Extremal Problems for Transformed Families of Sets

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Abstract

A fundamental theorem in extremal set theory is the so-called shadow theorem, which describes the size of transformed families of sets. The shadow theorem, independently discovered by Kruskal in 1963 and Katona in 1968, has thenceforth numerous variations and applications in many fields of mathematics. This article serves as a survey for this topic. We will first give the basic statement, some corollaries, and a few elegant and instructive proofs. Then we collect various results related to it, which were developed over the past few decades. Finally, some open problems are also discussed.

The structure of this article is as follows.
In Chapter 1, we give the basic statement and a few direct corollaries of the shadow theorem. A simplified version by Lovász is also discussed.
In Chapter 2, we firstly present an important technique named shift operation. Then three different proofs of shadow theorem, by Daykin, Frankl, and Keevash separately, are presented.
In Chapter 3, we show a few beautiful applications of shadow theorem. We will discuss the extremal problems about the density of triangles, size of intersecting family, and number of independent sets.
In Chapter 4, we collect some variations and generalizations of the shadow theorem. We will consider the shadow of the intersecting family, the balanced version of shadow theorem, some analogues of shadow of other mathematical objects. Finally, we list a few related open problems.

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Shadow Theorem

In the first chapter, we will introduce the shadow theorem.

1.1 Statement

Let $[n]$ be $\{1, 2 \cdots, n\}$. For any non-negative integer k, denote by $\binom{[n]}{k}$ the family of all the k-subsets of $[n]$.

Definition 1.1. *(**Shadow**) Given a family of sets $\mathcal{F} \subseteq \binom{[n]}{k}$, for $H \in \mathcal{F}$, define $\triangle H$, the shadow of H to be*

$$\triangle H = \left\{ M \in \binom{[n]}{k-1} \;\middle|\; M \subseteq H \right\}.$$

And define the shadow of $\triangle \mathcal{F}$ to be

$$\triangle \mathcal{F} = \bigcup_{H \in \mathcal{F}} \triangle H.$$

Therefore the shadow $\triangle \mathcal{F}$ is just the set family which consists of all the $(k-1)$ subsets of the sets in $\mathcal{F}$. For example, if $\mathcal{F} = \{\{1,2,3\},\{1,2,4\},\{2,4,5\}\}$, then $\triangle \mathcal{F} = \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{2,5\},\{4,5\}\}$. It should be noted that $\triangle \mathcal{F}$ is fully determined by $\mathcal{F}$ and independent of the underlying set $[n]$. So when talking about the shadow, we sometimes will just omit the underlying set and assume $[n]$ is large enough.

After introducing this definition, natural questions arise:

Question 1.2. *Given $|\mathcal{F}|$, how big or small $|\triangle \mathcal{F}|$ can be.*

Question 1.3. *Given $|\mathcal{F}|$, choose $\mathcal{F}$ randomly from $\binom{[n]}{k}$, what is $|\triangle\mathcal{F}|$ in average?*

It turns out that the upper bound can be easily answered. Firstly for any $\mathcal{F} \subseteq \binom{[n]}{k}$, $|\triangle\mathcal{F}| \leqslant k\,|\mathcal{F}|$ since for any $H \in \mathcal{F}$, the number of $(k-1)$-subsets of H is exactly k. If and only if all these subsets of size $(k-1)$ each $S \in \mathcal{F}$ contains are different will this bound be achieved. And it's obvious that it can be achieved indeed, for example, when all $H \in \mathcal{F}$ are disjoint. The second question is also easy to answer, simply by the linearity of the expectation.

Therefore our main thinking is put on the lower bound. Based on the discussion in the last paragraph, a very intuitive idea is that if we want to make the $|\triangle\mathcal{F}|$ as small as possible, we should require the sets in $\mathcal{F}$ to overlap with each other as much as possible and in this case will the lower bound be achieved. It turns out that this is correct, which was proven by Kruskal [26] and Katona [24] independently. To fully describe what we call the Kruskal-Katona shadow theorem, we need another notation.

Definition 1.4. ***(k-binomial decomposition)*** *Given two positive integers m and k, the k-binomial representation of m is*

$$m = \binom{a_0}{k} + \binom{a_1}{k-1} + \binom{a_2}{k-2} + \cdots + \binom{a_t}{k-t},$$

where $a_0 > a_1 > \cdots > a_t \geqslant k-t \geqslant 1$.

For example, the followings are some 5-binomial representations:

$$\begin{aligned}
1 &= \binom{5}{5} \\
2 &= \binom{5}{5} + \binom{4}{4} \\
26 &= \binom{7}{5} + \binom{5}{4} \\
35 &= \binom{7}{5} + \binom{5}{4} + \binom{4}{3} + \binom{3}{2} + \binom{2}{1}.
\end{aligned}$$

Lemma 1.5. *For any positive integers m and k, the k-binomial representation of m exists and is unique.*

Proof. Prove by induction on k. If $k = 1$, it's trivial. Assume that it's true for $k-1$.

For the existence, choose a_0 to be the largest integer such that $m \geqslant \binom{a_0}{k}$. If equality holds, we are done. Otherwise, by induction, there is a $(k-1)$-binomial representation of $m - \binom{a_0}{k}$,

$$m - \binom{a_0}{k} = \binom{a_1}{k-1} + \binom{a_2}{k-2} + \cdots + \binom{a_t}{k-t}$$

where $a_1 > a_2 > \cdots > a_t \geqslant k-t \geqslant 1$. So we only need to confirm $a_0 > a_1$. If not, then

$$m \geqslant \binom{a_0}{k} + \binom{a_1}{k-1} \geqslant \binom{a_0}{k} + \binom{a_0}{k-1} = \binom{a_0+1}{k}$$

which contradicts our choice of a_0.

Assume that there are two different k-binomial representation of m:

$$m = \binom{a_0}{k} + \binom{a_1}{k-1} + \binom{a_2}{k-2} + \cdots + \binom{a_t}{k-t} = \binom{b_0}{k} + \binom{b_1}{k-1} + \binom{b_2}{k-2} + \cdots + \binom{b_r}{k-r}.$$

a_0 can't equal b_0, otherwise $m - \binom{a_0}{k} = m - \binom{b_0}{k}$ has two different $(k-1)$-binomial representations. Assume $a_0 < b_0$. Then

$$\begin{aligned}
m &= \binom{a_0}{k} + \binom{a_1}{k-1} + \binom{a_2}{k-2} + \cdots + \binom{a_t}{k-t} \\
&\leqslant \binom{a_0}{k} + \binom{a_0-1}{k-1} + \binom{a_0-2}{k-2} + \cdots + \binom{a_0-t}{k-t} \\
&\leqslant \binom{a_0}{k} + \binom{a_0-1}{k-1} + \binom{a_0-2}{k-2} + \cdots + \binom{a_0-t}{k-t} + \cdots + \binom{a_0-(k-1)}{1} \\
&= \binom{a_0+1}{k} - 1 \\
&< \binom{b_0}{k} \\
&\leqslant \binom{b_0}{k} + \binom{b_1}{k-1} + \binom{b_2}{k-2} + \cdots + \binom{b_r}{k-r} \\
&= m.
\end{aligned}$$

□

Note that the proof of uniqueness can also similarly confirm the following statement.

Claim 1.6. *For any two positive integer m and n, whose k-binomial representations are*

$$m = \binom{a_0}{k} + \binom{a_1}{k-1} + \cdots + \binom{a_t}{k-t} \quad n = \binom{b_0}{k} + \binom{b_1}{k-1} + \cdots + \binom{b_r}{k-r}.$$

Then $m > n$ if and only if

- $t > r$ *and* $a_i = b_i$ *for* $0 \leqslant i \leqslant r$ *or*
- $t \leqslant r$ *and* $\exists i \left(0 \leqslant i \leqslant t\right),\ a_i > b_i$.

Now we are finally ready for the statement of the Kruskal-Katona shadow theorem, which answers the question of the lower bound for the size of shadow.

Theorem 1.7. *If $\mathcal{F} \subseteq \binom{[n]}{k}$ and the k-binomial representation of $|\mathcal{F}|$ is*

$$|\mathcal{F}| = \binom{a_0}{k} + \binom{a_1}{k-1} + \binom{a_2}{k-2} + \cdots + \binom{a_t}{k-t},$$

then

$$|\triangle\mathcal{F}| \geqslant \binom{a_0}{k-1} + \binom{a_1}{k-2} + \binom{a_2}{k-3} + \cdots + \binom{a_t}{k-t-1}.$$

We will discuss the equality condition and several proofs in the following sections.

For any positive integer m whose k-binomial representation is $\binom{a_0}{k} + \binom{a_1}{k-1} + \cdots + \binom{a_t}{k-t}$, for convenience, we define $KK_k\left(m\right) = \binom{a_0}{k-1} + \binom{a_1}{k-2} + \cdots + \binom{a_t}{k-t-1}$, the lower bound given in Theorem 1.7.

Firstly, a very natural question arises after reading this theorem is that is this lower bound given in the theorem increasing with $|\mathcal{F}|$? The answer is affirmative, simply according to Claim 1.6. Namely, $KK_k\left(m\right)$ is non-decreasing with m.

Note that $\binom{a}{0} = 1$ and when s is negative or $a < s$, $\binom{a}{s} = 0$. We can actually extend Theorem 1.7 to the following form, which says that it doesn't matter that there are some tail terms in the expression of $|\mathcal{F}|$. This extended form can sometimes help us avoid many minor troubles.

Theorem 1.8. *If $\mathcal{F} \subseteq \binom{[n]}{k}$ and*

$$|\mathcal{F}| = \binom{a_0}{k} + \binom{a_1}{k-1} + \binom{a_2}{k-2} + \cdots + \binom{a_{k-1}}{1} + \binom{a_k}{0} + \binom{a_{k+1}}{-1} + \cdots \binom{a_{k+r}}{-r}$$

where $a_0 > a_1 > \cdots a_{k-1} \geqslant 1$, $a_k, a_{k+1}, \cdots, a_{k+r} \geqslant 0$, *then*

$$|\triangle \mathcal{F}| \geqslant \binom{a_0}{k-1} + \binom{a_1}{k-2} + \binom{a_2}{k-3} + \cdots + \binom{a_{k-1}}{0} + \binom{a_k}{-1} + \binom{a_{k+1}}{-2} + \cdots \binom{a_{k+r}}{-r-1}.$$

Note that it is possible that this expression of $|\mathcal{F}|$ is not unique, due to the existence of $\binom{a_k}{0}$. But it can be easily seen that this won't influence the lower bound given.

Now let's look at a few direct corollaries of this theorem.

For any $\mathcal{F} \subseteq \binom{[n]}{k}$, define $\triangle^r \mathcal{F} = \left\{X \in \binom{[n]}{k-r} \;\middle|\; \exists H \in \mathcal{F} \text{ such that } X \subseteq H\right\}$. So $\triangle^r \mathcal{F}$ is just the set family obtained by taking shadow of $\mathcal{F}$ for r times. By repeatedly applying the shadow theorem, we can get

Corollary 1.9. *If* $\mathcal{F} \subseteq \binom{[n]}{k}$, *the* k*-binomial representation of* $|\mathcal{F}|$ *is*

$$|\mathcal{F}| = \binom{a_0}{k} + \binom{a_1}{k-1} + \binom{a_2}{k-2} + \cdots + \binom{a_t}{k-t},$$

then

$$|\triangle^r \mathcal{F}| \geqslant \binom{a_0}{k-r} + \binom{a_1}{k-1-r} + \binom{a_2}{k-2-r} + \cdots + \binom{a_t}{k-t-r}.$$

Now let's turn our focus to another kind of transformation of a family of sets. For any $\mathcal{F} \subseteq \binom{[n]}{k}$, define $\triangledown \mathcal{F} = \left\{X \in \binom{[n]}{k+1} \;\middle|\; \exists H \in \mathcal{F} \text{ such that } H \subseteq X\right\}$. So now, instead of considering the subsets contained in some $H \in \mathcal{F}$, we consider the bigger sets which contain some H. Note that it's different from $\triangle \mathcal{F}$ that $\triangledown \mathcal{F}$ depends on the underlying set $[n]$. Besides, $\mathcal{F} \subseteq \triangle(\triangledown \mathcal{F}) = \triangledown(\triangle \mathcal{F})$. One may also ask the following question.

Question 1.10. *Given* $\mathcal{F}$, *how big or small* $|\triangledown \mathcal{F}|$ *can be.*

The upper bound is easy, by setting sets in $\mathcal{F}$ to be disjoint at all. For the lower bound, it turns out, somewhat surprisingly, this can also be answered completedly by the shadow theorem. We will discuss this in Section 1.2.

Sometimes for the applications, the Kruskal-Katona theorem is not easy to apply directly due to the k-binomial representation. Lovász [27] (Problem 13.31) raised the following version of the theorem.

Theorem 1.11. *If* $\mathcal{F} \subseteq \binom{[n]}{k}$ *and* $|\mathcal{F}| = \binom{x}{k} = \dfrac{x(x-1)(x-2)\cdots(x-k+1)}{k!}$ *for some real number* $x \geqslant k$*, then*

$$|\triangle \mathcal{F}| \geqslant \binom{x}{k-1}.$$

We will also prove this theorem later. This theorem is, in many situations, weaker than Theorem 1.7. For example, if

$$|\mathcal{F}| = 35 = \binom{7}{5} + \binom{5}{4} + \binom{4}{3} + \binom{3}{2} + \binom{2}{1} \approx \binom{7.49442}{5},$$

Theorem 1.7 guarantees that

$$|\triangle \mathcal{F}| \geqslant \binom{7}{4} + \binom{5}{3} + \binom{4}{2} + \binom{3}{1} + \binom{2}{0} = 55$$

while Theorem 1.11 says

$$|\triangle \mathcal{F}| \geqslant \binom{7.49442}{4} \approx 50.0798.$$

However, it can be more handy sometimes due to the use of $\binom{x}{k}$ instead of k-binomial representations.

The Corollary 1.9 can also be stated similarly.

Corollary 1.12. *If* $\mathcal{F} \subseteq \binom{[n]}{k}$ *and* $|\mathcal{F}| = \binom{x}{k}$ *for some real number* $x \geqslant k$*, then*

$$|\triangle^r \mathcal{F}| \geqslant \binom{x}{k-r}.$$

1.2 Equality Condition

Natural questions of Theorem 1.7 are whether it is tight and if so when this bound can be achieved. To answer these questions, we first need to introduce the notion of **co-lexicographical order**.

Definition 1.13. ***(Co-lexicographical order*[1]*)*** *For two sets* $M, H \in \binom{[n]}{k}$*, define the co-lexicographical order as*

$$M <_{col} H \iff \max\left((M-H)\cup(H-M)\right) \in H.$$

In all the following chapters, we will only use the co-lexicographical order so hereafter we use $<$ for $<_{col}$, and whenever we say set M is smaller than set H or set M is the smallest one in a family of sets, we always refer to the co-lexicographical order. Besides, denote by $\mathcal{F}(m,k)$ the family [2] of m smallest sets in $\binom{[n]}{k}$.

Example 1.14. $\{2,3\} < \{1,4\}$.

Example 1.15. *For* $\binom{[5]}{3}$*,*

$$\{1,2,3\} < \{1,2,4\} < \{1,3,4\} < \{2,3,4\} < \{1,2,5\} \\ < \{1,3,5\} < \{2,3,5\} < \{1,4,5\} < \{2,4,5\} < \{3,4,5\}.$$

Example 1.16. $\mathcal{F}(4,4) = \{\{1,2,3,4\},\{1,2,3,5\},\{1,2,4,5\},\{1,3,4,5\}\}$.

By the definition, we immediately get the following properties of the co-lexicographical order and $\mathcal{F}(m,k)$.

Lemma 1.17. *For* $M, H \in \binom{[n]}{k}$*,* $M < H \iff (M-H) < (H-M) \iff [n]-H < [n]-M$*.*

Lemma 1.18. $M, H \in \binom{[n]}{k}$*,* $M < H$ *and* $M \cap H = \varnothing$*. Then for any* $i \in M, j \in H$ *where* j *is not the biggest element in* H*,* $M - \{i\} < H - \{j\}$*.*

Lemma 1.19. *For* m *whose* k*-binomial representation is*

$$m = \binom{a_0}{k} + \binom{a_1}{k-1} + \binom{a_2}{k-2} + \cdots + \binom{a_t}{k-t},$$

[1] It is called co-lexicographical because the usual lexicographical order is defined as $M <_{lex} H \iff \min\left((M-H)\cup(H-M)\right) \in M$. For example, $\{1,2,3\}$ is smaller than $\{1,3,4\}$. $M <_{lex} H$ if and only if $\{n+1-a \mid a \in H\} <_{col} \{n+1-b \mid b \in M\}$.

[2] Strictly speaking, $\mathcal{F}(m,k)$ also depends on the underlying set $[n]$, but n is always large enough in our discussions so we don't need to include n in this notation.

we have

$$\mathcal{F}(m,k) = \binom{[a_0]}{k} \bigcup \left\{ H \cup \{a_0+1\} \mid H \in \mathcal{F}\left(m - \binom{a_0}{k}, k-1\right)\right\}$$

where $\mathcal{F}(0,k) = \varnothing$ *naturally.*

Recall our definition of $KK_k(m)$. Now we can answer our questions by the following theorem.

Theorem 1.20. *If* $\mathcal{F} = \mathcal{F}(m,k)$*, then* $\triangle\mathcal{F} = \mathcal{F}(KK_k(m), k-1)$*.*

Proof. This can be shown by the structure of $\mathcal{F}(m,k)$ according to Lemma 1.19.
For the smallest $\binom{a_0}{k}$ sets, they are actually just $\binom{[a_0]}{k}$, whose contribution to $\triangle\mathcal{F}$ is $\binom{[a_0]}{k-1}$.
For the next $\binom{a_1}{k-1}$ sets, they are $\left\{H \cup \{a_0+1\} \mid H \in \binom{[a_1]}{k-1}\right\}$, whose new contribution to $\triangle\mathcal{F}$ is $\left\{M \cup \{a_0+1\} \mid M \in \binom{[a_1]}{k-2}\right\}$.
For the next $\binom{a_2}{k-2}$ sets, they are $\left\{H \cup \{a_0+1, a_1+1 \mid H \in \binom{[a_2]}{k-2}\right\}$, whose new contribution to $\triangle\mathcal{F}$ is $\left\{M \cup \{a_0+1, a_1+1 \mid M \in \binom{[a_2]}{k-3}\right\}$.
Continue this process so on and so forth. Finally we can confirm $\triangle\mathcal{F} = \mathcal{F}(KK_k(m), k-1)$. □

Let's call $\mathcal{F}$ **extremal** if $|\triangle\mathcal{F}|$ achieves the bound in Theorem 1.7 ($|\triangle\mathcal{F}| = KK_k(|\mathcal{F}|)$) and **r-extremal** if $|\triangle^r\mathcal{F}|$ achieves the bound in the Corollary 1.9. This theorems tells us that the bound in Theorem 1.7 is tight and $\mathcal{F}(m,k)$ is extremal. In other words, Theorem 1.7 can be rephrased as follows.

Theorem 1.21. *If* $\mathcal{F} \subseteq \binom{[n]}{k}$ *and* $|\mathcal{F}| = m$ *then*

$$|\triangle\mathcal{F}| \geqslant |\triangle\mathcal{F}(m,k)|.$$

Note that by using this theorem repeatedly, we can also confirm that $\mathcal{F}(m,k)$ is also r-extremal.

We now can also give the answer to Question 1.10, if we admit Theorem 1.7. In general, define $\nabla^r \mathcal{F} = \left\{ X \in \binom{[n]}{k+r} \,\middle|\, \exists H \in \mathcal{F} \text{ such that } H \subseteq X \right\}$. Let $\mathcal{F}^b(m,k)$[1] to be the biggest m sets in $\binom{[n]}{k}$, namely $\mathcal{F}^b(m,k) = \binom{[n]}{k} - \mathcal{F}\left(\binom{n}{k} - m, k\right)$.

Theorem 1.22. *If $\mathcal{F} \subseteq \binom{[n]}{k}$ and $|\mathcal{F}| = m$ then*

$$|\nabla^r \mathcal{F}| \geqslant \left|\nabla^r \mathcal{F}^b(m,k)\right|.$$

Proof. This theorem follows from considering the complement of each set in $\mathcal{F}$. Let $\mathcal{M} = \left\{ M \in \binom{[n]}{n-k} \,\middle|\, M = [n] - H \text{ for some } H \in \mathcal{F} \right\}$. By Lemma 1.17, if $\mathcal{F} = \mathcal{F}^b(m,k)$, $\mathcal{M}$ is actually $\mathcal{F}(m, n-k)$, and vice versa. Now recall the definition of $\triangle\mathcal{F}$ and $\nabla\mathcal{F}$. We get

$$A \in \nabla^r \mathcal{F} \iff [n] - A \in \triangle^r \mathcal{M}$$

Therefore $|\nabla^r \mathcal{F}| = |\triangle^r \mathcal{M}| \geqslant |\triangle^r \mathcal{F}(m, n-k)| = \left|\nabla^r \mathcal{F}^b(m,k)\right|$. ☐

Actually, we have also proved the following theorem.

Theorem 1.23. *If $\mathcal{F} = \mathcal{F}^b(m,k)$, then $\nabla\mathcal{F} = \mathcal{F}^b(KK_{n-k}(m), k+1)$.*

Now, let's go back to $\triangle\mathcal{F}$ and one may still ask the following question:

Question 1.24. *Is $\mathcal{F}(m,k)$ the unique extremal family?*

We may hope this to be true, however it turns out that the general answer is no.

Example 1.25. $\mathcal{F} = \{\{1,2\},\{2,3\},\{1,4\},\{2,4\}\}$. $|\mathcal{F}| = 4 = \binom{3}{2} + \binom{1}{1}$. *Then*

$$\triangle\mathcal{F} = \{\{1\},\{2\},\{3\},\{4\}\}.$$

So $|\triangle\mathcal{F}| = 4 = \binom{3}{1} + \binom{1}{0}$. However, let $\mathcal{F}_0 = \mathcal{F}(4,2) = \{\{1,2\},\{1,3\},\{2,3\},\{1,4\}\}$; $\mathcal{F}$ can actually be converted to $\mathcal{F}_0$ by the permutation $(12)(34)$ on $[4]$. So strictly speaking, $\mathcal{F}$ is not different from $\mathcal{F}(4,2)$. Actually, by the definition of shadow, the image of $\mathcal{F}(m,k)$ under any permutation of $[n]$ is still extremal.

[1] We still leave out n for convenience whenever there is no ambiguity.

Example 1.26. *Let $\mathcal{F}$ be*

$$\{\{1,2,4\},\{1,3,4\},\{2,3,4\},\{1,2,5\},\{1,3,5\},\{2,3,5\},\{1,4,5\},\{2,4,5\}\}.$$

$|\mathcal{F}| = 8 = \binom{4}{3} + \binom{3}{2} + \binom{1}{1}$. *Then*

$$\triangle\mathcal{F} = \{\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{2,3\},\{2,4\},\{2,5\},\{3,4\},\{3,5\},\{4,5\}\}.$$

So $\triangle\mathcal{F} = 10 = \binom{4}{2} + \binom{3}{1} + \binom{1}{0}$. *Let* $\mathcal{F}_0 = \mathcal{F}(8,3)$*, which is*

$$\{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},\{1,2,5\},\{1,3,5\},\{2,3,5\},\{1,4,5\}\}.$$

At this time, $\mathcal{F}$ can NOT be converted to $\mathcal{F}_0$ by any permutation on $[n]$, because 1 appears in $\mathcal{F}_0$ 6 times while any $i \in [5]$ appears in $\mathcal{F}$ at most 5 times.

In fact, Füredi and Griggs [20] noticed that we can generate counter-examples for infinite different $|\mathcal{F}|$ very easily.

Example 1.27. *Note that for every positive integer a, $\binom{a}{0} = 1$. Therefore, for m_1 whose k-binomial representation is*

$$m_1 = \binom{a_0}{k} + \binom{a_1}{k-1} + \binom{a_2}{k-2} + \cdots + \binom{a_{k-2}}{2} + \binom{a_{k-1}}{1}$$

and m_2 whose k-binomial representation is

$$m_2 = \binom{a_0}{k} + \binom{a_1}{k-1} + \binom{a_2}{k-2} + \cdots + \binom{a_{k-2}}{2} + \binom{b_{k-1}}{1}$$

where $a_{k-1} > b_{k-1}$, there is $KK_k(m_1) = KK_k(m_2)$. This means that if $\mathcal{F}$ is an extremal family of size m_1, such as $\mathcal{F}(m_1,k)$, we can delete arbitrary $m_1 - m_2$ sets from $\mathcal{F}$ and the family we get is still extremal.

Theorem 1.28. *([20]) Let $\mathcal{F} = \mathcal{F}(m_1,k)$, then Example 1.27 gives more than one extremal family of size m_2 with respect to permutations of $[n]$.*

But let's also cite without proof the following theorem in the same paper, which shows that in many cases $\mathcal{F}(m,k)$ is indeed the unique family.

Theorem 1.29. *For positive integer m, whose k-binomial representation is*

$$m = \binom{a_0}{k} + \binom{a_1}{k-1} + \binom{a_2}{k-2} + \cdots + \binom{a_t}{k-t}$$

if $t < k-1$, the extremal family of size m is unique.

Remark. *For those m where $a_{k-1} = a_{k-2} - 1$, there can also exist some extremal family that are not $\mathcal{F}(m,k)$, which of course can not be given by Example 1.27. One example given in [20] is the following. Let $n = s+3$, $H_0 \subseteq [n]$, $|H_0| = s$. Define*

$$\mathcal{F} = \left\{ H \in \binom{[n]}{3} \;\middle|\; |H \cap H_0| \geqslant 2 \right\}.$$

Then $|\mathcal{F}| = m = \binom{s}{3} + 3\binom{s}{2} = \binom{s+2}{3} + \binom{s-2}{2} + \binom{s-3}{1}$ and

$$\triangle\mathcal{F} = \left\{ M \in \binom{[n]}{2} \;\middle|\; |M \cap H_0| \geqslant 1 \right\}.$$

Therefore $|\triangle\mathcal{F}| = \binom{s}{2} + 3\binom{s}{1} = \binom{s+2}{2} + \binom{s-2}{1} + \binom{s-3}{0}$. Hence $\mathcal{F}$ is extremal. It's different from $\mathcal{F}(m,3)$ because there are s elements in $\mathcal{F}$ which appear $\binom{s-1}{2} + 3\binom{s-1}{1}$ times but there are only $s-3$ such elements in $\mathcal{F}(m,3)$.

Besides, they also give the following result for $\triangle^r\mathcal{F}$.

Theorem 1.30. *For integer $r^{'} > r \geqslant 1$,*

$$\mathcal{F} \text{ is } r\text{-extremal} \implies \mathcal{F} \text{ is } r'\text{-extremal.}$$

Therefore the examples above actually gives different r-extremal families for any $r \geqslant 1$.

At last, we turn to Theorem 1.11, the simplified version of Lovász. It is clear that if x is an integer, the equality holds. It turns out that this is the only case.

Theorem 1.31. *The equality in Theorem 1.11 holds if and only if x is a positive integer and $\mathcal{F} = \binom{[x]}{r}$.*

We will prove this theorem in Section 2.4.

2

Proof of Shadow Theorem

In this chapter, we will present a few simple and elegant proofs for Theorem 1.7 as well as Theorem 1.11.

2.1 Shift

Recall that our first feeling of the lower bound for the shadow is that if we want to make the $|\triangle\mathcal{F}|$ as small as possible, we should require the sets in $\mathcal{F}$ to overlap with each other as much as possible. Shift is just such an operation that can be applied to a family of sets to compress it. So before going into proofs, we first talk about the shift operation.

Definition 2.1. ***(Shift)*** *Given* $\mathcal{F} \subseteq \binom{[n]}{k}$, $H \in \mathcal{F}$, *two subsets* $A, B \in \binom{[n]}{k}$ *where* $A \cap B = \varnothing$, *define* S_{AB}[1], *the shift operation from* B *to* A *for* H *as*

$$S_{AB}(H) = \begin{cases} (H - B) \cup A & \text{if } B \subseteq H, A \cap H = \varnothing, (H - B) \cup A \notin \mathcal{F} \\ H & \text{otherwise} \end{cases}.$$

Define the shift operation from B *to* A *for* $\mathcal{F}$ *as*

$$S_{AB}(\mathcal{F}) = \{S_{AB}(H) \mid H \in \mathcal{F}\}.$$

[1]Strictly speaking, $S_{AB}(H)$ depends on $\mathcal{F}$, but we omit this when it is clear which $\mathcal{F}$ we are referring to.

So shift is actually the operation that we try to replace some elements in sets of a family with the other ones whenever it's possible. Note that $|S_{AB}(\mathcal{F})| = |\mathcal{F}|$. We call S_{AB} an r-shift if $|A| = |B| = r$. For convenience, we use S_{ij} to stand for the 1-shift $S_{\{i\}\{j\}}$ and use S_j to stand for S_{1j} for $1 < j \leqslant n$.

We call $H \in \mathcal{F}$ **stable** under S_{AB} if $S_{AB}(H) = H$, call $\mathcal{F}$ **stable** under S_{AB} if all the sets in $\mathcal{F}$ are stable under S_{AB} and call $\mathcal{F}$ ***r*-stable**[1] if it is stable under all S_{AB} where A and B are disjoint subsets of size r and $A < B$. Note that r-stable doesn't imply r'-stable, no matter $r' > r$ or $r' < r$.

Table 2.1 and Table 2.2 are a few examples of shift.

Table 2.1: Examples of shift

$\mathcal{F}$	$S_2(\mathcal{F})$
$\{\ \{1,2\}\ \}$	$\{\ \{1,2\}\ \}$
$\{\ \{2,3\},\{2,4\}\ \}$	$\{\ \{1,3\},\{1,4\}\ \}$
$\{\ \{1,2\},\{1,3\},\{2,3\},\{2,4\}\ \}$	$\{\ \{1,2\},\{1,3\},\{2,3\},\{1,4\}\ \}$

Table 2.2: Examples of shift

$\mathcal{F}$	$S_{\{1,2,3\}\{4,5,6\}}(\mathcal{F})$
$\{\ \{2,3,4,5\},\{3,4,5,6\}\ \}$	$\{\ \{2,3,4,5\},\{3,4,5,6\}\ \}$
$\{\ \{4,5,6,7\},\{1,2,3,8\},\{4,5,6,8\}\ \}$	$\{\ \{1,2,3,7\},\{1,2,3,8\},\{4,5,6,8\}\ \}$

The most important property of shift we will use is the following lemma.

Lemma 2.2. *If $\mathcal{F}$ is $(r-1)$-stable, then for any A, B where $A < B$ and $A \cap B = \varnothing$ and r-shift S_{AB}*

$$\triangle\left(S_{AB}(\mathcal{F})\right) \subseteq S_{AB}(\triangle\mathcal{F}).$$

[1] Every $\mathcal{F}$ is 0-stable.

Proof. For any $M \in \triangle(S_{AB}(\mathcal{F}))$, there exists $H \in S_{AB}(\mathcal{F})$ such that $H = M \cup \{l\}$. There are two possibilities:

- $H \in \mathcal{F}$. So $M \in \triangle\mathcal{F}$. M must be also in $S_{AB}(\triangle\mathcal{F})$ too. Otherwise, $B \subseteq M \subseteq H, A \cap M = \varnothing, (M - B) \cup A \notin \triangle\mathcal{F}$.
  - If $l \in A$, let j be an element in B which is not the biggest one in B. By Lemma 1.18, $A - \{l\} < B - \{j\}$. By the assumption that $\mathcal{F}$ is $(r-1)$ stable, $(H - (B - \{j\})) \cup (A - \{l\}) \in \mathcal{F}$. Dropping j from this set, we get $(M - B) \cup A \in \triangle\mathcal{F}$.
  - If $l \notin A$, $(H - B) \cup A$ must be in $\mathcal{F}$ since H is in $\mathcal{F}$ and in $S_{AB}(\mathcal{F})$. Dropping l from $(H - B) \cup A$, we get $(M - B) \cup A \in \triangle\mathcal{F}$.
- $H \notin \mathcal{F}$. Then there exists $K \in \mathcal{F}\,(A \cap K = \varnothing, B \subseteq K)$ such that $H = (K - B) \cup A$.
  - If $l \in A$, let j be an element in B which is not the biggest one in B. Then $(K - (B - \{j\})) \cup (A - \{l\}) \in \mathcal{F}$. Dropping j from this set, we get $M \in \triangle\mathcal{F}$. M must also be in $S_{AB}(\triangle\mathcal{F})$ since $B \cap M = \varnothing$.
  - If $l \notin A$, then $K - \{l\}$ is in $\triangle\mathcal{F}$. Then $M = ((K - \{l\}) - B) \cup A$ must be in $S_{AB}(\triangle\mathcal{F})$.

□

Note that the assumption $\mathcal{F}$ is $(r-1)$-stable is necessary. An easy but good example is $\mathcal{F} = \{\{1,2\},\{1,5\}\}$ and consider $S_{\{3,4\}\{1,5\}}$.

The previous lemma immediately implies the following two lemmas.

Lemma 2.3. *If $\mathcal{F}$ is $(r-1)$-stable, then for any A, B where $A < B$ and $A \cap B = \varnothing$ and r-shift S_{AB}*

$$|\triangle(S_{AB}(\mathcal{F}))| \leqslant |\triangle\mathcal{F}|.$$

Lemma 2.4. *For any 1-shift S_{ij} (not necessarily $i < j$),*

$$\triangle(S_{ij}(\mathcal{F})) \subseteq S_{ij}(\triangle\mathcal{F}), \text{ therefore } |\triangle(S_{ij}(\mathcal{F}))| \leqslant |\triangle\mathcal{F}|.$$

2.2 Proof of Daykin

The first proof we will present is by Daykin [11] (see also [6] for a very similar proof). In this proof, we can see how the shift method works to derive the lower bound for the shadow.

Proof. Note that we have shown that $\triangle\mathcal{F}(m,k) = \mathcal{F}(KK_k(m), k-1)$ in Section 1.2. Now for any $\mathcal{F} \subseteq \binom{[n]}{k}$ where $|\mathcal{F}| = m$, we just need to show that $|\triangle\mathcal{F}| \geqslant |\triangle\mathcal{F}(m,k)|$.

If $\mathcal{F}$ is not $\mathcal{F}(m,k)$, let $r \geqslant 1$ be the minimum value of $|K-H|$ over $K \in \left(\binom{[n]}{k} - \mathcal{F}\right)$, $H \in \mathcal{F}$ and $K < H$. Note that $\mathcal{F}$ is $(r-1)$-stable and there exist K_0, H_0 for K, H such that $r = |K_0 - H_0|$. Let $A = K_0 - H_0$ and $B = H_0 - K_0$, so $A \cap B = \varnothing$, $r = |A| = |B|$ and $A < B$. Note that every set in $\mathcal{F}$ can not become bigger and at least H_0 becomes strictly smaller, under S_{AB}. Do S_{AB} to $\mathcal{F}$.

Repeat the process above until $\mathcal{F}$ finally becomes $\mathcal{F}(m,k)$. By Lemma 2.3, this process won't increase the shadow of $\mathcal{F}$, therefore we confirm $|\triangle\mathcal{F}| \geqslant |\triangle\mathcal{F}(m,k)|$. □

2.3 Proof of Frankl

The second proof we will present here is by Frankl [15], which is also elegant and uses only 1-shift and induction.

We first give the proof of Theorem 1.11, the simplified version of Lovász.

Proof. Prove this by induction on k and $|\mathcal{F}|$. For $k = 1$ and arbitrary $|\mathcal{F}|$, it's trivial.

For any $\mathcal{F} \subseteq \binom{[n]}{k}$ where $|\mathcal{F}| = \binom{x}{k}$, repeatedly do $S_j\,(1 < j \leqslant n)$ to $\mathcal{F}$ until it is 1-stable. This process is finite because, whenever $\mathcal{F}$ is not stable under S_j, the number of 1 in sets of $S_j(\mathcal{F})$ will be strictly bigger than the number of 1 in sets of $\mathcal{F}$. Recall Lemma 2.4 which guarantees that this process won't increase the size of the shadow. So let's still call the final family of sets after this process $\mathcal{F}$.

Now let

$$\mathcal{F}_1 = \{H \in \mathcal{F} \mid 1 \in H\}, \quad \mathcal{F}_2 = \{H \in \mathcal{F} \mid 1 \notin H\}.$$

The key observation is

$$\triangle\mathcal{F} = \triangle\mathcal{F}_1$$

because for any $M \in \triangle\mathcal{F}_2$, there exists some j such that $M \cup \{j\} \in \mathcal{F}_2\,(1 < j \leqslant n)$, then $M \cup \{1\} = ((M \cup \{j\}) - \{j\}) \cup \{1\} \in F_1$, since $\mathcal{F}$ is stable under S_j. Thus M is also in $\triangle\mathcal{F}_1$.

Another important claim is $|\mathcal{F}_1| \geqslant \binom{x-1}{k-1}$. Otherwise, if $|\mathcal{F}_1| < \binom{x-1}{k-1}$, then $|\mathcal{F}_2| >$

$\binom{x-1}{k}$. By induction

$$|\triangle\mathcal{F}_2| = \left|\{H-\{j\} \bigm| H \in \mathcal{F}_2,\ 2 \leqslant j \leqslant n\}\right| \geqslant \binom{x-1}{k-1}.$$

But then

$$|\mathcal{F}_1| \geqslant \left|\{(H-\{j\}) \cup \{1\} \bigm| H \in \mathcal{F}_2,\ 2 \leqslant j \leqslant n\}\right| = \left|\{H-\{j\} \bigm| H \in \mathcal{F}_2,\ 2 \leqslant j \leqslant n\}\right|.$$

We get $\binom{x-1}{k-1} > |\mathcal{F}_1| \geqslant \binom{x-1}{k-1}$.

Now, divide $\triangle\mathcal{F}_1$ into two parts:

$$\mathcal{P} = \{H-\{1\} \bigm| H \in \mathcal{F}_1\},$$

$$\mathcal{Q} = \{H-\{j\} \bigm| H \in \mathcal{F}_1,\ 2 \leqslant j \leqslant n\}.$$

$|\mathcal{P}| = |\mathcal{F}_1| \geqslant \binom{x-1}{k-1}$ and $|\mathcal{Q}| = \left|\triangle\left\{M-\{1\} \bigm| M \in \mathcal{F}_1\right\}\right| \geqslant \binom{x-1}{k-2}$ by induction.

Therefore

$$|\triangle\mathcal{F}_1| = |\mathcal{P}| + |\mathcal{Q}| \geqslant \binom{x-1}{k-1} + \binom{x-1}{k-2} = \binom{x}{k-1}.$$

□

The proof for Theorem 1.7 is almost the same.

Proof. Prove this still by induction on k and $|\mathcal{F}|$. Still, it's trivial whenever $k = 1$.

Let $\mathcal{F} \subseteq \binom{[n]}{k}$ whose k-binomial representation of $|\mathcal{F}|$ is

$$|\mathcal{F}| = \binom{a_0}{k} + \binom{a_1}{k-1} + \binom{a_2}{k-2} + \cdots + \binom{a_t}{k-t}.$$

Again, we can assume $\mathcal{F}$ is stable under S_j for $2 \leqslant j \leqslant n$.

By the same proof and argument, we get

$$|\mathcal{F}_1| \geqslant \binom{a_0-1}{k-1} + \binom{a_1-1}{k-2} + \binom{a_2-1}{k-3} + \cdots + \binom{a_t-1}{k-t-1}$$

and then

$$|\mathcal{P}| \geqslant \binom{a_0-1}{k-1} + \binom{a_1-1}{k-2} + \binom{a_2-1}{k-3} + \cdots + \binom{a_t-1}{k-t-1}$$

$$|\mathcal{Q}| \geqslant \binom{a_0-1}{k-2} + \binom{a_1-1}{k-3} + \binom{a_2-1}{k-4} + \cdots + \binom{a_t-1}{k-t-2}.$$

Therefore

$$|\triangle\mathcal{F}| = |\triangle\mathcal{F}_1| = |\mathcal{P}| + |\mathcal{Q}| \geqslant \binom{a_0}{k-1} + \binom{a_1}{k-2} + \binom{a_2}{k-3} + \cdots + \binom{a_t}{k-t-1}.$$

□

2.4 Proof of Keevash

The last proof we will present is by Keevash [25]. It only works for the Theorem 1.11, but different from the previous two proofs, it does not use shift operation. Instead, it interprets this problem as a counting problem in hypergraphs and solves it by pure induction.

A k-hypergraph $G = (V, E)$ is a hypergraph, each edge of which has exactly k elements. Denote by K_n^k the complete k-hypergraph on n vertices. Let $d(v) = |\{S \in E(G) \mid v \in S\}|$ and denote by $L(v)$ the $(k-1)$-hypergraph (V, A), where $A = \{S \subseteq V(G) \mid |S| = k-1, S \cup \{v\} \in E(G)\}$. Note that $|A| = d(v)$. Finally, let $\triangle G$ be the $(k-1)$-hypergraph $(V, \triangle E)$.

Here comes the key observation: there exists a one-to-one correspondence between copies of K_k^{k-1} in $\triangle G$ and E. This can be easily checked by the definition of the shadow. Denote by $K_r^k(G)$ the number of copies of K_r^k in G. By this observation, if given $\triangle G$, we can have an upper bound for $K_k^{k-1}(\triangle G)$ in terms of $|\triangle E|$, then we actually get an upper bound for $|E|$ in terms of $|\triangle E|$. By taking it around, we get a lower bound for $|\triangle E|$ in terms of $|E|$.

Lemma 2.5. *Suppose $r \geqslant 1$ and $G = (V, E)$ is an r-hypergraph with $\binom{x}{r}$ edges for some real number $x \geqslant r$. Then*

$$K_{r+1}^r(G) \leqslant \binom{x}{r+1},$$

with equality if and only if x is an integer and $G = K_x^r$.

Proof. Prove by induction on r. The case $r = 1$ is trivial.
Firstly, we may assume that $d(v) > 0$ for every vertex v. Note that any $S \subseteq V$ whose size is r, $S \cup \{v\}$ spans a K_{r+1}^r in G if and only if S is an edge of G and spans a K_r^{r-1} in $L(v)$. Let $K_{r+1}^r(v)$ be the number of copies of K_{r+1}^r in G which cover v. By these two conditions,

we get

$$K_{r+1}^{r}(v) \leqslant |E| - d(v) = \binom{x}{r} - d(v),$$
$$K_{r+1}^{r}(v) \leqslant K_{r}^{r-1}(L(v)).$$

Then we claim

$$K_{r+1}^{r}(v) \leqslant \left(\frac{x}{r} - 1\right) d(v)$$

and equality holds when $d(v) = \binom{x-1}{r-1}$. To see this, consider $d(v)$. If $d(v) \geqslant \binom{x-1}{r-1}$,

$$K_{r+1}^{r}(v) \leqslant \binom{x}{r} - \binom{x-1}{r-1} = \binom{x-1}{r} = \left(\frac{x}{r} - 1\right)\binom{x-1}{r-1} \leqslant \left(\frac{x}{r} - 1\right) d(v).$$

If $d(v) \leqslant \binom{x-1}{r-1}$, let $x_v\,(r \leqslant x_v \leqslant x)$ be the real number such that $d(v) = \binom{x_v - 1}{r-1}$. Then by induction,

$$K_{r+1}^{r}(v) \leqslant K_{r}^{r-1}(L(v)) \leqslant \binom{x_v - 1}{r} = \left(\frac{x_v}{r} - 1\right) d(v) \leqslant \left(\frac{x}{r} - 1\right) d(v).$$

From the proof, the equality condition is also clear. Now

$$(r+1)\,K_{r+1}^{r}(G) = \sum_{v \in V} K_{r+1}^{r}(v) \leqslant \left(\frac{x}{r} - 1\right) \sum_{v \in V} d(v) = \left(\frac{x}{r} - 1\right) r \binom{x}{r} = (r+1)\binom{x}{r+1}.$$

Thus $K_{r+1}^{r}(G) \leqslant \binom{x}{r+1}$. Equality holds if and only if $d(v) = \binom{x-1}{r-1}$ for every vertex $v \in V$. Then

$$|V|\binom{x-1}{r-1} = \sum_{v \in V} d(v) = r|E| = r\binom{x}{r},$$

so $|V| = x$ and $G = K_x^r$. □

Now we just need to take this lemma around and then can get Theorem 1.11.

Proof. (Theorem 1.11) If $|\triangle \mathcal{F}| < \binom{x}{k-1}$, let $x'\,(k-1 \leqslant x' < x)$ be the real number such that $\binom{x'}{k-1} = |\triangle \mathcal{F}|$. Then consider the $(k-1)$-hypergraph $H = ([n], \triangle \mathcal{F})$ and apply the previous lemma, we get

$$|\mathcal{F}| = K_k^{k-1}(H) \leqslant \binom{x'}{k} < \binom{x}{k} = |\mathcal{F}|.$$

□

Remark 2.6. *This proof definitely can be stated in only the language of set. However, as we can see, these terminologies of hypergraphs make the proof much more intuitive and easier to understand.*

At the end of this chapter, we want to mention that a stable version of the previous lemma is also proved in [25].

Theorem 2.7. *For any $\varepsilon > 0$ and $k \geqslant 1$, there exists $\delta > 0$ such that if G is a k-graph with $\binom{x}{k}$ edges and $K_{k+1}^k(G) > (1-\delta)\binom{x}{k+1}$, then there is a set $\mathcal{F}$ of $\lceil x \rceil$ vertices so that all but at most $\varepsilon\binom{x}{k}$ edges of G are contained in S.*

3

Applications of Shadow Theorem

In this chapter, we will present a few applications of shadow theorem. It should be noted that these results are selected by personal preference and should never be regarded as a complete list. Indeed, the shadow theorem, as a fundamental theorem in extremal set theory, has numerous elegant applications in many fields of mathematics. Many results can not be included here, just either because they are based on many other theorems too, or due to the length of their statement.

3.1 Number of Simplices and Density of Triangles

Given six sticks, all of the same size, how can you put them together to make four triangles of the same size? The answer, of course, is a tetrahedron. This question can be generalized further as follows.

Question 3.1. *Given a complex which has exactly m k-dimensional simplices, what is the maximum number of r-dimensional simplices ($r > k$) this complex can have?*

This question is actually the one considered by Kruskal in his paper of shadow theorem [26]. Recall that a simplex can be labelled by its vertices. We can use a set of size k to stand for a $(k-1)$-dimensional simplex and then the containment of simplices is just the containment of sets. Hence the answer can be given by theorems in Chapter 1. This result also has some applications in reliability theory. See [9].

Similarly, we can also ask this question in the terminology of graph theory. Recall that in Section 2.4 we prove an upper bound for $K_{r+1}^{r}(G)$, the number of copies of the complete

r-hypergraph on $r+1$ vertices in G, in terms of $E(G)$, the number of edges. In particular, setting $r = 2$, we get the following corollary of the number of triangles in G.

Corollary 3.2. *If $G = (V, E)$ has $\binom{x}{2}$ edges for some real number $x \geqslant 2$, then*

$$\#\text{ triangles in } G \leqslant \binom{x}{3},$$

and equality holds if and only if x is an integer and $G = K_x$.

This means that the number of triangles in G is of $\mathrm{O}\left(E(G)^{3/2}\right)$.

We also want to mention that the lower bound of the number of triangles is also of interest and turns out to be harder. See [21, 5] for some early works, [28] for the complete conjecture, and finally [30] for the proof of the conjecture.

3.2 Size of Intersecting Family

We call a family of sets $\mathcal{F}$ to be **intersecting** if any two sets in this family are not disjoint. A natural question is how large an intersecting family can be.

The answer depends on whether we require these sets to have the same size. If not, the maximum of $|\mathcal{F}|$ is obviously 2^{n-1}, by considering every set and its complement. This bound is tight, for example, let $\mathcal{F} = \{\{n\} \cup H \mid H \subseteq [n-1]\}$. On the other hand, if we require $\mathcal{F} \subseteq \binom{[n]}{k}$, the following answer was given by Erdös, Ko, and Rado [12].

Theorem 3.3. *Suppose $\mathcal{F} \subseteq \binom{[n]}{k}$ and $n \geqslant 2k$. If $\mathcal{F}$ is intersecting, then*

$$|\mathcal{F}| \leqslant \binom{n-1}{k-1}$$

This is also tight by $\mathcal{F} = \{[n] \cup H \mid H \in \binom{[n-1]}{k-1}$.

It was Daykin [10] who first noticed that this classical theorem is actually almost a direct corollary of Theorem 1.7.

Proof. (using shadow theorem) If $|\mathcal{F}| > \binom{n-1}{k-1} = \binom{n-1}{n-k}$, define $\mathcal{M} = \{[n] - H \mid H \in \mathcal{F}\} \subseteq \binom{[n]}{n-k}$. Note that $|\mathcal{M}| = |\mathcal{F}| > \binom{n-1}{n-k}$. By Corollary 1.12, $\left|\triangle^{(n-k)-k}(\mathcal{M})\right| >$

$\binom{n-1}{k}$. Then

$$\left|\mathcal{F}\right| + \left|\triangle^{(n-k)-k}\left(\mathcal{M}\right)\right| > \binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}.$$

Therefore there exists $A \in \mathcal{F} \cap \triangle^{(n-k)-k}\left(\mathcal{M}\right)$. However, this means that there exist $A, B \in \mathcal{F}$ such that $A \subseteq [n] - B$, which contradicts the assumption that $\mathcal{F}$ is intersecting. □

Here we also want to present the original proof in paper [12], because it made use of the shift operation we defined in Section 2.1. We first need a lemma.

Lemma 3.4. *For $\mathcal{F} \subseteq \binom{[n]}{k}$, $i \neq j \in [n]$, if $\mathcal{F}$ is intersecting, then $S_{ij}\left(\mathcal{F}\right)$ is also intersecting.*

Proof. For $M, H \in \mathcal{F}$ (therefore $M \cap H \neq \varnothing$), we prove that $S_{ij}\left(M\right) \cap S_{ij}\left(H\right) \neq \varnothing$. If both M and H are stable or none of them are stable, it's obviously true. So assume M is not stable while H is stable. If $\left(\left(M - \{j\}\right) \cup \{i\}\right) \cap H = \varnothing$, this means that $M \cap H = \{j\}$ and $i \notin H$. Then $\left(H - \{j\}\right) \cup \{i\}$ must be in $\mathcal{F}$ since H is stable. But $M \cap \left(\left(H - \{j\}\right) \cup \{i\}\right) = \varnothing$. □

Note that this lemma isn't true for r-shift when $r \geqslant 2$. A simple example is $\mathcal{F} = \{\{1,2\},\{1,5\}\}$ and $S_{\{3,4\}\{1,5\}}$.

Proof. (using shift operation) Prove by induction on k and n. Whenever $k = 1$, it's trivial. If $n = 2k$, $\left|\mathcal{F}\right| \leqslant \frac{1}{2}\binom{2k}{k} = \binom{2k-1}{k-1}$ because for any set $H \in \mathcal{F}$, its complement can not be in $\mathcal{F}$.

If $n \geqslant 2k+1$, by the previous lemma, we can assume that $\mathcal{F}$ is 1-stable. Let

$$\mathcal{F}_n = \{H \in \mathcal{F} \mid n \in H\}.$$

Note that $\mathcal{F} - \mathcal{F}_n$ can be viewed as an intersecting family in $\binom{[n-1]}{k}$. By induction, $\left|\mathcal{F} - \mathcal{F}_n\right| \leqslant \binom{n-2}{k-1}$. Let

$$\mathcal{P} = \{H - \{n\} \mid H \in \mathcal{F}_n\}.$$

We claim $\mathcal{P}$ is also intersecting. Otherwise, there exist two sets $A, B \in \mathcal{P}$ such that $A \cap B = \varnothing$. Since $|A| = |B| = k-1$ and $n \geqslant 2k+1$, there exist $i \in [n-1]$ where $i \notin A \cup B$. By the assumption that $\mathcal{F}$ is 1-stable, $B \cup \{i\} = \left(\left(B \cup \{n\}\right) - \{n\}\right) \cup \{i\}$ is in $\mathcal{F}$. But then $\left(A \cup \{n\}\right) \cap \left(B \cup \{i\}\right) = \varnothing$, which contradicts the assumption that $\mathcal{F}$ is intersecting.

Now by induction, $|\mathcal{F}_n| = |\mathcal{P}| \leqslant \binom{n-2}{k-2}$. Therefore

$$|\mathcal{F}| = |\mathcal{F} - \mathcal{F}_n| + |\mathcal{F}_n| \leqslant \binom{n-2}{k-1} + \binom{n-2}{k-2} = \binom{n-1}{k-1}.$$

□

Remark. *In fact, the proof based on the shadow theorem can be viewed as a generalization of considering a set and its complement. The shadow theorem offers us a bound for the transformation between families of sets of different sizes.*

At last, we want to mention that Frankl [16] proves the following similar theorem with the condition on maximal degree, using Theorem 1.7 and shift operation.

For $\mathcal{F} \subseteq \binom{[n]}{k}$, define $d(\mathcal{F})$, the maximal degree of $\mathcal{F}$ to be $\max_{i\in[n]} |\{H \in \mathcal{F} \mid i \in H\}|$. Let $[i, j] = \{i, i+1, \cdots, j\}$ for $1 \leqslant i \leqslant j \leqslant n$. Define $\mathcal{F}_i\,(i \in [3, k+1])$ to be

$$\mathcal{F}_i = \left\{ H \in \binom{[n]}{k} \middle| \, 1 \in H, H \cap [2, i] \neq \varnothing \right\} \cup \left\{ H \in \binom{[n]}{k} \middle| \, 1 \notin H, [2, i] \subseteq H \right\}.$$

It's obvious that $\mathcal{F}_i$ are all intersecting.

Theorem 3.5. *Suppose $\mathcal{F} \subseteq \binom{[n]}{k}$ and $n \geqslant 2k$. If $\mathcal{F}$ is intersecting and $d(\mathcal{F}) \leqslant d(\mathcal{F}_i)$ for some $i \in [3, k+1]$, then*

$$|\mathcal{F}| \leqslant |\mathcal{F}_i|.$$

Moreover, equality holds if and only if either $\mathcal{F}$ is isomorphic to $\mathcal{F}_i$ or $i = 4$ and $\mathcal{F}$ is isomorphic to $\mathcal{F}_3$.

See also [22] and [18] for similar results.

3.3 Maximum Number of Independent Sets

For the last application, let's consider the following question.

Question 3.6. *Given n and m, what is the maximum number of independent sets among all the simple graphs on n vertices with m edges?*

At first sight, this question seems not to be easy to answer and one may expect the answer is rather long and involved. However the answer to this problem can be given, a little bit surprisingly, by shadow theorem quite easily [8]. Actually, we can get an even more general result.

For any graph G, let $\mathcal{I}(G)$ be the collection of all the independent sets in G. Given any weight function $w : \mathbb{N} \cup \{0\} \to [0, \infty)$, define

$$i_w(G) = \sum_{I \in \mathcal{I}(G)} w(|I|).$$

Theorem 3.7. *For any graph $G = ([n], E)$ where $|E| = m$ and weight function $w : \mathbb{N} \cup \{0\} \to [0, \infty)$,*

$$i_w(G) \leqslant i_w\left(\left([n], F^b(m, 2)\right)\right).$$

The proof is surprisingly simple, based on Theorem 1.22.

Proof. For any $k \in \mathbb{N} \cup \{0\}$, let

$$w_k(i) = \begin{cases} 1 & , \quad i = k \\ 0 & , \quad \text{otherwise} \end{cases}.$$

We only need to prove the theorem for every w_k.

Recall the definition of independent set. A set $I \subseteq V$ of size k is an independent set if and only if there is no edge between any pair of vertices in I. In our word, this means that

$$I \notin \triangledown^{k-2}(E).$$

Therefore

$$i_k(G) = \binom{n}{k} - \left|\triangledown^{k-2}(E)\right| \leqslant \binom{n}{k} - \left|\triangledown^{k-2} F^b(m, 2)\right| = i_k\left(\left([n], F^b(m, 2)\right)\right).$$

□

4

Variations and Generalizations of Shadow Theorem

In this chapter, we will first briefly give several variations and generalizations of the shadow theorem. And then, to end this article, we talk about a few open problems.

4.1 Shadow of Intersecting Family

Recall that in Theorem 1.7, we don't pose any requirement on the structure of $\mathcal{F}$. Then a natural question is what the lower bound of the shadow will be if we do.

One of such theorems is by Katona [23], who talked about the shadow of intersecting families. In Section 3.2, we have defined $\mathcal{F}$ to be intersecting if any two sets in $\mathcal{F}$ are not disjoint. In general, we call $\mathcal{F}$ **t-intersecting** if for $M, H \in \mathcal{F}$, $|M \cap H| \geqslant t$.

Theorem 4.1. *For integers $1 \leqslant r \leqslant t \leqslant k \leqslant n$, if $\mathcal{F} \subseteq \binom{[n]}{k}$ and $\mathcal{F}$ is t-intersecting, then*

$$|\triangle^r \mathcal{F}| \geqslant |\mathcal{F}| \frac{\binom{2k-t}{k-r}}{\binom{2k-t}{k}}.$$

The equality holds if and only if $\mathcal{F} = \binom{[2k-t]}{k}$ (up to permutation of $[n]$) [1]. For some extensions of this theorem, see [17].

4.2 Balanced Shadow Theorem

A far more difficult question is the balanced version of the shadow theorem. Recall that in Chapter 1, our first feeling of the shadow is that it will be small if $\mathcal{F}$ is strongly compressed. It is also confirmed that only $\mathcal{F}(m,k)$ can achieve the lower bound in shadow theorem in many situations (Theorem 1.29). Then we may ask, what can we say about the shadow if the sets in $\mathcal{F}$ don't actually overlap with each other too much? There are a few results for this question. We may call them sparse or balanced version of the shadow theorem.

Frankl, Füredi, and Kalai [19] gave the lower bound for the shadow of the r-colored family of sets. This result looks quite similar to Theorem 1.7 we have in Chapter 1. First, we need a few more notations. For $\mathcal{F} \subseteq \binom{[n]}{k}$, it is **r-colored** if there is a partition of $[n] = V_1 \cup V_2 \cup \cdots \cup V_r$ such that for any $H \in \mathcal{F}$ and $i\,(1 \leqslant i \leqslant r)$, $|H \cap V_i| \leqslant 1$.

For integers $n \geqslant r \geqslant k > 0$, always let $a = \lfloor n/r \rfloor$, $r_1 = n - ra$. Let $M_1, M_2, \cdots, M_r$ be a partition of $[n]$ where $|M_i| = a+1$ for $1 \leqslant i \leqslant r_1$ and $|M_i| = a$ for $r_1 < i \leqslant r$. Define $C(n,k,r)$ by

$$C(n,k,r) = \{H \subseteq [n] \mid |H| = k, |H \cap M_i| \leqslant 1, \text{ for } i = 1, \cdots, r\}.$$

Define $\binom{n}{k}_r = |C(n,k,r)|$. Note that if $r = n$, $\binom{n}{k}_r$ is just $\binom{n}{k}$. In general,

$$\binom{n}{k}_r = \sum_{j=0}^{k} \binom{r_1}{j} \binom{r-r_1}{k-j} (a+1)^j a^{k-j}.$$

We will see that $\binom{n}{k}_r$ acts as a similar role with $\binom{n}{k}$ in the shadow theorem.

Lemma 4.2. *For any $r \geqslant k > 0$, every positive integer m can be written uniquely in the form*

$$m = \binom{a_0}{k}_r + \binom{a_1}{k-1}_{r-1} + \binom{a_2}{k-2}_{r-2} + \cdots + \binom{a_t}{k-t}_{r-t},$$

where $a_0 > a_1 > \cdots > a_t \geqslant k-t \geqslant 1$.

Theorem 4.3. *If r-colored $\mathcal{F} \subseteq \binom{[n]}{k}$ and $|\mathcal{F}| = m$,*

$$m = \binom{a_0}{k}_r + \binom{a_1}{k-1}_{r-1} + \binom{a_2}{k-2}_{r-2} + \cdots + \binom{a_t}{k-t}_{r-t},$$

where $a_0 > a_1 > \cdots > a_t \geqslant k-t \geqslant 1$, *then*

$$|\triangle \mathcal{F}| \geqslant \binom{a_0}{k-1}_r + \binom{a_1}{k-2}_{r-1} + \binom{a_2}{k-3}_{r-2} + \cdots + \binom{a_t}{k-t-1}_{r-t}.$$

This result is indeed a generalization of Theorem 1.7, by setting $r = n$.

The next result is by Alon, Moshkovitz, and Solomon [2]. They proved the following theorem in their discussion of the trace of hypergraphs.

For any $I \subseteq [n]$, let $\mathcal{F}[I] = \{H \in \mathcal{F} \mid H \subseteq I\}$. In the language of hypergraphs, $\mathcal{F}[I]$ are the edges induced by I. Then let

$$\operatorname{span}(\mathcal{F}, i) = \max_{\substack{I \subseteq [n] \\ |I| = i}} |\mathcal{F}[I]|.$$

This parameter can be viewed as a measure of the sparseness of $\mathcal{F}$. If it is not too big, this means that $\mathcal{F}$ doesn't have a large proportion of sets in a small subset of $[n]$.

Theorem 4.4. *Let* $t \geqslant 1, \alpha \in (0,1]$. *Let* $\mathcal{F} \subseteq \binom{[n]}{k}$ *and* $|\mathcal{F}| = n^t$. *If*

$$\operatorname{span}(\mathcal{F}, \alpha n) \leqslant \min\left\{\binom{x}{k-\lceil t \rceil} n,\ \frac{1}{2}|\mathcal{F}|\right\}$$

for some real number $x \geqslant 2k$, *then for every* $r\,(0 \leqslant r < k-t)$,

$$|\triangle^r \mathcal{F}| \geqslant \frac{1}{C} \cdot \frac{\binom{x}{k-r}}{\binom{x}{k}} |\mathcal{F}|,$$

where $C = (8k/\alpha)^{\lceil 5t \rceil} \log n$.

It's sharp in the following sense.

Theorem 4.5. *Let* n, k, x *be positive integers,* $t \geqslant 1$ *and* $0 < \alpha \leqslant 1$ *with* $3t \leqslant k \leqslant x \leqslant n^{\frac{1}{6}}$ *and* $n \leqslant \alpha^k n^t \leqslant \binom{x}{k} n$. *There exists* $\mathcal{F} \subseteq \binom{[n]}{k}$ *where* $|\mathcal{F}| = n^t$ *and* $\operatorname{span}(\mathcal{F}, \alpha n) \leqslant O\left(\binom{x}{k} n\right)$ *such that for every* $0 \leqslant r \leqslant k$, *we have*

$$|\triangle^r \mathcal{F}| \leqslant \frac{\binom{x}{k-r}}{\binom{x}{k}} |\mathcal{F}|.$$

We also recommend [14] for some results of the trace of sets, which are based on shadow theorem and the shift operation.

O'Donnell and Wimmer [29] have the following theorem, which is about $\nabla\mathcal{F}$. For $\mathcal{F} \subseteq \binom{[n]}{k}$, let $\mu_k(\mathcal{F}) = \frac{|\mathcal{F}|}{\binom{n}{k}}$. We use $H \sim \binom{[n]}{k}$ to stand for H is drawn uniformly from $\binom{[n]}{k}$.

Theorem 4.6. *For any $\varepsilon > 0$, there exists $\delta > 0$ such that the following holds. If $\mathcal{F} \subseteq \binom{[n]}{k}$, $\varepsilon \leqslant \frac{k}{n}, \mu_k(\mathcal{F}) \leqslant 1 - \varepsilon$, then*

$$\mu_{k+1}(\nabla\mathcal{F}) \geqslant \mu_k(\mathcal{F}) + \delta \cdot \frac{\log n}{n},$$

unless there exists $i \in [n]$ such that

$$\mathbb{P}_{H\sim\binom{[n]}{k}}(H \in \mathcal{F} \mid i \in H) - \mathbb{P}_{H\sim\binom{[n]}{k}}(H \in \mathcal{F} \mid i \notin H) \geqslant 1/n^{\varepsilon}.$$

This means that if $\mathcal{F}$ is not strongly correlated with a single coordinate, $|\nabla\mathcal{F}|$ will be large.

4.3 Shadow of Other Mathematical Objects

The idea of shadow has also been generalized to other mathematical objects and analogues to Theorem 1.7 have been raised. Here we briefly present two of them.

Chowdhury and Patkós generalized shadow to the vector space [7]. Let V denote an n-dimensional vector space over a finite field of order q. For any positive integer k, let $\begin{bmatrix} V \\ k \end{bmatrix}_q$ be the family of all k-dimensional subspaces of V. Besides, for $a \in \mathbb{R}$ and positive integer k, define the Gaussian binomial coefficient by

$$\begin{bmatrix} a \\ k \end{bmatrix}_q = \prod_{i=0}^{k-1} \frac{q^{a-i} - 1}{q^{k-i} - 1}.$$

Then we know $\left| \begin{bmatrix} V \\ k \end{bmatrix}_q \right| = \begin{bmatrix} n \\ k \end{bmatrix}_q$. We can naturally generalize the definition of shadow to vector spaces. For any family $\mathcal{F} \subseteq \begin{bmatrix} n \\ k \end{bmatrix}_q$, define $\triangle\mathcal{F}$ to be

$$\triangle\mathcal{F} = \left\{ X \in \begin{bmatrix} V \\ k-1 \end{bmatrix}_q \,\middle|\, \exists H \in \mathcal{F} \text{ such that } X \subseteq H \right\}.$$

Then a similar theorem to Theorem 1.11 can be proved.

Theorem 4.7. *If* $\mathcal{F} \subseteq \begin{bmatrix} V \\ k \end{bmatrix}_q$ *and* $|\mathcal{F}| = \begin{bmatrix} x \\ k \end{bmatrix}_q$ *for a real number* $x \geqslant k$*, then*

$$|\triangle \mathcal{F}| \geqslant \begin{bmatrix} x \\ k-1 \end{bmatrix}_q .$$

The equality holds if and only if x *is an integer and* $\mathcal{F} = \begin{bmatrix} X \\ k \end{bmatrix}_q$*, where* X *is an* x*-dimensional subspace of* V*.*

Actually, the proof for this theorem in [7] is a generalization to the one of Keevash in Section 2.4.

In [3], Bollobás, Brightwell, and Morris raised the notion of shadow for ordered graphs. An **ordered graph** $G = (V, E)$ is a graph together with a linear order on V. We regard two ordered graphs $G_1 = (V_1, E_1)$ with order $a_1 < a_2 < \cdots < a_n$ and $G_2 = (V_2, E_2)$ with order $b_1 < b_2 < \cdots < b_n$ as the same ordered graph if for every $i, j\,(1 \leqslant i < j \leqslant n)$, $\{a_i, a_j\} \in E_1 \iff \{b_i, b_j\} \in E_2$. Therefore, we can always assume the vertex set of an ordered graph G with n vertices to be $[n]$, with the usual order. Given an ordered graph $G = (V, E)$ and $U \subseteq V$, let $G[U]$ denote the ordered graph induced by U, with the inherited order. Besides, write $G - v$ for $G[V - \{v\}]$.

The shadow of an ordered graph $G = (V, E)$ is defined[1] as

$$\triangle G = \{H \mid H = G - v \text{ for some } v \in V\}$$

and if $\mathcal{G}$ is a collection of ordered graphs then the shadow of $\mathcal{G}$ is

$$\triangle \mathcal{G} = \bigcup_{G \in \mathcal{G}} \triangle G.$$

For example, if $\mathcal{G} = \{\ ([3], \{\{1,2\},\{2,3\}\})\,,\ ([3], \{\{1,2\},\{1,3\},\{2,3\}\})\ \}$, then

$$\triangle \mathcal{G} = \{\ ([2], \{1,2\})\,, ([2], \varnothing)\ \}\,.$$

Now a theorem for the lower bound of the shadow is proved, when $|\mathcal{G}|$ is not too big.

[1]In section 2.4, we used the notation $\triangle G$ once. But that one is still the shadow of sets in essence while now we are really making a new definition of shadow for graphs.

Theorem 4.8. *For $n \geqslant 135$ and $\mathcal{G}$ a collection of ordered graphs on $[n]$, if $|\mathcal{G}| < n$, then $|\triangle\mathcal{G}| \geqslant |\mathcal{G}|$. Moreover, this bound is sharp.*

The authors explain that the condition $n \geqslant 135$ in the theorem is an artifact of the proof and is almost certainly unnecessary. Besides, note that although this result seems simple, it can actually be difficult to derive. In this paper, the authors even encourage the readers to prove this theorem by themselves in the case where each ordered graph has at most two edges, or in the case $n = 4$, in order to gain an appreciation of the complexity of this problem.

4.4 Open Problems

In [4], Bollobás and Eccles ask several questions about some variations of shadow.

For $\mathcal{F} \subseteq \binom{[n]}{k}$, $\mathcal{M} \subseteq \binom{[n]}{k-1}$, we call $\mathcal{M}$ an **r-deficient shadow** of $\mathcal{F}$ if for every $H \in \mathcal{F}$, we have

$$|\triangle H - \mathcal{M}| \leqslant r.$$

So an r-deficient shadow doesn't have to contain the shadow of all the sets in $\mathcal{F}$, but only most of them. Note that if $r = 0$, it is just the shadow.

Question 4.9. *For given m, k, r, what is $f(m, k, r)$, the minimal possible size of an r-deficient shadow of a family of m k-element sets?*

A partial answer was given by Fitch [13], but it is still far from a complete answer. And as far as we know, all other cases remain open.

Another slightly different question we can ask is what will happen if instead of demanding that each set in $\mathcal{F}$ has at most r sets in its shadow missing in $\mathcal{M}$, we ask for many pairs (M, H) where $M \in \mathcal{M}, H \in \mathcal{F}$, and $M \in \triangle H$.

Question 4.10. *Given integers k, m_1 and m_2, what is $g(k, m_1, m_2)$, the maximum number of pairs (M, H) where $M \in \mathcal{M}, H \in \mathcal{F}$, $M \in \triangle H$ and $\mathcal{M}$ is a family of m_1 $(k-1)$-elements sets, $\mathcal{F}$ is a family of m_2 k-elements sets?*

This question is perhaps even more interesting if we do not specify the size of the sets in $\mathcal{M}$ and $\mathcal{F}$ while just still requiring $|\mathcal{M}| = m_1$ and $|\mathcal{F}| = m_2$.

We also have the following generalization of shadow. For $\mathcal{F} \subseteq \binom{[n]}{k}$, let's define the $\underline{\triangle} H$, **semi-shadow** of $H \in \mathcal{F}$ to be

$$\underline{\triangle} H = \left\{ M \in \binom{[n]}{k-1} \;\middle|\; |M - H| \leqslant 1 \right\}.$$

And define **semi-shadow** of $\mathcal{F}$ to be

$$\underline{\triangle}\mathcal{F} = \bigcup_{H \in \mathcal{F}} \underline{\triangle} H.$$

Note that $\triangle\mathcal{F} \subseteq \underline{\triangle}\mathcal{F}$. Again we can ask a similar question.

Question 4.11. *Given n, k, m, what is the minimum of $\left|\underline{\triangle}\mathcal{F}\right|$ where $\mathcal{F} \subseteq \binom{[n]}{k}$ and $|\mathcal{F}| = m$?*

Our first guess is $\mathcal{F}(m, k)$ will still achieve the optimum. This is true for $k = 3$, simply due to the following observation.

Lemma 4.12.

$$\underline{\triangle}\mathcal{F} = \nabla\left(\triangle\left(\triangle\mathcal{F}\right)\right)$$

However, for $k \geqslant 4$, this is not true. We show this by simple counting. Let $m = \binom{n-1}{k}$. If $\mathcal{F}(m, k)$ indeed achieves the lower bound, then for any $\mathcal{F} \subseteq \binom{[n]}{k}$ whose size is m, the semi-shadow should be $\binom{[n]}{k-1}$. Consider set $\{1, 2, \cdots, k-1\}$. If it is not in $\underline{\triangle}\mathcal{F}$, then there are $(n-k+1) + (k-1)\binom{n-k+1}{2}$ sets that can't be in $\mathcal{F}$, and if $\mathcal{F}$ doesn't contain any of these sets, $\{1, 2, \cdots, k-1\}$ indeed won't be in its shadow. So the maximum cardinality of $\mathcal{F}$ is

$$M = \binom{n}{k} - (k-1)\binom{n-k+1}{2} - (n-k+1)$$

We compare it with $\binom{n-1}{k}$.

$$M - \binom{n-1}{k} = \binom{n-1}{k-1} - (k-1)\binom{n-k+1}{2} - (n-k+1)$$

Note that $k - 1 > 2$, so when n is large enough, it's positive. This means that we indeed can find such an $\mathcal{F}$ with cardinality $\binom{n-1}{k}$ but its semi-shadow doesn't contain $\{1, 2, \cdots, k-1\}$.

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