Eötvös Loránd University, Faculty of Science

Tamás Kátay

Typical compact sets

MSc Thesis

Supervisor:
Tamás Keleti
professor
Department of Analysis

May 22, 2020

Contents

Acknowledgements

I would like to thank several people for their help.

First I would like to express my gratitude to those who helped me through my bachelor studies. That was a really difficult period and I truly needed their help. They are Tamás Szőke, János Wolosz, András Mihálykó, Donát Nagy, Gergely Zábrádi, Kata Kosztolányi, Emil Kiss and my family.

In my master studies I was also supported by a great number of people. I am truly grateful to Tamás Keleti for his help and guidance. I thank Márton Elekes for inspiring me. His lectures are awesome. I deeply appreciate the various discourses with Márk Poór. I am thankful to Boglárka Gehér for her tremendous help.

Introduction

This thesis is a glimpse into the theory of typical compact sets. Our goal is to demonstrate how to exploit the sheer power of the Baire category theorem (hereafter referred to as BCT) in a special setting.

The BCT is one of the important theorems which provide non-constructive methods. As the random method does in graph theory, the BCT allows analysts to reach out for objects which are quite complicated to construct explicitly. If we say that a property P is typical in a given space, then this means intuitively that almost all elements satisfy P. Another huge advantage of proving typicality is that if one accumulates several (even countably many) typical properties, then their conjunction is still typical.

Let us introduce two examples. In 1872 Weierstrass published the first example of a continuous nowhere differentiable function. Since then mathematicians have constructed several other examples. However, the proof of nowhere differentiability usually involves tedious calculations. A more elegant and much more meaningful approach is to prove that the property of being nowhere differentiable is typical in the complete metric space $C[0,1]$ (which is the set of all continuous functions on the interval $[0,1]$ with the usual supremum metric $d(f,g) = \sup_{x\in[0,1]} |f(x)-g(x)|$). We discuss this in Section 1.2. The same applies for nowhere monotonic continuous functions.

Another interesting problem is the existence of Nikodym sets. A Nikodym set is a set $N \subseteq \mathbb{R}^n$ of Lebesgue measure zero which contains a punctured hyperplane through every point of $\mathbb{R}^n$. The first example of such a subset in the plane was constructed by Otto Nikodym in 1927 [1]. Higher dimensional Nikodym sets were found by Kenneth Falconer [2]. More than the existence may be proved roughly as follows. We code the arrangement of hyperplanes in a suitable way, and then we use typical code sets. For a precise proof see [3]. This method illustrates a stunning possibility. Even when the objects we are interested in do not form a complete metric space naturally we may find a way to transform the problem into another to which the BCT is applicable. See [4] for a survey on the applications of the BCT.

This thesis is divided into three chapters.

Chapter 1 contains all the **groundwork**. First we introduce the basic notions and the BCT itself. Then we present the above-mentioned typicality argument about continuous functions as a nice and direct application. In Section 3 we prove Alexandroff's theorem about topological completeness which allows one to apply the BCT for a wide class of subspaces of a complete metric space. In Section 4 we introduce the hyperspace of compact sets, where most of our latter arguments take place. As we will see, for any complete metric space X the set of all nonempty compact subsets of X form a complete metric space with respect to a natural metric. In the end of Chapter 1 we sketch the basics of Hausdorff dimension and box dimension which show up in Chapter 2.

In Chapter 2 we go for a journey of discovery. One question leads us: **How does a typical compact set look like?** We explore the hyperspace of compact sets by asking several natural questions and answering them. First we look at the size: we investigate measure and dimension. After that we look for patterns and order. Then we turn to topological properties and we characterize typical compact sets up to homeomorphism. We encounter distance sets on the way which leads us to product spaces. Finally we prove a theorem of Mycielski and Kuratowski. In this chapter the author presents his own proofs for almost all results. Most of them are well-known and many of them can be found in textbooks. We do not claim that our proofs are new.

In Chapter 3 we discuss a more subtle application which takes advantage of the above-mentioned code set method. First we introduce the motivation: Besicovitch sets and the famous Kakeya conjecture. Then we explain the duality (or code set) method in details and we present some important observations. We use a result of M. Talagrand [5] and a theorem of K. Simon and B. Solomyak [6] to obtain a typical property for code sets. Finally we prove a theorem on Besicovitch sets. **In this last chapter the author presents his own results** based on his paper [7].

Notations are listed on the next page. Note that we use some non-standard notations throughout the thesis.

Notations

$\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$	the set of natural, rational and real numbers respectively
$\mathbb{N}^+$	the set of positive natural numbers
$:=$	equals by definition
$\dot{\cup}$	disjoint union
A^c	the complement of A
$B(a,r)$	the open ball of radius r and center a
$\dot{B}(a,r)$	$B(a,r) \setminus \{a\}$
$B^d(a,r)$	the open ball of radius r and center a with respect to the metric d
$B_H(A,\delta)$	the open ball of radius δ and center A with respect to the Hausdorff metric
$\overline{B}(a,r)$	the closed ball of radius r and center a
$\overline{A}$	the closure of A
A_ε	the open ε-neighbourhood of the set A
A^*_ε	the closed ε-neighbourhood of the set A
$A_{\varepsilon,d}$	A_ε with respect to the metric d
$A^*_{\varepsilon,d}$	A^*_ε with respect to the metric d
$x_n \xrightarrow{d} x$	x_n converges to x with respect to the metric d
$C[a,b]$	the space of continuous functions on the interval $[a,b]$
$[A]^{<\omega}$	the set of finite subsets of A
$\cong$	homeomorphic to
$\lambda(A)$	the 1-dimensional Lebesgue measure of A
$\lambda^d(A)$	the d-dimensional Lebesgue measure of A

Chapter 1

Preliminaries

1.1 The Baire category theorem

The Baire category theorem (BCT) is a theorem in general topology which has numerous important applications in many fields of mathematics, especially in functional analysis. However, in this thesis we focus on direct applications of BCT yielding sets of interesting properties.

For the sake of clarity we start with basic definitions.

Definition 1.1.1. Let X be a topological space and $E \subseteq X$.

- E is **dense** in X if its closure is X.
- E is **nowhere dense** in X if its closure has empty interior.
- E is of **first category** in X if it is the countable union of nowhere dense sets.
- E is of **second category** in X if it is not of first category.
- E is **residual** in X if its complement is of first category.

Remark 1.1.2. It is easy to check that:

(1) E is dense in $X \iff$ it meets every nonempty open subset of X.

(2) E is nowhere dense $\iff$ every nonempty open subset of X contains an open set disjoint to $E \iff X \setminus E$ contains a dense open set.

Definition 1.1.3. A topological space X is a **Baire space** if every nonempty open subset of X is of second category in X.

Definition 1.1.4. A topological space X is **topologically complete** (or completely metrizable) if it is homeomorphic to a complete metric space.

The Baire category theorem has many formulations, we present two of them here.

Theorem 1.1.5. ***(Baire category theorem)***

(1) *Every topologically complete space is a Baire space.*

(2) *Every locally compact Hausdorff space is a Baire space.*

Proof. (1) Let X be topologically complete space and d a metric which makes it a complete metric space. Let $U \subseteq X$ be a nonempty open set. We need to show that U cannot be covered by countably many nowhere dense sets $N_i \subseteq X$ ($i \in \mathbb{N}^+$).

Let $B_0 \subseteq U$ be a closed ball. We define a nested sequence of closed balls B_i such that $B_i \cap \bigcup_{j \leq i} N_i = \emptyset$. Suppose that B_j ($j \leq i$) are defined. Since N_{i+1} is nowhere dense, we can choose a closed ball $B_{i+1} \subseteq B_i$ of diameter at most $\frac{1}{i+1}$ which is disjoint to N_{i+1}. Pick $x_i \in B_i$ for every $i \in \mathbb{N}$. The points x_i form a Cauchy-sequence because of the assumption on diameters, hence $x_i \to x$ for some $x \in X$ by the completeness of X. Now $x \in B_i$ for every i since B_i is closed. This gives us $x \in U \setminus (\bigcup_{i=1}^{\infty} N_i)$.

(2) The proof goes similarly as the previous one. Let X be a locally compact Hausdorff space, U and N_i as before. Let $K_0 \subseteq U$ be an arbitrary compact set with nonempty interior (the local compactness is used here). We define a nested sequence of compact sets K_i with nonempty interior such that $K_i \cap \bigcup_{j \leq i} N_j = \emptyset$. Suppose that K_j ($j \leq i$) are defined. Now $\operatorname{int}(K_i)$ contains an open set U_i which is disjoint to N_{i+1} because the latter is nowhere dense. K_i is a compact Hausdorff space; therefore we may choose an open set $V_i \subseteq U_i$ such that $\overline{V_i} \subseteq U_i$. Let $K_{i+1} = \overline{V_i}$. Now $\bigcap_{i=0}^{\infty} K_i \subseteq U$ is the intersection of nested nonempty compact sets, so it is nonempty and clearly disjoint to $\bigcup_{i=1}^{\infty} N_i$. □

Note that we need the notion of topological completeness because completeness is not a topological property. For example, $(0, 1)$ is not complete but it is homeomorphic to $\mathbb{R}$. However, being a Baire space is a topological property by definition.

Definition 1.1.6. In a Baire space X the property P is **typical** (or generic) if $\{x \in X : P(x)\}$ is residual.

By the definition of Baire space a residual set is nonempty. Consequently, if P is typical in X, then there exists $x \in X$ such that $P(x)$ holds. We often use a less formal phrasing: instead of saying that P is typical in X we simply say that a typical element x in X is of property P.

A simple way to check if a set is residual is given by the following:

Proposition 1.1.7. *In a Baire space X, a set E is residual if and only if E contains a dense G_δ set.*

Proof. Note that N is nowhere dense $\iff$ $\overline{N}$ is nowhere dense.

E is residual $\overset{def}{\iff}$ E^c is of first category $\overset{def}{\iff}$ $E^c = \bigcup_{i=1}^{\infty} N_i$ for some nowhere dense sets N_i $\iff$ $E^c \subseteq \bigcup_{i=1}^{\infty} F_i$ for some closed nowhere dense sets F_i $\iff$ $E \supseteq \bigcap_{i=1}^{\infty} G_i$ for some open dense sets G_i $\overset{BCT}{\iff}$ E contains a dense G_δ set. $\square$

Before going deeper in the theory we discuss a nice direct application which shows the strength of BCT.

1.2 Typical continuous functions

In this section we show that nowhere differentiable continuous functions exist. Let $C[0,1]$ denote the set of all real-valued continuous functions on the interval $[0,1]$. It is a complete metric space with respect to the usual supremum metric $d(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|$. We prove more than nowhere differentiability:

Theorem 1.2.1. *A typical continuous function in* $C[0,1]$ *does not have bounded difference quotients on either side at any point.*

Proof. The union of two sets of first category is of first category, therefore it suffices to verify that

$$B := \left\{ f \in C[0,1] : \ \exists x_0 \in (0,1] \sup_{x\in[0,x_0)} \left| \frac{f(x) - f(x_0)}{x - x_0} \right| < \infty \right\}$$

is of first category. This may be written as a countable union:

$$B = \bigcup_{N=1}^{\infty} \underbrace{\left\{ f \in C[0,1] : \ \exists x_0 \in \left[\frac{1}{N}, 1\right] \sup_{x\in[0,x_0)} \left| \frac{f(x) - f(x_0)}{x - x_0} \right| \leq N \right\}}_{B_N}$$

We want to prove that B_N is nowhere dense in $C[0,1]$.

Claim 1.2.2. B_N *is closed.*

Proof. If $f \notin B_N$, then for every $x_0 \in \left[\frac{1}{N}, 1\right]$ we have

$$\sup_{x\in[0,x_0)} \left| \frac{f(x) - f(x_0)}{x - x_0} \right| > N.$$

Thus there exists $x_0' \in [0, x_0)$ and $\eta > 0$ such that $\left| \frac{f(x_0') - f(x_0)}{x_0' - x_0} \right| > N + 2\eta$. By continuity there exists $|x_0' - x_0| > \delta > 0$ such that for all $y \in (x_0 - \delta, x_0 + \delta)$ we have $\left|\frac{f(x_0')-f(y)}{x_0'-y}\right| > N+\eta$. Now the absolute value of the denominator is bounded from below, so there exists $\varepsilon_0 > 0$ such that if $\sup|f-g| < \varepsilon_0$, then $\left|\frac{g(x_0')-g(y)}{x_0'-y}\right| > N$ for all $y \in (x_0-\delta, x_0+\delta)$. Since $\left[\frac{1}{N}, 1\right]$ is compact, it is covered by finitely many intervals of the form $(x_0-\delta, x_0+\delta)$ giving us finitely many conditions. Hence we can choose $\varepsilon > 0$ such that if $\sup|f-g| < \varepsilon$, then for every $x_0 \in \left[\frac{1}{N}, 1\right]$

$$\sup_{x\in[0,x_0)} \left|\frac{g(x)-g(x_0)}{x-x_0}\right| > N.$$

This means that a neighbourhood of f lies in the complement of B_N. Consequently, B_N is closed. ■

Now we need to show that $\operatorname{int}(B_N) = \emptyset$. In fact, it is easy to prove that B_N^c is dense in $C[0,1]$. The following lemma is a useful tool in general.

Lemma 1.2.3. *The set of piecewise linear functions is dense in* $C[0,1]$.

Proof. Let $f \in C[0,1]$ and $\varepsilon > 0$ be arbitrary. We have to find a piecewise linear function in the 2ε-neighbourhood of f. Since f is uniformly continuous on $[0,1]$, there exists $M \in \mathbb{N}$ such that for every interval of the form $\left[\frac{i}{M}, \frac{i+1}{M}\right]$ $(0 \le i < M)$ we have

$$\sup_{x\in\left[\frac{i}{M},\frac{i+1}{M}\right]} \left|f(x) - f\left(\frac{i}{M}\right)\right| < \varepsilon.$$

Simply let g be the function which takes the same values at $\frac{i}{M}$ $(0 \le i \le M)$ as f and linear between these points. Now let $x \in [0,1]$ be arbitrary and $\left[\frac{i}{M}, \frac{i+1}{M}\right]$ is the interval containing x.

$$|f(x)-g(x)| \le \left|f(x) - f\left(\tfrac{i}{M}\right)\right| + \left|f\left(\tfrac{i}{M}\right) - g(x)\right| < \varepsilon + \varepsilon$$

■

Thus it suffices to approximate an arbitrary piecewise linear function g. By the previous proof we may assume that its partition is of the form $\left[\frac{i}{M}, \frac{i+1}{M}\right]$ $(0 \le i < M)$ for some $M \in \mathbb{N}$. Fix $\varepsilon > 0$. Let $K \in \mathbb{N}$ be such that $K\varepsilon$ is larger in absolute value than all the slopes occurring in the graph of g and larger than N as well. Let $h : \mathbb{R} \to \mathbb{R}$ be the "sawtooth" function: $h(x)$ is the distance of x from the nearest integer. Now $h^*(x) := 2\varepsilon h(Kx)$ is a piecewise linear continuous function with supremum ε consisting only of segments of slope $2\varepsilon K$. Therefore $g + h^*$ is a piecewise linear continuous function with slopes strictly larger in absolute value than $2\varepsilon K - \varepsilon K \ge N$. For such a function it is clear that

$$\sup_{x\in[0,x_0)} \left|\frac{(g+h^*)(x)-(g+h^*)(x_0)}{x-x_0}\right| > N$$

for all $x_0 \in \left[\frac{1}{N}, 1\right]$. In other words, $g + h^* \in B_N^c$ and it is ε-close to g. Consequently, B_N^c is dense in $C[0,1]$, and this completes the proof. □

A similar proof can be found in [8] Chapter 11.

1.3 Topological completeness

In this section we prove the theorem of Alexandroff and its converse to obtain a criterion on topological completeness. We altered the proofs of [8] Chapter 12 to improve clarity.

Theorem 1.3.1. *In a complete metric space (X, d) the subspace E is topologically complete if and only if E is G_δ.*

Proof. **(1) G_δ implies topologically complete:**

Let $E = \bigcap_{i=1}^{\infty} G_i$ for some open sets G_i. We shall introduce a compatible metric (i.e., a metric which induces the same topology) which makes E a complete metric space. We may assume that G_i^c is nonempty for every i.

We want to prevent the Cauchy sequences in E to converge to a point not in E. So we should remetrize E to avoid the Cauchy property of these "wrong" sequences while not changing the topology. Let

$$\sigma(x, y) := d(x, y) + \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \min\left(1, \left|\frac{1}{d(x, G_i^c)} - \frac{1}{d(y, G_i^c)}\right|\right) \qquad (\forall x, y \in E).$$

Here $d(x, G_i^c)$ denotes the distance of x from the closed set G_i^c. The idea behind this formula is to enlarge the distances near the boundaries.

First of all, we need to check that σ is a metric on E. The symmetry is clear, and $\sigma(x, y) = 0 \implies d(x, y) = 0 \implies x = y$. The triangle inequality holds because it is satisfied by each term (it is very easy to check).

Now we show that σ is compatible with d. A simple characterization of compatible metrics is the following:

Lemma 1.3.2. *Let d_1 and d_2 be metrics on the set X. They induce the same topology on X if and only if $(x_n \xrightarrow{d_1} x) \iff (x_n \xrightarrow{d_2} x)$ holds for every sequence $\{x_n : n \in \mathbb{N}\} \subseteq X$ and point $x \in X$.*

Proof. Convergence is a topological property, hence one implication is obvious.

For the converse, recall that in a metric space (X, d) the set F is closed if and only if for every sequence $\{x_n : n \in \mathbb{N}\} \subseteq F$ and point $x \in X$ we have $(x_n \xrightarrow{d} x) \implies x \in F$. ■

It is clear that $x_n \xrightarrow{\sigma} x$ implies $x_n \xrightarrow{d} x$.

To prove the converse fix a convergent sequence $x_n \xrightarrow{d} x$ in E and $\varepsilon > 0$. We should find $N \in \mathbb{N}$ such that $\sigma(x_n, x) < \varepsilon$ for all $n \geq N$. Let $N_1 \in \mathbb{N}$ be such that $d(x_n, x) < \frac{\varepsilon}{3}$ whenever $n \geq N_1$ and $\frac{1}{2^{N_1}} < \frac{\varepsilon}{3}$ which makes

$$\sum_{i=N_1+1}^{\infty} \frac{1}{2^i} \cdot \min\left(1, \left|\frac{1}{d(x, G_i^c)} - \frac{1}{d(y, G_i^c)}\right|\right) < \frac{\varepsilon}{3}.$$

Now we have to deal with finitely many terms. Clearly it suffices to find N such that

$$\left|\frac{1}{d(x, G_i^c)} - \frac{1}{d(x_n, G_i^c)}\right| < \frac{\varepsilon}{3N_1}$$

for every $1 \leq i \leq N_1$ and $n \geq N$. This is possible since every $\frac{1}{d(x, G_i^c)}$ is a continuous function on E (with respect to d).

Let us prove that we really unmade the "wrong" Cauchy sequences. Let $x_n \xrightarrow{d} x$ for some $\{x_n : n \in \mathbb{N}\} \subseteq E$ and $x \notin E$. Then $x \in G_i^c$ for some $i \in \mathbb{N}^+$, and for any $m \in \mathbb{N}$

$$\frac{1}{2^i} \cdot \min\left(1, \left|\frac{1}{d(x_m, G_i^c)} - \frac{1}{d(x_n, G_i^c)}\right|\right) = \frac{1}{2^i}$$

if n is sufficiently large since $\frac{1}{d(x_n, G_i^c)}$ diverges. Therefore $\{x_n : n \in \mathbb{N}\}$ is not Cauchy with respect to σ.

(2) Topologically complete implies G_δ:

Let $f : E \to Y$ be a homeomorphism onto some complete metric space (Y, ϱ). By continuity for every $n \in \mathbb{N}^+$ and $x \in E$ there exists $0 < \delta(x, n) < \frac{1}{n}$ such that

$$f\left(B^d(x, \delta(x, n)) \cap E\right) \subseteq B^\varrho\big(f(x), \tfrac{1}{n}\big).$$

Let

$$G_n := \bigcup_{x \in E} B^d\left(x, \frac{\delta(x, n)}{2}\right).$$

Each G_n is an open set containing E. Hence it suffices to show $E \supseteq \bigcap_{n=1}^{\infty} G_n$.

Pick $x \in \bigcap_{n=1}^{\infty} G_n$. For every $n \in \mathbb{N}^+$ there exists a point $x_n \in E$ such that $d(x, x_n) < \frac{\delta(x_n, n)}{2} < \frac{1}{2n}$ which implies $x_n \xrightarrow{d} x$. Thus for n, m large enough we have

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) \leq \tfrac{\delta(x_n, n)}{2} + \tfrac{\delta(x_m, m)}{2} < \max(\delta(x_n, n), \delta(x_m, m)).$$

Therefore $f(x_n)$ is Cauchy in Y. Let y be its limit. Then $x_n \xrightarrow{d} f^{-1}(y)$ because f^{-1} is continuous. However we already noted $x_n \xrightarrow{d} x$ which gives us $x = f^{-1}(y) \in E$. The proof is complete. □

1.4 The hyperspace of compact sets

In this section we introduce the space where most of our latter arguments take place. A slightly different approach is presented in [9].

Definition 1.4.1. Let (X, d) be a metric space and $A, B \subseteq X$ bounded subsets. The **Hausdorff distance** of A and B is

$$d_H(A, B) := \inf\{\delta > 0 : \ B \subseteq A_\delta, \ A \subseteq B_\delta\}.$$

As one notes easily $d_H((0, 1), [0, 1]) = 0$ for $X = \mathbb{R}$, so d_H is not a metric in general on the set of bounded subsets. It turns out to be a pseudometric. However we do not need this general setting, so we focus our attention to compact subsets. Let $\mathcal{K}(X)$ denote the set of nonempty compact subsets of X.

Proposition 1.4.2. *$\mathcal{K}(X)$ is a metric space with respect to the Hausdorff distance.*

Proof. To simplify calculations observe that we get an equivalent definition of d_H if we switch the open neighbourhoods to closed ones. Let A^*_δ denote the closed δ-neighbourhood of the set A. Moreover, if we use closed neighbourhoods, then infimum becomes minimum since $A^*_\delta = \bigcap_{\varepsilon>\delta} A^*_\varepsilon$.

(1) d_H is non-negative. Also $d_H(K_1, K_2) = 0$ means $K_1 = K_2$: if one of them would have a point not contained in the other, then that point would have a positive distance from the other set because that set is compact.

(2) The definition is symmetric.

(3) To check the triangle inequality let K, M and N be nonempty compact sets, $d_1 := d_H(K, M)$ and $d_2 := d_H(M, N)$. We need to verify that $M \subseteq K^*_{d_1+d_2}$ and $K \subseteq M^*_{d_1+d_2}$. The triangle inequality for (X, d) gives us

$$N \subseteq M^*_{d_2} \subseteq (K^*_{d_1})^*_{d_2} \subseteq K^*_{d_1+d_2}.$$

The other inclusion follows by symmetry. □

There are two easy observations both of which are of key importance.

Proposition 1.4.3. *For any metric space (X, d) the set of finite subsets of X is dense in $(\mathcal{K}(X), d_H)$.*

Proof. For any $K \in \mathcal{K}(X)$ and $\varepsilon > 0$ there exists a finite ε-net $F \subseteq K$. By the definition of Hausdorff distance, we have $d(F, K) \leq \varepsilon$. □

Proposition 1.4.4. *If (X, d) is a metric space, $U \subseteq X$ is an open set, then $A_U = \{K \in \mathcal{K}(X) : \ K \subseteq U\}$ and $B_U = \{K \in \mathcal{K}(X) : \ K \cap U \neq \emptyset\}$ are open sets in $\mathcal{K}(X)$.*

Proof. Fix $K \in A_U$. Since K and U^c are disjoint, they have positive distance. Also $K' \in B^{d_H}(K, \varepsilon)$ implies $K' \subseteq K_\varepsilon$ for any $\varepsilon > 0$, hence K is an interior point.

Fix $K \in B_U$. There is some $x \in K \cap U$ which has positive distance from U^c. However, for any $\varepsilon > 0$ and $K' \in B^{d_H}(K, \varepsilon)$ the set K' must contain a point ε-close to x. Thus for suitably small ε we have $B^{d_H}(K, \varepsilon) \subseteq B_U$. Consequently, B_U is open. □

This could serve as a starting point: for any topological space X one can define the topology on $\mathcal{K}(X)$ by taking sets of the form A_U and B_U for a base. Again, we do not need this general setting because our focus is on metric spaces.

We will see in the following theorems that $(\mathcal{K}(X), d_H)$ inherits several properties of (X, d).

Theorem 1.4.5. *Let (X, d) be a metric space. Then (X, d) is separable if and only if $(\mathcal{K}(X), d_H)$ is separable.*

Proof. (1) Let S be a countable dense set in X. Let $\mathcal{S} := [S]^{<\omega}$, that is, the set of finite subsets of S. We claim that $\mathcal{S}$ is dense in $\mathcal{K}(X)$. By Proposition 1.4.3 it suffices to approximate finite subsets of X. Let $\{x_1, \ldots, x_N\} \subseteq X$. Since S is dense, there exists $x_i' \in S$ for all $1 \leq i \leq N$ such that $d(x_i, x_i') < \varepsilon$. Then $d(\{x_1, \ldots, x_N\}, \{x_1', \ldots, x_N'\}) < \varepsilon$ and $\{x_1', \ldots, x_N'\} \in \mathcal{S}$.

(2) Let $\mathcal{S}$ be a countable dense set in $\mathcal{K}(X)$. Form S by picking a point from each $K \in \mathcal{S}$. Now S is dense in X since for any $x \in X$ and $\varepsilon > 0$ there exists $K \in \mathcal{S}$ such that $d(x, y) \leq d_H(\{x\}, K) < \varepsilon$ holds for any $y \in K$, specifically, it holds for some $y \in S$. □

Remark 1.4.6. Observe that the singletons constitute a closed subspace in $\mathcal{K}(X)$, and $x \mapsto \{x\}$ is an isometry from X onto this subspace.

Theorem 1.4.7. *The metric space (X, d) is complete if and only if $(\mathcal{K}(X), d_H)$ is complete.*

Proof. (1) If $\mathcal{K}(X)$ is complete, then X is complete by Remark 1.4.6.

(2) Let X be complete and K_n $(n \in \mathbb{N})$ be a Cauchy sequence in $\mathcal{K}(X)$. By the Cauchy property there is a subsequence K_{n_i} $(i \in \mathbb{N})$ such that

$$d_H(K_{n_i}, K_m) < \frac{1}{2^{i+1}} \tag{1.1}$$

for each i and $m \geq n_i$. Specifically, for $i < j$ we have $K_{n_j} \subseteq (K_{n_i})_{\frac{1}{2^{i+1}}}$ which implies $(K_{n_j})^*_{\frac{1}{2^j}} \subseteq (K_{n_i})^*_{\frac{1}{2^i}}$. A natural candidate for the limit is the intersection

$$\widetilde{K} := \bigcap_{i=0}^{\infty} (K_{n_i})^*_{\frac{1}{2^i}}.$$

To verify our foreknowledge fix $\varepsilon > 0$. Clearly $\widetilde{K}$ is closed. It is totally bounded because for any $\delta > 0$ a finite $\frac{1}{2^i}$-net in K_{n_i} is a δ-net in $(K_{n_i})^*_{\frac{1}{2^i}} \supseteq \widetilde{K}$ if $\frac{1}{2^i} < \frac{\delta}{2}$. Thus $\widetilde{K}$ is compact.

It suffices to show that $d_H(K_{n_i}, \widetilde{K}) \leq \frac{1}{2^i}$ since this and (1.1) together say that if $\frac{1}{2^{i-1}} < \varepsilon$, then n_i is a suitable threshold for ε. We have $\widetilde{K} \subseteq (K_{n_i})^*_{\frac{1}{2^i}}$ by the definition of K. To prove $K_{n_i} \subseteq \widetilde{K}_{\frac{1}{2^i}}$ fix any point $x_i \in K_{n_i}$. Let $x_j := x_i$ for all $j \leq i$. We can pick $x_j \in K_{n_j}$ for each $j > i$ such that $d(x_j, x_{j+1}) < \frac{1}{2^{j+1}}$ for every $j \in \mathbb{N}$ because of (1.1). The Cauchy sequence x_i converges to some point $x \in X$ since X is complete. Clearly $d(x_j, x) < \frac{1}{2^j}$ for each $j \in \mathbb{N}$ which means $x \in \widetilde{K}$. For $i = j$ this gives us $x_i \in \widetilde{K}_{\frac{1}{2^i}}$. Note that this proves $\widetilde{K} \neq \emptyset$ as well. □

Theorem 1.4.8. *The metric space (X, d) is compact if and only if $(\mathcal{K}(X), d_H)$ is compact.*

Proof. (1) If $\mathcal{K}(X)$ is compact, then X is compact by Remark 1.4.6.

(2) Recall that a metric space is compact if and only if it is complete and totally bounded. Let X be compact and thereby complete. By Theorem 1.4.7 $\mathcal{K}(X)$ is complete as well. Let $\varepsilon > 0$ and N be an ε-net in X. Observe that $\mathcal{P}(N)$ is a ε-net in $\mathcal{K}(X)$ since $d_H(K_\varepsilon \cap N, K) < \varepsilon$ as one checks easily. □

Since we already dedicated a section to a remetrization problem, it is quite natural to ask whether compatible metrics d_1 and d_2 on X give rise to compatible metrics d_H^1 and d_H^2 on $\mathcal{K}(X)$. To investigate this we need another criterion.

Lemma 1.4.9. *The metrics d_1 and d_2 on the set X are compatible if and only if for every $x \in X$ and $\varepsilon > 0$ there exist $\delta_1, \delta_2 > 0$ such that*

$$B^{d_1}(x, \delta_1) \subseteq B^{d_2}(x, \varepsilon) \quad \textit{and} \quad B^{d_2}(x, \delta_2) \subseteq B^{d_1}(x, \varepsilon).$$

Proof. (1) Suppose that d_1 and d_2 are compatible. Then $B^{d_2}(x, \varepsilon)$ is open with respect to d_1 as well, hence some ball witnesses that x is an interior point. The other inclusion follows by symmetry.

(2) For the converse recall that $G \subseteq X$ is open if and only if all of its points are interior points (and balls witness this). Notice that our criterion says that witnesses exist for d_1 if and only if they exist for d_2. ■

Theorem 1.4.10. *If the metrics d_1 and d_2 on the set X are compatible, then d_H^1 and d_H^2 are compatible metrics on $\mathcal{K}(X)$.*

Proof. We rely on Lemma 1.4.9. By symmetry it suffices to show that for any $K \in \mathcal{K}(X)$ and $\varepsilon > 0$ there exists $\delta > 0$ such that $B^{d_1}(K, \delta) \subseteq B^{d_2}(K, \varepsilon)$. Fix K and ε.

Let us denote the ε-neighbourhood of A with respect to the metric d by $A_{\varepsilon,d}$. We need $\delta > 0$ such that

$$(1)\ K' \subseteq K_{\varepsilon,d_2} \quad \text{and} \quad (2)\ K \subseteq K'_{\varepsilon,d_2} \quad \text{whenever} \quad d^1_H(K,K') < \delta.$$

Note that the set $B_0 := \{K' \in (\mathcal{K}(X), d^1_H) :\ K' \subseteq K_{\varepsilon,d_2}\}$ is open in $(\mathcal{K}(X), d^1_H)$ by Proposition 1.4.4. Consider a finite $\frac{\varepsilon}{2}$-net $\{x_1, \dots, x_N\}$ in K. Now $\bigcup_{i=1}^N B^{d_2}\big(x_i, \frac{\varepsilon}{2}\big)$ is a cover of K with the following property: if K' intersects each of these balls, then $K \subseteq K'_{\varepsilon,d_2}$. Note that the set $B_i := \big\{K' \in (\mathcal{K}(X), d^1_H) :\ K' \cap B^{d_2}\big(x_i, \frac{\varepsilon}{2}\big) \neq \emptyset\big\}$ is open in $(\mathcal{K}(X), d^1_H)$ by Proposition 1.4.4. Thus $\bigcap_{i=0}^N B_i$ is an open neighbourhood of K such that for all $K' \in \bigcap_{i=0}^N B_i$ (1) and (2) holds. Hence there exists a suitable δ. □

1.5 Hausdorff, box and similarity dimensions

For the sake of clarity we present a brief outline for the notion of Hausdorff and box-counting dimensions. Much more detailed discussion may be found in [10].

The problem of investigating the notion of dimension may arise when one encounters fractals, or more concretely, self-similar sets. If we apply a similarity of ratio c to a d-dimensional set, we get a set whose d-dimensional Lebesgue measure is c^d-times larger. Let us informally assume the existence of some nice translation-invariant measure which assigns a finite positive value to the triadic Cantor set C. If we apply a similarity of ratio 3 to C, we get a bigger Cantor set which is the disjoint union of 2 copies of C. This suggests $3^{\dim(C)} = 2$ which results $\dim(C) = \frac{\log 2}{\log 3}$.

More generally, let K be a self-similar compact set generated by similarities f_i of ratio $0 < q_i < 1$ for $1 \le i \le m$. That is, $K = \bigcup_{i=1}^m f_i(K)$. The unique $s \ge 0$ for which $\sum_{i=1}^m q_i^s = 1$ holds is the **similarity dimension** of K.

To back up arguments of the previous kind we introduce a measure. Let (X, ϱ) be a metric space and denote the diameter of $E \subseteq X$ by $|E|$.

Definition 1.5.1. For every $s \ge 0$ and $\delta > 0$ we define an s-**dimensional Hausdorff premeasure** as $\mathcal{H}^s_\delta : \mathcal{P}(X) \to \mathbb{R}$,

$$\mathcal{H}^s_\delta(E) := \inf\left\{\sum_{i=1}^\infty |A_i|^s :\ \bigcup_{i=1}^\infty A_i \supseteq E,\ |A_i| < \delta\right\}.$$

Definition 1.5.2. The s-**dimensional Hausdorff (outer) measure** is

$$\mathcal{H}^s : \mathcal{P}(X) \to \mathbb{R}, \qquad \mathcal{H}^s(E) := \lim_{\delta \to 0} \mathcal{H}^s_\delta(E).$$

The outer measure $\mathcal{H}^s$ restricted to the σ-algebra of $\mathcal{H}^s$-measurable sets (including all Borel sets) is called the **s-dimensional Hausdorff measure**.

Remark 1.5.3. Notice that for fixed E and s the function $\delta \mapsto \mathcal{H}^s_\delta(E)$ is monotonically decreasing. Thus

$$\mathcal{H}^s(E) = 0 \iff \forall \delta > 0 \quad \mathcal{H}^s_\delta(E) = 0.$$

Remark 1.5.4. Another quite useful fact is that in the definition of $\mathcal{H}^s_\delta$ one may suppose that all sets A_i are open, or all of them are closed.

We enumerate some important properties without proof to show that $\mathcal{H}^s$ is an appropriate generalization of the Lebesgue measure. Some of these are very easy and none of them is hard to prove. Let λ^d denote the d-dimensional Lebesgue measure.

Proposition 1.5.5. *The Hausdorff measure has the following properties:*
(1) $\mathcal{H}^d(E) = C(d) \cdot \lambda^d(E)$ *for each* $d \in \mathbb{N}^+$ *if* $X = \mathbb{R}^d$.
(2) $\mathcal{H}^0$ *is the counting measure.*
(3) $\mathcal{H}^s$ *is a metric*[1] *outer measure.*
(4) $\mathcal{H}^s$ *is translation-invariant if* $X = \mathbb{R}^d$.
(5) $\mathcal{H}^s(\lambda E) = \lambda^s \cdot \mathcal{H}^s(E)$.
(6) *For any set* $E \subseteq X$ *if* $\mathcal{H}^s(E) < \infty$, *then* $\mathcal{H}^t(E) = 0$ *for all* $t > s$.

Property (6) implies that for any $E \subseteq X$ there exists $s \geq 0$ such that

$$\mathcal{H}^t(E) = \begin{cases} \infty & \text{if } t < s \\ 0 & \text{if } t > s. \end{cases}$$

This "break point" is the dimension of E.

Definition 1.5.6. The **Hausdorff dimension** of $E \subseteq X$ is

$$\dim_H(E) := \sup\{s \geq 0 : \mathcal{H}^s(E) = \infty\} = \inf\{t \geq 0 : \mathcal{H}^t(E) = 0\}.$$

The following are easy consequences of the definition.

Corollary 1.5.7. *The Hausdorff dimension is σ-stable. That is, for sets $E_i \subseteq X$ of Hausdorff dimension at most s we have* $\dim_H \left(\bigcup_{i=1}^\infty E_i\right) \leq s$.

Corollary 1.5.8. *Lipschitz maps do not increase the Hausdorff dimension. In separable metric spaces this holds for locally Lipschitz maps as well.*

[1] An outer measure μ on a metric space (X, ϱ) is metric if $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever $A, B \subseteq X$ and $\varrho(A, B) > 0$.

Hausdorff dimension meets most of our expectations. For simple sets like linear subspaces and balls it gives the expected value. For self-similar sets with disjoint parts it agrees with the similarity dimension.

However, Hausdorff dimension may be difficult to calculate. This gives rise to other dimension concepts. One of them is box dimension. Relations between different dimension concepts are intensively studied.

Box dimension applies to bounded sets $E \subseteq \mathbb{R}^d$. The idea is to count the diadic cubes of fixed side length $\frac{1}{2^k}$ intersecting E and "compare" this number (in order of magnitude) to 2^k. As $k \to \infty$ we see an increasingly accurate picture.

Let $\mathcal{Q}_k$ be the set of closed diadic cubes of side length $\frac{1}{2^k}$ in $\mathbb{R}^d$, that is,

$$\mathcal{Q}_k := \left\{ \left[\tfrac{i_1}{2^k}, \tfrac{i_1+1}{2^k}\right] \times \ldots \times \left[\tfrac{i_d}{2^k}, \tfrac{i_d+1}{2^k}\right] : \ i_1, \ldots, i_d \in \mathbb{Z} \right\}.$$

Let $N_k(E) := |\{Q \in \mathcal{Q}_k : \ Q \cap E \neq \emptyset\}|$.

Definition 1.5.9. The **upper box dimension** of a bounded set $E \subseteq \mathbb{R}^d$ is

$$\overline{\dim_B}(E) := \limsup_{k\to\infty} \frac{\log N_k(E)}{\log 2^k}.$$

The **lower box dimension** is

$$\underline{\dim_B}(E) := \liminf_{k\to\infty} \frac{\log N_k(E)}{\log 2^k}.$$

If the limit exists, then it is the **box dimension** of E denoted by $\dim_B(E)$.

Box dimension lacks an important property which makes it second fiddle to Hausdorff dimension: it is not σ-stable. Follows from the definition that $\mathbb{Q} \cap [0,1]^d$ has box dimension d despite being a countable set.

Straightforward calculations show that $\dim_H(E) \leq \underline{\dim_B}(E)$ for every bounded set $E \subseteq \mathbb{R}^d$.

Chapter 2

Properties of typical compact sets

In this chapter we study the behaviour of typical compact sets. More precisely, we look for typical properties in complete metric spaces of the form $\mathcal{K}(X)$ for some complete metric space X. A series of naturally occurring questions leads our investigations.

How does a typical compact set look like? We have already observed in Proposition 1.4.3 that the finite sets constitute a dense set in $\mathcal{K}(X)$. Following this clue we inspect concepts of size.

Propositions 1.1.7, 1.4.3 and 1.4.4 together serve as essential tools.

The results in this chapter are known. However, we present our own proofs for most of them.

2.1 Measure and dimension

First let us consider Lebesgue measure as a property in $\mathcal{K}(\mathbb{R}^d)$. Recall that Lebesgue measure is regular, that is, any measurable set $A \subseteq \mathbb{R}^d$ may be approximated in measure by compact sets from the inside and by open sets from the outside. Specifically, if A is compact, then $\lim_{\varepsilon\to 0} \lambda^d(A_\varepsilon) = \lambda^d(A)$.

Observe that for any $c \geq 0$ the set

$$G_c := \{K \in \mathcal{K}(\mathbb{R}^d) : \lambda^d(K) < c\}$$

is open. Indeed, for every $K \in G_c$ there exists a suitably small $\varepsilon > 0$ such that $\lambda^d(K_\varepsilon) < c$, and therefore $\lambda^d(K') < c$ for every element of $B^{d_H}(K, \varepsilon)$.

Proposition 2.1.1. *A typical $K \in \mathcal{K}(\mathbb{R}^d)$ is of Lebesgue measure zero.*

Proof. We shall prove that $C := \{K \in \mathcal{K}(\mathbb{R}^d) : \lambda^d(K) = 0\}$ is residual. It is dense because even the finite sets constitute a dense set by Proposition 1.4.3. It is G_δ since $C = \bigcap_{n=1}^{\infty} G_{\frac{1}{n}}$. Thus C is residual by Proposition 1.1.7. □

This illustrates a general rule: **Typical compact sets are small.**

The following theorem strengthens and generalizes Proposition 2.1.1 at the same time.

Theorem 2.1.2. *Let (X, ϱ) be a complete metric space. Then a typical $K \in \mathcal{K}(X)$ has Hausdorff dimension 0.*

Proof. We may use the same method as in the previous proof. Let us define $C := \{K \in \mathcal{K}(X) : \dim_H(K) = 0\}$. Proposition 1.4.3 guarantees that C is dense. We show that it is G_δ.

Claim. For any $s \geq 0$ the set $U_s := \{K \in \mathcal{K}(X) : \dim_H(K) \leq s\}$ is G_δ.

Observe that

$$\dim_H(K) \leq s \iff \forall k \in \mathbb{N}^+ \quad \mathcal{H}^{s+\frac{1}{k}}(K) = 0.$$

By Remark 1.5.3 we have

$$\mathcal{H}^{s+\frac{1}{k}}(K) = 0 \iff \forall m \in \mathbb{N}^+ \quad \mathcal{H}^{s+\frac{1}{k}}_{\frac{1}{m}}(K) = 0.$$

Furthermore, it is clear that

$$\mathcal{H}^{s+\frac{1}{k}}_{\frac{1}{m}}(K) = 0 \iff \forall n \in \mathbb{N}^+ \quad \mathcal{H}^{s+\frac{1}{k}}_{\frac{1}{m}}(K) < \frac{1}{n}.$$

Thus

$$U_s = \bigcap_{k=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcap_{n=1}^{\infty} \underbrace{\left\{K \in \mathcal{K}(X) : \mathcal{H}^{s+\frac{1}{k}}_{\frac{1}{m}}(K) < \frac{1}{n}\right\}}_{U_s(k,m,n)}.$$

We shall show that for fixed k, m and n the set $V := U_s(k, m, n)$ is open. If $K \in V$, then by Remark 1.5.4 there is an open cover $\bigcup_{i=1}^{\infty} A_i \supseteq K$ which witnesses this. By Proposition 1.4.4 $\{K' \in \mathcal{K}(X) : K' \subseteq \bigcup_{i=1}^{\infty} A_i\} \subseteq V$ is an open neighbourhood of K. Consequently, V is open which proves the claim.

Finally, note that $C = U_0$. □

Remark 2.1.3. Notice that we have already used each of our "essential tools" 1.1.7, 1.4.3 and 1.4.4 in one proof.

The following theorem provides a nice counterpoint to the previous one.

Theorem 2.1.4. *A typical $K \in \mathcal{K}(\mathbb{R}^d)$ has upper box dimension d.*

Proof. Again, the key is to prove that $U_s := \{K \in \mathcal{K}(\mathbb{R}^d) : \overline{\dim}_B(K) \geq s\}$ is G_δ for any $0 \leq s \leq d$.

It is easy to see that in the definition of lower and upper box dimension one may replace $N_k(E)$ by a variant $\tilde{N}_k(E)$ which is defined as follows. Instead of counting the number of diadic cubes intersecting E, we scale up every cube by ratio 2 and count the intersections with the interiors of these bigger cubes. Unravelling the new definition we get

$$\overline{\dim}_B(K) \geq s \iff \limsup_{k\to\infty} \frac{\log \tilde{N}_k(K)}{\log 2^k} \geq s \iff$$

$$\iff \inf_{n\in\mathbb{N}} \sup_{k\geq n} \frac{\log \tilde{N}_k(K)}{\log 2^k} \geq s \iff \forall n \in \mathbb{N} \quad \sup_{k\geq n} \frac{\log \tilde{N}_k(K)}{\log 2^k} \geq s \iff$$

$$\iff \forall n \in \mathbb{N}\ \forall m \in \mathbb{N}^+\ \exists k \geq n \quad \frac{\log \tilde{N}_k(K)}{\log 2^k} \geq s - \frac{1}{m}.$$

Thus we may write U_s as

$$\bigcap_{n=0}^{\infty} \bigcap_{m=1}^{\infty} \bigcup_{k=n}^{\infty} \underbrace{\left\{ K \in \mathcal{K}\big(\mathbb{R}^d\big) : \ \frac{\log \tilde{N}_k(K)}{\log 2^k} \geq s - \frac{1}{m} \right\}}_{U_s(n,m,k)}.$$

It suffices to show that for fixed n, m and k the set $V := U_s(n, m, k)$ is open. We rearrange the inequality:

$$\frac{\log \tilde{N}_k(K)}{\log 2^k} \geq s - \frac{1}{m} \iff \tilde{N}_k(K) \geq \left(2^k\right)^{s-\frac{1}{m}}.$$

Now we take advantage of the technical convenience given by our alternative definition. Fix $K \in V$ and suppose that $Q_1, \dots, Q_l$ are the modified diadic open cubes which meet K. Then the set $\{K' \in \mathcal{K}\big(\mathbb{R}^d\big) : \ K' \cap Q_i\}$ is open for each $1 \leq i \leq l$ by Proposition 1.4.4. Hence the intersection $\bigcap_{i=1}^{l} \{K' \in \mathcal{K}\big(\mathbb{R}^d\big) : \ K' \cap Q_i\}$ is an open neighbourhood of K which witnesses that K is an interior point of V. Therefore V is open and U_d is G_δ.

Also U_d is dense because even

$$\left\{ K \in \mathcal{K}\big(\mathbb{R}^d\big) : \ \exists \varepsilon > 0\ \exists x_1, \dots, x_n \in \mathbb{R}^d \quad K = \bigcup_{i=1}^{n} \overline{B}(x_i, \varepsilon) \right\}$$

is dense (this follows from Proposition 1.4.3), which completes the proof. □

We will dedicate the third chapter to a construction which takes heavy use of the intuitive rule that smallness is a typical property. Now we turn to explore another general rule.

2.2 Disorder

How organized is a typical compact set? Let us inspect algebraic dependency first.

Theorem 2.2.1. *A typical $K \in \mathcal{K}(\mathbb{R})$ is algebraically independent over $\mathbb{Q}$.*

Proof. Consider the set

$$B := \left\{ \begin{array}{ll} K \in \mathcal{K}(\mathbb{R}) : & \exists n \in \mathbb{N}^+ \ \exists p \in \mathbb{Q}[z_1, \dots, z_n] \setminus \{0\} \\ & \exists x_1, \dots, x_n \in K \text{ pairwise distinct } \quad p(x_1, \dots, x_n) = 0 \end{array} \right\}.$$

For $x_1, \dots, x_n$ pairwise distinct there exists $M \in \mathbb{N}^+$ such that $|x_k - x_l| \geq \frac{1}{M}$ for all $k \neq l$. Since there are only countably many such (n, p, M) triple, it suffices to show that for arbitrary fixed n, p and M the set

$$C := \left\{ K \in \mathcal{K}(\mathbb{R}) : \ \exists x_1, \dots, x_n \in K \quad |x_k - x_l| \geq \tfrac{1}{M} \text{ for } k \neq l, \ p(x_1, \dots, x_n) = 0 \right\}$$

is nowhere dense. We show that (1) C is closed and (2) it has no interior points.

(1) Let $K_i \overset{d_H}{\to} \widetilde{K}$ and $K_i \in C$. Then for every $i \in \mathbb{N}$ there are $x_1^i, \dots, x_n^i$ in K_i, having at least $\frac{1}{M}$ distance from each other, such that $p(x_1^i, \dots, x_n^i) = 0$. For every $1 \leq k \leq n$ and $\varepsilon > 0$ we have $x_k^i \in K_i \subseteq \widetilde{K}_\varepsilon$ if i is suitably large. Thus every x_k^i $(i \in \mathbb{N})$ is bounded in $\mathbb{R}$, so it has a convergent subsequence. Moreover, we can find a common subsequence i_m $(m \in \mathbb{N})$ such that $x_k^{i_m}$ converges for every k. Let $x_k := \lim_{m \to \infty} x_k^{i_m}$.

Now $x_k \in \widetilde{K}$ because for every $\varepsilon > 0$ we have $x_k^i \in \widetilde{K}_\varepsilon^*$ for all except finitely many i. Clearly $|x_k - x_l| \geq \frac{1}{M}$ holds as well. On the other hand, p represents a continuous function on $\mathbb{R}^n$, so $(x_1^{i_m}, \dots, x_n^{i_m}) \to (x_1, \dots, x_n)$ implies $p(x_1^{i_m}, \dots, x_n^{i_m}) \to p(x_1, \dots, x_n)$. This gives us $p(x_1, \dots, x_n) = 0$ which means that $x_1, \dots, x_n$ witnesses $\widetilde{K} \in C$.

(2) We know that the finite sets constitute a dense set in $\mathcal{K}(\mathbb{R})$. Therefore, it suffices to approximate finite sets by finite independent sets. Fix $\{y_1, \dots, y_N\} \subseteq \mathbb{R}$ and $\varepsilon > 0$. We can pick algebraically independent elements $y_j' \in B(y_j, \varepsilon)$ over $\mathbb{Q}$ for each $1 \leq j \leq N$ because for $y_1', \dots, y_j'$ already chosen they exclude only countably many possibilities for y_{j+1}'. □

This suggests another rule. **Typical compact sets are irregular.**

Proposition 2.2.2. *A typical $K \in \mathcal{K}(\mathbb{R}^d)$ has no $d+1$ points in a hyperplane. Specifically, it has no 3 collinear points.*

Proof. We need to show that

$$B := \left\{ K \in \mathcal{K}(\mathbb{R}^d) : \ \exists x_0, \dots, x_d \in K \text{ distinct } \det(x_1 - x_0, \dots, x_d - x_0) = 0 \right\}$$

is of first category. We may write B as the countable union of the sets

$$B_n := \left\{ \begin{array}{ll} K \in \mathcal{K}(\mathbb{R}^d) : & \exists x_0, \ldots, x_d \in K \quad d(x_i, x_j) \geq \frac{1}{n} \quad (i \neq j) \\ & \det(x_1 - x_0, \ldots, x_d - x_0) = 0 \end{array} \right\}.$$

We prove that (1) B_n is closed and (2) its complement is dense in $\mathcal{K}(\mathbb{R}^d)$.

(1) Let $K_m \overset{d_H}{\to} \widetilde{K}$ and $K_m \in B_n$. We may assume $K_m \subseteq \widetilde{K}_{\frac{1}{m}}$ (replace K_m with a subsequence if necessary). Let $x_0^m, \ldots, x_d^m \in K_m$ be points which witness $K_m \in B_n$. Again, by taking a subsequence finitely many times if necessary, we may assume that for every $0 \leq i \leq d$ the sequence x_i^m $(m \in \mathbb{N}^+)$ converges to some point $x_i \in \widetilde{K}$.

Now $d(x_i, x_j) \geq \frac{1}{n}$ for $i \neq j$ since $d(x_i, x_j) < \frac{1}{n}$ would imply $d(x_i^m, x_j^m) < \frac{1}{n}$ for large m which contradicts $K_m \in B_n$. Specifically, they are distinct. Since $\det(y_1 - y_0, \ldots, y_d - y_0)$ is a continuous function in $d+1$ variables, $(x_0^m, \ldots, x_d^m) \to (x_0, \ldots, x_d)$ implies $\det(x_1^m - x_0^m, \ldots, x_d^m - x_0^m) \to \det(x_1 - x_0, \ldots, x_d - x_0)$. Thus $\det(x_1 - x_0, \ldots, x_d - x_0) = 0$ which witnesses $\widetilde{K} \in B_n$.

(2) Finite sets constitute a dense set in $\mathcal{K}(\mathbb{R}^d)$. Hyperplanes are nowhere dense in $\mathbb{R}^d$, therefore we can reach general position by arbitrarily small perturbation of a finite set. □

Now we turn to topological properties.

2.3 Topological characterization

Theorem 2.3.1. *If (X, d) is a complete metric space with no isolated points, then a typical $K \in \mathcal{K}(X)$ is perfect.*

Proof. Let $\dot{B}(a, r)$ denote the punctured ball $B(a, r) \setminus \{a\}$.

It suffices to verify that $B := \{K \in \mathcal{K}(X) : K \text{ has an isolated point}\}$ is of first category since every $K \in \mathcal{K}(X)$ is closed. Clearly

$$B = \bigcup_{n=1}^{\infty} \underbrace{\left\{ K \in \mathcal{K}(X) : \exists x \in K \quad \dot{B}(x, \tfrac{1}{n}) \cap K = \emptyset \right\}}_{B_n}.$$

We show that (1) B_n is closed and (2) it has no interior points.

(1) Let $K_m \overset{d_H}{\to} \widetilde{K}$ and $K_m \in B_n$ for every $m \in \mathbb{N}$. We may suppose that $K_m \subseteq \widetilde{K}_{\frac{1}{m}}$ (replace K_m with a subsequence if necessary). By the definition of B_n there exist $x_m \in K_m$ such that $\dot{B}(x_m, \frac{1}{n}) \cap K_m = \emptyset$. Pick $x'_m \in \widetilde{K}$ such that $d(x_m, x'_m) < \frac{1}{m}$ for every $m \in \mathbb{N}$. Now x'_m is a sequence in the compact set $\widetilde{K}$, so

we may suppose that it is convergent (once more replacing K_m with a subsequence if necessary): $x'_m \to x'$ for some $x' \in \widetilde{K}$.

Claim. The point x' witnesses $\widetilde{K} \in B_n$.

Suppose that there exists some $y \in \dot{B}(x', \frac{1}{n}) \cap \widetilde{K}$. Let $\varepsilon > 0$ be such that $d(x', y) + 3\varepsilon < \frac{1}{n}$. For suitably large m we have

$$d(x'_m, x') < \varepsilon, \qquad d(x_m, x'_m) < \tfrac{1}{m} < \varepsilon \quad \text{and} \quad y_m \in B(y, \varepsilon)$$

for some $y_m \in K_m$. Thus

$$d(x_m, y_m) \leq d(x_m, x'_m) + d(x'_m, x') + d(x', y) + d(y, y_m) < d(x', y) + 3\varepsilon < \tfrac{1}{n}$$

contradicting $\dot{B}(x_m, \frac{1}{n}) \cap K_m = \emptyset$ which proves the claim.

(2) We can approximate any $\widetilde{K} \in \mathcal{K}(X)$ by a finite set, which can be approximated by another one not in B_n. (Change every point to two nearby points. Note that isolated points would ruin this argument.) □

One might have thought that the real reason behind the smallness rule is that a typical compact set is countable. Our result on perfectness dispels doubts as we show in the following theorem.

Theorem 2.3.2. *In a metric space (X, d) every compact perfect set K has continuum cardinality.*

Proof. First we show that $|K| \geq c := 2^{\aleph_0}$. Let x_0 and x_1 be distinct points in K. Let $\overline{B}(x_0, \varepsilon_0)$ and $\overline{B}(x_1, \varepsilon_1)$ be disjoint closed balls separating x_0 and x_1.

Now we proceed as follows. If x_s and $\overline{B}(x_s, \varepsilon_s)$ are already defined for some finite binary sequence s, then we can find distinct points $x_{s0}, x_{s1} \in B(x_s, \varepsilon_s) \cap K$ since x_s is not isolated, and we can separate them by disjoint closed balls $\overline{B}(x_{s0}, \varepsilon_{s0})$ and $\overline{B}(x_{s1}, \varepsilon_{s1})$ contained in $\overline{B}(x_s, \varepsilon_s)$.

For any infinite binary sequence s let us form the intersection of elements of the nested sequence of nonempty compact sets associated to s:

$$\bigcap_{i=1}^{\infty} \left(\overline{B}(x_{s|i}, \varepsilon_{s|i}) \cap K\right).$$

Clearly these intersections are pairwise disjoint nonempty subsets of K for each $s \in 2^\omega$. Thus $|K| \geq c$.

On the other hand, in a metric space every compact set is a separable since it has a finite $\frac{1}{n}$-net for every $n \in \mathbb{N}^+$. A separable metric space must have cardinality at most c: if S is a countable dense set, then every point may be identified by a countable sequence of elements of S. Consequently, $|K| \leq {\aleph_0}^{\aleph_0} = c$. □

At this point it is quite natural to ask about connectedness. In fact, we can settle this problem without further investigations. However, we need the notion of topological dimension (or more precisely, the small inductive dimension).

Definition 2.3.3. Let (X, d) be a metric space. Let -1 be the topological dimension of the empty set. We define the topological dimension by induction. If $(n-1)$-dimensional sets are defined, then $E \subseteq X$ has **topological dimension** n, if E is not at most $(n-1)$-dimensional but E has a basis $\mathcal{B}$ such that for every $B \in \mathcal{B}$ the boundary of B is at most $(n-1)$-dimensional. Let $\dim_t(E)$ denote the topological dimension of E.

The following theorem is well-known, see [11] for example.

Theorem 2.3.4. *Let (X, d) be a metric space. Then for any $E \subseteq X$ we have $\dim_t(E) \le \dim_H(E)$.*

Note that by Definition 2.3.3 the metric space X is zero dimensional if and only if it is nonempty and it has a basis consisting of clopen sets. Therefore every zero dimensional metric space is totally disconnected. We already discovered that a typical compact set has Hausdorff dimension zero, hence Theorem 2.3.4 answers our question.

Corollary 2.3.5. *Let (X, d) be a complete metric space. Then a typical $K \in \mathcal{K}(X)$ has topological dimension 0. Specifically, it is totally disconnected.*

It might be of some interest that we do not need the full joint power of Theorem 2.1.2 and Theorem 2.3.4. It is possible to detect the existence of a clopen basis directly. For this reason we present a proof for Corollary 2.3.5.

Our guess is as simple as possible: we show that in a typical compact set clopen balls form a basis. More explicitly, around every point of a typical compact set K there are *spheres* of arbitrarily small radius which are disjoint to K. These spheres witness the clopen property of the corresponding balls. However, it is easy to prove even more: K lacks arbitrarily small distances.

Now we make this precise.

Definition 2.3.6. Let (X, d) be a metric space and $E \subseteq X$. The **distance set** of E is

$$D(E) := \{d(x, y) : \ x, y \in E\}.$$

Or equivalently, $D(E)$ is the image of $E \times E$ by the distance function.

Proof of Corollary 2.3.5. We show that for every $n \in \mathbb{N}^+$

$$B_n := \left\{K \in \mathcal{K}(X) : \ D(K) \supseteq \left[0, \tfrac{1}{n}\right]\right\}$$

is nowhere dense.

(1) The set B_n is closed:

Let $K_i \xrightarrow{d_H} \widetilde{K}$ and $K_i \in B_n$. Note that $D(\widetilde{K})$ is compact because the distance function is continuous, so it suffices to find distances in $\widetilde{K}$ arbitrarily close to any $z \in \left[0, \frac{1}{n}\right]$. Fix $\varepsilon > 0$. From every K_i we can pick x_i and y_i such that $d(x_i, y_i) = z$. If i is large enough, then there exist $x_i', y_i' \in \widetilde{K}$ such that $d(x_i, x_i') < \frac{\varepsilon}{2}$ and $d(y_i, y_i') < \frac{\varepsilon}{2}$. Thus $z - \varepsilon < d(x_i', y_i') < z + \varepsilon$ by the triangle inequality.

(2) The interior of B_n is empty since finite sets are not in B_n.

Consequently, a typical $K \in \mathcal{K}(X)$ lacks arbitrarily small distances, which provides us arbitrarily small clopen balls around every point of K. $\square$

Now we gained enough information to characterize typical compact sets up to homeomorphism. The following result is Theorem 7.4 in [9] and it is due to Brouwer. For completeness we present a proof.

Theorem 2.3.7. *The Cantor space $\mathcal{C} = 2^\omega$ is the unique, up to homeomorphism, perfect nonempty, compact metrizable, zero-dimensional space.*

Proof. It is well-known that the Cantor space has these properties. Let X be a topological space with these properties and d a compatible metric. We construct a Cantor scheme $\{C_s : s \in 2^{<\omega}\}$ on X such that

(1) $C_\emptyset := X$.

(2) C_s is a nonempty clopen set for every $s \in 2^{<\omega}$.

(3) $C_s = C_{s0} \mathbin{\dot\cup} C_{s1}$.

(4) $\lim_{n\to\infty} |C_{x|n}| = 0$ for every $x \in 2^\omega$ (here $|\cdot|$ denotes the diameter as in Section 1.5).

Assuming this can be done, let $f : \mathcal{C} \to X$, $\{f(x)\} = \bigcap_{n=0}^{\infty} C_{x|n}$. Then f is well-defined by (4). It is injective by (3) and surjective by (1) and (3). Furthermore, f is an open map because it maps the canonical basis of $\mathcal{C}$ to open sets of the form C_s by (2). Since f is a bijection between compact Hausdorff spaces, there is no need to check the continuity.

At first we partition X into nonempty disjoint clopen sets $X_1, \ldots, X_n$ of diameter $< \frac{1}{2}$. (We may take a finite cover of X by clopen sets of diameter $< \frac{1}{2}$ and make them disjoint.) Split X into $C_0 = X_2 \cup \cdots \cup X_n$ and $C_1 = X_1$. Then split C_0 into $C_{00} = X_3 \cup \cdots \cup X_n$ and $C_{01} = X_2$, and so on: split C_{0^i} into $C_{0^{i+1}} = X_{i+2} \cup \cdots \cup X_n$ and $C_{0^i 1} = X_{i+1}$ for every $0 \le i \le n-2$.

Repeat this process within each X_i using diameters $< \frac{1}{3}$, and so on by induction. Now along every branch the diameter converges to zero, which gives us (4). The other conditions are clearly satisfied. $\square$

Corollary 2.3.8. *If (X, d) is a complete metric space without isolated points, then a typical $K \in \mathcal{K}(X)$ is homeomorphic to the Cantor space.*

Proof. A typical $K \in \mathcal{K}(X)$ is perfect by Theorem 2.3.1 and zero dimensional by Corollary 2.3.5, therefore it is homeomorphic to the Cantor space by Theorem 2.3.7. □

2.4 Products

In the (direct) proof of Corollary 2.3.5 we used the distance set as a tool. However, it is quite an interesting problem in itself to say something about the size of the distance set. In this section we show that even the distance set of a typical compact set has Hausdorff dimension 0. This is nontrivial since sets of Hausdorff dimension 0 may have much larger distance sets as the following example shows.

Example 2.4.1. We construct sets $A, B \subseteq [0,1]$ of Hausdorff dimension 0 such that $A + B = [0,1]$. This is equivalent to $A - (-B) = [0,1]$ which means that the set $A \cup (-B)$ has Hausdorff dimension 0 and its distance set covers the $[0,1]$ interval. Let x_i denote the ith decimal digit of the number $x \in [0,1]$. If the decimal representation is not unique, then we choose the one with a trailing infinite sequence of zeros.

Let A be the set of those $x \in [0,1]$ such that for every $n \in \mathbb{N}^+$ we have $x_i = 0$ for all $(2n)^{2n} \leq i < (2n+1)^{2n+1}$.

Similarly, B is the set of those $x \in [0,1]$ such that for every $n \in \mathbb{N}^+$ we have $x_i = 0$ for all $(2n-1)^{2n-1} \leq i < (2n)^{2n}$.

Follows directly from the construction that exactly the elements of $[0,1]$ can be written as $a + b$ for some $a \in A$ and $b \in B$. Straightforward calculation shows that $\dim_H(A) = \dim_H(B) = 0$.

Remark 2.4.2. Recall that if (X,d) is a metric space, then the topological product X^n can be metrized by

$$d' : X^n \times X^n \to \mathbb{R}, \quad d'((x_1,\dots,x_n),(y_1,\dots,y_n)) = d(x_1,y_1) + \ldots + d(x_n,y_n).$$

The following theorem strengthens Theorem 2.1.2.

Theorem 2.4.3. *Let (X,d) be a complete metric space and $n \in \mathbb{N}^+$. Then for a typical $K \in \mathcal{K}(X)$ even $K^n \subseteq X^n$ has Hausdorff dimension 0.*

Proof. We need only a small modification of the proof of Theorem 2.1.2. Using the original notations, we should show that

$$V := \left\{ K \in \mathcal{K}(X) : \mathcal{H}^{s+\frac{1}{k}}_{\frac{1}{m}}(K^n) < \frac{1}{n} \right\}$$

is open. Fix an open set U covering K^n which witnesses $K \in V$ as in the original proof. The set $\{K' \in \mathcal{K}(X^n) : K' \subseteq U\}$ is open by Proposition 1.4.4. On the other hand, it is easy to check that the map $K \mapsto K^n$ from $\mathcal{K}(X)$ to $\mathcal{K}(X^n)$ is continuous (use Remark 2.4.2). Consequently, V is open.

Direct products of finite sets are finite, so they have Hausdorff dimension 0. Thus compact sets of zero dimensional power constitute a dense set in $\mathcal{K}(X)$. This completes the proof. □

Corollary 2.4.4. *If (X, d) is a complete metric space, then the distance set of a typical $K \in \mathcal{K}(X)$ has Hausdorff dimension 0.*

Proof. Consider the distance function $d : X \times X \to \mathbb{R}$, $(x, y) \mapsto d(x, y)$. If we endow $X \times X$ with the sum metric as above, then the distance function is Lipschitz-1. That is, for $(x_1, x_2), (y_1, y_2) \in X \times X$ we have

$$|d(x_1, x_2) - d(y_1, y_2)| \leq d(x_1, y_1) + d(x_2, y_2),$$

because of the triangle inequality. Then Theorem 2.4.3 and Corollary 1.5.8 together imply that for a typical $K \in \mathcal{K}(X)$ the image of $K \times K$, which is the distance set, has Hausdorff dimension 0. □

The following theorem, which is due to Mycielski and Kuratowski, can be found in [9].

Theorem 2.4.5. *Let (X, d) be a complete metric space and $U \subseteq X^n$ a dense open set. Define $(A)^n := \{\mathbf{x} \in A^n : \ x_i \neq x_j \text{ whenever } i \neq j\}$.*

(1) *Then $B(U) := \{K \in \mathcal{K}(X) : \ (K)^n \subseteq U\}$ is residual in $\mathcal{K}(X)$.*

(2) *Moreover, if $R_i \subseteq X^{n_i}$ are residual sets for every $i \in \mathbb{N}$, then even the set $\{K \in \mathcal{K}(X) : \ \forall i \ (K)^{n_i} \subseteq R_i\}$ is residual in $\mathcal{K}(X)$.*

(3) *Specifically, if X has no isolated points, then there exists a Cantor set $C \subseteq X$ such that $(C)^{n_i} \subseteq R_i$ for all i.*

Proof. (1) We show that $B(U)$ is a dense G_δ set.

Let $D := \{\mathbf{x} \in X^n : \ x_i = x_j \text{ for some } i \neq j\}$. Note that $(K)^n \subseteq U \iff K^n \subseteq U \cup D$. Clearly D is closed; therefore $U \cup D$ is G_δ. By Proposition 1.4.4 the set $\{K' \in \mathcal{K}(X^n) : \ K' \subseteq U \cup D\}$ is G_δ. As we have already noted in the proof of Theorem 2.4.3 the map $K \mapsto K^n$ from $\mathcal{K}(X)$ to $\mathcal{K}(X^n)$ is continuous, so $\{K \in \mathcal{K}(X) : \ K^n \subseteq U \cup D\}$ is G_δ as well.

It suffices to show that for any $\varepsilon > 0$ and finite set $L = \{x_1, \dots, x_m\} \subseteq X$ there exists $L' \in B^{d_H}(L, \varepsilon)$ such that $(L')^n \subseteq U$. Enumerate all subsets of the form $\{x_{i_1}, \dots, x_{i_n}\} \subseteq \{x_1, \dots, x_m\}$. Since U is dense, for any fixed subset we can achieve $(x_{i_1}, \dots, x_{i_n}) \in U$ by an arbitrarily small change in the coordinates. Also U is open, which gives us some space: if we do suitably small changes, then we do not unmake what we have already achieved. Thus we are done in finitely many steps.

(2) If R_i is residual, then it contains a dense G_δ set: $\bigcap_{j=0}^{\infty} U_i^j \subseteq R_i$. Here every U_i^j is dense open, so we may apply (1): $B(U_i^j)$ is residual. Taking countable intersections twice we get that $\{K \in \mathcal{K}(X) : \forall i \ (K)^{n_i} \subseteq R_i\}$ is residual.

(3) Theorem 2.3.8 and (2) implies that a typical $K \in \mathcal{K}(X)$ satisfies the prescribed properties. □

Chapter 3

An application: Besicovitch sets

3.1 Short history

In 1917 Sōichi Kakeya asked if there is a minimum area of convex planar sets in which one can turn a needle through 360°. The answer is positive. However, it turned out that the problem is much more interesting if we drop the restriction about convexity. Then it is possible to turn a needle in a set of arbitrarily small area. There is an elementary construction called Perron tree which gives a solution. But there exists another approach which uses an important tool.

A Besicovitch set is a set $B \subseteq \mathbb{R}^d$ $(d \geq 2)$ which contains a unit line segment in every direction. Besicovitch showed that there exists a Besicovitch set of measure zero in $\mathbb{R}^2$ ([12], see also [13] Chapter 7). It is easy to see that this gives us a Besicovitch nullset in every dimension $d \geq 2$. Knowing the existence of a Besicovitch nullset it was natural to ask if it is possible to make it even smaller.

Kakeya conjecture: A Besicovitch set in $\mathbb{R}^d$ necessarily has Hausdorff dimension d.

This conjecture is still **open** except for $d = 2$ in which case it turned out to be true ([14] Davies 1971). The Kakeya conjecture is connected to several famous open questions in various fields of mathematics such as number theory, geometric combinatorics, arithmetic combinatorics, oscillatory integrals, and even the analysis of dispersive and wave equations [15].

Tom Körner proved that if we consider a well-chosen closed subspace of $\mathcal{K}(\mathbb{R}^2)$ in which every element contains a unit segment in every direction between $\frac{\pi}{3}$ and $\frac{2\pi}{3}$, then a typical element in this subspace is of measure zero ([16] Theorem 2.3). The union of three rotated copies of such a set is a Besicovitch set of measure zero. In this sense it is typical for a Besicovitch set to have measure zero.

There is a variation of the definition of Besicovitch set:

Definition 3.1.1. A **Besicovitch set** is a set $B \subseteq \mathbb{R}^d$ $(d \geq 2)$ which contains a line in every direction.

This gives us a variation of the Kakeya conjecture which is open as well. It is conjectured to be equivalent to the previous form. We will work with Definition 3.1.1 throughout this chapter.

It is clear from Fubini's theorem that if we intersect a planar Besicovitch nullset with lines of a fixed direction, then almost every intersection is of measure zero. We will use Baire category arguments combined with duality methods to obtain Besicovitch sets with stronger properties.

3.2 Duality and special code sets

We denote the orthogonal projection of the set $A \subseteq \mathbb{R}^2$ in the direction v by $pr_v(A)$ (where v is a nonzero vector or sometimes just its angle if it leads to no confusion). Similarly

$$P_v(A) := \left\{ \frac{x - v}{|x - v|} \in S^1 : \ x \in A \setminus \{v\} \right\}$$

is the radial projection of A from the point v. We may refer to elements of S^1 as angles causing no confusion. Let A_x denote the vertical section of the set A corresponding to x.

Definition 3.2.1. Let $l(a, b)$ denote the line which corresponds to the equation $y = ax + b$. We say that $\mathcal{L}$ is the **dual** of $K \subseteq \mathbb{R}^2$ (or $\mathcal{L}$ is coded by K) if $\mathcal{L} = \{l(a, b) : \ (a, b) \in K\}$.

A well-known consequence of this definition is the following.

Proposition 3.2.2. *Let $K \subseteq \mathbb{R}^2$ be a set and $\mathcal{L}$ its dual. Then the vertical sections of $L := \bigcup \mathcal{L}$ are scaled copies of the corresponding orthogonal projections of K. More precisely, $L_x = |(x, 1)| \cdot pr_{(-1,x)}(K)$.*

Proof. By Definition 3.2.1 we have

$$L_x = \{ax + b : \ (a, b) \in K\} = \left\{ |(x, 1)| \frac{(x, 1)}{|(x, 1)|} \cdot (a, b) : \ (a, b) \in K \right\}.$$

And this is exactly the orthogonal projection of K in the direction $(-1, x)$ scaled by the constant $|(x, 1)|$. □

We need to prove a generalization of the previous observation. This generalization will play a key role in the main proof.

Proposition 3.2.3. *Let $\mathcal{L}$ be the dual of the set $K \subseteq \mathbb{R}^2$, $L := \bigcup \mathcal{L}$, and let $e \notin \mathcal{L}$ be a line in $\mathbb{R}^2$. Then the intersection $e \cap L$ is*

(1) *a scaled copy of an orthogonal projection of K if e is vertical,*

(2) *otherwise it is the image of $P_{(a_0,b_0)}(K) \setminus \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$ by a locally Lipschitz function, where the equation of e is $y = a_0 x + b_0$.*

Proof. (1) is just the previous proposition.

(2): Note that $\mathcal{L}$ does not contain vertical lines because it is the dual of K. Then

$$e \cap L = \{(x, y) \in \mathbb{R}^2 : \ \exists (a, b) \in K \quad y = a_0 x + b_0 = ax + b\}.$$

So in the intersection $x = \frac{b-b_0}{a_0-a}$ holds (we have $a \neq a_0$ because e does not intersect lines parallel to itself). It is enough to determine the projection of $e \cap L$ to the x-axis since $e \cap L$ is the image of this projection by a Lipschitz function.

On the other hand, the projection of $e \cap L$ to the x-axis is

$$\left\{\frac{b-b_0}{a_0-a} : \ (a, b) \in K\right\} = \left\{(-1) \cdot \frac{b-b_0}{a-a_0} : \ (a, b) \in K\right\},$$

which is the set of slopes of the lines connecting points of K to (a_0, b_0) multiplied by (-1). It is easy to see that this set is the image of $P_{(a_0,b_0)}(K) \setminus \ \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$ by the function $-\tan(\varphi)$ which is locally Lipschitz. ☐

We will need the following.

Proposition 3.2.4. *The union of the dual of a compact set is closed.*

Proof. Let K be a compact set, $\mathcal{L}$ be its dual and $L = \bigcup \mathcal{L}$.

Let $(x_n, y_n) \to (x, y)$ be a convergent sequence such that $(x_n, y_n) \in L$ for all $n \in \mathbb{N}$. Here $(x_n, y_n) \in L$ means that there exists a pair $(a_n, b_n) \in K$ such that $y_n = a_n x_n + b_n$. Then the sequence (a_n, b_n) has a convergent subsequence because K is compact. Let $(a, b) \in K$ be its limit. Now $y = ax + b$ holds since $a_n \to a$, $b_n \to b$, $x_n \to x$, $y_n \to y$, and $y_n = a_n x_n + b_n$ along the subsequence. Hence (x, y) is on the line coded by $(a, b) \in K$ which means $(x, y) \in L$. Consequently, L is closed. ☐

For the main proof we need two compact sets with special properties.

The following theorem is due to Michel Talagrand [5]. For a direct proof in English, see [17] Appendix A.

Theorem 3.2.5. *For any non-degenerate rectangle $[a, b] \times [c, d] \subseteq \mathbb{R}^2$ there exists a compact set $K \subseteq [a, b] \times [c, d]$ such that its orthogonal projection to the x-axis is the whole $[a, b]$ interval, but in every other direction its projection is of measure zero.*

Definition 3.2.6. A set $A \subseteq \mathbb{R}^2$ is **invisible** from a point $a \in \mathbb{R}^2$ if $\lambda(P_a(A)) = 0$.

We will use a theorem of Károly Simon and Boris Solomyak [6]:

Theorem 3.2.7. *Let Λ be a self-similar set of Hausdorff dimension 1 in $\mathbb{R}^2$ satisfying the Open Set Condition, which is not contained in a line. Then, Λ is invisible from every $a \in \mathbb{R}^2$.*

It is an easy exercise to check that the four corner Cantor set of contraction ratio $\frac{1}{4}$ projects orthogonally to an interval in four different directions. It is well-known that this set satisfies the conditions of Theorem 3.2.7. Rotate it to have an interval as projection to the x-axis. Now by an affine transformation we can make it fit to the rectangle $[a, b] \times [c, d]$ while not losing its properties required by Theorem 3.2.7. By these easy observations we get the following corollary.

Corollary 3.2.8. *For any non-degenerate rectangle $[a, b] \times [c, d] \subseteq \mathbb{R}^2$ there exists a compact set $K \subseteq [a, b] \times [c, d]$ such that its projection to the x-axis is the whole $[a, b]$ interval, but it is invisible from every point of the plane.*

3.3 A theorem on Besicovitch sets

We could construct a Besicovitch set by simply taking the dual of the compact set given by Corollary 3.2.8. It would have intersections of measure zero with every non-vertical line not contained in it by Proposition 3.2.3. However, we will go further to obtain the following stronger result:

Theorem 3.3.1. *There exists a Besicovitch set $B = \bigcup \mathcal{L}$ (where $\mathcal{L}$ is a family of lines) in the plane such that:*

(1) *B is closed.*

(2) *B is of 2-dimensional Lebesgue measure zero.*

(3) *For every line $e \notin \mathcal{L}$ the intersection $B \cap e$ is of 1-dimensional Lebesgue measure zero.*

(4) *For every $e \in \mathcal{L}$ the intersection $e \cap \bigcup(\mathcal{L} \setminus \{e\})$ is of 1-dimensional Lebesgue measure zero.*

Moreover, we claim that these properties are typical in the sense described below.

We work in $\mathcal{K}([0, 1]^2)$ which is a complete metric space with the Hausdorff distance. Consider the subspace

$$\mathcal{C} := \left\{K \in \mathcal{K}\left([0, 1]^2\right) : \; pr_{\frac{\pi}{2}}(K) = [0, 1]\right\}.$$

It is easy to check that $\mathcal{C}$ is a closed subspace hence a complete metric space as well. The typicality in the main theorem means that a typical $K' \in \mathcal{C}$ codes a family of lines $\mathcal{L}'$ for which $L' = \bigcup \mathcal{L}'$ is an almost Besicovitch set: the union of four rotated copies of L' satisfies all the properties in Theorem 3.3.1.

The following theorem strengthens Theorem 3.2.5 and it is due to Alan Chang [18]. Here we present our own proof (found independently of Chang) to provide a useful analogue for the proof of the next theorem.

Theorem 3.3.2. *A typical element of $\mathcal{C}$ has orthogonal projections of measure zero in every non-vertical direction.*

We will need the following lemma.

Lemma 3.3.3. *Let A be a compact set and $f_A : S^1 \to \mathbb{R}$, $f_A(\varphi) = \lambda(pr_\varphi(A))$. Then f_A is upper semicontinuous.*

Talagrand proved in [5] that $\{f_A : \ A \in \mathcal{K}(\mathbb{R}^2)\}$ is the set of non-negative upper semicontinuous functions. We need only the easy direction, hence we present a proof only for that.

Proof. Let $c \in \mathbb{R}$ be arbitrary. We have to verify that $f_A^{-1}((-\infty, c))$ is open. Let φ be such that $\lambda(pr_\varphi(A)) < c$. Since $pr_\varphi(A)$ is compact as well, it can be covered by finitely many open intervals I_j $(1 \le j \le l)$ for which $\lambda\left(\bigcup_{j=1}^{l} I_j\right) < c$ holds. This cover shows that A can be covered by rectangles $R_1, \dots, R_l$ whose projections in the direction φ are the intervals $I_1, \dots, I_l$. But for the union of finitely many rectangles it is clear that if we change φ by a suitably small ($< \delta$) angle we keep the measure of its projection less than c. This implies that for any $\varphi' \in (\varphi - \delta, \varphi + \delta)$ we have

$$\lambda(pr_{\varphi'}(A)) \le \lambda\left(pr_{\varphi'}\left(\bigcup_{j=1}^{l} R_j\right)\right) < c.$$

In other words, a neighbourhood of φ also lies in $f_A^{-1}((-\infty, c))$, therefore the preimage is open. ■

Proof of Theorem 3.3.2. We have to prove that

$$\left\{K \in \mathcal{C} : \ \exists \varphi \in [0, \pi] \setminus \left\{\tfrac{\pi}{2}\right\} \quad \lambda(pr_\varphi(K)) > 0\right\}$$

is of first category. Let $T_n = \left\{\varphi \in [0, \pi] : \ |\varphi - \frac{\pi}{2}| \ge \frac{1}{n}\right\}$. It suffices to show that for every n

$$B_n := \left\{K \in \mathcal{C} : \ \exists \varphi \in T_n \quad \lambda(pr_\varphi(K)) \ge \tfrac{1}{n}\right\}$$

is nowhere dense in $\mathcal{C}$.

Fix a compact set $K \in \mathcal{C}$ and $\varepsilon > 0$. Denote the open ball of center A and radius δ by $B_H(A, \delta)$ (with respect to the Hausdorff distance). We need to find $K' \in \mathcal{C}$ and $\varepsilon' > 0$ such that $B_H(K', \varepsilon') \subseteq B_H(K, \varepsilon)$ and $B_H(K', \varepsilon') \cap B_n = \emptyset$.

At first we construct K'. Take a finite $\frac{\varepsilon}{3}$-net in K: $\{(x_1, y_1), \ldots, (x_N, y_N)\}$. Consider the squares of the form

$$Q_i := \left[x_i - \frac{\varepsilon}{3}, x_i + \frac{\varepsilon}{3}\right] \times \left[y_i - \frac{\varepsilon}{3}, y_i + \frac{\varepsilon}{3}\right] \qquad (1 \leq i \leq N).$$

Some of the squares may not lie in $[0,1]^2$. We cut off the parts sticking out of $[0,1]^2$ making Q_i a rectangle if necessary. Since it was created from an $\frac{\varepsilon}{3}$-net, $\bigcup_{i=1}^N Q_i$ covers K. Hence its projection to the x-axis is the whole $[0,1]$. For every rectangle Q_i, Theorem 3.2.5 gives us a compact set $K_i' \subseteq Q_i$ which has orthogonal projections of measure zero in every non-vertical direction and $pr_{\frac{\pi}{2}}(K_i') = pr_{\frac{\pi}{2}}(Q_i)$. Now let $K' = \bigcup_{i=1}^N K_i'$.

We need to check the following:

(1) $K' \in \mathcal{C}$,

(2) $K' \in B_H(K, \varepsilon)$ and

(3) $\lambda(pr_\varphi(K')) < \frac{1}{n}$ for all $\varphi \in T_n$.

(1) This is clear since $pr_{\frac{\pi}{2}}\left(\bigcup_{i=1}^N Q_i\right) = [0,1]$ and $pr_{\frac{\pi}{2}}(K_i') = pr_{\frac{\pi}{2}}(Q_i)$ for each Q_i.

(2) The following sequences of containments prove that $d_H(K, K') < \varepsilon$.

$$K' \subseteq \bigcup_{i=1}^N Q_i \subseteq \{(x_1, y_1), \ldots, (x_N, y_N)\}_{\frac{2}{3}\varepsilon} \subseteq K_{\frac{2}{3}\varepsilon}$$

$$K \subseteq \{(x_1, y_1), \ldots, (x_N, y_N)\}_{\frac{1}{3}\varepsilon} \subseteq \left(K'_{\frac{\sqrt{2}}{3}\varepsilon}\right)_{\frac{1}{3}\varepsilon} \subseteq K'_{\frac{\sqrt{2}+1}{3}\varepsilon}$$

(3) K' is the union of N sets whose projections are of measure zero in every non-vertical direction.

Now we have to find ε'.

Recall that A_δ^* denotes the closed δ-neighbourhood of the set A. It is very easy to check that for any compact set A, positive real number δ and angle φ the following holds: $pr_\varphi(A_\delta^*) = (pr_\varphi(A))_\delta^*$.

For every φ the projection $pr_\varphi(K')$ is compact, so we have

$$\lim_{\delta \to 0} \lambda\left((pr_\varphi(K'))_\delta^*\right) = \lambda(pr_\varphi(K')).$$

Hence there exists $\varepsilon_\varphi > 0$ for each $\varphi \in T_n$ such that

$$\lambda\left(pr_\varphi\left((K')_{\varepsilon_\varphi}^*\right)\right) = \lambda\left((pr_\varphi(K'))_{\varepsilon_\varphi}^*\right) < \tfrac{1}{n}.$$

The upper semicontinuity ensured by Lemma 3.3.3 for $A = (K')_{\varepsilon_\varphi}^*$ says that there exists $\delta_\varphi > 0$ such that for any $\varphi' \in (\varphi - \delta_\varphi, \varphi + \delta_\varphi)$ the projection is small enough: $\lambda\left(pr_{\varphi'}\left((K')^*_{\varepsilon_\varphi}\right)\right) < \frac{1}{n}$. On the other hand, T_n is compact; therefore it is covered by finitely many of these neighbourhoods. Hence we can choose ε' so that $\lambda(pr_\varphi(K'_{\varepsilon'})) < \frac{1}{n}$ for all $\varphi \in T_n$. Since every element of $B_H(K', \varepsilon')$ lies in $K'_{\varepsilon'}$, we proved $B_H(K', \varepsilon') \cap B_n = \emptyset$.

If necessary, we decrease ε' further to satisfy $B_H(K', \varepsilon') \subseteq B_H(K, \varepsilon)$. □

Theorem 3.3.4. *A typical $K \in \mathcal{C}$ is invisible from every point of the plane.*

Again, we will need a lemma about upper semicontinuity.

Lemma 3.3.5. *If $A \subseteq \mathbb{R}^2$ is compact, then the function $F_A : \mathbb{R}^2 \setminus A \to \mathbb{R}$, $F_A(v) = \lambda(P_v(A))$ is upper semicontinuous.*

Proof. Let $c \in \mathbb{R}$. We will check that $F_A^{-1}((-\infty, c))$ is open. Let v be a point such that $F_A(v) = \lambda(P_v(A)) < c$. Then by compactness we can take a finite cover of $P_v(A)$ by open arcs $I_1, \dots, I_l$ such that $\lambda\left(\bigcup_{j=1}^l I_j\right) < c$. This cover shows that A can be covered by l sectors $R_1, \dots, R_l$ of an annulus such that their radial projections from v are $I_1, \dots, I_l$. For the union of finitely many sectors of an annulus and a point which has a positive distance from them it is clear that if we move v by a suitably small distance we keep the measure of the radial projection of $\bigcup_{j=1}^l R_j$ less than c. In other words, a neighbourhood of v lies in $F_A^{-1}((-\infty, c))$, so it is open. ■

Proof of Theorem 3.3.4. The proof is very similar to the previous one. We need to prove that $\{K \in \mathcal{C} : \exists v \in \mathbb{R}^2 \quad \lambda(P_v(K)) > 0\}$ is of first category.

First observe that for any point $v \in \mathbb{R}^2$ and compact set $K \subseteq \mathbb{R}^2$

$$P_v(K) = \bigcup_{n=1}^{\infty} P_v\left(K \setminus B\left(v, \frac{1}{n}\right)\right),$$

which implies

$$\lambda(P_v(K)) = \lim_{n\to\infty} \lambda\left(P_v\left(K \setminus B\left(v, \tfrac{1}{n}\right)\right)\right).$$

Therefore, it suffices to show that

$$B_n := \left\{K \in \mathcal{C} : \exists v \in [-n, n] \times [-n, n] \quad \lambda\left(P_v\left(K \setminus B\left(v, \tfrac{1}{n}\right)\right)\right) \geq \tfrac{1}{n}\right\}$$

is nowhere dense.

Fix $K \in \mathcal{C}$ and $\varepsilon > 0$. Then take a finite $\frac{\varepsilon}{3}$-net $\{(x_1, y_1), \dots, (x_N, y_N)\}$ in K and consider the little squares of side length $\frac{2\varepsilon}{3}$ around them. After chopping off the parts outside $[0, 1]^2$ we get the rectangles $Q_1, \dots, Q_N$.

Now for every Q_i, Corollary 3.2.8 gives us a compact set $K_i' \subseteq Q_i$ which is invisible from every point of the plane and satisfies $pr_{\frac{\pi}{2}}(K_i') = pr_{\frac{\pi}{2}}(Q_i)$. Let

$K' = \bigcup_{i=1}^N K'_i$. Then K' is also invisible from every point of the plane. Exactly the same argument as in the previous proof shows that $K' \in \mathcal{C}$ and $d_H(K,K') < \varepsilon$.

Now we have to find ε'.

Claim 3.3.6. *For every $n \in \mathbb{N}^+$ and $v \in [-n,n] \times [-n,n]$ there exists ε_v such that $\lambda\left(P_v\left((K')^*_{\varepsilon_v} \setminus B(v,\frac{1}{2n})\right)\right) < \frac{1}{n}$.*

Proof. Fix n and v. Restricting the radial projection to an annulus of inner radius $\frac{1}{4n}$ centered at v makes it a Lipschitz function with Lipschitz constant $4n$. Since $P_v\left(K' \setminus B(v,\frac{1}{4n})\right)$ is a compact set of measure zero (recall that even K' is invisible from v), we know that

$$\lim_{\delta \to 0} \lambda\left(\left(P_v\left(K' \setminus B(v,\tfrac{1}{4n})\right)\right)_\delta\right) = \lambda\left(P_v\left(K' \setminus B(v,\tfrac{1}{4n})\right)\right) = 0.$$

Thus for a suitably small $\delta \leq 1$ we have $\lambda\left(\left(P_v\left(K' \setminus B(v,\frac{1}{4n})\right)\right)_\delta\right) < \frac{1}{n}$. Now we show that

$$P_v\left(K'_{\frac{\delta}{4n}} \setminus B(v,\tfrac{1}{2n})\right) \subseteq \left(P_v\left(K' \setminus B(v,\tfrac{1}{4n})\right)\right)_\delta. \tag{3.1}$$

Indeed, if $x \in K'_{\frac{\delta}{4n}} \setminus B(v,\frac{1}{2n})$, then there exists $y \in K' \setminus B(v,\frac{1}{4n})$ such that $|x-y| < \frac{\delta}{4n} \leq \frac{1}{4n}$. Therefore $|P_v(x) - P_v(y)| < \delta$ because of the Lipschitz property, and $P_v(y) \in P_v\left(K' \setminus B(v,\frac{1}{4n})\right)$, so $P_v(x) \in \left(P_v\left(K' \setminus B(v,\frac{1}{4n})\right)\right)_\delta$ which proves (3.1). By taking $\varepsilon_v = \frac{\delta}{5n}$, we prove the claim. ■

Observe that for every $v' \in B(v,\frac{1}{2n})$

$$(K')^*_{\varepsilon_v} \setminus B(v',\tfrac{1}{n}) \subseteq (K')^*_{\varepsilon_v} \setminus B(v,\tfrac{1}{2n})$$

and therefore

$$\lambda\left(P_{v'}\left((K')^*_{\varepsilon_v} \setminus B(v',\tfrac{1}{n})\right)\right) \leq \lambda\left(P_{v'}\left((K')^*_{\varepsilon_v} \setminus B(v,\tfrac{1}{2n})\right)\right).$$

For $A = (K')^*_{\varepsilon_v} \setminus B(v,\frac{1}{2n})$ the function F_A is upper semicontinuous on the complement of A by Lemma 3.3.5. Hence there exists an open neighbourhood $U_v \subseteq B(v,\frac{1}{2n})$ of v such that for all $v' \in U_v$

$$\lambda\left(P_{v'}\left((K')^*_{\varepsilon_v} \setminus B(v',\tfrac{1}{n})\right)\right) \leq \lambda\left(P_{v'}\left((K')^*_{\varepsilon_v} \setminus B(v,\tfrac{1}{2n})\right)\right) = F_A(v') < \tfrac{1}{n}.$$

Since $[-n,n] \times [-n,n]$ is compact, it can be covered by finitely many such neighbourhoods; therefore we may choose an ε' which is suitable for every point $v \in [-n,n] \times [-n,n]$.

We need to prove that $B_n \cap B_H(K',\varepsilon') = \emptyset$ holds. Let $L \in B_H(K',\varepsilon')$ and $v \in [-n,n] \times [-n,n]$. Then $L \subseteq K'_{\varepsilon'}$ hence

$$\lambda\left(P_v\left(L \setminus B(v,\tfrac{1}{n})\right)\right) \leq \lambda\left(P_v\left(K'_{\varepsilon'} \setminus B(v,\tfrac{1}{n})\right)\right) < \tfrac{1}{n}$$

by the choice of ε'. Consequently, $L \notin B_n$. □

Now we have two typical properties in $\mathcal{C}$ by Theorem 3.3.2 and Theorem 3.3.4, so we may merge them into one corollary.

Corollary 3.3.7. *A typical element $K \in \mathcal{C}$ has orthogonal projections of measure zero in every non-vertical direction, and it is invisible from every point of the plane.*

Proof of Theorem 3.3.1. Let $K' \in \mathcal{C}$ be a set satisfying the properties described in Corollary 3.3.7, $\mathcal{L}'$ be its dual and $L' := \bigcup \mathcal{L}'$. Then L' contains a line of slope m for every $m \in [0, 1]$ because the slope is coded by the first coordinate and $pr_{\frac{\pi}{2}}(K') = [0, 1]$.

(1) L' is closed by Proposition 3.2.4.

(3) Let e be any vertical line. Then its intersection with L' is similar to a non-vertical orthogonal projection of K' by Proposition 3.2.3. Therefore, it is of measure zero by Corollary 3.3.7. This implies (2) immediately.

Now let e be any non-vertical line not in $\mathcal{L}'$. Then its intersection with L' is the image of $P_v(K') \setminus \{\frac{\pi}{2}, \frac{3\pi}{2}\}$ by a locally Lipschitz function for some point $v \in \mathbb{R}^2 \setminus K'$ (Proposition 3.2.3 again). Therefore it is of measure zero by Corollary 3.3.7. (It is an easy exercise to show that locally Lipschitz maps from S^1 to $\mathbb{R}$ preserve nullsets.)

So L' has an intersection of measure zero with every line not contained in it.

(4) Let $e \in \mathcal{L}'$ and let $y = a_0 x + b_0$ be its equation. Now $\mathcal{L}' \setminus \{e\}$ is the dual of $K' \setminus \{(a_0, b_0)\}$, thus the intersection $e \cap \bigcup(\mathcal{L}' \setminus \{e\})$ is the image of the set $P_{(a_0,b_0)}(K' \setminus \{(a_0, b_0)\}) \setminus \{\frac{\pi}{2}, \frac{3\pi}{2}\}$ by a locally Lipschitz function (Proposition 3.2.3 again). Therefore it is of measure zero by Corollary 3.3.7.

Let B be the union of four rotated copies of L'. It contains a line in every direction and it still satisfies properties (1)–(4). The proof of the main theorem is complete. □

Bibliography

[1] O. Nikodym, *Sur la mesure des ensembles plans dont tous les points sont rectalineairement accessibles*, Fundam. Math., **10** (1927), 116–68.

[2] K. J. Falconer, *Sets with Prescribed Projections and Nikodym Sets*, Proc. London. Math. Soc., **53** (1986), 48–64.

[3] A. Chang, M. Csörnyei, K. Héra and T. Keleti, *Small unions of affine subspaces and skeletons via Baire category*, Adv. Math. **328** (2018), 801–821.

[4] S. H. Jones *Applications of the Baire Category Theorem*, Real Anal. Exch. **23(2)** (1997), 363–394.

[5] M. Talagrand, *Sur la mesure de la projection d'un compact et certaines familles de cercles*, Bull. Sci. Math. **104** (1980), 225–231.

[6] K. Simon and B. Solomyak, *Visibility for self-similar sets of dimension one in the plane*, Real Anal. Exch. **32** (2007), 67–78.

[7] T. Kátay, *The intersection of typical Besicovitch sets with lines*, Real Anal. Exch., to appear.

[8] J. C. Oxtoby, *Measure and Category*, Springer-Verlag, 1980.

[9] A. S. Kechris, *Classical descriptive set theory*, Springer-Verlag, New York, 1995.

[10] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces*, Cambridge University Press, 1995.

[11] W. Hurewicz and H. Wallman, *Dimension Theory*, Princeton University Press, 1948.

[12] A. S. Besicovitch, *On Kakeya's problem and a similar one*, Math. Z. **27** (1928), 312–20.

[13] K. J. Falconer, *The geometry of fractal sets*, Cambridge University Press, Cambridge, 1985.

[14] R. O. Davies, *Some remarks on the Kakeya problem*, Math. Proc. Camb. Philos. Soc. **69** (2008), 417–421.

[15] T. Tao, *From Rotating Needles to Stability of Waves: Emerging Connections between Combinatorics, Analysis, and PDE*, Notices Am. Math. Soc. **48** (2001), 294–303.

[16] T. W. Körner, *Besicovitch via Baire*, Stud. Math. **158** (2003), 65–78.

[17] W. Hansen, *Littlewood's one-circle problem, revisited*, Expo. Math. **26** (2008), 365–374.

[18] A. Chang, *Constructions of planar Besicovitch sets*, `http://math.uchicago.edu/~ac/alan_chang_topic_proposal_planar_besicovitch.pdf#page7`, unpublished topic proposal (2016).

NYILATKOZAT

Név: Kátay Tamás

ELTE Természettudományi Kar, szak: Matematikus MSc

NEPTUN azonosító: ERRKB4

Szakdolgozat címe:
Typical compact sets

A **szakdolgozat** szerzőjeként fegyelmi felelősségem tudatában kijelentem, hogy a dolgozatom önálló szellemi alkotásom, abban a hivatkozások és idézések standard szabályait következetesen alkalmaztam, mások által írt részeket a megfelelő idézés nélkül nem használtam fel.

Budapest, 2020. 05. 24.

Kátay

a hallgató aláírása