$ define a feasible LP solution such that $C_j^{LP} = p_{a(h)} = 2^{h-1-m}$ for $j \in J_h$ and $h = 1,\dots,m$, $C_{u(i)}^{LP} = 1+\frac{2}{m}$, for $i = 1,\dots,m$ and $C_{n-1}^{LP} = C_n^{LP} = M_n^{LP} = p_{a(m)} + p_{n-1} = \frac{1}{2} + \frac{2}{m}$. So the list in order of midpoints is given by $J_1, J_2, \dots, J_m$, where each of them processes before the medium job, then job $j_{n-1}$, after the unit jobs $u(i)$ for $i = 1,\dots,m$ and at the end job $j_n$. Then the algorithm, using a list in order of LP midpoints, works as follows:

- first it processes $J_1, J_2, \dots J_m$, where for $k = 1,\dots,m$:
  - a machine processes job $a_k$ with $C_{a_k}^H = \sum_{i=1}^k p_{a(i)} = 2^{h-m} - 2^{-m}$
  - jobs $b_k(h)$, for $h = 1,\dots,m$, are processed on the $m$ machines with $C_{b_k(h)}^H = C_{a_k}^H$,
- it processes job $j_{n-1}$ and jobs $u(i)$, $i = 1,\dots,m-1$ concurrently on the $m$ machines,
- it processes $u(m)$ on the same machine as $j_{n-1}$
- it processes job $j_n$, starting at time $s_n^H = 2 - 2^{-m}$

Let suppose now $w_n = 1$ and all the other weights are 0, then the optimum solution is $\sum w_jC_j^* = w_n(p_{a(m)}+p_{n-1}+p_n) = \frac{1}{2}+\frac{2}{m}$, while the LP midpoints

list produces a solution with value $\sum w_j C_j^H = 2 - 2^{-m}$. Hence, for $m$ sufficiently large we have $s_n^H = 2 - 2^{-m} \approx 2 + \frac{8}{m} = 4 \cdot M_n^{LP}$. So the bound is tight. $\square$

# 10 Conclusion

We considered some scheduling problems that arise in a wide range of applications and we shown some possible algorithms and their performances. These algorithms used two different concepts: the LP relaxation of an integer program and the $\alpha$-point scheduling, by a generic conversion algorithm.

Despite the work done on these scheduling problems there are still some open questions or conjectures that have to be proven.

In Chapter 5, for the problem $1|r_j|\sum w_jC_j$, we presented an approximation algorithm with garantee 1.6853, but the actual lower bound of 1.5819 is given in Section 5.5, so it would be interesting to close this gap.

Another interesting problem regards the problem studied in Chapter 7, namely $1|r_j, prec|\sum w_jC_j$. In a recent paper of Skutella, [41], was found a new performance guarantee of $\frac{\sqrt{e}}{\sqrt{e-1}} \approx 2.54$, that it is hardly the last word for this problem. By some non-approximability results it seems somewhat unlikely to achieve a performance ratio strictly better than 2. Hence Skutella formulated the conjecture that, for any $\epsilon > 0$, there is a $(2+\epsilon)$-approximation algorithm for $1|r_j, prec|\sum w_jC_j$.

We also tried to use the concept of $\alpha$-point for scheduling problems with parallel machines, the big problem, here, arises with the precedence constraints. In fact, while for $P|r_j|\sum w_jC_j$ the best approximation algorithm has a guarantee of 2, for the same problem with precedences, namely $P|r_j|\sum w_jC_j$, the best known performance is 4, and it is reached by an intricate algorithm which uses a fixed $\alpha = \frac{1}{2}$. Unfortunately, by a lack of time, we couldn't find no new results, but we think that, with a better understanding of the $\alpha$-point over parallel machines, it is possible to improve the best current bound, by modify the $\alpha$-conversion algorithm for the case with parallel machines and precedence constraints.

Our approach to Scheduling Theory was the survey Scheduling Algorithms of Karger et al., [23], where the authors, after the definition of Scheduling, wrote:

> *"the practice of this field dates to the first time two humans contended for a shared resource and developed a plan to share it without bloodshed"*

hence we hope that this thesis will head to less bloodshed.

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