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Packing common independent sets through reductions of matroids

— Diploma Thesis —

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Contents

Introduction

Matroids were introduced by Whitney [75] in 1935 and independently by Nakasawa [61] as abstract generalizations of linear independence in vector spaces. In combinatorial optimization, they are important tools for various graph characterization and optimization problems. In many cases matroids not only help to understand the true background of known problems, but they are often unavoidable in solving natural optimization problems in which matroids do not appear explicitly at all.

One of the most powerful results in matroid theory is the min-max theorem and algorithm of Edmonds for finding a common independent set of two matroids with maximum cardinality. Edmonds and Fulkerson gave a min-max formula and algorithm for partitioning the ground set of a matroid into a minimal number of independent sets, and also for finding the maximal number of disjoint bases of a matroid. These results motivate the analogous questions for two matroids on the same ground set: (A) find a partition of the ground set into a minimal number of common independent sets, and (B) find a maximal number of disjoint common bases of two matroids.

The importance of these problems is underpinned by a long list of well-studied conjectures that can be formalized as a packing common bases of matroids. Rota's basis conjecture [35] states that if M is a matroid of rank n whose ground set can be partitioned into n disjoint bases $B_1, \ldots, B_n$, then it is possible to rearrange the elements of these bases into an $n \times n$ matrix in such a way that the rows are exactly the given bases, and the columns are also bases of M. Woodall's conjecture [77] on packing disjoint dijoins in a directed graph is also a special case of packing common bases, as was shown by Frank and Tardos [23]. Given a directed graph D, a dijoin is a subset of arcs whose contraction results in a strongly connected digraph. The conjecture states that the maximum number of pairwise disjoint dijoins equals the minimum size of a directed cut.

In our joint work [3] with Kristóf Bérczi we showed that problems (A) and (B) are difficult for general matroids, i.e. it requires an exponential number of independence queries in the independence oracle model. These problems remain intractable even for matroids given by explicit linear representations. However, these complexity results motivate the study of tractable special cases and approximation results.

This thesis deals in more details with partitioning the ground set into common independent sets. We denote by $\chi(M)$ the minimum number of independent sets of M needed to cover its ground set, and analogously, $\chi(M_1, M_2)$ is used for the minimum number of common independent sets of M_1 and M_2 covering their common ground set. Davies and McDiarmid [16] showed that $\chi(M_1, M_2) = \max\{\chi(M_1), \chi(M_2)\}$ holds whenever M_1 and M_2 are so-called strongly bases orderable matroids. Aharoni and Berger [1] proved using topological tools that $\chi(M_1, M_2) \leq 2 \max\{\chi(M_1), \chi(M_2)\}$ holds for general matroids M_1 and M_2.

Recently, in [4] we proposed a conjecture with Kristóf Bérczi and Yutaro Yamaguchi, which strengthens the theorem of Aharoni and Berger. The conjecture states that for every loopless matroid $M = (S, \mathcal{I})$ there exists a partition matroid $N = (S, \mathcal{J})$ such that $\mathcal{J} \subseteq \mathcal{I}$ and $\chi(N) \leq 2\chi(M)$. We proved the conjecture for the special case when M is a transversal matroid, a paving matroid, a truncation of a graphic matroid or a gammoid. These special cases also provided new results about a list colouring problem of matroids proposed by Király [42]. Another interesting aspect of this new method is that various elementary colouring problems appear naturally, such as Gallai colourings of complete graphs or colouring the points of a projective plane with three colours such that no line contains all three colours.

The thesis is organized as follows. Chapter 1 introduces basic definitions and notations. Chapter 2 consists most of our complexity results [3]. Chapter 3 presents the result of Davies and McDiarmid [16] about strongly base orderable matroids, the results of Kotlar and Ziv [47] about matroids without $(k+1)$-spanned elements and a list of famous conjectures fitting our framework. Chapter 4 is about the theorem of Aharoni and Berger [1] with a detailed overview of the required topological tools. Chapter 5 is a slightly extended version of our paper [4] about reduction of matroids to partition matroids.

Chapter 1

Preliminaries

In this chapter we list some basic elements of matroid theory that will be used later. The reader is referred to books [22, 66, 73] for more details.

1.1 Definitions

A *matroid* M is a pair $(S, \mathcal{I})$ where S is the *ground set* of the matroid and $\mathcal{I} \subseteq 2^S$ is the family of *independent sets* that satisfies the following, so-called *independence axioms*: (I1) $\emptyset \in \mathcal{I}$, (I2) $X \subseteq Y \in \mathcal{I} \Rightarrow X \in \mathcal{I}$, (I3) $X, Y \in \mathcal{I}, |X| < |Y| \Rightarrow \exists e \in Y - X$ s.t. $X + e \in \mathcal{I}$. The *rank* of a set $X \subseteq S$ is the maximum size of an independent subset of X and is denoted by $r_M(X)$.

The maximal independent sets of M are called *bases*. Alternatively, simple properties of bases can be taken as axioms as well. In terms of bases, a matroid M is a pair $(S, \mathcal{B})$ where $\mathcal{B} \subseteq 2^S$ satisfies the *basis axioms*: (B1) $\mathcal{B} \neq \emptyset$, (B2) for any $B_1, B_2 \in \mathcal{B}$ and $x_1 \in B_1 - B_2$ there exists $x_2 \in B_2 - B_1$ such that $B_1 - x_1 + x_2 \in \mathcal{B}$. Property (B2) is known to be equivalent to the so-called *symmetric basis exchange property*: for any $B_1, B_2 \in \mathcal{B}$ and $x_1 \in B_1 - B_2$ there exists $x_2 \in B_2 - B_1$ such that $B_1 - x_1 + x_2 \in \mathcal{B}$ and $B_2 - x_2 + x_1 \in \mathcal{B}$. The following theorem shows that this property can be strengthened even more.

Theorem 1.1 (Greene [30])**.** *Let B_1 and B_2 be two bases of matroid M. For every $X_1 \subseteq B_1$ there is a subset $X_2 \subseteq B_2$ such that both $B_1 - X_1 + X_2$ and $B_2 - X_2 + X_1$ are bases of M.*

A *circuit* of a matroid is an inclusionwise minimal dependent subset of S. A *cut* is an inclusionwise minimal subset of S that intersects every basis. The *closure* or *span* of a subset $X \subseteq S$ is $\text{span}(X) = \{x \in S : r(X + x) = r(X)\}$. An element $x \in S$ is *spanned* by X if $x \in \text{span}(X)$. A *spanning set* or a *generator* of the matroid is a set $X \subseteq S$ such that $\text{span}(X) = S$. A *closed set* or a *flat* is a set $X \subseteq S$ such that $\text{span}(X) = X$. A *hyperplane* is a closed set of rank $r(S) - 1$. A *loop* is an element that is non-independent on its own. Two non-loop elements $e, f \in S$ are *parallel* if $\{e, f\}$ is non-independent.

Adding a parallel copy of an element $s \in S$ results in a matroid $M' = (S', \mathcal{I}')$ on ground set $S' = S + s'$ where $\mathcal{I}' = \{X \subseteq S' : \text{either } X \in \mathcal{I}, \text{ or } s \notin X,\ s' \in X \text{ and } X - s' + s \in \mathcal{I}\}$. Given a matroid $M = (S, \mathcal{I})$, its *restriction* to a subset $S' \subseteq S$ is the matroid $M|_{S'} = (S', \mathcal{I}')$ where $\mathcal{I}' = \{I \in \mathcal{I} : I \subseteq S'\}$. By *contracting* a subset $Z \subsetneq S$ of the ground set of a matroid M, we get the matroid M/Z on $S - Z$ defined by the rank function $r'(X) = r(X \cup Z) - r(Z)$ $(X \subseteq S - Z)$. The *dual* of M is the matroid $M^* = (S, \mathcal{I}^*)$ where $\mathcal{I}^* = \{X \subseteq S : S - X \text{ contains a basis of } M\}$. The k*-truncation* of a matroid $M = (S, \mathcal{I})$ is a matroid $(S, \mathcal{I}_k)$ with $\mathcal{I}_k = \{X \in \mathcal{I} : |X| \leq k\}$. We denote the k-truncation of M by $(M)_k$. The *direct sum* $M_1 \oplus M_2$ of matroids $M_1 = (S_1, \mathcal{I}_1)$ and $M_2 = (S_2, \mathcal{I}_2)$ on disjoint ground sets is the matroid $M = (S_1 \cup S_2, \mathcal{I})$ whose independent sets are the disjoint unions of an independent set of M_1 and an independent set of M_2. The *sum* $M_1 + M_2$ of $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ on the same ground set is the matroid $M = (S, \mathcal{I})$ whose independent sets are the disjoint unions of an independent set of M_1 and an independent set of M_2. For a matroid $M = (S, \mathcal{I})$ the sum $\sum_{i=1}^{k} M$ is denoted by kM.

For a loopless[1] matroid $M = (S, \mathcal{I})$, let $\chi(M)$ denote the *colouring number* of M, that is, the minimum number of independent sets into which the ground set can be decomposed in M. We call a matroid *k-colourable* if $\chi(M) \leq k$. Analogously, for two matroids $M_1 = (S, \mathcal{I}_1)$, $M_2 = (S, \mathcal{I}_2)$ on the same ground set, let $\chi(M_1, M_2)$ denote the minimum number of common independent sets needed to cover the ground set.

1.2 Matroid intersection and union

A central result of combinatorial optimization is Edmonds' matroid intersection theorem.

Theorem 1.2 (Edmonds' matroid intersection theorem [17]). *Matroids $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ have a common independent set of size k if and only if*

$$r_1(X) + r_2(S - X) \geq k \quad \text{for all } X \subseteq S.$$

The following theorem can be derived from the matroid intersection theorem, and vice versa.

Theorem 1.3 (Edmonds' and Fulkerson's matroid union theorem [18]). *Let $M_1 = (S, \mathcal{I}_1)$, ..., $M_k = (S, \mathcal{I}_k)$ be matroids on the same ground set S. Then the rank function of $M_1 + \cdots + M_k$ is given by*

$$r_{M_1 + \cdots + M_k}(Z) = \min_{X \subseteq Z} \{|Z - X| + \sum_{i=1}^{k} r_i(X)\} \quad (Z \subseteq S).$$

As a corollary, we get a characterization of the partitionability of the ground set into k independent sets.

Corollary 1.4. *Let $M = (S, \mathcal{I})$ be a matroid with rank function r. Then S can be partitioned into k independent sets if and only if $|X| \leq k \cdot r(X)$ holds for every $X \subseteq S$.*

In other words, the colouring number of a matroid $M = (S, \mathcal{I})$ is

$$\chi(M) = \lceil \max\{|X|/r(X) : \emptyset \neq X \subseteq S\} \rceil.$$

Another corollary is a characterization of the existence of k disjoint bases of a matroid.

Corollary 1.5. *Let $M = (S, \mathcal{I})$ be a matroid with rank function r. Then M has k pairwise disjoint bases if and only if $|S - X| \geq k \cdot (r(S) - r(X))$ holds for every $X \subseteq S$.*

Edmonds and Fulkerson also provided polynomial algorithms for these results, i.e., to find a common independent set of two matroids with maximum size, and to find independent sets $I_j \in M_j$ such that $I_1 \cup \cdots \cup I_k \subseteq Z$ and $|I_1 \cup \cdots \cup I_k| = r_{M_1 + \cdots + M_k}(Z)$. In order to speak about matroid algorithms and their complexity, it should be made clear how the matroids are given. Throughout we assume that a matroid $M = (S, \mathcal{I})$ is given by a so-called *independence oracle*, which is an algorithm testing whether a subset of S is independent in M. A matroid can be given by other oracles as well, for example the *rank oracle* tells the rank of any subset of S. However, the independence and rank oracles are polynomially equivalent, as well as circuit-finding, spanning, port, strong basis and certain closure oracles [63, 33, 13].

[1] The ground set of a matroid containing a loop cannot be decomposed into independent sets. Therefore, every matroid considered in the thesis is assumed to be loopless without explicitly mentioning this. Nevertheless, parallel elements might exist.

1.3 Notations from graph theory

Some of the matroids considered in this thesis arise from graphs, hence we list the required graph theoretical definitions and notations.

For a graph $G = (V, E)$ and a subset $X \subseteq V$ of vertices, the set of edges spanned by X is denoted by $E[X]$, while the graph spanned by X is denoted by $G[X]$. Given a connected component K of G, a *cut* of K is a subset of edges in $E[K]$ whose deletion disconnects K. The component is *k-edge-connected* if the minimum size of a cut in K is at least k. The graphs obtained by deleting a subset $X \subseteq V$ of vertices or a subset $F \subseteq E$ of edges are denoted by $G - X$ and $G - F$, respectively. The degree of a vertex v with respect to $F \subseteq E$ is denoted by $d_F(v)$. The symmetric difference of two sets P, Q is denoted by $P \triangle Q = (P - Q) \cup (Q - P)$.

Let $G = (A, B; E)$ be a bipartite graph and $F \subseteq E$ be a subset of edges. For a set $X \subseteq A$, the set of neighbours of X with respect to F is denoted by $N_F(X)$, that is, $N_F(X) = \{b \in B :$ there exists an edge $ab \in F$ with $a \in X\}$. We will drop the subscript F when F is the whole edge set. We denote the set of vertices in X incident to edges in F by $X(F)$. A forest (or tree) $F \subseteq E$ is a *B_2-forest* (or *B_2-tree*, respectively) if $d_F(b) = 2$ for every $b \in B$. The existence of a B_2-forest was characterized by Lovász [52].

Theorem 1.6 (Lovász)**.** *Let $G = (A, B; E)$ be a bipartite graph. Then there exists a B_2-forest in G if and only if the strong Hall condition holds for every nonempty subset of B, that is,*

$$|N(X)| \geq |X| + 1 \quad \textit{for all } \emptyset \neq X \subseteq B.$$

1.4 Constructions of matroids

We list several well-known examples of matroids that will be used later.

For a set S, the matroid in which every subset of S is independent is called a *free matroid* and is denoted by M_S^{free}.

A *laminar matroid* is a matroid $M = (S, \mathcal{I}) = \{X \subseteq S : |X \cap S_i| \leq g_i$ for $i = 1, \dots, q\}$ for a laminar family $\{S_1, \dots, S_q\}$ of subsets of S and nonnegative integers $g_1, \dots, g_q$. (A family $\{S_1, \dots, S_q\}$ is called *laminar*, if it has now two properly intersecting members, that is, every two members are comparable or disjoint.) An important special case arises when $S = S_1 \cup \dots \cup S_q$ is a partition and $g_1 = \dots = g_q = 1$, such matroids are called *partition matroids*. (In the literature partition matroids are usually defined for every $g_1, \dots, g_q$, we make the restriction $g_1 = \dots = g_q = 1$ since all partition matroids considered in this thesis are of this special form.)

A matroid $M = (S, \mathcal{I})$ is called *linear* (or *representable*) if there exists a matrix A over a field $\mathbb{F}$ and a bijection between the columns of A and S, so that $X \subseteq S$ is independent in M if and only if the corresponding columns in A are linearly independent over the field $\mathbb{F}$. It is not difficult to verify that the class of linear matroids is closed under duality, taking direct sum (when the field $\mathbb{F}$ for linear representations is common), taking minors and taking k-truncation. Moreover, if we apply any of these operations for a matroid (or a pair of matroids) given by a linear representation over a field $\mathbb{F}$, then a linear representation of the resulting matroid can be determined by using only polynomially many operations over $\mathbb{F}$. (See e.g. [51] for k-truncation.).

For a graph $G = (V, E)$, the *graphic matroid* $M = (E, \mathcal{I})$ of G is defined on the edge set by considering a subset $F \subseteq E$ to be independent if it is a forest, that is, $\mathcal{I} = \{F \subseteq E :$ F does not contain a cycle$\}$. It is not difficult to verify that graphic matroids are linear over any field. Nash-Williams [60] gave a characterization for G being decomposable into k forests, or in other words, for the graphic matroid of G being k-colourable.

Theorem 1.7 (Nash-Williams)**.** *Given a graph $G = (V, E)$, the edge set can be decomposed into k forests if and only if $|E[X]| \leq k(|X| - 1)$ for every nonempty subset X of V.*

Given a bipartite graph $G = (S, T; E)$, a set $X \subseteq S$ is *matchable* if there is a matching of G covering X. The *transversal matroid* of G is the matroid on S whose independent sets are the matchable subsets of S. It is an easy exercise to show that the size of T can be chosen to be r where r denotes the rank of the matroid. By the Frobenius-Kőnig-Hall theorem [46, 31, 24] the rank of a subset $X \subseteq S$ in the transversal matroid is $r(X) = \min\{|X| - |Y| + |N(Y)| : Y \subseteq X\}$.

A generalization of transversal matroids can be obtained with the help of directed graphs. Given a directed graph $D = (V, A)$ and two sets $X, Y \subseteq V$, we say that X is *linked* to Y if $|X| = |Y|$ and there exists $|X|$ vertex-disjoint directed paths from X to Y. Let $S \subseteq V$ be a set of starting vertices and $T \subseteq V$ be a set of destination vertices. Then the family $\mathcal{I} = \{Y \subseteq T : \exists X \subseteq S \text{ s.t. } X \text{ is linked to } Y\}$ forms the independent sets of a matroid that is called a *gammoid*. The gammoid is a *strict gammoid* if $T = V$. That is, a gammoid is obtained by restricting a strict gammoid to a subset of its elements. Ingleton and Piff [37] showed that strict gammoids are exactly the duals of transversal matroids, hence every gammoid is the restriction of the dual of a transversal matroid.

A matroid $M = (S, \mathcal{I})$ of rank r is called *paving* if every set of size at most $r - 1$ is independent, or in other words, every circuit of the matroid has size at least r. We will use the following technical statement about paving matroids.

Theorem 1.8. *Let $r \geq 2$ be an integer and S a set of size at least r. Let $\mathcal{H} = \{H_1, \ldots, H_q\}$ be a (possibly empty) family of proper subsets of S in which every set H_i has at least r elements and the intersection of any two of them has at most $r - 2$ elements. Then the set system $\mathcal{B}_{\mathcal{H}} = \{X \subseteq S : |X| = r, X \not\subseteq H_i \text{ for } i = 1, \ldots, q\}$ forms the set of bases of a paving matroid. Moreover, every paving matroid can be obtained in this form.*

A paving matroid is called *sparse paving* if its dual is also paving. Note that $\mathcal{B}_{\mathcal{H}}$ forms the set of the bases of a sparse paving matroid if in addition to the previous requirements, we require that each member of $\mathcal{H}$ has size r. The importance of (sparse) paving matroids comes from their large number. It is known that there are double-exponentially many sparse paving matroids on a ground set S, and it is conjectured that the asymptotic fraction of matroids on n elements that are sparse paving tends to 1 as n tends to infinity [57]. A similar statement on the asymptotic ratio of the logarithms of the numbers of matroids and sparse paving matroids has been proven in [62].

Chapter 2

Complexity results

This chapter contains two sections of our joint work with Kristóf Bérczi [3].

2.1 Hardness in the independence oracle model

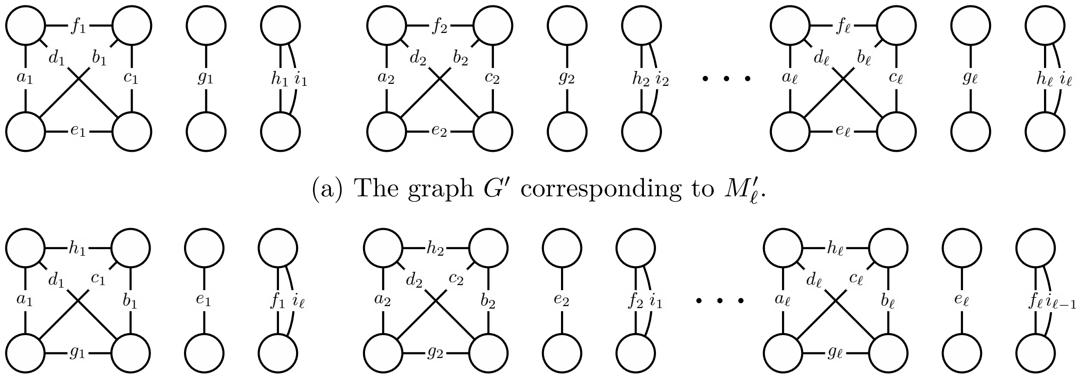
Given two matroids $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$, there are three different problems that can be asked: (A) Can S be partitioned into k common independent sets of M_1 and M_2? (B) Does S contain k disjoint common bases of M_1 and M_2? (C) Does S contain k disjoint common spanning sets of M_1 and M_2? These problems may seem to be closely related, and (A) and (B) are indeed in a strong connection, but (C) is actually substantially different from the others. We will concentrate on the following problem which is a special case of all three problems.

Definition 2.1. The PartitionIntoCommonBases problem is the following: Given matroids $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$, find a partition of S into common bases.

We show that PartitionIntoCommonBases is difficult under the rank oracle model. This immediately implies the all (A), (B) and (C) are intractable as well. We prove the difficulty of PartitionIntoCommonBases by a reduction from the so-called PartitionIntoModularBases problem.

Definition 2.2. Let $M = (S, \mathcal{I})$ be a matroid and let $\mathcal{P}$ be a partition of the ground set into nonempty subsets. Members of $\mathcal{P}$ are called *modules*, and a set $X \subseteq S$ is *modular* if it is the union of modules. In the special case when every module is a pair, modular sets are called *parity sets*.
The PartitionIntoModularBases problem is as follows: Given a matroid $M = (S, \mathcal{I})$ over a ground set S of size $2r(S)$ together with a partition $\mathcal{P}$ of S, find a partition of S into two modular bases. In the special case when every module is a pair, we refer to the problem as PartitionIntoParityBases.

The latter problem is the packing counterpart of the matroid parity problem which asks for a parity independent set of maximum size. This problem was introduced by Lawler [49] as a common generalization of graph matching and matroid intersection. Unfortunately, matroid parity is intractable for general matroids as it includes NP-hard problems, and requires an exponential number of queries if the matroid is given by an independence oracle [38, 53]. On the positive side, for linear matroids, Lovász developed a polynomial time algorithm [53] that is applicable if a linear representation is available. Although PartitionIntoParityBases seems to be closely related to matroid parity, the relationship between the two problems is unclear.



(a) The graph G' corresponding to M'_ℓ.

(b) The graph G'' corresponding to M''_ℓ.

Figure 2.1: The edge-labeled graphs defining M'_ℓ and M''_ℓ.

Theorem 2.3. *The* PartitionIntoParityBases *problem requires an exponential number of independence queries.*

Proof. Let S be a finite set of $4t$ elements and let $\mathcal{P}$ be an arbitrary partition of S into $2t$ pairs, forming the modules. Let $\mathcal{H} = \{X \subseteq S : |X| = 2t, X \text{ is a parity set}\}$. For a parity set X_0 with $|X_0| = 2t$, define $\mathcal{H}_0 = \mathcal{H} - \{X_0, S - X_0\}$. Both $\mathcal{H}$ and $\mathcal{H}_0$ satisfy the conditions of Theorem 1.8, hence $\mathcal{B}_\mathcal{H}$ and $\mathcal{B}_{\mathcal{H}_0}$ define two matroids M and M_0, respectively.

Clearly, the ground set cannot be partitioned into parity bases of M, while $X_0 \cup (S - X_0)$ is such a partition for M_0. For any sequence of independence queries which does not include X_0 or $S - X_0$, the result of those oracle calls are the same for M and M_0. That is, any sequence of queries which does not include at least one of the parity subsets X_0 or $S - X_0$ cannot distinguish between M and M_0, concluding the proof of the theorem. $\square$

We will use the following technical lemma to prove the difficulty of PartitionIntoCommonBases.

Lemma 2.4. *Let $\ell \in \mathbb{Z}_+$ and let S be a ground set of size 9ℓ. There exist two matroids M'_ℓ and M''_ℓ of rank 5ℓ satisfying the following conditions:*

(a) S can be partitioned into two common independent sets of M'_ℓ and M''_ℓ having sizes 5ℓ and 4ℓ;

(b) for every partition $S = S_1 \cup S_2$ into two common independent sets of M'_ℓ and M''_ℓ, we have $\{|S_1|, |S_2|\} = \{5\ell, 4\ell\}$, that is, one of the partition classes has size exactly 5ℓ while the other has size exactly 4ℓ.

Proof. Let $S = \bigcup_{j=1}^{\ell} W_j$ denote a ground set of size 9ℓ where $W_j = \{a_j, b_j, c_j, d_j, e_j, f_j, g_j, h_j, i_j\}$. Let M'_ℓ and M''_ℓ denote the graphic matroids defined by the edge-labeled graphs G' and G'' on Figures 2.1a and 2.1b, respectively. We first prove (a).

Claim 2.5. *S can be partitioned into two common independent sets of M'_ℓ and M''_ℓ having sizes 5ℓ and 4ℓ.*

Proof. It is not difficult to find a partition satisfying the conditions of the claim, for example, $S = S_1 \cup S_2$ where $S_1 = \bigcup_{j=1}^{\ell}\{d_j, e_j, f_j, g_j, h_j\}$ and $S_2 = \bigcup_{j=1}^{\ell}\{a_j, b_j, c_j, i_j\}$. $\square$

In order to verify (b), take an arbitrary partition $S = S_1 \cup S_2$ into common independent sets of M'_ℓ and M''_ℓ. Let $\hat{W}_j = W_j - i_j$.

Claim 2.6. *For each $j = 1, \ldots, \ell$, S_1 and S_2 partition $\hat{W}_j$ into common independent sets having sizes 5 and 3. Moreover, the elements e_j, f_j, g_j and h_j are contained in the same partition class.*

Proof. S_1 and S_2 necessarily partition the K_4 subgraphs spanned by $\hat{W}_j$ in G' and G'' into two paths of length 3, so $|S_k \cap \{a_j, b_j, c_j, d_j, e_j, f_j\}| = 3$ and $|S_k \cap \{a_j, b_j, c_j, d_j, g_j, h_j\}| = 3$ for $k = 1, 2$. This implies that either $|S_k \cap \{e_j, f_j\}| = |S_k \cap \{g_j, h_j\}| = 1$ for $k = 1, 2$, or e_j, f_j, g_j and h_j are contained in the same partition class.

In the former case, we may assume that $g_j \in S_1$ and $h_j \in S_2$. In order to partition the K_4 subgraph spanned by $\hat{W}_j$ in G'' into two paths of length 3, either $\{a_j, b_j\} \subseteq S_1$ and $\{c_j, d_j\} \subseteq S_2$ or $\{c_j, d_j\} \subseteq S_1$ and $\{a_j, b_j\} \subseteq S_2$ hold. However, these sets cannot be extended to two paths of length 3 in G', a contradiction.

Thus e_j, f_j, g_j and h_j are contained in the same partition class. Since $|S_k \cap \{a_j, b_j, c_j, d_j, e_j, f_j\}| = 3$ for $k = 1, 2$, the claim follows. □

Now we analyze how the presence of edges i_j affect the sizes of the partition classes. By Claim 2.6, we may assume that $\{e_1, f_1, g_1, h_1\} \subseteq S_1$, and so $i_\ell \in S_2$.

Claim 2.7. *$\{e_j, f_j, g_j, h_j\} \subseteq S_1$ and $i_j \in S_2$ for $j = 1, \ldots, \ell$.*

Proof. We prove by induction on j. By assumption, the claim holds for $j = 1$. Assume that the statement is true for j. As i_j is parallel to f_{j+1} in G'', $f_{j+1} \in S_1$. By Claim 2.6, $\{e_{j+1}, f_{j+1}, g_{j+1}, h_{j+1}\} \subseteq S_1$. As i_{j+1} is parallel to h_{j+1} in G', necessarily $i_{j+1} \in S_2$, proving the inductive step. □

Claims 2.6 and 2.7 imply that $|S_1| = 5\ell$ while $|S_2| = 4\ell$, concluding the proof of the lemma. □

It should be emphasized that, for our purposes, any pair of matroids satisfying the conditions of Lemma 2.4 would be suitable; we defined a specific pair, but there are several other choices that one could work with.

We are now in the position to prove the main result of this chapter.

Theorem 2.8. *The* PartitionIntoCommonBases *problem requires an exponential number of independence queries.*

Proof. We prove by reduction from PartitionIntoModularBases.[1] Let $M = (S, \mathcal{I})$ be a matroid together with a partition $\mathcal{P}$ of its ground set into modules. Recall that $|S| = 2r(S)$, that is, the goal is to partition the ground set into two modular bases.

We define two matroids as follows. For every set $P \in \mathcal{P}$, let $M'_P = (S_P, \mathcal{I}'_P)$ and $M''_P = (S_P, \mathcal{I}''_P)$ be copies of the matroids $M'_{|P|}$ and $M''_{|P|}$ provided by Lemma 2.4. We denote

$$S' = S \cup \left(\bigcup_{P \in \mathcal{P}} S_P \right).$$

[1]By Theorem 2.3, we could assume that every module has size 2. However, our construction in Section 2.2 for proving that the linear case is already difficult uses modules of larger sizes, hence we show reduction from the general version of PartitionIntoModularBases.

Note that $|S'| = 10|S|$, that is, the size of the new ground set is linear in that of the original. Let

$$M_1 = \left(M \oplus \left(\bigoplus_{P \in \mathcal{P}} M'_P \right) \right)_{\frac{|S'|}{2}}$$

$$M_2 = \bigoplus_{P \in \mathcal{P}} (M_P^{\mathit{free}} \oplus M''_P)_{5|P|}.$$

M_1 is defined as the $|S'|/2$-truncation of the direct sum of M and the matroids M'_P for $P \in \mathcal{P}$. For the other matroid, we first take the $5|P|$-truncation of the direct sum of M''_P and the free matroid M_P^{free} on P for each $P \in \mathcal{P}$, and then define M_2 as the direct sum of these matroids. We first determine the ranks of M_1 and M_2.

Claim 2.9. *Both M_1 and M_2 have rank $|S'|/2$.*

Proof. The rank of M_1 is clearly at most $|S'|/2$ as it is obtained by taking the $|S'|/2$-truncation of a matroid. Hence, it suffices to show that $M \oplus (\bigoplus_{P \in \mathcal{P}} M'_P)$ has an independent set of size at least $|S'|/2$. For each $P \in \mathcal{P}$, let B_P be a basis of M'_P. Then $\bigcup_{P \in \mathcal{P}} B_P$ is an independent set of M_1 having size $\sum_{P \in \mathcal{P}} 5|P| = 5|S| = |S'|/2$ as requested.

The rank of $(M_P^{\mathit{free}} \oplus M''_P)_{5|P|}$ is $5|P|$ for each $P \in \mathcal{P}$. This implies that the rank of M_2 is at most $\sum_{P \in \mathcal{P}} 5|P| = 5|S| = |S'|/2$. We get an independent set of that size by taking a basis B_P of M''_P for each $P \in \mathcal{P}$, and then taking their union $\bigcup_{P \in \mathcal{P}} B_P$. □

The main ingredient of the proof is the following.

Claim 2.10. *If $S' = B'_1 \cup B'_2$ is a partition of S' into two common bases of M_1 and M_2, then each module $P \in \mathcal{P}$ is contained completely either in B'_1 or in B'_2.*

Proof. For an arbitrary module P, let $I_1 = S_P \cap B'_1$ and $I_2 = S_P \cap B'_2$. Clearly, I_1 and I_2 are independent in both M'_P and M''_P. By Lemma 2.4, we may assume that $|I_1| = 4|P|$ and $|I_2| = 5|P|$. As the rank of $(M_P^{\mathit{free}} \oplus M''_P)_{5|P|}$ is $5|P|$, we get $P \subseteq B'_1$ as requested. □

The next claim concludes the proof of the theorem.

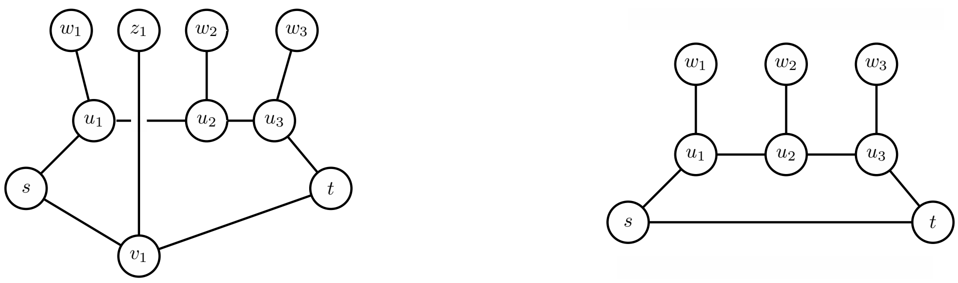
Claim 2.11. *S has a partition into two modular bases if and only if S' can be partitioned into two common bases of M_1 and M_2.*

Proof. For the forward direction, assume that there exists a partition $S' = B'_1 \cup B'_2$ of S' into two common bases of M_1 and M_2. By Claim 2.10, for every module $P \in \mathcal{P}$, the elements of P are all contained either in B_1 or in B_2. This implies that $B_1 = S \cap B'_1$ and $B_2 = S \cap B'_2$ are modular sets. By the definition of M_1, these sets are independent in M. As $|S| = 2r(S)$, B_1 and B_2 are modular bases of M.

To see the backward direction, let $S = B_1 \cup B_2$ be a partition of S into modular bases. For each $P \in \mathcal{P}$, let $I_P^1 \cup I_P^2$ be a partition of S_P into common independent sets of M'_P and M''_P having sizes $4|P|$ and $5|P|$, respectively. Recall that such a partition exists by Lemma 2.4. Then the sets

$$\begin{aligned} B'_1 &= B_1 \cup \{I_P^1 : P \subseteq B_1\} \cup \{I_P^2 : P \subseteq B_2\} \qquad \text{and} \\ B'_2 &= B_2 \cup \{I_P^1 : P \subseteq B_2\} \cup \{I_P^2 : P \subseteq B_1\} \end{aligned}$$

form common independent sets of M_1 and M_2 and partitions the ground set S'. By Claim 2.9, B'_1 and B'_2 are bases, concluding the proof of the claim. □



(a) Gadget $H[3, 1]$.　　(b) Gadget $H[3, 0]$.

Figure 2.2: Examples for variable gadgets.

The theorem follows by Claim 2.11. □

2.2 Hardness in the linear case

The aim of this section is to show that the PartitionIntoModularBases problem might be difficult to solve even when the matroid is given with a concise description, namely by an explicit linear representation over a field in which the field operations can be done efficiently. In order to do so, we consider the PartitionIntoModularBases problem for graphic matroids, which can be rephrased as follows.

Definition 2.12. The PartitionIntoModularTrees problem is the following: Given a graph $G = (V, E)$ and a partition $\mathcal{P}$ of its edge set, find a partition of E into two spanning trees consisting of partition classes.

Theorem 2.13. PartitionIntoModularTrees *is NP-complete.*

Proof. We prove by reduction from Not-All-Equal Satisfiability, abbreviated as NAE-SAT: Given a CNF formula, decide if there exists a truth assignment not setting all literals equally in any clause. It is known that NAE-SAT is NP-complete, see [64].[2]

Let $\Phi = (U, \mathcal{C})$ be an instance of NAE-SAT where $U = \{x_1, \ldots, x_n\}$ is the set of variables and $\mathcal{C} = \{C_1, \ldots, C_m\}$ is the set of clauses. We construct an undirected graph $G = (V, E)$ as follows. We may assume that no clause contains a variable and its negation simultaneously, as for such a clause every assignment has a true value and no assignment sets all literals equally.

First we construct the variable gadget. Let $H[p, q]$ denote an undirected graph on node set $\{s, t\} \cup \{u_i, w_i : i = 1, \ldots, p\} \cup \{v_j, z_j : j = 1, \ldots, q\}$ consisting of the two paths $su_1, u_1u_2, \ldots, u_pt$ and $sv_1, v_1v_2, \ldots, v_qt$, together with edges u_iw_i for $i = 1, \ldots, p$ and v_jz_j for $j = 1, \ldots, q$. If any of p or q is 0, then the corresponding path simplifies to a single edge st (see Figure 2.2).

We construct an undirected graph $G = (V, E)$ as follows. With each variable x_j, we associate a copy of $H[p_j, q_j]$ where the literal x_j occurs p_j times and the literal $\bar{x}_j$ occurs q_j times in the clauses. These components are connected together by identifying t^j with s^{j+1} for $j = 1, \ldots, n-1$. We apply the notational convention that in the gadget corresponding to a variable x_j, we add j as an upper index for all of the nodes. For a variable x_j, the ordering of the clauses naturally induces an ordering of the occurrences of x_j and $\bar{x}_j$. For every clause C_i, we do the following.

[2]In [65], Schmidt proved that NAE-SAT remains NP-complete when restricted to the class LCNF^3_+, that is, for monotone, linear and 3-regular formulas. Although the construction appearing in our reduction could be slightly simplified based on this observation, we stick to the case of NAE-SAT as it appears to be a more natural problem.

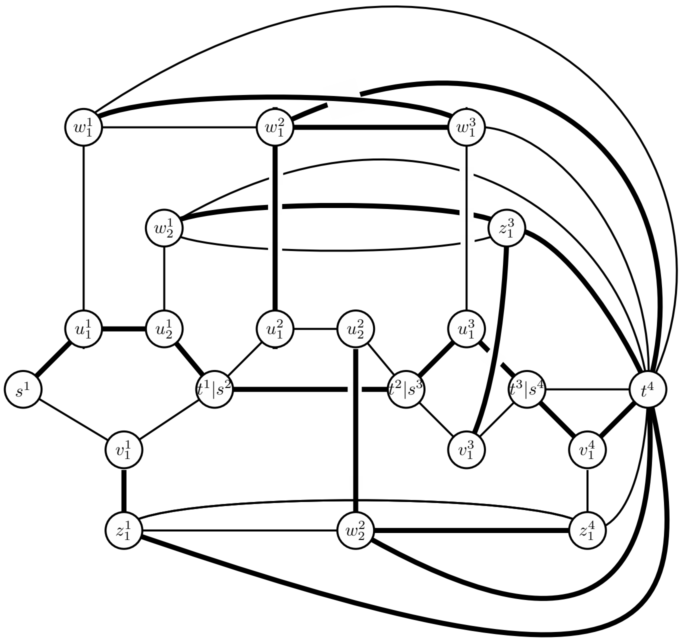


Figure 2.3: The graph corresponding to $\Phi = (x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_2 \vee \bar{x}_4)$. Thick and normal edges form modular spanning trees T_1 and T_2, respectively. Both the assignment $x_1 = x_3 = 1$, $x_2 = x_4 = 0$ corresponding to T_1 and the assignment $x_1 = x_3 = 0$, $x_2 = x_4 = 1$ corresponding to T_2 are solutions for NAE-SAT.

Assume that C_i involves variables $x_{j_1}, \dots, x_{j_\ell}$. Recall that no clause contains a variable and its negation simultaneously, hence ℓ is also the number of literals appearing in C_i. If C_i contains the literal x_{j_k} and this is the rth occurrence of the literal x_{j_k} with respect to the ordering of the clauses, let $y^i_{j_k} := w^{j_k}_r$. If C_i contains the literal $\bar{x}_{j_k}$ and this is the rth occurrence of the literal $\bar{x}_{j_k}$ with respect to the ordering of the clauses, let $y^i_{j_k} := z^{j_k}_r$. Then we add the edges of the cycle $y^i_{j_1}, \dots, y^i_{j_\ell}$ to the graph. Finally, we close the construction by adding edges $t^n w^j_k$ for $j = 1, \dots, n$, $k = 1, \dots, p_j$, and adding edges $t^n z^j_k$ for $j = 1, \dots, n$, $k = 1, \dots, q_j$ (see Figure 2.3). An easy computation shows that the number of edges is $|E| = 2|U| + 4\sum_{C \in \mathcal{C}} |C|$, while the number of nodes is $|V| = |U| + 2\sum_{C \in \mathcal{C}} |C| + 1$, that is, $|E| = 2|V| - 2$.

Now we partition the edge set of G into modules. For every variable x_j, if $p_j > 0$ then the path $P_j = \{s^j u^j_1, u^j_1 u^j_2, \dots, u^j_{p_j} t^j\}$ form a module. Similarly, if $q_j > 0$ then the path $N_j = \{s^j v^j_1, v^j_1 v^j_2, \dots, v^j_{q_j} t^j\}$ form a module. Finally, the pairs $M^j_k = \{u^j_k w^j_k, w^j_k t_n\}$ form modules of size two for $k = 1, \dots, p_j$, and similarly, the pairs $N^j_k = \{v^j_k z^j_k, z^j_k t_n\}$ form modules of size two for $k = 1, \dots, q_j$. All the remaining edges of G form modules consisting of a single element.

We claim that Φ has a truth assignment not setting all literals equally in any clause if and only if G can be partitioned into two modular spanning trees. For the forward direction, let $E = T_1 \cup T_2$ be a partition of E into two modular spanning trees. Then

$$\varphi(x_j) = \begin{cases} 1 & \text{if } p_j > 0 \text{ and } P_j \subseteq T_1, \text{ or } p_j = 0 \text{ and } s^j t^j \in T_1, \\ 0 & \text{otherwise.} \end{cases}$$

is a truth assignment not setting all literals equally in any clause. To verify this, observe that for a variable x_j, if $x_j = 1$ then $M^j_k \subseteq T_2$ for $k = 1, \dots, p_j$. This follows from the fact that T_2 has to span the node u_k^j and $\{u_k^j w_k^j, w_k^j t_n\}$ form a module for $k = 1, \ldots, p_j$. Similarly, if $x_j = 0$ then $N_k^j \subseteq T_2$ for $k = 1, \ldots, q_j$. Let now C_i be a clause involving variables $x_{j_1}, \ldots, x_{j_\ell}$ and recall the definition of $y_{j_1}^i, \ldots, y_{j_\ell}^i$. If all the literals in C_i has true value then, by the above observation, the cycle $y_{j_1}^i, \ldots, y_{j_\ell}^i$ has to lie completely in T_1, a contradiction. If all the literals in C_i has false value then, again by the above observation, the cycle $y_{j_1}^i, \ldots, y_{j_\ell}^i$ has to lie completely in T_2, a contradiction. A similar reasoning shows that T_2 also defines a truth assignment not setting all literals equally in any clause.

To see the backward direction, consider a truth assignment φ of Φ not setting all literals equally in any clause. We define the edges of T_1 as follows. For each variable x_j with $\varphi(x_j) = 1$, we add P_j and N_k^j for $k = 1, \ldots, q_j$ to T_1. For each variable x_j with $\varphi(x_j) = 0$, we add N_j and M_k^j for $k = 1, \ldots, p_j$ to T_1. Finally, for each clause C_i involving variables $x_{j_1}, \ldots, x_{j_\ell}$ do the following: for $k = 1, \ldots, \ell$, if C_i contains the literal x_{j_k} and $\varphi(x_{j_k}) = 1$ or C_i contains the literal $\bar{x}_{j_k}$ and $\varphi(x_{j_k}) = 0$, then add the edge $y_{j_k}^i y_{j_{k-1}}^i$ to T_1 (indices are meant in a cyclic order). By the assumption that φ does not set all literals equally in any clause, this last step will not form cycles in T_1. It is not difficult to see that both T_1 and its complement T_2 are modular spanning trees, thus concluding the proof of the theorem. □

As a consequence, we got the following.

Theorem 2.14. PartitionIntoCommonBases *includes NP-complete problems.*

Proof. The proof of Theorem 2.8 shows that PartitionIntoModularBases can be reduced to PartitionIntoCommonBases. As PartitionIntoModularTrees is a special case of the former problem, the theorem follows by Theorem 2.13. □

As the matroids M'_ℓ, M''_ℓ given in the proof of Lemma 2.4 are graphic, they are linear. If we apply the reduction described in the proof of Theorem 2.8 for a graphic matroid M, then the matroids M_1 and M_2 can be obtained from graphic matroids by using direct sums and truncations, hence they are linear as well and an explicit linear representation can be given in polynomial time [51]. This in turn implies that PartitionIntoCommonBases is difficult even when both matroids are given by explicit linear representations.

Harvey, Király and Lau [32] showed that the computational problem of common base packing reduces to the special case where one of the matroids is a partition matroid. Their construction involves the direct sum of M_1 and the matroid obtained from the dual of M_2 by replacing each element by k parallel elements. This means that if both M_1 and M_2 are linear, then the common base packing problem reduces to the special case where one of the matroids is a partition matroid and the other one is linear. Concluding these observations, we get the following.

Corollary 2.15. *The* PartitionIntoCommonBases *problem includes NP-complete problems even when $r(S) = 2|S|$, one of the matroids is a partition matroid and the other is a linear matroid given by an explicit linear representation.*

In [3] we also showed that PartitionIntoParityBases includes NP-complete problems even when restricted to transversal matroids given by a bipartite graph representation. However, one of the simplest cases when one of the matroids is a partition matroid while the other one is graphic remains open.

Chapter 3

Interesting special cases

Let $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ be partition matroids defined by partitions $S_1 \cup \cdots \cup S_q$ and $T_1 \cup \cdots \cup T_s$. Consider the bipartite graph $G = (A, B; E)$ where $A = \{a_1, \ldots, a_q\}$, $B = \{b_1, \ldots, b_s\}$ and the number of edges between a_i and b_j is $|S_i \cap T_j|$. Then an edge of G corresponds to an element of S, and a matching of G correspond to a common independent sets of M_1 and M_2. The classical result of Kőnig [46] states that the edge colouring number of a bipartite graph equals to its maximum degree. In matroidal terms this translates to $\chi(M_1, M_2) = \max\{\chi(M_1), \chi(M_2)\}$. In this chapter we present other pairs of matroids with this property.

3.1 Strongly base orderable matroids

The following definition requires a stronger basis exchange property than what Theorem 1.1 guarantees.

Definition 3.1. A matroid $M = (S, \mathcal{I})$ is *strongly base orderable* if for every two bases B_1, B_2 there is a bijection $f \colon B_1 \to B_2$ with the property that $B_1 - X + f(X)$ is a basis for any $X \subseteq B_1$.

If f satisfies this definition, then $B_2 - f(X) + X = B_1 - (B_1 - X) + f(B_1 - X)$ is also a basis for every $X \subseteq B_1$. It also follows that $f^{-1} : B_2 \to B_1$ is a bijection between B_2 and B_1 with the required property. Notice that f is necessarily identical on $B_1 \cap B_2$. Indeed, let $x \in B_1 \cap B_2$, then $B_2 - f(x) + x$ is a basis, in particular, $|B_2 - f(x) + x| = |B_2|$. Since x and $f(x)$ are elements of B_2, we get that $x = f(x)$.

One can easily check that the graphic matroid of K_4 is not strongly base orderable. However, we show that each gammoid is strongly base orderable.

Proposition 3.2. *The family of strongly base orderable matroids is closed for restrictions, contractions and duals.*

Proof. Let $M = (S, \mathcal{I})$ be a strongly base orderable matroid and $X \subseteq S$. First we prove that the contraction M/X is strongly base orderable. Let B_1, B_2 be bases of M/X. Let $Y \subseteq X$ be an independent set of M of size $r(X)$, then $B_1 \cup Y$ and $B_2 \cup Y$ are bases of M. Restricting the bijection $B_1 \cup Y \to B_2 \cup Y$ which the strongly base orderability of M guarantees, we get a bijection $B_1 \to B_2$ proving the strongly base orderability of M.

Now we prove that the dual M^* is strongly base orderable. Let B_1, B_2 be bases of M. Then $S - B_1$ and $S - B_2$ are bases of M, hence there exists a bijection $f : S - B_1 \to S - B_2$ such that $S - B_1 - X + f(X)$ is a basis of M for $X \subseteq S - B_1$. Thus, $B_1 + X - f(X) = B_2 - X + f(X)$ is a basis of M^* for $X \subseteq B_2 - B_1$, hence restricting f to $B_2 - B_1$ we get a bijection $B_2 - B_1 \to B_1 - B_2$ which proves that M^* is strongly base orderable.

Finally, the strongly base orderability of the restriction $M|_X$ follows from $M|_X = (M^*/X)^*$ and the two previous results. $\square$

One can show that the family of strongly base orderable matroids is also closed for the other matroid operations we defined in Section 1.1: adding parallel copies of an element, k-truncations, direct sums and sums.

The next theorem provides a large family of strongly base orderable matroids.

Theorem 3.3. *[9] Every gammoid is strongly base orderable.*

Proof. As gammoids are the restrictions of strict gammoids, and strict gammoids are the duals of transversal matroids, it is enough to show that each transversal matroid is strongly base orderable. Let $M = (S, \mathcal{I})$ be a transversal matroid and $G = (S, T; E)$ be a bipartite graph such that $|T| = r(S)$. Let $B_1, B_2 \subseteq S$ be two bases of M, then there exists matchings $F_1, F_2 \subseteq E$ covering B_1 and B_2, respectively. Since $|T| = r(S)$, both matchings cover T. Then $F_1 \cup F_2$ consists of alternating cycles and alternating paths, and each alternating path is between a vertex in $B_1 - B_2$ and a vertex in $B_2 - B_1$. These alternating paths define a bijection f between $B_1 - B_2$ and $B_2 - B_1$, and (letting f be identity on $B_1 \cap B_2$) it is not difficult to check that $B_1 - X + f(X)$ is a basis of M for every $X \subseteq B_1$. $\square$

It is worth mentioning that the class of gammoids is also closed for matroid operations such as taking restrictions, contractions, duals, direct sums and k-truncations. Nevertheless, it is not difficult to show that the converse of Theorem 3.3 does not hold, e.g. there exists a matroid on seven elements which is strongly base orderable but not a gammoid [36].

As partition matroids are strongly base orderable, the next result generalizes Kőnig's edge colouring theorem to a much larger class of matroids.

Theorem 3.4 (Davies and McDiarmid [16])**.** *Let $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ be k-colourable strongly base orderable matroids. Then S can be partitioned into k common independent sets.*

Proof [66]. Suppose for contradiction that S cannot be decomposed into k common independent sets. Let $S = X_1 \cup \cdots \cup X_k$ and $S = Y_1 \cup \cdots \cup Y_k$ be partitions of the ground set into independent sets of M_1 and M_2, respectively, such that $|X_1 \cap Y_1| + \cdots + |X_k \cap Y_k|$ is maximal. By our assumption, the two partitions are different, hence there are indices $i \neq j$ such that $X_i \cap Y_j$ is nonempty. Let $X_i \subseteq C_i$, $X_j \subseteq C_j$ be bases of M_1 and $Y_i \subseteq D_i$, $Y_j \subseteq D_j$ bases of M_2. Let $f\colon C_i \to C_j$ and $g\colon D_i \to D_j$ be exchange bijections guaranteed by the strongly base orderability of M_1 and M_2. Consider the graph $G = (C_i \cup C_j \cup D_i \cup D_j, E)$ with edge set

$$E = \{\{x, f(x)\} : x \in C_i \setminus C_j\} \cup \{\{y, g(y)\} : y \in D_i \setminus D_j\}.$$

E is the union of two matchings, hence G is bipartite. Let A and B denote the two colour classes of G, and consider the sets

$$X_i' = (X_i \cup X_j) \cap A, \quad X_j' = (X_i \cup X_j) \cap B,$$
$$Y_i' = (Y_i \cup Y_j) \cap A, \quad Y_j' = (Y_i \cup Y_j) \cap B.$$

We prove that $X_i', X_j' \in \mathcal{I}_1$ and $Y_i', Y_j' \in \mathcal{I}_2$. We have

$$X_i' \subseteq (C_i \cup C_j) \cap A = C_i - (C_i \setminus C_j) \cap B + (C_j \setminus C_i) \cap A.$$

For $x \in (C_i \setminus C_j) \cap B$, $\{x, f(x)\}$ is an edge of G, hence $f(x) \in (C_j \setminus C_i) \cap A$. Similarly, $f^{-1}(y) \in (C_i \setminus C_j) \cap A$ for $y \in (C_j \setminus C_i) \cap A$, hence $f((C_i \setminus C_j) \cap B) = (C_j \setminus C_i) \cap A$. Then $(C_i \cup C_j) \cap A = C_i - (C_i \setminus C_j) \cap B + f((C_i \setminus C_j) \cap B)$ is a basis of M_1, hence $X'_i \in \mathcal{I}_1$. $X'_j \in \mathcal{I}_1$ and $Y'_i, Y'_j \in \mathcal{I}_2$ follow by symmetry.

Since $X'_i \cup X'_j = X_i \cup X_j$ and $X'_i \cap X'_j = \emptyset$, we get a partition of S into independent sets of M_1 by replacing X_i with X'_i, and X_j with X'_j. Similarly, we get a new partition of S into independent sets of M_2 by replacing Y_i with Y'_i, and Y_j with Y'_j. The maximality of $|X_1 \cap Y_1| + \cdots + |X_k \cap Y_k|$ implies that

$$\begin{aligned}|X_i \cap Y_i| + |X_j \cap Y_j| &\geq |X'_i \cap Y'_i| + |X'_j \cap Y'_j| = |(X'_i \cup X'_j) \cap (Y'_i \cup Y'_j)| \\ &= |(X_i \cup Y_i) \cap (X_j \cup Y_j)| = |X_i \cap Y_i| + |X_j \cap Y_j| + |X_i \cap Y_j| + |X_j \cap Y_i|,\end{aligned}$$

using that $X'_i \cap Y'_j = \emptyset$ and $X'_j \cap Y'_i = \emptyset$. This contradicts that $X_i \cap Y_j$ is nonempty. □

The proof also shows a polynomial algorithm to find a required partition given a polynomial algorithm to find exchange bijections.

3.2 Matroids without $(k+1)$-spanned elements

We present the results of Kotlar and Ziv [47] about matroids without $(k+1)$-spanned elements.

Definition 3.5. An element $x \in S$ is $(k+1)$-*spanned* in a matroid $M = (S, \mathcal{I})$ if x has $k+1$ disjoint spanning sets.

As one of the sets can be $\{x\}$, we get that the following are equivalent: (i) x is $(k+1)$-spanned; (ii) there exists k disjoint subsets of $S - x$ that span x; (iii) there exists k circuits of M whose pairwise intersection is $\{x\}$.

Proposition 3.6. *If no element of a matroid $M = (S, \mathcal{I})$ is $(k+1)$-spanned, then M is k-colourable.*

Proof. We prove by induction on $|S|$. If $|S| \geq 2$, then choose an arbitrary element $x \in S$. $M|_{S-x}$ is k-colourable by the induction hypothesis, that is, there exist independent sets $X_1, \ldots, X_k$ such that $X_1 \cup \cdots \cup X_k = S - x$. As x is not $(k+1)$-spanned in M, x is not spanned by X_i for some i. We get a k-colouring of M by adding x to X_i. □

The converse of this statement is not true: every element of the graphic matroid of K_{2k} is $(2k-1)$-spanned and the matroid is k-colourable. However, it is easy to check that the converse holds for partition matroids, that is, a k-colourable partition matroid does not contain any $(k+1)$-spanned element. Therefore, the following conjecture generalizes Kőnig's edge colouring theorem.

Conjecture 3.7. *Let $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ be two matroids on S. If no element of S is $(k+1)$-spanned in either M_1 or M_2, then S can be partitioned into k common independent sets.*

In the following regular case the conjecture is a consequence of the theorems of Section 1.2.

Theorem 3.8. *Let $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ be matroids of common rank $r_1(S) = r_2(S) = r$ such that $|S| = kr$. If no element of S is $(k+1)$-spanned in either M_1 or M_2, the S can be partitioned into k common independent sets.*

Proof. We prove by induction on k, the statement is trivial for $k=1$. M_1 and M_2 are k-colourable by Proposition 3.6, hence assumption $|S|=kr$ implies that S is the union of k disjoint bases in either M_1 or M_2. Applying Corollary 1.5 to M_1 and Corollary 1.4 to M_2 we get that

$$r \leq r_1(S) + \frac{|S-X|}{k} \leq r_1(S) + r_2(S-X) \quad (X \subseteq S),$$

hence M_1 and M_2 have a common basis B by Theorem 1.2. B spans every element of $S-B$, hence the restriction of matroids M_1 and M_2 to $S-B$ does not contain any k-spanned element. By the induction hypothesis $S-B$ can be partitioned into $k-1$ common bases of M_1 and M_2, thus adding B to the partition we get a partition of S into k common bases. □

Kotlar and Ziv also proved their conjecture for the special case $k=2$.

Theorem 3.9. *Let $M_1=(S,\mathcal{I}_1)$ and $M_2=(S,\mathcal{I}_2)$ be two matroids on S. If no element of S is 3-spanned in either M_1 or M_2, then S can be partitioned into 2 common independent sets.*

Proof. We will rely on the following lemma.

Lemma 3.10. *Let $A,B \subseteq S$ be disjoint common independent sets of M_1 and M_2 such that $|A\cup B|$ is maximal. Then for every $x \notin A\cup B$*

$$A+x \in \mathcal{I}_1 \setminus \mathcal{I}_2,\ B+x \in \mathcal{I}_2 \setminus \mathcal{I}_1 \quad \textit{or} \quad A+x \in \mathcal{I}_2 \setminus \mathcal{I}_1,\ B+x \in \mathcal{I}_1 \setminus \mathcal{I}_2.$$

Proof. As $|A\cup B|$ is maximal, $A+x, B+x \notin \mathcal{I}_1\cap\mathcal{I}_2$ as otherwise $A+x$ and B, or A and $B+x$ would be common independent sets. Suppose that $A+x \notin \mathcal{I}_j$. Then A spans x in M_j, and since x is not 3-spanned, we get that B does not span x in M_j, that is, $B+x\in\mathcal{I}_j$. As $B+x \notin \mathcal{I}_1\cap\mathcal{I}_2$, we get that $A+x\in\mathcal{I}_1\cup\mathcal{I}_2$. Thus, we may assume that $A+x\in\mathcal{I}_1\setminus\mathcal{I}_2$, then $B+x\in\mathcal{I}_2$ by the previous argument. Since $B\notin\mathcal{I}_1\cap\mathcal{I}_2$, we get that $B+x\in\mathcal{I}_2\setminus\mathcal{I}_1$. □

Let A and B as in the lemma, and suppose for contradiction that $A\cup B\neq S$. Let $x\in S\setminus(A\cup B)$, then we may assume by the lemma that $A+x\in\mathcal{I}_1\setminus\mathcal{I}_2$ and $B+x\in\mathcal{I}_2\setminus\mathcal{I}_1$. Let $a_1\in A$ be an element such that $A+x-a_1\in\mathcal{I}_2$. Applying the lemma to $A+x-a_1, B\in\mathcal{I}_1\cap\mathcal{I}_2$ and the element a_1, we get that $B+a_1\in\mathcal{I}_2\setminus\mathcal{I}_1$ (using that $A+x\in\mathcal{I}_1\setminus\mathcal{I}_2$). We will keep constructing different elements $a_1,\dots,a_n\in A$ and $b_1,\dots,b_{n-1}\in B$ such that the followings hold. This will contradict the finiteness of A and B.

$$\begin{aligned}
A+x-\{a_1,\dots,a_{n-1},a_n\}&+\{b_1,\dots,b_{n-1}\}\in\mathcal{I}_1\cap\mathcal{I}_2\\
A+x-\{a_1,\dots,a_{n-1}\}\quad&+\{b_1,\dots,b_{n-1}\}\in\mathcal{I}_1\setminus\mathcal{I}_2\\
B+\{a_1,\dots,a_{n-1}\}\quad&-\{b_1,\dots,b_{n-1}\}\in\mathcal{I}_1\cap\mathcal{I}_2\\
B+\{a_1,\dots,a_{n-1},a_n\}&-\{b_1,\dots,b_{n-1}\}\in\mathcal{I}_2\setminus\mathcal{I}_1
\end{aligned}$$

For $n=1$, the a_1 above satisfies these conditions (we define $\{a_1,\dots,a_0\}$ and $\{b_1,\dots,b_0\}$ to be empty). Suppose that we have $a_1,\dots,a_n$ and $b_1,\dots,b_{n-1}$ with the required properties, we prove the existence of b_n and a_{n+1}. Let $A'=A-\{a_1,\dots,a_{n-1}\}+\{b_1,\dots,b_{n-1}\}$ and $B'=B+\{a_1,\dots,a_{n-1}\}-\{b_1,\dots,b_{n-1}\}$. As $B'\in\mathcal{I}_1$ and $B'+a_n\notin\mathcal{I}_1$, the latter set contains exactly one circuit C_1 of M_1. Since $\{a_1,\dots,a_n\}\subseteq A$ is independent in M_1, circuit C_1 intersects B. Let $b_n\in C_1\cap B$, then $B'+a_n-b_n\in\mathcal{I}_1\cap\mathcal{I}_2$. Applying the lemma to $A'+x-a_n, B'+a_n-b_n\in\mathcal{I}_1\cap\mathcal{I}_2$ and the element b_n, we get that $A'+x-a_n+b_n\in\mathcal{I}_1\setminus\mathcal{I}_2$ (using that $B'+a_n\in\mathcal{I}_2\setminus\mathcal{I}_1$). As $A'+x-a_n\in\mathcal{I}_2$ and $A'+x-a_n+b_n\notin\mathcal{I}_2$, the latter set contains exactly one circuit C_2 of M_2. Since $\{b_1,\dots,b_n\}+x\subseteq B+x$ is independent in M_2, circuit C_2 intersects A. Let $a_{n+1}\in C_2\cap A$, then $A'+x-a_n+b_n-a_{n+1}\in\mathcal{I}_1\cap\mathcal{I}_2$. Applying the lemma to $A'+x-a_n+b_n-a_{n+1}, B'+a_n-b_n$

and the element a_{n+1}, we get that $B'+a_n-b_n+b_{n+1} \in \mathcal{I}_2 \setminus \mathcal{I}_1$ (using that $A'+x-a_n+b_n \in \mathcal{I}_1 \setminus \mathcal{I}_2$). This proves that $a_1, \ldots, a_{n+1}$ and $b_1, \ldots, b_n$ satisfies the required conditions, concluding the proof of the theorem. □

Note that the proofs of these two theorems also provide polynomial algorithms for partitioning the ground set into k common independent sets.

Recently, Takazawa and Yokoi [71] proposed a generalized-polymatroid approach, which yields unified proofs for Theorems 3.8, 3.9 and also Theorem 3.4 in case of laminar matroids. They also showed that if $M_1 = (S, \mathcal{I}_1)$ satisfies the conditions of either Theorem 3.8 or Theorem 3.9, and $M_2 = (S, \mathcal{I}_2)$ is a k-colourable laminar matroid, then S can be partitioned into k common independent sets of M_1 and M_2. Later, Fujishige, Takazawa and Yokoi [25] strengthened these results and proved that under the same assumptions there exists a nearly uniform partition of the ground set into k common independent sets, that is, a partition such that difference of the cardinalities of any two partition classes is at most 1.

3.3 Open problems

We end this chapter with a list of some well-studied conjectures that can be formulated as $\chi(M_1, M_2) = \max\{\chi(M_1), \chi(M_2)\}$ for specific pairs of matroids M_1, M_2.

3.3.1 Rota's basis conjecture

Rota made the following conjecture in 1989, and published it with Huang in 1994 [35].

Conjecture 3.11. *If M is a matroid of rank n whose ground set can be partitioned into n disjoint bases $B_1, \ldots, B_n$, then it is possible to rearrange the elements of these bases into an $n \times n$ matrix in such a way that the rows are exactly the given bases, and the columns are also bases of M.*

If N denotes the partition matroid defined by the partition $B_1 \cup \cdots \cup B_n$, then M and N are n-colourable matroids and the conjecture states that the ground set can be partitioned into n common bases. As partition matroids are strongly base orderable, Theorem 3.4 implies that the conjecture holds if M is strongly base orderable as well. Geelen and Humphries [28] proved the conjecture for paving matroids. For other partial results see e.g. [29, 11, 32, 10].

3.3.2 Woodall's conjecture on packing dijoins

A *directed cut* of a directed graph $D = (V, A)$ is the set of arcs entering a set $X \subseteq V$ with out-degree 0. A *dijoin* is a set of arcs whose contraction results in a strongly connected digraph, or equivalently, a set of arcs intersecting each directed cut. Woodall [77] made the following conjecture.

Conjecture 3.12. *In a directed graph the maximum number of pairwise disjoint dijoins equals the minimum size of a directed cut.*

The conjecture was known to be true for $k = 2$, for source-sink connected digraphs by Schrijver [67] and independently by Feofiloff and Younger [21], for series-parallel digraphs by Lee and Wakabayashi [50]. Recently Mészáros [58] proved that if k is a prime power, then the conjecture holds if the underlying undirected graph is $(k-1, 1)$-partition-connected. Note that the dual problem is the theorem of Lucchesi and Younger [56]: the maximum number of pairwise disjoint directed cuts equals the minimum size of a dijoin.

Frank and Tardos [23] observed that Woodall's conjecture can be formulated as packing common bases of two matroids. Let $D = (V, A)$ be a directed graph and $k \geq 2$ an integer. For every arc $e \in A$ let $H(e)$ and $T(e)$ be disjoint sets of size 1 and $k-1$, respectively, and for a node $v \in V$ let $S(v) = \bigcup\{H(e) : e = uv \in A\} \cup \bigcup\{T(e) : e = vw \in A\}$. For a subset $X \subseteq V$ we use the notation $S(X) = \bigcup_{v \in X} S(v)$, and we set $S = S(V)$. Let $M_1 = (S, \mathcal{I}_1)$ denote the partition matroid with partition classes $H(e) \cup T(e)$ for $e \in A$. The matroid $M_2 = (S, \mathcal{I}_2)$ is defined by the following set of bases:

$$\{B \subseteq S : |B| = |A|,\ |B \cap S(X)| \geq i(X) + 1 \text{ for every } X \subseteq V \text{ with out-degree } 0\},$$

where $i(X)$ denotes the number of edges induced by X in D. Frank and Tardos showed that this family forms the set of bases of a matroid indeed.

For a basis B of the partition matroid M_1 let us consider the subset of arcs $F = \{e \in A : H(e) \subseteq B\}$. Notice that B is a basis of M_2 if and only if F is a dijoin. Indeed, if B is a common basis of M_1 and M_2, then $|B \cap S(X)| \geq i(X) + 1$ for every set $X \subseteq V$ with out-degree 0. Then $H(e) \subseteq B$ for at least one arc e entering X, thus $e \in F$. This proves that F intersects every directed cut of D, that is, F is a dijoin. The reverse direction is similar. (See Figure 3.1 for an illustration.)

It is not difficult to see that the previous observation implies that D has k pairwise disjoint dijoins if and only if M_1 and M_2 have k pairwise disjoint common bases. (Note that $|S| = k|A| = kr_{M_1}(S) = kr_{M_2}(S)$, hence this is equivalent to saying that S can be partitioned into k common independent sets of M_1 and M_2.) Thus, Conjecture 3.12 states that M_1 and M_2 have k pairwise disjoint common bases where k denotes the minimum size of a directed cut of D. It is clear that S can be partitioned into k bases of M_1, and it can be shown using the base polyhedron of M_2 given by Frank and Tardos [23] that S can be partitioned into k bases of M_2 if and only if k is not less than the minimum size of a directed cut [32]. This means that Woodall's conjecture can be formulated as $\chi(M_1, M_2) = \chi(M_1) = \chi(M_2)$.

3.3.3 Equitability of matroids

The following conjecture would follow from several well-studied conjectures. [19] calls matroids with the following property *equitable*.

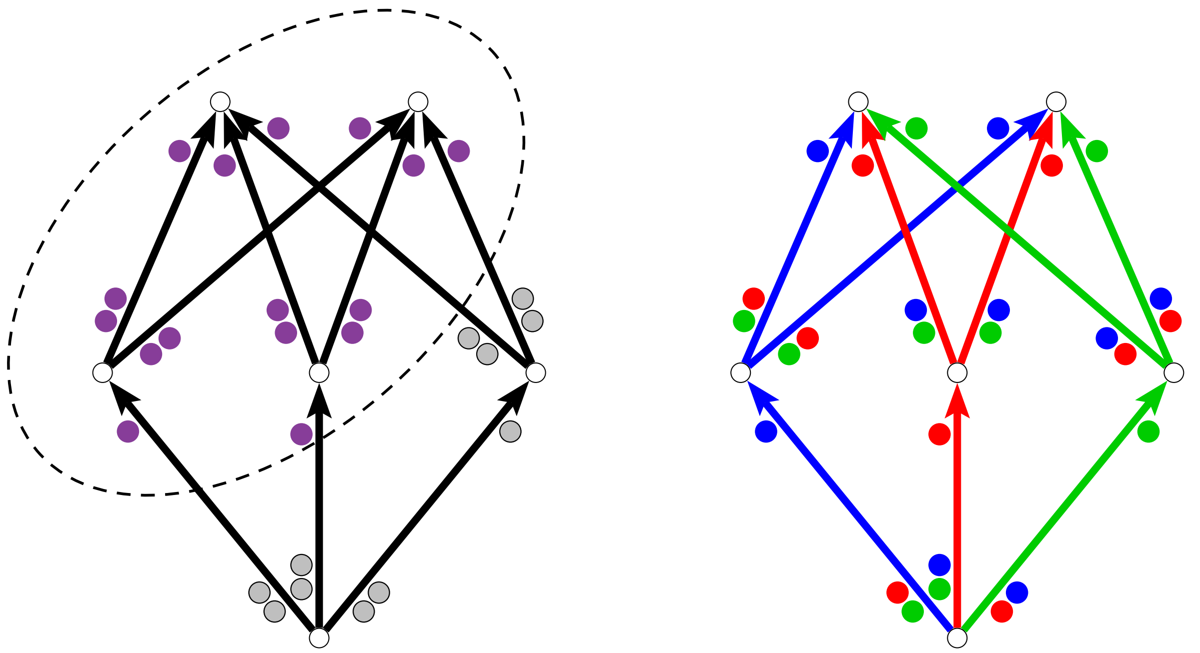
Conjecture 3.13. *Suppose that the ground set of a matroid $M = (S, \mathcal{I})$ can be partitioned into two bases. Then for any set $X \subseteq S$ the ground set can be partitioned into bases B_1 and B_2 such that*

$$\lfloor |X|/2 \rfloor \leq |B_i \cap X| \leq \lceil |X|/2 \rceil \quad (i = 1, 2).$$

Let N denote the partition matroid $\left(M_X^{\textit{free}}\right)_{\lceil |X|/2 \rceil} \oplus M_{X-S}^{\textit{free}}$. (Here the original definition of partition matroids is used where upper bounds different from 1 are allowed on partition classes.) N is clearly a 2-colourable matroid, and Conjecture 3.13 states that $\chi(M, N) = 2$ whenever $\chi(M) = 2$ and $|S| = 2r_M(S)$. Fekete and Szabó [20] showed that the conjecture holds for graphic matroids.

A much general conjecture is due to White [74]. Let $B_1, \ldots, B_m$ be bases of a matroid $M = (S, \mathcal{I})$. If $1 \leq i < j \leq m$, $x \in B_i - B_j$ and $y \in B_j - B_i$, then we say that the basis sequence

$$(B_1, \ldots, B_{i-1}, B_i - x + y, B_{i+1}, \ldots, B_{j-1}, B_j - y + x, B_{j+1}, \ldots, B_m)$$



(a) An example of a set X with out-degree 0. The elements of $S(X)$ are the coloured ones.

(b) The three colours show the three disjoint dijoins and the three disjoint common bases of the two matroids.

Figure 3.1: An example of a directed graph where the minimum size of a directed cut is $k = 3$. The ground set of the associated matroids is represented by circles next to the arcs.

is obtained from $(B_1, \dots, B_m)$ by a *symmetric exchange*. The conjecture states that for any two basis sequences $(B_1, \dots, B_m)$ and $(B'_1, \dots, B'_m)$ such that $|\{i : s \in B_i\}| = |\{i : s \in B'_i\}|$ holds for every element $s \in S$, $(B'_1, \dots, B'_m)$ can be obtained from $(B_1, \dots, B_m)$ by a series of symmetric exchanges. Bonin [8] proved that the conjecture holds for sparse paving matroids, and observed that it follows from the results of Blasiak [7] and Kashiwabara [41] that the conjecture holds for graphic matroids, matroids of rank at most three, and also the duals of these. Conjecture 3.13 would follow from the $m = 2$ case of White's conjecture. Indeed, let $S = B'_1 \cup B'_2$ denote any partition of S into two bases and suppose that (B'_2, B'_1) can be obtained from (B'_1, B'_2) by a series of symmetric exchanges. Consider $|B_1 \cap X|$ for members (B_1, B_2) of the series. This sequence starts with $|B'_1 \cap X|$, ends with $|B'_2 \cap X| = |X| - |B'_1 \cap X|$ and the difference of adjacent members is at most one, hence there is a basis pair (B_1, B_2) such that $\lfloor |X|/2 \rfloor \le |B_1 \cap X| \le \lceil |X|/2 \rceil$.

Another much general conjecture than Conjecture 3.13 is about cyclically orderable matroids. A matroid $M = (S, \mathcal{I})$ is called *cyclically orderable* if S has a cyclic permutation such that all sets of $r(S)$ cyclically consecutive elements are bases of M. Kajitani, Ueno and Miyano [40] conjectured that M is cyclically orderable if and only if $|X|/r(X) \le |S|/r(S)$ holds for $X \subseteq S$. Van den Heuvel and Thomassé [34] proved the conjecture for the case when $(|S|, r(S)) = 1$. Bonin [8] proved the conjecture for sparse paving matroids. The inequalities $|X|/r(X) \le |S|/r(S)$ clearly hold if $|S| = kr(S)$, that is, a special case of the conjecture states that every matroid is cyclically orderable whose ground set can be partitioned into k bases. Notice that Conjecture 3.13 would easily follow from the $k = 2$ case. Indeed, let $M = (S, \mathcal{I})$ be a matroid of rank r such that its ground set can be partitioned into 2 bases. Suppose that M is cyclically orderable and let $B_1, \dots, B_{2r}$ denote the sets of r cyclically consecutive elements (each of which is a basis of M). Since $\frac{1}{2r} \sum_{i=1}^{2r} |X \cap B_i| = |X| r / 2r = |X|/2$ and

$||X \cap B_i| - |X \cap B_{i+1}|| \leq 1$ for $i = 1, \ldots, 2r$, it follows that $\lfloor |X|/2 \rfloor \leq |B_i \cap X| \leq \lceil |X|/2 \rceil$ for some i. Thus, B_i and $S - B_i = B_{i+r}$ satisfy the requirements of Conjecture 3.13.

A strengthening of the previous $k = 2$, $|S| = 2r(S)$ case of the conjecture about cyclically orderable matroids is the following: for any two disjoint bases B_1, B_2 of a matroid there is a cyclic permutation $(b_1, \ldots, b_{2r})$ of the elements of $B_1 \cup B_2$ such that each cyclically consecutive r elements form a base of M, $B_1 = \{b_1, \ldots, b_r\}$ and $B_2 = \{b_{r+1}, \ldots, b_{2r}\}$ [26, 12]. Cordovil and Morerira [12] proved the conjecture for graphic matroids and Bonin [8] showed it for sparse paving matroids. The conjecture holds for strongly base orderable matroids as well: if $f\colon B_1 \to B_2$ is a bijection such that $B_1 - X + f(X)$ is a base for $X \subseteq B_1$, then $(b_1, \ldots, b_r, f(b_1), \ldots, f(b_r))$ is a good cyclic ordering for any ordering $(b_1, \ldots, b_r)$ of the elements of B_1.

Chapter 4

Upper bound on common independent set cover

In this chapter we present the results of Aharoni and Berger [1]. We prove Theorem 4.20, a generalization of the nontrivial direction of Edmond's Theorem 1.2. As a consequence, we obtain the upper bound $\chi(M_1, M_2) \leq 2\max\{\chi(M_1), \chi(M_2)\}$ for any pair of matroids M_1, M_2 on the same ground set. The methods used are mostly topological, thus we begin with a detailed overview of the required topological concepts.

4.1 Topological tools

4.1.1 Simplicial complexes

We will use more general set systems than the independent sets of a matroid: the following definition only requires property (I2).

Definition 4.1. A *simplicial complex* (or plainly *complex*) $\mathcal{C}$ is a hereditary family of finite sets, that is, $Y \subseteq X \in \mathcal{C}$ implies $Y \in \mathcal{C}$. The *vertex set* of $\mathcal{C}$ (denoted by $V(\mathcal{C})$) is the union of all members of $\mathcal{C}$.

In what follows we will associate a topological space to any complex.

Definition 4.2. An *(n-dimensional) simplex* is the convex hull of $n+1$ affinely independent points (called *vertices*) in some space $\mathbb{R}^d$ $(d \geq n)$. A *face* of the simplex is the convex hull of a (possibly empty) subset of its vertices.

A *geometric simplicial complex* $\mathcal{G}$ is a family of simplices (in some $\mathbb{R}^d$) such that (1) the face of any simplex of $\mathcal{G}$ is in $\mathcal{G}$, and (2) the intersection of two simplices of $\mathcal{G}$ is a common face of them. We write

$$||\mathcal{G}|| = \bigcup_{\Delta \in \mathcal{G}} \Delta \subseteq \mathbb{R}^d.$$

It is clear that the vertex sets of simplices of a geometric simplicial complex form a complex. Conversely, for any complex $\mathcal{C}$ we construct a geometric simplicial complex $\mathcal{G}$.

Definition 4.3. The *geometric representation* of a complex $\mathcal{C}$ is an injective mapping $p\colon V(\mathcal{C}) \to \mathbb{R}^d$ such that for every $X, Y \in \mathcal{C}$

$$\operatorname{conv}(p(X)) \cap \operatorname{conv}(p(Y)) = \operatorname{conv}(p(X \cap Y)), \tag{4.1}$$

where $\operatorname{conv}(S)$ denotes the convex hull of $S \subseteq \mathbb{R}^d$. We often write $p_v = p(v)$ for $v \in V(\mathcal{C})$.

It is clear, that every complex $\mathcal{C}$ has a geometric representation: let $d = |V(\mathcal{C})| - 1$ and choose p_v ($v \in V(\mathcal{C})$) to be affinely independent.

It is not difficult to show that the injectivity of p and (4.1) implies that for $X \in \mathcal{C}$ the points of $p(X)$ are affinely independent, thus $\mathrm{conv}(p(X))$ is an $(|X| - 1)$-dimensional simplex. We write

$$||X|| = \mathrm{conv}(p(X)) \ (X \in \mathcal{C}),$$

then (4.1) implies that $\{||X|| : X \in \mathcal{C}\}$ is a geometric simplicial complex. We define

$$||\mathcal{C}|| = \bigcup_{X \in \mathcal{C}} \mathrm{conv}(p(X)) \subseteq \mathbb{R}^d.$$

Clearly, $||\mathcal{C}||$ depends on the chosen geometric representation of $\mathcal{C}$. However, we show that $||\mathcal{C}||$ as a topological space (with the subspace topology from $\mathbb{R}^d$) is unique up to homeomorphism.

Definition 4.4. Let $\mathcal{A}$ and $\mathcal{B}$ be complexes and $f : V(\mathcal{A}) \to V(\mathcal{B})$ a mapping. f is *simplicial* if $f(A) \in \mathcal{B}$ for every $A \in \mathcal{A}$. A bijective simplicial mapping whose inverse is also a simplicial is called an *isomorphism*.

Proposition 4.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be complexes with geometric representations p and q, respectively. If $f : V(\mathcal{A}) \to V(\mathcal{B})$ is simplicial, then*

$$||f|| : ||\mathcal{A}|| \to ||\mathcal{B}||, \quad \sum_{i=1}^{k} \alpha_i p_{v_i} \mapsto \sum_{i=1}^{k} \alpha_i q_{f(v_i)}$$

is a well-defined continuous map where $\{v_1, \ldots, v_k\} \in \mathcal{A}$, $0 \le \alpha_1, \ldots, \alpha_k$ and $\sum_{i=1}^{k} \alpha_i = 1$. Moreover, if f is an isomorphism, then $||f||$ is a homeomorphism.

Proof. As f is simplicial, $\{f(v_1), \ldots, f(v_k)\} \in \mathcal{B}$ for $\{v_1, \ldots, v_k\} \in \mathcal{A}$, hence $||f|| \left(\sum_{i=1}^{k} \alpha_i p_{v_i}\right) \in ||\mathcal{B}||$. Thus $||f||$ is well-defined, and it is also clear that $||f||$ is continuous, as it is continuous on each simplex. If f is an isomorphism, then it is not difficult to check that $||\mathcal{B}|| \to ||\mathcal{A}||$, $\sum_{i=1}^{k} \alpha_i q_{v_i} \mapsto \sum_{i=1}^{k} \alpha_i p_{f^{-1}(v_i)}$ is the inverse of $||f||$ and it is continuous, hence $||f||$ is a homeomorphism. □

Corollary 4.6. *For a complex $\mathcal{C}$, $||\mathcal{C}||$ is unique up to homeomorphism.*

Proof. $||\,\mathrm{id}_{V(\mathcal{C})}\,||$ is a homeomorphism between any two spaces obtained from geometric representations of $\mathcal{C}$. □

It will be easier to formulate some results using the following definition.

Definition 4.7. Let $\mathcal{C}$ be a complex with geometric representation p. The *support* of $x \in ||\mathcal{C}||$ is the smallest set $\mathrm{supp}_{\mathcal{C}}(x) = \{v_1, \ldots, v_k\} \in \mathcal{C}$ such that $x \in \mathrm{conv}\{p_{v_1}, \ldots, p_{v_k}\}$.

4.1.2 Barycentric subdivisions

Definition 4.8. The *barycentric subdivision* of a complex $\mathcal{C}$ is the complex

$$\beta(\mathcal{C}) = \{\{S_1, \ldots, S_k\} : S_1, \ldots, S_k \in \mathcal{C}, \emptyset \neq S_1 \subsetneq S_2 \subsetneq \cdots \subsetneq S_k, k \ge 0\}.$$

Clearly, $\beta(\mathcal{C})$ is a complex and $V(\beta(\mathcal{C})) = \mathcal{C} \setminus \{\emptyset\}$. The following statement shows the geometric meaning of this construction.

Proposition 4.9. *Let $p\colon V(\mathcal{C}) \to \mathbb{R}^d$ be a geometric representation of a complex $\mathcal{C}$. Then*

$$q\colon V(\beta(\mathcal{C})) \to \mathbb{R}^d,\ S \mapsto \frac{1}{|S|}\sum_{v\in S} p_v$$

is a geometric representation of $\beta(\mathcal{C})$ such that $||\mathcal{C}|| = ||\beta(\mathcal{C})||$. Moreover, for $x \in ||\mathcal{C}||$

$$\operatorname{supp}_{\beta(\mathcal{C})}(x) = \{S_1, \dots, S_k\} \text{ for some } S_1, \dots, S_k \subseteq \operatorname{supp}_{\mathcal{C}}(x).$$

Proof [76]. We prove by induction on $|\mathcal{C}|$. If $\mathcal{C}$ consists of only singletons (and the empty set), then the statements are trivial. Otherwise, let $S \in \mathcal{C}$ be a set of the complex such that $|S|$ is maximal (then $|S| \geq 2$ by our assumption). Let Δ denote the simplex $\operatorname{conv}(p(S))$, $\partial\Delta$ its boundary and $\operatorname{int}\Delta$ its interior. We will rely on the following basic geometric fact:

$$\forall x \in \operatorname{int}\Delta, x \neq q_S\ \exists! t \in (0,1), y \in \partial\Delta:\ x = t\cdot q_S + (1-t)\cdot y. \tag{4.2}$$

As $|S|$ is maximal, $\mathcal{C}' = \mathcal{C} \setminus \{S\}$ is a complex as well, hence $q|_{\mathcal{C}'}$ and $\beta(\mathcal{C}')$ satisfy the statements above by the induction hypothesis. We have $q_S \in \operatorname{int}\Delta$ and $q_{S'} \in ||\beta(\mathcal{C}')|| = ||\mathcal{C}'||$ for $S' \in \mathcal{C}'$, hence the injectivity of q follows from $||\mathcal{C}'|| \cap \operatorname{int}\Delta = \emptyset$ and the injectivity of $q|_{\mathcal{C}'}$. To prove that q is a geometric representation of $\beta(\mathcal{C})$, we need to show that $\operatorname{conv}(q(X)) \cap \operatorname{conv}(q(Y)) = \operatorname{conv}(q(X\cap Y))$ for every $X, Y \in \beta(\mathcal{C})$. As q is a geometric representation of $\mathcal{C}'$, we may assume that $X \in \beta(\mathcal{C}) \setminus \beta(\mathcal{C}')$, that is, $S \in X$. Then X is a chain with maximal element S, hence $\operatorname{conv}(q(X)) \subseteq \Delta$. Suppose first that $S \notin Y$. Then $\operatorname{conv}(q(Y)) \subseteq ||\beta(\mathcal{C}')|| = ||\mathcal{C}'||$, hence $\operatorname{conv}(q(Y)) \cap \operatorname{int}\Delta = \emptyset$. Thus, $\operatorname{conv}(q(X)) \cap \operatorname{conv}(q(Y)) = (\operatorname{conv}(q(X)) \cap \partial\Delta) \cap \operatorname{conv}(q(Y)) = \operatorname{conv}(q(X-S)) \cap \operatorname{conv}(q(Y))$, which equals to $\operatorname{conv}(q((X-S)\cap Y)) = \operatorname{conv}(q(X\cap Y))$ as the restriction of q is a geometric representation of $\beta(\mathcal{C}')$. Assume now that $S \in Y$. Let $F = \operatorname{conv}(q(X-S))$ and $G = \operatorname{conv}(q(Y-S))$, then (4.2) implies that $\operatorname{conv}(q(X)) \cap \operatorname{conv}(q(Y)) = \operatorname{conv}(F, q_S) \cap \operatorname{conv}(G, q_S) = \operatorname{conv}(F\cap G, q_S)$. As the restriction of q is a geometric representation of $\beta(\mathcal{C}')$, $F \cap G = \operatorname{conv}(q(X\cap Y - S))$, thus $\operatorname{conv}(q(X)) \cap \operatorname{conv}(q(Y)) = \operatorname{conv}(q(X\cap Y))$ follows. This proves that q is a geometric representation of $\beta(\mathcal{C})$.

We show that $||\beta(\mathcal{C})|| = ||\mathcal{C}||$. If $S \in X \in \beta(\mathcal{C})$, then we noticed above that $||X|| \subseteq \Delta \subseteq ||\mathcal{C}||$, hence $||\beta(\mathcal{C})|| \subseteq ||\mathcal{C}||$ follows from $||\beta(\mathcal{C}')|| = ||\mathcal{C}'||$. To prove the other direction we only need to show that $\operatorname{int}\Delta \subseteq ||\beta(\mathcal{C}')||$ (using that $||\beta(\mathcal{C}')|| = ||\mathcal{C}'||$). It is clear that $q_S \in ||\beta(\mathcal{C}')||$, and for $x \in \operatorname{int}\Delta$, $x \neq q_S$ there exists y and t as in (4.2). As $y \in \partial\Delta \subseteq ||\mathcal{C}'|| = ||\beta(\mathcal{C}')||$, there exists an $X' \in \beta(\mathcal{C}')$ such that $y \in ||X'||$. Then $X = X' \cup \{S\} \in \beta(\mathcal{C})$ and $x \in ||X||$, concluding the proof of our claim.

Now we prove the last statement about $\operatorname{supp}_{\beta(\mathcal{C})}(x)$. As the claim holds for $\mathcal{C}'$, we may assume that $x \in ||\beta(\mathcal{C})|| \setminus ||\beta(\mathcal{C}')|| = \operatorname{int}\Delta$. If $x = q_S$, then $\operatorname{supp}_{\beta(\mathcal{C})}(x) = \{S\}$ and the claim is true. If $x \neq q_S$, then there exists y and t as in (4.2), and $\operatorname{supp}_{\beta(\mathcal{C})}(x) = \operatorname{supp}_{\beta(\mathcal{C}')}(y) \cup \{S\}$. As $\operatorname{supp}_{\beta(\mathcal{C}')}(y) = \{S_1, \dots, S_k\}$ for some $S_1, \dots, S_k \in \operatorname{supp}_{\mathcal{C}'}(y) \subseteq \operatorname{supp}_{\mathcal{C}}(x)$, it follows that $\operatorname{supp}_{\beta(\mathcal{C})}(x) = \{S_1, \dots, S_k, S\}$ where $S_1, \dots, S_k, S \in \operatorname{supp}_{\beta(\mathcal{C})}(x)$. This concludes the proof of the proposition. □

4.1.3 k-connectivity

The most important concept we will use in this chapter is the connectivity of a complex. We denote the d-dimensional closed unit ball by B^d and its boundary (the $(d-1)$-dimensional sphere) by S^{d-1}.

Definition 4.10. A topological space X is *k-connected* ($k \geq 0$), if for every $0 \leq r \leq k$, every continuous map $f\colon S^r \to X$ extends to a continuous map $\bar{f}\colon B^{r+1} \to X$.
X is *(-1)-connected* if it is nonempty.

In particular, 0-connectivity means path-connectivity.

We say that a complex $\mathcal{C}$ is k-connected if $||\mathcal{C}||$ is k-connected. (Note that k-connectivity is homeomorphism invariant.) To handle k-connectivity, we need some more definitions.

Definition 4.11. Let X and Y be topological spaces and $f, g\colon X \to Y$ continuous mappings. f and g are *homotopic* ($f \sim g$), if there exists a continuous mapping $H\colon X \times [0,1] \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for every $x \in X$. f is *0-homotopic* ($f \sim 0$ in Y), if it is homotopic to a mapping of X onto a single point of Y.

Definition 4.12. Let $X \subseteq Y$ be topological spaces. A continuous mapping $f\colon Y \to X$ is called a *retraction*, if $f|_X = \mathrm{id}_X$. X is a *retract* of Y if there exists a retraction $Y \to X$.

It is straightforward to check that a retract of a k-connected space is k-connected.

The next definition corresponds to the truncation of matroids but the notation is slightly different.

Definition 4.13. The *k-dimensional skeleton* of a complex $\mathcal{C}$ is $\mathcal{C}_k = \{X \in \mathcal{C} : |X| \le k+1\}$.

Using these definitions one can directly prove the following characterization of k-connectivity (see e.g. [6] for a proof).

Proposition 4.14. *For any complex $\mathcal{C}$ and $k \ge 0$, the followings are equivalent:*

(1) $\mathcal{C}$ is k-connected,

(2) $\mathcal{C}_{k+1}$ is k-connected,

(3) $||\mathcal{C}_{k+1}||$ is a retract of $||(2^{V(\mathcal{C})})_{k+1}||$,

(4) $id_{||\mathcal{C}_k||} \sim 0$ in $||C||$.

The following theorem (in a more general setting) follows from algebraic topological results, but we give here an elementary proof using only Proposition 4.14.

Theorem 4.15. *If $\mathcal{A}$ and $\mathcal{B}$ are k-connected complexes and $\mathcal{A} \cap \mathcal{B}$ is $(k-1)$-connected, then $\mathcal{A} \cup \mathcal{B}$ is k-connected.*

Proof [6]. The proof is based on the following lemma.

Lemma 4.16. *Let $\mathcal{C}$ be a complex and U any finite set.*

(a) If $\mathcal{C}$ is k-connected and $\mathcal{C} \cap 2^U$ is $(k-1)$-connected, then $\mathcal{C} \cup 2^U$ is k-connected.

(b) If $\mathcal{C} \cap 2^U$ and $\mathcal{C} \cup 2^U$ are k-connected, then $\mathcal{C}$ is k-connected.

Proof. (a) $\mathcal{C} \cap 2^U$ is $(k-1)$-connected, hence there exists a retraction $f\colon ||(2^U)_k|| \to ||(\mathcal{C} \cap 2^U)_k|| = ||\mathcal{C}_k|| \cap ||(2^U)_k||$. (By Proposition 4.14, $||(\mathcal{C} \cap 2^U)_{k+1}||$ is a retract of $||(2^{U'})_{k+1}||$ where $U' = V(\mathcal{C} \cap 2^U) \subseteq U$. Since $||(2^{U'})_{k+1}||$ is a retract of $||(2^U)_{k+1}||$, $||(\mathcal{C} \cap 2^U)_{k+1}||$ is a retract of $||(2^U)_{k+1}||$ as well.) f can be extended to a retraction $\bar{f}\colon ||\mathcal{C}_k|| \cup ||(2^U)_k|| \to ||\mathcal{C}_k||$ by letting it be identity on $||\mathcal{C}_k||$. Clearly, $||\mathcal{C}_k|| \cup ||(2^U)_k|| = ||(\mathcal{C} \cup 2^U)_k||$ and

$$\bar{f} \sim \mathrm{id}_{||(\mathcal{C} \cup 2^U)_k||} \text{ in } ||\mathcal{C} \cup 2^U||$$

(since for $x \in ||(2^U)_k||$, points x and $\bar{f}(x)$ are both contained in the convex set $||2^U||$). Since $\mathcal{C}$ is k-connected, $\mathrm{id}_{\mathcal{C}_k} \sim 0$ in $||\mathcal{C}||$, hence $\bar{f} = \mathrm{id}_{\mathcal{C}_k} \circ \bar{f} \sim 0$ in $||\mathcal{C}||$ and so $\bar{f} \sim 0$ in $||\mathcal{C} \cup 2^U||$ as well. Then $\mathrm{id}_{||(\mathcal{C} \cup 2^U)_k||} \sim 0$ in $||\mathcal{C} \cup 2^U||$, thus $\mathcal{C} \cup 2^U$ is k-connected.

(b) $\mathcal{C}\cap 2^U$ is k-connected, which implies (as we have seen in part (a)) that $||\mathcal{C}_{k+1}||$ is a retract of $||(\mathcal{C}\cup 2^U)_{k+1}||$. Since $||(\mathcal{C}\cup 2^U)_{k+1}||$ is k-connected, its retract $||\mathcal{C}_{k+1}||$ is k-connected as well, thus $\mathcal{C}$ is k-connected. $\square$

Consider the barycentric subdivisions $\beta(\mathcal{A})$ and $\beta(\mathcal{B})$. It is not difficult to check that $\beta(\mathcal{A}\cap\mathcal{B}) = \beta(\mathcal{A})\cap\beta(\mathcal{B})$ and $\beta(\mathcal{A}\cup\mathcal{B}) = \beta(\mathcal{A})\cup\beta(\mathcal{B})$ (the latter follows from the hereditary property of $\mathcal{A}$ and $\mathcal{B}$). Hence, by Proposition 4.9 $\mathcal{A}' := \beta(\mathcal{A})$ and $\mathcal{B}' := \beta(\mathcal{B})$ are k-connected, $\mathcal{A}'\cap\mathcal{B}'$ is $(k-1)$-connected, and it is sufficient to show that $\mathcal{A}'\cup\mathcal{B}'$ is k-connected.

Let $U = V(\mathcal{B}') = \mathcal{B}$. As $\mathcal{A}'$ is k-connected and $\mathcal{A}'\cap 2^U = \mathcal{A}'\cap\mathcal{B}'$ is $(k-1)$-connected, part (a) of Lemma 4.16 yields that $\mathcal{A}'\cup 2^U$ is k-connected. Let $\mathcal{C} = \mathcal{A}'\cup\mathcal{B}'$, then $\mathcal{C}\cup 2^U = \mathcal{A}'\cup 2^U$ and $\mathcal{C}\cap 2^U = (\mathcal{A}'\cap 2^U)\cup(\mathcal{B}'\cap 2^U) = (\mathcal{A}'\cap\mathcal{B}')\cup\mathcal{B}' = \mathcal{B}'$ are k-connected, thus by part (b) of Lemma 4.16, $\mathcal{C} = \mathcal{A}\cup\mathcal{B}$ is k-connected as well.

$\square$

We introduce the following notation for a complex $\mathcal{C}$:

$$\eta(\mathcal{C}) = 2 + \sup\{k : \mathcal{C} \text{ is } k\text{-connected}\},$$

where we define every topological space to be (-2)-connected (that is, $\eta(\mathcal{C}) = 0$ if $||\mathcal{C}|| = \emptyset$). In particular, if $\mathcal{C}$ is k-connected for every k, then $\eta(\mathcal{C}) = \infty$. The addition of 2 corresponds to the number of vertices of the simplex homeomorphic to S^k in the definition of k-connectivity.

Using this notation the previous theorem can be formulated as

$$\eta(\mathcal{A}\cup\mathcal{B}) \geq \min\{\eta(\mathcal{A}), \eta(\mathcal{B}), \eta(\mathcal{A}\cap\mathcal{B}) + 1\}. \tag{4.3}$$

We shall only use this result in a special case. For a vertex x of a complex $\mathcal{C}$ consider the analogue of restriction to $V(\mathcal{C}) - x$ and contraction of x:

$$\mathcal{C} - x = \{X \in \mathcal{C} : x \notin X\},$$
$$\mathcal{C}/x = \{X \in \mathcal{C} : x \notin X, X + x \in \mathcal{C}\}.$$

For complexes $\mathcal{C}/x$ is usually called the *link* of x.

Corollary 4.17. *For every vertex x of a complex $\mathcal{C}$*

$$\eta(\mathcal{C}) \geq \min\{\eta(\mathcal{C} - x), \eta(\mathcal{C}/x) + 1\}.$$

Proof. We may assume that $\{x\} \in \mathcal{C}$ as otherwise $\mathcal{C} = \mathcal{C} - x$ and $\eta(\mathcal{C}) = \eta(\mathcal{C} - x)$. Let $\mathcal{A} = \mathcal{C} - x$ and $\mathcal{B} = \{X : X + x \in \mathcal{C}\}$, then $\mathcal{A}\cup\mathcal{B} = \mathcal{C}$ and $\mathcal{A}\cap\mathcal{B} = \mathcal{C}/x$. Notice that $||\mathcal{B}||$ is a nonempty star-convex set (the line segment between the point corresponding to x and any point of $||\mathcal{B}||$ lies completely in $||\mathcal{B}||$), hence $\eta(\mathcal{B}) = \infty$. Using inequality (4.3) the desired result follows. $\square$

As complexes are generalizations of matroids, it is natural to ask the relationship between the rank function r and the connectivity parameter η of a matroid $M = (S, \mathcal{I})$. It can be shown that they are more or less the same:

$$\eta(\mathcal{I}) = \begin{cases} \infty, & \text{if } M \text{ contains a co-loop,} \\ r(S), & \text{otherwise.} \end{cases}$$

The inequality $\eta(\mathcal{I}) \geq r(S)$ can be derived from Corollary 4.17; later we will prove the more general Proposition 4.21.

4.1.4 The Knaster-Kuratowski-Mazurkiewicz lemma

The last geometrical tool we will need is a result equivalent to Sperner's following classical lemma.

Theorem 4.18 (Sperner's lemma [70]). *Let Δ be an n-dimensional simplex with vertices $p_1, \ldots, p_{n+1}$ and $\mathcal{G}$ a geometric simplicial complex such that $||\mathcal{G}|| = \Delta$. Consider a colouring $c\colon V(\mathcal{G}) \to \{1, \ldots, n+1\}$ such that for every $v \in V(\mathcal{G})$*

$$v \in \operatorname{conv}\{p_{i_1}, \ldots, p_{i_k}\} \Rightarrow c(v) \in \{i_1, \ldots, i_k\}.$$

Then there exists an odd number of n-dimensional simplices of $\mathcal{G}$, whose vertices are coloured with all $n+1$ colours.

The next theorem formulates two closely related results. The first one, with $A_1, \ldots, A_{n+1}$ closed is the classical theorem of Knaster, Kuratowski and Mazurkiewicz [45] from 1929, which we prove as a consequence of Sperner's lemma. The second one, with $A_1, \ldots, A_{n+1}$ open is a consequence of the closed version, for this we follow the proof of [69].

Theorem 4.19 (KKM lemma). *Let Δ be an n-dimensional simplex with vertices $p_1, \ldots,\ p_{n+1}$. Suppose that $A_1, \ldots, A_{n+1}$ are closed (resp. open) subsets of Δ such that*

$$\operatorname{conv}\{p_{i_1}, \ldots, p_{i_k}\} \subseteq A_{i_1} \cup \cdots \cup A_{i_k} \tag{4.4}$$

for every subset $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n+1\}$. Then $A_1 \cap \cdots \cap A_{n+1}$ is nonempty.

Proof. Suppose first that $A_1, \ldots, A_{n+1}$ are closed. Let $\varepsilon > 0$, and construct a geometric simplicial complex $\mathcal{G}$ such that $||\mathcal{G}|| = \Delta$ and the diameter of every simplex of $\mathcal{G}$ is at most ε. (As Δ is a standard n-dimensional simplex, it is not difficult to give an elementary construction – a more general way to do this would be to consider repeated barycentric subdivisions.) For $v \in V(\mathcal{G})$ let $\operatorname{conv}\{p_{i_1}, \ldots, p_{i_k}\}$ be the smallest face of Δ containing v, then by (4.19) there exists a colour $c(v) \in \{i_1, \ldots, i_k\}$ such that $v \in A_{c(v)}$. The colouring c satisfies the conditions of Sperner's lemma, hence there exists an n-dimensional simplex in $\mathcal{G}$ with vertices $q_1^\varepsilon, \ldots, q_{n+1}^\varepsilon$ such that $c(q_i^\varepsilon) = i$, that is, $q_i^\varepsilon \in A_i$. Consider a sequence ε_m converging to 0, then it has a subsequence ε'_m such that $q_1^{\varepsilon'_m}$ converges to a point q. As the diameter of $\{q_1^{\varepsilon'_m}, \ldots, q_{n+1}^{\varepsilon'_m}\}$ is at most ε'_m, we get that $q_i^{\varepsilon'_m} \to q$ for $i = 1, \ldots, n+1$. As $q_i^{\varepsilon'_m} \in A_i$ and A_i is closed, $q \in A_i$ as well, thus $q \in A_1 \cap \cdots \cap A_{n+1}$.

Assume now that $A_1, \ldots, A_{n+1}$ are open. We show that there exists closed sets $A'_1, \ldots, A'_{n+1}$ satisfying (4.19) such that $A'_i \subseteq A_i$, then the result will follow by $\emptyset \neq A'_1 \cap \cdots \cap A'_{n+1} \subseteq A_1 \cap \cdots \cap A_{n+1}$. For $x \in \Delta$ there exists an open set U_x such that $x \in U_x \subseteq \overline{U_x} \subseteq \bigcap_{x \in A_i} A_i$, since $\bigcap_{x \in A_i} A_i$ is open (here $\overline{U_x}$ denotes the closure of U_x). For every $i_1, \ldots, i_k$ the union $\bigcup\{U_x : x \in A_{i_1} \cup \cdots \cup A_{i_k}\} = A_{i_1} \cup \cdots \cup A_{i_k}$ is an open cover of the compact set $\operatorname{conv}\{p_{i_1}, \ldots, p_{i_k}\}$, hence there exists a finite subset $X_{i_1,\ldots,i_k} \subseteq A_{i_1} \cup \cdots \cup A_{i_k}$ such that $\operatorname{conv}\{p_{i_1}, \ldots, p_{i_k}\} \subseteq \bigcup\{U_x : x \in X_{i_1,\ldots,i_k}\}$. Let $X = \bigcup_{i_1,\ldots,i_k} X_{i_1,\ldots,i_k}$, then X is a finite set such that

$$\operatorname{conv}\{p_{i_1}, \ldots, p_{i_k}\} \subseteq \bigcup\{U_x : x \in X, U_x \subseteq A_{i_1} \cup \cdots \cup A_{i_k}\}. \tag{4.5}$$

Now let

$$A'_i = \bigcup\{\overline{U_x} : x \in X, U_x \subseteq A_i\},$$

then A'_i is closed as it is a finite union of closed sets. By $\overline{U_x} \subseteq \bigcap_{x \in A_i} A_i$, $U_x \subseteq A_i$ implies $\overline{U_x} \subseteq A_i$, hence $A'_i \subseteq A_i$. Moreover, if $U_x \subseteq A_{i_1} \cup \ldots A_{i_k}$, then $U_x \subseteq A_{i_j}$ for some j, hence (4.5) implies that $A_1, \ldots, A_{n+1}$ satisfies (4.19). This concludes the proof of the theorem. $\square$

4.2 The intersection of a matroid and a simplicial complex

We are ready to prove the main result of [1]. For a set X and complex $\mathcal{C}$, we write $\mathcal{C}|_X = \{C \in \mathcal{C} : C \subseteq X\}$.

Theorem 4.20. *Let $k \geq 0$, $\mathcal{C}$ be a complex and $M = (S, \mathcal{I})$ a matroid with rank function r such that $V(\mathcal{C}) \subseteq S$. If*

$$\eta(\mathcal{C}|_X) + r(S - X) \geq k \quad (X \subseteq S), \tag{4.6}$$

then there exists an independent set $I \in \mathcal{C} \cap \mathcal{I}$ of size k.

Proof. We may assume that $k = r(S)$ by considering the k-truncation of M. We may also assume that $V(\mathcal{C}) = S$.

Let $\mathcal{H}$ denote the complex whose vertices are the hyperplanes[1] of the matroid M and which consists of all subsets of hyperplanes of M. Clearly, $||\mathcal{H}||$ is a simplex. In what follows, we construct a continuous map $f\colon ||\mathcal{H}|| \to ||\mathcal{C}||$ with some special properties.

By Proposition 4.9, we have $||\mathcal{H}|| = ||\beta(\mathcal{H})||$. We define the *flat complex* of M by

$$\mathcal{F}(M) = \{\{F_1, \ldots, F_k\} : \quad F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k \subsetneq S \text{ are closed in } M,\ k \geq 0\},$$

then the vertices of $\mathcal{F}(M)$ are the closed sets of M excluding S. The vertices of $\beta(\mathcal{H})$ are of the form $\{H_1, \ldots, H_k\}$ for some hyperplanes $H_1, \ldots, H_k$ of M. We define

$$g\colon V(\beta(\mathcal{H})) \to V(\mathcal{F}(M)),\ g(\{H_1, \ldots, H_k\}) = \bigcap_{i=1}^{k} H_i.$$

Notice that g is simplicial: the image of a chain of vertices of $\beta(\mathcal{H})$ is a chain of vertices of $\mathcal{F}(M)$ (the inclusion is reversed). Thus, by Proposition 4.5 we have a continuous mapping $||g||\colon ||\beta(\mathcal{H})|| \to ||\mathcal{F}(M)||$.

Now we construct a continuous map $h\colon ||\mathcal{F}(M)|| \to ||\mathcal{C}||$. We define h for points of the interior of each simplex $||\{F_1, \ldots, F_k\}||$ at a time (here $\{F_1, \ldots, F_k\} \in \mathcal{F}(M)$, $F_1 \subsetneq \cdots \subsetneq F_k$). If $k = 1$, then we define h on the point $||\{F_1\}||$ to be the point corresponding to a vertex in $S \setminus F_1$ (note that we did not allow $F_1 = S$ in the definition of $\mathcal{F}(M)$). Assume that we defined h on the interiors for $1, 2, \ldots, k-1$, then it is defined on the boundary of $||\{F_1, \ldots, F_k\}||$, now we define it on its interior. As $F_1 \subsetneq \cdots \subsetneq F_k \subsetneq S$ is a chain of closed sets, we have $r(F_1) \leq r(F_2) - 1 \leq r(F_3) - 2 \leq \cdots \leq r(S) - k$. Then by (4.6) we have

$$\eta(\mathcal{C}|_{S-F_1}) \geq r(S) - r(F_1) \geq k,$$

thus $\mathcal{C}|_{S-F_1}$ is $(k-2)$-connected. The boundary of the $(k-1)$-dimensional simplex $||\{F_1, \ldots, F_k\}||$ is homeomorphic to the sphere S^{k-2}, hence by the definition of $(k-2)$-connectivity h extends continuously to the interior of $||\{F_1, \ldots, F_k\}||$. This concludes the definition of $h\colon ||\mathcal{F}(M)|| \to ||\mathcal{C}||$. By the construction, we have the following property:

$$\forall y \in ||\mathcal{F}(M)||\ \exists F \in \operatorname{supp}_{\mathcal{F}(M)}(y) : F \cap \operatorname{supp}_{\mathcal{C}}(h(y)) = \emptyset. \tag{4.7}$$

Indeed, if $\operatorname{supp}_{\mathcal{F}(M)}(y) = \{F_1\}$, then $\operatorname{supp}_{\mathcal{C}}(h(y))$ is a vertex in $S - F_1$, hence $F = F_1$ satisfies (4.7). Otherwise, $\operatorname{supp}_{\mathcal{F}(M)}(y) = \{F_1, \ldots, F_k\}$ for some $k \geq 2$, then $h(y) \in ||\mathcal{C}|_{S-F_1}||$, hence $F = F_1$ satisfies (4.7).

[1] [1] considered cuts (the complement of hyperplanes) as vertices of the complex. This minor modification is due to [59] and it will not make any difference.

Let $f = h \circ ||g||$, then it is a continuous map $||\mathcal{H}|| \to ||\mathcal{C}||$:

$$f : ||\mathcal{H}|| = ||\beta(\mathcal{H})|| \xrightarrow{||g||} ||\mathcal{F}(M)|| \xrightarrow{h} ||\mathcal{C}||.$$

For a hyperplane $H \in V(\mathcal{H})$ let

$$A_H = \{x \in ||\mathcal{H}|| : \mathrm{supp}_{\mathcal{C}}(f(x)) \not\subseteq H\}.$$

A_H is an open subset of $||\mathcal{H}||$, as

$$A_H = \bigcup_{s \in S \setminus H} \{x \in ||\mathcal{H}|| : s \in \mathrm{supp}_{\mathcal{C}}(f(x))\} = \bigcup_{s \in S \setminus H} f^{-1}\left(\{y \in ||\mathcal{C}|| : s \in \mathrm{supp}_{\mathcal{C}}(y)\}\right)$$

is the union of open sets since f is continuous and $\{y \in ||\mathcal{C}|| : s \in \mathrm{supp}_{\mathcal{C}}(y)\}$ is open for every s.

We claim that the sets A_H $(H \in V(\mathcal{H}))$ satisfy (4.4), which can be rephrased as follows:

$$\forall x \in ||\mathcal{H}||\ \exists H \in \mathrm{supp}_{\mathcal{H}}(x) : x \in A_H.$$

Let $x \in ||\mathcal{H}||$, then by (4.7) there exists an $F \in \mathrm{supp}_{\mathcal{F}(M)}(||g||(x))$ such that $F \cap \mathrm{supp}_{\mathcal{C}}(f(x)) = \emptyset$. By the definition of $||g||$, $F \in \mathrm{supp}_{\mathcal{F}(M)}(||g||(x))$ implies that $\beta(\mathcal{H})$ has a vertex $v \in \mathrm{supp}_{\beta(\mathcal{H})}(x)$ such that $g(v) = F$. By Proposition 4.9, v is of the form $v = \{H_1, \dots, H_k\}$ for some hyperplanes of $\mathcal{H}$ such that $\{H_1, \dots, H_k\} \subseteq \mathrm{supp}_{\mathcal{H}}(x)$. As $F = g(v) = \bigcap_{i=1}^k H_i$ is disjoint from $\mathrm{supp}_{\mathcal{C}}(f(x))$, we get that $\mathrm{supp}_{\mathcal{C}}(f(x)) \not\subseteq H_i$ for some $i \in \{1, \dots, k\}$. Thus $x \in A_{H_i}$ and $H_i \in \mathrm{supp}_{\mathcal{H}}(x)$, which concludes the proof of our claim.

Therefore, we can apply Theorem 4.19, which yields that there exists an $x \in \bigcap_{H \in V(\mathcal{H})} A_H$. This means that $\mathrm{supp}_{\mathcal{C}}(f(x)) \not\subseteq H$ for every hyperplane H, thus $\mathrm{supp}_{\mathcal{C}}(f(x))$ is a generator of M (that is, it has rank $r(S)$). Then $\mathrm{supp}_{\mathcal{C}}(f(x))$ contains a basis of M which is in $\mathcal{C}$ by $\mathrm{supp}_{\mathcal{C}}(f(x)) \in \mathcal{C}$. This concludes the proof of the theorem. □

4.3 2-approximation

To apply Theorem 4.20, we need a lower bound on the connectivity parameter η.

Proposition 4.21. *For matroids $M_1 = (S, \mathcal{I}_1), \dots, M_k = (S, \mathcal{I}_k)$ on common ground set S*

$$\eta(\mathcal{I}_1 \cap \dots \cap \mathcal{I}_k) \geq \max\{|I| : I \in \mathcal{I}_1 \cap \dots \cap \mathcal{I}_k\}/k.$$

Proof. For a complex $\mathcal{C}$ we will us use the notation $\nu(\mathcal{C}) = \max\{|I| : I \in \mathcal{C}\}$.

We prove by induction on $|S|$. Let $\mathcal{C} = \mathcal{I}_1 \cap \dots \cap \mathcal{I}_k$ and $I \in \mathcal{C}$ a set of size $\nu(\mathcal{C})$. Let $x \in S - I$ be an element which is not a loop in any of the matroids (if there is no such x then $\eta(\mathcal{C}) = \eta(2^I) = \infty$ and the inequality is trivial). Notice that $\nu(\mathcal{I}_1/x \cap \dots \cap \mathcal{I}_k/x) \geq \nu(\mathcal{C}) - k$. Indeed, for $j = 1, \dots, k$ we can pick an element $x_j \in I$ such that $I - x_j + x \in \mathcal{I}_j$ (assuming $I \neq \emptyset$), and then $I - \{x_1, \dots, x_k\} \in \mathcal{I}_1/x \cap \dots \cap \mathcal{I}_k/x$ is a set of size at least $|I| - k$. Applying induction to $\mathcal{C} - x$ and $\mathcal{C}/x$ we get

$$\eta(\mathcal{C} - x) = \eta((\mathcal{I}_1 - x) \cap \dots \cap (\mathcal{I}_k - x)) \geq \nu((\mathcal{I}_1 - x) \cap \dots \cap (\mathcal{I}_k - x))/k = \nu(\mathcal{C})/k,$$
$$\eta(\mathcal{C}/x) = \eta(\mathcal{I}_1/x \cap \dots \cap \mathcal{I}_k/x) \geq \nu(\mathcal{I}_1/x \cap \dots \cap \mathcal{I}_k/x)/k \geq (\nu(\mathcal{C}) - k)/k = \nu(\mathcal{C})/k - 1,$$

and $\eta(\mathcal{C}) \geq \min\{\eta(\mathcal{C} - x)/k, \eta(\mathcal{C}/x)/k + 1\} \geq \nu(\mathcal{C})/k$ by Corollary 4.17.

□

Corollary 4.22. *Let M_1, M_2, M_3 be matroids on the same ground set S such that*

$$\frac{1}{2}r_1(X_1) + \frac{1}{2}r_2(X_2) + r_3(X_3) \geq k \tag{4.8}$$

whenever $S = X_1 \cup X_2 \cup X_3$. Then they have a common independent set of size k.

Proof. Let $\mathcal{C} = \mathcal{I}_1 \cap \mathcal{I}_2$, then by Theorem 4.20 it suffices to show that $\eta(\mathcal{C}|_X) + r_3(S - X) \geq k$ holds for every $X \subseteq S$. Applying Proposition 4.21, Theorem 1.2 and assumption (4.8) we get the desired result:

$$\begin{aligned}\eta(\mathcal{C}|_X) &\geq \frac{1}{2}\max\{|I| : I \in \mathcal{I}_1 \cap \mathcal{I}_2, I \subseteq X\} = \frac{1}{2}\min\{r_1(X_1) + r_2(X_2) : X = X_1 \cup X_2\} \\ &\geq k - r_3(S - X).\end{aligned}$$

□

Corollary 4.22 implies the following result just as Theorem 1.2 implies Theorem 1.3.

Theorem 4.23. *For matroids $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$*

$$\chi(M_1, M_2) \leq 2\max\{\chi(M_1), \chi(M_2)\}.$$

Proof. Let $k = 2\max\{\chi(M_1), \chi(M_2)\}$, $S_1, \ldots, S_k$ be disjoint copies of S and $S' = \bigcup_{i=1}^k S_i$. For $s \in S$ we write s_i for the copy of s in S_i. Consider the matroids $M'_1 = \bigoplus_{i=1}^k M_1$, $M'_2 = \bigoplus_{i=1}^k M_2$ and M'_3 on ground set S where M'_3 is the partition matroid defined by the partition $\{\{s_1, \ldots, s_k\} : s \in S\}$. We claim that S can be partitioned into k common independent sets of M_1 and M_2 if and only if M'_1, M'_2 and M'_3 have a common independent set of size $|S|$. Indeed, if I' is such a set than $\{\{s \in S : s_i \in I'\} : i = 1, \ldots, k\}$ is a partition of S into k common independent sets of M_1 and M_2, and conversely, if $S = I_1 \cup \cdots \cup I_k$ is such a partition then $\bigcup_{i=1}^k \{s_i : s \in I_i\}$ is a common independent set of M'_1, M'_2, M'_3 of size $|S|$. Hence, by Corollary 4.22 it suffices to show that

$$\frac{1}{2}r'_1(X'_1) + \frac{1}{2}r'_2(X'_2) + r'_3(X'_3) \geq |S|$$

holds for any partition $S' = X'_1 \cup X'_2 \cup X'_3$ where r'_j denotes the rank function of M'_j. We may assume that X_3 is closed in M'_3, that is, it is of the form $X'_3 = \{s_1, \ldots, s_k : s \in X_3\}$ for a set $X_3 \subseteq S$. Then

$$\begin{aligned}\frac{1}{2}r'_1(X'_1) + \frac{1}{2}r'_2(X'_2) + r'_3(X'_3) &= \frac{1}{2}\sum_{i=1}^{k}\left(r'_1(X'_1 \cap S_i) + r'_2(X'_2 \cap S_i)\right) + |X_3| \\ &\geq \frac{k}{2}\min\{r_1(X_1) + r_2(X_2) : X_1 \cup X_2 = S - X_3\} + |X_3| \\ &\geq \min\{|X_1| + |X_2| : X_1 \cup X_2 = S - X_3\} + |X_3| = |S|,\end{aligned}$$

using that $\frac{k}{2}r_1(X_1) \leq |X_1|$ and $\frac{k}{2}r_2(X_2) \leq |X_2|$ for every $X_1, X_2 \subseteq S$ since M_1 and M_2 are $\frac{k}{2}$-colourable. This concludes the proof of the theorem. □

4.4 A slightly stronger result

We prove a generalization of Proposition 4.21 for the case of two matroids. This will also imply a generalization of Corollary 4.22.

Proposition 4.24. *Let $\ell \geq 1$ and $M = (S, \mathcal{I})$ and $N = (S, \mathcal{J})$ be matroids on common ground set S. Let $\mathcal{J}^\ell$ denote the system of independent sets of matroid ℓN. Then*

$$\eta(\mathcal{I} \cap \mathcal{J}) \geq \max\{|I| : I \in \mathcal{I} \cap \mathcal{J}^\ell\}/(\ell + 1).$$

Proof. We prove by induction on $|S|$. As in the proof of Proposition 4.21, let $\mathcal{C} = \mathcal{I} \cap \mathcal{J}$ and $I \in \mathcal{C}$ a set of size $\nu(\mathcal{C})$. We may assume that there is an element $x \in S - I$ which is not a loop in M or N. We claim that $\nu(\mathcal{I}/x \cap (\mathcal{J}/x)^\ell) \geq \nu(\mathcal{C}) - \ell - 1$. If $I \neq \emptyset$, then there exists an element $x_0 \in I$ such that $I - x_0 + x \in \mathcal{I}$. As $I \in \mathcal{J}^\ell$, it is the union some independent sets $I_1, \ldots, I_\ell$ of N, and there exists elements $x_1, \ldots, x_\ell$ such that $I_1 - x_1 + x, \ldots, I_\ell - x_\ell + x \in \mathcal{J}$. Then $I - \{x_1, \ldots, x_\ell\} = (I_1 - x_1) \cup \cdots \cup (I_\ell - x_\ell) \in (\mathcal{J}/x)^\ell$, thus $I - \{x_0, \ldots, x_\ell\} \in \mathcal{I}/x \cap (\mathcal{J}/x)^\ell$ is a set of size $\nu(\mathcal{C}) - \ell - 1$ proving our claim. Applying induction to $\mathcal{C} - x$ and $\mathcal{C}/x$ we get

$$\eta(\mathcal{C} - x) = \eta((\mathcal{I} - x) \cap (\mathcal{J} - x)) \geq \nu((\mathcal{I} - x) \cap (\mathcal{J} - x)^\ell)/(\ell + 1) = \nu(\mathcal{C})/(\ell + 1),$$
$$\eta(\mathcal{C}/x) = \eta(\mathcal{I}/x \cap \mathcal{J}/x) \geq \frac{\nu(\mathcal{I}/x \cap (\mathcal{J}/x)^\ell)}{\ell + 1} \geq \frac{\nu(\mathcal{C}) - \ell - 1}{\ell + 1} = \frac{\nu(\mathcal{C})}{\ell + 1} - 1,$$

and $\eta(\mathcal{C}) \geq \nu(\mathcal{C})/(\ell + 1)$ follows by Corollary 4.17. □

Corollary 4.25. *Let $\ell \geq 1$ and M_1, M_2, M_3 be matroids on the same ground set S such that*

$$\frac{1}{\ell + 1} \cdot r_1(X_1) + \frac{\ell}{\ell + 1} \cdot r_2(X_2) + r_3(X_3) \geq k \tag{4.9}$$

whenever $S = X_1 \cup X_2 \cup X_3$. Then they have a common independent set of size k.

Proof. Let $\mathcal{C} = \mathcal{I}_1 \cap \mathcal{I}_2$, then by Theorem 4.20 it suffices to show that $\eta(\mathcal{C}|_X) + r_3(S - X) \geq k$ holds for every $X \subseteq S$. Let r_2^ℓ denote the rank function of the matroid $\ell M_2 = (S, \mathcal{I}_2^\ell)$, then Proposition 4.24 and Theorem 1.2 yield

$$\eta(\mathcal{C}|_X) \geq \frac{1}{\ell + 1} \max\{|I| : I \in \mathcal{I}_1 \cap \mathcal{I}_2^\ell, I \subseteq X\} = \frac{1}{\ell + 1} \min\{r_1(Y_1) + r_2^\ell(Y_2) : X = Y_1 \cup Y_2\}.$$

Let $X = Y_1 \cup Y_2$, then by Theorem 1.3 there exists a set $X_2 \subseteq Y_2$ such that $r_2^\ell(Y_2) = \ell r_2(X_2) + |Y_2 - X_2|$. Then

$$r_1(Y_1) + r_2^\ell(Y_2) = r_1(Y_1) + \ell r_2(X_2) + |Y_2 - X_2| \geq r_1(X_1) + \ell r_2(X_2),$$

where $X_1 = Y_1 \cup (Y_2 - X_2)$. Combining the previous inequalities and assumption (4.9) we get the desired result:

$$\eta(\mathcal{C}|_X) \geq \frac{1}{\ell + 1} \min\{r_1(X_1) + \ell r_2(X_2) : X = X_1 \cup X_2\} \geq k - r_3(S - X).$$

□

From this we get the following result by a minor modification of the proof of Theorem 4.23.

Theorem 4.26. *Let $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ matroids such that $\chi(M_1) \leq p$ and $\chi(M_2) \leq pq$. Then*

$$\chi(M_1, M_2) \leq p + pq.$$

In particular, we proved the following conjecture for the case where one of $\chi(M_1)$ and $\chi(M_2)$ is a multiple of the other.

Conjecture 4.27. *For matroids $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$*

$$\chi(M_1, M_2) \leq \chi(M_1) + \chi(M_2).$$

4.5 The best possible upper bound

Aharoni and Berger [2] proposed the following strengthening of Conjecture 4.27.

Conjecture 4.28. *For matroids $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$*

$$\chi(M_1, M_2) \leq \max\{\chi(M_1), \chi(M_2)\} + 1.$$

Moreover, if $\chi(M_1) \neq \chi(M_2)$, then $\chi(M_1, M_2) = \max\{\chi(M_1), \chi(M_2)\}$.

The next example shows that this bound is the best possible (except for the trivial $\chi(M_1) = \chi(M_2) = 1$ case).

Proposition 4.29. *For every $k \geq 2$ there exists matroids $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ such that $\chi(M_1) = \chi(M_2) = k$ and $\chi(M_1, M_2) = k + 1$.*

Proof [2]. Consider the complete graph on vertices v_1, v_2, v_3, v_4 and replace each of the edges v_1v_2, v_1v_3 and v_1v_4 with $k-1$ parallel edges. Let S denote the set of edges, M_1 the graphic matroid of this graph and M_2 the partition matroid on S where a set of edges is independent if it contains neither two parallel edges nor two disjoint edges. Clearly, $r_1(S) = 3$ and $|S| = 3k$ implies that $\chi(M_1) \geq k$, hence $\chi(M_1) = k$ as S can be partitioned into $k-2$ stars with centre v_1 and two paths of length 3. M_2 is defined by a partition where each of the three classes has size k, hence $\chi(M_2) = k$.

The common independent sets of M_1 and M_2 are exactly the stars of the graph. It is clear that S can be partitioned into $k+1$ stars: consider $\{v_2v_3, v_2v_4\}$, $\{v_3v_4\}$ and $k-1$ stars centred at v_1. Suppose that S can be partitioned into k stars, then $|S| = 3k$ implies that each star has 3 edges. Looking at triangle $v_2v_3v_4$, we get that at most one of the stars is centred at v_2, v_3 or v_4. Then one of the edges of this triangle remains uncovered, a contradiction. Thus, we proved that $\chi(M_1, M_2) = k + 1$. □

The following observation of Aharoni, Berger and Ziv [2] shows that in Conjecture 4.28 we may assume that the ground set can be partitioned into bases in both matroids.

Observation 4.30. *Let M_1 and M_2 be two matroids on common ground set S. Then there exists matroids M_1' and M_2' on common ground set S' such that $\chi(M_1') = \chi(M_1)$, $\chi(M_2') = \chi(M_2)$, $\chi(M_1', M_2') = \chi(M_1, M_2)$ and in each of the matroids M_1' and M_2', S' can be partitioned into bases.*

Proof. Let $n = |S|$, $k = \chi(M_1)$, $\ell = \chi(M_2)$, and S' a superset of S such that $|S'| = nk\ell$. Let $\mathcal{I}_1' = \{X \subseteq S' : |X| \leq n\ell, X \cap S \in \mathcal{I}_1\}$ and $\mathcal{I}_2' = \{X \subseteq S' : |X| \leq nk, X \cap S \in \mathcal{I}_2\}$, then $M_1' = (S, \mathcal{I}_1')$ and $M_2' = (S, \mathcal{I}_2')$ are matroids whose restriction to S is M_1 and M_2, respectively. Consider a covering of S by k independent sets of M_1, then if we extend each of these sets arbitrarily into subsets of S' of size $\ell|S|$, then we get a covering of S by k independent sets of M_1'. This shows that $\chi(M_1') = \chi(M_1)$, and a similar reasoning yields $\chi(M_2') = \chi(M_2)$ and $\chi(M_1', M_2') = \chi(M_1, M_2)$. Moreover, $|S'| = k\ell n = kr_1'(S')$ implies that S' can be partitioned into k bases of M_1', and similarly S' can be partitioned into ℓ bases of M_2'. □

Aharoni, Berger and Ziv proved Conjecture 4.28 for $\chi(M_1) = \chi(M_2) = 2$.

Theorem 4.31. *If $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ are two matroids such that $\chi(M_1) = \chi(M_2) = 2$, then $\chi(M_1, M_2) \leq 3$.*

Proof [2]. By Observation 4.30 we may assume that S can be partitioned into two bases of M_1 and also into two bases of M_2. Let B_1, B_2 be bases of M_1 and B'_1, B'_2 bases of M_2 such that $S = B_1 \cup B_2 = B'_1 \cup B'_2$ and $|B_1 \cap B'_1| + |B_2 \cap B'_2|$ is maximal. Let $X_1 = B_1 \cap B'_2$, then by Theorem 1.1 there exists a subset $X_2 \subseteq B_2$ such that both $B_1 - X_1 \cup X_2$ and $B_2 - X_2 \cup X_1$ are bases of M_1. As $|B_1 \cap B'_1| + |B_2 \cap B'_2|$ is maximal and $(B_1 - X_1 \cup X_2) \cup (B_2 - X_2 \cup X_1)$ is a partition of S into bases of M_1, we have

$$\begin{aligned}|B_1 \cap B'_1| + |B_2 \cap B'_2| &\geq |(B_1 - X_1 \cup X_2) \cap B'_1| + |(B_2 - X_2 \cup X_1) \cap B'_2| \\ &= |B_1 \cap B'_1| + |B'_1 \cap X_2| + |B_2 \cap B'_2| - |B_2 \cap B'_2 \cap X_2| + |X_1| \\ &\geq |B_1 \cap B'_1| + |B_2 \cap B'_2| - |X_2| + |X_1| = |B_1 \cap B'_1| + |B_2 \cap B'_2|,\end{aligned}$$

thus $|B'_1 \cap X_2| = 0$, that is, $X_2 \subseteq B'_2$. Then $(B_1 \cap B'_2) \cup (B'_1 \cap B_2) \subseteq B_2 - X_2 \cup X_1$, hence $(B_1 \cap B'_2) \cup (B'_1 \cap B_2)$ is independent in M_1. Similarly, this set is also independent in M_2, thus $S = (B_1 \cap B'_1) \cup (B_2 \cap B'_2) \cup ((B_1 \cap B'_2) \cup (B'_1 \cap B_2))$ is a partition of S into three common independent sets of M_1 and M_2. $\square$

Note that this proof can be converted into a polynomial algorithm, as Theorem 1.1 can be proven algorithmically.

In case of $\chi(M_1) = 2$ and $\chi(M_2) = 3$, Aharoni, Berger and Ziv showed with a slight generalization of this proof that $\chi(M_1, M_2) \leq 4$. However, Conjecture 4.28 remains open even in this special case.

Chapter 5

Reduction to partition matroids

In this chapter we present the novel approach of reducing a matroid to a partition matroid without increasing its colouring number too much. This method might serve as a useful tool for problems related to packing common independent sets of two matroids. In particular, we propose a conjecture which strengthens Theorem 4.23 and prove its first special cases. These special cases provide new results for a question of Király [43] about list colouring of two matroids. Sections 5.1-5.5 are mainly from our joint work [4] with Kristóf Bérczi and Yutaro Yamaguchi.

5.1 Reductions of matroids

Definition 5.1. Let $M = (S, \mathcal{I})$ and $N = (S, \mathcal{J})$ be two matroids on ground set S. We say that N is a *reduction* of M and use the notation $N \preceq M$, if $\mathcal{J} \subseteq \mathcal{I}$. The reduction is *rank preserving* if $r_M(S) = r_N(S)$ holds, and is denoted by $N \preceq_r M$.

The idea of reducing a matroid to a simpler one goes back to the late 60's. In [14], Crapo and Rota introduced the notion of weak maps. Given two matroids M and N on the same ground set, N is a *weak map* of M if every independent set of N is also independent in M. Using our terminology, N is a weak map of M if and only if N is a reduction of M. Weak maps were further investigated by Lucas [54, 55] who characterized rank preserving weak maps for linear matroids. However, these results did not consider the possible increase in the colouring number of the matroid that plays a crucial role in our investigations. We find the name 'map' slightly misleading as it suggests that there is a function in the background, although the 'mapping' in question is simply the identity map between the ground sets of the matroids. Hence, we stick to the term 'reduction' throughout the chapter.

We present results about reductions to partition matroids and prove special cases of the following conjecture.

Conjecture 5.2. *Every k-colourable matroid can be reduced to a $2k$-colourable partition matroid.*

In particular, we show that matroid a M is reducible to a partition matroid N such (A) $\chi(N) = \chi(M)$ if M is a transversal matroid, (B) $\chi(N) \leq 2\chi(M) - 1$ if M is a graphic matroid, (C) $\chi(N) \leq \lceil kr/(r-1) \rceil$ if M is a paving matroid of rank r, and (D) $\chi(N) \leq 2\chi(M) - 2$ if M is a gammoid. It should be emphasized that in cases (A), (B) and (D) the reduction is rank preserving. Our main result is the proof for gammoids.

Notice that Theorem 4.23 would easily follow from Conjecture 5.2 using Theorem 3.4 for partition matroid (that is, the classical result of Kőnig about the edge colouring number of

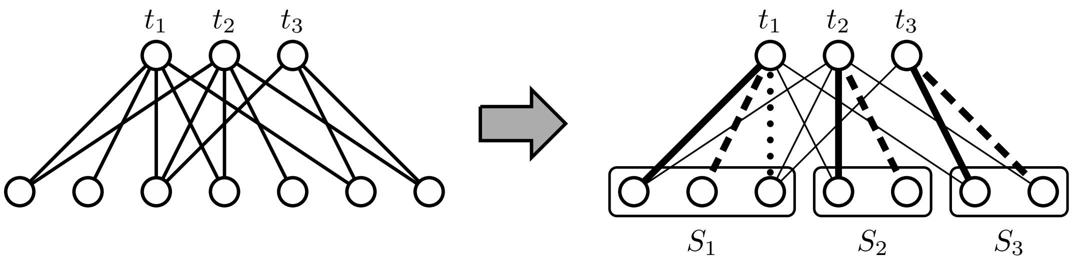


Figure 5.1: An illustration of the proof of Theorem 5.3. Thick, dashed and dotted edges are corresponding to three matchings covering S.

bipartite graphs). Indeed, if matroids M_1 and M_2 are reducible to matroids N_1 and N_2 such that $\chi(N_1) \leq 2\chi(M_1)$ and $\chi(N_2) \leq \chi(M_2)$, then $\chi(M_1, M_2) \leq \chi(N_1, N_2) = \max\{\chi(N_1), \chi(N_2)\} \leq 2\max\{\chi(M_1), \chi(M_2)\}$.

It is worth mentioning that every matroid $M = (S, \mathcal{I})$ has a reduction to a partition matroid $N = (S, \mathcal{J})$ of the same rank. The sketch of the proof is as follows. Fix an arbitrary basis $B = \{s_1, \ldots, s_r\}$ of M, and add s_i to the ith partition class. Then for an arbitrary element $s \in S - B$, consider the fundamental circuit $C(s, B)$ of s with respect to B, and add s to the partition class containing the element of $C(s, B) \cap B$ with the smallest index. If we pick exactly one element from every class of the partition thus obtained, we get a basis of the matroid. This can be verified using the circuit axioms, not discussed in this thesis. Nevertheless, this algorithm has no control over the sizes of the partition classes. It can happen that some of the classes have a large size compared to the colouring number of the original matroid, and such a reduction is not suitable for our purposes.

5.2 Transversal, graphic and paving matroids

As a warm-up, we first consider three basic cases: transversal, graphic, and paving matroids. Although the proofs are simple, they might help the reader to get familiar with the notion of reduction. Also, we show the connection to some earlier results such as Gallai colourings of complete graphs.

5.2.1 Transversal matroids

Theorem 5.3. *Let $M = (S, \mathcal{I})$ be a k-colourable transversal matroid. Then there exists a k-colourable partition matroid N with $N \preceq_r M$.*

Proof. Let $G = (S, T; E)$ a bipartite graph where $T = \{t_1, \ldots, t_r\}$, r being the rank of the transversal matroid on S. By assumption, the transversal matroid is k-colourable, so there exist k matchings $F_1, \ldots, F_k$ covering every vertex in S exactly once. We may assume that none of these matchings is empty. Let $S_i = \bigcup_{j=1}^{k} N_{F_j}(t_i)$ for $i = 1, \ldots, r$ (see Figure 5.1). Then $S_1 \cup \cdots \cup S_r$ is a partition of S with classes of size at most k. Pick an arbitrary element $s_j \in S_j$ for $j = 1, \ldots, r$. The edge set $\{t_j s_j : j = 1, \ldots, r\}$ shows that the picked elements form a matchable set, hence the partition matroid defined by the partition is a k-colourable rank preserving reduction of the transversal matroid. $\square$

5.2.2 Graphic matroids

Now we turn to graphic matroid. Notice that the colouring number of the graphic matroid of graph G, is the smallest number of forests covering the edge set (hence the notation χ might be slightly confusing). Observe that reducing the graphic matroid of G to a partition matroid is equivalent to colouring the edges of the graph in such a way that there is no cycle whose edges are coloured with completely different colours.

Theorem 5.4. *Let $M = (S, \mathcal{I})$ be a k-colourable graphic matroid. Then there exists a $(2k-1)$-colourable partition matroid N with $N \preceq_r M$, and the bound for the colouring number of N is tight.*

Proof. Let $G = (V, E)$ be a graph whose graphic matroid $M = (E, \mathcal{I})$ is k-colourable and let $K \subseteq V$ be a connected component of G of size at least 2. We claim that there exists a cut in K of size at most $2k-1$. Indeed, if every cut of K contains at least $2k$ edges then K is a $2k$-edge-connected component and so $|E[K]| \geq k|K|$ by counting the edges around each vertex in K. By Theorem 1.7, this contradicts the k-colourability of M.

Set $S_0 := \emptyset$ and $i := 0$. As long as there exists a connected component K in $G - \bigcup_{j=0}^{i} S_j$ of size at least 2, let $S_{i+1} \subseteq E$ be a minimum cut of K (see Figure 5.2), and update $i := i + 1$. By the above, $|S_{i+1}| \leq 2k - 1$. Let $E = S_1 \cup \cdots \cup S_q$ denote the partition thus obtained. We claim that the partition matroid corresponding to this partition is a reduction of M. In order to see this, we have to show that every cycle of G intersects at least one of the partition classes in at least two elements. Given a cycle C, let i be the smallest index with $|S_i \cap C| > 0$. Then $C \subseteq \bigcup_{j \geq i}^{q} S_j$ and S_i is a cut in $\bigcup_{j \geq i}^{q} S_j$, hence $|S_i \cap C| \geq 2$. As the deletion of S_i increases the number of components of $G - \bigcup_{j=0}^{i-1} S_j$ by exactly one for $i = 1, \ldots, q$, the rank of the partition matroid thus obtained is the same as the rank of the graphic matroid of G, concluding the first half of the theorem.

To show that the given bound is tight, let $G = (V, E)$ be a complete graph on $2k$ vertices. By Nash-Williams' theorem, the colouring number of the graphic matroid of G is k. An edge colouring of a complete graph is called a *Gallai colouring* if no triangle is coloured with three distinct colours, which is a weaker restriction than the above. Bialostocki, Dierker and Voxman [5] showed that every Gallai colouring contains a monochromatic spanning tree. This means that for any reduction of the graphic matroid of G to a partition matroid, there is a partition class of size at least $2k - 1$. $\square$

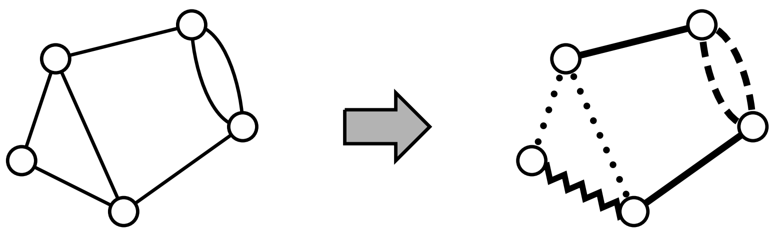


Figure 5.2: An illustration of the proof of Theorem 5.4. The graph $G = (V, E)$ can be decomposed into three forests. Let S_1, S_2, S_3 and S_4 denote the sets of thick, dashed, dotted and zigzag edges, respectively. Then S_{i+1} is a minimum cut in one of the components of $G - \bigcup_{j=1}^{i} S_j$ for $i = 0, \ldots, 3$. Observe that there is no rainbow coloured cycle in G (in which any two edges receive different colours).

Remark 5.5. Theorem 5.4 can be proved in a similar way by observing that any graph that can be decomposed into k forests contains a vertex of degree at most $2k-1$. The advantage of the proof based on cuts is twofold: it provides a rank preserving reduction, and it can be straightforwardly extended to arbitrary matroids in the following sense.

Theorem 5.6. *If $M=(S,\mathcal{I})$ is a matroid so that $M|_{S'}$ has a cut of size at most k for any $S'\subseteq S$, then M can be reduced to a k-colourable partition matroid.*

The proof of Theorem 5.6 is based on the fact that the intersection of a circuit and a cut in a matroid cannot have size 1.

5.2.3 Paving matroids

The next results are about paving matroids.

Theorem 5.7. *Let $M=(S,\mathcal{I})$ be a k-colourable paving matroid of rank $r\geq 2$. Then there exists a $\lceil\frac{rk}{r-1}\rceil$-colourable partition matroid N with $N\preceq M$.*

Proof. Consider any partition $S=S_1\cup\cdots\cup S_{r-1}$ into $r-1$ parts of almost equal sizes, that is, $|S_i|=\lfloor|S|/(r-1)\rfloor$ or $|S_i|=\lceil|S|/(r-1)\rceil$ for $i=1,\ldots,r-1$. As M is k-colourable, we have $|S|\leq kr$ and so $|S_i|\leq\lceil kr/(r-1)\rceil$. As M is paving, any set of size at most $r-1$ is independent, hence the partition matroid N defined by the partition $S_1\cup\cdots\cup S_{r-1}$ is a $\lceil kr/(r-1)\rceil$-colourable reduction of M, as required. $\square$

The bound on the colouring number can be improved when $r=2$, and the reduction can be chosen to be rank preserving. It is not difficult to see that every loopless matroid of rank 2 is paving, hence the next theorem gives a tight bound on the colouring number of the reduction of such matroids.

Theorem 5.8. *Let $M=(S,\mathcal{I})$ be a k-colourable paving matroid of rank 2. Then there exists a $\lfloor\frac{4k}{3}\rfloor$-colourable partition matroid N with $N\preceq_r M$, and the bound for the colouring number of N is tight.*

Proof. Let $S=T_1\cup\cdots\cup T_q$ denote the partition of the ground set into classes of parallel elements, that is, for every $x\in T_i$ and $y\in T_j$ the set $\{x,y\}$ is independent if and only if $i\neq j$. We may assume that $|T_1|\geq\cdots\geq|T_q|$. Note that $|T_1|\leq k$ as the matroid is k-colourable. Let i denote the smallest index such that $|T_1\cup\cdots\cup T_i|\geq|S|/3$ holds, and consider the partition $S=S_1\cup S_2$ where $S_1=T_1\cup\cdots\cup T_i$ and $S_2=T_{i+1}\cup\cdots\cup T_q$. If $i=1$, then $|S_1|=|T_1|\leq k$, otherwise

$$|S_1|=(|T_1|+\cdots+|T_{i-1}|)+|T_i|<\frac{|S|}{3}+|T_i|\leq\frac{|S|}{3}+|T_1|<\frac{2|S|}{3}\leq\frac{4k}{3},$$

where we used that $|S|\leq 2k$ holds as M is k-colourable and $r=2$. By the definition of i, we have $|S_2|\leq 2|S|/3\leq 4k/3$ as well. Thus $\max\{|S_1|,|S_2|\}\leq 4k/3$ always holds, hence the partition matroid N defined by the partition $S_1\cup S_2$ is a $\lfloor 4k/3\rfloor$-colourable reduction of M.

The bound $\lfloor 4k/3\rfloor$ on the colouring number of N is tight. Let S be a set of size $2k$ and take a partition $S=S_1\cup S_2\cup S_3$ where $\lceil|S|/3\rceil=|S_1|\geq|S_2|\geq|S_3|=\lfloor|S|/3\rfloor$. Consider the laminar matroid $M=(S,\mathcal{I})$ defined by the laminar family $\{S,S_1,S_2,S_3\}$ where $X\subseteq S$ is independent if and only if $|X|\leq 2$ and $|X\cap S_i|\leq 1$ for $i=1,2,3$. It is not difficult to see that the colouring number of M is k. Suppose that M is reducible to a partition matroid N. The rank of N is either 1 or 2, as M has rank 2. In the former case $\chi(N)=2k$, while in the latter case N is defined by a partition $S = P_1 \cup P_2$. Then every S_i is a subset of either P_1 or P_2, as otherwise there exists two elements $x, y \in S_i$ such that $x \in P_1$ and $y \in P_2$, implying that $\{x, y\}$ is independent in N but dependent in M, a contradiction. Thus, P_1 or P_2 contains at least two of the S_i's, and so has size at least $|S_2| + |S_3| = |S| - |S_1| = 2k - \lceil 2k/3 \rceil = \lfloor 4k/3 \rfloor$, proving $\chi(N) \geq \lfloor 4k/3 \rfloor$. $\square$

For $r = 3$, we can provide rank preserving reductions at the price of increasing the colouring number of N. For this purpose we need the following simple characterization of the colouring number of paving matroids.

Lemma 5.9. *Let $\mathcal{H} = \{H_1, \ldots, H_q\}$ be a (possibly empty) family satisfying the conditions of Theorem 1.8, and let $M = (S, \mathcal{I})$ be the paving matroid defined by $\mathcal{H}$. Then*

$$\chi(M) = \max\left\{ \left\lceil \frac{|S|}{r} \right\rceil, \left\lceil \frac{|H_1|}{r-1} \right\rceil, \ldots, \left\lceil \frac{|H_q|}{r-1} \right\rceil \right\}.$$

Proof. Corollary 1.4 implies that $\chi(M) = \max\{\lceil |X|/r(X) \rceil : \emptyset \neq X \subseteq S\}$. Since $r(S) = r$ and $r(H_i) = r - 1$ (as every set of size at most $r - 1$ is independent), we get that $\chi(M) \geq \max\{\lceil |S|/r \rceil, \lceil |H_1|/(r-1) \rceil, \ldots, \lceil |H_q|/(r-1) \rceil\}$.

To see the reverse inequality, take an arbitrary subset $X \subseteq S$. If $|X| \leq r-1$, then $r(X) = |X|$ holds as the matroid is paving, therefore $|X|/r(X) = 1$. If $|X| \geq r$ and $X \subseteq H_i$ for some i, then $r(X) = r - 1$ and so $|X|/r(X) \leq |H_i|/(r-1)$. Finally, if $|X| \geq r$ and none of the H_i's contains X, then $r(X) = r$ and so $|X|/r(X) \leq |S|/r$, proving our claim. $\square$

Theorem 5.10. *Let $M = (S, \mathcal{I})$ be a k-colourable paving matroid of rank 3. Then there exists a $(2k-1)$-colourable partition matroid N with $N \preceq_r M$.*

Proof. Let $\mathcal{H} = \{H_1, \ldots, H_q\}$ be a (possibly empty) family satisfying the conditions of Theorem 1.8 that defines M. Without loss of generality, we may assume that $|H_1| \geq \cdots \geq |H_q|$. We distinguish two cases.

Case 1. $|S|/r \leq |H_1|/(r-1)$.

Consider any partition $H_1 = S_1 \cup \cdots \cup S_{r-1}$ into $r - 1$ parts of almost equal sizes, that is, $|S_i| = \lfloor |H_1|/(r-1) \rfloor$ or $|S_i| = \lceil |H_1|/(r-1) \rceil$ for $i = 1, \ldots, r-1$, and let $S_r = S - H_1$. Note that none of $S_1, \ldots, S_r$ is empty since H_1 is a proper subset of S of size at least $r - 1$. Taking any elements $s_1 \in S_1, \ldots, s_r \in S_r$ the set $X = \{s_1, \ldots, s_r\}$ is independent in M as $X \not\subseteq H_i$ $(i = 1, \ldots, q)$ by $|X \cap H_1| = r - 1$ and $|H_1 \cap H_i| \leq r - 2$ $(i = 2, \ldots, q)$. Thus, the partition matroid $N = (S, \mathcal{J})$ defined by the partition $S_1 \cup \cdots \cup S_r$ is a rank preserving reduction of M. N is clearly $\chi(M)$-colourable as $|S_i| \leq \lceil |H_1|/(r-1) \rceil = \chi(M)$ for $i = 1, \ldots, r-1$ and $|S_r| = |S| - |H_1| \leq r|H_1|/(r-1) - |H_1| = |H_1|/(r-1) \leq \chi(M)$.

Case 2. $|S|/r > |H_1|/(r-1)$.

Pick an arbitrary $s \in S$, let $H_{i_1}, \ldots, H_{i_l}$ denote the sets of the family $\mathcal{H}$ containing s and let $H'_j = H_{i_j} - s$ for $j = 1, \ldots, l$. The sets $H'_1, \ldots, H'_l$ are disjoint as $|H_i \cap H_j| \leq r - 2 = 1$ for $i \neq j$. We may assume that $|H'_1| \geq \cdots \geq |H'_l|$. Note that for any set $T \subseteq S - s$ which does not intersect any H'_j properly, the partition $S = \{s\} \cup T \cup (S - T - s)$ defines a partition matroid $N = (S, \mathcal{J})$ which is a reduction of M.

If $|H'_1| + \cdots + |H'_l| < |S|/3$, let $T \subseteq S - s$ be a set of size $\lfloor |S|/2 \rfloor$ containing $H'_1 \cup \cdots \cup H'_l$. Then $\chi(N) = \max\{|T|, |S| - |T| - 1\} \leq |S|/2 < 2|S|/3 \leq 2\chi(M)$. If $|H'_1| + \cdots + |H'_l| \geq |S|/3$, then let j denote the smallest index such that $|H'_1| + \cdots + |H'_j| \geq |S|/3$ and let $T = H'_1 \cup \cdots \cup H'_j$.

If $j = 1$, then $|T| = |H'_1| < 2|S|/3$ by our assumption $|S|/r > |H_1|/(r-1)$ and $r = 3$. Otherwise, $|H_1| < |S|/3$ and so $|T| \leq |S|/3 + |H'_j| \leq |S|/3 + |H'_1| < 2|S|/3$. Thus $\chi(N) = \max\{|T|, |S| - |T| - 1\} < 2|S|/3 \leq 2\lceil |S|/3 \rceil = 2\chi(M)$. $\square$

Remark 5.11. Note that Case 1 of the proof does not rely on the fact that $r = 3$. That is, any paving matroid satisfying the assumption of Case 1 has a rank preserving reduction $N \preceq_r M$ with $\chi(N) = \chi(M)$.

While Theorem 5.10 provides a rank preserving reduction, Theorem 5.7 gives a better bound on the colouring number of the reduction for $r = 3$. The bound $\lceil 3k/2 \rceil$ is not necessarily tight. A computer-assisted case checking shows that the tight bound for $k = 3$ is 4, an extremal example being the Fano matroid. However, we show that $\lceil 3k/2 \rceil$ is tight for infinitely many values of k.

A *finite projective plane* is a pair $(S, \mathcal{L})$, where S is a finite set of *points* and $\mathcal{L} \subseteq 2^S$ is the family of *lines* that satisfies the following axioms: (P1) any two distinct points are on exactly one line, (P2) any two distinct lines have exactly one point in common, (P3) there exists four points, no three of which are collinear. For every projective plane there exists a number q called the *order*, such that (1) each line in the plane contains $q+1$ points, (2) $q+1$ lines pass through each point of the plane, (3) the plane contains q^2+q+1 points and q^2+q+1 lines [73].

The family of lines satisfies the conditions of Theorem 1.8, thus every projective plane defines a paving matroid $M = (S, \mathcal{I})$ of rank 3. A partition matroid $N = (S, \mathcal{J})$ is a reduction of M if and only if the colouring of S defined by the partition classes of N satisfies the conditions of the following theorem.

Theorem 5.12. *Consider any 3-colouring of the points of a projective plane of order q such that each line contains at most 2 colours. Then at least one of the following cases holds:*

(i) there exists an empty colour class,

(ii) there exists a colour class of size 1,

(iii) one of the colour classes is the complement of a line.

Proof. Let 1, 2 and 3 denote the three colours and S_1, S_2, S_3 the corresponding colour classes. The proof is based on the following claim.

Claim 5.13. *There exists a colour class which is a subset of a line.*

Proof. Suppose indirectly that each of the three colour classes contains three non-collinear points. Pick arbitrary points p_1, p_2, p_3 from colour classes S_1, S_2, S_3, respectively. Let L_i denote the line through p_{i+1} and p_{i+2}, and set $m_{i,i+1} = |L_i \cap S_{i+1}|$ and $m_{i,i+2} = |L_i \cap S_{i+2}|$ (all indices are meant in a cyclic order). As every line of the plane has $q+1$ points, we have $m_{i,i+1} + m_{i,i+2} = q+1$. Each line through a fixed point of S_i has exactly one common point with the line L_i, hence $m_{i,i+1}$ of them contain colours i and $i+1$ and $m_{i,i+2}$ of them contain colours i and $i+2$. Since p_1, p_2, p_3 were arbitrary, we get that each line containing colours $i+1$ and $i+2$ has $m_{i,i+1}$ points of colour $i+1$ and $m_{i,i+2}$ points of colour $i+2$.

As S_{i+1} contains three non-collinear points, there exists a point $p'_{i+1} \in S_{i+1} - L_i$. By changing i to $i+1$ in the previous paragraph, we get that exactly $m_{i+1,i+2}$ lines through p'_{i+1} contain colours $i+1$ and $i+2$. As the $m_{i,i+2}$ lines through p'_{i+1} and one of the points of $L_i \cap S_{i+2}$ contain colours $i+1$ and $i+2$, and the number of lines through p'_{i+1} with these colours is $m_{i+1,i+2}$, we get that $m_{i,i+2} \leq m_{i+1,i+2}$. By symmetry, we obtain $m_{i+1,i+2} = m_{i,i+2}$. Therefore,

$$m_{1,2} = m_{3,2} = q+1-m_{3,1} = q+1-m_{2,1} = m_{2,3} = m_{1,3} = q+1-m_{1,2},$$

hence $m_{i,i+1} = m_{i,i+2} = (q+1)/2$ for all i. We get that all lines through p_{i+1} contain $(q+1)/2$ points of colour i, hence $|S_i| = (q+1)^2/2$. Therefore, $|S_1|+|S_2|+|S_3| = 3(q+1)^2/2 > q^2+q+1$, a contradiction. $\square$

By Claim 5.13, we may assume that $S_1 \subseteq L$ for a line L. Suppose indirectly that none of the cases *(i)*, *(ii)* and *(iii)* hold. As $|S_1| \geq 2$, we can pick two distinct points $p_1, p_1' \in S_1$. As none of S_2 and S_3 is the complement of L, there exists $p_2 \in S_2 - L$ and $p_3 \in S_3 - L$. The points of the line through p_1 and p_2 have colour 1 or 2 and the points of the line through p_1' and p_3 have colour 1 or 3, hence the intersection of these lines have colour 1. This intersection point cannot lie on L, hence $S_1 \not\subseteq L$, a contradiction. $\square$

Corollary 5.14. *Let $M = (S, \mathcal{I})$ be a paving matroid of rank 3 defined by the lines of a projective plane of order q. Suppose that $N = (S, \mathcal{J})$ is a partition matroid such that $N \preceq M$. Then*

$$\chi(N) \geq \begin{cases} (|S|-1)/2, & \text{if } q \text{ is odd,} \\ (|S|+1)/2, & \text{if } q \text{ is even.} \end{cases}$$

In particular, if $q \equiv 4 \pmod 6$ then $\chi(N) \geq \left\lceil \frac{3\chi(M)}{2} \right\rceil$, and if equality holds then N is not a rank preserving reduction of M.

Proof. As $N \preceq M$, the colouring defined by the partition classes of N satisfies the conditions of Theorem 5.12. If there exists an empty colour class, then one of the colour classes has size at least $\lceil |S|/2 \rceil = (|S|+1)/2$. If there exists a colour class containing only one point p, then each of the $q+1$ lines through p are monochromatic except for p, hence one of the colour classes has size at least $q\lceil (q+1)/2 \rceil$, that is, $q(q+1)/2 = (|S|-1)/2$ if q is odd, and $q(q+2)/2 = (|S|+q-1)/2$ if q is even. If one of the colour classes is the complement of a line, then it has size $q^2 > (|S|+1)/2$. In all three cases, we get a colour class of size at least $(|S|-1)/2$ if q is odd, and $(|S|+1)/2$ if q is even, proving our bound on the colouring number of N.

Assume now that $q \equiv 4 \pmod 6$. Lemma 5.9 implies that $\chi(M) = \max\{\lceil (q^2+q+1)/3 \rceil, \lceil (q+1)/2 \rceil\} = (q^2+q+1)/3$, and so $\lceil 3\chi(M)/2 \rceil = \lceil (q^2+q+1)/2 \rceil = (|S|+1)/2 \leq \chi(N)$. If equality holds then N has rank 2, that is, one of the colour classes is empty, since we have strict inequalities above in the other two cases. $\square$

Corollary 5.14 implies that the bound $\lceil 3k/2 \rceil$ for paving matroids of rank 3 is tight for infinitely many values of k. Indeed, consider projective planes of order $q = 4^\ell$ for $\ell \in \mathbb{Z}_{>0}$ and set $k = \frac{q^2+q+1}{3}$.

The next example shows another family of paving matroids of rank three which prove that the bound $\lceil 3k/2 \rceil$ for some other values of k as well.

Example 5.15. Let $G = (V, E)$ denote the complete graph on vertices $V = \{v_0, \ldots, v_{n-1}\}$. Choose n to be even and consider a proper edge colouring $c : E \to \{c_1, c_2, \ldots, c_{n-1}\}$ of G with $n-1$ colours (that is, adjacent edges receive different colours). Let

$$\mathcal{H} = \{\{v_i, v_j, c(v_iv_j)\} : v_iv_j \in E\} \cup \{\{c_1, \ldots, c_{n-1}\}\}.$$

Then we have $|H_1 \cap H_2| \leq 1$ for every $H_1, H_2 \in \mathcal{H}$, $H_1 \neq H_2$ as two edges with a common vertex have different colours. As every set of the system $\mathcal{H}$ has at least 3 elements, Theorem 1.8 implies that $\mathcal{H}$ defines a paving matroid $M = (S, \mathcal{I})$ of rank 3 on ground set $S = \{v_0, \ldots, v_{n-1}, c_1, \ldots, c_{n-1}\}$. We have $|S| = 2n-1$ and $\chi(M) = \lceil |S|/3 \rceil = \lceil (2n-1)/3 \rceil$ by Lemma 5.9. Note that for $n = 4$ this matroid coincides with the Fano matroid (which is the matroid defined by the projective plane of order 2). It is not difficult to prove that for $n = 6$ the 4-colourable matroid M is not reducible to any 5-colourable partition matroid, thus the bound $\lceil 3k/2 \rceil$ is tight for $k = 4$.

In general, it remains open for what (even) values of n there exists a proper edge colouring c such that M is not reducible to any $(n-1)$-colourable partition matroid. A computer-assisted case checking shows that the edge colouring $c(v_i v_{n-1}) = c(v_{(i+j) \bmod (n-1)} v_{(i-j) \bmod (n-1)}) = i$ (for $i = 0, \ldots, n-2$, $j = 1, \ldots, \frac{n}{2} - 1$) has this property for $n = 4, 6, 10, 12, 14, 16$ but does not have this property for $n = 8$. Nevertheless, it is not difficult to check that if a good colouring exists for an even value of n such that $n \not\equiv 1 \pmod 3$, then the bound $\lceil 3k/2 \rceil$ is tight for $k = \lceil (2n-1)/3 \rceil$.

5.3 Gammoids

The aim of this section is to prove our main result, Theorem 5.16.

Theorem 5.16. *Let $M = (S, \mathcal{I})$ be a k-colourable gammoid ($k \geq 2$). Then there exists a $(2k-2)$-colourable partition matroid N with $N \preceq_r M$, and the bound for the colouring number of N is tight.*

Proof. Let $M = (S, \mathcal{I})$ be a k-colourable gammoid where $k \geq 2$. By the result of Ingleton and Piff, M can be obtained as the restriction of the dual of a transversal matroid. Let R be such a transversal matroid, and choose R in such a way that its rank is as small as possible. Let $G = (A, B; E)$ be a bipartite graph defining R with $S \subseteq A$ and $|B|$ being the rank of R.

The high-level idea of the proof is the following. First we show that there exists a B_2-forest F in G. Then, by using an alternating structure on the components of F, we prove that F can be chosen in such a way that every component contains at most $2k-2$ vertices from S. Let $\mathcal{C}$ denote the set of the connected components of F, and let $N = (S, \mathcal{J})$ be the partition matroid corresponding to partition classes $S(C)$ for $C \in \mathcal{C}$. Every component C is a B_2-tree, hence it contains a perfect matching between $B(C)$ and $A(C) - a$ for any $a \in A(C)$. That is, if we leave out exactly one vertex from $A(C)$ for each $C \in \mathcal{C}$, the remaining vertices of A form a basis of R, and so the set of deleted vertices form a basis in the strict gammoid that is the dual of R. This implies that $N \preceq M$ with $\chi(N) \leq 2k-2$, thus proving the theorem.

We start with an easy observation.

Claim 5.17. *G contains k matchings of size $|B|$ such that every vertex in S is covered by at most $k-1$ of them.*

Proof. Observe that a set $X \subseteq S$ is independent in M if and only if $A - X$ contains a basis of R, that is, $G - X$ has a matching covering B. The assumption that M is k-colourable is equivalent to the condition that S can be partitioned into k independent sets of M, and the claim follows. □

The following claim proves an inequality that we will rely on.

Claim 5.18. $k \cdot (|A| - |B|) - |S - X| \geq k \cdot \max\{|Y| - |N(Y)| :\ Y \subseteq X\}$ *for every* $X \subseteq A$.

Proof. Let $\bar{R}$ be the matroid that is obtained from R by adding $k-1$ parallel copies of every element in $A-S$, and adding $k-2$ parallel copies of every element in S. The ground set A' of $\bar{R}$ has size $(k-1)|S|+k|A-S|$. Then Claim 5.17 states that $\bar{R}$ has k pairwise disjoint bases.

Let $X \subseteq A$ be an arbitrary set and let X' be the set consisting of all the parallel copies of the elements of X. Then $|X'| = (k-1)\cdot|X\cap S| + k\cdot|X-S|$ and $r_{\bar{R}}(X') = r_R(X) = \min\{|X|-|Y|+|N(Y)| :\ Y \subseteq X\}$. Recall that $|A'| = (k-1)\cdot|S|+k\cdot|A-S|$ and $r_{\bar{R}}(A') = |B|$, hence

$$\begin{aligned}|A'|-|X'| &= (k-1)\cdot|S| + k\cdot|A-S| - (k-1)\cdot|X\cap S| - k\cdot|X-S|\\ &= (k-1)\cdot|A| + |A-S| - (k-1)\cdot|X| - |X-S|\\ &= (k-1)\cdot|A-X| + |A-S-X|,\end{aligned}$$

and

$$\begin{aligned}r_{\bar{R}}(A') - r_{\bar{R}}(X') &= |B| - \min\{|X|-|Y|+|N(Y)| :\ Y\subseteq X\}\\ &= |B| - |X| + \max\{|Y|-|N(Y)| :\ Y \subseteq X\}.\end{aligned}$$

By Corollary 1.5 and Claim 5.17, $|A'|-|X'| \geq k\cdot(r_{\bar{R}}(A') - r_{\bar{R}}(X'))$, thus we get

$$(k-1)\cdot|A-X| + |A-S-X| \geq k\cdot(|B|-|X|+\max\{|Y|-|N(Y)| :\ Y\subseteq X\}).$$

After rearranging, we obtain

$$k\cdot(|A|-|B|) - |S-X| \geq k\cdot\max\{|Y|-|N(Y)| :\ Y \subseteq X\}$$

as stated. □

Our next goal is to show that there exists a B_2-forest in G.

Claim 5.19. *$G=(A,B;E)$ contains a B_2-forest.*

Proof. As G has a matching of size $|B|$, the Hall condition holds for every subset of B, thus $|N(U)| \geq |U|$ for every $U \subseteq B$. Let us call a set $U \subseteq B$ *tight* if $|N(U)| = |U|$. Assume that G does not have a B_2-forest. Then, by Theorem 1.6, there exists a nonempty tight set in B. For arbitrary tight sets $U, W \subseteq B$, we get

$$\begin{aligned}|U|+|W| &= |N(U)|+|N(W)| = |N(U)\cap N(W)| + |N(U)\cup N(W)|\\ &\geq |N(U\cap W)| + |N(U\cup W)| \geq |U\cap W| + |U\cup W|\\ &= |U|+|W|,\end{aligned}$$

hence equality holds throughout, and so $U\cap W$ and $U \cup W$ are also tight. This implies that there is a unique maximal tight set $\emptyset \neq Z \subseteq B$.

Let $X = A - N(Z)$. As Z is a tight set, $\max\{|Y|-|N(Y)| :\ Y\subseteq X\} \geq |X|-|N(X)| \geq |A-N(Z)|-|B-Z| = |A|-|B|$, thus $S-X = N(Z)\cap S = \emptyset$ by Claim 5.18. Furthermore, every matching of size $|B|$ provides a perfect matching between Z and $N(Z)$. That is, R is the direct sum of the transversal matroids R' and R'' defined by $G[Z\cup N(Z)]$ and $G[(B-Z)\cup(A-N(Z))]$, respectively. Therefore M is the restriction of the dual of R'' to S, contradicting the minimal choice of R. □

Take an arbitrary B_2-forest F in G. We will need the following technical claim.

Claim 5.20. *Every leaf of F is in S.*

Proof. Suppose to the contrary that F has a leaf vertex $a \in A - S$. Let $b \in B$ be the unique neighbour of a in F, and let $G' = G - \{a, b\}$ denote the graph obtained by deleting vertices a and b form G. Let $M' = (S, \mathcal{I}')$ denote the restriction of the dual of the transversal matroid defined by G' to S. As the strong Hall condition holds for G, the maximum size of a matching of G' is $|B| - 1$. We claim that $M = M'$, contradicting the minimality of G.

Take an arbitrary set $X \in \mathcal{I}'$. By definition, $G' - X$ has a matching P' covering $B - b$. Then $P' + ab$ is a matching of $G - X$ covering B, showing that $\mathcal{I}' \subseteq \mathcal{I}$.

To see the opposite direction, consider any set $X \in \mathcal{I}$. By definition, $G - X$ has a matching P covering B. Take an arbitrary matching P' of G' covering $B - b$. Now $|P| = |B| = |B - b| + 1 = |P'| + 1$, hence the symmetric difference $P \triangle P'$ contains an alternating path Q whose first and last edges are in P, and one of the end vertices of Q is b. Then $P \triangle Q$ is a matching of $G' - X$ covering $B - b$, implying $X \subseteq \mathcal{I}'$. $\square$

We denote the difference $|A| - |B|$ by q. As M is the restriction of R to S, $r_M(S) \leq q$ is clearly satisfied. Moreover, equality holds since, by Claim 5.20, every leaf of F is in S, and taking an arbitrary leaf in every component of F results a basis of M.

Let $\mathcal{C}$ denote the set of connected components of F. Note that the forest might have components consisting of a single vertex of A. We have $|\mathcal{C}| = |A| - |B| = q$ as $|A(C)| = |B(C)| + 1$ for each $C \in \mathcal{C}$. We call a component $C \in \mathcal{C}$ *large* if $|S(C)| \geq 2k - 1$, *normal* if $k \leq |S(C)| \leq 2k - 2$, and *small* if $|S(C)| \leq k - 1$. We say that a component $C' \in \mathcal{C}$ is *reachable* from a component $C'' \in \mathcal{C}$ if there exists an alternating sequence $C_1, b_1a_2, C_2, b_2a_3, \ldots, b_{p-2}a_{p-1}, C_{p-1}, b_{p-1}a_p, C_p$ of components and edges such that $C_1 = C''$, $C_p = C'$, and $b_i \in B(C_i)$, $a_{i+1} \in A(C_{i+1})$ hold for $i = 1, \ldots, p - 1$. Such an alternating sequence is called a *path*, the *length* of the path being $p - 1$. The *distance* of C' from C'' is the minimum length of a path from C'' to C'.

We define a potential function on the set of B_2-forests as follows. Let $\nu \gg \mu_1 \gg \lambda_1 \gg \mu_2 \gg \lambda_2 \gg \cdots \gg \mu_{q-1} \gg \lambda_{q-1}$ be a decreasing sequence of $2q - 1$ positive numbers such that the ratio between any two consecutive ones is at least $|A| + 2$. Recall that $|\mathcal{C}| = q$. For a component $C \in \mathcal{C}$, the minimum distance of C from a large component is denoted by $\mathrm{dist}(C)$. We define $\mathrm{dist}(C)$ to be $+\infty$ if C is not reachable from any of the large components. The *potential* of the B_2-forest F is defined as

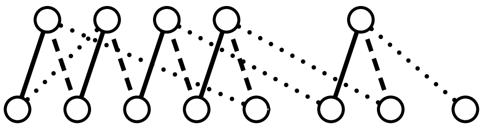
$$
\begin{aligned}
\varphi(F) = \nu \cdot \sum_{C \in \mathcal{C}} \max\{|S(C)| - (2k-2), 0\} \qquad & \text{(total violation)} \\
- \sum_{i=1}^{q-1} \mu_i \cdot |\{C \in \mathcal{C} : \ \mathrm{dist}(C) = i\}| \qquad & \text{(number of components at distance } i\text{)} \\
+ \sum_{i=1}^{q-1} \lambda_i \cdot \sum_{\substack{C \in \mathcal{C} \\ \mathrm{dist}(C) = i}} |S(C)|. \qquad & \text{(number of } S\text{-vertices in components at distance } i\text{)}
\end{aligned}
$$

Let F be a B_2-forest for which $\varphi(F)$ is as small as possible. The following claim concludes the proof of the theorem.

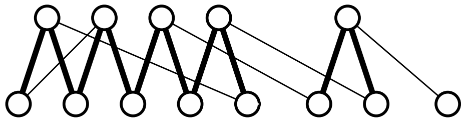
Claim 5.21. *F has no large components.*

Proof. Suppose indirectly that there exists a large component. By applying Claim 5.18 with $X = \emptyset$, $|S| \leq k \cdot (|A| - |B|) = k \cdot |\mathcal{C}|$, hence, by the pigeonhole principle, there exists a small component as well.

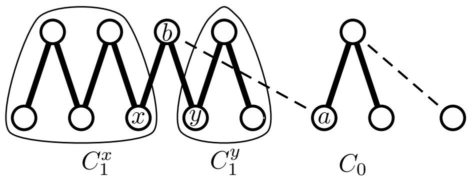
First we show that there exists a small component that is reachable from a large component. Suppose indirectly that this is not true, and let $\mathcal{C}' \subseteq \mathcal{C}$ denote the set of components that are



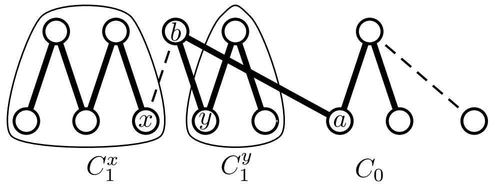
(a) A graph $G = (A, B; E)$ with three matchings of size $|B|$ such that every vertex in A is covered by at most two of them.



(b) A B_2-forest F of G. For simplicity, every component of F is chosen to be a path.



(c) Alternating structure on the connected components of F.



(d) The B_2-forest we obtain by substituting C_0 and C_1 by $C_0 + ab + by + C_1^y$ and C_1^x.

Figure 5.3: An illustration of the proof of Theorem 5.16. In the example, $k = 3$ and $S = A$. The only large component of F is C_1, all the other components are small.

not reachable from a large component. Note that $\mathcal{C}'$ consists of normal and small components. Define $X = \bigcup\{A(C) : C \in \mathcal{C}'\}$. By the definition of reachability, $N(X) = \bigcup\{B(C) : C \in \mathcal{C}'\}$ and so $|X| - |N(X)| = |\mathcal{C}'|$. As every component in $\mathcal{C} - \mathcal{C}'$ is either normal or large and there is at least one large component, $|S - X| \geq k \cdot |\mathcal{C} - \mathcal{C}'| + 1$. Then

$$\begin{aligned} k \cdot \max\{|Y| - |N(Y)| : Y \subseteq X\} &\geq k \cdot (|X| - |N(X)|) = k \cdot |\mathcal{C}'| = k \cdot (|\mathcal{C}| - |\mathcal{C} - \mathcal{C}'|) \\ &= k \cdot (|A| - |B|) - k \cdot |\mathcal{C} - \mathcal{C}'| \geq k \cdot (|A| - |B|) - |S - X| + 1, \end{aligned}$$

contradicting Claim 5.18.

Let C_0 be a small component with $\operatorname{dist}(C_0)$ being minimal. By the above, $\operatorname{dist}(C_0) < +\infty$. Consider a shortest path from the set of large components to C_0, and let C_1 be the last component on the path before C_0. By the definition of a path, there exists an edge ab with $a \in A(C_0)$ and $b \in B(C_1)$. Let $x, y \in A(C_1)$ denote the neighbours of b in C_1. The deletion of b from C_1 results in two connected components C_1^x and C_1^y such that $x \in C_1^x$ and $y \in C_1^y$ (see Figure 5.3).

Assume first that C_1 is a large component. As $|S(C_1)| \geq 2k - 1$ and $|S(C_0)| \leq k - 1$, either $|S(C_0 + ab + bx + C_1^x)| < |S(C_1)|$ or $|S(C_0 + ab + by + C_1^y)| < |S(C_1)|$ by Claim 5.20. Hence substituting C_0 and C_1 either by $C_0 + ab + bx + C_1^x$ and C_1^y or by $C_0 + ab + by + C_1^y$ and C_1^x decreases the total violation in $\varphi(F)$, a contradiction.

Therefore, C_1 is a normal component, and there is another non-small component C_2 before C_1 on the shortest path from the set of large components to C_0, together with an edge $b'a'$ with $a' \in A(C_1)$ and $b' \in A(C_2)$. We may assume that $a' \in C_1^x$. We distinguish two cases.

Case 1. $|S(C_1^x)| \geq |S(C_1)| - |S(C_0)|$

Modify F by substituting components C_0 and C_1 by $C_0 + ab + by + C_1^y$ and C_1^x, respectively. By the assumption, $|S(C_0 + ab + by + C_1^y)| = |S(C_0)| + |S(C_1)| - |S(C_1^x)| \leq 2 \cdot |S(C_0)| \leq 2k - 2$, thus no new large component appears. Furthermore, the set of components with distance less than $\operatorname{dist}(C_1)$ does not change. The distance of C_1^x remains $\operatorname{dist}(C_1)$ because of the edge $b'a'$.

If the distance of $C_0 + ab + by + C_1^y$ is $\text{dist}(C_1)$, then the number of components at distance $\text{dist}(C_1)$ increases. Otherwise, the distance of $C_0 + ab + by + C_1^y$ is at least $\text{dist}(C_1) + 1$, hence the number of S-vertices in components at distance $\text{dist}(C_1)$ decreases by Claim 5.20. In both cases, $\varphi(F)$ decreases, a contradiction.

Case 2. $|S(C_1^x)| < |S(C_1)| - |S(C_0)|$

Modify F by substituting components C_0 and C_1 by $C_0 + ab + bx + C_1^x$ and C_1^y, respectively. By the assumption, $|S(C_0 + ab + bx + C_1^x)| \leq |S(C_0)| + |S(C_1)| - |S(C_0)| = |S(C_1)|$. As C_1 is normal, no new large component appears. Furthermore, the set of components with distance less than $\text{dist}(C_1)$ does not change. The distance of $C_0 + ab + bx + C_1^x$ remains $\text{dist}(C_1)$ because of the edge $b'a'$. The distance of C_1^y is either $\text{dist}(C_1)$ or $\text{dist}(C_0)$. In the former case, the number of components at distance $\text{dist}(C_1)$ increases, while in the latter case, the number of S-vertices in components at distance $\text{dist}(C_1)$ decreases as $|S(C_1^x)| + |S(C_0)| < |S(C_1)|$. In both cases, $\varphi(F)$ decreases, a contradiction. □

By Claim 5.21, F has no large component. As we have seen before, the partition matroid $N = (S, \mathcal{J})$ corresponding to partition classes $S(C)$ for $C \in \mathcal{C}$ is a reduction of the original gammoid M with colouring number at most $2k-2$. By Claim 5.20, $S(C)$ is nonempty for every $C \in \mathcal{C}$ and $r_M(S) = |\mathcal{C}| = q$, hence the reduction is rank preserving.

The bound on the colouring number of N is tight. Consider the laminar matroid $M = (S, \mathcal{I})$ defined by the laminar family $\{S, S_1, \ldots, S_k\}$ where $S_1 \cup \cdots \cup S_k$ is a partition of S into subsets of size $k-1$. That is, the size of the ground set S is $k^2 - k$. We define a set $X \subseteq S$ to be independent in M if $|X \cap S_i| \leq 1$ for $i = 1, \ldots, k$, and $|X| \leq k-1$. It is not difficult to see that M is a strict gammoid with colouring number k.

We claim that if $N \preceq M$ is a partition matroid, then $\chi(N) \geq 2k-2$. Let $P_1 \cup \cdots \cup P_q$ denote the partition defining N. Then every S_i is a subset of some P_j, as otherwise there exists two elements $x, y \in S_i$ such that $x \in P_a$ and $y \in P_b$ for $a \neq b$, implying that $\{x, y\}$ is independent in N but dependent in M, a contradiction. As the rank of M is $k-1$, we have $q \leq k-1$. By the above, there exists a class P_j that contains at least two of the S_i's, and so has size at least $2k-2$, proving $\chi(N) \geq 2k-2$. □

For the first sight, the proof seems to provide a polynomial-time algorithm for determining the partition matroid, assuming that a digraph $D = (V, A)$ representing the gammoid is given. A bipartite graph $G = (A, B; E)$ representing R can be constructed from D (see e.g. [22]). The reductions appearing in the proofs of Claims 5.19 and 5.20 can be performed in polynomial time, hence we may assume that G contains a B_2-forest F. Such a forest can be found by [52]. By using the alternating structure described in the proof of Claim 5.21, we can modify F to get a B_2-forest in which every component contains at most $2k-2$ vertices from S. However, it is not clear how to bound the number of augmentation steps as the coefficients in the potential function can be exponential. An interesting question is whether this procedure terminates after a polynomial number of steps.

5.4 Truncation and reducibility

The following theorem provides new examples of matroids for which Conjecture 5.2 holds. In fact, it provides new results only for truncations of graphic matroids, as the truncation of a paving matroid is either a free matroid or itself, and the truncation of a gammoid is a gammoid again.

Theorem 5.22. *The family of matroids M that can be reduced to a $2\chi(M)$-colourable partition matroid is closed for truncation.*

Proof. Let $M = (S, \mathcal{I})$ denote a matroid of rank r that is reducible to a $2\chi(M)$-colourable partition matroid N. As every k-truncation of M can be obtained by a series of $r-1, r-2, \ldots, k$-truncations, it suffices to prove that the $(r-1)$-truncation M' of M is reducible to a $2\chi(M')$-colourable partition matroid.

Let $S = S_1 \cup \cdots \cup S_q$ denote the partition that defines N. We may assume that $|S_1| \geq \cdots \geq |S_q|$. If $q \leq r-1$, then N is already a $2\chi(M)$-colourable reduction of M' and the claim follows by $\chi(M') \geq \chi(M)$. Hence, assume that $q = r$. Consider the partition matroid N' defined by the partition classes $S_1, S_2, \ldots, S_{r-2}, S_{r-1} \cup S_r$. Then N' is a reduction of M', hence it is sufficient to prove that N' is $2\chi(M')$-colourable.

If $|S_{r-1}| + |S_r| \leq |S_1|$, then $\chi(N') = |S_1| = \chi(N) \leq 2\chi(M) \leq 2\chi(M')$. Otherwise $|S_{r-1}| + |S_r| > |S_1|$, and so $\chi(N') = |S_{r-1}| + |S_r|$. Using $|S| = |S_1| + \cdots + |S_r|$ and $|S_i| \geq (|S_{r-1}| + |S_r|)/2$ for $i = 1, 2, \ldots, r-2$, we get

$$|S| \geq (r-1) \cdot \frac{|S_{r-1}| + |S_r|}{2} + |S_{r-1}| + |S_r| = \frac{r+1}{2} \cdot (|S_{r-1}| + |S_r|) = \frac{r+1}{2} \cdot \chi(N').$$

That is,

$$\chi(N') \leq \frac{2|S|}{r+1} < 2 \cdot \frac{|S|}{r-1} \leq 2\chi(M'),$$

concluding the proof of the theorem. $\square$

Remark 5.23. Note that an analogous statement holds if we replace $2\chi(M)$ by $2\chi(M) - 1$, as we proved $\chi(N') < 2\chi(M')$ in the second case. As laminar matroids can be obtained from free matroids by taking direct sums and truncations, Theorem 5.22 provides a simple proof that every k-colourable laminar matroid is reducible to a $(2k-1)$-colourable partition matroid. As laminar matroids form a subclass of gammoids, Theorem 5.16 implies that the bound can be improved to $2k-2$. However, it is not clear whether the analogue of Theorem 5.22 holds if we replace $2\chi(M)$ by $2\chi(M) - 2$.

5.5 An application: list colouring of two matroids

In this section we show how our techniques can be applied to the list colouring problem of two matroids.

Assume that a list L_e of colours is given for each edge $e \in E$ of a graph $G = (V, E)$. A *proper list edge colouring* of G is a proper edge colouring such that every edge e receives a colour from its list L_e. The *list edge colouring number* is the smallest integer k for which G has a proper list edge colouring whenever $|L_e| \geq k$ for every $e \in E$. The List Colouring Conjecture [39, 72] states that for any graph, the list edge colouring number equals the edge colouring number. The conjecture is widely open, and only partial results are known. The probably most famous one is the celebrated result of Galvin [27] who showed that the conjecture holds for bipartite multigraphs.

Theorem 5.24 (Galvin)**.** *The list edge colouring number of a bipartite graph is equal to its edge colouring number, that is, to its maximum degree.*

Matchings in bipartite graphs are forming the common independent sets of two matroids, hence one might consider matroidal generalizations of list colouring. If a list L_s of colours is given for each element $s \in S$, then a *proper list edge colouring* of M is a colouring such that every element s receives a colour from its list L_s. The *list colouring number* is the smallest integer k for which M has a proper list colouring whenever $|L_s| \geq k$ for every $s \in S$. Analogously, we define the *proper list edge colouring* of the intersection of M_1 and M_2, and define the *list colouring number* $\chi_\ell(M_1, M_2)$ to be the smallest integer k for which the intersection of M_1 and M_2 has a proper list colouring whenever $|L_s| \geq k$ for every $s \in S$. Hence, Theorem 5.24 states that if both M_1 and M_2 are partition matroids then $\chi_\ell(M_1, M_2) = \max\{\chi(M_1), \chi(M_2)\}$.

Seymour observed [68] that the list colouring theorem holds for a single matroid.

Theorem 5.25 (Seymour)**.** *The list colouring number of a matroid is equal to its colouring number.*

Lasoń [48] gave a generalisation of the theorem when the sizes of the lists are not necessarily equal. As a common generalisation of Theorems 5.24 and 5.25, it is tempting to conjecture that $\chi(M_1, M_2) = \chi_\ell(M_1, M_2)$ holds for every pair of matroids [43]. No pair M_1, M_2 is known for which the conjecture fails. Nevertheless, there are only a few matroid classes for which the problem was settled. Király and Pap [44] verified the conjecture for transversal matroids, for matroids of rank two, and if the common bases are the arborescences of a digraph which is the disjoint union of two spanning arborescences rooted at the same vertex.

In [42], Király proposed a weakening of the problem where the aim is to find a constant c such that if the colouring number is k, then the list colouring number is at most $c \cdot k$. For spanning arborescences, it was observed by Kobayashi [42] that the constructive characterization of k-arborescences implies that lists of size $\frac{3}{2}k+1$ are sufficient. As $\chi_\ell(N_1, N_2) = \max\{\chi(N_1), \chi(N_2)\}$ holds for partition matroids N_1 and N_2, lists of size $2k$ are sufficient whenever Conjecture 5.2 holds for matroids M_1 and M_2. In particular, Theorem 5.16 implies that list of size $2k - 2$ are sufficient if M_1 and M_2 are gammoids.

5.6 Some negative results

In this section we give counterexamples to some possible conjectures about reductions of matroids. Many of them were only checked by computer and it remains open to prove them. For computer searches involving matroids, database [15] was a useful tool in many cases.

5.6.1 Reduction to strongly base orderable matroids

To use Galvin's theorem in the previous list colouring problem, it was crucial that we considered reductions only to partition matroids. In other applications, by Theorem 3.4 of Davies and McDiarmid it would be sufficient to find a reduction to a strongly base orderable matroid without increasing its colouring number too much.

In particular, if every k-colourable matroid were reducible to a $(k + 1)$-colourable strongly base orderable matroid, then the first claim of Conjecture 4.28 would follow by Theorem 3.4. Notice that for $k = 2$ this is equivalent to the conjecture that every 2-colourable matroid is reducible to a 3-colourable partition matroid. This equivalence follows from the fact that every 2-colourable strongly base orderable matroid $M = (S, \mathcal{I})$ is reducible to a 2-colourable partition matroid. Indeed, if $S = B_1 \cup B_2$ for bases B_1, B_2 of M and $f \colon B_1 \to B_2$ is a bijection guaranteed by the definition of strongly base orderability, then $\{\{x, f(x)\} : x \in B_1\}$ forms the classes of a 2-colourable partition matroid N such that $N \preceq M$.

For $k = 3$, a complicated computer-assisted case checking shows that the graphic matroid of K_6 is not reducible to any 4-colourable strongly base orderable matroid. We outline the proof of the weaker statement that it is not reducible to any 3-colourable strongly base orderable matroid, or more generally, the graphic matroid $M = (S, \mathcal{I})$ of K_{2k} is not reducible to any k-colourable strongly base orderable matroid. If $N \preceq M$ for a k-colourable matroid $N = (S, \mathcal{J})$, then $|S| = kr_M(S)$ implies that there exists disjoint bases $B_1, \dots, B_k$ of M which are bases of N. It can be proved that if a graph is the union of two disjoint spanning trees then its graphic matroid is not reducible to any 2-colourable partition matroid. This implies that there is no bijection $f\colon B_1 \to B_2$ such that $B_1 - X + f(X)$ is a spanning tree for every $X \subseteq B_1$, that is, for every bijection $f\colon B_1 \to B_2$ there exists $X \subseteq B_1$ such that $B_1 - X + f(X) \notin \mathcal{I}$, and so $B_1 - X + f(X) \notin \mathcal{J}$. Thus the restriction $N|_{B_1 \cup B_2}$ is not strongly base orderable, and so neither is N.

However, the following question remains open.

Question 5.26. *Is is true that the graphic matroid of K_{2k} is not reducible to any $(2k-2)$-colourable strongly base orderable matroid?*

5.6.2 Reduction of matroids without $(k+1)$-spanned elements

It is natural to ask whether Conjecture 3.7 of Kotlar and Ziv can be proved by reduction to partition matroids. In particular, if every matroid without $(k+1)$-spanned elements were reducible to a k-colourable partition matroid, then the conjecture would follow. The following example shows that this is not the case.

Recall that finite projective planes define paving matroids of rank three. The Fano matroid is the matroid defined by the Fano plane, the projective plane with seven points.

Proposition 5.27. *The dual of the Fano matroid is a matroid without 3-spanned elements which is not reducible to any 2-colourable partition matroid.*

Proof. Let $M = (S, \mathcal{I})$ denote the dual of the Fano matroid. M is a paving matroid of rank 4, where the dependent sets of size 4 are the complements of the lines of the Fano plane. Suppose that $s \in S$ is a 3-spanned element in M, that is, s is spanned by disjoint sets $X, Y \subseteq S - s$. As every set of size 3 is independent in M, we have $4 \leq |X + s|$ and $4 \leq |Y + s|$, thus $|X| = |Y| = 3$ and $S = X \cup Y \cup \{s\}$. As $X + s$ and $Y + s$ are dependent sets of size 4, the complement of these sets are lines of the Fano plane, thus Y and X are disjoint lines. This is a contradiction since X and Y are disjoint.

Suppose that $N \preceq M$ for a 2-colourable partition matroid N. As $|S| = 7$ and M has rank 4, N is defined by a partition $S = \{s\} \cup S_1 \cup S_2 \cup S_3$ with $|S_1| = |S_2| = |S_3| = 2$. Among the seven lines of the Fano plane three contain s, and for each $i = 1, 2, 3$ exactly one contains both points of S_i, thus there is at least one line containing exactly one point from each S_i. The complement of this line is independent in M and dependent in N, a contradiction. ☐

As suggested in the previous subsection, using Theorem 3.4 of Davies and McDiarmid it would be sufficient to find a reduction of a matroid without $(k+1)$-spanned elements to a k-colourable strongly base orderable matroid. Unfortunately, a computer-assisted case checking shows that every 2-colourable reduction of the dual M of the Fano matroid is either equals to M or isomorphic to the direct sum of the one element matroid and the graphic matroid of K_4. As neither of these two matroids is strongly base orderable, it follows that M is not reducible to any 2-colourable strongly base orderable matroid.

5.6.3 Reduction to matroids without $(k+2)$-spanned elements

Concerning the relationship of the conjecture of Kotlar and Ziv and the conjecture of Aharoni and Berger, the following question arises. Is every k-colourable matroid reducible to a matroid without $(k+2)$-spanned elements? If this were the case, then the first statement of Conjecture 4.28 would follow from Conjecture 3.7. Unfortunately, a computer-assisted case checking shows that the graphic matroid of K_6 is a counterexample to this possible conjecture.

Let $G = (V, E)$ denote the complete graph K_n and M its graphic matroid. Clearly, every element of M is $(n-1)$-spanned, as each edge of K_n is contained in $n-2$ triangles. Let $N \preceq M$ be any reduction of M. Colour the edges of G such that two edges have the same colour if and only if they are parallel elements in N, and let c denote the colouring obtained. Suppose that there exists an edge $e = uv \in E$ such that

$$|\{f \in E : c(e) = c(f)\}| + |\{w \in V : c(uv) \neq c(vw) \neq c(wu) \neq c(uv)\} \geq n-1. \qquad (\star)$$

It is clear that e is spanned by $\{f\}$ whenever $c(e) = c(f)$, that is, e and f are parallel. Notice that e is also spanned by $\{f, g\}$ whenever $c(f) \neq c(g)$ and e, f, g are the edges of a triangle of G, as we have $2 \leq r_N(\{f, g\}) \leq r_N(\{e, f, g\}) \leq r_M(\{e, f, g\}) = 2$ in this case. Thus $(\star)$ implies that $n-1$ disjoint sets span e. A computer-assisted case checking shows that for $n \leq 6$ and every (not necessarily proper) edge colouring c, K_n has an edge e satisfying $(\star)$. This implies that for $n \leq 6$ every reduction of M contains an $(n-1)$-spanned element, in particular, the 3-colourable graphic matroid of K_6 is not reducible to any matroid without 4-spanned elements.

5.7 Conclusions

In this chapter we proved Conjecture 5.2 for the case of transversal matroids, paving matroids, truncations of graphic matroids and gammoids. However, there is more work to do even in these special cases. For paving matroids of rank $r \geq 4$ our reduction is not rank preserving, and we do not know whether the bound $\lceil \frac{rk}{r-1} \rceil$ is tight for infinitely many values of k. For gammoids it is not clear whether our algorithm is polynomial or not. Another class of matroids whose reductions might be of special interest is the family of matroids appearing in Woodall's Conjecture 3.12.

We find it possible that our algorithm for gammoids can be extended to the general case. If this is not the case and Conjecture 5.2 turns out to be false, a weaker conjecture might still be true where we allow the reduction N to be any $2k$-colourable strongly base orderable matroid. As the proof of Theorem 4.23 does not provide any algorithm for partitioning the ground set into $2\max\{\chi(M_1), \chi(M_2)\}$ common independent sets, a polynomial algorithm for finding such a strongly base orderable reduction would have many applications.

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