Eötvös Loránd university

Faculty of science

Lê Thi Thanh Tú

RIGIDITY THEORY AND ITS APPLICATIONS

MSc in Mathematics

Supervisor:

Prof. Tibor Jordán
Department of Operations Research

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Abstract

Back to 1766, Euler conjectured that "a closed spacial figure allows no changes, as long as it is not ripped apart" which led to one of the oldest and most beautiful unsolved problems in geometry at that time. The closed spacial figures in Euler's thinking were closed polyhedral surfaces made up of rigid polygon plates that are hinged along the edges where plates meet. The conjecture that all such closed surfaces are rigid drew great attentions over hundreds of years. The major contribution was made by Cauchy in 1813 when he proved that strictly convex surfaces were rigid. Then in 1975, Gluck indicated that Euler's conjecture is almost always true for closed simply connected polyhedral surfaces and Connelly finally disproved Euler's conjecture by giving a counterexample that is a closed polyhedral surface (topologically a sphere), embedded in three-space, which flexes. The thing making difference here is that Gluck concentrated on generic graphs, in which there is no algebraic dependencies between the coordinates of vertices, while Connelly's construction is non-generic. The history shows the significant impact of being generic or non-generic toward the graphs' rigidity. In my thesis, I only deal with generic frameworks, in which case rigidity and global rigidity only depend on the underlying graph.

Although rigidity has been studied for such a long time, it is only the last 45 years that it began to find applications in the basic sciences. Laman's theorem (1970), which made the combinatorial approach to the subject rigorous in 2-dimensions, can be seen as the foundation for multiple important applications arising from rigidity theory. The miraculous development of computer science as well as human's demand make the applications of rigidity theory more and more abundant and practical. In my thesis, I try to refer to some applications that we can see how interesting rigidity theory is in reality.

The structure of my thesis is as follows. After preliminaries in chapter 1, existing results about minimal k-vertex-rigidity are provided in chapter 2. Chapter 3 continues with minimal k-vertex-global-rigidity but especially, the second part of this chapter is my own work on strongly and weakly minimal 3-vertex-global-rigidity in the two-dimensional space. Some partial results about minimal k-edge-rigidity and minimal k-edge-global-rigidity are presented in chapter 4. Finally, my thesis ends with some applications of rigidity theory in biomolecules, sensor networks, formation control, statics and truss structures. Since rigidity theory is still challenging with many open questions, I shall list few of them in each chapter and hopefully, we will have the answer for those questions in the near future.

Contents

Chapter 1

Preliminaries

Consider finite graphs without loops, multiple edges or isolated vertices. A **d-dimensional bar-joint framework** (G, p) is a graph $G = (V, E)$ and a configuration $p : V \to \mathbb{R}^d$. We consider the framework to be a straight line realization of G in $\mathbb{R}^d$.
Given frameworks (G, p) and (G, q), we say that:

- (G, p) and (G, q) are **equivalent** if $\|p(v_i) - p(v_j)\| = \|q(v_i) - q(v_j)\|$ for all $v_i v_j \in E$, where $\|.\|$ denotes the distance.
- (G, p) and (G, q) are **congruent** if $\|p(v_i) - p(v_j)\| = \|q(v_i) - q(v_j)\|$ for all $v_i, v_j \in V$.
- (G, p) is **rigid** if there exists an $\epsilon > 0$ such that every framework (G, q) which is equivalent to (G, p) and satisfies $\|p(v) - q(v)\| < \epsilon$ for all $v \in V$, is congruent to (G, p).
- (G, p) is **globally rigid** if every framework that is equivalent to (G, p) is congruent to (G, p).

Roughly speaking, a rigid framework is one that preserves its shape during a smooth motion, while a globally rigid framework keeps its unique shape throughout all motions.

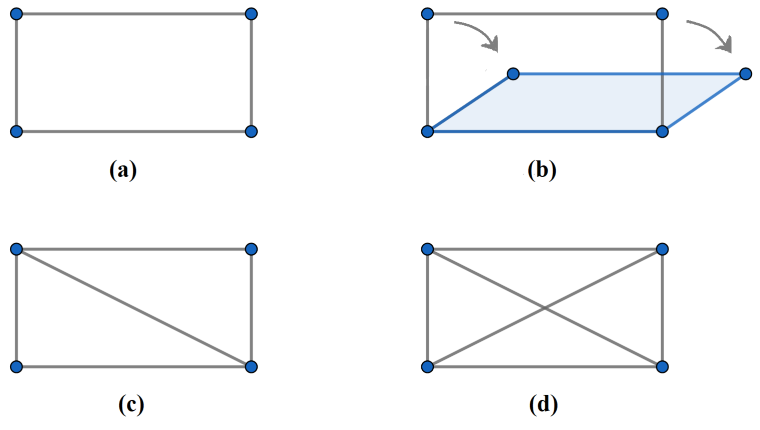


Figure 1.1: The framework in (a) is nonrigid since it can be deformed by a smooth motion without affecting the distance between the vertices connected by edges, as shown in (b). The frameworks represented in (c) and (d) are rigid and moreover, the formation (d) is globally rigid.

The rigidity matrix $R_d(G,p)$ of a framework (G,p) in $\mathbb{R}^d$ is the $|E| \times d|V|$ matrix, whose rows correspond to the edges and whose columns correspond to the coordinates of the vertices, where $|.|$ denotes the cardinality of a set. If $e = v_iv_j \in E$, then the entry in the row e and d columns of v_i is $p(v_i) - p(v_j)$, the entry in the row e and d columns of v_j is $p(v_j) - p(v_i)$ and the other entries in the row e are zeros.
For integers $n \geq 2$ and $d \geq 1$, let

$$S(n,d) = \begin{cases} dn - \binom{d+1}{2} & \text{if } n \geq d+2 \\ \binom{n}{2} & \text{if } n \leq d+1. \end{cases} \tag{1.1}$$

Theorem 1.1. [15] Let (G,p) be a d-dimensional framework with $n \geq 2$ vertices. Then rank $R_d(G,p) \leq S(n,d)$. Furthermore, if equality holds, then (G,p) is rigid.

Proof. We consider the case when (G,p) is properly embedded in $\mathbb{R}^d$ i.e. the affine hull of the points $p(v), v \in V$, is equal to $\mathbb{R}^d$. (This implies in particular that $n \geq d+1$.) Let $Z(G,p)$ be the null space of the matrix $R_d(G,p)$ and consider vectors in $Z(G,p)$ as instantaneous motions of the framework (G,p). Each translation and rotation of $\mathbb{R}^d$ gives rise to a smooth motion of (G,p) and hence to an instantaneous motion of (G,p). Let $Z_0(G,p)$ be the subspace of $Z(G,p)$ generated by these special instantaneous motions. The subspace $Z_0(G,p)$ contains a linearly independent set of instantaneous motions corresponding to the translations along each vector in the standard basis, and the rotations about the $(d-2)$ dimensional subspaces containing each set of $(d-2)$ vectors in the standard basis. Thus

$$\dim Z(G,p) \geq \dim Z_0(G,p) \geq d + \binom{d}{d-2} = \binom{d+1}{2} \tag{1.2}$$

and hence rank $R_d(G,p) \leq dn - \binom{d+1}{2}$.
To indicate why the second part of the theorem holds we suppose that (G,p) is not rigid. One may use the definition of rigidity to show that this assumption will imply that there exists a smooth motion $P(t,v)$ of (G,p) such that $\|P(t,x) - P(t,y)\|^2 \neq \|p(x) - p(y)\|^2$ for all $t > 0$ and some fixed $x, y \in V$. Differentiating with respect to t and putting $t = 0$ we deduce that there exists an instantaneous motion q of (G,p) such that $[p(x) - p(y)][q(x) - q(y)] \neq 0$. Since translations and rotations preserve distances between all points of $\mathbb{R}^d, q \notin Z_0(G,p)$. Thus strict inequality must occur in (1.2) and rank $R_d(G,p) < dn - \binom{d+1}{2}$. ♠

A framework (G,p) is called **infinitesimally rigid** if rank $R_d(G,p) = d|V| - \binom{d+1}{2}$. Infinitesimal rigidity of (G,p) implies rigidity. The converse is not true in general. However, infinitesimal rigidity is equivalent to rigidity for **generic frameworks** (G,p) where the (multi)set containing the coordinates of all the points $p(v), v \in V$, is algebraically independent over $\mathbb{Q}$. Hence, for generic frameworks, infinite rigidity as well as rigidity depends only on graph G. Also, it is proved that the global rigidity of d-dimensional frameworks is a generic property for all $d \geq 1$ [11]. Therefore, from now on, we only consider generic frameworks. A graph G is called rigid in $\mathbb{R}^d$ if every (or equivalently, if some) generic realization of G in $\mathbb{R}^d$ is rigid. Similarly, a graph G is called globally rigid in $\mathbb{R}^d$ if every (or equivalently, if some) generic realization of G in $\mathbb{R}^d$ is globally rigid.
From the rigidity matrix $R_d(G,p)$, we consider a special matroid. In this matroid, independent sets is edge sets F of G such that the rows of the rigidity matrix indexed from F is linearly independent. Since the entries of the rigidity matrix are polynomial

functions with integer coefficients, any two generic frameworks (G, p) and (G, q) have the same rigidity matroid. We call this the rigidity matroid of graph G, denoted by $\mathcal{M}_d(G)$. Denote $r_d(G)$ be the rank of $\mathcal{M}_d(G)$. It follows from the above that a graph G on n vertices is rigid in $\mathbb{R}^d$ if and only if $r_d(G) = S(n, d)$. We say that a graph $G = (V, E)$ is M-independent if E is independent in $\mathcal{M}_d(G)$ and G is M-circuit if E is a circuit (i.e. minimal dependent set) in $\mathcal{M}_d(G)$. The following is a corollary from theorem 1.1.

Theorem 1.2. If $G = (V, E)$ is M-independent in $\mathbb{R}^d$ then $i_G(X) \leq d|X| - \binom{d+1}{2}$ for all $X \subseteq V$ with $|X| \geq d + 2$ where $i_G(X)$ is the number of edges in the graph G induced by the vertex set X.

Definition 1.1. A graph $G = (V, E)$ is called **minimally rigid** if it is rigid and $G - e$ is nonrigid for all $e \in E$.

Hence, a graph $G = (V, E)$ on n vertices is minimally rigid in $\mathbb{R}^d$ if and only if G is M-independent and $|E| = S(n, d)$. Also, if G is rigid in $\mathbb{R}^d$, then the edge sets of the minimally rigid spanning subgraphs of G are bases of $\mathcal{M}_d(G)$. This leads to the following result.

Theorem 1.3. [39] Let $G = (V, E)$ be minimally rigid in $\mathbb{R}^d$. If $|V| \geq d + 1$ then $|E| = d|V| - \binom{d+1}{2}$.

We can easily see that in $\mathbb{R}$, a graph is rigid if and only if it is connected. Thus, one-dimensional minimally rigid graphs are trees.
In $\mathbb{R}^2$, Laman (1970) gave a fully combinatorial characterization of minimally rigid graphs, marking an important milestone for the whole development of rigidity theory later.

Theorem 1.4. [28] (Laman's Theorem) A graph $G = (V, E)$ is minimally rigid in $\mathbb{R}^2$ if and only if $|E| = 2|V| - 3$ and for all $X \subseteq V$ with $|X| \geq 2$, $i_G(X) \leq 2|X| - 3$. The graph is rigid if and only if it has a minimally rigid spanning subgraph.

According to that, a graph with n vertices is minimally rigid in $\mathbb{R}^2$ if and only if it has exactly $2n - 3$ edges and every nonempty subgraphs induced by n_0 vertices have at most $2n_0 - 3$ edges. This is also known as the $2n - 3$ edge count condition.

Theorem 1.5. [35] [14] A graph is minimally rigid in $\mathbb{R}^2$ if and only if it can be constructed from the complete graph on two vertices K_2 by a sequence of vertex addition operations (2-dimensional Henneberg 0-extensions) and edge splitting operations (2-dimensional Henneberg 1-extensions).

It is involved to see the full characterization of rigidity in higher dimensions. The problem of characterizing when a graph is rigid in $\mathbb{R}^d$ for $d \geq 3$ remains a major open problem of rigidity theory so far.

Remark. There are some operations known to preserve rigidity in $\mathbb{R}^d$ such as d-dimensional 0-extension, d-dimensional 1-extension and d-dimensional simplex-based X-replacement [35] [22] but which operations that are necessary and sufficient to build and construct all minimally rigid graphs in three-space or higher dimensions is still a matter of conjecture.

- A d-dimensional Henneberg 0-extension adds a new vertex to a graph and connects it to d distinct vertices of the graph.

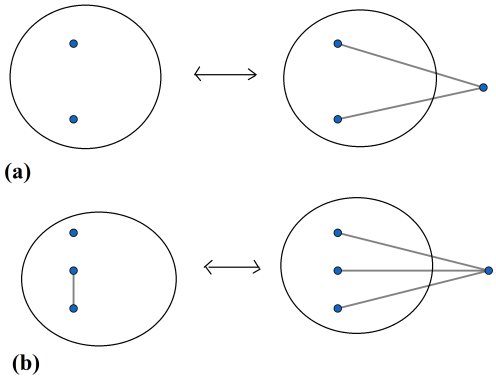


Figure 1.2: Representation of (a) vertex addition operation and (b) edge splitting operation.

- A d-dimensional Henneberg 1-extension deletes an edge of a graph, adds a new vertex v and connects it to two vertices of the deleted edge and $d-1$ other vertices of the graph.
- Let $a, b, w_1, ..., w_{d-2}$ be a complete subgraph of d-dimensional graph $G = (V, E)$ and $cd \in E$ be an edge which is vertex-disjoint from the simplex. The d-dimensional simplex-based X-replacement extension deletes ab, cd, adds a new vertex v and connects it to $a, b, c, d, w_1, ..., w_{d-2}$.

Obviously, global rigidity is stricter than rigidity. Hendrickson gave the necessary conditions for global rigidity in all dimensions.

Theorem 1.6. [13] Let G be a globally rigid graph in $\mathbb{R}^d$ with at least $(d+1)$ vertices. Then G is $(d+1)$-connected and redundantly rigid in $\mathbb{R}^d$.

Note that G is $(d+1)$-connected if at least $d+1$ vertices must be removed to disconnect the graph. "Redundantly rigid" means the graph maintains rigidity after the deletion of an arbitrary edge. More formally, a graph $G = (V, E)$ is called **redundantly rigid** if $G - e$ is rigid for all $e \in E$.

Proof. Since we can reflect the vertices on one side of a hyperplane through any separating set of d vertices, the first condition $(d+1)$-vertex connectivity must hold for a unique realization.
The second condition is more subtle. Here we only mention the general idea. Suppose by contradiction, $G - e$ is not rigid for some edges $e \in G$. Let the resulting framework flex, then there exists an equivalent, but not congruent framework to the initial framework. Hence, G is not globally rigid, which contracts with the hypothesis. ♠

These conditions are not suffice for a graph to be globally rigid in three or higher dimensional spaces. One example showing their failure in $\mathbb{R}^d$ $(d \geq 3)$ is a class of complete bipartite graphs $K_{a,b}$ with $a + b = \binom{d+1}{2}$ and $a, b \geq d+2$. It is $(d+1)$-connected and redundantly rigid but not globally rigid [6]. So characterizing the globally rigid graphs in three-space and in higher dimensions is another important future work.
In one and two dimensions, Hendrickson's conditions give the complete characterization of globally rigid graphs.

Theorem 1.7. [15] A graph G is a globally rigid on the line if and only if either $G = K_2$ or G is 2-connected. In the latter case, there is a construction from K_3, using only 1-dimensional Henneberg 1-extensions and edge additions.

Note that K_n denotes the complete graph on n vertices.

Theorem 1.8. [16] A graph G be a globally rigid in the plane if and only if G is a complete graph on at most three vertices or G is 3-connected and redundantly rigid. Moreover, if G has more than 3 vertices then it can be obtained from K_4 by a sequence of 2-dimensional Henneberg 1-extensions and edge additions.

As we can see above, Henneberg 1-extension operation preserves global rigidity in $\mathbb{R}$ and $\mathbb{R}^2$. Connelly answered the question of whether or not it holds in higher dimensional spaces.

Theorem 1.9. [6] Let G be a graph obtained from a globally rigid graph H in $\mathbb{R}^d$ by a d-dimensional Henneberg 1-extensions. Then G is globally rigid in $\mathbb{R}^d$.

There is a special operation that augments the (global) rigidity in d-dimensional space to the (global) rigidity in $d+1$-dimensional space, called the coning. The cone graph of G is the graph that arises from G by adding a new vertex v and connecting v to all the vertices of G. We denote this graph by $G * v$.

Theorem 1.10. (Coning theorem)
[37] A graph G is rigid in $\mathbb{R}^d$ if and only if the cone graph $G * v$ is rigid in $\mathbb{R}^{d+1}$.
[8] A graph G is globally rigid in $\mathbb{R}^d$ if and only if the cone graph $G * v$ is globally rigid in $\mathbb{R}^{d+1}$.

We say that a graph $G = (V, E)$ is **vertex-redundantly rigid** in $\mathbb{R}^d$ if $G - v$ is rigid in $\mathbb{R}^d$ for all $v \in V$.

Theorem 1.11. [17] If G is rigid in $\mathbb{R}^{d+1}$ then it is vertex-redundantly rigid in $\mathbb{R}^d$.

Proof. For a contradiction, suppose that $G - v$ is not rigid in $\mathbb{R}^d$ for some vertex $v \in V$. It follows from the above theorem that the cone graph $(G - v) * u$ is not rigid in $\mathbb{R}^{d+1}$. Since G is a spanning subgraph of $(G - v) * u$, we obtain that G is not rigid in $\mathbb{R}^{d+1}$, a contradiction. ♠

Theorem 1.12. [33] If G is vertex-redundantly rigid in $\mathbb{R}^d$ then it is globally rigid in $\mathbb{R}^d$.

Two above theorems implies the sufficient condition of global rigidity.

Theorem 1.13. [17] If G is rigid in $\mathbb{R}^{d+1}$ then it is globally rigid in $\mathbb{R}^d$.

Another operation related to (global) rigidity is the gluing operation, which is stated below.

Theorem 1.14. (Gluing theorem)
[39] If $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are rigid graphs in $\mathbb{R}^d$ sharing at least d vertices, then $G = (V_1 \cup V_2, E_1 \cup E_2)$ is rigid in $\mathbb{R}^d$.
[5] If $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are globally rigid graphs in $\mathbb{R}^d$ sharing at least $d + 1$ vertices. Then $G = (V_1 \cup V_2, E_1 \cup E_2 - G_1[V_1 \cap V_2])$ is globally rigid in $\mathbb{R}^d$.

Chapter 2

Minimal k-vertex-rigidity

Definition 2.1. A graph $G = (V, E)$ is called k**-vertex-rigid** in $\mathbb{R}^d$ if $|V| \geq k + 1$ and after deleting any set of at most $k - 1$ vertices, the resulting graph is rigid in $\mathbb{R}^d$.

Definition 2.2. A k-vertex-rigid graph G is called **minimally** k**-vertex-rigid** if the omission of an arbitrary edge results in a graph that is not k-vertex-rigid.

Actually, the expression "deleting any set of at most $k - 1$ vertices" can be replaced by deleting any set of $k - 1$ vertices due to the following theorem.

Theorem 2.1. Suppose that G is a graph on at least $k + 2$ vertices that satisfies $G - S$ is (globally) rigid in $\mathbb{R}^d$ for every subset S of vertices with cardinality exactly k. Then $G - S'$ is (globally) rigid in $\mathbb{R}^d$ for every subset S' of vertices with cardinality at most k.

Proof. If $|V| \leq k + d + 1$, then for any subset $S \subseteq V$ with $|S| = k$ we have that $G - S$ is a (globally) rigid graph on at most $d + 1$ vertices, thus it is complete. Since $|V| \geq k + 2$, for any pair of vertices, we can choose S such that it avoids the given pair. These observations together imply that in this case G is complete, so the statement is true. Now suppose that $|V| \geq k + d + 2$. It is clearly sufficient to prove the statement for $S' \subseteq V$ with $|S'| = k - 1$. Fix some such S'. Pick two vertices v_1, v_2 in $G - S'$ and let $G_i = G - S' - v_i$ for $i = 1, 2$. By our assumption, G_1 and G_2 are both (globally) rigid. Now $G - S' = G_1 \cup G_2$, while G_1 intersects G_2 at $|V| - (k - 1) - 2 = |V| - k - 1 \geq d + 1$ vertices. Since the gluing operation is known to preserve (global) rigidity, $G - S'$ is (globally) rigid, as desired. ♠

For $k = 1$, minimal 1-vertex-rigidity becomes minimal rigidity, which we already mentioned in chapter 1. Recall that the edge set of a minimally rigid graph G in $\mathbb{R}^d$ corresponds to a base of the rigidity matroid $\mathcal{M}_d(G)$. Thus, the edge sets of d-dimensional minimally rigid graphs (that is, minimal 1-vertex-rigid graphs) on the same vertex set have the same cardinality. However, this is not true for $k \geq 2$, there are minimally k-vertex-rigid graphs for any $k \geq 2$ with different edge numbers [22], which leads to two different concepts: Strongly minimal k-vertex-rigidity and weakly minimal k-vertex-rigidity.

2.1 Strongly minimal k-vertex-rigidity

Definition 2.1.1. A k-vertex-rigid graph is called **strongly minimal** if it has the minimum number of edges on a given number of vertices.

As we know, k-vertex-rigidity in $\mathbb{R}$ corresponds to k-connectivity. Searching for strongly minimally k-vertex-rigid graphs in $\mathbb{R}$, we need to find k-vertex-connected graphs that have minimum number of edges on a given number of vertices. A family of that was given by Frank Harary in his paper [12]. For detail, every vertex in a k-vertex-connected graph has degree at least k. Then a k-vertex-connected graph with n vertices ($n \geq k+1$) has at least $\lceil \frac{kn}{2} \rceil$ edges. The Harary graph $H_{n,k}$ (with $\lceil \frac{kn}{2} \rceil$ edges) is a graph on the n vertices $v_1, v_2, \ldots, v_n$ defined by the following construction:

- If k is even, then each vertex v_i is adjacent to $v_{i\pm 1}, v_{i\pm 2}, \ldots, v_{i\pm \frac{k}{2}}$, where the indices are subjected to the wraparound convention that $v_i \equiv v_{i+n}$ (e.g. v_{n+3} represents v_3).
- If k is odd and n is even, then $H_{n,k}$ is $H_{n,k-1}$ with additional adjacencies between each v_i and $v_{i+\frac{n}{2}}$ for each i.
- If k and n are both odd, then $H_{n,k}$ is $H_{n,k-1}$ with additional adjacencies $\{v_1, v_{1+\frac{n-1}{2}}\}$, $\{v_1, v_{1+\frac{n+1}{2}}\}, \{v_2, v_{2+\frac{n+1}{2}}\}, \{v_3, v_{3+\frac{n+1}{2}}\}, \ldots, \{v_{\frac{n-1}{2}}, v_n\}$.

Let k_H be the connectivity of the Harary graph H. The connectivity of a connected graph is at most the minimum of the degrees of its vertices, so $k_H \leq k$. To show that $k_H \geq k$, we merely have to show that at least k vertices must be removed to disconnect the graph. In the case of k is even, it is necessary (and sufficient) to remove two separate subsets of $\frac{k}{2}$ consecutive vertices each, along the circumference of the polygon. For k is odd and n is even, it is still necessary to remove two such subsets of $\frac{k-1}{2}$ vertices each to break the circumferential connections, but at least one more vertex must also be removed to break the diametral connection. Similar argument applies to the last case when k and n are both old. Hence, $k_H = k$ and Harary graph $H_{n,k}$ ($n \geq k+1$) is an example of strongly minimal k-vertex-rigid graphs in $\mathbb{R}$.
Before going to other higher dimensional spaces, we recall a result that is proved by authors in paper [22].

Theorem 2.1.1. If a graph $G = (V, E)$ is k-rigid in $\mathbb{R}^d$ with $|V| \geq d^2 + d + k$ then $|E| \geq d|V| - \binom{d+1}{2} + (k-1)d + \max\left\{0, \lceil k - 1 - \frac{d+1}{2} \rceil\right\}$. (See proof in Theorem 4.1 [22].)

This lower bound is sharp for all 2-vertex-rigid graphs in all dimensions, and for 3-vertex-rigid graphs in $\mathbb{R}^2$ and $\mathbb{R}^3$, as we will see later.
A complete characterization of strongly minimal 2-vertex-rigid graphs in $\mathbb{R}^2$ follows from the next theorems.

Theorem 2.1.2. [31] In $\mathbb{R}^2$, let $G = (V, E)$ be a strongly minimal 2-vertex-rigid graph on 5 or more vertices. Then G has exactly two vertices with degree 3 and the remaining have degree 4, which implies $|E| = 2|V| - 1$. On 4 or fewer vertices, G must be complete to be strongly minimally 2-vertex-rigid.

Theorem 2.1.3. [31] In $\mathbb{R}^2$, a graph $G = (V, E)$ is strongly minimally 2-vertex-rigid if and only if G has exactly two vertices of degree 3 and there is a partition of the edge set E

$$E = E_1 \cup E_2 \cup \cdots \cup E_k$$

such that the graph induced by $E \setminus E_i$ is minimally redundantly rigid for all i, and either

- E_1 and E_2 are the edges incident to the two non-adjacent vertices of degree 3, respectively, and E_i is a single edge for $3 \leq i \leq k$, or

- E_1 is the union of the edges incident to the two adjacent vertices of degree 3, and E_i is a single edge for $2 \leq i \leq k$.

This can be thought of as a Laman-type characterization, analogous to minimal rigidity: There must be a minimum number of edges ($|E| = 2|V| - 1$), and the edges must be properly distributed, as described in the conditions of the theorem. The two possible partitions of the edge set correspond to whether or not the two degree 3 vertices are adjacent.

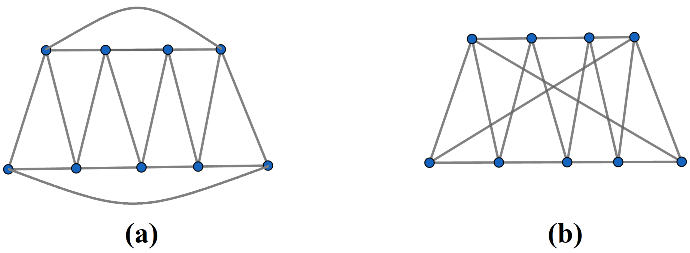


Figure 2.1: Examples of the two possible partitions of the edge set for strongly minimal 2-vertex-rigid graphs: (a) the degree three vertices are adjacent, and (b) the degree three vertices are non-adjacent.

Remark. Servatius also provides a way to "grow" all strongly minimal 2-vertex-rigid graphs in $\mathbb{R}^2$ using an operation (which is referred as "1-extendability" in [31]) similar to the edge splitting operation mentioned previously. Thus, we have a complete characterization of 2-dimensional strongly minimal 2-vertex-rigid graphs and a way to construct them.

Theorem 2.1.4. [22] If $G = (V, E)$ is a strongly minimal 2-vertex-rigid graph in $\mathbb{R}^d$ with $|V| \geq d^2 + d + 2$ then $|E| = d|V| - \binom{d}{2}$.

Proof. It follows from theorem 2.1.1 that all strongly minimal 2-vertex-rigid graphs $G = (V, E)$ in $\mathbb{R}^d$ satisfy $|E| \geq d|V| - \binom{d}{2}$. Consider graph C_n^d (the dth power of C_n, or equivalently, $E(C_n^d) = v_i v_j : i - d \leq j \leq i + d$ where $v_{n+i} := v_i$) and its subgraph L_d induced by vertices $v_{n-d+1}, ..., v_n$. $H_{n,2}^d = C_n^d - E(L_d)$ denotes the graph we get from C_n^d after deleting the edge set of L_d. It is easy to see that $|E(H_{n,2}^d)| = d|V| - \binom{d}{2}$ and moreover, if $n \geq 3d$, $H_{n,2}^d$ is 2-vertex-rigid in $\mathbb{R}^d$ (see proof in Lemma 6.1 [22]). ♠

For strongly minimal 3-vertex-rigidity in $\mathbb{R}^2$, we have the following result.

Theorem 2.1.5. [30] There exists a strongly minimal 3-vertex-rigid graph $G = (V, E)$ in $\mathbb{R}^2$ with $|E| = 2|V| + 2$ for any $|V| \geq 6$. In G, there are exactly 4 vertices with degree of 5 which are all adjacent (forming a complete K_4 subgraph) and all other vertices have degree of 4.

Proof. It follows from theorem 2.1.1 that a 2-dimensional minimal 3-vertex-rigid graph on n vertices must have $2n + 2$ edges at least. The existence of 2-dimensional strongly minimal 3-vertex-rigid graphs satisfying the lower bound follows from the fact that there is a strongly minimal 3-vertex-rigid graph on 6 vertices with 14 edges (shown in figure 2.2) and one operation preserving the 3-vertex-rigidity in $\mathbb{R}^2$ (mentioned later). Now, we prove the necessary condition of strongly minimal 3-vertex-rigidity in $\mathbb{R}^2$.

The condition $|E| = 2n+2$ implies that the graph has an average degree of $4+\frac{4}{n}$ (n denotes the number of vertices of the graph). Therefore, there is at least one vertex with degree of more than 4. Now suppose $v_1, v_2 \in V$ with degrees k_1, k_2, respectively. Then, $G - v_1 - v_2$ has $n-2$ vertices and at most $|E(G-v_1-v_2)| = 2n+2-(k_1+k_2-1) = 2(n-2)-(k_1+k_2-7)$ edges (when v_1, v_2 are neighbors). Since $G-v_1-v_2$ is rigid, $|E(G-v_1-v_2)| \geq 2(n-2)-3$ holds. Hence, $k_1 + k_2 - 7 \leq 3$ or $k_1 + k_2 \leq 10$. By considering $k_1 \geq 4, k_2 \geq 4$ we conclude that $4 \leq k_1 \leq 6$ and $4 \leq k_2 \leq 6$. Finally, if there are m vertices of degree 6 and t vertices of degree 5, we have $6m + 5t + 4(n - m - t) = 2(2n + 2)$, which gives $2m + t = 4$. There are 3 possible cases:

a. $m = 1, t = 2$: In this case if we remove the 6-degree vertex in addition to a 5-degree one, the number of edges will be at most $|E(G-v_1-v_2)| = 2n+2-(6+5-1) = 2n-8 = 2(n-2)-4$ which contradicts the fact that $G-v_1-v_2$ is rigid and $|E(G-v_1-v_2)| \geq 2(n-2)-3$. Therefore, this case cannot occur.

b. $m = 2, t = 0$: With the same argument as the case a, by removing two vertices with degree of 6 we have $|E(G - v_1 - v_2)| = 2n + 2 - (6 + 6 - 1) = 2n - 9 = 2(n - 2) - 5$ and it is obvious that the resulting graph is not rigid. Hence again, this case cannot occur.

c. $m = 0, t = 4$: The proof of this case is trivial. The only important condition is that all 5-degree vertices should be adjacent. Otherwise, removing any 2 of them results in a nonrigid graph ($|E(G - v_1 - v_2)| = 2(n - 2) - 4 < 2(n - 2) - 3$). ♠

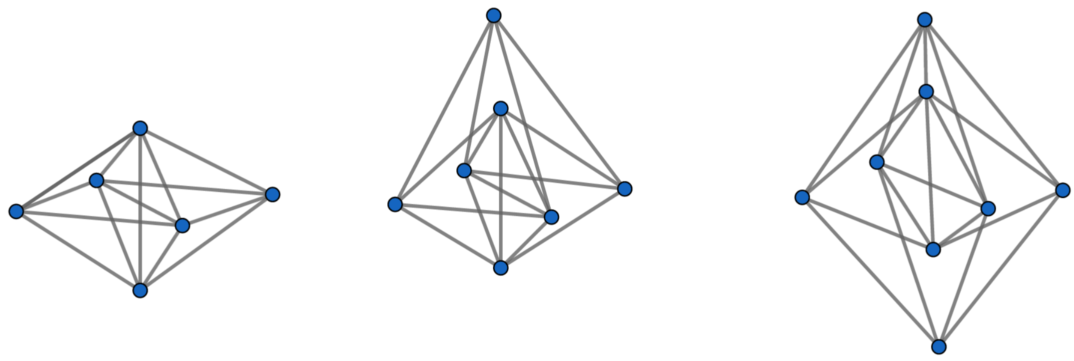


Figure 2.2: Examples of 2-dimensional strongly minimal 3-vertex-rigid graphs.

Remark. [30] So, all strongly minimal 3-vertex-rigid graphs in $\mathbb{R}^2$ must have exactly 4 vertices with degree 5 which form a complete subgraph and other vertices having degree 4. However, that condition is not sufficient to ensure strongly minimal 3-vertex-rigidity in $\mathbb{R}^2$. Consider the graph G shown in Figure 2.3a. It is easy to observe that this graph satisfies Theorem 2.1.5. However, as shown in Figure 2.3b, removing two vertices from this graph results in a nonrigid graph. Therefore, G is not 3-vertex-rigid.

For growing strongly minimal 3-vertex-rigid graphs in $\mathbb{R}^2$, we consider 4-5 X-replacement: Suppose that the original graph is $G = (V, E)$. Choose two edges $e_1 = ab$ and $e_2 = cd$ and $e_1, e_2 \in E$ so that a, c have degree 4 and are non-adjacent and b, d have degree 5 (such edges choice always exists) (see proof in Lemma 12 [30]). Remove e_1, e_2 and add a new vertex called z. Connect z to a, b, c, d. In the 4-5 X-Replacement operation, the degree of the original vertices remains the same and a new vertex of degree 4 is added to G. Therefore, the new graph satisfies the conditions of theorem 2.1.5.

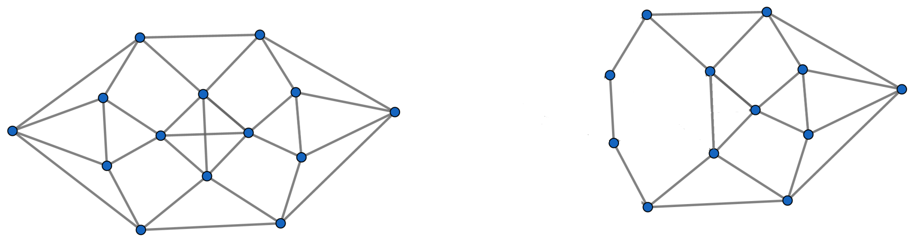


Figure 2.3: The left graph is not 3-vertex-rigid but still satisfies the conditions in theorem 2.1.5.

Theorem 2.1.6. Suppose $G = (V, E)$ is a strongly minimal 3-vertex-rigid graph in $\mathbb{R}^2$. After applying the 4-5 X-replacement operation on G, the resulting graph G is strongly minimal 3-vertex-rigid on $|V| + 1$ vertices in $\mathbb{R}^2$ (see proof in Theorem 13 [30]).

Remark. Two above theorems give partial results about strongly minimal 3-vertex-rigidity in $\mathbb{R}^2$. The sufficient condition for a graph to be strongly minimally 3-vertex-rigid in $\mathbb{R}^2$ and "necessary and sufficient" operations to build all such graphs remain open questions.

In $\mathbb{R}^3$, strongly minimal 3-vertex-rigidity is more complicated. However, theorem 2.1.1 implies that for a strongly minimal 3-vertex-rigid graph $G = (V, E)$, the condition $|E| \geq 3|V|$ must hold. In addition, C_n^3 $(n \geq 9)$, the third power of C_n, has $3n$ edges and is proved to be 3-dimensional 3-vertex-rigid (Lemma 7.1 [22]). This gives us the following.

Theorem 2.1.7. [22] If $G = (V, E)$ is a strongly minimally 3-vertex-rigid graph in $\mathbb{R}^3$ with $|V| \geq 15$, then $|E| = 3|V|$.

The question of full characterization of strongly minimal 3-vertex-rigid graphs in three-space or higher dimensions as well as operations to build them is still open. There is a conjecture of the authors in paper [22] (Conjecture 8.1) about that question.

Conjecture 2.1.8. If a graph $G = (V, E)$ is strongly minimal 3-vertex-rigid graph in $\mathbb{R}^d, d > 3$ with $|V| \geq d^2 + d + 3$ then $|E| = d|V| - \binom{d+1}{2} + 2d$.

2.2 Weakly minimal k-vertex-rigidity

Definition 2.2.1. A k-vertex-rigid graph is called **weakly minimal** if it is minimally k-vertex-rigid graph but not strongly minimally k-vertex-rigid. Equivalently, it has more edges compared to strongly minimal graphs but still has the property that removing any edge destroys k-vertex-rigidity.

Theorem 2.2.1. [22] Let $G = (V, E)$ be a minimally k-vertex-rigid graph in $\mathbb{R}$ with $|V| \geq 3k - 1$. Then $|E| \leq k|V| - k^2$.

It comes out that the number of edges a 1-dimensional weakly minimal k-vertex-rigid graph with n vertices $(n \geq 3k - 1)$ is between $\lceil \frac{1}{2}nk \rceil + 1$ and $kn - k^2$. The complete bipartite graph $K_{k,n-k}$ $(n \geq 3k - 1)$ is an example for this. Indeed, the complete bipartite graph $K_{k,n-k}$ is a 1-dimensional weakly minimal k-vertex-rigid graph as long as $n > 2k$.

When $n = 2k$, the complete bipartite graph $K_{k,n-k} = K_{k,k}$ is an example for strongly minimal k-vertex-rigidity in $\mathbb{R}$.
As proved in paper [22] (Section 8.1), for $k, d \geq 2$, there exists minimal k-vertex-rigid graphs on $\mathbb{R}^d$ with the same number of vertices but different number of edges. Such a pair of graphs show that the graph with larger number of edges has to be weakly minimally rigid and then there always exists weakly minimally k-vertex-rigid graphs in $\mathbb{R}^d$ for all $k, d \geq 2$.
Some examples for weakly minimal 2-vertex-rigid graphs in $\mathbb{R}^2$ are given in paper [32].

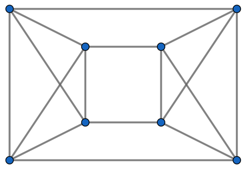


Figure 2.4: An example of 2-dimensional weakly minimal 2-vertex-rigid graph consists of two complete subgraphs on four vertices connected by four edges.

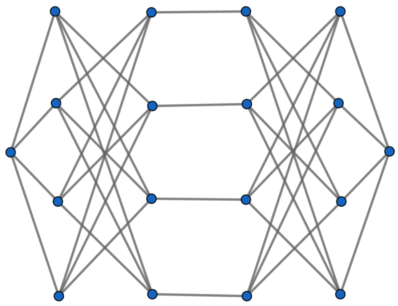


Figure 2.5: Another example of weakly minimal 2-vertex-rigid graph in $\mathbb{R}^2$.

Theorem 2.2.2. [32] The graph $G = (V, E)$ in Figure 2.5 is weakly minimally 2-vertex-rigid in $\mathbb{R}^2$.

Proof. Every vertex has degree 4; thus, $|E| = 2|V|$, which is excess three (excess $= |E| - (2|V| - 3)$). Observe that the subgraphs induced by the left and right nine vertices (call them G_L and G_R) both have excess one. Removing any vertex in G results in a graph of excess one that has exactly one subgraph of excess one, which is rigid. Thus, the graph is 2-vertex-rigid. Now remove any edge outside of G_L; call the new graph G'. Then remove any degree 4 vertex in $G' \setminus G_L$. The resulting graph, call it G", has an excess of zero with a subgraph of excess one (viz G_L), and so G" is not rigid. Therefore, G' is not 2-vertex-rigid. Obviously, the same argument applies if we remove any edge outside of G_R. Hence, removing any edge in G destroys 2-vertex-rigidity, and thus G is weakly minimally 2-vertex-rigid. ♠

Remark. The question of finding an inductive construction for the class of 2-dimensional weakly minimal 2-vertex-rigid graphs remains unanswered. But at least, according to papers [3] [31], we know that 2-dimensional X-replacement operation, degree 3 vertex addition operation (under certain conditions) preserve weakly minimal 2-vertex-rigidity in $\mathbb{R}^2$.

- Given two non-adjacent edges ux and wy in a graph $G(V, E)$, an X-replacement adds a degree 4 vertex z to construct the graph $G'(V', E')$, where $V' = V \cup z$ and $E' = E \setminus \{ux, wy\} \cup \{uz, wz, xz, yz\}$.
- Let i, j, and k be three distinct vertices in a graph $G(V, E)$. A degree 3 vertex addition operation adds a vertex l and edges il, jl, and kl to the graph G.

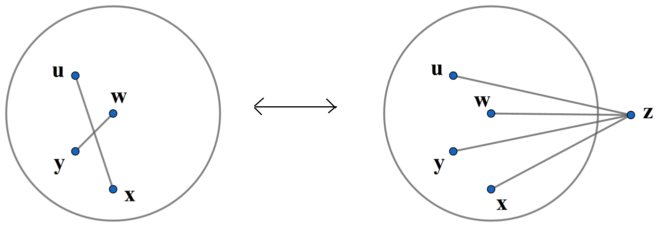


Figure 2.6: Representation of the X-replacement operation.

Since the X-replacement preserves 2-vertex-rigidity, it can be applied successively to each 2-vertex-rigid subgraph in figure 2.4 (that is, each complete subgraph on four vertices) to create a class of weakly minimal 2-vertex-rigid graphs with excess three, which includes the example in figure 2.5. Indeed, one can easily verify that the graph in figure 2.4 can be obtained by repeatedly applying the reverse X-replacement operation on the left and right subgraphs in figure 2.5.

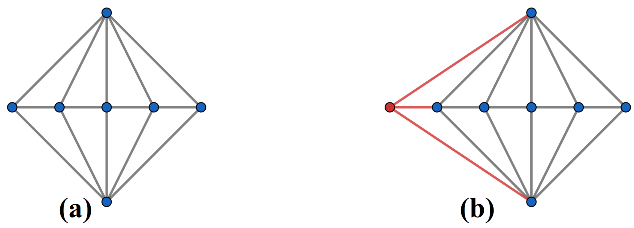


Figure 2.7: (a) A 2-dimensional weakly minimal 2-vertex-rigid graph with excess three, (b) Adding a degree 3 vertex to create a 2-dimensional weakly minimal 2-vertex-rigid graph with excess four. By successively adding a degree 3 vertex to either end, one can obtain weakly minimal 2-vertex-rigid graphs with arbitrarily large excess in $\mathbb{R}^2$.

Theorem 2.2.3. [32] The class of graphs illustrated in figure 2.7 (which is obtained by successively adding a degree 3 vertex) is weakly minimally 2-vertex-rigid.

Proof. Let $G = (V, E)$ be the graph in figure 2.7(a). One can easily verify that $|E| = 2|V|$ (excess three) and that removing any vertex from G results in a rigid graph; thus, G is 2-vertex-rigid. Now, remove any edge not incident to the top vertex, then remove the top vertex, resulting in a graph $G' = (V', E')$. We have $|E'| = 2|V'| - 4$, which implies that G' is not rigid. The same argument holds when removing any edge not incident to the bottom vertex, then removing the bottom vertex. Thus, removing any edge in G destroys 2-vertex rigidity, and therefore G is weakly minimally 2-vertex-rigid. The same analysis holds for the graph in figure 2.7(b) and all other graphs in this class. ♠

In [32], Summers, Yu and Anderson conjectured that the degree 3 vertex addition and the 2-dimensional X-replacement operations are sufficient to build up every weakly minimally 2-vertex-rigid graph in $\mathbb{R}^2$ with at least nine vertices.

Conjecture 2.2.4. [32] Let $G(V, E)$ be a weakly minimal 2-vertex-rigid graph in $\mathbb{R}^2$ with at least 9 vertices. Then there exists either
(a) a degree 4 vertex on which a reverse X-replacement operation can be performed to obtain a weakly minimal 2-vertex-rigid graph or
(b) there exists a degree three vertex on which a reverse degree 3 vertex addition can be performed to obtain a weakly minimal 2-vertex-rigid graph.

Later, this one is disproved by authors in paper [22] by using the $\mathcal{K}_4$-extension operation. Figure 2.8 illustrates a 2-dimensional weakly minimal 2-vertex-rigid graph, which does not have a vertex at which the reverse degree 3 vertex addition or the reverse X-replacement can be performed.

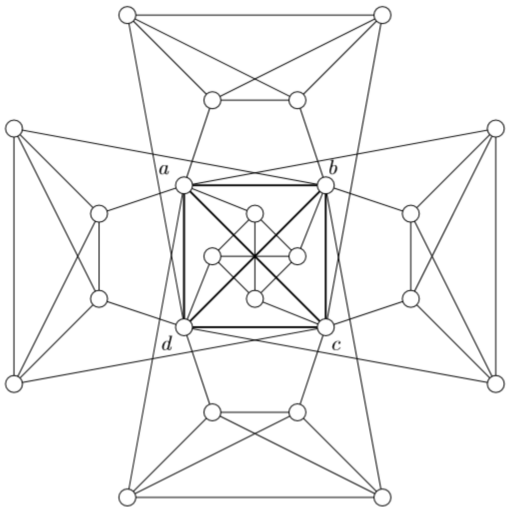


Figure 2.8: [22] A graph H is built by performing five $\mathcal{K}_4$-extensions on the subgraph induced by vertices a, b, c, d. The complete K_4 on a, b, c, d is minimally 2-vertex-rigid in $\mathbb{R}^2$ hence H is 2-vertex-rigid in $\mathbb{R}^2$. It can be easily seen that deleting any of the edges bc, cd, db from graph $H - a$ results in a nonrigid graph. By symmetry the deletion of any edge in the complete graph $\{a, b, c, d\}$ results in a graph that is not 2-vertex-rigid in $\mathbb{R}^2$. Combining with theorem 2.2.5, this implies that H is weakly minimal 2-vertex-rigid in $\mathbb{R}^2$.

Remark. Let v_1,v_2,v_3,v_4 be four distinct vertices in a graph G. The $\mathcal{K}_4$-extension adds four new vertices u_1,u_2,u_3,u_4 to G, connects v_i to u_i for every $1 \leq i \leq 4$ and u_k to u_l for every pair $1 \leq k,\ l \leq 4$. It is easy to see that the $\mathcal{K}_4$-extension operation preserves 2-vertex-rigidity in $\mathbb{R}^2$.

Theorem 2.2.5. [22] If $G = (V, E)$ is 2-vertex-rigid in $\mathbb{R}^2$ then $G_0 = (V_0, E_0)$ obtained by applying a $\mathcal{K}_4$-extension from G is also 2-vertex-rigid in $\mathbb{R}^2$. Furthermore $G_0 - e$ is not 2-vertex-rigid for any $e \in E_0 \setminus E$.

Proof. Clearly, $G_0 - v$ is rigid for any $v \in V_0$. Consider the graph $G_0 - e$ for some $e \in E_0 \setminus E$. Let $u_i \in V_0 \setminus V$ be such that e is not incident to u_i. We claim that $G_{00} = G_0 - u_i - e$ is not rigid. G_{00} consist of G and a set of three vertices that is incident to five edges only. Hence, there are only $2|V| - 3 + 5 = 2|V_0| - 4$ independent edges in G_{00}. Thus, G_{00} is not rigid as we claimed. ♠

Now let $G_0 = (V_0, E_0)$ be a 2-vertex-rigid graph in $\mathbb{R}^2$ with $|V_0| \geq 4$. Apply some $\mathcal{K}_4$-extensions to vertices of V_0, let the resulting graph be $G_1 = (V_1, E_1)$ (see figure 2.8). Suppose that every vertex in V_0 is incident to at least five edges from $E_1 - E_0$. After the extensions, delete edges from E_1 (if necessary) to obtain a minimally 2-vertex-rigid graph $G_2 = (V_1, E_2)$ in $\mathbb{R}^2$. By the above theorem, deleting any edge from $E_1 - E_0$ results in a graph that is not 2-vertex-rigid in $\mathbb{R}^2$. Hence, the minimum degree of vertices in G_2 is four (as we can choose applying more than 4 extensions $\mathcal{K}_4$ at the beginning) and all the degree four vertices are in $V_1 - V_0$. Clearly we cannot perform the reverse degree 3 vertex addition in G_2. Every vertex in $V_1 - V_0$ is contained in a complete K_4 subgraph of G_2 and every reverse X-replacement on one of these vertices creates a parallel pair of edges. Thus no reverse X-replacement operation preserves minimal 2-vertex-rigidity in $\mathbb{R}^2$ of G_2. This disproves the conjecture.

Remark. For any positive integer t, graph G_1 can be constructed in a way such that every vertex in V_0 is incident to at least t edges from $E_1 - E_0$. Hence the minimum degree in G_2 is 4 and the vertices in V_0 have degree at least t. Since t can be arbitrarily large, it implies that it may not be easy to find a constructive characterization of 2-dimensional weakly minimal 2-vertex-rigid graphs that only uses operations that add low-degree vertices.

We have searched for other weakly minimal k-vertex-rigid examples so far. The following result probably helps us in the future.

Theorem 2.2.6. Let $G = (V, E)$ be a minimally k-vertex-rigid graph in $\mathbb{R}^d$. Then

$$|E| \leq (d + k - 1)|V| - \binom{d+k}{2}.$$

This bound is sharp for $d \geq 2$. (See the detail in Theorem 5.2 [22].)

2.3 Open questions

1. Combinatorial characterization of strongly minimal rigid graphs in $\mathbb{R}^3$.
2. Full characterization/ Examples of strongly minimal k-vertex-rigid graphs in three-space or higher dimensions.
3. Sufficient conditions of strongly minimal 3-vertex-rigid graphs in $\mathbb{R}^2$ and "necessary and sufficient" operations to build them.
4. Strongly minimal k-vertex-rigidity in $\mathbb{R}^d$, $k \geq 4$ and $d \geq 2$.
5. Inductive construction for weakly minimally k-vertex-rigid graphs in $\mathbb{R}^2$, $k \geq 2$.
6. Weakly minimal k-vertex-rigidity in three-space or higher dimensions.

Chapter 3

Minimal k-vertex-global-rigidity

3.1 About existing results

Definition 3.1.1. A graph $G = (V, E)$ is called **k-vertex-globally-rigid** if $|V| \geq k + 1$ and it remains globally rigid after deleting any set of at most $k - 1$ vertices.

Definition 3.1.2. A graph $G = (V, E)$ is called **minimal k-vertex-globally-rigid** if it is k-vertex-globally-rigid but $G - e$ is not k-vertex-globally-rigid for any $e \in E$.

It follows from theorem 2.1 that the expression "deleting any set of at most $k-1$ vertices" in the above definition can be replaced by deleting any $k - 1$ vertices.

Definition 3.1.3. A graph is called **strongly minimally k-vertex-globally-rigid** if it is k-vertex-globally-rigid and there exists no k-vertex-globally-rigid graph with the same number of vertices and a smaller number of edges.

Definition 3.1.4. A graph is called **weakly minimally k-vertex-globally-rigid** if it is minimally k-vertex-globally-rigid but not strongly minimally k-vertex-globally-rigid.

In $\mathbb{R}$, a graph is globally rigid if and only if it is 2-vertex-connected. So a graph is k-vertex-globally-rigid in $\mathbb{R}$ if and only if it is $k+1$-vertex-connected. Hence, Harary graph $H_{n,k}$ $(n \geq k+1)$ and the complete bipartite graph $K_{k,k}$ are examples for strongly minimal $(k-1)$-vertex-global-rigidity in $\mathbb{R}$. Further, complete bipartite graph $K_{k,n-k}(n > 2k)$ is an example for weakly minimal $k-1$-vertex-global-rigidity in $\mathbb{R}$.
In $\mathbb{R}^2$, every strongly minimal global rigid graph $G = (V, E)$ with more than 4 vertices has $|E| = 2|V| - 2$. This is because the fact that global rigidity in $\mathbb{R}^2$ implies redundant rigidity, which implies $|E| \geq 2|V| - 2$, and moreover, wheel graph is a model of global rigid graph in $\mathbb{R}^2$ with $|E| = 2|V| - 2$. It is also known that every strongly minimally global rigid graph in $\mathbb{R}^2$ can be built up by repeatedly applying 2-dimensional Henneberg 1-extension operation from the initial complete graph K_4 [4].
In $\mathbb{R}^2$, a 2-vertex-globally-rigid graph $G = (V, E)$ on 5 or more vertices has the property that every vertex has degree 4 at least, which results in $|E| \geq 2|V|$. Moreover, a cycle on n vertices $(n \geq 5)$ with 2-hop neighbors (Harary graph $H_{n,4}$) is a 2-vertex-globally-rigid graph in $\mathbb{R}^2$. Hence, strongly minimal 2-vertex-globally-rigid graphs $G = (V, E)$ on 5 or more vertices in $\mathbb{R}^2$ have the following properties:

- $|E| = 2|V|$.
- Every vertex in G has degree 4.

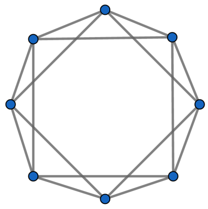


Figure 3.1: A cycle on 8 vertices with 2-hop neighbors.

Moreover, we also have a complete characterization of strongly minimal 2-vertex-global-rigidity in $\mathbb{R}^2$.

Theorem 3.1.1. [32] A graph $G(V, E)$ is strongly minimally 2-vertex-globally-rigid in $\mathbb{R}^2$ if and only if the following conditions hold:

- $|E| = 2|V|$;
- G is 4-connected;
- G is redundantly strongly minimal 2-vertex-rigid (i.e. removing any edge results in a strongly minimal 2-vertex-rigid graph).

Proof. For sufficiency, suppose the conditions hold for a graph G. Note first that since G is 4-connected, a graph obtained by removing a vertex and its incident edges is 3-connected. Further, 4-connectivity implies that every vertex has degree at least 4, and since $|E| = 2|V|$ then every vertex has precisely degree 4. Now choose any vertex v in G and remove any edge incident to this vertex. The resulting graph is 2-vertex-rigid by the third condition. Remove another edge incident to v. Via the edge partition in Theorem 2.1.3, the resulting graph consists of v with degree 2 attached to a redundantly rigid graph. Now we can remove v and the resulting graph is redundantly rigid. By the second condition, it is also 3-vertex-connected and therefore is globally rigid. The argument holds for any vertex v in G, which proves that G is 2-vertex-globally-rigid, and thus the conditions are sufficient. The 4-connectivity of G is obviously necessary because G minus any vertex must be 3-connected. Further, $|E| = 2|V|$ is obviously necessary. Now we need to prove the necessity of the final condition. To obtain a contradiction, suppose G is a 2-vertex-globally-rigid graph with an edge e that when removed does not result in a 2-vertex-rigid graph. Remove such an edge e and call the resulting graph G'. This implies that there exists a vertex v in G' that when removed results in a non-rigid graph G". There are two cases. First, if e is incident to v, then effectively we have removed v from G to obtain a non-rigid graph G". Thus, G is not 2-vertex-globally-rigid, contradicting our assumption. Second, if e is not incident to v, then if G is 2-vertex-globally-rigid, we should be able to reinsert e into G" to obtain a globally rigid graph. However, it is impossible to add a single edge to a nonrigid graph to make it redundantly rigid. This again contradicts our assumption, which proves the necessity of the final condition and completes the proof. ♠

Remark. Another example of strongly minimal 2-vertex-globally-rigid graphs is a cycle with 3-hop neighbors. The smallest strongly minimal 2-vertex-globally-rigid graph is the complete graph on 5 vertices.

For 2-dimensional weakly minimal 2-vertex-global-rigidity, we can take the family of complete graphs $K_{4,n-4}$ ($n \geq 9$) as an example. However, inductive constructions for weakly minimal 2-vertex-globally-rigid graphs in $\mathbb{R}^2$ remains an open question.

It is easy to see that in 2-dimensional 3-vertex-globally-rigid graph $G(V, E)$, each vertex has degree 5 at least since if one vertex has degree 4 at most, then after we remove 2 vertices incident with it, the vertex has degree 2 at most in the remaining graph that cannot happen in a 2-dimensional globally rigid graph. It implies that if $|V|$ is even, $|E| \geq \frac{5}{2}|V|$ and if $|V|$ is odd, then $|E| \geq \frac{5|V|+1}{2}$. The authors in paper [30] gave examples of 2-dimensional strongly minimal 3-vertex-globally-rigid graphs, when that strict bound holds (see figure 3.2).

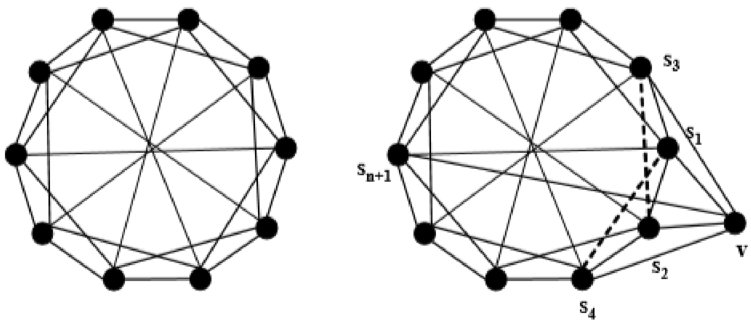


Figure 3.2: [30] (a) A strongly minimal 3-vertex-globally-rigid graph of size 10 in $\mathbb{R}^2$.
(b) A strongly minimal 3-vertex-globally-rigid graph of size 11 in $\mathbb{R}^2$.

Basically, the authors divided into 2 cases, when the graph has even number of vertices and when the graph has odd number of vertices. In the former case, the graph is actually Harary graph $H_{n,5}$ with n even, $n \geq 3$. In the latter case, it is first formed from Harary graph $H_{n,5}$ (n even, $n \geq 3$), and then we apply an extension that is called "2-extension" in [32]. The idea of their proof is first to show that the Harary graph $H_{n,5}$ is 3-vertex-globally-rigid in $\mathbb{R}^2$ and then to prove that the 2-extension operation applied to the Harary graph $H_{n,5}$ would preserve 3-vertex-global-rigidity. As we can see above, the Harary graph $H_{n,5}$ is 5-connected. So we only need to prove that it becomes redundantly rigid after the removal of any 2 vertices and the 2-extension operation preserves 3-vertex-global-rigidity in $\mathbb{R}^2$. But detailed proofs are omitted due to space limitation of the paper. Complete bipartite graphs $K_{5,n-5}$ ($n \geq 11$) are examples of weakly minimal 3-vertex-global-rigidity in $\mathbb{R}^2$.

3.2 More about strongly and weakly minimal 3-vertex-global-rigidity in $\mathbb{R}^2$

In this section, I will give another proof for the 2-dimensional 3-vertex-global-rigidity of Harary graph $H_{2n,5}$ ($n \geq 3$) and then give another family of 2-dimensional strongly minimal 3-vertex-globally-rigid graphs. Some families of weakly minimal 3-vertex-globally-rigid graphs in $\mathbb{R}^2$ are given at the end.

Theorem 3.2.1. A Harary graph $H_{2n,5}$ ($n \geq 3$) is 3-vertex-globally-rigid in $\mathbb{R}^2$.

Proof. One can check it holds for $n = 3, n = 4$ by removing two arbitrary vertices from the graph and realizing the remaining graph is still globally rigid.
For cases $n \geq 5$: Since the graph is symmetric, w.o.l.g. we can assume that we delete v_1 and another vertex (between v_2 and v_{n+1}) and we shall prove that the remaining graph

is globally rigid since it contains a spanning subgraph that is built from one K_4 complete graph by applying a sequence of Henneberg 1-extensions and degree 3 vertex operations, hence the initial graph is 3-vertex-globally-rigid.
The subgraph is constructed as follows:
Case 1: v_1 and v_2 are removed.
Vertices v_i $(2n-4 \geq i \geq n)$ are added in turn (v_{2n-4} first and v_n last) by 1-extensions from the complete graph $v_{2n}v_{2n-1}v_{2n-2}v_{2n-3}$ (delete $v_{i+1}v_{2n}$ and add $v_iv_{2n}, v_iv_{i+1}v_{i+2}$). Then vertices v_k $(n-1 \geq k \geq 3)$ are added in turn (v_{n-1} first and v_3 last) by degree 3 vertex operations (add $v_kv_{k+1}, v_kv_{k+2}, v_kv_{n+k}$). Hence, the subgraph is globally rigid.
Case 2: v_1 and v_{n+1} are removed.
Vertices v_i $(2n-4 \geq i \geq n+2)$ are added in turn (v_{2n-4} first and v_{n+2} last) by 1-extensions from the complete graph $v_{2n}v_{2n-1}v_{2n-2}v_{2n-3}$ (delete $v_{i+1}v_{2n}$ and add $v_iv_{i+1}, v_iv_{i+2}, v_iv_{2n}$). (In case $n = 5$, we only start with the complete graph $v_{10}v_9v_8v_7$ and do not add v_i as above). Then, v_n is added from this by 1-extension operation (delete $v_{n+2}v_{2n}$ and add $v_nv_{2n}, v_nv_{n+2}, v_nv_{2n-1}$). Vertex v_{n-1} is added from this by 1-extension operation (delete v_nv_{2n-1} and add new edges $v_{n-1}v_{2n}, v_{n-1}v_n, v_{n-1}v_{2n-1}$). Vertices v_k $(n-2 \geq k \geq 2)$ are added in turn (v_{n-2} first and v_2 last) by 1-extensions (delete $v_{k+1}v_{2n}$ and add $v_kv_{k+1}, v_kv_{2n}, v_kv_{k+2}$). Hence, the subgraph is globally rigid.
Case 3: v_1 and v_i $(3 \leq i \leq n)$ are removed.
Vertices v_k $(2n-4 \geq k \geq i+1)$ are added in turn (v_{2n-4} first and v_{i+1} last) by 1-extensions from complete graph $v_{2n}v_{2n-1}v_{2n-2}v_{2n-3}$ (delete $v_{k+1}v_{2n}$ and add $v_kv_{2n}, v_kv_{k+1}, v_kv_{k+2}$). Then v_{i-1} is added by 1-extension (delete $v_{i+1}v_{2n}$ and add $v_{i-1}v_{i+1}, v_{i-1}v_{2n}, v_{i-1}v_{n+i-1}$). Vertices v_k $(i-2 \geq k \geq 2)$ are added in turn (v_{i-2} first and v_2 last) by 1-extensions (delete $v_{k+1}v_{2n}$ and add $v_kv_{2n}, v_kv_{k+1}, v_kv_{k+n}$). Hence, the subgraph is globally rigid. ♠

For the case of odd number of vertices, one example of 2-dimensional strongly minimal 3-vertex-global-rigid graphs on $2n+1$ vertices $(n \geq 3)$ can be constructed as follows: Start with the Harary graph $H_{2n,5}$, remove three edges connecting three consecutive vertices and their opposite vertices, add one more vertex v_{2n+1} to the graph and connect it with these six vertices (see firgure 3.3). It is easy too see that it satisfy the strict bound $|E| = \frac{5|V|+1}{2}$.

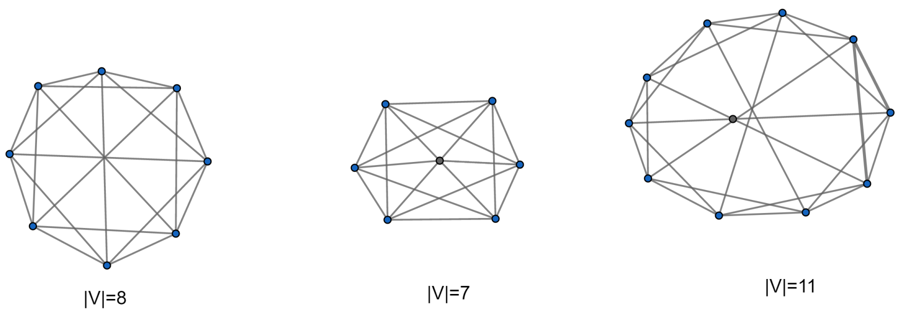


Figure 3.3: Examples for strongly minimal 3-vertex-globally-rigid graphs in $\mathbb{R}^2$.

Theorem 3.2.2. The constructed graph is 3-vertex-globally-rigid in $\mathbb{R}^2$.

Proof. It suffices show that after removing 2 vertices, the remaining graph is still globally rigid in $\mathbb{R}^2$. One can check that it holds for cases $n = 3, n = 4, n = 5, n = 6$. Now we concentrate for cases $n > 6$. If we remove v_{2n+1}, then the remaining graph contains a

subgraph that is a cycle with 2-hop neighbors, which is 2-vertex-globally-rigid. Hence, if one of two removed vertices is v_{2n+1} then we are done. Now, we only consider the cases when 2 removed vertices are from the main cycle (Harary graph).
For easy understanding, in each case, number vertices in the main cycle in a way such that the first removed vertex is marked with 1 and from that point, other vertices are marked clockwise. So we have vertices in the original main cycle numbering from 1 to $2n$, starting from the first removed vertex and increasing with the clockwise. Assume that according to that numbering, $v_k, v_{k+1}, v_{k+2}, v_{k+n}, v_{k+n+1}, v_{k+n+2}$ are vertices connecting to v_{2n+1} and v_b is the second removed vertex.

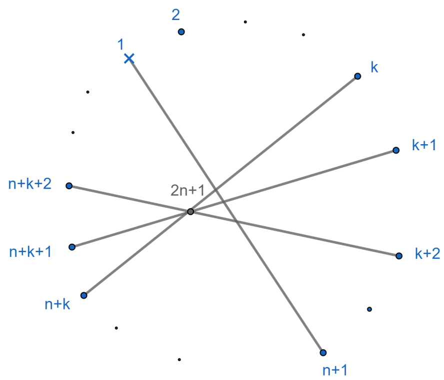


Figure 3.4: Numbering vertices according to the clockwise.

Because of the symmetry, it suffices to consider three cases.

1. $v_1, v_b \notin \{v_k, v_{k+1}, v_{k+2}, v_{k+n}, v_{k+n+1}, v_{k+n+2}\}$.

2. $v_1 \equiv v_{k+n+2}$.

3. $v_1 \equiv v_{k+n+1}$.

Now, in each case, we will prove that after v_1, v_b are removed, the remaining graph contains a globally rigid spanning subgraph which is built from one complete graph K_4 by applying a sequence of Henneberg 1-extensions and degree 3 vertex operations, thus it is globally rigid. The construction of the subgraph is as follows:
Case 1: $v_1, v_b \notin \{v_k, v_{k+1}, v_{k+2}, v_{k+n}, v_{k+n+1}, v_{k+n+2}\}$.
First, assume that v_b belongs to left side of v_1 in the original cycle, i.e, $n + 1 \leq b \leq 2n$.
Case 1.1: $b \neq 2n$.

- Vertices v_i $(6 \leq i \leq b - 1)$ are added in turn by 1-extensions from the complete graph $v_2v_3v_4v_5$ (delete $v_{i-1}v_2$ and add $v_iv_2, v_iv_{i-1}, v_iv_{i-2}$);

- v_{2n+1} is added with 3 edges: $v_{2n+1}v_k, v_{2n+1}v_{k+1}, v_{2n+1}v_{k+2}$;

- v_{b+1} is added by 1-extension operation (delete $v_{b-1}v_2$ and add either $v_{b+1}v_{b-1}, v_{b+1}v_2$, $v_{b+1}v_{b+1-n}$ if $v_{b+1} \notin \{v_{k+n}, v_{k+n+1}, v_{k+n+2}\}$ or $v_{b+1}v_{b-1}, v_{b+1}v_2, v_{b+1}v_{2n+1}$, otherwise);

- Vertices v_j $(b + 2 \leq j \leq 2n)$ are added in turn by 1-extensions (delete $v_{j-1}v_2$ and add either $v_jv_2, v_jv_{j-1}, v_jv_{j-n}$ if $v_j \notin \{v_{k+n}, v_{k+n+1}, v_{k+n+2}\}$ or $v_jv_2, v_jv_{j-1}, v_jv_{2n+1}$, otherwise).

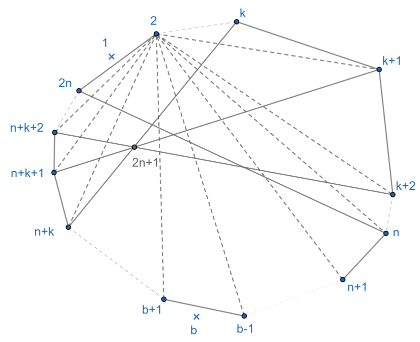


Figure 3.5: Illustration for case 1.1.

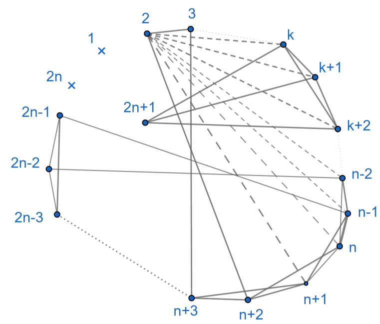


Figure 3.6: Illustration for case 1.2.

Case 1.2: $b = 2n$.

- Vertices v_i $(6 \leq i \leq n+2)$ are added in turn by 1-extensions from the complete graph $v_2v_3v_4v_5$ (delete $v_{i-1}v_2$ and add $v_iv_2, v_iv_{i-1}, v_iv_{i-2}$);
- v_{2n+1} is added with 3 edges $v_{2n+1}v_k, v_{2n+1}v_{k+1}, v_{2n+1}v_{k+2}$;
- Vertices v_j $(n+3 \leq j \leq 2n-1)$ are added in turn by degree 3 vertex operations (add $v_jv_{j-1}, v_jv_{j-2}, v_jv_{j-n}$).

The case when v_b belongs to right side of v_1 $(v_b \notin \{v_k, v_{k+1}, v_{k+2}\})$ is similar since in that situation, we can consider v_1 belongs to left side of v_b and interchange their roles to return previous cases.
Case 2: $v_1 \equiv v_{k+n+2}$.
Case 2.1: $v_b \equiv v_{n+k+1}$.
So $b = 2n$ in this case.

- Vertices v_i $(6 \leq i \leq n+2)$ are added in turn by 1 extensions from the complete graph $v_2v_3v_4v_5$ (delete $v_{i-1}v_2$ and add $v_iv_2, v_iv_{i-1}, v_iv_{i-2}$);
- v_{2n+1} is added with 3 edges $v_{2n+1}v_{n-1}, v_{2n+1}v_n, v_{2n+1}v_{n+1}$;
- Vertices v_j $(n+3 \leq j \leq 2n-2)$ are added in turn by degree 3 vertex operations (add $v_jv_{j-1}, v_jv_{j-2}, v_jv_{j-n}$);
- Vertex v_{2n-1} is added with three edges $v_{2n-1}v_{2n-2}, v_{2n-1}v_{2n-3}, v_{2n+1}v_{2n-1}$.

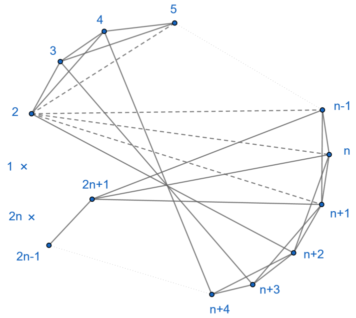


Figure 3.7: Illustration for case 2.1.

Case 2.2: $v_b \equiv v_{n+k}$.
In this case $b = 2n - 1$.

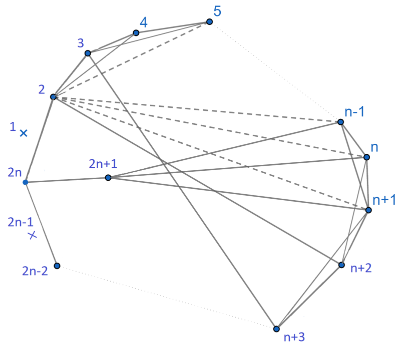


Figure 3.8: Illustration for case 2.2.

- Vertices v_i $(6 \leq i \leq n+2)$ are added in turn by 1-extensions from the complete graph $v_2v_3v_4v_5$ (delete $v_{i-1}v_2$ and add $v_iv_2, v_iv_{i-1}, v_iv_{i-2}$);
- v_{2n+1} is added with 3 edges $v_{2n+1}v_{n-1}, v_{2n+1}v_n, v_{2n+1}v_{n+1}$;
- Vertices v_j $(n+3 \leq j \leq 2n-2)$ are added in turn by degree 3 vertex operations $v_jv_{j-1}, v_jv_{j-2}, v_jv_{j-n}$;
- v_{2n} is added with 3 edges $v_{2n}v_{2n-2}, v_{2n}v_{2n+1}, v_{2n}v_2$.

Case 2.3: $v_b \equiv v_k$ or $v_b \equiv v_{k+1}$.
Here we have two cases: $v_b \equiv v_k \equiv v_{n-1}$ or $v_b \equiv v_{k+1} \equiv v_n$.

- Vertices v_i $(2n-3 \geq i \geq n+1)$ are added in turn by 1-extensions from the complete graph $v_{2n+1}v_{2n}v_{2n-1}v_{2n-2}$ (delete $v_{i+1}v_{2n+1}$ and add $v_iv_{2n+1}, v_iv_{i+1}, v_iv_{i+2}$);
- v_2 is added with 3 edges $v_2v_{2n}, v_2v_{2+n}, v_2v_{2n+1}$;
- Vertices v_j $(3 \leq j \leq n-2)$ are added in turn by 1-extensions (delete $v_{j-1}v_{2n+1}$ and add $v_jv_{j-1}, v_jv_{2n+1}, v_jv_{j+n}$);
- Finally, vertex v_l $(v_l \in \{v_n, v_{n-1}\})$ is added by 1-extension operation from this (delete $v_{n-2}v_{2n+1}$ and add $v_lv_{n-2}, v_lv_{2n+1}, v_lv_{n+1}$).

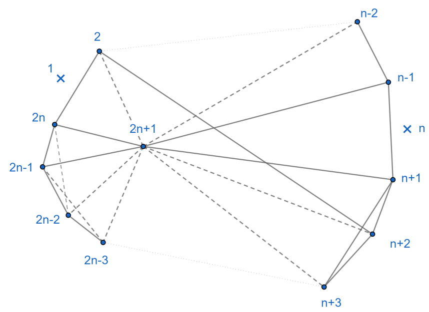


Figure 3.9: Illustration for case 2.3 when $v_b \equiv v_{k+1}$.

Case 2.4: $v_b \equiv v_{k+2}$.
In this case, $b = n + 1$.

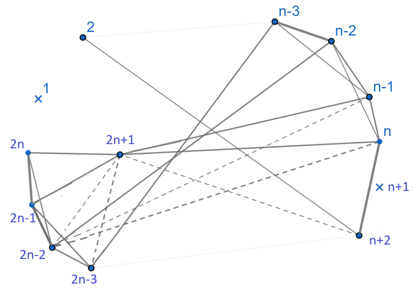


Figure 3.10: Illustration for case 2.4.

- Vertices v_i $(2n-3 \geq i \geq n+2)$ are added in turn by 1-extensions from the complete graph $v_{2n+1}v_{2n}v_{2n-1}v_{2n-2}$ (delete $v_{i+1}v_{2n+1}$ and add $v_iv_{i+1}, v_iv_{i+2}, v_iv_{2n+1}$);
- v_n is added from by 1-extension operation (delete $v_{2n+1}v_{n+2}$ and add v_nv_{2n+1}, v_nv_{n+2}, v_nv_{2n-2});
- v_{n-1} is added by 1-extension operation (delete v_nv_{2n-2} and add $v_{n-1}v_{2n-2}, v_{n-1}v_n$, $v_{n-1}v_{2n+1}$);
- v_{n-2} is added by 1-extension operation (delete $v_{n-1}v_{2n-2}$ and add $v_{n-2}v_{2n-2}, v_{n-2}v_{n-1}$, $v_{n-2}v_n$);
- Vertices v_j $(n-3 \geq j \geq 2)$ are added in turn by degree 3 vertex operations (add $v_jv_{j+1}, v_jv_{j+2}, v_jv_{j+n}$).

Case 2.5: v_b belongs to left side of v_1 in the main cycle, $v_b \notin \{v_{n+k}, v_{n+k+1}, v_{k+2}\}$.
Case 2.5.1: $b \neq 2n-2$.

- Vertices v_i $(b-5 \geq i \geq b-1-n)$ are added in turn by 1-extensions from the complete graph $v_{b-1}v_{b-2}v_{b-3}v_{b-4}$ (delete $v_{i+1}v_{b-1}$ and add $v_iv_{b-1}, v_iv_{i+1}, v_iv_{i+2}$);
- v_{2n+1} is added with 3 edges $v_{2n+1}v_{n-1}, v_{2n+1}v_n, v_{2n+1}v_{n+1}$;
- Vertices v_j $(b-n-2 \geq j \geq 2)$ are added in turn by degree 3 vertex operations (add $v_jv_{j+n}, v_jv_{j+1}, v_jv_{j+2}$);

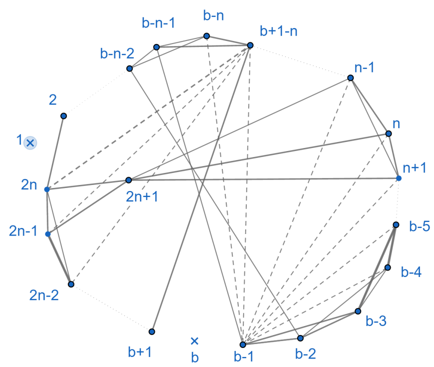


Figure 3.11: Illustration for case 2.5.1.

- v_{2n} is added with 3 edges $v_{2n}v_{2n+1}, v_{2n}v_2, v_{2n}v_{b+1-n}$;
- v_{2n-1} is added by 1-extension operation (delete $v_{2n}v_{b+1-n}$ and add $v_{2n-1}v_{2n}$, $v_{2n-1}v_{2n+1}$, $v_{2n-1}v_{b+1-n}$);
- Vertices v_j $(2n-2 \geq j \geq b+1)$ are added in turn by 1-extensions (delete $v_{j+1}v_{b+1-n}$ and add either $v_jv_{b+1-n}, v_jv_{j+1}, v_jv_{j-2}$ in case $v_j = v_{b+1}$ or $v_jv_{b+1-n}, v_jv_{j+1}, v_jv_{j+2}$, otherwise).

Case 2.5.2: $b = 2n-2$.

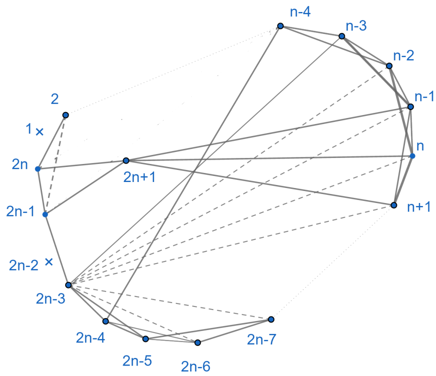


Figure 3.12: Illustration for case 2.5.2.

- Vertices v_i $(2n-7 \geq i \geq n-3)$ are added in turn by 1-extensions from the complete graph $v_{2n-3}v_{2n-4}v_{2n-5}v_{2n-6}$ (delete $v_{i+1}v_{2n-3}$ and add $v_iv_{2n-3}, v_iv_{i+1}, v_iv_{i+2}$);
- v_{2n+1} is added with 3 edges $v_{2n+1}v_{n-1}, v_{2n+1}v_n, v_{2n+1}v_{n+1}$;
- Vertices v_j $(n-4 \geq j \geq 2)$ are added in turn with degree 3 vertex operations (add $v_jv_{j+1}, v_jv_{j+2}, v_jv_{j+n}$);
- v_{2n-1} is added with 3 edges $v_{2n-1}v_{2n+1}, v_{2n-1}v_{2n-3}, v_{2n-1}v_2$;
- v_{2n} is added by 1-extension operation (delete $v_{2n-1}v_2$ and add $v_{2n}v_2, v_{2n}v_{2n+1}, v_{2n}v_{2n-1}$).

Case 2.6: v_b belongs to right side of v_1 $(v_b \notin \{v_k, v_{k+1}, v_{k+2}\})$.
Case 2.6.1: $b = 2$

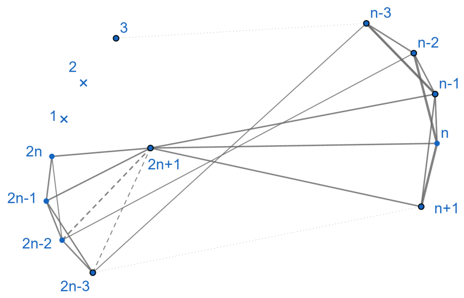


Figure 3.13: Illustration for case 2.6.1.

- Vertices v_i $(2n-3 \geq i \geq n+1)$ are added in turn by 1-extensions from the complete graph $v_{2n+1}v_{2n}v_{2n-1}v_{2n-2}$ (delete $v_{i+1}v_{2n+1}$ and add $v_iv_{2n+1}, v_iv_{i+1}, v_iv_{i+2}$);
- Vertices v_j $(n \geq j \geq 3)$ are added in turn by degree 3 vertex operations (add either $v_jv_{j+1}, v_jv_{j+2}, v_jv_{2n+1}$ if $v_j \notin \{v_n, v_{n-1}\}$ or $v_jv_{j+1}, v_jv_{j+2}, v_jv_{j+n}$, otherwise).

Case 2.6.2: $b \neq 2$.

- Vertices v_i $(2n-3 \geq i \geq n)$ are added in turn by 1-extensions from the complete graph $v_{2n+1}v_{2n}v_{2n-1}v_{2n-1}$ (delete $v_{i+1}v_{2n+1}$ and add $v_iv_{i+1}, v_iv_{2n+1}, v_iv_{i+2}$);
- v_2 is added with 3 edges $v_2v_{2n}, v_2v_{n+2}, v_2v_{2n+1}$;

Case 2.6.2.1: $b = 3$

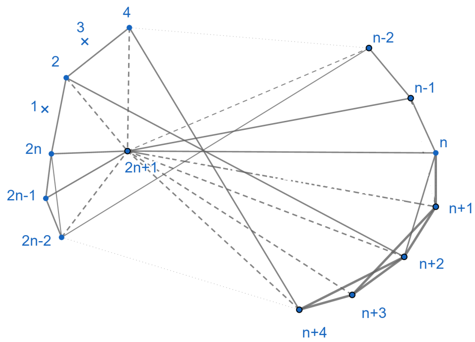


Figure 3.14: Illustration for case 2.6.2.1.

- v_4 is added from above by 1-extension operation (delete v_2v_{2n+1} and add v_4v_{2n+1}, v_4v_2, v_4v_{n+4});
- Vertices v_j $(5 \leq j \leq n-2)$ are added in turn 1-extensions (delete $v_{j-1}v_{2n+1}$ and add $v_jv_{2n+1}, v_jv_{j-1}, v_jv_{j+n}$);
- v_{n-1} is added by 1-extension operation (delete $v_{n-2}v_{2n+1}$ and add $v_{n-1}v_{n-2}, v_{n-1}v_{2n+1}$, $v_{n-1}v_n$).

Case 2.6.2.2: $b \neq 3$.

- Vertices v_i $(3 \leq i \leq b-1)$ are added in turn from above by 1-extensions (delete $v_{i-1}v_{2n+1}$ and add $v_iv_{2n+1}, v_iv_{i-1}, v_iv_{i+n}$);

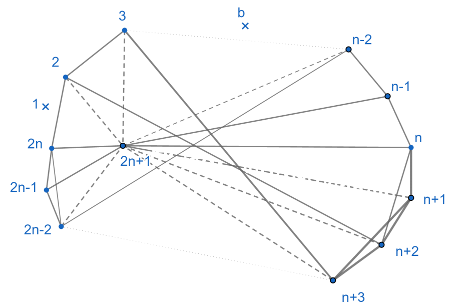


Figure 3.15: Illustration for case 2.6.2.2.

- v_{b+1} is added 1-extension operation (delete $v_{b-1}v_{2n+1}$ and add $v_{b+1}v_{b-1}, v_{b+1}v_{2n+1}$, $v_{b+1}v_{b+1+n}$. Skip this step in case $b = n-3$;
- Vertices v_j $(b+2 \leq j \leq n-2)$ are added in turn by 1-extensions (delete $v_{j-1}v_{2n+1}$ and add $v_jv_{2n+1}, v_jv_{j-1}, v_jv_{j+n}$). Skip this step in case $b = n-3$;
- v_{n-1} is added by 1-extension operation (delete $v_{n-2}v_{2n+1}$ and add $v_{n-1}v_{2n+1}, v_{n-1}v_n$, $v_{n-1}v_{n-2}$).

Case 3: $v_1 \equiv v_{n+k+1}$.
Case 3.1: $v_b \equiv v_{n+k}$.
Similar to the case $v_1 \equiv v_{k+n+2}, v_b \equiv v_{k+n+1}$ because of the symmetry.
Case 3.2 $v_b \equiv v_k$ or $v_b \equiv v_{k+1}$.
There are two cases: $v_b \equiv v_k \equiv v_n$ or $v_b \equiv v_{k+1} \equiv v_{n+1}$.

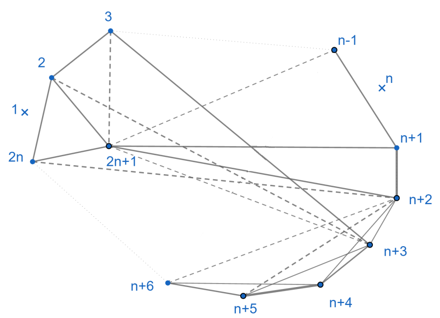


Figure 3.16: Illustration for the case 3.2 when $v_b \equiv v_k$.

- Vertices v_i $(n+6 \leq i \leq 2n)$ are added in turn by 1-extensions from the complete graph $v_{n+2}v_{n+3}v_{n+4}v_{n+5}$ (delete $v_{i-1}v_{n+2}$ and add $v_iv_{i-1}, v_iv_{n+2}, v_iv_{i-2}$);
- v_{2n+1} is added by 1-extension operation (delete $v_{2n}v_{n+2}$ and add $v_{2n+1}v_{2n}, v_{2n+1}v_{n+2}$, $v_{2n+1}v_{n+3}$);
- v_2 is added by 1-extension operation (delete $v_{2n+1}v_{n+3}$ and add $v_2v_{2n+1}, v_2v_{n+3}, v_2v_{2n}$);
- v_3 is added by 1-extension operation (delete v_2v_{n+3} and add $v_3v_{n+3}, v_3v_{2n+1}, v_3v_2$);
- Vertices v_j $(4 \leq j \leq n-1)$ are added in turn by 1-extensions (delete $v_{j-1}v_{2n+1}$ and add $v_jv_{2n+1}, v_jv_{j-1}, v_jv_{j+n}$);

- Finally, v_l ($v_l \in \{v_n, v_{n+1}\}$) is added by 1-extension operation (delete $v_{n-1}v_{2n+1}$ and add $v_lv_{n-1}, v_lv_{2n+1}, v_lv_{n+2}$).

Case 3.3: v_b belongs to left side of v_1 and $v_b \notin \{v_{n+k}, v_{k+1}, v_{k+2}\}$.

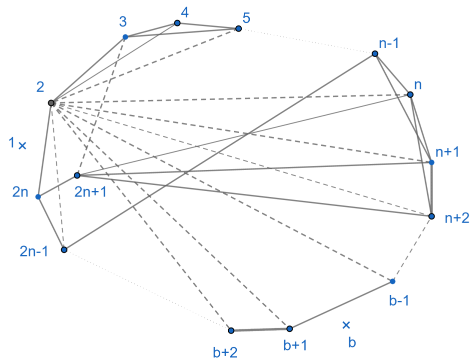


Figure 3.17: Illustration for case 3.3.

- Vertices v_i ($6 \leq i \leq b-1$) are added in turn by 1-extensions from the complete graph $v_2v_3v_4v_5$ (delete $v_{i-1}v_2$ and add $v_iv_2, v_iv_{i-1}, v_iv_{i-2}$);
- v_{2n+1} is added with 3 edges $v_{2n+1}v_n, v_{2n+1}v_{n+1}, v_{2n+1}v_{n+2}$;
- v_{b+1} is added by 1-extension operation (delete $v_{b-1}v_2$ and add $v_{b+1}v_{b-1}, v_{b+1}v_2$, $v_{b+1}v_{b+1-n}$). Skip this step in case $b = 2n-1$;
- Vertices v_j ($b+2 \leq j \leq 2n-1$) are added in turn by 1-extensions (delete $v_{j-1}v_2$ and add $v_jv_2, v_jv_{j-1}, v_jv_{j-n}$). Skip this step in cases $b = 2n-1$; $b = 2n-2$;
- Vertex v_{2n} is added by 1-extension operation (delete $v_{2n-1}v_2$ and add $v_{2n}v_{2n-1}, v_{2n}v_2$, $v_{2n}v_{2n+1}$ in the case $b \neq 2n-1$ or delete $v_{2n-2}v_2$ and add $v_{2n}v_{2n-2}, v_{2n}v_{2n+1}, v_{2n}v_2$ in case $b = 2n-1$).

Because the graph that we constructed is symmetric through $v_{k+n+1}v_{k+1}$, other cases can be proved similarly as we did above.
Since all subgraphs that we constructed above are globally rigid in $\mathbb{R}^2$ and contain all other $2n-1$ vertices after the removal of two vertices v_1, v_b, the original graph is 3-vertex-globally rigid in $\mathbb{R}^2$. ♠

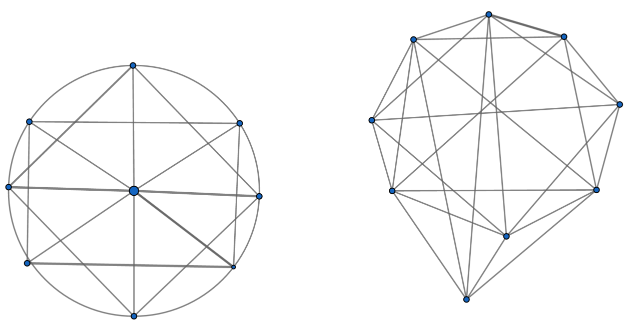


Figure 3.18: Two weakly minimal 3-vertex-global-rigid graphs with 9 vertices in $\mathbb{R}^2$.

Figure 3.18 shows examples of weakly minimal 3-vertex-global-rigidity in $\mathbb{R}^2$. In fact, one family of weakly minimal 3-vertex-globally-rigid graphs with n vertices ($n > 6$) can be built from Harary graph $H_{n-1,4}$ and then connect all vertices with one new vertex. The number of edges in this case is $3n - 3$. If n is odd, another family of weakly minimal 3-vertex-globally-rigid graphs can be constructed by Harary graph $H_{n-1,5}$ first and then add one more vertex and connect it with five other existed vertices. This type of graph has $\frac{5n+5}{2}$ edges.

3.3 Open questions

1. Operations to build all strongly minimal 2-vertex-globally-rigid graphs in $\mathbb{R}^2$.
2. Full characterization of strongly/weakly minimal k-vertex-global-rigidity in $\mathbb{R}^2$, $k \geq 3$ and operations to build them.
3. Complete characterization of minimal k-vertex-global-rigidity in three-space or higher dimensions.

Chapter 4

Minimal k-edge-(global) rigidity

Due to the higher severity of vertex loss than edge loss, research attempts pay less attention to redundancy of edges. Some results about minimal k-edge-rigidity and minimal k-edge-global-rigidity are mentioned below.

4.1 Minimal k-edge-rigidity

Definition 4.1.1. A graph $G = (V, E)$ is called **k-edge-rigid** if it is rigid after deleting any $k - 1$ edges.

Definition 4.1.2. A graph $G = (V, E)$ is called **minimal k-edge-rigid** if it is k-edge-rigid and $G - e$ is not k-edge-rigid for any $e \in E$.

It implies that in a minimal k-edge-rigid graph, for each edge, there is a set of k edges containing it, which makes the graph nonrigid after removing the edge set.

Definition 4.1.3. A graph is called **strongly minimally k-edge-rigid** if it is minimal k-edge-rigid and there exists no k-edge-rigid graph with the same number of vertices and a smaller number of edges.

Definition 4.1.4. A graph is called **weakly minimally k-edge-rigid** if it is minimal k-edge-rigid but not strongly minimal k-edge-rigid.

Firstly, we can easily see that for $k = 1$, there is no differences between strongly minimal 1-edge-rigidity and weakly minimal 1-edge-rigidity since they become minimal rigidity. In a d-dimensional space, minimal rigid graphs $G = (V, E)$ must be complete graphs if $|V| \leq d$ or satisfy the condition $|E| = d|V| - \binom{d+1}{2}$, otherwise (theorem 1.3).
In $\mathbb{R}$, k-edge-rigidity is equivalent to k-edge-connectivity. Hence, strongly minimal k-edge-rigid graphs in $\mathbb{R}$ are k-edge-connected graphs with fewest possible edges. Since every vertex of a k-edge-connected graph has degree at least k, the minimum number of edges of a k-edge-connected graph with n vertices is $n - 1$ if $k = 1$ and $\lceil \frac{1}{2}kn \rceil$ otherwise. For $k = 1$, the graph is a tree. For $k > 1$, we can build strongly minimal k-edge-rigid graphs in $\mathbb{R}$ as follows:
Let C be a graph on $V := \{1, \ldots, n\}$ with edges $\{i, i + 1\}$ for $i \in V$ (with addition mod n). Let G be the graph obtained from C by replacing each edge by $\lfloor \frac{1}{2}k \rfloor$ parallel edges. If k is even, then G is k-edge-connected as required. If k is odd, add $\lceil \frac{1}{2}n \rceil$ edges $\{i, j\}$ to G, where i and j have distance $\lfloor \frac{1}{2}n \rfloor$ in C and such that these edges cover all vertices in V. So G has $\lceil \frac{1}{2}kn \rceil$ edges and in fact, G is k-edge-connected. Suppose that $d_G(U) < k$

for some nonempty proper subset U of V. By symmetry, we can assume that $|U| \geq \frac{1}{2}n$. Now $C[U]$ is connected (as otherwise $d_G[U] \geq 4\lfloor\frac{1}{2}k\rfloor \geq k$ since $k > 1$). So we can assume that $U = [1, s]$, with $s \geq \lceil\frac{1}{2}n\rceil$. However, $n \in V \setminus U$ is adjacent to $\lceil\frac{1}{2}n\rceil$. As this vertex belongs to U, we have $d_G(U) \geq 2\lfloor\frac{1}{2}k\rfloor + 1 = k$, a contradiction.
Specially, in the case $n \geq k+1$, then Harary graph $H_{n,k}$ is an example for strongly minimal k-edge-rigidity in $\mathbb{R}$ since as we know, it has $\lceil\frac{1}{2}kn\rceil$ edges and its k-vertex-connectivity implies its k-edge-connectivity. Note that the smallest vertex set that is incident to all edges of the minimal cut succeeds in disconnecting the graph.
Now we consider minimal 2-edge-rigidity. Because the loss of one edge does not make it nonrigid, a 2-edge-rigid graph on n vertices in $\mathbb{R}^d$ ($n > d$) must have $dn - \binom{d+1}{2} + 1$ edges at least. In fact, we have 2-edge-rigid graphs with this fewest possible edges. These can be constructed by applying a sequence of d-dimensional Henneberg 1-extension operations to the complete graph on $d+2$ vertices. The first reason for that is from the 2-edge-rigid property of a complete graph on $d+2$ vertices. After deleting an arbitrary edge e from a complete graph G on $d+2$ vertices, the remaining graph contains a complete subgraph on $d+1$ vertices, which connects with the last vertex by d edges; hence, $G-e$ is rigid and G is 2-edge-rigid. The second reason is from that fact that the d-dimensional Henneberg 1-extension operation preserves the 2-edge-rigidity in $\mathbb{R}^d$ (Easy to check that). Moreover, the initial graph, a complete graph on $d+2$ vertices, has $\binom{d+2}{2}$ vertices (which is equal to $dn - \binom{d+1}{2} + 1$ when $n = d+2$) and once a d-dimension 1-extension is applied, a new vertex and d new edges are added. Therefore, in $\mathbb{R}^d$, a strongly minimal 2-edge-rigid graph on n vertices has $dn - \binom{d+1}{2} + 1$ edges.
Applying for $d = 2$, a strongly minimal 2-edge-rigid graph on n vertices ($n > 2$) has $2n-2$ edges (e.g. a wheel graph). For weakly minimal 2-edge-rigidity in $\mathbb{R}^2$, we have the one result presented in paper [19]. Two-edge-rigidity in this paper is mentioned as redundant rigidity. We have the equivalent definition for minimal 2-edge-rigid graphs.

Definition 4.1.5. $G = (V, E)$ is **minimally redundantly rigid** if G is redundantly rigid but $G - e$ is not redundantly rigid for all $e \in E$.

Theorem 4.1.1. [19] Let $G = (V, E)$ be a minimally redundantly rigid graph in $\mathbb{R}^2$. Then $|E| \leq 3|V| - 6$.

In other words, all 2-dimensional weakly minimal 2-edge-rigid graphs $G = (V, E)$ satisfy the condition $|E| \leq 3|V| - 6$. Moreover, in case $|V| \geq 7$, Jordan showed us a tighter condition $|E| \leq 3|V| - 9$. The complete bipartite graphs $K_{3,m}$ ($m \geq 5$) are examples of weakly minimal 2-edge-rigid graphs in $\mathbb{R}^2$.
As far as I know, the characterization of minimal k-edge-rigidity, $k \geq 3$ in the plane or higher dimensional spaces is still an open question.

4.2 Minimal k-edge-global-rigidity

Definition 4.2.1. A graph $G = (V, E)$ is called **k-edge-globally-rigid** if it is still globally rigid after removal of any $k-1$ edges.

Definition 4.2.2. A graph $G = (V, E)$ is called **minimal k-edge-globally-rigid** if it is k-edge-globally-rigid and $G - e$ is not k-edge-global-rigid for any $e \in E$.

Definition 4.2.3. A graph is called **strongly minimally k-edge-globally-rigid** if it is minimal k-edge-globally-rigid and there exists no k-edge-globally-rigid graph with the same number of vertices and fewer edges.

Definition 4.2.4. A graph is called **weakly minimally k-edge-globally-rigid** if it is minimal k-edge-globally-rigid but not strongly minimally k-edge-globally-rigid.

Start with $k = 1$, strongly minimal 1-edge-global-rigidity is actually strongly minimal global rigidity. In fact, a graph G on at most $d + 1$ vertices is globally rigid in $\mathbb{R}^d$ if and only if G is complete. So we concentrate for the case when G has more than $d+1$ vertices. In this case, all strongly minimal globally rigid graphs on n vertices ($n \geq d + 2$) have $dn - \binom{d+1}{2} + 1$ edges. Again, a family of graphs that is constructed from a complete graph on $d+2$ vertices by applying a sequence of d-dimension Henneberg 1-extension operations is an example for that. Such graphs are globally rigid in $\mathbb{R}^d$ due to that fact that the d-dimension 1-extension operation preserves global rigidity (proved by Connelly in [6]).
Weakly minimal 1-edge-global-rigidity is weakly minimal global rigidity. In $\mathbb{R}$ and $\mathbb{R}^2$, we already have complete characterization of global rigidity. A graph is globally rigid in $\mathbb{R}$ if and only if it is the complete graph K_2 or it is 2-connected. A graph is globally rigid in $\mathbb{R}^2$ if and only if it is K_2 or K_3 or it is 3-connected and redundantly rigid. From that, we can find examples about weakly minimal globally rigid graphs. For example, complete bipartite graphs $K_{2,n-2}$ ($n \geq 5$), $K_{3,n-3}$ ($n \geq 8$) can be illustrations for weakly minimal global rigidity in $\mathbb{R}$ and $\mathbb{R}^2$, respectively.
For three-space and higher dimensions, the existence of weakly minimal globally rigid graphs is still mysterious somehow. There is a conjecture related to that.

Conjecture 4.2.1. [21] Let $G = (V, E)$ be minimally globally rigid in $\mathbb{R}^d$. Then

- $|E| \leq (d + 1)|V| - \binom{d+2}{2}$,
- the minimum degree of G is at most $2d + 1$.

4.3 Open questions

1. Minimal k-edge-rigidity in $\mathbb{R}^2$ with $k \geq 3$.
2. Weakly minimal k-edge-global-rigidity in $\mathbb{R}^2$ with $k \geq 2$.
3. Combinatorial characterization of minimal k-edge-(global) rigidity in three-space or higher dimensions.

Chapter 5

Applications of rigidity theory

Rigidity, global rigidity, redundant rigidity, and redundant global rigidity are not only terminologies in mathematics but also have origins from nature. They are found in a variety of practical applications. Although the definitions of rigidity and global rigidity in different fields have some slight changes with the pure definitions in mathematics, after all we see applications of rigidity theory in real life. In this chapter, among its applications such as architecture, engineering, materials science, medicine and biochemistry, statics, computer-aided design (CAD), network sensoring, motion planning, NMR reconstruction, and percolation theory, I shall focus on the ones I feel most interested.

5.1 Applications to sensor network localization and formation control

Some animals like birds, fishes, or ants often forage for food or move together. One reason for such group behaviors is that they can do sophisticated tasks that cannot be achieved by individual members. In this collective behavior, the relative positions between members are preserved, and the group moves as a cohesive whole.

Nowadays, artificial systems of robots, underwater vehicles, and autonomous or piloted airborne vehicles are being deployed to tackle specific missions without human involvement in both the civilian and military spheres, such as bush-fire control, surveillance, and underwater exploration. They also work in a cohesive whole to complete complicated works and are known as autonomous vehicle formations.

Autonomous vehicle formations and sensor networks have lately received considerable attention due to recent technological advances. A sensor network, a wider concept than an autonomous vehicle formation, is a collection of agents, each with sensing, communication, and computation capabilities that cooperate to accomplish a task.

Rigidity, global rigidity, and redundant rigidity are important properties of information architectures for these networks because of the crucial role they play in formation shape control, self-localization, and robustness of the whole system.

5.1.1 Formation shape control

A sensor network is modeled with a graph $G(V, E)$, where V is a set of vertices representing agents and E is a set of edges representing inter-agent distances to be actively held constant as moving. As we know from rigidity theory, if a suitably large and well-chosen set of inter-agent distances is held constant, then all remaining inter-agent distances will be constant as a consequence; thus, maintaining formation shape. Although an easy way to maintain a desired formation shape is to control every possible inter-agent distance, this is not an effective solution in real life. It is much better to control a number of necessary distances in a prescribed sensing and communication range. Rigidity theory (especially minimal rigidity) is the key to address that issue; it tells us the minimum sufficient sensing radius for each agent to guarantee the formation shape control in sensor networks.

5.1.2 Wireless sensor networks self-localization

A wireless sensor network consists of a small number of anchors and a large number of small, cheap ordinary nodes. Anchors can be aware of their own positions and ordinary nodes have no prior knowledge of their locations. If ordinary nodes were capable making measurements to multiple anchors, they could determine their positions. However, in some cases, several ordinary nodes cannot directly communicate with anchors because of power limitations or signal blockage. Hence, ordinary nodes not only make connections with anchors, but also they make measurements with other ordinary nodes.
Global rigidity addresses the self-localization of the network. When we know the exact position of three non-collinear anchors in the space, we can determine the position of whole system because of its unique realization.

5.1.3 Robustness of sensor networks

Another important property of sensor networks is their potential robustness to loss of some agent(s) or several link(s). These losses can occur due to enemy attack or jamming, due to random mechanical or electrical failure, or due to intentionally deploying an agent for a separate task. Therefore, we need to investigate the structure of graphs with the property that rigidity or global rigidity is preserved after removal of some vertices or some edges. This is the main reason why the previous chapters are about k-vertex-rigidity, k-vertex-global-rigidity, k-edge-rigidity and k-edge-global-rigidity.

5.2 Applications to biomolecules

Biomolecules are heterogeneously composed of rigid and flexible (nonrigid) regions. Here, flexibility and rigidity denote the possibility, or impossibility, of internal motions in an object under force without giving information about directions and magnitudes of movements. Understanding biomolecular flexibility and rigidity is instrumental in understanding of biomolecular function.
In fact, it is still challenging for us to find a full combinatorial characterization of rigid graphs in the three-dimensional space. We have Laman's theorem for complete characterization of rigidity in $\mathbb{R}^2$ (the $2n-3$ edge count condition). However, we cannot apply the $3n-6$ edge count condition in $\mathbb{R}^3$ (see the "double banana" graph).
The question is whether we can determine the rigid components of biological molecules while all of them exist in the three-dimensional space. Luckily, the answer is yes. This is because we can model biomolecules by special frameworks such as body-bar frameworks, body-hinge frameworks, or body-bar-hinge frameworks where nodes are rigid bodies, instead of working with bar-joint frameworks where nodes are points as usual. In a body-bar

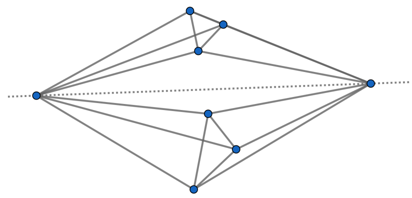


Figure 5.1: Double banana graph. It satisfies the $3n - 6$ edge count condition (the graph is on 8 nodes with 18 edges and every subgraph with n_0 nodes ($2 \leq n_0 \leq 7$) has at most $3n_0 - 6$ edges). However, internal motion within this graph is possible along the dashed line between the two 'banana' subgraphs.

framework, rigid bodies are connected by bars and are free to move continuously such that the distance between any two points joined by a bar is fixed. In a body-hinge framework, rigid bodies are connected by hinges and rotate about the hinges. In a body-bar-hinge framework, rigid bodies are connected by both bars and hinges and they are free to move continuously subject to the constraints that the distance between any two points joined by a bar is fixed and that the relative motion of any two bodies joined by a hinge is a rotation about the hinge. A framework is called rigid here if every such continuous motion preserves the distances between all pairs of points belonging to different rigid bodies. A framework is called generic if the coordinates of all vertices of all the bodies are algebraically linear independent. From a point to a rigid body, there must be huge differences. One of these is changes in the degree of freedoms. Originally from physics, the degrees of freedom of a system is the number of independent parameters that define its configuration. Hence, while a node (a point) in a three-dimensional bar-joint framework has three degrees of freedom (three coordinates), a node (a rigid body) in a three-dimensional body-bar framework has six degrees of freedom (three translations and three rotations). First consider the case when a biomolecule is modeled as a three-dimensional body-bar framework. Given a molecule as a set of atoms and covalent bonds (as graph M in figure 5.2a). Since the angles between the covalent bonds of an atom are also fixed, we need to add additional bond bending edges between the nearest neighbors in the graph M, to fix these angles. This forms **the square of the graph** M^2 (obtained from M by adding a new edge for each pair of vertices of distance two in M) (figure 5.2b). Now, an atom with its locked bonds creates a rigid body (figure 5.2c). So, in a 3-dimensional body-bar framework:

- each atom, with its locked bonds, is considered as a rigid body having six degrees of freedom;
- every bar in the framework takes one degree of freedom off the system;
- a covalent bond is a rotatable hinge leaving one degree of freedom between the two rigid bodies between the two rigid bodies, so it is replaced by five bars between these bodies;
- a double bond, or a peptide bond, is non-rotatable locking all six degrees of freedom of the two atoms into a single body;
- additional types of constraints can be modeled by any number of bars between 1 and 6.

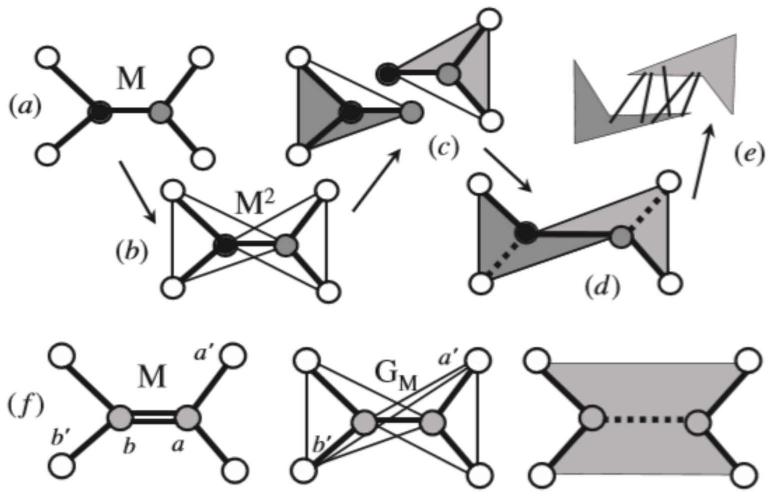


Figure 5.2: [38] (a) A biomolecule M, (b) with bond bending edges M^2, (c) each atom and its bonds become a full rigid body, (d) the bond becomes a hinge between bodies, (e) the hinge is represented by five constraints (bars) between the bodies. (f) A non-rotatable bond locks both atoms into a single body.

In the underlying graph of a body-bar framework, vertices correspond to rigid bodies and edges correspond to bars. Therefore, it is a multigraph where up to six edges are allowed between any pair of vertices. This is totally different with the bar-joint frameworks when we assume there is no multiple edges. However, similar to the rigidity matrix for a bar-joint framework, we can extract a new type of matrices. The rigidity matrix of multigraph $G = (V, E)$ here has $|E|$ rows and $6|V|$ columns. Working in this special matrix, Tay proved one surprising result, which is considered as Laman's theorem for rigidity of body-bar frameworks in the three-dimensional space.

Theorem 5.2.1. [34] Given a generic body–bar framework (G, b) on the multigraph G in $\mathbb{R}^3$, a subset of edges E is independent in the rigidity matrix $R(G, b)$ if and only if for all non-empty subsets $E' \subseteq E, |E'| \leq 6|V'| - 6$.

More clearly, a body-bar framework (G, b) is rigid in $\mathbb{R}^3$ if and only if its underlying graph G has a minimal subgraph $G'(V, E')$ in which $|E'| = 6|V| - 6$ and for every nonempty subgraphs induced by vertices $V_0 \subseteq V$, $i_{G'}(V_0) \leq 6|V_0| - 6$. This result is the base for the 3D body-bar pebble game running in some computer programs such as FIRST, KINARI that analyze the rigid/flexible components of biomolecules (see figure 5.3).
The next theorem gives a combinatorial property of "minimally rigid multigraphs".

Theorem 5.2.2. [36] (Tutte) A multigraph $G = (V, E)$ with $6|V| - 6$ edges, satisfies the count $|E_0| \leq 6|V_0| - 6$ on all subgraphs if and only if graph G is a union of six edge-disjoint spanning trees.

We also consider body-bar frameworks in an arbitrary d-dimensional space. Generally, a d-dimensional body-bar framework (G, b) can transfer to a d-dimensional bar-joint framework. The underlying graph of the resulting bar-joint framework is called a body-bar graph, denoted by G^B and it is defined as follows:

- G^B consists of $(d + 1)|V(G)| + 2|E(G)|$ vertices; for each $v \in V(G)$ we have $d + 1$ vertices $x_{v,1}, \ldots, x_{v,d+1}$ and for each $e = uv \in E(G)$ we have two vertices $x_{e,u}$ and $x_{e,v}$;

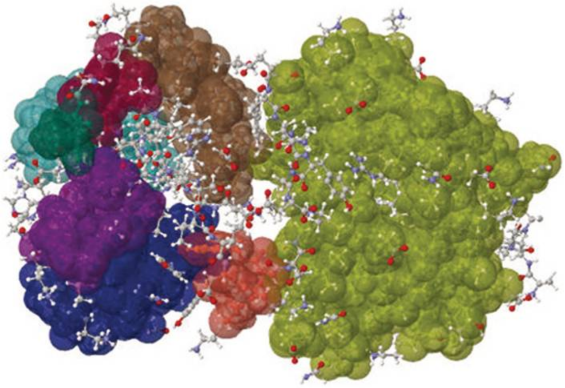


Figure 5.3: [25] Rigidity analysis results for 2LAO. Cluster colors, assigned at random, indicate groups of atoms which tend to move together.

- the core $C(v)$ of $v \in V(G)$ is the complete graph on $\{x_{v,1}, \ldots, x_{v,d+1}\}$;
- the body $B(v)$ of $v \in V(G)$ is the complete graph on $V(C(v)) \cup x_{e,v} : e \in \delta_G(v)$, where $\delta_G(v)$ denotes the set of edges in G incident to v;
- $E(G^B) := \bigcup_{v \in V(G)} E(B(v)) \cup \{\{x_{e,u}, x_{e,v}\} : e = uv \in E(G)\}$.

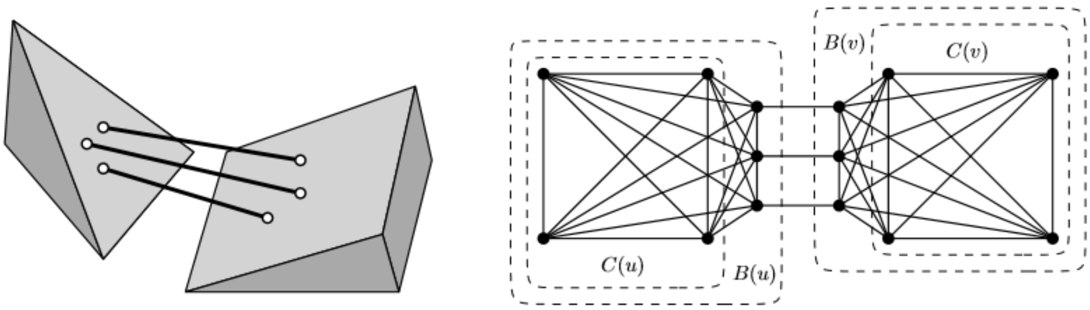


Figure 5.4: [24] A body-bar framework and the body-bar graph induced by the underlying graph in $\mathbb{R}^3$.

Hence, everything with a d-dimensional body-bar framework can transfer to a d-dimensional bar-joint framework. In all dimensions, the rigidity of generic body-bar frameworks depends only the underlying graph. The general result on combinatorial characterization of underlying graphs of rigid body-bar frameworks is presented in the next theorem.

Theorem 5.2.3. [34] (Tay) A generic body-bar framework (G, b) is rigid in $\mathbb{R}^d$ if and only if its underlying graph G contains $\binom{d+1}{2}$ edge-disjoint spanning trees.

The global rigidity of body-bar frameworks is also a generic property in all dimensions. The condition for the global rigidity of body-bar frameworks is as follows.

Theorem 5.2.4. [7] Let (G, b) be a generic body-bar framework with its underlying graph $G = (V, E)$, $|V| \geq 2$ and $|E| \geq 2$. Then the following are equivalent:
(a) (G, b) is globally rigid in $\mathbb{R}^d$,
(b) (G, b) is redundantly rigid in $\mathbb{R}^d$,
(c) $G - e$ contains $\binom{d+1}{2}$ edge-disjoint spanning trees on the vertices V for all $e \in E$.

Biomolecules can also be presented by body-hinge frameworks. A d-dimensional body-hinge framework (G, h) models a structure consisting of rigid bodies connected by hinges. Each hinge is a $(d-2)$-dimensional simplex that connects a pair of bodies. In the underlying graph $G = (V, E)$ of the body-hinge framework, vertices V correspond to bodies and edges E correspond to hinges. We can obtain an equivalent d-dimensional bar-and-joint framework by replacing each body by a bar-and-joint realization of a large enough complete graph in such a way that two bodies joined by a hinge share $d-1$ joints. The graph of such a bar-and-joint framework is called a body-hinge graph, denoted by G_H. It is defined as follows:

- G_H consists of $(d+1)|V(G)|+(d-1)|E(G)|$ vertices; for each $v \in V(G)$ we have $d+1$ vertices $x_{v,1}, \ldots, x_{v,d+1}$ and for each $e \in E(G)$ we have $d-1$ vertices $x_{e,1}, \ldots, x_{e,d-1}$;
- the hinge $H(e)$ of e is the complete graph on $\{x_{e,1}, \ldots, x_{e,d-1}\}$ for each $e \in E$;
- the core $C(v)$ of v is the complete graph on $\{x_{v,1}, \ldots, x_{v,d+1}\}$ for each $v \in V$;
- the body $B(v)$ of v is the complete graph on $V(C(v)) \cup \bigcup_{e \in \delta_G(v)} V(H(e))$;
- $E(G_H) := \bigcup_{v \in V} E(B(v))$.

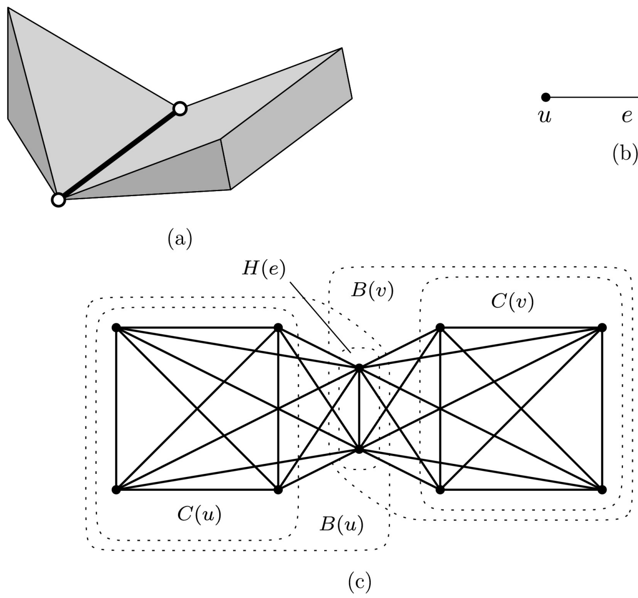


Figure 5.5: [24] (a) A body-hinge framework in $\mathbb{R}^3$, (b) the underlying graph H, and (c) the body-hinge graph G_H.

Each rigid body in $\mathbb{R}^d$ has $\binom{d+1}{2}$ degrees of freedom because of d translations and $\binom{d}{d+2}$ rotations. Since each hinge leaves one degree of freedom between two rigid bodies, each hinge can be presented by $\binom{d+1}{2} - 1$ bars. Hence, a d-dimensional body-hinge framework becomes a d-dimensional body-bar framework if every hinge is replaced by $\binom{d+1}{2} - 1$ bar. Combined with theorem 5.2.3, we obtain the following.

Theorem 5.2.5. [34] [40] (Tay and Whiteley) A generic body-hinge framework (G, b) is rigid with its underlying graph $G(V, E)$ in $\mathbb{R}^d$ $(d \geq 2)$ if and only if every edge $e \in E$ is replaced by $D-1$ parallel edges, the resulting multigraph contains D edge-disjoint spanning trees on the vertices V where $D = \binom{d+1}{2}$.

From hence on, for a graph G and a positive integer k, we denote kG be the graph obtained from G by replacing each edge of G by k parallel edges.
The combinatoiral characterization of globally rigid body-hinge frameworks in all dimensional is also studied.

Theorem 5.2.6. [20] A generic body-hinge framework (G, h) is globally rigid in $\mathbb{R}^d$ ($d \geq 3$) if and only if $(D-1)G - f$ contains D edge-disjoint spanning trees for any edge f of $(D-1)G$ where $D = \binom{d+1}{2}$.

For two-dimensional case, the authors of the paper gave one specific result.

Theorem 5.2.7. [20] A generic body-hinge framework (G, h) is globally rigid in $\mathbb{R}^2$ if and only if its underlying graph G is 3-edge-connected.

We can see from the above theorem that the 2-dimensional global rigidity of a generic body-hinge framework is equivalent to the 3-edge-connectivity of the underlying graph. Compared to bar-joint frameworks, it is far simpler since we do not need the "redundant rigidity" condition anymore.
Not only that, the body-hinge frameworks have splendid properties when redundant rigidity and redundant global rigidity of the generic frameworks can be characterized through the connectivity of underlying graph in all dimensions. This nice research is presented in paper [26]. For easy understanding, main notations of the paper is explained below.

- A graph G is called (k, h)-connected if removing any $k-1$ vertices from G results in a graph which is h-edge-connected.
- A body-hinge framework (G, h) is called (k, h)-rigid in $\mathbb{R}^d$ if removing any $k-1$ rigid bodies that are equivalent to $k-1$ vertices in the underlying graph and then any $h-1$ hinges from (G, h) results in a framework which is rigid in $\mathbb{R}^d$.
- A body-hinge framework G is called (k, h)-globally rigid in $\mathbb{R}^d$ if removing any $k-1$ rigid bodies that are equivalent to $k-1$ vertices in the underlying graph and then any $h-1$ hinges from (G, h) results in a framework which is globally rigid in $\mathbb{R}^d$.

Theorem 5.2.8. [26] Let k and h be integers such that $k \geq 1$ and $h \geq 2$, respectively.
(a) A generic body-hinge framework (G, h) is (k, h)-rigid in $\mathbb{R}^2$ if and only if its underlying graph is $(k, h+1)$-connected and (G, b) is (k, h)-globally rigid in $\mathbb{R}^2$ if and only if G is $(k, h+2)$-connected.
(b) For any $d \geq 3$, a generic body-hinge framework (G, h) is (k, h)-rigid in $\mathbb{R}^d$ if and only if it is (k, h)-globally rigid in $\mathbb{R}^d$ if and only if its underlying graph G is $(k, h+1)$-connected.

Explained from the beginning of this chapter, the rigidity of molecules related closely with the square of a graph. Hence, there are some research about squares of graphs.

Theorem 5.2.9. [18] Let G be a graph and suppose that G^2 is 7-connected. Then G^2 is rigid in the three-dimensional space.

Also, the authors of the paper showed that it is the best bound by giving one example, which is 6-connected but non-rigid.

Theorem 5.2.10. [23] Let G be a graph with minimum degree at least two. Then G^2 is rigid in $\mathbb{R}^3$ if and only if $5G$ contains six edge-disjoint spanning trees.

Conjecture 5.2.11. [7] [17] Suppose that G has no cycles of length at most four. Then G^2 is globally rigid in $\mathbb{R}^3$ if and only if G^2 is 4-connected and $5G-e$ contains 6 edge-disjoint spanning trees for every $e \in 5G$.

5.3 Application to statics

Figure 5.6: Quasicrystal framework COAST by Tony Robbin, installed at the Danish Technical University, 1994.

First, let's us consider an $m \times n$ square-grid framework in the plane. It consists of horizontal and vertical rods. Each rod has the same length and is rigid. Incident rods are attached together by joints which allows the rods to pivot. A square-grid framework can be deformed by rotating certain parts along certain joints. To prevent such things happen, you are allowed to add diagonal rods (which would force a rhombus to be square). A framework with extra rods is rigid if fixing the position of one rod in the plane, the positions of all other rods are uniquely determined. The question is *where should you place the additional rods, and what is the fewest number of diagonal rods needed to stabilize the structure (i.e. the grid framework is rigid)?*

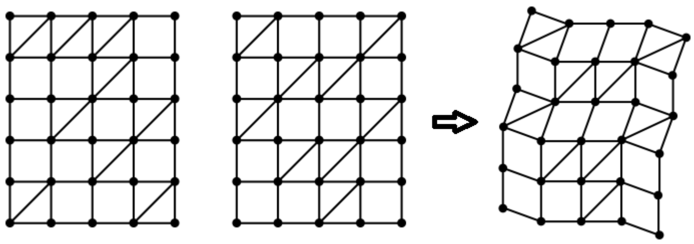
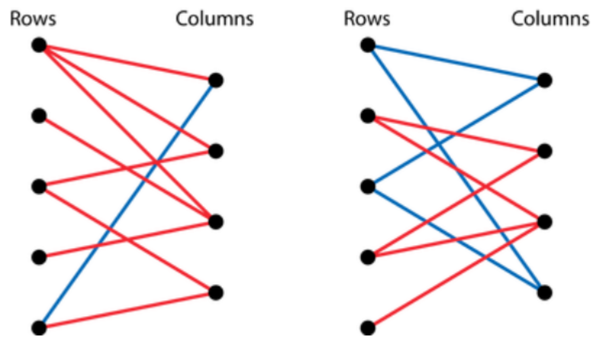


Figure 5.7: Both adding 9 extra rods but the left framework keep the same shape under continuous motions while the right one distorts. The left framework is even rigid after removing the left most bottom rod.

To answer that question, we define a bipartite graph where the vertices of the graph correspond to the rows and columns of the grid framework, respectively, and there is an edge between two vertices if and only if there is a diagonal rod in the intersection of the corresponding row and column.

Theorem 5.3.1. [1] (Bolker, Crapo 1979) A planar square-grid framework will be rigid if and only if the corresponding bipartite graph is connected.

We see that the bipartite graph on the left below corresponds to the left square-grid framework above. The left bipartite graph is connected, and in fact, we can remove the blue edge (which is equivalent with the left most bottom rod in the left grid framework) to obtain a tree. So the left square-grid framework is even rigid after the removal of the left most bottom rod. In contrast, the bipartite graph on the right below is not connected (the blue part disconnects with the red part), hence the right square-grid framework above is non-rigid.



What happens if we replace squares in a square-grid framework by rhombi?
A rhombic carpet is a planar arrangement of rhombi, which is connected and simply connected. This means every rhombus may be reached from any other by a contiguous succession of rhombi. Two rhombi are contiguous if they share a common side. Two diamonds touching at a vertex will not be considered as connected. By simply connected we mean that the carpet has no holes. And we do not allow two rhombi to overlap. Similar to the square-grid framework, now we can add diagonal rods to fix some rhombi. Here we call this process as bracing the rhombic carpet. The questions is *When is the braced rhombic carpet rigid ?*
To answer this, we try to do similar as before. We cannot consider columns and rows here, but we can consider ribbons. In a rhombic carpet, a maximal succession of contiguous rhombi, whose common edges are parallel, is called a ribbon.

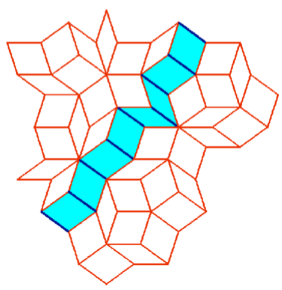


Figure 5.8: [9] Illustration for a ribbon.

We associate a rhombic carpet with a Wester graph, where the vertices of the graph are all of the ribbons in the carpet and the edges are defined by the rule that ab is an edge if and only if the ribbons a and b intersect.

Theorem 5.3.2. [9] (Wester) Let K be a rhombic carpet with associated Wester graph τ and let ϕ be a subgraph which is both spanning and connected. Then, bracing the rhombi corresponding to the edges of ϕ makes K rigid.

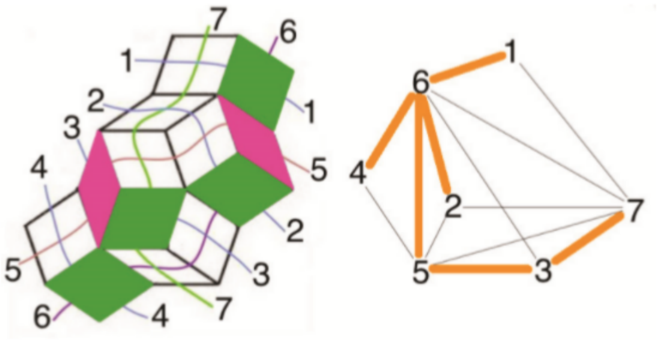


Figure 5.9: [9] Left: A rhombi carpet with 13 rhombi, 7 ribbons and 6 bracing plates. Right: the corresponding Wester graph and a maximal bracing tree.

The set of colored rhombi in figure 5.9 is the minimal rhombi set that need to be braced for making the whole rhombi carpet rigid since it is equivalent with the spanning and connected subgraph in the right hand side.
However, everything is not simple in three-dimensional space. We generalize the notion of a ribbon to the three-dimensional space as a maximal successions of contiguous rhombohedra which share a family of mutually parallel faces. To distinguish this from the two-dimensions case, they shall be called worms instead of ribbons. And rhombi carpets in the two-dimensional space become quasicrystal structures in the space. In figure 5.10, we consider a cubic framework, one of the simplest type of quasicrystal structures. It is easy too see that even if we brace all but one of the cubes, labeled (157), (267), (368), (469) for their crossing worms, the last red cube (058) distorts as its free edge slides forward into the negative Z-direction(blue). Therefore, the rigidity of quasicrystal structures in the space still needs further investigation.

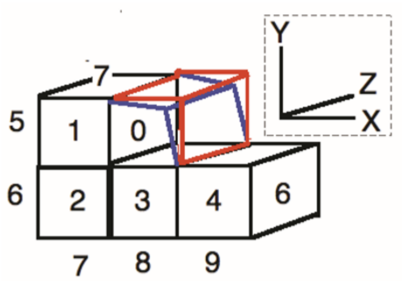


Figure 5.10: [9] A cubic framework with two horizontal worms (5,6), three vertical worms (7,8,9) and five worms into the Z-direction (0,1,2,3,4).

Now, come back again with square-grid frameworks but with another problem. Given a square-grid framework with some initial diagonal rods. We want to add more extra rods so that it becomes k-edge rigid, i.e. the square-grid framework remains rigid even after any $k' < k$ extra rods fail. Theorem 5.3.1 implies that finding a smallest set of new extra rods in this case corresponds exactly to the bipartite k-edge-connectivity augmentation problem. In the bipartite edge-connectivity augmentation problem, we are given a bipartite graph $G = (A, B, E)$ and a positive integer k; the goal here is to find a smallest set F of edges so that $G' = (A, B, E \cup F)$ is a new bipartite graph and k-edge-connected. Note that E and F maybe contain parallel edges. In fact, the bipartite edge-connectivity

augmentation problem is a special case of the general constrained edge-connectivity augmentation problems, which is explained clearly in paper [2].
Let $G = (V, E)$ be a graph and let $\mathcal{P} = \{P_1, P_2, \dots, P_r\}, r \geq 2$, be a partition of V. In the partition-constrained k-edge-connectivity augmentation problem, the goal is to find a smallest set F of new edges, such that every edge in F joins two distinct members of P and $G' = (V, E \cup F)$ is k-edge-connected. Recall that a graph $G = (V, E)$ is k-edge-connected if $d(X) \geq k$ for all $0 \neq X \subset V$ where $d(X)$ denotes the number of edges connecting X with $G - X$.
Let $OPT^k(G)$ and $OPT^k_{\mathcal{P}}(G)$ denote the size of a solution to the edge-connectivity augmentation problem with no constrains and partition constraints respectively. Obviously, $OPT^k_{\mathcal{P}}(G) \geq OPT^k(G)$. Let ϕ be the largest of the following quantities in G:

$$\alpha = \max \left\{ \left\lceil \frac{1}{2} \sum_{X \in \mathcal{F}} (k - d(X)) \right\rceil : \mathcal{F} \text{ a subpartition of } V \right\};$$

$$\beta_i = \max \left\{ \sum_{Y \in \mathcal{F}} (k - d(Y)) : \mathcal{F} \text{ a subpartition of } P_i \right\}.$$

.
It is proved that $OPT^k(G) = \alpha$. Since no new edge can be added between vertices in the same member P_i of P, it follows that $OTP^k_{\mathcal{P}}(G) \geq \beta_i$ for all $1 \leq i \leq r$. Hence, $OTP^k_{\mathcal{P}}(G) \geq \phi$.

Theorem 5.3.3. [2] If k is even then $OPT^k_{\mathcal{P}}(G) = \phi$.

In the case k is odd, we also have $OPT^k_{\mathcal{P}}(G) \leq \phi + 1$. But the final result depends on whether the graph contains a C_4- or C_6- configuration or not.

Definition 5.3.1. Let X_1, X_2, Y_1, Y_2 be a partition of V with the following properties in G:

- $d(A) < k$ for $A = X_1, X_2, Y_1, Y_2$;
- $d(A, B) = 0$ for $(A, B) = (X_1, X_2), (Y_1, Y_2)$;
- There exist subpartitions $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}'_1, \mathcal{F}'_2$ of X_1, X_2, Y_1, Y_2 respectively, such that for A ranging over X_1, X_2, Y_1, Y_2 and $\mathcal{F}$ the corresponding subpartition of A, then we have $k - d(A) = \sum_{U \in \mathcal{F}} (k - d(U))$. Furthermore for some $i \leq r$, P_i contains every set of either $\mathcal{F}_1 \cup \mathcal{F}_2$ or $\mathcal{F}'_1 \cup \mathcal{F}'_2$;
- $(k - d(X_1)) + (k - d(X_2)) = (k - d(Y_1)) + (k - d(Y_2)) = \phi$.

Such a partition is called a C_4-configuration of G.

We skip the definition of a C_6-configuration since it does not happen for the bipartite augmentation problem. (It only exists in graphs with $r \geq 3$ and $\phi = 3$.) Now, we come to the final result of paper [2].

Theorem 5.3.4. [2] Let $k \geq 2$ and let $G = (V, E)$ be a graph with a partition $\mathcal{P} = \{P_1, \dots, P_r\}$ of V into $r \geq 2$ nonempty classes. Then $OPT^k_{\mathcal{P}}(G) = \phi$ unless G contains a C_4- or C_6-configuration, in which case $OPT^k_{\mathcal{P}}(G) = \phi + 1$.

Applying this result for the specific bipartite edge-connectivity augmentation problem ($r = 2$), we can solve the highly rigid square-rigid problem.

Theorem 5.3.5. [2] Let $k \geq 2$, let $G = (A, B, E)$ be a bipartite graph and let $\mathcal{P} = \{A, B\}$. Let ϕ be the largest of the following three quantities in G:

$$\alpha = \max \left\{ \left\lceil \frac{1}{2} \sum_{X \in \mathcal{F}} (k - d(X)) \right\rceil : \mathcal{F} \text{ a subpartition of } A \cup B \right\};$$

$$\beta_1 = \sum_{v \in A} \max \{k - d(v), 0\};$$

$$\beta_2 = \sum_{v \in B} \max \{k - d(v), 0\}.$$

Then $OPT^k_{\mathcal{P}}(G) = \phi$ unless k is odd and G contains a C_4-configuration, in which case $OPT^k_{\mathcal{P}}(G) = \phi + 1$.

5.4 Application in truss structures

Figure 5.11: An example of through-truss steel bridges.

Truss topology design (TTD) deals with constructions like bridges, cantilevers and roof trusses supporting different loading scenarios. For example, a bridge should withstand forces corresponding to morning or evening rush hour traffic and even to an earthquake. Ground structure method is known as the most popular method of TTD. This method deals with minimizing the total volume of material while satisfying nodal equilibrium constraint and predefined stress limits. In this method, a ground structure with a sufficiently large number of members is given as the initial configuration and the optimal topology is obtained by solving the optimization problem in which design variables are the cross section areas of the members, and the members with no cross section area are removed. However, the results of this approach often contain very slender members and unstable nodes, therefore making the structure sensitive to instabilities.

Figure 5.12: [27] The left side is a ground structure and the right hand side is an optimal solution obtained from the ground structure method.

It is very important to take redundant rigidity into consideration in structural design since the structures can be against the force beyond our estimation. A truss structure is called a redundantly rigid truss if after the loss or the damage of one member, the structure remains rigid; therefore, it still keeps the shape and continues to work normally. A method of finding redundantly rigid TTD is presented below, based on two papers [27] [29]. We consider the ground structure as figure 5.12 with n nodes. There are $\binom{n}{2}$ members. The objective function to minimize is compliance, i.e., the work of the external force. We consider four constraints: the equilibrium of force, upper bound of the total volume, lower bound of the cross section area of each member and redundantly rigid constraint. Mathematically, we consider the following problem :

$$
\begin{aligned}
\text{Minimize} \quad & \boldsymbol{P}^T \boldsymbol{U}, \\
\text{subject to} \quad & \textstyle\sum_{i \in I} A_i K_i \boldsymbol{U} = \boldsymbol{P}, \\
& \textstyle\sum_{i \in I} A_i L_i \leq V^U, \\
& A_i \geq A_i^L \ (i \in I), \\
& I - e \in \mathcal{L}, \ \forall \, e \in I,
\end{aligned}
$$

where $\boldsymbol{P}$ and $\boldsymbol{U}$ denote the vectors of external force and displacement of nodes, A_i, K_i, L_i and A_i^L are respectively the cross section area, the stiffness matrix, the length and the lower bound of the cross section area of the member i, V^U denotes the upper bound of the total volume, I is a set of members in a topology and $\mathcal{L}$ is the set of two-dimensional minimally rigid graphs. I can be referred to as the topology. Design variables are both the topology I and the set of cross section areas in the topology $\{A_i \geq A_i^L \ (i \in L)\}$.
It is difficult to find an exact optimal solution for these above large-scale problems and it is not necessary in practical. So, we will find a redundantly rigid approximately optimal truss structure based on two-dimensional redundantly rigid augmentations.
The approximation method for redundantly rigid TTD problem consists of three steps:
Step 1: Solve the above problem as a relaxation problem: we will not consider the redundant rigidity constraint and replace lower bound constraints of the cross section areas of the members by non-negative constraints. More clearly, we find the solution for the following problem:

$$
\begin{aligned}
\text{Minimize} \quad & \boldsymbol{P}^T \boldsymbol{U}, \\
\text{subject to} \quad & \textstyle\sum_{i \in I_g} A_i K_i \boldsymbol{U} = \boldsymbol{P}, \\
& \textstyle\sum_{i \in I_g} A_i L_i \leq V^U, \\
& A_i \geq 0 \ (i \in I_g),
\end{aligned}
$$

where I_g denotes a set of members in a ground structure. Note that the number of elements of I_g can be extremely large depending on the size of the problem. This problem is a convex programming problem, hence the optimal solution can be found in polynomial time by using an appropriate way. It gives the lower bound of the optimal solution of the original problem.
Step 2: The optimal truss structure in step 1 is not redundantly rigid. Then in this step, we make a redundantly rigid truss topology by adding minimum number of edges to the topology of the optimal truss of step 1. The first approach to redundantly rigid

augmentation in the plane is proposed by Garcia and Tejel [10]. Another method is found recently by Andras Mihalyko [29]. Two papers addresses the question of **finding a minimum covering** to a minimally rigid graph to make it redundantly rigid, which can be done in polynomial time. We shall consider the newest approach here.
Consider a Laman graph $L = (V, E)$ (i.e. L is minimally rigid in $\mathbb{R}^2$). For every $i, j \in V$, the set of generated edges by uv is defined by $L(uv) = \{ij|\, G + uv - ij \text{ is rigid}\}$.

Lemma 5.4.1. If L is Lamman, $L(uv) = \cap\{L|u, v \in L, L \text{ Laman subgraph}\}$.

Let $L = (V, E)$ be a Laman graph. $C \subset V$ is called co-rigid if $V - C$ is rigid.
Equivalently, $|C| < |V| - 1$ and $e(C) = 2|C|$.
If $L + H$ is redundantly rigid, H must touch every co-rigid sets.

Lemma 5.4.2. Let L be a Laman graph, and H an edge set such that $L+H$ is redundantly rigid. Then $|H| \geq \left\{ \left\lceil \frac{|\mathcal{C}|}{2} \right\rceil \,|\, \mathcal{C} \text{ is the family of disjoint minimal co-rigid sets of } L \right\}$.

Lemma 5.4.3. [19] Let L be a Laman graph and $\mathcal{C}$ be the family of minimal co-rigid sets of L. Then the sets of $\mathcal{C}$ are pairwise disjoint or there exists $\{u, v\}$ such that $C \cap \{u, v\} \neq$ $\forall\, C \in \mathcal{C}$. Moreover, this u and v are not neighboring.

If there exists such a $\{u, v\}$ edge, we can augment the graph to redundantly rigid with one edge between u and v.
Now, suppose $\mathcal{C}$ consists of pairwise disjoint sets. Take $i_1, \ldots, i_{|\mathcal{C}|}$ be representative vertices of the minimal co-rigid sets. Let $N(X)$ denote the neighbors of vertices in set X in graph L.

Lemma 5.4.4. $C_1 \cup N(C_1) \cup C_2 \cup N(C_2) \subset L(i_1 i_2)$.

Lemma 5.4.5. Let L be a Laman graph with minimal co-rigid sets $\mathcal{C}$. For any connected graph H on the representative vertices, $L + H$ is redundantly rigid.

Lemma 5.4.6. Let $L' = L(i_1i_2) \cup L(i_1i_3) \cup L(i_1i_4)$. Then $L' = L(i_1i_2) \cup L(i_3i_4)$ or $L' = L(i_1i_4) \cup L(i_2i_3)$.

This whole procedure, described above, when we have just 2 edges to generate the same subgraph that was generated by 3 edges sharing one common vertex, is called reduction step. Assume that we start from a covering centered around i_1. This means we connect all i_j $(j > 1)$ to i_1 to make a connected graph H. Then using the reduction step, we replace three of these Laman graphs, $L(i_1i_h) \cup L(i_1i_{h+1}) \cup L(i_1i_{h+2})$ with just 2, say, $L(i_1i_h) \cup L(i_{h+1}i_{h+2})$ using the reduction step. And we can repeat this until fewer than 3 Laman graphs left containing i_1. (It is clear from the construction that none of the other representative vertices participates in more than one generated Laman.) So this lead to a $\left\lceil \frac{\mathcal{C}}{2} \right\rceil$ size optimal covering at the end.

Theorem 5.4.7. Let L be a Laman graph (i.e. minimally rigid graph). Then

$$\begin{aligned} &\min\,\{|H|\,|H \text{ is an edge set}, L + H \text{ is redundantly rigid}\} \\ = \;&\max\Big\{\Big\lceil \frac{|\mathcal{C}|}{2} \Big\rceil \,|\, \mathcal{C} \text{ is a set of pairwise disjoint co-rigid sets}\Big\}. \end{aligned}$$

There exists a polynomial algorithm that can find such an edge set.

Algorithm for the redundantly rigid augmentations:

- Check if L can be made redundant rigid using only one edge. This can be made by checking every pair of vertices whether adding that edge to L makes it redundantly rigid (polynomial time);

- Find a representative set. For any given set $S \subseteq V$, let us denote the minimal Laman graph with C_S for which $S \subseteq C_S$ holds. We start with $S = V$. Delete vertices greedily from S while $C_S = L$. The final S is a representative for the family of minimal co-rigid sets.

- Choose i_1 as the center, connect others i_j $(1 < j)$ with i_1 to make a connected graph H. Use the reduction step to decrease the number of edges in H. Each reduction step can be made in polynomial time, so we can reach the optimal size edge set H with a polynomial algorithm.

Coming back to the problem of finding a minimal covering for the relaxation problem of TTD problem that is mentioned in step 1, we need to apply the algorithm with a slight modification. Since the above algorithm does not consider the lengths of added edges, there is a possibility that long edges are added in the above algorithm. However, this is undesirable with the TTD problem since it will increase the total volume. So here, when applying the above algorithm, we choose representative vertices in a way such that the shorter edges $i_j i_k$ are prioritized. Recall that no matter how representative vertices are chosen, $L + H$ is redundantly rigid as long as H is connected on the representative vertices. Moreover, we can choose i_1 arbitrarily among representative vertices.
Step 3: Determine the optimal cross section areas of the members of the truss structure under the truss topology of step 2. The problem is formulated as follows:

$$\begin{aligned} \text{Minimize} \quad & \boldsymbol{P}^T\boldsymbol{U}, \\ \text{subject to} \quad & \textstyle\sum_{i\in I_2} A_i K_i \boldsymbol{U} = \boldsymbol{P}, \\ & \textstyle\sum_{i\in I_2} A_i L_i \leq V^U, \\ & A_i \geq A_i^L \ (i \in I_2), \end{aligned}$$

where I_2 denotes the set of members in the truss topology of step 2. The optimal value of this problem is the upper bound of the original problem.
Hence, by this method, we can achieve upper and lower bound solutions for global optimal solution of the TTD problem. Numerical examples in paper [27] of the ground structure with 200 vertices and 19900 members showed that the upper bound is about one percent greater than the lower bound. The solution in step 3 is redundantly rigid, in which the truss structure is more stable and it is done by polynomial time. Therefore, the algorithm for the redundantly rigid augmentation problem mentioned in step 2 is very useful for large-scale TTD problems with certain redundancy when the exact solutions are hardly obtained.

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DECLARATION OF AUTHORSHIP

Name: Le Thi Thanh Tu
Program: MSc in Mathematics, Faculty of Science, ELTE
NEPTUN ID: EM3HGT
Title of the thesis: Rigidity theory and its applications.

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