EÖTVÖS LORÁND UNIVERSITY


MASTER'S THESIS

# Numerical Solution of Non–Linear Conservation Laws

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# *Abstract*

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Conservation laws, which are conservative systems of hyperbolic differential equations result from continuum mechanics (e.g. the equations of shallow water waves, fluid dynamics, magneto-fluid dynamics and certain elasticity problems).
In this work, we will start with a general introduction and show how they arise from the basic principles of physics. We will discuss the non-uniqueness of the solution in general, and later we will show how the entropy condition can deal with this problem. We will study an other interesting fact, which is the non-smoothness of the solution, that makes the numerical treatment hard.
Besides these two difficulties, we will find a special difficulty when we will construct the numerical schemes, which is the physical structure of the original problem namely the conservation, that leads us to define the conservative schemes in order to preserve this property.
We will discuss the Lax–Wendroof and Godunov schemes and their properties, turning to the high order nonlinear schemes MUSCL, which will be obtained by applying a non-oscillatory first order accurate scheme to an appropriately modified flux function. We will study van Leer's scheme and we want to make it more accurate by avoiding oscillations. For this we will use TVD schemes, which are free of oscillation and define the slope limiters. We will state most popular second order limiters: van Leer, minmod and superbee limiters. We end this work with some numerical experiments which demonstrate the performance of these schemes.

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# Contents

# Chapter 1

# Introduction

Flow is a phenomena that occurs in many natural and technical environment, in physics, fluid dynamics is a sub-discipline of fluid mechanics that describes the flow of fluids (liquids and gases). Flows are everywhere even in the our body where flow-dependent transport processes that supply our body with the oxygen.
Fluid flows in the atmosphere are used to determine whether certain regions can be used for agriculture, if they are sufficiently supplied with rain. Other negative effects on our natural environment are the devastations that hurricanes and cyclones can cause. This makes it clear that humans not only depend on fluid flows in the positive sense, but also have to learn to live with the effects of such fluid flows that can destroy the entire environment.
The above paragraph clearly shows that without fluid flows, life as we know it will not be possible on earth, our natural and technical world would be different and might not even exist at all.

In physics, several principles state that certain physical properties (measurable quantities) do not change in the course of time within an isolated physical system.
In classical physics, laws of this type govern energy, momentum, angular momentum, mass and electric charge.
In particle physics, other conservation laws are applied to properties of subatomic particles that are invariant during interactions.
An important function of conservation laws is that they make possible to predict the macroscopic behavior of a system without having to consider the microscopic details of the course of physical processes or chemical reactions.
To analyze and solve problems that involve fluid flows, we need to use and study the conservation laws which can be formulated mathematically by integral or differential equations. The main difficulties faced when solving these problems are the non-uniqueness of the solution which makes it hard to distinguish the physical solution from non-physical onces, the lake of smoothness of solution which is not required physically and the type of domain ( convex or concave).

# Chapter 2

# Preliminaries to Conservation Laws

## 2.1 Conservation laws

In mathematics, conservation laws are written as a time-dependent hyperbolic system of partial differential equations, which is usually non-linear insert this in one space dimension the equations take the form

$$\frac{\partial}{\partial t}u(t,x) + \frac{\partial}{\partial x}f(u(t,x)) = 0. \tag{2.1}$$

Here $u\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}^m$ is an $m$-dimensional vector function of conserved quantities and $f\colon \mathbb{R}^m \to \mathbb{R}^m$ is called the flux function.
The main assumption underlying the equation (2.1) is that knowing the value of $u(t,x)$ at a given point and time allows us to determine the rate of flow or flux of each state variable $j$.
The equation (2.1) must be augmented by some initial conditions and also possibly boundary conditions on a bounded spatial domain.
In two space dimensions a system of conservation laws takes the form

$$\frac{\partial}{\partial t}u(t,x,y) + \frac{\partial}{\partial x}f(u(t,x,y)) + \frac{\partial}{\partial y}g(u(t,x,y)) = 0. \tag{2.2}$$

Here the flux function is given with the pair $(f,g)$, where $f,g\colon \mathbb{R}^m \to \mathbb{R}^m$.
There are several reasons for studying this particular class of equations on their own in some depth:

- Many practical problems in science and engineering involve conserved quantities and lead to PDEs of this class.
- There are special difficulties associated with solving these systems (e.g. shock formation) that are not seen elsewhere and must be dealt with carefully in developing numerical methods.
- Although few exact solutions are known, a great deal is known about the mathematical structure of these equations and their solution.

We want to exploit this theory to develop special methods that overcome some of the above and the numerical difficulties encountered with a more naive approach.

### 2.1.1 Integral and differential form

Conservation laws are formulated originally in an integral form. After some differential operations, we can arrive to the differential form.

To see how the conservation laws arise from physical principles let us state the following physical example of conservation mass in a one-dimensional gas dynamics problem [2].
We assume that we have tube where its walls are impermeable and the density $\rho$ and velocity $v$ of the gas are constants across each cross section of the tube.
Let $x$ represent the distance along the tube, we can define the mass of gas in the section $[x_1, x_2]$ by the following equation:

$$\text{mass in } [x_1, x_2] \text{at time } t = \int_{x_1}^{x_2} \rho(t, x)\, \mathrm{d}x. \tag{2.3}$$

Then the rate of flux of gas past the point $x$ at time $t$ is given by:

$$\text{mass flux at } (t, x) = \rho(t, x)v(t, x).$$

The rate of change of mass in $[x_1, x_2]$ is given by the difference in fluxes at $x_1$ and $x_2$

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho(t, x)\, dx = \rho(t, x_1)v(t, x_1) - \rho(t, x_2)v(t, x_2), \tag{2.4}$$

which gives the integral form of conservation laws.
In order to derive the differential form of the conservation laws, we assume that $\rho(t, x)$ and $v(t, x)$ are differentiable functions and integrate the equation (2.4) with respect to time between $t_1$ and $t_2$

$$\int_{x_1}^{x_2} \rho(t_2, x)\, dx = \int_{x_1}^{x_2} \rho(t_1, x)\, dx + \int_{t_1}^{t_2} \rho(t, x_1)v(t, x_1)\, dt - \int_{t_1}^{t_2} \rho(t, x_2)v(t, x_2)\, dt.$$

Using the following equations

$$\rho(t_2, x) - \rho(t_1, x) = \int_{t_1}^{t_2} \frac{d}{dt}\rho(t, x)\, dt,$$

$$\rho(t, x_2)v(t, x_2) - \rho(t, x_1)v(t, x_1) = \int_{x_1}^{x_2} \frac{d}{dx}\rho(t, x)v(t, x)\, dx,$$

we find

$$\int_{t_1}^{t_2}\int_{x_1}^{x_2} \frac{d}{dt}\rho(t, x) + \frac{d}{dx}\rho(t, x)v(t, x)\, dx\, dt = 0.$$

Since this must hold for any section $[x_1, x_2]$ and any time interval $[t_1, t_2]$, we obtain

$$\rho_t + (\rho v)_x = 0. \tag{2.5}$$

In order to solve this equation the velocity, $v(t, x)$ is assumed to be given or known as a function of the density $\rho(t, x)$, then we can write the equation

(2.5) as follows

$$\rho_t + f(\rho)_x = 0, \tag{2.6}$$

which gives the differential form of conservation laws.

## 2.2 Scalar conservation laws

The study of linear conservation laws is important for understanding the behavior of a numerical scheme, but it is also very important to consider that the introduction of the nonlinearity changes dramatically the nature of the problem because it induces a loss of the uniqueness of the weak solution.

### 2.2.1 The linear advection equation

In this subsection, we mainly follow the exposition in [2].
Let us consider the following Cauchy problem for the linear advection equation on the domain $x \in (-\infty, \infty)$, $t \in \mathbb{R}^+$

$$\begin{aligned} &u_t + (a(x)u)_x = 0. \\ &u(t,x) = u_0(x). \end{aligned} \tag{2.7}$$

Here $a(x)$ is a smooth function representing the velocity.
Let us assume that $a$ is a constant and $u_0$ is a given function.
The exact solution for this problem is:

$$u(t,x) = u_0(x - at).$$

As time evolves, the initial data simply propagates unchanged to the right if the velocity $a > 0$ or to the left if $a < 0$ .
The solution $u(t,x)$ is constant along each ray $x - at = x_0$, which are known as the characteristics of the equation.
The characteristics are curves $t \to (t, x(t))$ which satisfy the ordinary differential equation

$$\begin{cases} x'(t) = a, \\ x(0) = x_0. \end{cases}$$

If we differentiate $u(t,x)$ along one of these curves to find the rate of change of $u$ along the characteristics, we find that

$$\begin{aligned} \frac{d}{dt}u(t,x(t)) &= \frac{\partial}{\partial x}u(t,x(t)) + \frac{\partial}{\partial t}u(t,x(t))x'(t) \\ &= u_t + au_x \\ &= 0, \end{aligned}$$

confirming that $u$ is constant along these characteristics.
If the initial data $u_0$ is non differentiable then $u$ is not a classical solution for the PDE any more, however this function $u$ does satisfy the integral form.
One can suggest to approximate the non smooth initial data $u_0$ to a sequence

of smooth functions, but unfortunately this does not work for nonlinear problems.

### 2.2.2 Burgers's equation

The most famous case is Burgers's equation, in which the flux $f(u) = \frac{1}{2}u^2$, then the problem can be written as

$$u_t + uu_x = 0. \tag{2.8}$$

We can construct the solution of this problem with a smooth initial condition by the following characteristics

$$x'(t) = u(t, x(t)),$$

where $u$ is constant along each characteristic, such that the slope $x'(t)$ is constant. So the characteristics are straight lines determined by the initial data. If $u(t, \cdot)$ is smooth then the characteristics do not cross and we can solve the equation $x = \xi + u(0, \xi)t$ for $\xi$ and then $u(t, x) = (\xi, 0)$.
This is true for small $t$ but for large $t$ the equation $x = \xi + u(0, \xi)t$ may have more solutions. This happens when the characteristics cross each other and the PDE does not have any more classical solution and the weak solution (we hope to determined) becomes discontinuous, (for more details, see [1], and [2]).
For some initial condition the function can become multivalued in one point, which is impossible to happen in the physics.

## 2.3 Classical solution of conservation laws

Before turning to the numerical approximation of the solution, we have to see whether the problem is well posed or not (existence and uniqueness).
We start with the classical solution.

### 2.3.1 Classical solution

Consider the following Cauchy problem of conservation laws

$$\begin{cases} u_t + f(u,t)_x = 0 & x \in \mathbb{R},\ t > 0, \\ u(0,x) = u_0 & x \in \mathbb{R},\ t = 0. \end{cases} \tag{2.9}$$

If $u_0(x)$ is increasing, (decreasing) and $f(u)$ is convex, ( concave), the classical solution of this problem (2.9) well defined for all $t > 0$.
However, in the general case, classical solutions fail to exist for all $t > 0$ even if $u_0$ is very smooth. This happens when $\inf_x u_0'(x)\ f''(u_0(x)) < 0$, then

classical solutions exist only for $t$ in $[0, T^*]$ where

$$T^* = \frac{1}{\inf_x u_0'(x) f''(u_0(x))}.$$

At the time $t = T^*$ the characteristics first cross which lead the function $u(x,t)$ to have an infinite slope and forms a shock [2].

*Proof.* Since along characteristics $u(x(t),t)$ is equal to $u_0(x_0)$, we can write $x(t) = x_0 + tf'(u_0(x_0))$. We can calculate the first time when two different characteristics arrive at same point $(x,t)$.
In this case there are two points $x_0$ and $\hat{x}_0$ such that

$$x = x_0 + tf'(u_0(x_0)) = \hat{x}_0 + tf'(u_0(\hat{x}_0)),$$

which implies that

$$\begin{aligned} t &= \frac{\hat{x}_0 - x_0}{f'(u_0(\hat{x}_0)) - f'(u_0(x_0))} \\ &= \frac{1}{\frac{f'(u_0(\hat{x}_0)) - f'(u_0(x_0))}{\hat{x}_0 - x_0}} \\ &= \frac{1}{u_0'(\xi) f''(u_0(\xi))}, \end{aligned}$$

where $\xi$ lies between $x_0$ and $\hat{x}_0$. Obviously, this expression for $t$ makes sense when $\frac{1}{u_0'(\xi) f''(u_0(\xi))}$ is negative. Thus, the blow up occurs if $u_0'(\xi) f''(u_0(\xi))$ is somewhere negative at $T^*$ and the solution forms a shock wave. □

Since the classical solution may not exist for large $t$, let us consider the weak solution.

### 2.3.2 Weak solution

To define a generalized solution which does not require differentiability is to go back to the integral form, and assume that $u(t,x)$ is a generalized solution if it satisfies the following equation for all $x_1$, $x_2$, $t_1$, $t_2$

$$\int_{t_1}^{t_2} \int_{x_1}^{x_2} \frac{d}{dt} u(t,x) + \frac{d}{dx} f(u(t,x))\, dx\, dt = 0.$$

In our case we will use a continuously differentiable test function with compacted support $\phi \in C_0^1(\mathbb{R} \times \mathbb{R})$, we multiply our PDE by this test function and integrate once with respect to the space and once with respect to the time

$$\int_0^{+\infty} \int_{-\infty}^{+\infty} \phi u_t + \phi f_x(u) = 0.$$

Using integration by part,

$$\int_0^{+\infty} \int_{-\infty}^{+\infty} \phi_t u + \phi_x f(u) = \int_{-\infty}^{+\infty} \phi(0,x) u(0,x). \tag{2.10}$$

**Definition 1.** The measurable and bounded function $u(t,x)$ called a weak solution of the conservation laws if it satisfies the equation (2.10) for all functions $\phi \in C_0^1(\mathbb{R}\times\mathbb{R}^+)$.

Rewrite the integral form of conservation laws which is the original equation we want to solve,

$$\int_{x_1}^{x_2} u(t_2,x)\,dx = \int_{x_1}^{x_2} u(t_1,x)\,dx + \int_{t_1}^{t_2} f(t,x_1)\,dt - \int_{t_1}^{t_2} f(t,x_2)\,dt. \tag{2.11}$$

We give the difference between the original integral and the integral of weak solution in the following table,

| Integral | Domain | check that |
|---|---|---|
| Original (2.11) | arbitrary rectangle | Holds for all $x_1, x_2, t_1, t_2$ |
| Weak solution (2.10) | fixed domain | Holds for all $\phi \in C_0^1(\mathbb{R}\times\mathbb{R}^+)$ |

Mathematically the two integral forms are equivalent and we should rightly expect a more direct connection between the two that does not involve the differential equation [2]. This can be achieved by considering special test functions $\phi(x,t)$ with the property that

$$\phi(t,x) = \begin{cases} 1 & \text{for } (t,x) \in [t_1,t_2]\times[x_1,x_2] \\ 0 & \text{for } (t,x) \notin [t_1-\epsilon, t_2+\epsilon]\times[x_1-\epsilon, x_2+\epsilon], \end{cases} \tag{2.12}$$

with $\phi$ smooth in the intermediate strip of width $\epsilon$. Then $\phi_x = \phi_t = 0$ except in this strip and so the integral (2.10) reduces to an integral over this strip. Unfortunately, it turns out that the weak solutions are often not unique and we can not distinguish between the physical solution and non-physical solution [2].

### 2.3.3 Riemann problem and Rankine–Hugoniot jump condition

The conservation law together with piecewise constant data having a single discontinuity is known as Riemann problem, (see [1] and [2]).
Consider Burgers's equation with piecewise constant initial data

$$u_t + uu_x = 0\,; \qquad u(0,x) = \begin{cases} u_l & x < 0 \\ u_r & x > 0. \end{cases} \tag{2.13}$$

We can distinct two cases.

- First case is when $u_l < u_r$: we find infinity many weak solutions, one of them is the rarefaction wave

$$\begin{cases} u_l & x < u_l t \\ \frac{x}{t} & u_l t \leq x \leq u_r t \\ u_r & x > u_r t. \end{cases}$$

- Second case is when $u_l > u_r$, we find unique weak solution which is "The propagating shock solution"

$$u(t,x) = \begin{cases} u_l & x < st \\ u_r & x > st. \end{cases}$$

Here $s$ is the shock speed

$$\begin{aligned} s &= \frac{f(u_l) - f(u_r)}{u_l - u_r} \\ &= \frac{\frac{u_l^2}{2} - \frac{u_r^2}{2}}{u_l - u_l} \\ &= \frac{u_l + u_r}{2}. \end{aligned} \tag{2.14}$$

This is coming from the fact that the solution satisfies

$$\int_{-M}^{M} u(t,x) = (M + st)u_l + (M - st)u_r.$$

Taking the derivative with respect to $t$, we obtain

$$\frac{d}{dt}\int_{-M}^{M} u(t,x) = s(u_l - u_r). \tag{2.15}$$

In the other hand we have that

$$\frac{d}{dt}\int_{-M}^{M} u(t,x) = f(u_l) - f(u_r). \tag{2.16}$$

Then from (2.16) and (2.15) we can conclude

$$f(u_l) - f(u_r) = s(u_l - u_r), \tag{2.17}$$

which is called the Rankine–Hugoniot jump condition.
Only certain jumps $u_l - u_r$ are allowed namely those for which the vectors $f(t,u) - f(t,u)$ and $u_l - u_r$ are linearly dependent.

## 2.4 Manipulating conservation laws

One danger to observe in dealing with conservation laws is that transforming the differential form into what appears to be an equivalent differential equation may not give an equivalent equation in the context of weak solutions [2].

**Example 2.4.1.** *If we multiply Burgers's equation*

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0 \tag{2.18}$$

*by* $2u$*, we obtain an equivalent equation* $2uu_t + 2u^2u_x = 0$ *which can be rewritten as*

$$(u^2)_t + \left(\frac{2}{3}u^3\right)_x = 0. \tag{2.19}$$

*This is again a conservation law for* $u^2$ *rather than* $u$ *itself, with flux function* $f(u^2) = \frac{2}{3}(u^2)^{\frac{3}{2}}$*. The differential equations* (2.18) *and* (2.19) *have precisely the same smooth solutions, However, they have different weak solutions, as we can see by considering the Riemann problem with* $u_l > u_r$*. The unique weak solution of Burgers's equation is a shock traveling at speed*

$$s_1 = \frac{2}{3}\left(\frac{u_r^3 - u_l^3}{u_r^2 + u_l^2}\right) \tag{2.20}$$

*whereas the unique weak solution to manipulated Burgers's equation is a shock traveling at speed*

$$s_2 = \frac{1}{2}(u_l + u_r), \tag{2.21}$$

*and so* $s_1 \neq s_2$ *when* $u_l \neq u_r$*, and the two equations have different weak solutions.*

This example clearly shows that to manipulate a conservation laws problem the smoothness is required to have equivalent week solution and do not contradict with the fact that they have same smooth solution but different week solution.

# Chapter 3

# Numerical Methods for Conservation Laws

The previous chapter gave an overlook about the structure of conservation laws and its classical solution, and the difficulties which we faced when we want to solve this kind of problems. This leads us to think on the numerical solution.
Solving such a problem numerically means that, we find approximation to its solution at a finite number of points of the intervals $[0, T]$ and $[x_l, x_r]$, say at the grid points of the following uniform discretization

$$\{0 = t_0 < t_1 < \cdots < t_M = T \mid \delta = \Delta t = t_{n+1} - t_n\},\ \forall n = 1, \dot{M} - 1$$
$$\{x_l = x_0 < x_1 < \cdots < x_N = x_r \mid h = \Delta x = x_{j+1} - x_j\}\ \forall\ j = 1, \dot{N} - 1.$$

Denote by $u_j^n \approx u(t_n, x_j)$ the approximation of the function $u$ at the point $(t_n, x_j)$.

The numerical methods have their own difficulties as well, where can give either a catastrophic result or work with rate of accuracy at most one.
In this chapter we will stat different numerical methods, and explain in details the physical reason of their results, and improve these methods in order to get more accurate method, which preserve the physical structure of our problem.
Some features we would like such a method to possess are:

- At least second order accuracy on smooth solutions, and also in smooth regions of a solution even when discontinuities are present elsewhere.
- Sharp resolution of discontinuities without excessive smearing.
- The absence of spurious oscillations in the computed solution.
- An appropriate form of consistency with the weak form of the conservation law, required if we hope to converge to weak solutions.
- Nonlinear stability bounds that, together with consistency, allow us to prove convergence as the grid is refined.

## 3.1 Finite difference method

The finite difference methods, are a well known methods for solving numerically a partial differential equations. In the following, we will discuss an example to show that the finite difference methods can be useless for conservation laws.

**Example 3.1.1.** *Consider Burgers equation with the following initial condition*

$$\begin{aligned} &\partial_t u(t,x) + u(t,x)\partial_x u(t,x) = 0 \\ &u(0,x) = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{if } x > 0. \end{cases} \end{aligned} \tag{3.1}$$

*We want to approximate this problem using the backward finite difference scheme for this using the finite differences*

$$\begin{aligned} \partial_t u &\approx \frac{u_k^{n+1} - u_k^n}{\delta}, \\ \partial_x u &\approx \frac{u_k^n - u_{k-1}^n}{h}. \end{aligned} \tag{3.2}$$

*We obtain*

$$\frac{u_k^{n+1} - u_k^n}{\delta} + u_k^n \frac{u_k^n - u_{k-1}^n}{h} = 0. \tag{3.3}$$

*Which can be rearranged to get the scheme*

$$u_k^{n+1} = u_k^n - \frac{\delta}{h} u_k^n \left(u_k^n - u_{k-1}^n\right).$$

*Compute this scheme with the above initial condition*

$$u_k^1 = u_k^0 - \frac{\delta}{h} u_k^0 \left(u_k^0 - u_{k-1}^0\right) = 0.$$

*If* $u_k^0 = 1 \Rightarrow u_{k-1}^0 = 1 \Rightarrow \frac{\delta}{h} u_k^0 \left(u_k^0 - u_{k-1}^0\right) = 0.$
*If* $u_k^0 = 1 \Rightarrow \frac{\delta}{h} u_k^0 \left(u_k^0 - u_{k-1}^0\right) = 0.$
*So for this initial condition our scheme for the first iteration will be* $u_k^1 = u_k^0$ *which is not correct.*

**Remark 1.** We can find the exact solution of this problem (3.1) using Riemann problem.

The above example clearly shows that the finite difference methods can break down for conservation, which is not surprising result due to its physical structure and properties which are the conservation and the discontinuity of the solution. Where this method do not preserve these properties.

## 3.2 Conservative schemes

The previous section gave us the main fault we have to avoid when we are constructing a method for solving conservation laws. So, we have to construct schemes which preserve the structure of conservation in order to guaranty that the solution will not be catastrophic, and later we will work on the rate of convergence.

We have the integral form of conservation laws

$$\frac{d}{dt}\int_{x_1}^{x_2} u(t,x)\,dx = f(t,x_1) - f(t,x_2). \tag{3.4}$$

Let us introduce the computational cells; the $j^{th}$ cell is $I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$,

$$x_j = j\Delta x$$
$$t^n = n\Delta t$$

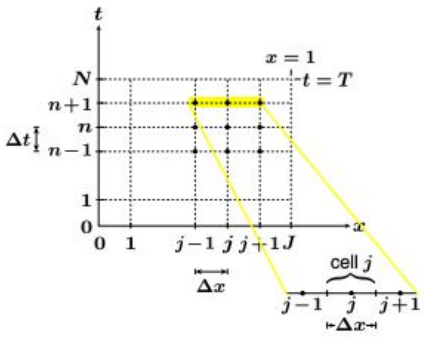



FIGURE 3.1: Computational cell on the grid [4],

and we think on $\hat{u}_j^n$ as a representing cell averages

$$\hat{u}_j^n \approx \frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(t^n,x)\,\mathrm{d}x. \tag{3.5}$$

Taking its derive with respect to $t$ between $(t_n, t_{n+1})$

$$\frac{d}{dt}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(t,x)\,\mathrm{d}x \approx \Delta x\left(\frac{\hat{u}_j^{n+1} - \hat{u}_j^n}{\Delta t}\right). \tag{3.6}$$

On the other hand by applying the integral form on the $I_j$ cell

$$\frac{d}{dt}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(t,x)\,\mathrm{d}x = -\left[f(u(t,x_{j+\frac{1}{2}})) - f(u(t,x_{j-\frac{1}{2}}))\right]. \tag{3.7}$$

From the equations (3.6) (3.5) we find the following scheme

$$\hat{u}_j^{n+1} = \hat{u}_j^n - \frac{\Delta t}{\Delta x}\left[\hat{f}_{j+\frac{1}{2}}^n - \hat{f}_{j-\frac{1}{2}}^n\right].$$

Since this scheme is derived from the integral form of conservation laws, it preserves the conservation property , (see [5] and [4]).

**Definition 2.** A scheme to solve conservation laws is called conservative scheme if and only if it can be written as

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}\left[\hat{f}_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}}\right]. \tag{3.8}$$

with the flux function

$$\hat{f}_{j+\frac{1}{2}} = f(\hat{u}_{j-l}, \hat{u}_{jl+1}, ..., \hat{u}_j ..., \hat{u}_{j+r}),$$

which approximates the flux $f$ on the interface of the $I_j$ and the $I_{j+1}$ cells and satisfies the following properties:

- Lipschitz condition which is sufficient for consistency, i.e. there is a constant $L$ such that

$$|\hat{f}(u_{j-l+1}, u_{j+r}, ..., u_{j-l}) - f(u)| \leq L \max_{-k+1 \leq i \leq k} |u_{j+i} - u|.$$

- It reduces to the original flux $f$ for the case of constant flow,

$$\hat{f}(u, u, ..., u) = f(u).$$

The main advantage of conservative and consistent schemes, is when they converge to solutions whose shocks or discontinuities satisfy automatically the jump conditions, that is, the discontinuities always travel at the correct velocity. This important result, which is not true for non conservative or non consistent schemes.

**Theorem 1.** Lax–Wendroff theorem
Assume that the conserved scheme (3.8) is consistent with the conservation law and that it generates a sequence that converge to a function $u^*$ as the grid sizes $\Delta x$, $\Delta t$ attend to zero. Then $u^*$ is a weak solution of the conservation law.

### 3.2.1 Lax–Wendroff's method

We want to use natural ideas to construct a conservative schemes. Consider the differential form of conservation laws

$$u_t = -f(u)_x \approx -\frac{f(u^n_{j+1}) - f(u^n_{j-1})}{\Delta x}. \tag{3.9}$$

This implies that

$$\begin{aligned} u_{tt} &= -f(u)_{xt} = -(f(u)_t)_x = -(f(u)_t)_x = -(f'(u)u_t)_x \\ &= (f'(u)f(u)_x)_x \approx \left( \frac{f'(u^n_{j+\frac{1}{2}})f(u^n_{j+\frac{1}{2}}) - f'(u^n_{j-\frac{1}{2}})f(u^n_{j-\frac{1}{2}})}{\Delta x} \right). \end{aligned} \tag{3.10}$$

We use

$$f(u_{j+\frac{1}{2}}) \approx \frac{f(u_{j+1}) - f(u_j)}{\Delta x}, \quad f(u_{j-\frac{1}{2}}) \approx \frac{f(u_j) - f(u_{j-1})}{\Delta x}$$

Write the Taylor expansion at the point $(n, j)$ with respect to the time variable

$$u^{n+1}_j = u^n_j + \Delta t \partial_t u^n_j + \frac{\Delta t^2}{2} \partial_{tt} u^n_j.$$

We substitute the derivatives $u_t$ and $u_{tt}$ by their approximation written in the equations (3.9) and (3.10), we obtain Lax–Wendroff scheme [5],

$$\begin{aligned} u^{n+1}_j &= u^n_j - \frac{\Delta t}{2\Delta x} \left( f(u^n_{j+1}) - f(u^n_{j-1}) \right) + \\ &\frac{1}{2} \left( \frac{\Delta t}{\Delta x} \right)^2 \left[ f'(u^n_{j+\frac{1}{2}}) \left( f(u^n_{j+1}) - f(u^n_j) \right) - f'(u^n_{j-\frac{1}{2}}) \left( f(u^n_j) - f(u^n_{j-1}) \right) \right]. \end{aligned}$$

Here $u^n_{j+\frac{1}{2}} = \frac{u^n_j + u^n_{j+1}}{2}$.
This scheme is a conservative scheme with the following flux $\hat{f}$

$$\hat{f}_{j+\frac{1}{2}} = \frac{1}{2} \left[ f(u_j) + f(u_{j+1}) - \frac{\Delta t}{\Delta x} f'(u_{j+\frac{1}{2}}) \left( f(u_{j+1}) - f(u_j) \right) \right].$$

**Theorem 2.** Lax–Wendroff scheme is second order consistent with the original equation.

### 3.2.2 Godunov's method

Godunov constructed a numerical way to calculate the flux, which is a modification to the upwind flux, guarantees that the solution satisfies the entropy condition [4]. In this method, the solution is considered as piecewise constant over each cell at a fixed time, the evolution of the solution to the next time step results from the wave interactions originating, at the boundaries between adjacent cells. The cell interfaces separate two different states at the left and at the right side, and the resulting local interaction can be exactly resolved since the initial conditions at time $t = n\Delta t$ correspond to the Riemann problem. In order to define completely the interaction between adjacent cells, the time interval over which the waves are allowed to propagate should be limited by the condition that adjacent Riemann problems do not interfere. This leads to a form of CFL condition [3].

1. Given data $u_j^n$ at time $t_n$, construct a piecewise constant function over each cell
$$\hat{u}_j^n(t_n, x) = u_j^n \quad x_{j-\frac{1}{2}} \leq x \leq x_{j+\frac{1}{2}}.$$

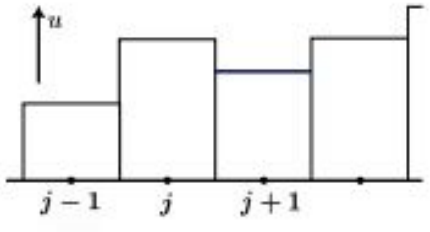



FIGURE 3.2: A piecewise constant approximation with the cell-midpoints [4].

2. Compute the interface flux by solving the local Riemann problem at the cell interfaces, that is, on each subinterval $I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$
$$\partial_t \hat{u}_j^n + \partial_x f(\hat{u}_j^n) = 0$$
$$\hat{u}_j^n(t_n, x) = \begin{cases} u_j^n & x_{j-\frac{1}{2}} \leq x \leq x_j \\ u_{j+1}^n & x_{j-\frac{1}{2}} \leq x \leq x_{j+\frac{1}{2}} \end{cases}$$

3. Define the approximation $u_j^{n+1}$ at time $t_{n+1}$ by averaging the Riemann problem solution $\hat{u}_j^{n+1}$ at the time $t_{n+1}$, so
$$u_j^{n+1} = \frac{1}{\Delta x} \int_{I_j} \hat{u}_j^{n+1}(t_{n+1}, x)\, \mathrm{d}x. \tag{3.11}$$
These values are then used to define new piecewise constant data $\hat{u}_j^{n+1}(t_{n+1}, x)$ and the process repeats.

Is conservative scheme with the flux

$$\hat{f}_{j+\frac{1}{2}} = \begin{cases} \min_{u_j \le u \le u_{j+1}} f(u_j) & u_j < u_{j+1}, \\ \max_{u_j \le u \le u_{j+1}} f(u_j) & u_j \ge u_{j+1}, \end{cases} \tag{3.12}$$

which is called the Godunov scheme, (see [5]). Note that this is equal with the first-order upwind scheme for linear advection equations.

**Theorem 3.** Godunov scheme is first order consistent with the original problem.

**Advantage of Godunov scheme**

- Gives the exact solution for the Riemann problem and is valid for any scaler conservation laws with convex or concave flux.
- Gives the correct flux corresponding to the weak solution satisfying entropy condition. More in general, provided we assume the CFL condition.
- Can deal the discontinuity without oscillation.

**Remark 2.** The solution is exact for short times; in fact its valid until the waves generated from the solution of Riemann problem start interacting with the waves generated by the neighboring interfaces, which is due to the particular form of the solution of Riemann problem. i.e. although the general solution of the equations is a function of $x$ and $t$, the solution to the Riemann problem can be expressed as a function of a single variable namely $x/t$. This property is known as similarity [5].

**The CFL condition**

In order to have stability when using explicit numerical schemes, we are required to apply the necessary condition known as the Courant–Friedrichs–Lewy condition. It is often referred to as the CFL which is:

$$\left| \frac{\partial f}{\partial u} \frac{\Delta t}{\Delta x} \right| \le 1$$

# Chapter 4

# Higher order and TVD methods

The main difficulty in the construction of any high-order method is the resolution of two contradictory requirements namely high-order of accuracy and absence of spurious oscillations.
The high-order linear schemes, which are schemes with second or higher order spatial accuracy in smooth parts of the solution and also around shocks and discontinuities (see [6] and [3]), produce unphysical oscillations, on the other hand, the class of monotone methods do not produce unphysical oscillations. However, monotone methods are at most first order accurate and are therefore of limited use.
One way of resolving the contradiction between linear schemes of high-order of accuracy and absence of spurious oscillations is by constructing non-linear methods.

In the following we will state a natural idea to build a non-linear scheme in order to have the high order accuracy and avoid oscillation.

## 4.1 MUSCL- higher order methods

We follow in this section the presentation of in [3] and [6].
Godunov gave the basic idea of the one-order conservative schemes which deal the oscillations, to achieve higher order of accuracy, van Leer introduced the idea of modifying the first step of Godunov's method.
This approach has become known as the Monotone Upstream-centered Scheme for Conservation Laws " MUSCL" or Variable Extrapolation approach. The MUSCL approach allows the construction of very high order methods, fully discrete, semi–discrete and also implicit methods.
Back to the Godunov's method which depends on three steps:

- Construct constant data over each cell.
- Solve Riemann problem with this data.
- Take the solution as the cell average.

The first step can be modified without influencing the physical input. Van Leer's method is based on modifying Godunov scheme in the following way.

## Steps of van Leer's method

Van Leer's method is a second-order slope limiter method which is based on the concept of non linear discretizations leads to a non linear scheme even when applied to linear equation under the form of limiters, which controls the gradient of the computed solution.
There are three main steps in the van Leer's scheme:

1. **Reconstruction step**:
   Reconstruction piecewise linear functions $u_j(x)$ over each $j^{th}$ cell $I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$

$$u_j^n(x) = u_j^n + \frac{s_j^n}{\Delta x}(x - x_j) \qquad x_{j-\frac{1}{2}} < x < x_{j+\frac{1}{2}}. \tag{4.1}$$

   Here $s_j^n$ is slope on the $I_j$ cell.
   The center of the $I_j$ cell is the point $x_j$ so $u_j^n(x_j) = u_j^n$.
   The values of $u_j(x)$ at the extreme points of each cell play a fundamental role; they are given by

$$u_j^L = u_j^n(x_{j-\frac{1}{2}}) = u_j^n - \frac{s_j^n}{2}\ ; \quad u_j^R = u_j^n(x_{j+\frac{1}{2}}) = u_j^n + \frac{s_j^n}{2}. \tag{4.2}$$

   Note that the integral of $u_j(x)$ in cell $I_j$ is identical to that of $u_j^n$ and thus the reconstruction process retains conserved.

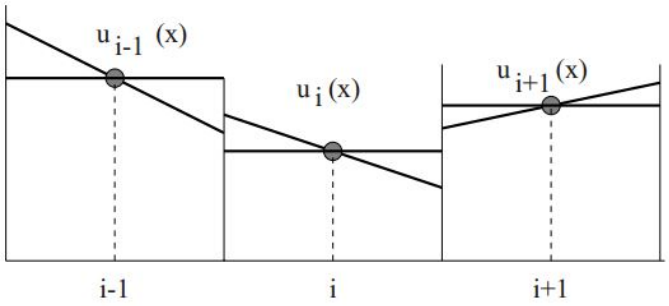



FIGURE 4.1: Piece–wise linear MUSCL reconstruction for three successive computing cells [6].

2. **Evolution step**:
   Evolution of $u_j^L$ and $u_j^R$ over a half time $\frac{1}{2}\Delta t$ according to

$$\begin{cases} \bar{u}_j^L = u_j^L + \frac{1}{2}\frac{\Delta t}{\Delta x}\left(f(u_j^L) - f((u_j^R)\right), \\ \bar{u}_j^R = u_j^R + \frac{1}{2}\frac{\Delta t}{\Delta x}\left(f(u_j^L) - f((u_j^R)\right). \end{cases} \tag{4.3}$$

   Solve the piecewise constant data Riemann problem

$$\begin{aligned} &\bar{u}_t + f(\bar{u})_x = 0, \\ &\bar{u}(x,0) = \begin{cases} \bar{u}_j^R, & x < x_j \\ \bar{u}_{j+1}^L & x > x_j. \end{cases} \end{aligned} \tag{4.4}$$

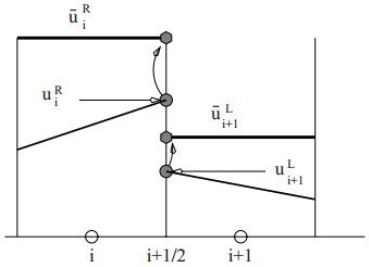



FIGURE 4.2: Boundary extrapolated values. At each interface $i+\frac{1}{2}$ boundary extrapolated values $u_i^R$ , $u^L i+1$ are evolved to form the piece-wise constant data for a conventional Riemann problem at the inter-cell boundary [6].

3. **A cell averaging step**:
   project the solution $\bar{u}(t_{n+1}, x)$ onto piecewise constant functions in order to find $u_j^{n+1}$

$$u_j^{n+1} = \frac{1}{\Delta x} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \bar{u}(t_{n+1}, x)\, \mathrm{d}x. \tag{4.5}$$

   The cell average of $u^n(x)$ over $[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$ is equal to $u_j^n$ for any choice of $s_j^n$.

In the following we will discuss each step in more details.
The first step is clear, we construct a linear functions (4.1), which depend on parameter namely the slope $s_j^n$ over each cell $j$ at time $n$. We create this parameter in order to control the approximation to be more accurate to be discussed later. Using these linear data we want to construct constant data; a natural idea is to divide the $j^{th}$ cell $I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$. using the values of extreme points (4.2) and predictor corrector by half time stepping we can find constant data (4.3) which we solve with Riemann problem (4.4), in order to have more explicit form of the scheme, we integrate the equation (4.4) over the $I_j$ cell $(x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$ and time interval $(t_n, t_{n+1})$,

$$\int_{t_n}^{t_{n+1}} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \partial_t \bar{u} + \partial_x f(\bar{u}) = 0 \tag{4.6}$$

we obtain

$$\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \bar{u}(t_{n+1}, x) - \bar{u}(t_n, x)\, dx = \int_{t_n}^{t_{n+1}} f(\bar{u}(t, x_{j-\frac{1}{2}})) - f(\bar{u}(t, x_{j+\frac{1}{2}}))\, dt. \tag{4.7}$$

Since the flux is continuous, and by (4.5) we find

$$\Delta x \left(u_j^{n+1} - u_j^n\right) = \int_{t_n}^{t_{n+1}} f(\bar{u}(t, x_{j-\frac{1}{2}})) - f(\bar{u}(t, x_{j+\frac{1}{2}}))\, dt. \tag{4.8}$$

The idea here is to approximate the flux, we assume that the numerical flux is

$$\hat{f}^n_{j-\frac{1}{2}} = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(\bar{u}(t, x_{j+\frac{1}{2}}))\, dt. \tag{4.9}$$

The equation (4.8) becomes

$$u^{n+1}_j = u^n_j + \frac{\Delta t}{\Delta x}\left[\hat{f}^n_{j+\frac{1}{2}} - \hat{f}^n_{j-\frac{1}{2}}\right]. \tag{4.10}$$

Using the midpoint rule, we find

$$\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(\bar{u}(t, x_{j-\frac{1}{2}})) = f(\bar{u}(t_n + \frac{\Delta t}{2}, x_{j+\frac{1}{2}})) + O(\Delta t^2). \tag{4.11}$$

To approximate the numerical flux $\hat{f}$ we have to approximate $f(\bar{u}(t_n+\frac{\Delta t}{2}, x_{j+\frac{1}{2}}))$ in each cell.
Then, we solve the Riemann problem at the point $x_{j+\frac{1}{2}}$ with piecewise constant initial data.
After solving Riemann problem we have to find the proper slope for each cell. For better understanding the way of calculating the slope we will study this method for advection equation.

### 4.1.1 Van Leer's method for advection and Burger's equations

#### Van Leer's method for advection equation

Let us consider the following advection equation

$$u_t + cu_x = 0,\ \ c > 0. \tag{4.12}$$

with nonlinear initial condition and boundary condition.
Step I: Gives the boundary extrapolated values (4.2).
Step II: we need the fluxes $f(u^L_j)$ and $f(u^R_j)$. As for the linear advection equation $f(u) = cu$, then we have

$$f(u^L_j) = c\left(u^n_j - \frac{s^n_j}{2}\right);\ \ f(u^R_j) = c\left(u^n_j + \frac{s^n_j}{2}\right) \tag{4.13}$$

Then

$$f(u^L_j) - f(u^R_j) = c\left(u^n_j - \frac{s^n_j}{2} - u^n_j - \frac{s^n_j}{2}\right) = -cs^n_j \tag{4.14}$$

Substitute of these into the equation (4.3) gives the evolved boundary extrapolated values in $I_j$ cell, [6]

$$\begin{aligned} \bar{u}^L_j &= u^n_j - \frac{1}{2}(1+R)\, s^n_j \\ \bar{u}^R_j &= u^n_j + \frac{1}{2}(1-R)\, s^n_j \end{aligned} \tag{4.15}$$

We solve the conventional Riemann problem at the interface $j+\frac{1}{2}$ with data $(\bar{u}_j^R, \bar{u}_j^L)$ the solution of which is

$$u_{j+\frac{1}{2}}^{n+1} = \begin{cases} \bar{u}_j^R = u_j^n + \frac{1}{2}(1-R)\, s_j^n & x < c\Delta t, \\ \bar{u}_{j+1}^L = u_{j+1}^n - \frac{1}{2}(1+R)\, s_{j+1}^n & x > c\Delta t \end{cases} \tag{4.16}$$

**Remark 3.** We can check easily that by choosing the slope to be $s_j = u_{j+1} - u_j$ then van Leer's method is identical to Lax-Wendroff method.

The above remark affirms that the choice of the slope $s_j$ play a main role for avoiding the oscillation.

### Van Leer method for Burgers's equation

We solve numerically the following Burgers's equation with non-linear initial condition

$$\partial_t u(t,x) + u(t,x)\partial_x u(t,x) = 0 \tag{4.17}$$

From the equation (4.3) we find

$$\begin{cases} \bar{u}_j^L = u_j^L + \frac{1}{4}\frac{\Delta t}{\Delta x}\left((u_j^L)^2 - ((u_j^R)^2\right), \\ \bar{u}_j^R = u_j^R + \frac{1}{4}\frac{\Delta t}{\Delta x}\left((u_j^L)^2 - ((u_j^R)^2\right). \end{cases}$$

$$\begin{cases} \bar{u}_j^L = u_j^n - \frac{s_j^n}{2} + \frac{1}{4}\frac{\Delta t}{\Delta x}\left((u_j^n - \frac{s_j^n}{2})^2 - (u_j^n + \frac{s_j^n}{2})^2\right), \\ \bar{u}_j^R = u_j^n + \frac{s_j^n}{2} + \frac{1}{4}\frac{\Delta t}{\Delta x}\left((u_j^n - \frac{s_j^n}{2})^2 - (u_j^n + \frac{s_j^n}{2})^2\right). \end{cases}$$

$$\begin{cases} \bar{u}_j^L = u_j^n - \frac{s_j^n}{2} + \frac{1}{2}\frac{\Delta t}{\Delta x}\left(s_j^n u_j^n\right), \\ \bar{u}_j^R = u_j^n + \frac{s_j^n}{2} + \frac{1}{2}\frac{\Delta t}{\Delta x}\left(s_j^n u_j^n\right). \end{cases}$$

$$\begin{cases} \bar{u}_j^L = \left(1 + \frac{\Delta t}{2\Delta x} s_j^n\right) u_j^n - \frac{s_j^n}{2}, \\ \bar{u}_j^R = \left(1 + \frac{\Delta t}{2\Delta x} s_j^n\right) u_j^n + \frac{s_j^n}{2}. \end{cases}$$

We solve Riemann problem with the following condition

$$\begin{cases} \bar{u}_{j+1}^L = \left(1 + \frac{\Delta t}{2\Delta x} s_{j+1}^n\right) u_{j+1}^n - \frac{s_{j+1}^n}{2}, \\ \bar{u}_j^R = \left(1 + \frac{\Delta t}{2\Delta x} s_j^n\right) u_j^n + \frac{s_j^n}{2}, \end{cases}$$

and take the solution as cell average.

Among the schemes which are free of numerical oscillations is the Total Variation Diminishing TVD scheme which is free of oscillation whenever stability is assured.

**Definition 3.** A numerical method to solve hyperbolic conservation laws is called total variation diminishing if

$$TV(u^{n+1}) \leq TV(u^n).$$

In other words,

$$\sum_{-\infty}^{\infty} |u_{j+1}^{n+1} - u_j^{n+1}| \leq \sum_{-\infty}^{\infty} |u_{j+1}^n - u_j^n|.$$

we want to choose the slope $s_j^n$ in such way that the scheme becomes a TVD scheme.

**Theorem 4.** Suppose $u_j$ is generated by a numerical method in conservation form with a Lipschitz continuous numerical flux, consistent with some scalar conservation law. If the method is $TV$-stable, i.e., $TV(u^n) = \sum_{-\infty}^{\infty} |u_{j+1}^n - u_j^n|$ is uniformly bounded for all the approximations at different time; $\forall n, j$ with $j < j_0$, $nj \leq T$, then the method is convergent, where $T$ is the final time, (for proof see [2]).

### 4.1.2 Slope Limiter

We define the slope with the following function

$$s_j^n = (u_{j+1}^n - u_j^n)\phi_j^n. \tag{4.18}$$

Here $\phi_j^n = \phi(r_j^n)$ is a function of $r_j^n$ which represents the ratio of consecutive gradients

$$r_j^n = \frac{u_j^n - u_{j-1}^n}{u_{j+1}^n - u_j^n}, \tag{4.19}$$

while $\phi$ is limiter function defined to obtain a TVD method.
We set the following conditions which are sufficient to obtain TVD scheme,

$$\phi(r) = 0 \quad r \leq 0,$$

$$0 \leq \phi(r) \leq 2r.$$

To obtain second order accuracy, additional conditions are to be imposed, for example,

$$\phi(1) = 1.$$

**Remark 4.** We can observe that
If $\phi_j = \phi_{j-1} = 1 \Rightarrow$ Lax–Wendroff (not TVD).
If $\phi_j = \phi_{j-1} = 0 \Rightarrow$ Upwind (TVD).

Various limiter functions have been defined in the literature. Actually, we consider three limiter functions [3] :

**Van Leer's limiter**

$$\phi(r) = \frac{|r| + r}{1 + |r|}. \tag{4.20}$$

In this case we might write the slope as follows

$$\begin{aligned} s_j^n &= (u_{j+1}^n - u_j^n)\frac{|r_j^n| + r_j^n}{1 + |r_j^n|} \\ &= (u_{j+1}^n - u_j^n)\frac{|\frac{u_j^n - u_{j-1}^n}{u_{j+1}^n - u_j^n}| + \frac{u_j^n - u_{j-1}^n}{u_{j+1}^n - u_j^n}}{1 + |\frac{u_j^n - u_{j-1}^n}{u_{j+1}^n - u_j^n}|} \end{aligned}$$

- First case : $u_{j-1}^n < u_j^n < u_{j+1}^n$ or $u_{j-1}^n > u_j^n > u_{j+1}^n$

$$s_j^n = 2\frac{(u_{j+1}^n - u_j^n)(u_j^n - u_{j-1}^n)}{u_{j+1}^n - u_{j-1}^n}. \tag{4.21}$$

- Second case : $u_{j-1}^n < u_j^n$ and $u_j^n > u_{j+1}^n$ or $u_{j-1}^n > u_j^n$ and $u_j^n < u_{j+1}^n$

$$S_j^n = 0. \tag{4.22}$$

**Minmod limiter**

Minmod limiter that represents the lowest boundary of the second-order TVD region:

$$\phi(r) = \begin{cases} \min(r, 1) & r > 0 \\ 0 & r \leq 0. \end{cases} \tag{4.23}$$

It is a particular case of the minmod function, defined as the function that selects the number with the smallest modulus from a series of numbers when they all have the same sign, and zero otherwise. For two arguments:

$$\text{minmod}\,(x, y) = \begin{cases} x & |x| < |y|,\ xy > 0, \\ y & |x| > |y|,\ xy > 0, \\ 0 & xy < 0. \end{cases} \tag{4.24}$$

**Superbee limiter**

The superbee limiter, that represents the upper limit of the second-order TVD region and has been introduced by Roe:

$$\phi(r) = \max[0, \min(2r, 1), \min(r, 2)]. \tag{4.25}$$

These three limiters produce second order scheme when the solution is smooth, and reduce to upwind at the discontinuity.

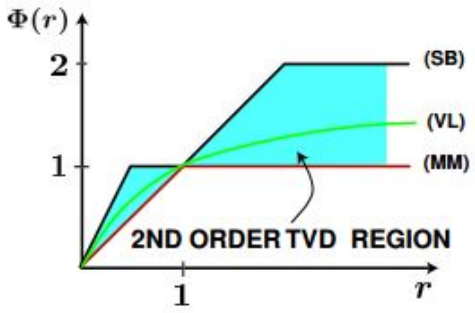



FIGURE 4.3: Popular choices for limiter function [4].

**Remark 5.** These three limiters possess the following property

$$\frac{\phi(r)}{r} = \phi(\frac{1}{r}).$$

This property ensures that the top corner of a discontinuity is related symmetrically to a bottom corner [4].

# Chapter 5

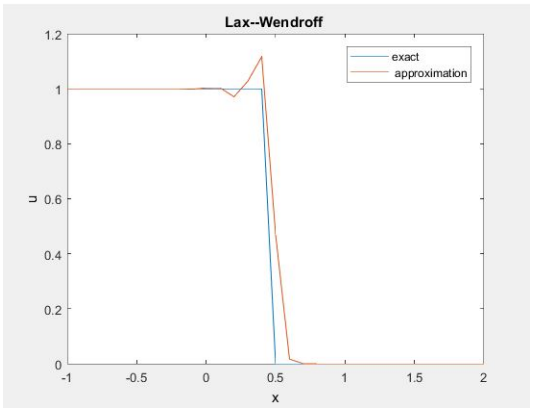
# Numerical Result

In this chapter we will see some results of different numerical methods for conservation laws, using Matlab.

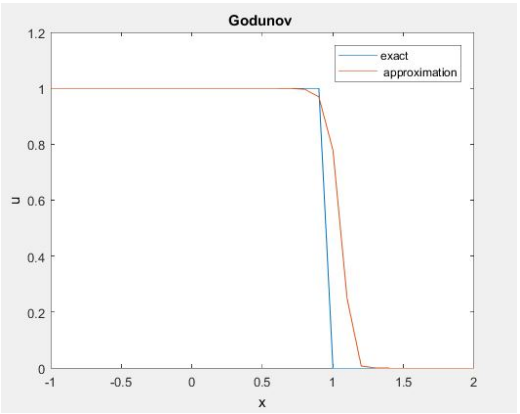
**Example 5.0.1.** *Consider the Burgers equation with discontinuous initial condition and boundary condition,*

$$\begin{cases} u_t + (\frac{u^2}{2})_x = 0 & x \in (-1,2),\ t \in (0,2) \\ u(t,-1) = 1 & t \in (0,2) \\ u(t,2) = 0 & t \in (0,2) \\ u(0,x) = \begin{cases} 1 & \text{if } -1 \leq x < 0\ , \\ 0 & \text{if } 0 \leq x \leq 2. \end{cases} \end{cases}$$

*The following numerical result clearly shows that the Lax–Wendroff's method oscillates at the discontinuity, where Godunov's method does not oscillate.*

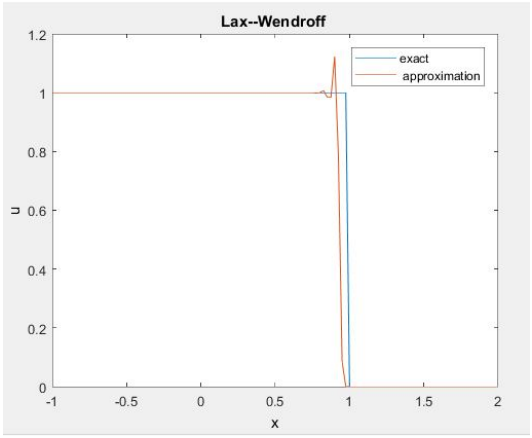



Error =0.0358

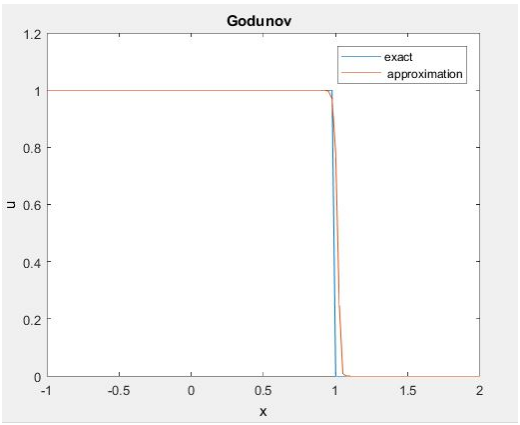



Error=0.1069

Lax–Wendroff and Godunov's methods for $\Delta x = 0.1$, $\Delta t = 0.04$ and $t = 2$.

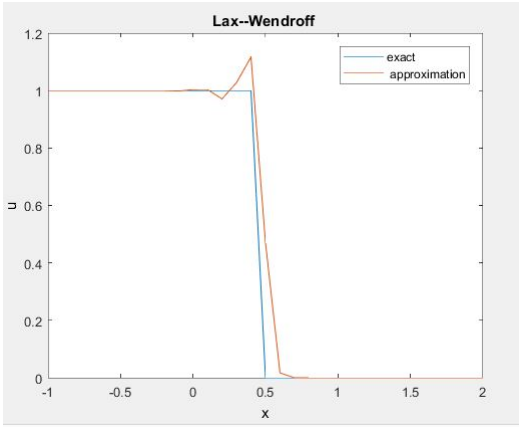



Error =0.0574

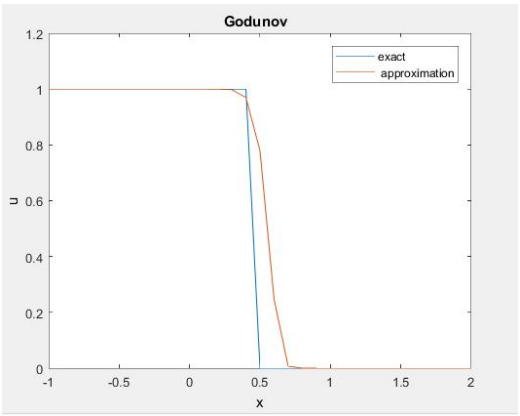



Error=0.0267

Lax–Wendroff and Godunov's methods for $\Delta x = 0.025$, $\Delta t = 0.01$ and $t = 2$.

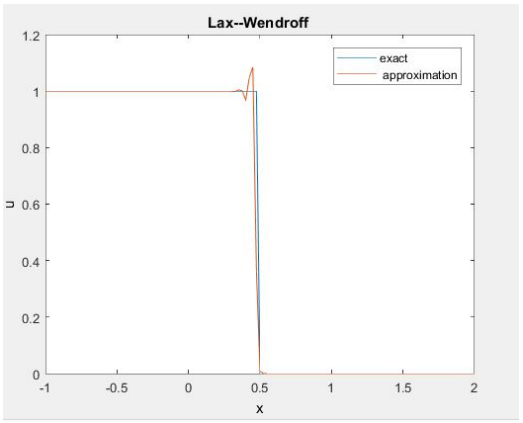



Error =0.0669

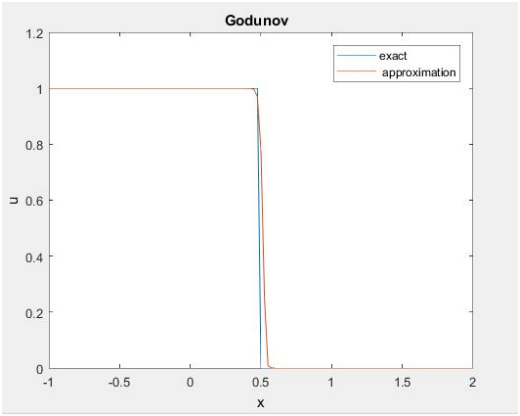



Error=0.1065

Lax–Wendroff and Godunov's methods for $\Delta x = 0.1$, $\Delta t = 0.04$ for $t = 1$.




Error =0.0204




Error=0.0267

Lax–Wendroff and Godunov's methods for $\Delta x = 0.025$, $\Delta t = 0.01$ and $t = 1$.

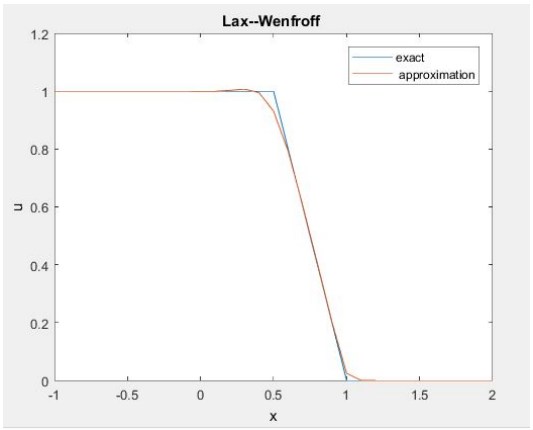
**Example 5.0.2.** *Shock waves*
*We consider Burgers equation with the following initial condition*

$$\begin{cases} u_t + (\frac{u^2}{2})_x = 0 & x \in (-1,2),\ t \in (0,T) \\ u(t,-1) = 1 & t \in (0,T) \\ u(t,2) = 0 & t \in (0,T) \\ u(0,x) = \begin{cases} 1 & x \le 0 \\ 1-x & 0 \le x \le 1 \\ 0 & x \ge 1 \end{cases} \end{cases}$$

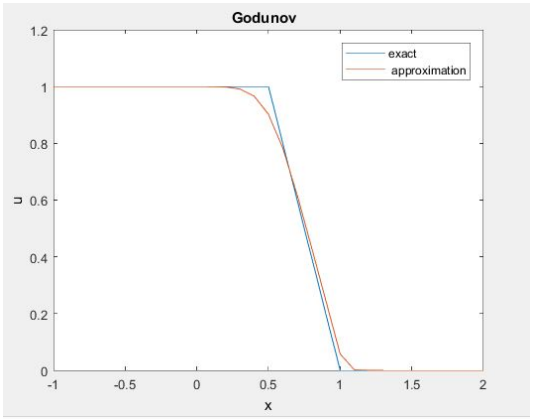
*The exact solution for this problem is:*

$$u(t,x) = \begin{cases} \begin{cases} 1 & x \le t \\ \frac{1-x}{1-t} & t \le x \le 1 \\ 0 & x \ge 1 \end{cases} & t < 1, \\ \begin{cases} 1 & x < s(t) \\ 0 & x > s(t). \end{cases} & t \ge 1 \end{cases}$$

*Here $s$ is the speed $s(t) = \frac{1+t}{2}$.*
*The following numerical result clearly shows that the Lax–Wendroff's method is second order of accuracy on the smooth solutions, where Godunov's method is first order of accuracy.*

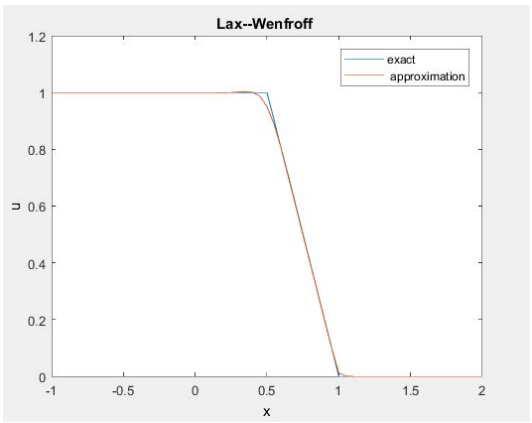



Error =0.0128

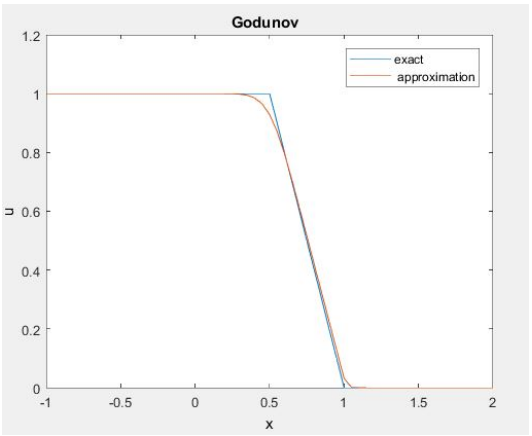



Error=0.0317

Lax–Wendroff and Godunov's methods for $\Delta x = 0.1$, $\Delta t = 0.05$ and $t = 0.5$.

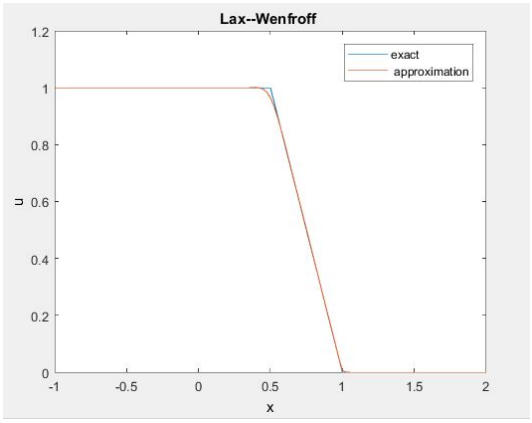



Error =0.0056

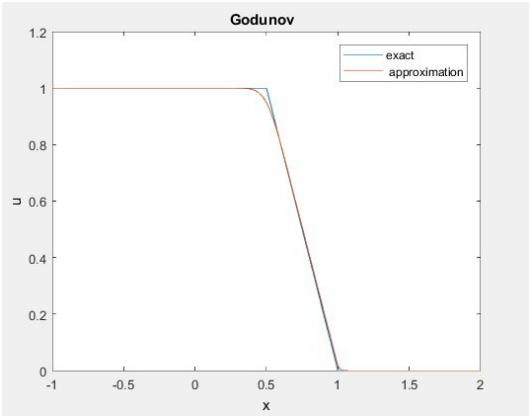



Error=0.0162

Lax–Wendroff and Godunov's methods for $\Delta x = 0.05$, $\Delta t = 0.025$ and $t = 0.5$.




Error= 0.0027




Error=0.0083

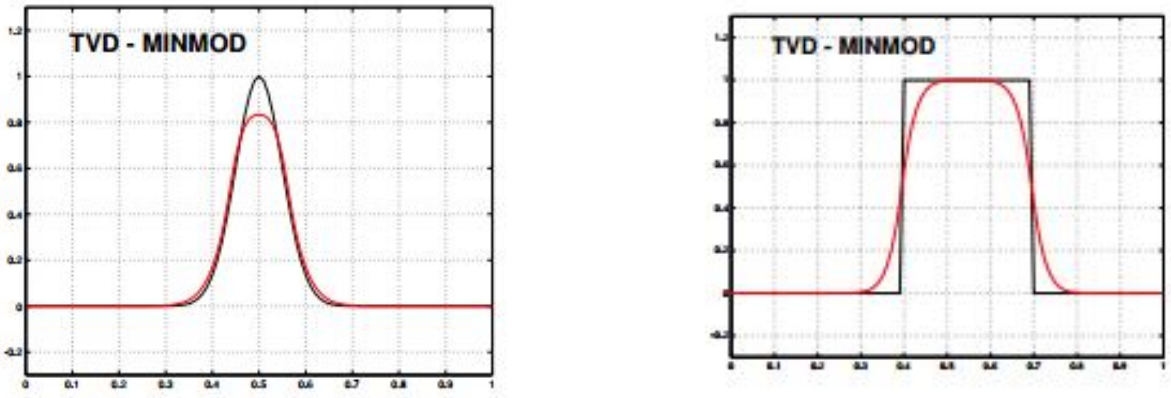
Lax–Wendroff and Godunov's methods for $\Delta x = 0.025$, $\Delta t = 0.0125$ and $t = 0.5$.

**Example 5.0.3.** *Consider the linear advection equation*

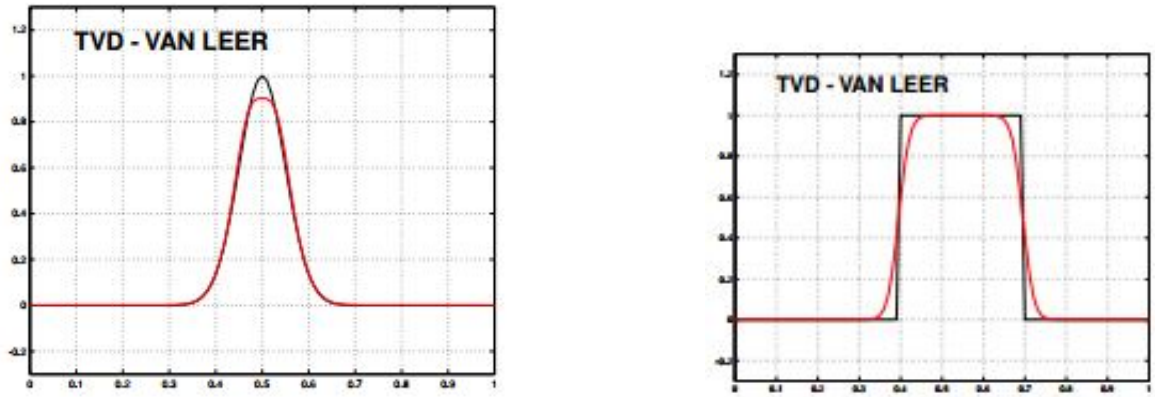
$$u_t + au_x = 0,$$

*with boundary conditions is solved in the unit interval. the domain is subdivided into 100 equal subintervals [4].*

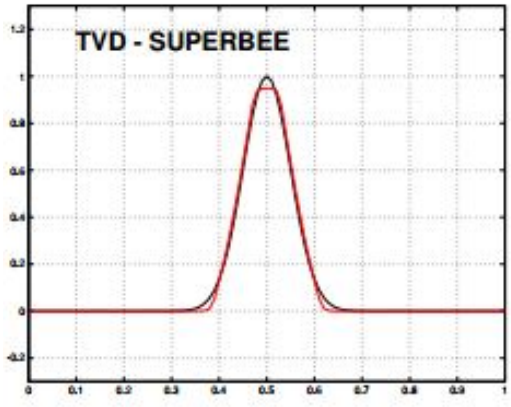
*Below we show the numerical result of this problem correspond to three TVD schemes: the minmod, van Leer and superbee limiters. In the figure we show the exact solution and the computed solution at* $\frac{a\Delta t}{\Delta x} = \frac{1}{2}$*. Two initial conditions with smooth (left) and discontinuous (right) data are considered*

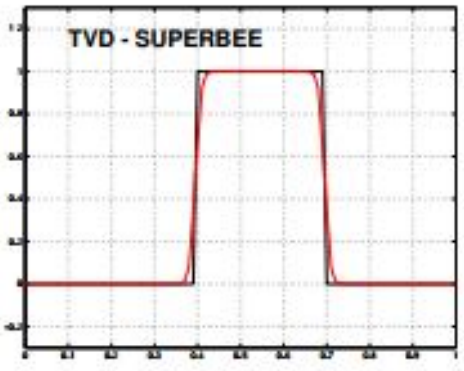



Van Leer's method with minmod limiter [4].




Van Leer's method with van Leer's limiter [4].




Van Leer's method with superbee limiter [4].

# General Conclusion

## *conclusion:*

In this work, we have studied and simulated the solution of conservation laws, with three schemes with continuous and discontinuous initial conditions. The results show that the Lax–Wendroff's scheme oscillates at the discontinuity (not TVD scheme), even if it is second order accurate, differently than Godunov scheme which is first order of accuracy but a TVD scheme. The third scheme was van Leer's scheme, where we have found that this scheme depends on the choice of the slope. For this, We defined slope functions, and stated three popular slope limiters: minmod, van Leer and superbee, which make the scheme to be TVD and second order of accuracy.

## *Recommendations for future research:*

Through this research experience with the topic of conservation laws, we have encountered some points that we found interesting and worth more investigation. We ambition to extend our work to if time allows.
It would be interesting to study the multi-dimensional extension of the presented methods and the extension to systems. Practically, the computational efficiency is the most important question. If the slope limiters increase the computational costs and the complexity of the computation, would it not be better to try an implicit method?.
Also, it would be interesting to see how the slope limiters effect the qualitative properties of the numerical solution, for example their positivity.

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