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Introduction to Hodge theory on compact Kähler manifolds

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Introduction

The main goal of this thesis is to prove the Hodge and Lefschetz decompositions of the cohomology groups of a compact Kähler manifold using harmonic theory. Both of these are standard and beautiful theorems with many applications and generalisations.

The thesis mainly follows the books of [7] Wells, R. O. - *Differential Analysis on Complex Manifolds* and [3] Huybrechts, D. - *Complex Geometry: An Introduction.* We assume that the reader is familiar with the basic theory of smooth manifolds and Lie groups.

The thesis consists of two parts. The first is the foundational material, which consists of definitions and results used in the second part. There are five sections in the foundational material. In the first we introduce the notion of almost complex structures and compatible metrics on a real vector space V. We will show how these structures decompose the exterior algebra of V. We define the Hodge $*$-star operator on the exterior algebra of V, which is uniquely determined by the almost complex structure and compatible metric, and show that it is compatible with the decomposition of the exterior algebra. We will finish by showing that the almost complex structure and the compatible metric also define a Lie-algebra representaion of $\mathfrak{sl}(2, \mathbb{C})$ on the exterior algebra.

The second section starts with sketching the representation theory of $\mathfrak{sl}(2, \mathbb{C})$, where we show that every finite dimensional representation can be broken down to irreducible representations and then characterise the finite dimensional irreducible representations. With this tool at our disposal we return to the representation defined in the previous section, and prove the Lefschetz decomposition theorem, which further decomposes the exterior algebra of V. After that we connect the representation of $\mathfrak{sl}(2, \mathbb{C})$ and the Hodge $*$-operator.

The third and fourth section is a collection of required definitions and results about sheaf cohomology and complex manifolds. These sections can not serve as a proper introduction to the subjects, but we hope that it is enough to get the feeling of them and understand the main objects and results and how we use them later.

The last section of the first part is about harmonic theory on compact oriented manifolds. This section is halfway between the first and second two sections. We prove most of the theorems but leave out the most technical ones. Here we introduce the notion of differential operators between vector bundles. Differential operators between vector bundles naturally occur when one studies smooth manifolds, as the most standard example of a differential operator between vector bundles is the exterior derivative. We will also see that complex manifolds naturally come with a differential operator which is very closely related to holomorphic functions defined on them. We will define the symbol of a differential operator which captures the most important proterties of a differential operator. With the symbol we define elliptic differential operators, and prove the elliptic regularity theorem. After that we define the elliptic complexes, which are the generalisation of the de Rham complex and prove the Hodge decomposition theorem of elliptic complexes which will be one of our main tool later.

The second part is about applications of the Hodge decomposition on complex manifolds. It consists of two sections. In the first we will deal with general compact complex manifolds and prove the Poincaré and Serre duality theorem. If a complex manifold is equiped with a Riemannian metric compatible with the complex structure, then we can naturally assign three different Laplace operators. We will show that these three operators are not related to each other on a general compact complex manifold.

The second section is about Kähler manifolds, which are complex manifolds with a Riemannian structure that has a more subtle relation with the complex structure, than simple compatibility. We will prove some basic properties of Kähler manifolds, and show that not every complex manifold is Kähler. After that we show that on compact Kähler manifolds, the three differential operators are very strongly related (they are constant multiples of each other) and give a lot of corollaries such as the Hodge and Lefschetz decompositions of the cohomology of a Kähler manifold.

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1 Foundational material

In the sections Complex and Hermitian structures, Representation theory of $\mathfrak{sl}(2,\mathbb{C})$ and Complex manifolds and vector bundles we follow the book of [3] Huybrechts, D. - *Complex Geometry: An Introduction.* In the section Sheaves and sheaf cohomology we follow the book of [5] Ramanan, S. - *Global Calculus.* Lastly in the section Harmonic theory on compact manifolds we mainly follow the books of [7] Wells, R. O. - *Differential Analysis on Complex Manifolds.* and [1] Gilkey, P. B. - *Invariance Theory: The Heat Equation and the Atiyah-Singer Index Theorem.*

1.1 Complex and Hermitian structures

In this section we shall study additional structures on a real finite-dimensional vector space, such as almost complex structures, scalar products and we will be interested in the interactions between these structures.

In the following V shall denote a real finite-dimensional vector space.

Definition 1.1. A linear map $I : V \to V$ such that $I^2 = -\mathrm{id}$ is called an almost complex structure on V.

If V is a complex vector space, then it is a real vector space too, and the multiplication with i is an almost complex structure on V. The next lemma says that the converse is also true.

Lemma 1.2. *Let V be a finite-dimensional vector space with an almost complex structure I, then I induces a complex vector space structure on V.*

Proof. It is easy to check, that $(a + bi)v = a \cdot v + b \cdot I(v)$ will give a complex vector space structure on V. □

Corollary 1.3. *If V is an m-dimensional vector space with an almost complex structure I, then $m = 2n$ and I induces an orientation on V.*

Proof. By the previous lemma I induces a complex vector space structure on V, so we can choose a complex basis $e_1, \dots, e_n$. It is clear that $e_1, Ie_1, \dots, e_n, Ie_n$ is a real basis for V so $m = 2n$, and this basis gives the required orientation. Indeed, if we take another complex basis $f_1, \dots, f_n$ then there exists $A \in GL_{\mathbb{C}}((V, I))$ such that $A(e_i) = f_i$. Since the determinant of A as a real operator is equal to $\det_{\mathbb{C}}(A)\overline{\det_{\mathbb{C}}(A)} > 0$, we see that the orientation is well defined. □

For a real vector space V let's denote the complex vector space $V \otimes_{\mathbb{R}} \mathbb{C}$ by $V_{\mathbb{C}}$. It is clear that $V \to V_{\mathbb{C}}$, $v \mapsto v \otimes 1$ is an injective linear map. If $w = v \otimes c \in V_{\mathbb{C}}$ then the conjugate of w is $\overline{w} = \overline{v \otimes c} = v \otimes \bar{c}$. It is easy to see, that the image of V in $V_{\mathbb{C}}$

is precisely those elements that stay fixed under conjugation, i.e. $w \in V_{\mathbb{C}}$ such that $\overline{w} = w$.

Suppose that V is endowed with an almost complex structure I. Then we will also denote by I its complex linear extension to an endomorphism $V_{\mathbb{C}} \to V_{\mathbb{C}}$. Clearly, the only eigenvalues of I on $V_{\mathbb{C}}$ are $\pm i$.

Definition 1.4. Let I be an almost complex structure on the real vector space V, and let $I : V_{\mathbb{C}} \to V_{\mathbb{C}}$ be its complexification. Then define the following subspaces:

$$V^{1,0} = \{v \in V_{\mathbb{C}} \mid Iv = iv\} \quad V^{0,1} = \{v \in V_{\mathbb{C}} \mid Iv = -iv\}.$$

So $V^{1,0}$ and $V^{0,1}$ are the eigenspaces of I.

Lemma 1.5. *Let V be a vector space equipped with an almost complex structure I. Then*

$$V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}.$$

The conjugation on $V_{\mathbb{C}}$ gives a complex antilinear isomorphism between $V^{1,0}$ and $V^{0,1}$, i.e. $\overline{V^{1,0}} = V^{0,1}$.

Proof. Because $V^{1,0}$ and $V^{0,1}$ are different eigenspaces of I we have $V^{1,0} \cap V^{0,1} = 0$, so the natural map:

$$\begin{aligned} V^{1,0} \oplus V^{0,1} &\to V_{\mathbb{C}} \\ (v_1, v_2) &\mapsto v_1 + v_2 \end{aligned}$$

is injective. We will prove the first part by giving an inverse map, which is is given by:

$$\begin{aligned} V_{\mathbb{C}} &\to V^{1,0} \oplus V^{0,1} \\ v &\mapsto \left(\frac{v - iI(v)}{2}, \frac{v + iI(v)}{2} \right). \end{aligned}$$

It is easy to see that these two maps are inverses to each other.

Let $v \in V_{\mathbb{C}}$, then $v = x + iy = x \otimes 1 + y \otimes i$, where $x, y \in V$, then

$$\begin{aligned} \overline{v - iI(v)} &= x - iy + iI(x) + I(y) \\ &= x - iy + i(I(x - iy)) = \overline{v} + iI(\overline{v}) \end{aligned}$$

thus conjugation gives a complex antilinear isomorphism between $V^{1,0}$ and $V^{0,1}$. $\square$

One should be aware of the existence of two almost complex structures on $V_{\mathbb{C}}$. One is given by I and the other is given by multiplication with i. They coincide on the subpsace $V^{1,0}$ but differ by a sign on $V^{0,1}$. Obviously, $V^{1,0}$ and $V^{0,1}$ are complex subpsaces of $V_{\mathbb{C}}$ with respect to both almost complex structure. From now on, we will always regard $V_{\mathbb{C}}$ as the complex vector space with respect to i.

If $V^{1,0}$ and $V^{0,1}$ are complex vector spaces with respect to i, then the following compositions:

$$\begin{array}{ll} V \to V_{\mathbb{C}} \to V^{1,0} & V \to V_{\mathbb{C}} \to V^{0,1} \\ v \mapsto v \mapsto \dfrac{v - iI(v)}{2} & v \mapsto v \mapsto \dfrac{v + iI(v)}{2} \end{array}$$

are complex linear respectively complex antilinear isomorphisms.

If V has an almost complex structure I, then I induces an almost complex structure on $V^* = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$ by $I(f)(v) = f(I(v))$. This means, that $V^* \otimes \mathbb{C} = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) = \mathrm{Hom}_{\mathbb{C}}(V_{\mathbb{C}}, \mathbb{C}) = V_{\mathbb{C}}^*$ also has a decomposition:

$$(V^*)^{1,0} = \{f \in V_{\mathbb{C}}^* \mid If = if\} = (V^{1,0})^*$$
$$(V^*)^{0,1} = \{f \in V_{\mathbb{C}}^* \mid If = -if\} = (V^{0,1})^*$$

such that $V_{\mathbb{C}}^* = (V^*)^{1,0} \oplus (V^*)^{0,1}$. Also note that $(V^{1,0})^* = \mathrm{Hom}_{\mathbb{C}}((V, I), \mathbb{C})$.

If V is an m dimensional real vector space, then the natural decomposition of its exterior algebra is of the form:

$$\bigwedge^* V = \bigoplus_{i=0}^{m} \bigwedge^i V.$$

Analogously, $\bigwedge^* V_{\mathbb{C}}$ denotes the exterior algebra of the complex vector space $V_{\mathbb{C}}$, which also decomposes as:

$$\bigwedge^* V_{\mathbb{C}} = \bigoplus_{i=0}^{m} V_{\mathbb{C}}.$$

Also note that $\bigwedge^* V_{\mathbb{C}} = \bigwedge^* V \otimes \mathbb{C}$ and $\bigwedge^* V$ is the real subspace of $\bigwedge^* V_{\mathbb{C}}$ that is left invariant under conjugation.

If V is also endowed with an almost complex structure I, then $\dim_{\mathbb{R}}(V) = 2n$ and $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$ with $\dim_{\mathbb{C}}(V^{1,0}) = \dim_{\mathbb{C}}(V^{0,1}) = n$.

Definition 1.6. One defines

$$\bigwedge^{p,q} V = \bigwedge^p V^{1,0} \otimes_{\mathbb{C}} \bigwedge^q V^{0,1},$$

where the exterior products of $V^{1,0}$ and $V^{0,1}$ are taken as exterior products of complex vector spaces. An $\alpha \in \bigwedge^{p,q} V$ is of bidegree (p, q).

Proposition 1.7. *Suppose that V is equipped with an almost complex structure I, then one has:*

a) $\bigwedge^{p,q} V$ *is naturally a subspace of* $\bigwedge^{p+q} V_{\mathbb{C}}$.

b) $\bigwedge^k V_{\mathbb{C}} = \bigoplus_{p+q=k} \bigwedge^{p,q} V$.

c) Complex conjugation on $\bigwedge^* V_{\mathbb{C}}$ *defines a complex antilinear isomorphism between* $\bigwedge^{p,q} V$ *and* $\bigwedge^{q,p} V$ *i.e.* $\overline{\bigwedge^{p,q} V} = \bigwedge^{q,p} V$.

d) The exterior product is of bidegree $(0,0)$ *i.e. if* $\alpha \in \bigwedge^{p,q} V$ *and* $\beta \in \bigwedge^{p',q'}$, *then* $\alpha \wedge \beta \in \bigwedge^{p+p',q+q'} V$.

Proof. Let $v_1, \dots, v_n$ and $w_1, \dots, w_n$ be a basis of $V^{1,0}$ and $V^{0,1}$ respectively. If $\mathrm{I} = (i_1, \dots, i_p)$ where $i_j \in \{1, \dots, n\}$, then let's denote $v_{i_1} \wedge \cdots \wedge v_{i_p}$ by v_{I}. With these notations we get that $v_I \otimes w_{\mathrm{J}}$, where $1 \leq i_1 < \cdots < i_p \leq n$ and $1 \leq j_1 < \cdots < j_q \leq n$, forms a basis of $\bigwedge^{p,q} V$, and $v_I \otimes w_{\mathrm{J}} \mapsto v_I \wedge w_{\mathrm{J}}$ is an injective linear map. This proves $a), b)$ and $d)$.

To prove $c)$ we have to notice that the conjugation on $\bigwedge^* V_{\mathbb{C}}$ is multiplicative, i.e. $\overline{v \wedge w} = \overline{v} \wedge \overline{w}$, and with that in mind, $c)$ follows from $b)$ and Lemma 1.5.

$\square$

Suppose that $z_j = \frac{1}{2}(x_j - iy_j) \in V^{1,0}$ is a complex basis of $V^{1,0}$, where $x_j, y_j \in V$. Since $z_j \in V^{1,0}$ we have $Iz_j = iz_j$, which implies that $I(x_j) = y_j$ and $I(y_j) = -x_j$, and $x_j, y_j = I(x_j)$ forms a real basis of V. Also note that $\overline{z}_j = \frac{1}{2}(x_j + iy_j)$ is a complex basis of $V^{0,1}$.

Conversely if $v \in V$, then $\frac{1}{2}(v - iI(v)) \in V^{1,0}$, therefore if $x_j, y_j = I(x_j)$ is a real basis of V, then $z_j = \frac{1}{2}(x_j - y_j)$ is a complex basis of $V^{1,0}$.

With these notations we have the following.

Lemma 1.8. *For any* $l \leq \dim_{\mathbb{C}} V^{1,0}$ *we have the following:*

$$(-2i)^l (z_1 \wedge \overline{z}_1) \wedge \cdots \wedge (z_l \wedge \overline{z}_l) = (x_1 \wedge y_l) \wedge \cdots \wedge (x_l \wedge y_l)$$

if $l = \dim_{\mathbb{C}} V^{1,0}$, *then this gives the positive orientation induced by the almost complex structure.*

Remark 1.9. If V is a vector space with $\dim(V) = m$, then choosing an orientation is the same thing as choosing a nonzero element in $\bigwedge^m V$.

Proof. If $z_j = \frac{1}{2}(x_j - iy_j)$, then

$$z_j \wedge \overline{z}_j = \frac{1}{4}(x_j - iy_j) \wedge (x_j + iy_j) = \frac{1}{4}(ix_j \wedge y_j - iy_j \wedge x_j) = \frac{i}{2} x_j \wedge y_j$$

Now we just have to substitute this into the left side of the lemma and the assertation follows. $\square$

We can do the same with V^*. Suppose that $x_j, y_j = I(x_j)$ is a basis of V, then lets denote the (real) dual basis by x^j, y^j. Notice, that we have $Ix^j = -y^j$ and $Iy^j = x^j$. It follows that $z^j = x^j + iy^j$ and $\overline{z}^j = x^j - iy^j$ are the basis of $V^{1,0^*}$ and $V^{0,1^*}$ (complex) dual to z_j respectively $\overline{z}_j$. A similar computation as above yields the formula

$$\left(\frac{i}{2}\right)^l (z^1 \wedge \overline{z}^1) \wedge \cdots \wedge (z^l \wedge \overline{z}^l) = (x^1 \wedge y^1) \wedge \cdots \wedge (x^l \wedge y^l).$$

If we have an I almost complex structure on V, then we can extend this operator to an endomorphish $\mathbb{I} : \bigwedge^* V \to \bigwedge^* V$ as follows, if $\alpha = \alpha_1 \wedge \cdots \wedge \alpha_k \in \bigwedge^* V$, then

$$\mathbb{I}(\alpha) = I(\alpha_1) \wedge \cdots \wedge I(\alpha_k).$$

It is not hard to see that this is a well defined endomorphism of $\bigwedge^* V$. If we extend $\mathbb{I}$ to $\bigwedge^* V \otimes \mathbb{C} = \bigwedge^* V_{\mathbb{C}}$ and still denote the extension by $\mathbb{I}$, then it is easy to see, that $\mathbb{I}(\alpha) = i^{p-q}\alpha$, with $\alpha \in \bigwedge^{p,q} V$.

Definition 1.10. With respect to the direct sum decompositions, one defines the natural projections:

$$\pi^k : \bigwedge^* V_{\mathbb{C}} \to \bigwedge^k V_{\mathbb{C}} \quad \pi^{p,q} : \bigwedge^* V_{\mathbb{C}} \to \bigwedge^{p,q} V.$$

With these notations we see that $\mathbb{I} = \sum_{p,q} i^{p-q}\pi^{p,q}$.

Remark 1.11. We denote the corresponding operators on the dual space $\bigwedge^* V_{\mathbb{C}}^*$ also by $\pi^k, \pi^{p,q}$ and $\mathbb{I}$. Notice, that $\mathbb{I}(\alpha)(v_1, \ldots, v_k) = \alpha(I(v_1), \ldots, I(v_k))$ with $\alpha \in \bigwedge^k V_{\mathbb{C}}^*$ and $v_j \in V_{\mathbb{C}}$.

Suppose that V is also endowed with a scalar product $\langle\ ,\ \rangle$, i.e. $\langle\ ,\ \rangle$ is a positive definite symmetric biliniear form.

Definition 1.12. An almost complex structure I on V is compatible with the scalar product $\langle\ ,\ \rangle$ if $\langle I(v), I(w)\rangle = \langle v, w\rangle$ for all $v, w \in V$.

Definition 1.13. The fundamental form associated to $(V, \langle\ ,\ \rangle, I)$ is the form:

$$\omega = \langle I(\ ), (\ )\rangle = -\langle(\ ), I(\ )\rangle.$$

Lemma 1.14. *The associated form ω is real of type $(1,1)$, i.e. $\omega \in \bigwedge^2 V^* \cap \bigwedge^{1,1} V^*$.*

Proof. Let $v, w \in V$ arbitrary, then:

$$\omega(v,w) = \langle I(v), w\rangle = \langle v, -I(w)\rangle = -\langle I(w), v\rangle = -\omega(w,v)$$

This proves that ω is a real 2-form. To see that it is of type $(1,1)$ we compute as follows:

$$\mathbb{I}\omega(v,w) = \omega(I(v), I(w)) = \langle I^2(v), I(w)\rangle = \langle I(v), w\rangle = \omega(v,w)$$

Thus ω is of type $(1,1)$. $\square$

Lemma 1.15. *Let $(V, \langle\ ,\ \rangle, I)$ as before. Then $(\ ,\ ) = \langle\ ,\ \rangle - i\omega$ is a positive definite Hermitian form on the complex vector space (V, I).*

Proof. It is clear that the form $(\ ,\ )$ is real linear, and for any $v \in V$, with $v \neq 0$, we have $(v,v) = \langle v, v\rangle > 0$. Moreover $(v,w) = \overline{(w,v)}$, and

$$\begin{aligned}(Iv, w) &= \langle I(v), w\rangle - i\omega(I(v), w)\\ &= \langle I(v), w\rangle + i\langle v, w\rangle\\ &= i(-i\langle I(v), w\rangle + \langle v, w\rangle)\\ &= i(v,w)\end{aligned}$$

$\square$

We can extend $\langle\ ,\ \rangle$ to a positive definite Hermitian form on $V_{\mathbb{C}}$ the following way, let $v \otimes \mu, w \otimes \lambda \in V_{\mathbb{C}}$, then:

$$\langle v \otimes \mu, w \otimes \lambda\rangle_{\mathbb{C}} = \mu\overline{\lambda}\langle v, w\rangle.$$

Lemma 1.16. *Let $(V, \langle\ ,\ \rangle, I)$ as before, then $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$ is an orthogonal decomposition with respect to $\langle\ ,\ \rangle_{\mathbb{C}}$.*

Proof. let $v, w \in V$, then $v - iI(v) \in V^{1,0}$ and $w + iI(v) \in V^{0,1}$, and

$$\langle v - iI(v), w + iI(w)\rangle_{\mathbb{C}} = \langle v, w\rangle - \langle I(v), I(w)\rangle - i\langle I(v), w\rangle - i\langle v, I(w)\rangle = 0$$

$\square$

Lemma 1.17. *Let $(V, \langle\ ,\ \rangle, I)$ as before, then we have the canonical complex isomorphism $(V, I) \to (V^{1,0}, i)$. Under this isomorphism one has $\langle\ ,\ \rangle_{\mathbb{C}} = \frac{1}{2}(\ ,\ )$.*

Proof. The canonical isomorphism was $v \mapsto \frac{v-iI(v)}{2}$, and the computation goes as follows:

$$\Big\langle \frac{v-iI(v)}{2}, \frac{w-iI(w)}{2} \Big\rangle_{\mathbb{C}} = \frac{1}{4}(\langle v,w\rangle + \langle I(v), I(w)\rangle + i\langle v, I(w)\rangle - i\langle I(v), w\rangle$$
$$= \frac{1}{4}(2\langle v,w\rangle - 2i\langle I(v), w\rangle = \frac{1}{2}(v,w).$$

□

It is useful to compute in coordinates, so lets see how the above looks once a basis have been chosen.

Let $z_1, \ldots, z_n$ be a $\mathbb{C}$ basis of $V^{1,0}$, then $z_j = \frac{1}{2}(x_j - iI(x_j))$, with $x_j \in V$. Then $x_1, y_1 = I(x_1), \ldots, x_n, y_n = I(x_n)$ is a real basis of V and $x_1, \ldots, x_n$ is a complex basis of (V, I). Then we compute as follows:

$$\Big\langle \sum_{r^1}^{n} a^r z_r, \sum_{s=1}^{n} b^s z_s \Big\rangle_{\mathbb{C}} = \frac{1}{2}\sum_{r,s} a^r \overline{b}^s h_{r,s}.$$

Using the previous lemma we get that $(x_r, x_s) = h_{r,s}$. Since $(\ ,\ )$ is Hermitian on (V, I) we have that $(x_r, y_s) = -h_{r,s}$ and $(y_r, y_s) = h_{r,s}$. By definition of $(\ ,\ )$. one has that $-\mathrm{Im}(\ ,\ ) = \omega$ and $\mathrm{Re}(\ ,\ ) = \langle\ \rangle$. Hence $\omega(x_r, x_s) = \omega(y_r, y_s) = -\mathrm{Im}(h_{r,s})$, $\omega(x_r, y_s) = \mathrm{Re}(h_{r,s})$, $\langle x_r, x_s\rangle = \langle y_r, y_s\rangle = \mathrm{Re}(h_{r,s})$ and $\langle x_r, y_s\rangle = \mathrm{Im}(h_{r,s})$. Thus

$$\omega = -\sum_{r<s} \mathrm{Im}(h_{r,s})(x^r \wedge x^s + y^r \wedge y^s) + \sum_{r,s} \mathrm{Re}(h_{r,s})(x^r \wedge y^s).$$

Using that $z_r \wedge \overline{z}_s = (x^r + iy^r) \wedge (x^s + iy^s) = x^r \wedge x^s - i(x^r \wedge y^s + x^s \wedge y^r) + y^r \wedge y^s$ yields the following:

$$\omega = \frac{i}{2}\sum_{r,s} h_{r,s} z^r \wedge \overline{z}^s.$$

If $x_1, y_1, \ldots, x_n, y_n$ is an orthonormal basis of V with respect to $\langle\ ,\ \rangle$, then:

$$\omega = \frac{i}{2}\sum_{r} z^r \wedge \overline{z}^r = \sum_{r} x^r \wedge y^r.$$

Note that there always exists an orthonormal basis like above because we can pick x_1 arbitrarily, then $\langle x_1, Ix_1\rangle = \langle Ix_1, x_1\rangle = -\langle x_1, Ix_1\rangle$, thus $\langle x_1, y_1\rangle = 0$ and we can continue on the orthogonal complement of $\mathrm{Span}(x_1, y_1)$.

Corollary 1.18. *If $x_1, y_1, \ldots, x_n, y_n$ is an orthonormal basis as above, then:*

$$\omega^n = n!(x^1 \wedge y^1 \wedge \cdots \wedge x^n \wedge y^n)$$

Notice that by definition $x_1, y_1, \ldots, x_n, y_n$ is a positively oriented orthonormal basis of (V, I), thus $x^1 \wedge y^1 \wedge \cdots \wedge x^n \wedge y^n$ is the unique $2n$-form which takes value 1 on any positively oriented orthonormal basis.

Proposition 1.19. *Let $z_1, \ldots, z_n$ be a basis of $V^{1,0}$, and let ω be a $(1,1)$-form. Then by definition $\omega = \frac{i}{2}\sum_{r,s} h_{r,s} z_r \wedge \overline{z_s}$. We claim, that ω is a real form if and only if the matrix $(h_{r,s})$ is Hermitian, moreover if this matrix is also positive definite, then ω is the fundamental form of a scalar product compatible with the almost complec structure.*

Proof. The form ω is real if and only if $\overline{\omega} = \omega$. With that in mind we compute as follows:

$$\frac{i}{2} h_{r,s} = \omega(z_r, \overline{z}_s) = \overline{\omega(\overline{z}_r, z_s)} = \overline{-\frac{i}{2} h_{s,r}} = \frac{i}{2}\overline{h_{s,r}}$$

Hence $\overline{\omega} = \omega$ if and only if the matrix $(h_{r,s})$ is Hermitian.

Suppose that $(h_{r,s})$ is positive definite Hermitian form. Clearly if such a scalar product exists, then it has to be equal to $\omega(-, I-)$. It is then an easy excercise to show that $\omega(-, I-)$ satifsies the properties of the proposition, which we will omit. $\square$

If V is a finite-dimensional vector space with a scalar product $\langle\ ,\ \rangle$, then $\langle\ ,\ \rangle$ induces a scalar product on $\bigwedge^k V$ for every k the following way: if $v_1, \ldots, v_k, w_1, \ldots, w_k \in V$, then

$$(v_1, \ldots, v_k, w_1, \ldots, w_k) \mapsto \det(\langle v_i, w_j \rangle)$$

defines multilinear map $g : V^k \oplus V^k \to \mathbb{R}$. It is easy to see that this map is antilinear in the first and second k variables, thus by the universal property of exterior product, we get a map:

$$\langle\ ,\ \rangle_k : \bigwedge^k V \bigoplus \bigwedge^k V \to \mathbb{R}$$

It is easy to see that this map is symmetric. Note that $\det(\langle v_i, v_j \rangle)$ equals 0 if and only if $v_1, \ldots, v_k$ are linearly dependant. If $v_1, \ldots, v_k$ are linearly independent, the previous determinant is just the square of the k-dimensional volume of the parallelepiped formed by $v_1, \ldots, v_k$. Thus $\langle\ ,\ \rangle_k$ is positive definite.

Corollary 1.20. *If $e_1, \ldots, e_m$ is an orthonormal basis of V, then*

$$\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k} \mid 1 \leq i_1 < i_2 < \cdots < i_k \leq m\}$$

is an orthonormal basis of $\bigwedge^k V$ with respect to $\langle\ ,\ \rangle_k$.

We define a scalar product on $\bigwedge^* V$ by stating that $\bigwedge^i V$ is orthogonal to $\bigwedge^j V$ if $i \neq j$, and on $\bigwedge^k V$ it is $\langle\ ,\ \rangle_k$. We also denote this scalar product by $\langle\ ,\ \rangle$.

Suppose that $(V, \langle\ ,\ \rangle)$ also has an orientation. If $e_1, \ldots, e_n$ is a positively oriented orthonormal basis, then lets denote $e_1 \wedge \cdots \wedge e_n$ by Vol. Then the Hodge $*$-operator is defined by:

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle \mathrm{Vol}$$

for $\alpha, \beta \in \bigwedge^* V$. This determines $*$, for the exterior product defines a nondegenerate pairing $\bigwedge^k V \oplus \bigwedge^{m-k} V \to \bigwedge^m V = \mathbb{R}\mathrm{Vol}$. It is easy to see that $* : \bigwedge^k V \to \bigwedge^{m-k} V$.

Proposition 1.21. *Let $(V, \langle\ ,\ \rangle)$ be an oriented euclidian vector space. Let $e_1, \ldots, e_m$ be a positively oriented orthonormal basis, and let $e_1 \wedge \cdots \wedge e_m =$ Vol. The Hodge $*$-operator associated to $(V, \langle\ ,\ \rangle$, Vol) satisfies the following conditions:*

a) If $\{i_1, i_2, \ldots, i_k, j_1, \ldots, j_{m-k}\} = \{1, \ldots, m\}$, then

$$*e_{i_1} \wedge \cdots \wedge e_{i_k} = \varepsilon e_{j_1} \wedge \cdots \wedge e_{j_{m-k}}$$

where $\varepsilon = sgn(i_1, \ldots, i_k, j_1, \ldots, j_{m-k})$. In particular $*1 = Vol$.

b) The $*$-operator is selfadjoint up to sign i.e. if $\alpha \in \bigwedge^k V$, $\beta \in \bigwedge^{m-k} V$, then

$$\langle \alpha, *\beta \rangle = (-1)^{k(m-k)} \langle *\alpha, \beta \rangle.$$

c) If $W = \sum_r (-1)^{mr+r} \pi^r : \bigwedge^* V \to \bigwedge^* V$, then

$$** = W.$$

In particular $* : \bigwedge^k \to \bigwedge^{m-k}$ is an isomorphism.

d) The $*$-operator is an isometry with respect to $\langle\ ,\ \rangle$.

Proof. For a) we first note, that for $\langle e_I, e_I \rangle = 1$ for any $I = (i_1, \ldots, i_k)$, thus

$$e_I \wedge *e_I = \langle e_I, e_I \rangle \mathrm{Vol} = e_1 \wedge e_2 \wedge \cdots \wedge e_m$$

This means that $*e_I = \mathrm{sgn}(I, J) e_J$, where $J = (j_1, \ldots, j_{m-k})$ and $I \cup J = \{1, \ldots, m\}$. This also proves proves d) because we see that the $*$-operator sends orthonormal basis to orthonormal basis. Let $\alpha \in \bigwedge^k V$, $\beta \in \bigwedge^{m-k} V$, then we have

$$\begin{aligned} \langle \alpha, *\beta \rangle \mathrm{Vol} &= \langle *\beta, \alpha \rangle \mathrm{Vol} = *\beta \wedge *\alpha \\ &= (-1)^{k(m-k)} * \alpha \wedge *\beta = (-1)^{k(m-k)} \langle *\alpha, \beta \rangle \mathrm{Vol} \end{aligned}$$

This proves $b)$. Lastly let $I = (i_1, \dots, i_k)$, by $a)$ it is trivial, that $** e_I = \alpha e_I$, and we compute α as follows

$$\alpha = \langle e_I, ** e_I\rangle = (-1)^{k(m-k)}\langle *e_I, *e_I\rangle = (-1)^{k(m-k)}$$

□

Now we want to move to $\bigwedge^* V_{\mathbb{C}}$, we first note that there are two ways to give a Hermitian metric on $\bigwedge^* V_{\mathbb{C}}$. First we can extend $\langle\ ,\ \rangle$ to positive definite Hermitian form on $\bigwedge^* V_{\mathbb{C}} = \bigwedge^* V \otimes \mathbb{C}$, or, we can $\langle\ ,\ \rangle_{\mathbb{C}}$ extend from $V_{\mathbb{C}}$ to $\bigwedge^* V_{\mathbb{C}}$ the same way we extended $\langle\ ,\ \rangle$ from V to $\bigwedge^* V$. It is not hard to see that we get the same metric on $\bigwedge^* V_{\mathbb{C}}$ either way. We denote this new Hermitian form by $\langle\ ,\ \rangle_{\mathbb{C}}$.

The Hodge $*$-operator associated to $(V, \langle\ ,\ \rangle, \mathrm{Vol})$ is extended complex linearly to $* : \bigwedge^k V_{\mathbb{C}} \to \bigwedge^{2n-k} V_{\mathbb{C}}$. On $\bigwedge^* V_{\mathbb{C}}$ these two operators are related by:

$$\alpha \wedge *\overline{\beta} = \langle \alpha, \beta\rangle_{\mathbb{C}}\mathrm{Vol}.$$

Lemma 1.22. *Let $\langle\ ,\ \rangle_{\mathbb{C}}$ and $*$ be as above. Then*

a) $\bigwedge^k V_{\mathbb{C}} = \bigoplus_{p+q=k} \bigwedge^{p,q} V$ *is an orthogonal decomposition with respect to* $\langle\ ,\ \rangle_{\mathbb{C}}$.

b) $* : \bigwedge^{p,q} V \to \bigwedge^{n-q,n-p} V$, *where* $= dim_{\mathbb{C}}((V, I)) = n$

c) $** |_{\bigwedge^{p,q} V} = (-1)^{p+q} id$.

Proof. To prove part $a)$ let $\alpha = v_1 \wedge \cdots \wedge v_p \wedge w_1 \wedge \cdots \wedge w_q \in \bigwedge^{p,q} V$ and let $\beta = v'_1 \wedge \cdots \wedge v'_{p'} \wedge w'_1 \wedge \cdots \wedge w'_{q'} \in \bigwedge^{p',q'}$, with $p+q = p'+q' = k$ and $p' < p$. We want to show that these two elements are orthogonal with respect to $\langle\ ,\ \rangle_{\mathbb{C}}$, which means, that we have to show that the following matrix has zero determinant:

$$M = \begin{pmatrix} \langle v_i, v'_j\rangle_{\mathbb{C}} & \langle v_i, w'_j\rangle_{\mathbb{C}} \\ \langle w_i, v'_j\rangle_{\mathbb{C}} & \langle w_i, w'_j\rangle_{\mathbb{C}} \end{pmatrix} = \begin{pmatrix} \langle v_i, v'_j\rangle_{\mathbb{C}} & 0 \\ 0 & \langle w_i, w'_j\rangle_{\mathbb{C}} \end{pmatrix}$$

Here we used lemma 1.16 to see that $\langle v_i, w'_j\rangle_{\mathbb{C}} = \langle w_i, v'_j\rangle_{\mathbb{C}} = 0$ for all i, j. The upper left block of M is a $p \times p'$ matrix and the lower left block is a $q \times q'$ matrix. We know that the determinant is nonzero if and only if the columns are linearly independent. Now look at the last q' columns, all of them are elements of a q dimensional subspace, but $q' > q$ so they must be linearly dependant, thus $\langle \alpha, \beta\rangle_{\mathbb{C}} = \det(M) = 0$.

Now we want prove part $b)$. First notice, that if $\alpha_1 \in \bigwedge^{p_1,q_1} V$ and $\alpha_2 \in \bigwedge^{p_2,q_2}$ with $p_1 + q_1 + p_2 + q_2 = 2n$ and $\alpha_1 \wedge \alpha_2 \neq 0$, then $(p_1 + p_2, q_1 + q_2) = (n, n)$. Now let $\beta \in \bigwedge^{p,q} V$, then one has that $\alpha \wedge *\overline{\beta} = \langle \alpha, \beta\rangle_{\mathbb{C}}\mathrm{Vol}$, and we know that $\langle \alpha, \beta\rangle_{\mathbb{C}} \neq 0$ implies that $\alpha \in \bigwedge^{p,q} V^*$. This implies that $*\overline{\beta} \in \bigwedge^{p',q'} V^*$ for some (p', q'). We also know that $\beta\wedge *\overline{\beta} \neq 0$, thus by the previous remark we get that $(p+p', q+q') = (n, n)$, thus $*\overline{\beta} \in \bigwedge^{n-p,n-q} V^*$, hence $*\beta = \overline{*\overline{\beta}} \in \bigwedge^{n-q,n-p} V^*$.

Lastly $c)$ is an easy consequence of Proposition 1.21. □

Definition 1.23. If we have $(V, \langle\ ,\ \rangle, I)$ as above, then the Lefschetz operator $L : \bigwedge^* V^*_{\mathbb{C}} \to \bigwedge^* V^*_{\mathbb{C}}$ is given by $\alpha \mapsto \omega \wedge \alpha$, where ω is the associated fundamental form.

It is easy to see, that L is the complexification of the real map $\alpha \mapsto \omega \wedge \alpha$, and that L is of bidegree $(1, 1)$.

Note, that an inner product $\langle\ ,\ \rangle$ on V induces an inner product on V^* by the following: let $e_1, \ldots, e_m$ an orthonormal basis in V and $e^1, \ldots, e^m$ the dual basis in V^*, then we define the induced inner product $\langle\ ,\ \rangle^*$ by stating that $e^1, \ldots, e^m$ is an orthonormal basis. One can show that this is actually independent of the choices we made as follows; the map $v \mapsto \langle\ , v\rangle$ gives a $V \to V^*$ isomorphism. With this isomorphism we can pullback the metric on V, and it is easy to see that the pullback coincides with the metric defined above. It is also easy to see that if $(V, \langle\ ,\ \rangle)$ is a euclidean space and I is a compatible almost complex structure, then I^* on $(V^*, \langle\ ,\ \rangle^*)$ is a compatible almost complex structure. This means that we have an innerproduct on $\bigwedge^* V^*_{\mathbb{C}}$ and thus we have an adjoint of L denoted by Λ. We claim that $\Lambda = *^{-1}L*$, to see that, let $\alpha \in \bigwedge^k V^*_{\mathbb{C}}$ and $\beta \in \bigwedge^{k+2} V^*_{\mathbb{C}}$, then

$$\begin{aligned}(L\alpha, \beta)Vol =& L\alpha \wedge *\overline{\beta} = \omega \wedge \alpha \wedge *\overline{\beta} = \alpha \wedge \omega \wedge *\overline{\beta} = \alpha \wedge L * \overline{\beta} \\ =& \alpha \wedge *(*^{-1}L * \overline{\beta}) = \alpha \wedge *(\overline{*^{-1}L * \beta}) = (\alpha, *^{-1}L * \beta)Vol.\end{aligned}$$

This implies that Λ is of bidegree $(1, 1)$, and it is easy to see, that Λ is the complexification of the real map $*^{-1}L|_{\bigwedge^* V^*}*$.

Definition 1.24. Let $H : \bigwedge^* V^* \to \bigwedge^* V^*$ be the counting operator, i.e.

$$H = \sum_{k=0}^{2n} (n-k)\pi^k,$$

where $2n = \dim(V)$

We can extend H complex linearly to $\bigwedge^* V^*_{\mathbb{C}}$, it is also denoted by H.

Theorem 1.25. *We have the following commutation relation between the real operators L, Λ and H:*

$$a)\ [H, L] = -2L,\ \ b)\ [H, \Lambda] = 2\Lambda, \textit{ and } c)\ [\Lambda, L] = H.$$

Proof. See [3] Proposition 1.2.26. □

Definition 1.26. Let k be a field, then $\mathfrak{sl}(n,k)$ denotes the Lie-algebra of $n \times n$ matrices with k entries and trace 0.

Corollary 1.27. *The operators L, Λ, H gives a representation of $\mathfrak{sl}(2,\mathbb{R})$ on $\bigwedge^* V^*$ and $\mathfrak{sl}(2,\mathbb{C})$ on $\bigwedge^* V^*_{\mathbb{C}}$.*

Proof. The Lie-algebras $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{sl}(2,\mathbb{C}) = \mathfrak{sl}(2,\mathbb{R}) \otimes \mathbb{C}$ have the following basis:

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

It is an easy computation, that

$$[B,Y] = -2Y, \; [B,X] = 2X, \; \text{and } [X,Y] = B,$$

thus the map

$$Y \mapsto L, \; X \mapsto \Lambda, \; \text{and } B \mapsto H$$

is a representation of $\mathfrak{sl}(2,\mathbb{R})$ on $\bigwedge^* V^*$, and if we tensor everything with $\mathbb{C}$ then we get a representation of $\mathfrak{sl}(2,\mathbb{C})$ on $\bigwedge^* V^*_{\mathbb{C}}$. ∎

1.2 Representation theory of $\mathfrak{sl}(2,\mathbb{C})$

In this section we will sketch the representation theory of $\mathfrak{sl}(2,\mathbb{C})$, and then study the representation we got on $\bigwedge^* V_{\mathbb{C}}^*$. We assume that the reader is familiar with the basic theory of Lie-algebras and Lie groups.

Proposition 1.28. *$\mathfrak{sl}(2,\mathbb{C})$ is a simple Lie-algebra.*

Proof. Let $I \trianglelefteq \mathfrak{sl}(2,\mathbb{C})$ be a non-zero ideal. First notice that if $B \in I$, then $I = \mathfrak{sl}(2,\mathbb{C})$. To see that let $B \in I$, then since I is an ideal we get that

$$[X,B] = -2X \in I, \text{ and } [Y,B] = 2Y \in I.$$

Thus $X, Y, B \in I$ and X, Y, B generate $\mathfrak{sl}(2,\mathbb{C})$. Now let $aX+bY+cB \in I$ a non-zero element. Applying $[B,-]$ we get that:

$$[B, aX + bY + cB] = 2aX - 2bY \in I.$$

This means that $2aX + cB, 2bY + cB \in I$, applying $[B,-]$ again, we see that $4aX, 4bY \in I$. If $a \neq 0$ or $b \neq 0$, then $B \in I$. If $a = b = 0$, then $cB \in I$, i.e. if $aX + bY + cB \neq 0$, then $B \in I$ thus $I = \mathfrak{sl}(2,\mathbb{C})$. □

Corollary 1.29. *Every finite dimensional representation of $\mathfrak{sl}(2,\mathbb{C})$ is completely reducible, i.e. direct sum of irreducible representations.*

Proof. By Weyl's theorem if $\mathfrak{g}$ is a semisimple Lie-algebra over a field of characteristic 0, then every finite dimensional representation of $\mathfrak{g}$ is completely reducible. We just showed that $\mathfrak{sl}(2,\mathbb{C})$ is simple, hence it is semisimple. □

Definition 1.30. Let $\rho : \mathfrak{sl}(2,\mathbb{C}) \to \mathrm{End}_{\mathbb{C}}(V)$ be a finite dimensional representation. Let $V^\lambda = \{v \in V \mid \rho(B)v = \lambda v\}$. We say that $v \in V^\lambda$ is of weight λ. We say that an element $v \in V$ is primitive if $v \neq 0$ and $\rho(X)v = 0$.

Lemma 1.31. *Let's fix a representation ρ. Then:*

a) $\sum_{\lambda\in\mathbb{C}} V^\lambda$ is a direct sum decomposition.

b) If $v \in V^\lambda$, then $\rho(X)v \in V^{\lambda+2}$ and $\rho(Y)v \in V^{\lambda-2}$.

Proof. a) We only have to show that if $\lambda \neq \lambda'$ then $V^\lambda \cap V^{\lambda'} = 0$, but this is trivial since V^λ is just the eigenspace of $\rho(B)$.

b) Let $v \in V^\lambda$, then:

$$\begin{aligned}\rho(B)\rho(X)v =& (\rho(B)\rho(X) - \rho(X)\rho(B))v + \rho(X)\rho(B)v \\ =& \rho([B,X])v + \lambda\rho(X)v = 2\rho(X)v + \lambda\rho(X)v = (\lambda+2)\rho(X)v.\end{aligned}$$

$\rho(Y)$ follows the same way. $\square$

Proposition 1.32. *Every finite dimensional representation ρ admits a primitive element.*

Proof. Let $v_0 \in V$ be a non-zero eigenvector of $\rho(B)$. Look at the following sequence of vectors:

$$v_0, \rho(X)v_0, \rho(X)^2 v_0, \ldots, \rho(X)^n v_0, \ldots$$

The non-zero vectors are all linearly independent, since they correspond to different eigenvalues. Thus there exists a $k \geq 0$ such that $\rho(X)^k v \neq 0$ but $\rho(X)^{k+1}(X)v = 0$. Hence, $\rho(X)^k v$ is a primitive vector, and we also got that v is of weight λ for some $\lambda \in \mathbb{C}$. $\square$

Theorem 1.33. *Let ρ be a finite dimensional irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$. Let v_0 be a primitive vector of weight λ. Let $v_{-1} = 0$ and $v_n = (\frac{1}{n!})\rho(Y)^n v_0$. Then for all $n \geq 0$ one has that:*

a) $\rho(B)v_n = (\lambda - 2n)v_n$.

b) $\rho(Y)v_n = (n+1)v_{n+1}$.

c) $\rho(X)v_n = (\lambda - n + 1)v_{n-1}$.

Also, $\lambda = m$, where $m + 1 = \dim_{\mathbb{C}}(V)$, and $\rho(Y)^n v_0 = 0$ for all $n > m$.

Proof. $a)$ we use induction on n. We have already seen for $n = 1$, suppose that we know for all $m < n$. Then

$$\begin{aligned}\rho(B)\rho(Y)^n v_0 =& (\rho(B)\rho(Y) - \rho(Y)\rho(B))\rho(Y)^{n-1}v_0 + \rho(Y)\rho(B)\rho(Y)^{n-1}v_0 \\ =& -2\rho(Y)^n v_0 + (\lambda + 2(n-1))\rho(Y)^n v_0 = (\lambda - 2n)\rho(Y)^n v_0.\end{aligned}$$

$b)$

$$\rho(Y)v_n = \frac{1}{n!}\rho(Y)^{n+1}v_0 = (n+1)\frac{1}{(n+1)!}\rho(Y)^{n+1}v_0 = (n+1)v_{n+1}$$

$c)$ We use induction on n. Let $n = 0$, then v_0 is primitive, thus $\rho(X)v_0 = 0 = v_{-1}$. Suppose that we know for all $m < n$. Then

$$\begin{aligned}n\rho(X)v_n =& \rho(X)\rho(Y)v_{n-1} \\ =& \rho([X,Y])v_{n-1} + \rho(Y)\rho(X)v_{n-1} \\ =& \rho(B)v_{n-1} + \rho(Y)(\lambda - n + 2)v_{n-2} \\ =& (\lambda - 2n + 2)v_{n-1} + (n-1)(\lambda - n + 2)v_{n-1} \\ =& n(\lambda - n + 1)v_{n-1}.\end{aligned}$$

To finish the proof we start with showing that $\lambda \in \mathbb{N}$. Since V is finite dimensional there exists $m \geq 0$ such that

$$\begin{aligned} &v_0, \ldots, v_m \text{ are all non-zero} \\ &v_{m+1}, \ldots \quad \text{are all zero.} \end{aligned}$$

Now use $c)$ on v_{m+1}:

$$0 = \rho(X)v_{m+1} = (\lambda - (m+1) + 1)v_m = (\lambda - m)v_m.$$

We know that v_m is non-zero, thus $\lambda = m \in \mathbb{N}$. Now we show that $m+1 = \dim_{\mathbb{C}}(V)$. Let $V_m = \mathrm{Span}\{v_0, \ldots, v_m\}$. We claim that V_m is an invariant subspace of V. Indeed, let $v = \sum_{i=0}^{m} \alpha^i v_i \in V_m$, then:

$$\begin{aligned} \rho(B)v &= \sum_{i=0}^{m} \alpha^i (m - 2n) v_n \in V_m \\ \rho(X)v &= \sum_{i=0}^{m} \alpha^i (m - n + 1) v_{n-1} \in V_m \\ \rho(Y)v &= \sum_{i=0}^{m} \alpha^i (n+1) v_{n+1} v_{n+1} \in V_m \end{aligned}$$

Here we used that $v_1 = v_{m+1} = 0$. This means that V_m is an invariant non-zero subspace of V, but we assumed that ρ is irreducible, hence $V_m = V$ and $\dim_{\mathbb{C}}(V) = \dim_{\mathbb{C}}(V_m) = m + 1$. $\square$

The theorem states that every irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$ looks like the following drawing:

$$\begin{tikzcd} 0 & v_m \arrow[l, "Y"'] \arrow[loop above, "B"] \arrow[r, shift left, "X"] & v_{m-1} \arrow[l, shift left, "Y"] \arrow[loop above, "B"] \arrow[r, shift left, "X"] & \cdots \arrow[l, shift left, "Y"] \arrow[r, shift left, "X"] & v_1 \arrow[l, shift left, "Y"] \arrow[loop above, "B"] \arrow[r, shift left, "X"] & v_0 \arrow[l, shift left, "Y"] \arrow[loop above, "B"] \arrow[r, "X"] & 0 \end{tikzcd}$$

Theorem 1.34. *Up to isomorphism there is only one $(m+1)$-dimensional irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$, and it is of the following form: let $v_0, \ldots, v_m$ be a basis of a vector space V. Then*

a) $\rho(B)v_n = (\lambda - 2n)v_n$.

b) $\rho(Y)v_n = (n+1)v_{n+1}$.

c) $\rho(X)v_n = (\lambda - n + 1)v_{n-1}$.

Where $n = 0, \ldots, m$, and $v_{-1} = v_{m+1} = 0$.

Proof. It is easy to check that this really is a representation of $\mathfrak{sl}(2,\mathbb{C})$. Now to show that this is an irreducible representation, suppose that $V_0 \subseteq V$ is a non-zero invariant subspace. Then $\rho(B) : V_0 \to V_0$ has an eigenvector in V_0. Since every eigenvector (up to non-zero scalar) of $\rho(B)$ is of the form v_j for some $0 \le j \le m$, we get that $v_k \in V_0$ for some k. Again since V_0 is an invariant subspace we get that $\rho(X)^k v_k = cv_0 \in V_0$ with $c \neq 0$. Now we use that $v_r = \frac{1}{r!}\rho(Y)^r v_0 \in V_0$, thus $V \subset V_0$, so $V = V_0$.

The previous theorem stated, that if ρ is an irreducible representation, then it is isomorphic to this representation. □

Corollary 1.35. *Let ρ be an irreducible representation. If $\varphi \in V^\lambda$, then there exists an $r \in \mathbb{N}$ such that φ_0 is a primitive element of weight $\lambda + 2r$ such that*

$$\varphi = \rho(Y)^r \varphi_0 \textit{ and } \varphi_0 = \frac{(m-r)!}{m!r!}\rho(X)^r \varphi_0,$$

where $m + 1 = \dim_{\mathbb{C}}(V)$.

Proof. Let $v_0, \dots, v_m$ be a basis of V like in theorem 1.33. Let's fix r, where $0 \le r \le m$. Then

$$\begin{aligned}
\rho(X)v_r &= (m-r+1)v_{r-1} \\
\rho(X)^2 v_r &= (m-r+1)(m-r+2)v_{r-2} \\
&\vdots \\
\rho(X)^r v_r &= \frac{m!}{(m-r)!}v_0 \\
\rho(Y)\rho(X)^r v_r &= \frac{m!}{(m-r)!}v_1 \\
\rho(Y)^2\rho(X)^r v_r &= \frac{m!2}{(m-r)!}v_2 \\
&\vdots \\
\rho(Y)^r\rho(X)^r v_r &= \frac{m!r!}{(m-r)!}v_r.
\end{aligned}$$

If φ is an eigenvector of $\rho(B)$, then $\varphi = \alpha v_r$ for some $r \in \mathbb{N}$ and α is a non-zero scalar. Hence

$$\varphi = \frac{(m-r)!}{m!r!}\rho(Y)^r\rho(X)^r\varphi,$$

with $\varphi_0 = \frac{(m-r)!}{m!r!}\rho(X)^r\varphi$ is a primitive element of weight $\lambda+2r$ and $\varphi = \rho(Y)^r\varphi_0$. □

One can get all the irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$ in one neat representation. Let $\mathbb{C}[x,y]$ be the polynomial ring in two variables over the complex numbers.

Then one can easily compute that the following map

$$
\begin{aligned}
X &\mapsto x\partial_y \\
Y &\mapsto y\partial_x \\
B &\mapsto x\partial_x - y\partial_y,
\end{aligned}
$$

is a Lie-algebra homomorphism. Let's denote this map by π. Since $\pi(X), \pi(Y), \pi(B)$ are all degree zero map one can restrict these maps to $\mathbb{C}[x, y]_m$ the vector space of polynomials of degree m to get an $m+1$-dimensional irreducible representation.

Now let $(V, \langle\ ,\ \rangle)$ be an euclidean space of dimension $2n$ with a compatible almost complex structure I. By Corollary 1.27 we have a representation of $\mathfrak{sl}(2, \mathbb{C})$ on $\bigwedge^* V_{\mathbb{C}}^*$ given by

$$X \mapsto \Lambda = L^*,\ Y \mapsto L,\ B \mapsto H = \sum_{p=0}^{2n} (n-p)\pi^p.$$

Let's denote this representation by α. We say that $\varphi \in \bigwedge^* V_{\mathbb{C}}^*$ is a primitive if $\varphi \neq 0$ and $\Lambda\varphi = \alpha(X)\varphi = 0$.

Proposition 1.36. *Let $\varphi \in \bigwedge^p V_{\mathbb{C}}^*$ be a primitive p-form, then $L^q(\varphi) = 0$ for all $q \geq \max(0, n-p+1)$.*

Proof. Let $V_\varphi = \operatorname{Span}\{L^i\varphi \mid i \in \mathbb{N}\}$ be the invariant subspace generated by φ. Restricting α to this subspace gives an irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$. By Theorem 1.33 we know that $\alpha(B)\varphi = m\varphi$, where $m+1 = \dim(V_\varphi)$, but we also know that $\alpha(B)\varphi = H\varphi = (n-p)\varphi$, thus $m = n-p$. This means that $\alpha(Y)^q\varphi = L^q\varphi = 0$ for all $q \geq \max(0, n-p+1)$. □

Corollary 1.37. *There are no primitive p-forms for $p > n$.*

Proof. Suppose that φ is a primitive p-form where $p > n$. Then, by the previous proposition, $\dim(V_\varphi) = (n-p+1) \leq 0$, thus $\varphi = 0$, which is a contradiction. □

Theorem 1.38 (Lefschetz decomposition)**.** *Let $(V, \langle\ ,\ \rangle, I)$ be as before, and let $\varphi \in \bigwedge^p V_{\mathbb{C}}^*$. Then*

a) φ can be written uniquely in the form

$$\varphi = \sum_{r \geq \max(0, p-n)} L^r \varphi_r,$$

where φ_r is a primitive $(p-2r)$-form or zero. We call this the primitive decomposition of φ.

b) If $L^m\varphi = 0$, then the primitive $(p-2r)$-forms φ_r in the decomposition of φ vanish for all $r \geq \max(0, p-n+m)$.

c) If $p \le n$, then $L^{n-p}\varphi = 0$ if and only if $\varphi = 0$.

Proof. *a*) Every finite dimensional representation of $\mathfrak{sl}(2,\mathbb{C})$ are completely reducible, thus

$$\bigwedge^{*} V_{\mathbb{C}}^{*} = V_1 \bigoplus \dots V_l,$$

where V_i are all invariant subspaces, and the restriction of α to V_i gives an irreducible representation. This means that

$$\varphi = \psi^1 + \dots + \psi^l,$$

where $\psi^i \in V_i$. Since every ψ^i is a p-form they are all eigenvectors of $\alpha(B) = H$ with eigenvalue $(n-p)$. Thus by Corollary 1.35 for all j, we get that

$$\psi^j = L^{r_j}\xi_j,$$

where ξ_j is a primitive $(p-2r_j)$-form. Now if we collect the primitive forms of the same degree we get the decomposition

$$\varphi = \sum_{r \ge \max(0,p-n} L^r \varphi_r.$$

To get that the decomposition is unique, we only have to show that $0 = \varphi_0 + L\varphi_1 + \dots + L^m\varphi_m$ implies $\varphi_m = 0$ for all m. Suppose that there is a non-trivial element in this decomposition, and let m be the largest such that $\varphi_m \neq 0$. Again by Corollary 1.35 we know that $\Lambda^k L^k \varphi_k = c_k\varphi_k$, for some $0 \neq c_k \in \mathbb{Q}$. Then

$$0 = \underbrace{\Lambda^m\varphi_0}_{0} + \underbrace{\Lambda^{m-1}(\overbrace{\Lambda L\varphi_1}^{\text{primitive}})}_{0} + \dots + \underbrace{\Lambda(\overbrace{\Lambda^{m-1}L^{m-1}\varphi_{m-1}}^{\text{primitive}})}_{0} + \Lambda^m L^m \varphi_m.$$

Thus $0 = \Lambda^m L^m \varphi_m = c_m\varphi_m$, with $c_m \neq 0$, but this is a contradiction since we assumed that $\varphi_m \neq 0$.

b) Let $\varphi \in \bigwedge^p V_{\mathbb{C}}^*$ and suppose that $L^m\varphi = 0$. By part *a*) we know that $\varphi = \sum L^r\varphi_r$, thus

$$0 = L^m\varphi = \sum_{r \ge \max(0,p-n)} L^{r+m}\varphi_r.$$

Since φ_r is a primitive $(p-2r)$-form we can use Proposition 1.36 see that $L^q\varphi_r = 0$ for all $q \ge \max(0, n-(p-2r)+1)$. This means that $L^{r+m}\varphi_r = 0$ for all $r < \max(0, p-n+m)$, hence

$$0 = \sum_{r \ge \max(0,n-p+m)} L^{r+m}\varphi_r$$

is a primitive decomposition of 0. By part a) we get, that $\varphi_r = 0$ for all $r \geq \max(0, p-n+m)$.

c) Let $p \leq n$, and suppose that $L^{n-p}\varphi = 0$. By part b) we get that $\varphi_r = 0$ for all $r \geq \max(0, p-n+n-p) = 0$, thus $\varphi = 0$. $\square$

Corollary 1.39. *Let $\varphi \in \bigwedge^* V_{\mathbb{C}}^*$. Then φ is primitive if and only if $p \leq n$ and $L^{n-p+1}\varphi = 0$.*

Proof. If φ is primitive, then we have already seen that $p \leq n$. We also know that $H\varphi = (n-p)\varphi$ and $\dim(V_\varphi) = n-p+1$, where $V_\varphi = \mathrm{Span}\{L^i\varphi \mid i \geq 0\}$. Hence $L^{n-p+1}\varphi = 0$.

Now suppose that $p \leq n$ and that $L^{n-p+1}\varphi = 0$. By Theorem 1.38 part a) we know that $\varphi = \sum_{r\geq 0} L^r\varphi_r$ and by part b) we know that $\varphi_r = 0$ for all $r \geq p-n+n-p+1 = 1$, thus $\varphi = \varphi_0$. $\square$

Corollary 1.40. *Let $P^k = \{\alpha \in \bigwedge^k V_{\mathbb{C}}^* \mid L^{n-k+1}\alpha = 0\}$ for all $k \leq n$ and let $P^k = 0$ for all $k > n$. Then*

a) $\bigwedge^k V_{\mathbb{C}}^* = \bigoplus_{i\geq 0} L^i(P^{k-2i})$ is an orthogonal decomposition with respect to $\langle\ ,\ \rangle_{\mathbb{C}}$.

b) $L^{n-k} : \bigwedge^k V_{\mathbb{C}}^* \to \bigwedge^{2n-k} V_{\mathbb{C}}^*$ is an isomorphism.

Proof. a) The fact that this is a direct sum decomposition follows from Corollary 1.39 and Theorem 1.38 part a). To see that this is an orthogonal decomposition, let $L^i\alpha_i, L^j\alpha_j \in \bigwedge^k V_{\mathbb{C}}^*$, with $i < j$ and α_i, α_j primitive. Then

$$\langle L^i\alpha_i, L^j\alpha_j\rangle_{\mathbb{C}} = \langle \overbrace{\Lambda(\Lambda^i L^i\alpha_i)}^{0}, \Lambda^{j-(i+1)}\alpha_j\rangle_{\mathbb{C}} = 0.$$

b) By Theorem 1.38 part c) we know that L^{n-k} restricted to $\bigwedge^k V_{\mathbb{C}}^*$ is injective, also $\dim(\bigwedge^k V_{\mathbb{C}}^*) = \dim(\bigwedge^{2n-k} V_{\mathbb{C}}^*)$, so we get that this is an isomorphism. $\square$

Remark 1.41. Since L, Λ and H are the complexifications of real operators, they map $\bigwedge^* V^*$ to $\bigwedge^* V^*$. Thus let $P_{\mathbb{R}}^k = P^k \cap \bigwedge^* V^*$. Then

a) $\bigwedge^k V^* = \bigoplus_i L^i(P_{\mathbb{R}}^{k-2i})$ is an orthogonal decomposition with respect to $\langle\ ,\ \rangle$.

b) $L^{n-k} : \bigwedge^k V^* \to \bigwedge^{2n-k} V^*$ is an isomorphis.

We also know that L, Λ and H respects the (p,q) decomposition, thus if we denote $P^k \cap \bigwedge(p,q)$ by $P^{p,q}$ we get the following:

a) $P^k = \bigoplus_{p+q=k} P^{p,q}$ is an orthogonal decomposition with respect to $\langle\ ,\ \rangle_{\mathbb{C}}$.

b) If $p+q = k$, then $L^{n-k} : \bigwedge^{p,q} V^* \to \bigwedge^{n-q,n-p} V^*$ is an isomorphism.

In the following we will connect the representation of $\mathfrak{sl}(2,\mathbb{C})$ with the Hodge $*$-operator. The results will be important in the proof of the Kähler identities. The following statements and proofs require some knowledge of Lie groups which we will use only.

Let $e_1, Ie_1, \dots, e_n, Ie_n$ be an orthonormal basis in V. Then $x^i = \langle -, e_i\rangle$ and $y^i = \langle -, Ie_i\rangle$ and $\omega = \sum_{i=1}^n x^i \wedge y^i$. Let $\varphi, \eta \in \bigwedge^* V_{\mathbb{C}}^*$, then $e(\eta)\varphi = \eta\wedge\varphi$. We claim, that if η is a real 1-form, then $e^*(\eta) = e(\eta)^* = *e(\eta)*$. Indeed, let $\alpha \in \bigwedge^{k-1} V_{\mathbb{C}}^*$ and $\beta \in \bigwedge^k V_{\mathbb{C}}^*$, then

$$\begin{aligned}\langle e(\eta)\alpha, \beta\rangle =&\eta\wedge\alpha\wedge*\overline{\beta} = (-1)^{k-1}\alpha\wedge\eta\wedge*\overline{\beta} = (-1)^{k-1}\alpha\wedge(-1)^{k-1}**\eta\wedge*\overline{\beta}\\ =&\alpha\wedge*(\overline{*\eta\wedge*\beta}) = \langle\alpha, *\eta\wedge*\beta\rangle = \langle\alpha, *e(\eta)*\beta\rangle.\end{aligned}$$

Now let $f^1, \dots, f^n$ be an oriented orthonormal basis in V^*. Then one can compute easily that

$$e^*(f^i)(f^{i_1}\wedge\dots\wedge f^{i_k})$$

is zero if and only if $i \notin \{i_i, \dots, i_k\}$, and if $i = i_a$ for some a, then it is equal to $(-1)^{a+1}f^{i_1}\wedge\dots\wedge f^{i_{a-1}}\wedge f^{i_{a+1}}\wedge\dots\wedge f^{i_k}$. Also notice, that $L = e(\omega) = \sum_{i=1}^n e(x^i)e(y^i)$ and $\Lambda = e^*(\omega) = \sum_{i=1}^n e^*(y^i)e^*(x^i)$. Since ω is a 2-form, we get that for all $\eta \in \bigwedge^* V_{\mathbb{C}}^*$ $[L, e(\eta)] = 0$.

Proposition 1.42. *Let $e_1, Ie_1, \dots, e_n, Ie_n$ be an orthonormal basis, $x^j = \langle -, e_j\rangle$ and $y^j = \langle -, Ie_j\rangle$. Then*

a) $[\Lambda, e(x^i)] = e^*(y^j)$

b) $[\Lambda, e(y^j)] = -e^*(x^j)$

Proof. a) We compute as follows:

$$\begin{aligned}[\Lambda, e(x^j)] =&\sum_{i=1}^n e^*(y^i)e^*(x^i)e(x^j) - e(x^j)e^*(y^i)e^*(x^i)\\ =&e^*(y^j)e^*(x^j)e(x^j) - e(x^j)e^*(y^j)e^*(x^j).\end{aligned}$$

Now let ψ be a monom which does not contain x^j and y^j. Then

$$\begin{aligned}e^*(y^j)\psi =&0\\ e^*(y^j)x^j\wedge\psi =&0\\ e^*(y^j)y^j\wedge\psi =&\psi\\ e^*(y^j)x^j\wedge y^j\wedge\psi =&-x^j\wedge\psi,\end{aligned}$$

and

$$\begin{aligned}
e^*(y^j)e^*(x^j)e(x^j) - e(x^j)e^*(y^j)e^*(x^j)\psi =&0\\
e^*(y^j)e^*(x^j)e(x^j) - e(x^j)e^*(y^j)e^*(x^j)x^j \wedge \psi =&0\\
e^*(y^j)e^*(x^j)e(x^j) - e(x^j)e^*(y^j)e^*(x^j)y^j \wedge \psi =&\psi\\
e^*(y^j)e^*(x^j)e(x^j) - e(x^j)e^*(y^j)e^*(x^j)x^j \wedge y^j \wedge \psi =&- x^j \wedge \psi.
\end{aligned}$$

The proof of *b*) is very similar to the previous computation. □

Corollary 1.43. *Let* $\eta \in \bigwedge^{1,0} V^*$*, then*

a) $[\Lambda, e(\eta)] = -ie^*(\overline{\eta})$.

b) $[\Lambda, e(\overline{\eta})] = ie^*(\eta)$.

Proof. One can assume, that $\eta = x^1 + iy^1$. Then

$$\begin{aligned}
[\Lambda, e(\eta)] =&[\Lambda, e(x^1) + ie(y^1)] = e^*(y^1) - ie^*(x^1) = -i(e^*(x^1) + ie^*(y^1))\\
=&- i(e^*(x^1) + e^*(-iy^1)) = -ie^*(x^1 - iy^1) = -ie^*(\overline{\eta})
\end{aligned}$$

b) is similar. □

Corollary 1.44. *Let* η *be a real* 1*-form, then*

$$[\Lambda, e(\eta)] = -\mathbb{I}e^*(\eta)\mathbb{I}^{-1},$$

Where $\mathbb{I} = \sum_{p,q} i^{p-q}\pi^{p,q}$.

Proof. Since η is a real 1-form there exists a φ $(1,0)$-form, such that $\eta = \varphi + \overline{\varphi}$. Then

$$[\Lambda, e(\eta)] = [\Lambda, e(\varphi)] + [\lambda, e(\overline{\varphi})] = -ie^*(\overline{\varphi}) + ie^*(\varphi) = e^*(i\overline{\varphi}) + e^*(-i\varphi).$$

Now suppose that *a*) $e(i\overline{\varphi}) = -\mathbb{I}e(\overline{\varphi})\mathbb{I}^{-1}$ and *b*) $e(-i\varphi) = \mathbb{I}e(\varphi)\mathbb{I}^{-1}$. Since $\mathbb{I}^{-1} = \mathbb{I}^*$ we get that

$$e^*(i\overline{\varphi}) = e(i\overline{\varphi})^* = (-\mathbb{I}e(\overline{\varphi})\mathbb{I}^{-1})^* = -\mathbb{I}e^*(\overline{\varphi})\mathbb{I}^{-1},$$

and similarly $e^*(-\varphi) = -\mathbb{I}e^*(\varphi)\mathbb{I}^{-1}$. This finishes our proof, since we got that

$$\begin{aligned}
[\Lambda, e(\eta)] =&e^*(i\overline{\varphi}) + e^*(-i\varphi) = -\mathbb{I}e^*(\overline{\varphi})\mathbb{I}^{-1} - \mathbb{I}e^*(\varphi)\mathbb{I}^{-1}\\
=&- \mathbb{I}e^*(\overline{\varphi} + \varphi)\mathbb{I}^{-1} = -\mathbb{I}e^*(\eta)\mathbb{I}^{-1}.
\end{aligned}$$

Now to see $a)$ let $\alpha \in \bigwedge^{p,q} V^*$. Then

$$-\mathbb{I}e(\overline{\varphi})\mathbb{I}^{-1}\alpha = (-1)i^{q-p}\mathbb{I}e(\overline{\varphi})\alpha = (-1)i^{q-p}\mathbb{I}\underbrace{\overline{\varphi} \wedge \alpha}_{(p,q+1)} = (-1)i^{-1}\overline{\varphi} \wedge \alpha = e(i\overline{\varphi})\alpha.$$

$b)$ follows in a similar way. □

It is known that $\mathfrak{sl}(2,\mathbb{C})$ is the Lie-algebra of the Lie group $SL(2,\mathbb{C})$. Let G be an arbitrary Lie group with Lie-algebra $\mathfrak{g}$. Since $SL(2,\mathbb{C})$ is simply connected there is a one-one correspondence between the Lie-algebra homomorphisms $\mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{g}$ and Lie group homomoprhisms $SL(2,\mathbb{C}) \to G$. Moreover if $\rho : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{g}$ is a Lie-algebra homomorphism and $\pi_\rho : SL(2,\mathbb{C}) \to G$ is the corresponding Lie group homomorphism, then the following diagram is commutative:

$$\begin{tikzcd} \mathfrak{sl}(2,\mathbb{C}) \arrow[r, "\rho"] \arrow[d, "\exp"'] & \mathfrak{g} \arrow[d, "\exp"] \\ SL(2,\mathbb{C}) \arrow[r, "\pi_\alpha"'] & G \end{tikzcd}$$

Definition 1.45. Let $\# = \exp(1/2i\pi(\Lambda + L)) = \exp(1/2i\pi(\alpha(X) + \alpha(L))$.

Proposition 1.46. *Let η be a real 1-form. Then*

$$\#e(\eta)\#^{-1} = -i\mathbb{I}e^*(\eta)\mathbb{I}^{-1}.$$

Proof. Let $t \in \mathbb{R}$ and define the following:

$$e_t(\eta) = \exp(it(\Lambda + L))e(\eta)\exp(-it(\Lambda + L)).$$

It is clear that $e_{\pi/2}(\eta) = \#e(\eta)\#^{-1}$. The idea is to show that $e_t(\eta)$ satfisfies a differential equation with initial condition $e_0(\eta) = e(\eta)$. We will solve the differential equation, eveluate it in $\pi/2$ and hopefully get what we wanted. First notice that we have the following commutative diagram:

$$\begin{tikzcd} \mathfrak{sl}(2,\mathbb{C}) \arrow[r, "\alpha"] \arrow[d, "\exp"'] & \mathrm{End}(\bigwedge^* V_{\mathbb{C}}^*) \arrow[d, "\exp"] \\ SL(2,\mathbb{C}) \arrow[r, "\pi_\alpha"'] & GL(\bigwedge^* V_{\mathbb{C}}^*) \end{tikzcd}$$

Where π_α corresponds to the Lie-algebra homomorphism α. Let $w_t = \exp(it(X+Y))$, then $e_t(\eta) = \pi_\alpha(w_t)e(\eta)\pi_\alpha(w_t)^{-1}$. We also have the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{End}(\bigwedge^* V_{\mathbb{C}}^*) & \xrightarrow{\mathrm{ad}} & \mathrm{End}(\mathrm{End}(\bigwedge^* V_{\mathbb{C}}^*)) \\
{\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\
GL(\bigwedge^* V_{\mathbb{C}}^*) & \xrightarrow[\mathrm{Ad}]{} & GL(\mathrm{End}(\bigwedge^* V_{\mathbb{C}}^*))
\end{array}$$

It is also known that if $g \in GL(\bigwedge^* V_{\mathbb{C}}^*)$, then $\mathrm{Ad}_g(e(\varphi)) = ge(\varphi)g^{-1}$. In our case, we get that

$$\begin{aligned}
e_t(\eta) =& \pi_\alpha(w_t)e(\eta)\pi_\alpha(w_t)^{-1} = \mathrm{Ad}_{\pi_\alpha(w_t)}e(\eta) = \mathrm{Ad}_{\exp(it(\Lambda+L))}(e(\eta)) \\
=& \exp(\mathrm{ad}(it(\Lambda+L))(e(\eta)) = \sum_{k\geq 0}\frac{1}{k!}[\mathrm{ad}(it(\Lambda+L))]^k e(\eta).
\end{aligned}$$

Since ad is complex linear, we get the following:

$$\begin{cases}
\frac{d}{dt}e_t(\eta) = i(\mathrm{ad}(\Lambda+L))(e_t(\eta)) \\
e_0(\eta) = e(\eta)
\end{cases}$$

One can easily check that $\cos(t)e(\eta) + i\sin(t)\mathrm{ad}(\Lambda)e(\eta)$ solves the differential equation above. Now eveluating in $\pi/2$ we get that

$$\# e(\eta)\#^{-1} = e_{\pi/2}(\eta) = \mathrm{ad}(\Lambda)(e(\eta)) = [\Lambda, e(\eta)] = -\mathbb{I}e^*(\eta)\mathbb{I}.$$

The third equality holds because $\mathrm{ad}(A) = [A, -]$ for all $A \in \mathrm{End}(\bigwedge^* V_{\mathbb{C}}^*)$ and for the last equality we used Corollary 1.44. $\square$

We had the representation $\pi_m : \mathfrak{sl}(2,\mathbb{C}) \to \mathrm{End}(\mathbb{C}[x,y]_m)$. One can easily compute that the corresponding Lie group homomorphism $\tilde{\pi}_m : SL(2,\mathbb{C}) \to GL(\mathbb{C}[x,y]_m)$ is defined as follows: let $A \in SL(2,\mathbb{C})$ and $f \in \mathbb{C}[x,y]$, then $\tilde{\pi}(A)f = f \circ A^T$. Notice that if $f, g \in \mathbb{C}[x,y]$, then $\tilde{\pi}(A)(fg) = \tilde{\pi}(A)f\tilde{\pi}(A)g$. It is easy to see that if f is polynomial of degree m then $\tilde{\pi}(A)f$ is of degree m, thus we can restrict $\tilde{\pi}(A)$ to a $\tilde{\pi}_m(A) : \mathbb{C}[x,y]_m \to \mathbb{C}[x,y]_m$ map. Let $w = \exp(1/2i\pi(X+Y))$, then $\tilde{\pi}_1(w)(x) = iy$ and $\tilde{\pi}_1(w)(y) = ix$, thus

$$\tilde{\pi}_m(w)x^{m-k}y^k = (\tilde{\pi}_1(w)x)^{m-k}(\tilde{\pi}_1(w)y)^k = i^m x^k y^{m-k}.$$

Let $\varphi_k = \pi_m(Y)^k x^m = (y\partial_x)^k x^m = \frac{m!}{(m-k)!}x^{m-k}y^k$, for all $0 \leq k \leq m$. Then

$$\tilde{\pi}_m(w)\varphi^k = \frac{m!}{(m-k)!}i^m y^{m-k}x^k = \frac{k!}{(m-k)!}i^m\varphi_{m-k},$$

thus

$$\tilde{\pi}_m(w)\pi_m(Y)^k\varphi_0 = i^m \frac{k!}{(m-k)!}\pi_m(Y)^{m-k}\varphi_0.$$

Our computation used concrete representations, but one sees that this last form holds for all $(m+1)$-dimensional irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$ and corresponding Lie group representation.

Corollary 1.47. *Let $\varphi_0 \in \bigwedge^* V_{\mathbb{C}}^*$ be a primitive form of weight m. Then*

$$\# L^k\varphi_0 = i^m \frac{k!}{(m-k)!}L^{m-k}\varphi_0,$$

for all $0 \leq k \leq m$.

Lemma 1.48. *Let $\varphi \in \bigwedge^p V_{\mathbb{C}}^*$, then $*\varphi = i^{p^2-n}\mathbb{I}^{-1}\#\varphi$.*

Proof. The Hodge *-star operator satisfies the following: $a)$ $*1 = Vol$, and $b)$ if η is a real 1-form, then $*e(\eta)|_{\bigwedge^p V_{\mathbb{C}}^*} = (-1)^p e^*(\eta)*$. Notice that $a)$ and $b)$ determine $*$ since if η is a real 1-form, then $*\eta = *e(\eta)(1) = e^*(\varphi)(Vol)$, and we can use induction to define it on p-forms for all p.

Let $\hat{*}|_{\bigwedge^p V_{\mathbb{C}}^*} = i^{p^2-n}\mathbb{I}^{-1}\#$. We want to show that $\hat{*} = *$. We will do that by proving $\hat{*}$ also satisfies $a)$ and $b)$. To prove $a)$ we first note that 1 is a primitve form of weight n. Thus $\#1 = \frac{i^n}{n!}L^n(1)$, and

$$\hat{*}(1) = i^{-n}\mathbb{I}^{-1}\frac{i^n}{n!}L^n(1) = \mathbb{I}^{-1}Vol = i^{n-n}Vol = Vol.$$

To prove $b)$ let η be a real 1-form and $\varphi \in \bigwedge^p V_{\mathbb{C}}^*$. Then

$$\begin{aligned}\hat{*}e(\eta)\varphi =& i^{(p+1)^2-n}\mathbb{I}^{-1}\#e(\eta)\varphi = i^{(p+1)^2-n}\mathbb{I}^{-1}\#e(\eta)\#^{-1}\#\varphi\\ =& i^{(p+1)^2-n}(-i)e^*(\eta)\mathbb{I}^{-1}\#\varphi = (-1)^p e^*(\eta)i^{p^2-n}\mathbb{I}^{-1}\#\varphi\\ =& (-1)^p e^*(\eta)\hat{*}\varphi.\end{aligned}$$

We just showed that $\hat{*}$ also satisfies $b)$, hence $* = \hat{*}$. □

Theorem 1.49. *Let φ be a primitive p-form. Then*

$$*L^r\varphi = (-1)^{p(p+1)/2}\frac{r!}{(n-p-r)!}L^{n-p-r}\mathbb{I}\varphi,$$

for all $0 \leq r \leq n-p$.

Proof. Since φ is a p-form it is of weight $n-p$. Thus by Corollary 1.47

$$\# L^r\varphi = i^{n-p}\frac{r!}{(n-p-r)!}L^{n-p-r}\varphi.$$

Now we use Lemma 1.48 to compute the following:

$$
\begin{aligned}
*L^r\varphi =& i^{(p+2r)^2-n}\mathbb{I}^{-1}\# L^r\varphi\\
=& i^{p^2-n}\mathbb{I}^{-1}i^{n-p}\frac{r!}{(n-p-r)!}L^{n-p-r}\varphi\\
=& i^{p^2-p}(\mathbb{I}^{-1})^2\frac{r!}{(n-p-r)!}L^{n-p-r}\mathbb{I}\varphi\\
=& i^{p^2-p}(-1)^p\frac{r!}{(n-p-r)!}L^{n-p-r}\mathbb{I}\varphi\\
=& (-1)^{p(p+1)/2}\frac{r!}{(n-p-r)!}L^{n-p-r}\mathbb{I}\varphi.
\end{aligned}
$$

Here we also used that $L^m\mathbb{I} = \mathbb{I}L^m$ for all m. □

1.3 Sheaves and sheaf cohomology

This section is devoted to collect some standard results and definitions that we will use later in the thesis. For a more detailed treatment we recommend the book [5] Ramanan, S. - *Global Calculus.*

Definition 1.50. Let X be a topological space, then $\mathcal{F}$ is a presheaf of sets over X if

a) For each open set $U \subseteq X$ corresponds a set $\mathcal{F}(U)$.

b) For all $U, V \subseteq X$ open, with $V \subseteq U$ a map $\mathrm{res}_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ is given satisfying the following properties: $\mathrm{res}_{UU} = \mathrm{id}_{\mathcal{F}(U)}$, and if $W \subseteq V \subseteq U$ are open sets in X, then
$$\mathrm{res}_{VW}\mathrm{res}_{UV} = \mathrm{res}_{UW}.$$

Elements of the set $\mathcal{F}(U)$ are called sections of $\mathcal{F}$ over U, and the maps res_{UV} are called restriction maps. If $\mathcal{F}$ maps open sets to sets with extra structures like vector spaces, abelian groups, rings, etc. and the restriction maps are linear maps, group homomorphisms, ring homomorphisms, etc. then we call $\mathcal{F}$ a presheaf of vector spaces, abelian groups, rings, etc.

Remark 1.51. One can look at the topological space X as a category by declaring that the objects are the open sets of X and the morphisms are the inclusions. Then a presheaf $\mathcal{F}$ over X is just a contravariant functor from this category to the category of sets Set.

Definition 1.52. A presheaf $\mathcal{F}$ is called a sheaf if it satisfies the following properties: let $U = \bigcup_{i \in I} U_i$ be an open cover of the open set U. Then

S1) If $s, t \in \mathcal{F}(U)$, with $\mathrm{res}_{UU_i}(s) = \mathrm{res}_{UU_i}(t)$ for all $i \in I$, then $s = t$.

S2) If $s_i \in \mathcal{F}(U_i)$ with $\mathrm{res}_{U_iU_i\cap U_j}(s_i) = \mathrm{res}_{U_jU_i\cap U_j}(s_j)$ for all $i, j \in I$, then there exists $s \in \mathcal{F}(U)$, such that $\mathrm{res}_{UU_i}(s) = s_i$ for all $i \in I$.

$S1$ says that if two elements in a sheaf are locally the same, then they are globally the same, and $S2$ says that in a sheaf you can glue together elements if they agree on overleaps.

It follows from the definitions, but we assume that in a sheaf $\mathcal{F}(\emptyset)$ consist of a single point.

Example 1.53. Since the definition is very abstract lets look at some examples.

1) Let X be a topological space, then the assigment $U \mapsto C(U)$ with the obvious restriction maps clearly defines a sheaf over X. It is called the sheaf of continuous

functions on X. Notice that $C(U)$ is an $\mathbb{R}$-algebra for all U and the restriction maps are $\mathbb{R}$-algebra homomorphisms.

2) Let X, Y be topological spaces, then $C(-, Y)$ the continuous maps to Y with obvious restriction maps is clearly a sheaf over X. Unlike above it is just a sheaf of sets since we cannot add or multiply maps to Y.

3) By a bundle we mean a triple $\xi = (E, \pi, B)$, where E, B are topological spaces and $\pi : E \to B$ is a continuous map. Define the sections of ξ over U as follows:

$$\Gamma(U, \xi) = \{s \in C(U, E) \mid \pi \circ s = \mathrm{id}_U\}$$

Clearly the assigment $U \mapsto \Gamma(U, \xi)$ with the obvious restriction maps is a sheaf over X. This sheaf is called the sheaf of sections of the bundle ξ.

4) Let X be a smooth manifold, and $U \subset X$ open. Denote by $C^k(U)$ the set of k-times continuously differentiable functions on U. The assigment $U \mapsto C^k(U)$ for all open set U, with the obvious restriction maps clearly defines a sheaf over X for all $k \in \mathbb{N} \cup \infty$.

5) Let $X = \mathbb{R}$ and let $\mathcal{F}(U)$ be the bounded continuous functions on U. Then $\mathcal{F}$ with the obvious restriction maps is clearly a presheaf over X but $\mathcal{F}$ is not a sheaf. It is clear that $\mathcal{F}$ satisfies $S1$) since $\mathcal{F}$ consists of functions. The problem is with $S2$). Indeed, let $U_i = (i-1, i+1)$, then clearly $\bigcup_{i \in \mathbb{Z}} U_i = \mathbb{R}$, and it is also clear that $x|_{U_i}$ is a bounded function on U_i, but x is not a bounded function on $\mathbb{R}$. The problem with this sheaf is that being bounded is not a local condition. Notice that if we define $\mathcal{F}(U)$ as the locally bounded functions on U, then $\mathcal{F}$ is a sheaf.

6) Let X be a topological space, and A a set of order at least two. Let $\mathcal{F}(U)$ be the constant A-valued functions on U if $U \neq \emptyset$ and let $\mathcal{F}(\emptyset)$ be a one point set. Then $\mathcal{F}$ with the obvious restriction maps is clearly a presheaf on $\mathcal{F}$ satisfying $S1$) but not $S2$). The problem comes again from the fact, that being constant is not a local property. Again if we localise the defining property, i.e. we set $\mathcal{F}(U)$ to be the locally constant A-valued functions on U, then clearly it will become a sheaf over X.

7) For a harder example let R be a commutative ring with unity. Denote by $\mathrm{Spec}(R)$ the set of prime ideals in R. First we define a topology on $\mathrm{Spec}(R)$. Let $S \subset R$, then $V(S) = \{P \in \mathrm{Spec}(R) \mid S \subset P\}$. We say that $V \subset \mathrm{Spec}(R)$ is closed, if there exists a set $S \subset R$ such that $V = V(S)$. To see that this will define a topology, let $S_\alpha \subset R$ for some index set A. Then clearly $\cap_{\alpha \in A} V(S_\alpha) = V(\cup_{\alpha \in A} S_\alpha)$, and $V(S_\alpha) \cup V(S_\beta) = V(S_\alpha S_\beta)$, where $S_\alpha S_\beta = \{u_\alpha v_\beta \in R \mid u_\alpha \in S_\alpha,\ v_\beta \in S_\beta\}$. Let $D(f) = \mathrm{Spec}(R) \setminus V(f)$, where $f \in R$. One can check that $\{D(f) \mid f \in R\}$ is a basis of the topology on $\mathrm{Spec}(R)$. We will define the structure sheaf of R only on a basis of the topology, but one can check that this indeed defines a sheaf over $\mathrm{Spec}(R)$.

Let $f \in R$, then $\mathcal{O}_R(D(f)) = R_f$, where $R_f = S^{-1}R$ and $S = 1, f, f^2, \ldots$. For more details see the book Algebraic geometry by R. Hartshorne.

Definition 1.54. Let $\mathcal{F}_1, \mathcal{F}_2$ be presheaves over the topological space X. A morphism of presheaves $f : \mathcal{F}_1 \to \mathcal{F}_2$ assigns to every open set U a morphism $f(U) : \mathcal{F}_1(U) \to F_2(U)$, making the following diagram commutative:

$$\begin{tikzcd} \mathcal{F}_1(U) \arrow[r, "f(U)"] \arrow[d, "\mathrm{res}_{UV}"] & \mathcal{F}_2(V) \arrow[d, "\mathrm{res}_{UV}"] \\ \mathcal{F}_1(V) \arrow[r, "f(V)"] & \mathcal{F}_2(V) \end{tikzcd}$$

for all $V \subset U$ open sets.

Remark 1.55. If we think of sheaves as contravariant functors from the category of open sets of X, then a morphism of sheaves is just a natural morphism between the two functor.

Definition 1.56. Let X be a topological space and $\mathcal{F}$ a presheaf over X. Let $x \in X$, then the stalk of $\mathcal{F}$ at the point x denoted by $\mathcal{F}_x$ is the equivelence class of pairs (U, s), where U is an open neighborhood of x and $s \in \mathcal{F}(U)$. Two pairs (U, s) and (V, t) are equivalent if there exists a open neighborhood W of x, with $W \subseteq U \cap V$, such that $\mathrm{res}_{UW}(s) = \mathrm{res}_{VW}(t)$. We denote the equivalence class of (U, s) by $[(U, s)]$.

If $\mathcal{F}$ is a presheaf then we can define the set $E = E(\mathcal{F}) = \bigsqcup_{x \in X} \mathcal{F}_x$. There is a natural map $\pi : E \to X$ that maps an element of $\mathcal{F}_x$ to x. Notice that if we have an element $s \in \mathcal{F}(U)$, then the equivalence class of the pair (U, s) defines an element in $\mathcal{F}_x$ denoted by s_x, for all $x \in U$. Hence an element $s \in \mathcal{F}(U)$ gives a section of the bundle $\pi : E \to X$ defined by $\tilde{s}(x) = s_x$. We want to define a topology on E that makes π continuous, and which makes the sections $\tilde{s}$ continuous. Look at the sets $\{\tilde{s}(U) \mid U \text{ open, and } s \in \mathcal{F}(U)\}$. One can check easily that this is a basis of a topology, and this topology satisfies what we wanted. We call $E(\mathcal{F})$ the Étale space of $\mathcal{F}$. We saw in example 3) that the sections of a bundle define a sheaf, hence given a presheaf $\mathcal{F}$ one can associate to it naturaly a sheaf $\tilde{\mathcal{F}}$, the sections of the Étale space, associated to $\mathcal{F}$. Also notice that one has a natural morphism $\mathcal{F} \to \tilde{\mathcal{F}}$, which maps $s \in \mathcal{F}(U)$, to $\tilde{s} \in \tilde{\mathcal{F}}(U) = \Gamma(U, E(\mathcal{F}))$.

If $\mathcal{F}, \mathcal{G}$ are presheaves over X and $f : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves, then f induces a map $f_x : \mathcal{F}_x \to \mathcal{G}_x$ by the following: let's represent an element of F_x by (U, s), then the image of this element is the class of $(U, f(s))$. It is easy to check, that this is a well defined map between the stalks. We claim that the map $\mathcal{F} \to \tilde{\mathcal{F}}$ induces an isomorphism $\mathcal{F}_x \to \tilde{\mathcal{F}}_x$ for all $x \in X$. Indeed, let $x \in X$ and $[(U, s)], [(V, t)] \in \mathcal{F}_x$. Suppose that $[(U, \tilde{s})] = [(V, \tilde{t})] \in \tilde{F}_x$, by definition this means that there exists an open set $W \subseteq U \cap V$ containing x, with $\tilde{s}|_W = \text{res}_{UW}(\tilde{s}) = \text{res}_{VW}(\tilde{t}) = \tilde{t}|_W$, hence $[(U,s)] = s_x = \tilde{s}(x) = \tilde{t}(x) = t_x = [(V,t)]$, so the map is injective. To see that it is surjective let $[(U,\gamma)] \in \tilde{F}_x$, then $\gamma(x) \in \mathcal{F}_x$, since γ is a section. Hence by definition there exists an open set V containing x and $t \in \mathcal{F}(U)$ such that $\gamma(x) = [(V,t)]$. Since γ is a continuous map and by definition $\tilde{t}(V)$ is open, we get that the open set $\gamma^{-1}(\tilde{t}(V))$ is not the empty set since it contains x. Thus there exists an open set W, with $x \in W$ such that $\gamma(y) = \tilde{t}(y)$ for all $y \in W$, hence the set $V \cap U \cap W = A$ is not empty and $\text{res}_{UA}(\gamma) = \text{res}_{VA}(\tilde{t})$ which by definition means that $[(U,\gamma)] = [(V,\tilde{t})]$.

Proposition 1.57. *Let $\mathcal{F}$ be a presheaf, then $\mathcal{F}$ is a sheaf if and only if the natural morphism $\mathcal{F}(U) \to \tilde{\mathcal{F}}(U)$ is an isomorphism for all U open.*

Definition 1.58. Let $\mathcal{F}, \mathcal{G}$ be sheaves over X. A morphism of sheaves is just a morphism of presheaves. We say that $\mathcal{F}$ is a subsheaf of $\mathcal{G}$ if there exists a morphism $\iota : \mathcal{F} \to \mathcal{G}$, with $\iota_x : \mathcal{F}_x \to \mathcal{G}_x$ being injective for all $x \in X$.

One can check that this is equivalent to saying that $\iota(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for all U open.

Definition 1.59. Let X be a topological space, let $\mathcal{R}$ be a sheaf of rings and $\mathcal{M}$ be a sheaf of abelian groups. We say that $\mathcal{M}$ is a sheaf of $\mathcal{R}$-modules, if for all open set U, the group $\mathcal{M}(U)$ is an $\mathcal{R}(U)$-module, and the restriction maps of $\mathcal{M}$ respects the module structure, i.e. for all $V \subseteq U$ open sets we have the following:

$$\text{res}_{UV}(fs) = \text{res}_{UV}(f)\text{res}_{UV}(s),$$

where $f \in \mathcal{R}(U)$ and $s \in \mathcal{M}(U)$.

Definition 1.60. Let $\mathcal{F}', \mathcal{F}, \mathcal{F}'$ be sheaves of abelian groups over X. Suppose we have morphisms $\mathcal{F}' \to \mathcal{F}$ and $\mathcal{F} \to \mathcal{F}'$. We say that that the sequence

$$\mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}''$$

is exact, if the sequence

$$\mathcal{F}'_x \longrightarrow \mathcal{F}_x \longrightarrow \mathcal{F}''_x$$

is exact, for all $x \in X$.

Remark 1.61. Let $\mathcal{F}, \mathcal{G}$ be sheaves, then the exactness of the sequence

$$\mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0$$

means that $\mathcal{F}_x \to \mathcal{G}_x$ is surjective for all $x \in X$, which basically means that if U is an open neighborhood of x and $s \in \mathcal{G}(U)$, then there exists a $V \subset U$ open neighborhood of x, and $t \in \mathcal{F}(V)$, such that the image of t in $\mathcal{G}(V)$ is $\text{res}_{UV}(s)$.

Proposition 1.62. *Suppose that that we have a short exact sequence of sheaves*

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

then

$$0 \longrightarrow \mathcal{F}'(X) \longrightarrow \mathcal{F}(X) \longrightarrow \mathcal{F}''(X)$$

is an exact sequence.

Usually the map $\mathcal{F}(X) \to \mathcal{F}''(X)$ is not surjective. For example let D be an open domain in $\mathbb{C}$, then we have the exponential sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2\pi i} \mathcal{O}_D \xrightarrow{\exp} \mathcal{O}_D^* \longrightarrow 0$$

where $\mathbb{Z}$ is the locally constant $\mathbb{Z}$-valued sheaf, $\mathcal{O}_D$ is the sheaf of holomorphic functions on D and $\mathcal{O}_D^*$ is the sheaf of nowhere zero holomorphic functions on D. Since locally we have complex logarithm this is an exact sequence of sheaves, however let $D = \mathbb{C} \setminus 0$, then $z \in \mathcal{O}_{\mathbb{C}\setminus 0}^*(\mathbb{C} \setminus 0)$ is not the exponential of a holomorphic function. Indeed, if it were, then we would have a complex logarithm $l(z)$ on $\mathbb{C} \setminus 0$. If we differentiate both side of $\exp(l(z)) = z$ we get that $l'(z) = 1/z$, hence we have a primitive function of $1/z$ which would imply that the integral of $1/z$ around the origin is zero, which is a contradiction.

The point of sheaf cohomology is to understand what happens at $\mathcal{F}(X) \to \mathcal{F}''(X)$. In some sense sheaf cohomology measures the nonexactness of this sequence.

Definition 1.63. Let $\mathcal{F}, \mathcal{G}, \mathcal{G}'$ be sheaf of $\mathcal{R}$-modules over X. Suppose that $\mathcal{G}'$ is a subsheaf of $\mathcal{G}$. We say that $\mathcal{F}$ is injective if arbitrary morphism from $\mathcal{G}'$ to $\mathcal{F}$ can be extended to a morphism from $\mathcal{G}$ to $\mathcal{F}$, i.e. we have the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{G}' \arrow[r] \arrow[d] & \mathcal{G} \arrow[dl, dashed, "\exists"] \\ & \mathcal{F} & \end{tikzcd}$$

Definition 1.64. Let $\mathcal{F}$ be a sheaf of abelian groups. We say that $\mathcal{F}$ is flabby/soft if an arbitrary section of $\mathcal{F}$ over an open/closed set can be extended to the whole $\mathcal{F}$.

Notice that if $\mathcal{F}$ is a sheaf, then we can identify $\mathcal{F}$ with $\tilde{\mathcal{F}}$ the sheaf of sections of the Ételе space, and then it makes sense to look at sections over a closed set K. We can look at this in a little bit more abstract way. Let $f : Y \to X$ be a continuous map, and $\mathcal{F}$ a sheaf over X, then we have $\pi : E(\mathcal{F}) \to X$. We can define the pullback of this bundle as having total space $f^*(E(\mathcal{F})) = \{(y, e) \in Y \times E(\mathcal{F}) \mid f(y) = \pi(e)\}$ and the map $\pi : f^*(E(\mathcal{F})) \to Y$ is just the restriction of the projection pr_1 to Y.

Define the inverse image sheaf $f^{-1}(\mathcal{F})$ as the sections of the bundle $(f^*(E(\mathcal{F})), \pi, Y)$. If K is a closed subset of X then this will give back what we just defined before. Let ι be the inclusion map of K to X, then we denote $\iota^{-1}\mathcal{F}$ by $\mathcal{F}|_K$.

Proposition 1.65. *If X is paracompact, and K is a closed subspace of X, then every element of $\mathcal{F}|_K(K)$ is the restriction of some $s \in \mathcal{F}(U)$, where $K \subset U$ open.*

Proposition 1.66. *Suppose that X is paracompact, and $\mathcal{R}$ is a soft sheaf of rings. Then every sheaf of $\mathcal{R}$-modules is soft.*

Let M be a smooth manifold, then it is known that M is paracompact. Let $E \to X$ be a smooth vector bundle over X, and denote it's smooth sections over U by $\Gamma(U, E)$. This is clearly a sheaf, moreover if we denote the sheaf of smooth real valued functions by C^∞, then $\Gamma(-, E)$ is clearly a C^∞-module. Suppose we have a section of $E(C^\infty)$ over a closed set K, then by Proposition 1.65 it is a restriction of a section over U where $K \subset U$, which is just a smooth function f on U. It is known from general topology that there exists a smooth function φ which is constant one on some neighborhood of K and $\mathrm{Supp}\varphi \subset U$. Hence we can define φf on whole X by defining it zero outside of U, and it is clear that the restriction of this map to K is the same as f restricted to K. Hence we just showed that C^∞ is a soft sheaf, which by the previous claim implies that $\Gamma(-, E)$ is soft.

Theorem 1.67. *If $\mathcal{F}$ is a sheaf of $\mathcal{R}$-modules, then there exists an injective sheaf $\mathcal{I}$ and an injective morphism $\mathcal{F} \to \mathcal{I}$, i.e. every $\mathcal{R}$-module is a subsheaf of an injective $\mathcal{R}$-module.*

If we have a morphism of presheaves $f : \mathcal{F} \to \mathcal{G}$, then one can define the presheaves $\ker(f)$, $\mathrm{im}(f)$ and $\mathrm{coker}(f)$ in a natural way, by assigning to U the set $\ker(f(U))$, $\mathrm{im}(f(U))$ and $\mathcal{G}(U)/\mathrm{im}(f(U))$. If $\mathcal{F}, \mathcal{G}$ are sheaves, then $\ker f$ is a subsheaf of $\mathcal{F}$, but $\mathrm{im}(f)$ and $\mathrm{coker}(f)$ are just presheaves. Still one can associate the sheaves $\widetilde{\mathrm{im}(f)}$ and $\widetilde{\mathrm{coker}(f)}$ to them. One can prove that in this case one has two exact sequnces of sheaves:

$$0 \longrightarrow \widetilde{\mathrm{im}(f)} \longrightarrow \mathcal{G} \longrightarrow \widetilde{\mathrm{Coker}(f)} \longrightarrow 0$$

and

$$0 \longrightarrow \ker(f) \longrightarrow \mathcal{F} \longrightarrow \widetilde{\mathrm{im}(f)} \longrightarrow 0$$

From now on we will not write out the ˜ sign, and by $\mathrm{im}(f)$ and $\mathrm{Coker}(f)$ we will always mean the associated sheaves.

By Theorem 1.67 one can always embed $\mathcal{F}$ into an injective sheaf $\mathcal{I}^0$. Then one can look at the factor sheaf $\mathcal{K}^1 = \mathcal{I}^0/\mathcal{F}$. Since $\mathcal{K}^1$ is also a sheaf of $\mathcal{R}$-modules, one can embed it to an injective sheaf $\mathcal{I}^1$, and then one has the following exact sequence:

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}^0 \longrightarrow \mathcal{I}^1 \longrightarrow \mathcal{K}^2 \longrightarrow 0$$

where $\mathcal{K}^2 = \mathcal{I}^1/\mathrm{im}(\mathcal{I}^0)$. By induction, we see that there is a long exact sequence of sheaves:

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}^0 \longrightarrow \mathcal{I}^1 \longrightarrow \mathcal{I}^2 \longrightarrow \ldots \longrightarrow \mathcal{I}^n \longrightarrow \ldots$$

where $\mathcal{F}$ is a subsheaf of $\mathcal{I}^0$, and $\mathcal{I}^n$ is injective for all $n \in \mathbb{N}$.

Definition 1.68. A sequence of sheaves of abelian groups

$$\ldots \longrightarrow \mathcal{F}^{n-1} \longrightarrow \mathcal{F}^n \longrightarrow \mathcal{F}^{n+1} \longrightarrow \ldots$$

is called a complex, if any two consecutive map is zero. We denote the complex by $\mathcal{F}^\bullet$. The maps are called differentials, usually denoted by d or δ.

Definition 1.69. An exact sequence of sheaves

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}^0 \longrightarrow \mathcal{I}^1 \longrightarrow \mathcal{I}^2 \longrightarrow \ldots$$

is called the resolution of $\mathcal{F}$, and the complex

$$\mathcal{I}^\bullet = \ \mathcal{I}^0 \longrightarrow \mathcal{I}^1 \longrightarrow \mathcal{I}^2 \longrightarrow \ldots$$

is called the resolving complex of $\mathcal{F}$. If in addition every sheaf in $\mathcal{I}^\bullet$ is an injective sheaf, then we call $\mathcal{I}^\bullet$ an injective resolution of $\mathcal{F}$.

By the reasoning above, we see that every sheaf $\mathcal{F}$ has an injective resolution.

Definition 1.70. A morphism of complexes $\varphi^\bullet : \mathcal{I}^\bullet \to \mathcal{J}^\bullet$ is a sequence of morphisms $\varphi^i : \mathcal{F}^i \to \mathcal{G}^i$, which is commuting with the differentials, i.e. we have the commutative diagram:

$$\begin{tikzcd}
\ldots \arrow[r] & \mathcal{I}^{n-1} \arrow[r] \arrow[d, "\varphi^{n-1}"'] & \mathcal{I}^n \arrow[r] \arrow[d, "\varphi^n"'] & \mathcal{I}^{n+1} \arrow[r] \arrow[d, "\varphi^{n+1}"'] & \ldots \\
\ldots \arrow[r] & \mathcal{J}^{n-1} \arrow[r] & \mathcal{J}^n \arrow[r] & \mathcal{J}^{n+1} \arrow[r] & \ldots
\end{tikzcd}$$

Proposition 1.71. *Let $0 \to \mathcal{F} \to \mathcal{I}^\bullet$ and $0 \to \mathcal{G} \to \mathcal{J}^\bullet$ be injective resolutions. Then every $\mathcal{R}$-module homomorphism $f : \mathcal{F} \to \mathcal{G}$ extends to a morphism of com-*

plexes $\varphi^\bullet : \mathcal{I}^\bullet \to \mathcal{J}^\bullet$, *making the following diagram commutative:*

$$\begin{tikzcd}
\mathcal{F} \arrow[r] \arrow[d, "f"'] & \mathcal{I}^\bullet \arrow[d, "\varphi^\bullet"'] \\
\mathcal{G} \arrow[r] & \mathcal{J}^\bullet
\end{tikzcd}$$

Definition 1.72. Let $f^\bullet, g^\bullet : \mathcal{I}^\bullet \to \mathcal{J}^\bullet$ be two morphisms of complexes. We say that $f^\bullet$ and $g^\bullet$ are homotopic, if there exists a sequence of morphisms $h^i : \mathcal{I}^i \to \mathcal{J}^{i-1}$, such that $d \circ k^i - k^{i-1} \circ d = f^i - g^i$, i.e. we have the following diagram:

$$\begin{tikzcd}
\dots \arrow[r] & \mathcal{I}^{i-1} \arrow[r] \arrow[dl, "k^{i-1}"'] \arrow[d, shift right, "f^{i-1}"'] \arrow[d, shift left, "g^{i-1}"] & \mathcal{I}^i \arrow[r] \arrow[dl, "k^i"'] \arrow[d, shift right, "f^i"'] \arrow[d, shift left, "g^i"] & \mathcal{I}^{i+1} \arrow[r] \arrow[dl, "k^{i+1}"'] \arrow[d, shift right, "f^{i+1}"'] \arrow[d, shift left, "g^{i+1}"] & \dots \arrow[dl, "k^{i+2}"'] \\
\dots \arrow[r] & \mathcal{J}^{i-1} \arrow[r] & \mathcal{I}^i \arrow[r] & \mathcal{I}^{i+1} \arrow[r] & \dots
\end{tikzcd}$$

Proposition 1.73. *Let* $0 \to \mathcal{F} \to \mathcal{I}^\bullet$ *be an injective resolution of* $\mathcal{F}$ *and* $0 \to \mathcal{G} \to \mathcal{J}^\bullet$ *a resolution of* $\mathcal{G}$. *Let* $f : \mathcal{F} \to \mathcal{G}$ *be a morphism of* $\mathcal{R}$*-modules and suppose* $\varphi^\bullet, \psi^\bullet : \mathcal{I}^\bullet \to \mathcal{J}^\bullet$ *are morphisms of complexes, making the diagram commutative*

$$\begin{tikzcd}
\mathcal{F} \arrow[r] \arrow[d, "f"'] & \mathcal{I}^\bullet \arrow[d, shift right, "\varphi^\bullet"'] \arrow[d, shift left, "\psi^\bullet"] \\
\mathcal{G} \arrow[r] & \mathcal{J}^\bullet
\end{tikzcd}$$

Then $\varphi^\bullet$ *and* $\psi^\bullet$ *are homotopic.*

Since we have morphism of complexes, we can define isomorphism of complexes. One hopes that the injective resolution of a sheaf is unique. This is not true, but some other kind of uniquenes holds.

Corollary 1.74. *Suppose that* $0 \to \mathcal{F} \to \mathcal{I}^\bullet$ *and* $0 \to \mathcal{F} \to \mathcal{J}^\bullet$ *are two injective resolutions of the sheaf* $\mathcal{F}$. *Then there exist morphisms* $\varphi^\bullet : \mathcal{I}^\bullet \to \mathcal{J}^\bullet$ *and* $\psi^\bullet : \mathcal{J}^\bullet \to \mathcal{I}^\bullet$ *which are inducing the identity map on* $\mathcal{F}$, *moreover the compositions* $\varphi^\bullet \circ \psi^\bullet$, *and* $\psi^\bullet \circ \varphi^\bullet$ *are homotopic to the identity map of the complex* $\mathcal{J}^\bullet$ *and* $\mathcal{I}^\bullet$ *respectively.*

Definition 1.75. Let $\mathcal{F}$ be a sheaf, and let $0 \to \mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution of $\mathcal{F}$. Then we define the cohomology groups of the sheaf $\mathcal{F}$, as the cohomologies of the complex

$$\mathcal{I}^0(X) \longrightarrow \mathcal{I}^1(X) \longrightarrow \mathcal{I}^2(X) \longrightarrow \dots$$

We denote the i-th cohomology group of $\mathcal{F}$ by $H^i(X, \mathcal{F})$.

This is well defined, since one can prove, that the homotopy between two injective resolution induces isomorphism between the cohomologies. One should also notice, that $H^0(X, \mathcal{F}) = \mathcal{F}(X)$. Indeed, by Proposition 1.62 we know that the sequence

$$0 \longrightarrow \mathcal{F}(X) \longrightarrow \mathcal{I}^0(X) \longrightarrow \mathcal{I}^1(X)$$

is exact, hence $H^0(X, \mathcal{F}) = \mathcal{F}(X)$.

Proposition 1.76. *Suppose that we have an exact sequence of sheaves*

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0$$

then there exisits injective resolutions $0 \to \mathcal{F}_i \to \mathcal{I}_i^\bullet$, making the diagram commutative.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{F}_1 \arrow[r] \arrow[d] & \mathcal{F}_2 \arrow[r] \arrow[d] & \mathcal{F}_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{I}_1^\bullet \arrow[r] & \mathcal{I}_2^\bullet \arrow[r] & \mathcal{I}_3^\bullet \arrow[r] & 0
\end{tikzcd}$$

Corollary 1.77. *A short exact sequence of sheaves*

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_2 \longrightarrow 0$$

induce a long exact sequence of cohomologies

$$\begin{tikzcd}
0 \arrow[r] & H^0(X,\mathcal{F}_1) \arrow[r] & H^0(X,\mathcal{F}_2) \arrow[r] & H^0(X,\mathcal{F}_3) \arrow[dll, out=-10, in=170] & \\
 & H^1(X,\mathcal{F}_1) \arrow[r] & H^1(X,\mathcal{F}_2) \arrow[r] & H^1(X,\mathcal{F}_3) \arrow[dll, dashed, out=-10, in=170] & \\
 & H^n(X,\mathcal{F}_1) \arrow[r] & H^n(X,\mathcal{F}_2) \arrow[r] & H^n(X,\mathcal{F}_3) \arrow[r] & \dots
\end{tikzcd}$$

As a corollary we get that, the sequence

$$0 \longrightarrow \mathcal{F}_1(X) \longrightarrow \mathcal{F}_2(X) \longrightarrow \mathcal{F}_3(X) \longrightarrow 0$$

is exact, if the first cohomology group of $\mathcal{F}_1$ is zero.

Injective resolutions are a really strong tools for proving these kind of statements, but it is very hard to do computations with them, so we want to look at other types of resolutions that will give back the cohomology groups of $\mathcal{F}$ but are easier to work with.

Lemma 1.78. *Let $0 \to \mathcal{F} \to \mathcal{I}^\bullet$ be an arbitrary resolution of $\mathcal{F}$. Suppose that $H^i(X, \mathcal{I}^j) = 0$ for all $i > 0$ and $j \geq 0$. Then the cohomologies of the complex $\mathcal{I}(X)^\bullet$ are naturally isomorphic with the cohomologies of $\mathcal{F}$.*

Now we need to find conditions that asserts that the cohologies of a sheaf $\mathcal{F}$ are all zero.

Proposition 1.79. *If $\mathcal{F}$ is a flabby sheaf, then $H^i(X, \mathcal{F}) = 0$ for all $i > 0$. The same result holds for soft sheaves over paracompact spaces.*

Proposition 1.80. *Let $0 \to \mathcal{F} \to \mathcal{I}^\bullet$ and $0 \to \mathcal{G} \to \mathcal{J}^\bullet$ be resolutions such that $H^i(X, \mathcal{I}^j) = H^i(X, \mathcal{J}^j) = 0$ for all $i > 0$ and $j \geq 0$. Suppose we have morphisms $f : \mathcal{F} \to \mathcal{G}$ and $\varphi^\bullet : \mathcal{I}^\bullet \to \mathcal{J}^\bullet$ making the diagram commutative*

$$\begin{tikzcd}
\mathcal{F}_1 \arrow[r] \arrow[d, "f"'] & \mathcal{I}^\bullet \arrow[d, "\varphi^\bullet"'] \\
\mathcal{G} \arrow[r] & \mathcal{J}^\bullet
\end{tikzcd}$$

Then we have the following commutative diagram:

$$\begin{tikzcd}
H^i(\mathcal{I}(X)^\bullet) \arrow[r, "H^i(\varphi^\bullet(X))"] \arrow[d] & H^i(\mathcal{J}(X)^\bullet) \arrow[d] \\
H^i(X, \mathcal{F}) \arrow[r, "H^i(f)"'] & H^i(X, \mathcal{G})
\end{tikzcd}$$

where the vertical maps are the isomorphisms from Lemma 1.78, $H^i(\varphi^\bullet(X))$ is the map induced by the maps $\varphi^i(X) : \mathcal{I}^i(X) \to \mathcal{J}^i(X)$, and $H^i(f)$ is the induced map on the sheaf cohomologies.

1.4 Complex manifolds and vector bundles

In this section we collect some standard definitions and results (mainly without proofs) that we need later on in the thesis. For a more detailed treatment we recommend the book [3] Huybrechts, D. *Complex Geometry: An Introduction.*

Definition 1.81. Let $\Omega \subset \mathbb{C}^n$ open and $F : \Omega \to \mathbb{C}^k$. We say that F is complex differentiable in $a \in \Omega$, if there exists an $L_a : \mathbb{C}^n \to \mathbb{C}^k$ linear, such that

$$F(z) = F(a) + L_a(z - a) + r_a(z),$$

with $\frac{r_a(z)}{||z-a||} \to 0$ if $z \to a$. We denote L_a by $F'(a)$.

Definition 1.82. We say that $F : \Omega \to \mathbb{C}^k$ is holomorphic, if F is complex differentiable for all $a \in \Omega$.

Proposition 1.83 (Chain rule)**.** *Let $\Omega \subset \mathbb{C}^p$ open, $G : \Omega \to \mathbb{C}^q$, with G complex differentiable in $a \in \Omega$, let $\tilde{\Omega} \subset \mathbb{C}^q$ open $G(a) \in \tilde{\Omega}$, $F : \tilde{\Omega} \to \mathbb{C}^s$, with F complex differentiable in $G(a)$. Then $F \circ G$ is complex differentiable in a and*

$$(F \circ G)'(a) = F'(G(a))G'(a).$$

Proposition 1.84. *Let $\partial_{z_j} = \frac{1}{2}(\partial_{x_j} - i\partial_{y_j})$ and $\partial_{\overline{z}_j} = \frac{1}{2}(\partial x_j + i\partial y_j)$. Let $F : U \to \mathbb{C}^m$ be a smooth map, where $U \subset \mathbb{C}^n$. Then F is holomorphic if and only if $\partial_{\overline{z}_j} F_k = 0$ for all j, k, and in this case $F'(a) = (\partial_{z_j} F_k(a))_{j,k}$.*

Definition 1.85. A holomorphic atlas on a smooth manifold M^{2n} is an atlas $\{(U_\alpha, \varphi_\alpha)\}$ of the form $\varphi_\alpha : U_\alpha \simeq \varphi_\alpha(U_\alpha) \subset \mathbb{C}^n$ such that whenever $U_\alpha \cap U_\beta \neq \emptyset$ the transition functions $\varphi_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ are holomorphic. The pair $(U_\alpha, \varphi_\alpha)$ is called a holomorphic chart. Two holomorphic atlases are compatible, if their union is a holomorphic atlas.

Definition 1.86. A complex manifold X of dimension n is a differentiable manifold M of (real) dimension $2n$ endowed with a maximal holomorphic chart.

A complex manifold is called compact, connected, simply connected, etc., if the underlying differentiable manifold has this property. By abuse of notation we will denote the underlying manifold M by X. It is clear that any open subset of X is a complex manifold.

Definition 1.87. A holomorphic function on a complex manifold X is a function $f : X \to \mathbb{C}$ such that $f \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha) \to \mathbb{C}$ is holomorphic for any $(U_\alpha, \varphi_\alpha)$ chart in the maximal holomorphic atlas.

Definition 1.88. Let X be a complex manifold, and let U be an open subset of X. Then

$$\mathcal{O}_X(U) = \Gamma(U, \mathcal{O}_X) = \{f : U \to \mathbb{C} \mid f \text{ is holomorphic}\},$$

with the obvious restrictions, defines a sheaf of rings over X. We call this the sheaf of holomorphic functions on X.

The following proposition shows the first difference between complex and differentiable manifolds.

Proposition 1.89. *Let X be a complex manifold. If X is compact and connected, then $\mathcal{O}_X(X) = \mathbb{C}$, i.e. the only global holomorphic functions are the constant functions.*

Proof. Let $f \in \mathcal{O}_X(X)$. Since X is compact $\|f\| : X \to \mathbb{R}$ attains it's maximum at some point $x \in X$. Let $U = f^{-1}(c)$, with $c = f(x)$. It is clear that $\emptyset \neq U$ is closed, and if $y \in U$, then by the maximum principle f is constant in a small neighborhood of y, thus U is open, hence $X = U$. $\square$

Definition 1.90. Let $\pi : E \to X$ be a holomorphic map between complex manifolds. We say that the triple (E, π, X) is a holomorphic vector bundle if it satisfy the following:

a) For all $x \in X$ the fiber $E_x = \pi^{-1}(x)$ is a d-dimensional complex vector space.

b) There exists $\{(U_\alpha, \phi_\alpha)\}_\alpha$, where $\{U_\alpha\}$ is an open cover of X, and for all α $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^d$ is a biholomorphism, such that $\mathrm{pr}_1(\phi_\alpha(v)) = \pi(v)$ for all $v \in E_x$, and for all $x \in X$ the composition $\mathrm{pr}_2 \circ \phi_\alpha|_{E_x} \to \mathbb{C}^d$ is a linear isomorphism.

$$\begin{tikzcd}
\pi^{-1}(U_\alpha) \arrow[rr, "\phi_\alpha"] \arrow[dr, "\pi"'] & & U_\alpha \times \mathbb{C}^d \arrow[dl, "\mathrm{pr}_1"] \\
& U_\alpha &
\end{tikzcd}$$

The pairs (U_α, ϕ_α) are called holomorphic charts and the set $\{(U_\alpha, \phi_\alpha)\}$ is called a holomorphic atlas of the bundle E.

Suppose that $U_\alpha \cap U_\beta \neq \emptyset$. Since ϕ_α and ϕ_β has to respect the fibers, one sees that that composition of $\phi_\alpha \phi_\beta^{-1} : U_\alpha \cap U_\beta \times \mathbb{C}^d \to U_\alpha \cap U_\beta \times \mathbb{C}^d$ is of the form

$$\psi_\alpha \psi_\beta^{-1}(x, v) = (x, g_{\alpha\beta}(x)(v))$$

where $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(d, \mathbb{C})$ is a holomorphic map. It is not hard to see that the maps $\{g_{\alpha\beta}\}$ satisfies the relations

a) $g_{\alpha\alpha}(x) = \mathrm{id}$, for all $x \in U_\alpha$.

b) $g_{\alpha\beta}(x)g_{\beta\alpha}(x) = \mathrm{id}$, for all $x \in U_\alpha \cap U_\beta$.

c) $g_{\alpha\gamma}(x) = g_{\alpha\beta}(x)g_{\beta\gamma}(x)$, for all $x \in U_\alpha \cap U_\beta \cap U_\gamma$.

We call the maps $\{g_{\alpha\beta}\}$ the transition functions of E associated to the atlas $\{(U_\alpha, \phi_\alpha)\}$.

One can show that if we have an open covering $\{U_\alpha\}$ of X, and holomorphic maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(d, \mathbb{C})$ satisfying $a), b)$ and $c)$, then there exists a holomorphic vector bundle $\pi : E \to X$, with holomorphic charts $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^d$ and transition functions $g_{\alpha\beta}$ associated to the holomorphic atlas $\{(U_\alpha, \phi_\alpha)\}$.

Theorem 1.91. *Let $E, F \to X$ be holomorphic vector bundles over X. Suppose that over the same covering E has $g_{\alpha\beta}$ and F has $h_{\alpha\beta}$ transition functions. Then*

1. *$E \oplus F \to X$ given by transition functions $g_{\alpha\beta} \oplus h_{\alpha\beta}$ is a holomorphic vector bundle with fibers $E_x \oplus F_x$ over $x \in X$.*

2. *$E \otimes F \to X$ given by transition functions $g_{\alpha\beta} \otimes h_{\alpha\beta}$ is a holomorphic vector bundle with fibers $E_x \otimes F_x$ over $x \in X$. If E and F are line bundles, then the transition functions of the tensor product is just $g_{\alpha\beta}h_{\alpha\beta}$.*

3. *$E^* \to X$ given by transition functions $(g_{\alpha\beta}^{-1})^T$ is a holomorphic vector bundle with fibers E_x^* over $x \in X$.*

4. *$\bigwedge^k E, S^k E \to X$ given by transition functiong $\bigwedge^k(g_{\alpha\beta}), S^k(g_{\alpha\beta})$ is a holomorphic vector bundle with fibers $\bigwedge^k E_x$ and $S^j E_x$ over $x \in X$. If the dimension of the fibers, called the rang of E, is r, then $\bigwedge^r E$ denoted by $det(E)$ is a line bundle, with transition functions $det(g_{\alpha\beta})$. It is called the determinant bundle of E.*

5. *If $Y \subset X$ a complex submanifold, then $\pi|_{\pi^{-1}(Y)} : \pi^{-1}(Y) \to Y$ is a holomorphic vector bundle over Y.*

Definition 1.92. Let $E, F \to X$ be vector bundles. We say that $\varphi : E \to F$ is a holomorphic vector bundle morphism, if:

a) φ is holomorphic.

b) φ is a bundle morphism, i.e. $\varphi(E_x) \subset F_x$.

c) $\varphi|_{E_x} : E_x \to F_x$ is complex linear.

d) $\mathrm{rank}(\varphi|E_x)$ is constant.

Definition 1.93. Let $E \to X$ be holomorphic vector bundle. We say that F is a holomorphic subbundle of E, denoted by $F < E$, if F is a complex submanifold of E, and F_x is a complex subspace of E_x for all $x \in X$ with $\dim(F_x)$ being constant on X.

Proposition 1.94. *Let E be a vector bundle then $F \subset E$ is a holomorphic subbundle of E if and only if there exists an holomorphic atlas $\{(U_\alpha, \varphi_\alpha)\}_\alpha$ of E making the diagram commutative*

$$\begin{tikzcd}
\pi^{-1}(U_\alpha) \cap F \arrow[r, "\phi_\alpha"'] \arrow[d] & U_\alpha \times \mathbb{C}^l \arrow[d] \\
\pi^{-1}(U_\alpha) \arrow[r, "\phi_\alpha"] & U_\alpha \times \mathbb{C}^d
\end{tikzcd}$$

where the first vertical map is the inclusion map and the second maps $(x, (c_1, \ldots, c_l))$ to $(x, (c_1, \ldots, c_l, 0, \ldots, 0))$.

Definition 1.95. Let $E, F \to X$ be holomorphic vector bundles, and $f : E \to F$ a holomorphic vector bundle morphism. Then $\ker(f) = \{v \in E \mid f(v) = 0 \in F_{\pi(v)}\}$, $\mathrm{Im}(f) = f(E)$. Let $v, w \in F$, we say that v is equivalent with w if $\pi(v) = \pi(w)$ and $v - w \in \mathrm{Im}(f)$. If two elements are equivalent, then they have to be in the same fibrum, hence we have a map $\pi : F/\sim \to X$, and this bundle is denoted by $\mathrm{Coker}(f)$.

Proposition 1.96. $\ker(f)$ *is a holomorphic subbundle of E, Im(f) is a holomorphic subbundle of F and Coker(f) is a holomorphic vector bundle. Moreover, if we have a holomorphic atlas of F like in Proposition 1.94, then the transition functions $\{g_{\alpha\beta}\}$ associated to this atlas is of the form*

$$g_{\alpha\beta} = \begin{pmatrix} h_{\alpha\beta} & * \\ 0 & k_{\alpha\beta} \end{pmatrix},$$

where $\{h_{\alpha\beta}\}$ and $\{k_{\alpha\beta}\}$ are transition functions of Im(f) and Coker(f) respectively.

Definition 1.97. Let $E \to X$ be a holomorphic vector bundle. Then

$$\mathcal{O}_E(U) = \{s : U \to \pi^{-1}(U) \mid s \text{ is holomorphic, and } \pi \circ s = \mathrm{id}_U\}$$

defines a sheaf over X. It is called the sheaf of holomorphic sections of E.

Notice that if $f \in \mathcal{O}_X(U)$ and $s \in \mathcal{O}_E(U)$, then fs makes sense, and it is an element of $\mathcal{O}_E(U)$. Hence $\mathcal{O}_E$ is a sheaf of $\mathcal{O}_X$-modules.

Let X be a complex manifold of dimension n, then $TX, T^*X, \det(T^*X)$ are complex vector bundles over X but not holomorphic vector bundles in a natural way. However let $\{(U_\alpha, \varphi_\alpha)\}$ be a holomorphic atlas of the complex manifold X. Suppose that $U_\alpha \cap U_\beta \neq \emptyset$, then by definition $\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha)$ is holomorphic, hence the complex derivative of $\varphi_\alpha \circ \varphi_\beta^{-1}$ is an element of $\mathrm{GL}(n, \mathbb{C})$. The maps $(\varphi_\alpha \circ \varphi_\beta^{-1})' \circ \varphi_\beta : U_\beta \to \mathrm{GL}(n, \mathbb{C})$ are clearly holomorphic, moreover they satisfy the relations $a), b)$ and $c)$ defined on page 36. Hence there exists a holomorphic vector bundle $\mathcal{T}X$ with the above defined transition functions. This bundle is called the holomorphic tangent bundle of X. It is clear that TX and $\mathcal{T}X$ are isomorphic as smooth complex vector bundles over X, but unlike TX, $\mathcal{T}X$ is a complex manifold. Define the holomorphic cotangent bundle as the dual of $\mathcal{T}X$, denoted by $\mathcal{T}^*X$ and denote the exterior powers by $\bigwedge^r \mathcal{T}^*X$. If $r = n$, then the line bundle $textdet(\mathcal{T}^*X)$ is called the canonical bundle of X denoted by K_X. We will see later how important this line bundle is, which justifies its fancy name.

Definition 1.98. Let M be a smooth manifold and $E \to M$ be a smooth vector bundle. A smooth bundle homomorphism $J : E \to E$ is called an almost complex structure if $J^2 = -\mathrm{id}$. An almost complex structure on a manifold M is just an almost complex structure on TM.

Notice that not every bundle has an almost complex structure, for example bundles of odd rank can not admit an almost complex structure.

Proposition 1.99. *If X is a complex manifold, then X admits an almost complex structure induced by the complex structure of X.*

Suppose that M is a smooth manifold with almost complex structure J, then one hopes that J defines a complex structure on M which induces J, however this is not true. There are examples of manifolds that admit an almost complex structure but do not have a single complex structure.

Corollary 1.100. *Let X be a complex manifold, and denote with I the almost complex structure induced by the complex structure of X. Let $TX \otimes \mathbb{C}$ denote the tensor product of TX with the trivial $\mathbb{C}$-bundle $X \times \mathbb{C}$, then $TX \otimes \mathbb{C} = T^{1,0}X \oplus T^{0,1}X$, where*

$$T^{1,0}X = \{v \in T_{\mathbb{C}}X \mid Iv = iv\}$$
$$T^{0,1}X = \{v \in T_{\mathbb{C}}X \mid Iv = -iv\}.$$

Sometimes we will denote $TX \otimes \mathbb{C}$ with $T_{\mathbb{C}}X$. $TX \otimes \mathbb{C}$ is called the complexified tangent bundle of X. Suppose that $(U_\alpha, \varphi_\alpha)$ is a holomorphic chart on X. Then this chart induces holomorphic coordinates on U_α denoted by $z_1, \ldots, z_n$, more precisely we have the natural projections $\pi_r : \mathbb{C}^n \to \mathbb{C}$, which sends $(c_1, \ldots, c_n)$ to c_r. Then $z_r = \pi_r \circ \varphi_\alpha$. It is clear that $z_r = x_r + iy_r$, and the functions $x_1, y_1, \ldots, x_n, y_n$ are smooth coordinate functions on U_α. If I is the almost complex structure induced by the complex structure on X, then

$$I\partial_{x_r} = \partial_{y_r} \; I\partial_{y_r} = -\partial_{x_r}$$

Hence $\partial_{x_1}, \partial_{y_1} \ldots, \partial_{x_n}, \partial_{y_n}$ gives a complex basis of $T_{\mathbb{C}}X_p$ over any $p \in U_\alpha$. But as we saw in page 3, there is a better choice of basis for $T_{\mathbb{C}}X$, which is more compatible with the almost complex structure, namely

$$\begin{aligned} \partial_{z_r} &= \frac{1}{2}(\partial_{x_r} - \partial_{y_r}) = \frac{1}{2}(\partial_{x_r} - I\partial_{x_r}) \\ \partial_{\overline{z}_r} &= \frac{1}{2}(\partial_{x_r} + \partial_{y_r}) = \frac{1}{2}(\partial_{x_r} + I\partial_{x_r}). \end{aligned}$$

By Lemma 1.5 we know that $\partial_{z_1}, \ldots, \partial_{z_n}$ is a basis of $T^{1,0}X_p$, and $\partial_{\overline{z}_1}, \ldots, \partial_{\overline{z}_n}$ is a basis of $T^{0,1}X_p$ for all $p \in U_\alpha$. Notice that the elements of $T_{\mathbb{C}}X$ act naturally on complex valued functions, defined by as follows: if $f = u + iv$ where u and v are smooth functions on X, and $w = r + is \in T_{\mathbb{C}}X$, then $w(h) = r(u) - s(v) + i(r(v) + s(u))$. Let X and Y be complex manifolds, and suppose that we have a smooth map $f : X \to Y$, then this induces a morphism of vector bundles $Tf : T_{\mathbb{C}}X \to T_{\mathbb{C}}Y$, which is just the complexification of the map Tf. Suppose that locally we have holomorphic coordinates $z_j = x_j + iy_j$ and $w_j = r_j + is_j$, then locally $f = (f_1, \ldots, f_m)$ and $f_j = u_j + iv_j$, where u_j, v_j are smooth functions. Then the Jacobian of f with respect to the basis $\partial_{x_1}, \ldots, \partial_{x_n}, \partial_{y_1}, \ldots, \partial_{y_n}$ and $\partial_{r_1}, \ldots, \partial_{r_m}, \partial_{s_1}, \ldots, \partial_{s_m}$ is just

$$J_{\mathbb{R}}(f) = \begin{pmatrix} \left(\frac{\partial u_i}{\partial x_j}\right)_{i,j} & \left(\frac{\partial u_i}{\partial y_j}\right)_{i,j} \\ \left(\frac{\partial v_i}{\partial x_j}\right)_{i,j} & \left(\frac{\partial v_i}{\partial y_j}\right)_{i,j} \end{pmatrix}.$$

Since $\partial_{z_1}, \partial_{z_n}, \ldots, \partial_{\overline{z}_1}, \ldots \partial_{\overline{z}_n}$ and $\partial_{w_1}, \ldots, \partial_{w_m}, \partial_{\overline{w}_1}, \ldots, \partial_{\overline{w}_m}$ are also bases of $T_{\mathbb{C}}X$ and $T_{\mathbb{C}}Y$ we can compute the Jacobian of f with respect to these bases, which is

$$J_{\mathbb{C}}(f) = \begin{pmatrix} \left(\frac{\partial f_i}{\partial z_j}\right)_{i,j} & \left(\frac{\partial f_i}{\partial \overline{z}_j}\right)_{i,j} \\ \left(\frac{\partial \overline{f}_i}{\partial z_j}\right)_{i,j} & \left(\frac{\partial \overline{f}_i}{\partial \overline{z}_j}\right)_{i,j} \end{pmatrix}.$$

This implies that the chain rule for ∂_{z_i} and $\partial_{\overline{z}_j}$ is the following. Let $g : U \to V$ and $f : V \to \mathbb{C}$ be smooth maps, where $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^m$ then

$$\frac{\partial}{\partial z_i}(f \circ g) = \sum_j^m \left(\frac{\partial f}{\partial z_j} \circ g\right) \frac{\partial g_j}{\partial z_i} + \sum_j^m \left(\frac{\partial f}{\partial \overline{z}_j} \circ g\right) \frac{\partial \overline{g}_j}{\partial z_i}$$
$$\frac{\partial}{\partial \overline{z}_i}(f \circ g) = \sum_j^m \left(\frac{\partial f}{\partial z_j} \circ g\right) \frac{\partial g_j}{\partial \overline{z}_i} + \sum_j^m \left(\frac{\partial f}{\partial \overline{z}_j} \circ g\right) \frac{\partial \overline{g}_j}{\partial \overline{z}_i}$$

Notice that if f is holomorphic, then

$$J_{\mathbb{C}}(f) = \begin{pmatrix} \left(\frac{\partial f_i}{\partial z_j}\right)_{i,j} & 0 \\ 0 & \overline{\left(\frac{\partial f_i}{\partial z_j}\right)_{i,j}} \end{pmatrix}.$$

This means that if f is holomorphic, then f respects the $(1,0)$ and $(0,1)$ decompositions. Also notice that if $f : U \to \mathbb{C}^m$ is holomorphic, where $U \subset C^n$ is open, then the complex derivative of f' is precisely $(\partial_{z_i} f_j)_{i,j}$. Hence if $\{(U_\alpha, \varphi_\alpha)\}$ is a holomorphic atlas of X, then the transition functions of $T_{\mathbb{C}}X$ are

$$g_{\alpha\beta} = \begin{pmatrix} (\varphi_\alpha \circ \varphi_\beta^{-1})' \circ \varphi_\beta & 0 \\ 0 & \overline{(\varphi_\alpha \circ \varphi_\beta^{-1})' \circ \varphi_\beta} \end{pmatrix}.$$

Hence we see that the transition functions of $T^{1,0}X$ are the same as the transition functions of $\mathcal{T}X$. This means that $T^{1,0}X$ is a holomorphic vector bundle over X and $\mathcal{T}X$ and $T^{1,0}X$ are isomorphic as holomorphic vector bundles.

Let's look at the cotangent bundle of X. Clearly it also has an almost complex structure, which acts on a 1-form by composing it with I. Let's denote this almost complex structure on T^*X also with I. Hence $T^*X \otimes \mathbb{C} = T^*_{\mathbb{C}}X$ also splits to $T^{*^{1,0}}X$ and $T^{*^{0,1}}X$, where

$$T^{*^{1,0}}X = \{\alpha \in T^*_{\mathbb{C}}X \mid I\alpha = \alpha \circ I = i\alpha\} = T^{1,0^*}X$$
$$T^{*^{0,1}}X = \{\alpha \in T^*_{\mathbb{C}}X \mid I\alpha = \alpha \circ I = -i\alpha\} = T^{0,1^*}X.$$

Let $\bigwedge^{p,q} T^*X = \bigwedge^p T^{1,0^*}X \otimes \bigwedge^q T^{0,1^*}X$. Then by Proposition 1.7 we know that

$$\bigwedge^k T^*_{\mathbb{C}}X = \bigoplus_{p+q=k} \bigwedge^{p,q} T^*X.$$

Definition 1.101. Let A^k_X be the sheaf of smooth sections of $\bigwedge^k T^*_{\mathbb{C}}X$, and $A^{p,q}_X$ be the sheaf of smooth sections of $\bigwedge^{p,q} T^*X$.

Clearly we have $A^k_X = \oplus_{p+q=k} A^{p,q}_X$. Notice that $\bigwedge^{p,0} T^*X$ is a holomorphic bundle, hence it makes sense to look at holomorphic sections of $\bigwedge^{p,0} T^*X$. Let $\pi^k : \bigwedge^* T^*_{\mathbb{C}}X \to \bigwedge^k T^*_{\mathbb{C}}X$ and $\pi^{p,q} : \bigwedge^* T^*X \to \bigwedge^{p,q} T^*X$ be the canonical projections coming from the direct sum decompositions. Then π^k and $\pi^{p,q}$ induce morphisms on the sheaves A^k_X, denote them also by π^k and $\pi^{p,q}$.

Definition 1.102. Let $d : A^k_X \to A^{k+1}_X$ be the complexification of the exterior derivative d. Then we define the morphisms:

$$\partial = \pi^{p+1,q} \circ d : A^{p,q}_X \to A^{p+1,q}_X$$
$$\overline{\partial} = \pi^{p,q+1} \circ d : A^{p,q}_X \to A^{p,q+1}_X.$$

We claim, that $d = \partial + \overline{\partial}$. Indeed, if $z_1, \dots, z_n$ are holomorphic coordinates, then

$$dz_j = dx_j + idy_j$$
$$d\overline{z_j} = dx_j - idy_j$$

is the dual basis of $\partial_{z_j}, \partial_{\overline{z}_k}$, hence $dz_1, \dots, dz_n$ and $d\overline{z_1}, \dots, d\overline{z_n}$ are bases of $T^{1,0^*}X$ and $T^{0,1^*}X$ respectively.

Let $f = u + iv$ be a smooth complex valued function on X. Then

$$df = \sum_j \partial_{x_j} f dx_j + \partial_{y_j} f dy_j = \sum_j (\partial_{x_j} u + i\partial_{x_j} v)dx_j + (\partial_{y_j} u + i\partial_{y_j} v)dy_j$$
$$= \sum \underbrace{\partial_{z_j} f dz_j}_{(1,0)-\text{form}} + \underbrace{\partial_{conj z_j} f d\overline{z}_j}_{(0,1)-\text{form}} = \partial f + \overline{\partial} f.$$

Hence on 0-forms $d = \partial + \overline{\partial}$. Now let φ be a (p,q)-form, then $\varphi = \sum_{I,J} \varphi_{I,J} dz_I \wedge d\overline{z}_j$, where $\varphi_{I,J}$ is a smooth complex valued map. Then we compute as follows:

$$d\varphi = \sum_{I,J} d\varphi_{I,J} \wedge dz_I \wedge d\overline{z}_J = \sum_{I,J} (\partial + \overline{\partial})\varphi_{I,J} \wedge dz_I \wedge d\overline{z}_J$$
$$= \sum_{I,J} \underbrace{\partial\varphi_{I,J} \wedge dz_I \wedge d\overline{z}_J}_{(p+1,q)-\text{form}} + \underbrace{\overline{\partial}\varphi_{I,J} \wedge dz_I \wedge d\overline{z}_J}_{(p,q+1)-\text{form}} = \partial\varphi + \overline{\partial}\varphi.$$

As a corollary we get that $\overline{\partial}^2 = \partial^2 = \partial\overline{\partial} + \overline{\partial}\partial = 0$, since

$$0 = d^2 = (\partial + \overline{\partial})(\partial + \overline{\partial}) = \partial^2 + \partial\overline{\partial} + \overline{\partial}\partial + \overline{\partial}^2,$$

and looking at the bidegrees of these maps the corollary follows.

Proposition 1.103. *Let $f : X \to Y$ be a holomorphic map. Then $f^* : A^{p,q}_Y \to A^{p,q}_X$, and $f^*\partial = \partial f^*$, $f^*\overline{\partial} = \overline{\partial} f^*$.*

Proposition 1.104. *Let's denote the sheaf of holomorphic p-forms by Ω^p, then Ω^p is a subsheaf of $A^{p,0}$, in fact $\Omega^p = \ker(\overline{\partial})$.*

This means that we have the following sequence of sheaves

$$\Omega^p \hookrightarrow A^{p,0} \xrightarrow{\overline{\partial}} A^{p,1} \xrightarrow{\overline{\partial}} A^{p,2} \xrightarrow{\overline{\partial}} \dots$$

This is clearly half-exact since $\overline{\partial}^2 = 0$. The next theorem will show that this sequence is actually exact.

Theorem 1.105 ($\overline{\partial}$-Poincaré lemma)**.** *Let U be an open neighborhood of the closure of a bounded polydisc $B_\epsilon \subset \overline{B_\epsilon} \subset U \subseteq \mathbb{C}^n$. If $\alpha \in A^{p,q}_{\mathbb{C}^n}(U)$ is $\overline{\partial}$-closed, then there exists a form $\beta \in A^{p,q-1}_{\mathbb{C}^n}(B_\epsilon)$, with $\overline{\partial}\beta = \alpha$ on B_ϵ.*

Hence the sequence of sheaves above, called the Doulbeault complex, is an exact sequence. Also notice that $A^{p,q}_X$ are soft sheaves for all (p,q), since they are all C^∞-modules which is soft as we saw in the previous section. Define the following groups:

$$H^{p,q}(X) = \frac{\ker(\overline{\partial} : A^{p,q}_X(X) \to A^{p,q+1}_X(X))}{\operatorname{Im}(\overline{\partial} : A^{p,q-1}_X(X) :\to A^{p,q}_X(X))}.$$

Clearly this is just the comohologies of the complex $(A^{p,*}(X), \overline{\partial}\ )$.

Theorem 1.106 (Dolbeault's thoerem)**.** *There exists a canonical isomorphism*

$$H^{p,q}(X) \simeq H^q(X, \Omega^p),$$

for all (p,q).

By Proposition 1.103 a holomorphic map $f : X \to Y$ induces a morphism of sheaves $f^* : A^{p,q}_Y \to A^{p,q}_X$. Since f^* commutes with $\overline{\partial}$, we see that f^* induces a morphism of complexes $f^* : A^{p,*}_Y \to A^{p,*}_X$, i.e. we have the following commutative diagram:

$$\begin{tikzcd}
\Omega^p_Y \arrow[r, hook] \arrow[d, "f^*"'] & A^{p,0}_Y \arrow[r, "\overline{\partial}"] \arrow[d, "f^*"'] & A^{p,1}_Y \arrow[r, "\overline{\partial}"] \arrow[d, "f^*"'] & \dots \\
\Omega^p_X \arrow[r, hook] & A^{p,0}_X \arrow[r, "\overline{\partial}"] & A^{p,1}_X \arrow[r, "\overline{\partial}"] & \dots
\end{tikzcd}$$

By Proposition 1.80 there is a commutative diagram

$$\begin{tikzcd}
H^{p,q}(Y) \arrow[r, "H^q(f^*(X))"] \arrow[d, leftrightarrow] & H^{p,q}(X) \arrow[d, leftrightarrow] \\
H^q(Y, \Omega^p_Y) \arrow[r, "H^q(f^*)"'] & H^q(X, \Omega^p_X)
\end{tikzcd}$$

where the vertical maps are the isomorphisms from the Doulbeault's theorem, $H^q(f^*(X))$ is the morphism induced by $f^* : A_Y^{p,q}(Y) \to A_X^{p,q}(X)$ and $H^q(f^*)$ is map induced by the sheaf morphism $f^* : \Omega_Y^p \to \Omega_X^p$.

Let $E \to X$ be a complex vector bundle, then denote by A_E^k and $A_E^{p,q}$ the sheaf of sections of the bundle $\bigwedge^k T_{\mathbb{C}}^* X \otimes E$ and $\bigwedge^{p,q} T^*X \otimes E$. Elements of A_E^k and $A_E^{p,q}$ are called E-valued k-forms and (p,q)-forms respectively.

Lemma 1.107. *If E is a holomorphic vector bundle, then there exists a natural complex linear operator $\overline{\partial}_E : A_E^{p,q} \to A_E^{p,q+1}$, such that $\overline{\partial}_E^2 = 0$ and satisfies the Leibniz product rule, i.e. $\overline{\partial}_E(f\alpha) = \overline{\partial} f \wedge \alpha + f \wedge \overline{\partial}_E \alpha$.*

If $s_1, \ldots, s_r$ are holomorphic sections of E over U, which are a basis of E_x for all $x \in U$, then an E-valued (p,q)-form over U can be written as $\sum_j \alpha_j \otimes s_j$, where α_j is a (p,q)-form for all j. One then defines $\overline{\partial}_E$ as

$$\overline{\partial}_E \alpha = \sum_j \overline{\partial} \alpha_j \wedge s_j.$$

Since $\overline{\partial}$ annihilate holomorphic functions, this is well defined. One can check that this will satisfy properties in the lemma.

Proposition 1.108. *Let's denote by Ω_E^p the sheaf of holomorphic E-valued p-forms. Then $\Omega_E^p = \ker(\overline{\partial}_E : A_E^{p,0} \to A_E^{p,1})$, and Ω^0 is naturally isomorphic the sheaf of holomorphic senctions of E.*

Since $\overline{\partial}_E$ locally is just r copies of $\overline{\partial}$ we see that the sequence

$$0 \longrightarrow \Omega_E^p \longrightarrow A_E^{p,0} \xrightarrow{\overline{\partial}_E} A_E^{p,1} \xrightarrow{\overline{\partial}_E} \ldots$$

is exact, hence the complex $(A^{p,*}, \overline{\partial}_E)$ is a soft resolving complex of Ω_E^p. Lets denote the q-th cohomology group of the complex $(A^{p,*}(X), \overline{\partial}_E)$ by $H^{p,q}(X, E)$.

Theorem 1.109 (Dolbeault's theorem)**.** *There exists a canonical isomorphism*

$$H^{p,q}(X, E) \to H^q(X, \Omega_E^p),$$

for all (p,q).

Notice that $H^0(X, \Omega_E^0) \simeq \mathcal{O}_E(X)$.

1.5 Harmonic theory on compact manifolds

The purpose of this chapter is to state the Hodge decomposition theorem and introduce the framework for the statement. For detailed proofs we recommend the book [7] Wells, R. O. - *Differential Analysis on Complex Manifolds*.

1.5.1 Sobolev spaces

In this section X denotes an n-dimensional, compact orientable smooth manifold. On X we fix a volume element $dVol$ which is just a nowhere zero smooth differential n-form.

Let E be a Hermitian (smooth) vector bundle over X. Let $\Gamma_k(X, E)$ be the set of kth order differentiable sections of E ($0 \leq k \leq \infty$), where $\Gamma(X, E) = \Gamma_\infty(X, E)$. Define an inner product (,) on $\Gamma(X, E)$ by setting

$$(\xi, \eta) = \int_X \langle \xi(x), \eta(x) \rangle_E \ dVol$$

Where $\langle \ , \ \rangle_E$ is the Hermitian metric on E. Let

$$\|\xi\|_0 = (\xi, \xi)^{\frac{1}{2}}$$

be the L^2-norm and let $W^0(X, E)$ be the completion of $\Gamma(X, E)$ in this norm. Let $\{U_\alpha, \varphi_\alpha\}$ be a finite trivialising cover of X, making the following diagram commutative,

$$\begin{array}{ccc} E|_{U_\alpha} & \xrightarrow{\varphi_\alpha} & \widetilde{U}_\alpha \times \mathbb{C}^p \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle pr_1} \\ U_\alpha & \xrightarrow{\widetilde{\varphi}_\alpha} & \widetilde{U}_\alpha \end{array}$$

Here φ_α is a bundle isomorphism, and $\widetilde{\varphi}_\alpha : U_\alpha \to \widetilde{U}_\alpha \subset R^n$ are a local coordinate system for the manifold X. Let

$$\varphi_{\alpha *} : \Gamma(U_\alpha, E|_{U_\alpha}) \to \Gamma(\widetilde{U}_\alpha, \widetilde{U}_\alpha \times \mathbb{C}^p) = [C^\infty(\widetilde{U}_\alpha)]^p$$

be the induced isomorphism. Let $\{\rho_\alpha\}$ be a finite partition of unity subordinate to $\{U_\alpha\}$ and define, for $\xi \in \Gamma(X, E)$

$$\|\xi\|_s = \sum_\alpha \|\varphi_{\alpha *} \rho_\alpha \xi\|_{\mathbb{R}^n, s},$$

where $\| \ \|_{\mathbb{R}^n,s}$ is the Sobolev norm. Let $f \in C_c^\infty(\mathbb{R}^n)$, then

$$\|f\|_{\mathbb{R}^n,s} = \int_{\mathbb{R}^n} (1+|y|^2)^s |\hat{f}(y)|^2 dy,$$

where

$$\hat{f}(y) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-i\langle y,x\rangle} f(x)dx$$

is the Fourier transform in $\mathbb{R}^n$. If $f \in C_c^\infty(R^n, \mathbb{C}^p)$ then

$$\|f\|^2_{R^n,s} = \|f_1\|^2_{\mathbb{R}^n,s} + ... + \|f_p\|^2_{\mathbb{R}^n,s}.$$

Let $W^s(X,E)$ be the completion of $\Gamma(X,E)$ with respect to the $\| \ \|_s$ norm. Notice that the $\| \ \|_s$ depends on the choices we made, but it can be shown that any two such norm is equivalent so $W^s(X,E)$ is a well defined space. For $s=0$ we have defined two norms, but these will be equivalent as well.

Remark 1.110. Intuitively $\|\xi\|_s < \infty$ for $s \in \mathbb{Z}_{>0}$ means that the first s derivatives of ξ are in $L^2(X)$. This follows from the fact, that on $C_c^\infty(R^n)$ the following norm is equivalent to $\| \ \|_{s,\mathbb{R}^n}$

$$\left[\sum_{|\alpha|\le s} \int_{\mathbb{R}^n} |D^\alpha f|^2 dx\right]^{\frac{1}{2}}$$

with $D^\alpha = (-i)^{|\alpha|}\partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}$. The equivalence of the two norms basically follows from $\widehat{D^\alpha f(y)} = y^\alpha \hat{f}(y)$.

Notice that for $t < s$ and $f \in C_c^\infty(\mathbb{R}^n)$ one has

$$\int_{R^n} (1+|y|^2)^t |\hat{f}(y)|^2 dy \le \int_{R^n} (1+|y|^2)^s |\hat{f}(y)|^2 dy.$$

Thus $W^t(E) \subseteq W^s(E)$.

Theorem 1.111. *Let $t < s$, then one has the following:*

a) (Rellich) $id : \Gamma(E) \to \Gamma(E)$ extends to an $i : W^s(E) \to W^t(E)$ compact, norm non-increasing linear operator.

b) (Sobolev) For $k + 1 + \frac{n}{2} < s$ we have $W^s(E) \subseteq \Gamma_k(E)$.

c) The L^2 pairing on $\Gamma(E)$ extends to a

$$W^s(E) \times W^{-s}(E) \to \mathbb{C}$$

perfect pairing, and $|(\xi,\eta)| \le \|\xi\|_s \|\eta\|_{-s}$ for all $\xi, \eta \in \Gamma(E)$.

The previous theorem says that if we want to prove that a section ξ is smooth we just have to show that $\xi \in W^s(E)$ for all s. Typically we will do the following. First we show that $\xi \in W^t(E)$ for some t, then we will show that if $\xi \in W^s(E)$ then $\xi \in W^{s+1}(E)$.

Definition 1.112. Let E, F be Hermitian vector bundles over X. Let $L : \Gamma(E) \to \Gamma(F)$ be a complex linear map. Then a complex linear map $S : \Gamma(F) \to \Gamma(E)$ is called the formal adjoint of L if:

$$(L\xi, \eta) = (\xi, S\eta),$$

for all $\xi \in \Gamma(E)$ and $\eta \in \Gamma(F)$. We denote S by L^*.

Remark 1.113. Since $\Gamma(E)$ is dense in $W^0(E)$ the formal adjoint is unique if it exists.

1.5.2 Differential operators

Let E, F be smooth complex vector bundles over X. We say that a complex linear map

$$L : \Gamma(E) \to \Gamma(F)$$

is a differential operator if it is locally a differential operator, i.e. if for any choice of local coordinates and local trivialisations, there exists a linear partial differential operator $\tilde{L}$, such that the following diagram commutes

$$\begin{array}{ccc}
\left[C^\infty(\tilde{U})\right]^p & \xrightarrow{\tilde{L}} & \left[C^\infty(\tilde{U})\right]^q \\
\| & & \| \\
\Gamma(\tilde{U}, \tilde{U} \times \mathbb{C}^p) & \longrightarrow & \Gamma(\tilde{U}, \tilde{U} \times \mathbb{C}^q) \\
\uparrow & & \uparrow \\
\Gamma(X, E)|_U & \xrightarrow{L} & \Gamma(X, F)|_U
\end{array}$$

where $U \subset X$ open and $\varphi : U \to \tilde{U} \subset R^n$ is a local coordinate system. That is, if $f = (f_1, \ldots, f_p) \in [C^\infty(U)]^p$, then

$$\tilde{L}(f)_i = \sum_{|\alpha| \leq k, j=1}^{p} a_\alpha^{i,j} D^\alpha f_j,$$

with $i = 1, \ldots, q$. We say that a differencial operator is of order k, if there are no derivatives of order $\geq k+1$ in a local representation. We denote by $\mathrm{Diff}_k(E, F)$ the vector space of all differential operators of order k.

Let's denote by $\mathrm{OP}_k(E, F)$ the vector space of all complex linear maps

$$T : \Gamma(E) \to \Gamma(F),$$

such that there is a continuous extension of T

$$T_s : W^s(E) \to W^{s-k}(F)$$

for all s. These are the operators of order k mapping E to F.

Proposition 1.114. *Suppose that $L \in OP_k(E, F)$, then L^* the formal adjoint of L exists and $L^* \in OP_k(F, E)$, and the extensions*

$$(L^*)_s : W^s(F) \to W^{s-k}(E)$$

is given by the adjoint map

$$(L_{k-s})^* : W^s(F) \to W^{s-k}(E).$$

Remark 1.115. Here we mean the adjoint of $L_{k-s} : W^{k-s}(E) \to W^{-s}(F)$ with respect the L^2 pairings in Theorem 1.111.

Proposition 1.116. *$\mathit{Diff}_k(E, F) \subset OP_k(E, F)$, and if $L \in \mathit{Diff}_k(E, F)$, then the formal adjoint of L is a differential operator, i.e. $L^* \in \mathit{Diff}_k(F, E)$.*

We now want to define the symbol of a differential operator. Let T^*X be the real cotangent bundle of X, and $\pi : T^*X \to X$ the projection map. Let's denote the non-zero covectors by $T'(X)$, i.e. $T'(X) = T^*(X) \setminus \{\text{zero section}\}$, thus $\pi : T'(X) \to X$ is a locally trivial bundle. If E, F are complex vector bundles over X, then $\pi^*(E)$ and $\pi^*(F)$ are complex vector bundles over $T'(X)$. We set, for any $k \in \mathbb{Z}$

$$\begin{aligned} \mathrm{Smbl}_k(E, F) = &\{\sigma \in \mathrm{Hom}(\pi^*(E), \pi^*(F)) \mid \sigma(x, \rho v) = \rho^k \sigma(x, v), \\ &\text{with } (x, v) \in T'(X),\ \rho > 0\}. \end{aligned}$$

We define a linear map

$$\sigma_k : \mathrm{Diff}_k(E, F) \to \mathrm{Smbl}_k(E, F),$$

where $\sigma_k(L)$ is called the k-symbol of the differential operator of L. To define $\sigma_k(L)$, we first note that $\sigma_k(L)(x, v)$ needs to be a linear mapping from E_x to F_x, where $(x, v) \in T'(X)$. Therefore let $(x, v) \in T'(X)$, and $e \in E_x$. Pick $g \in C^\infty(X)$ and $f \in \Gamma(E)$ such that $dg(x) = v$, and $f(x) = e$. Then we define

$$\sigma_k(L)(x, v)e = L\left(\frac{i^k}{k!}(g - g(x))^k f\right)(x) \in F_x.$$

This defines a linear mapping

$$\sigma_k(L)(x,v) : E_x \to F_x,$$

which then defines an element of $\mathrm{Smbl}_k(E,F)$, as is easily checked. Also, it is easy to see that $\sigma_k(L)(x,v)e$ doesn't depend on the choices we made.

Proposition 1.117. *The symbol map σ_k induces an exact sequence*

$$0 \longrightarrow \mathit{Diff}_{k-1}(E,F) \xrightarrow{j} \mathit{Diff}_k(E,F) \xrightarrow{\sigma_k} \mathit{Smbl}_k(E,F)\ ,$$

where j is the natural inclusion.

Proof. Locally L has the following form

$$L = \sum_{|\alpha|\leq k} A_\alpha D^\alpha,$$

where $\{A_\alpha\}$ are $q \times p$ matrices of C^∞ functions on U, with $U \subset X$ open. With these notations one has

$$\sigma_k(L) = \sum_{|\alpha|=k} A_\alpha \xi^\alpha,$$

with $v = \xi_1 dx_1 + \cdots + \xi_n dx_n$. For each fixed (x,v), $\sigma_k(L)(x,v)$ is a linear map from $x \times \mathbb{C}^p \to x \times \mathbb{C}^q$, given by the usual multiplication of a vector in $\mathbb{C}^p$ by the matrix

$$\sum_{|\alpha|=k} A_\alpha \xi^\alpha.$$

We now see, that $\sigma_k(L)$ is a smooth section of $\mathrm{Hom}(\pi^*E, \pi^*F)$, and that $\sigma_k(L) = 0$ iff L has no non-zero kth order term, i.e. $L \in \mathrm{Diff}_{k-1}(E,F)$. To see that the symbol $\sigma_k(L)$ is really of the form above, choose $g \in C^\infty(U)$ and $f \in \Gamma(E)$ such that $dg_x = \sum_{i=1}^n \partial_{x_i} g(x) dx_i = v$ and $f(x) = e$, then one has the following:

$$\begin{aligned}
\sigma_k(L)(x,v)e &= \sum_{|\alpha|\leq k} A_\alpha D^\alpha \left(\frac{i^k}{k!}(g - g(x))^k f\right)(x) \\
&= \sum_{|\alpha|\leq k} A_\alpha (-i)^k \partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n} \left(\frac{i^k}{k!}(g - g(x))^k f\right)(x)
\end{aligned}$$

Clearly, the evaluation at x of derivatives of order $\leq k-1$ will give zero, since there will be a $[g - g(x)]|_x = 0$ term left remaining. This means that the only terms remaining will be the following

$$\sum_{|\alpha|=k} A_\alpha \partial_{x_1} g(x)^{\alpha_1} \dots \partial_{x_n} g(x)^{\alpha_n} f(x) = \sum_{|\alpha|=k} A_\alpha \xi^\alpha e.$$

□

The most important properties of the symbol map σ_k are collected in the following proposition.

Proposition 1.118. *Let E, F, G be Hermitian vector bundles over X. Then we have the following*

a) Let $L_1 \in Diff_k(E,F)$ and $L_2 \in Diff_l(F,G)$, then $L_2 \circ L_1 \in Diff_{k+l}(E,G)$ and $\sigma_{k+l}(L_2 \circ L_1) = \sigma_l(L_2)\sigma_k(L_1)$.

b) Let $L \in Diff_k(E,F)$, then $\sigma_k(L^*) = \sigma_k(L)^*$.

Example 1.119. Let's see some example.

1) Consider the de Rham complex of a compact manifold X

$$A^0_X(X) \xrightarrow{d} A^1_X(X) \xrightarrow{d} A^2_X(X) \xrightarrow{d} \dots \xrightarrow{d} A^n_X(X) \ ,$$

We want to compute the associated 1-symbol mappings

$$\bigwedge^0 T^*_{\mathbb{C},x}X \xrightarrow{\sigma_1(d)(x,v)} \bigwedge^1 T^*_{\mathbb{C},x}X \xrightarrow{\sigma_1(d)(x,v)} \bigwedge^2 T^*_{\mathbb{C},x}(X) \xrightarrow{\sigma_1(d)(x,v)} \dots$$

We claim that for $e \in \bigwedge^k T^*_{\mathbb{C},x}X$, and $(x,v) \in T'(X)$ one has

$$\sigma_1(d)(x,v)e = iv \wedge e.$$

To see that, choose $g \in C^\infty(X)$ such that $dg(x) = v$, and $f \in \Gamma(X, \bigwedge^k T^*_{\mathbb{C}}X)$ such that $f(x) = e$, then one has

$$\sigma_1(d)(x,v)e = d\left(\frac{i}{1!}(g - g(x))f\right)(x) = i \cdot dg(x) \wedge f(x) = iv \wedge e.$$

2) The Dolbeault complex of a compact complex manifold X

$$A^{p,0}_X(X) \xrightarrow{\bar\partial} A^{p,1}_X(X) \xrightarrow{\bar\partial} A^{p,2}_X(X) \xrightarrow{\bar\partial} \dots \xrightarrow{\bar\partial} A^{p,n}_X(X) \ .$$

This has an associated symbol sequence

$$\bigwedge^{p,0} T^*_x X \xrightarrow{\sigma_1(\bar\partial)(x,v)} \bigwedge^{p,1} T^*_x X \xrightarrow{\sigma_1(\bar\partial)(x,v)} \bigwedge^{p,2} T^*_x X \xrightarrow{\sigma_1(\bar\partial)(x,v)} \dots$$

Let $v \in T_x^*X \subset T_{\mathbb{C},x}^*X$ be a nonzero covector, and $e \in \bigwedge^{p,q} T_x^*X$, then one has

$$\sigma_1(\overline{\partial})(x,v)e = iv^{0,1} \wedge e,$$

with $v = v^{1,0} + v^{0,1}$. Indeed, let g be a smooth function on X, such that $dg(x) = v$, and $f \in A^{p,q}(X) = \Gamma(X, \bigwedge^{p,q} T^*X)$ with $f(x) = e$. Since $d = \partial + \overline{\partial}$ we see that $v^{1,0} = \partial g(x)$ and $v^{0,1} = \overline{\partial} g(x)$, hence we have that

$$\begin{aligned}\sigma_1(\overline{\partial})(x,v)e &= i\overline{\partial}\big((g - g(x))f\big)(x) = i\big(\overline{\partial}(g - g(x)) \wedge f\big)(x) + \big((g - g(x))\overline{\partial} f\big)(x) \\ &= i\overline{\partial} g(x) \wedge f(x) = iv^{0,1} \wedge e\end{aligned}$$

3) For the last example, let E be a holomorphic vector bundle over X, then we have the following complex

$$A_E^{p,0}(X) \xrightarrow{\overline{\partial}_E} A_E^{p,1}(X) \xrightarrow{\overline{\partial}_E} A_E^{p,2}(X) \xrightarrow{\overline{\partial}_E} \dots \xrightarrow{\overline{\partial}_E} A_E^{p,n}(X) \ .$$

This has the following associated symbol sequence

$$\textstyle\bigwedge^{p,0} T_x^*X \otimes E_x \xrightarrow{\sigma_1(\overline{\partial}_E)(x,v)} \bigwedge^{p,1} T_x^*X \otimes E_x \xrightarrow{\sigma_1(\overline{\partial}_E)(x,v)} \bigwedge^{p,2} T_x^*X \otimes E_x \xrightarrow{\sigma_1(\overline{\partial}_E)(x,v)} \dots$$

We let $v = v^{1,0} + v^{0,1}$ as before, and $f \otimes e \in \bigwedge^{p,q} T_x^*X \otimes E_x$, then a similar computation as above shows that

$$\sigma_1(\overline{\partial}_E)(x,v) f \otimes e = (iv^{0,1} \wedge f) \otimes e.$$

Notice that every symbol sequence from above was an exact sequence. This can be seen easily by choosing a proper basis.

Definition 1.120. Let $\sigma \in \mathrm{Smbl}_k(E,F)$. We call σ elliptic if and only if for any $(x,v) \in T'(X)$ the linear map

$$\sigma(x,v) : E_x \to F_x$$

is an isomorphism.

Note that in this case E and F must have the same dimension.

Definition 1.121. Let $L \in \mathrm{Diff}_k(E,F)$, then L is said to be elliptic of order k if and only if $\sigma_k(L) \in \mathrm{Smbl}_k(E,F)$ is elliptic.

Notice that being elliptic depends only on the highest term of L. Also note that if L is an elliptic operator of order k, then it is an operator order $k+1$, but not an elliptic operator of order $k+1$, since $\sigma_{k+1}(L) = 0$.

Proposition 1.122. *Let $L \in \mathit{Diff}_k(E,F)$, then L is elliptic if and only if its formal adjoint $L^* \in \mathit{Diff}_k(F,E)$ is elliptic.*

Proof. We have to show that $\sigma_k(L)(x,v)$ is invertible if and only if $\sigma_k(L^*)(x,v)$ is invertible, but by Proposition 1.118 we know that $\sigma_k(L^*)(x,v) = \sigma_k(L)(x,v)^*$, and we know from classic linear algebra that a linear map A is invertible iff A^* is invertible. $\square$

Theorem 1.123. *Let $L \in \mathit{Diff}_k(E,F)$ elliptic, then there exists an $\tilde{L} \in OP_{-k}(E,F)$, such that*

$$L \circ \tilde{L} - id_F \in OP_{-1}(F)$$
$$\tilde{L} \circ L - id_E \in OP_{-1}(E)$$

Definition 1.124. Let $L \in \mathrm{OP}_k(E,F)$. We say that L is compact operator, if for all s the extension $L_s : W^s(E) \to W^{s-k}(F)$ is a compact operator.

Proposition 1.125. *Let $S \in OP_{-1}(E,E)$, then S is a compact operator of order* 0.

Proof. We have for any s the following commutative diagram,

$$\begin{tikzcd} W^s(E) \arrow[rr, "S_s"] \arrow[dr, "S_s"'] & & W^s(E) \\ & W^{s+1}(E) \arrow[ur, "j"'] & \end{tikzcd}$$

where j is natural inclusion, and by Theorem 1.111 a) this is a compact operator. $\square$

Definition 1.126. Let $L \in \mathrm{Diff}_k(E,F)$, then we set

$$\mathcal{K}_L = \{\xi \in \Gamma(E) \mid L(\xi) = 0\},$$

and we let

$$\mathcal{K}_L^{\perp} = \{\eta \in W^0(E) \mid (\xi,\eta) = 0 \text{ for all } \xi \in \mathcal{K}_L\}$$

denote the orthogonal complement of $\mathcal{K}_L$ in $W^0(E)$. It follows immediately that $\mathcal{K}_L^{\perp}$ is a closed subspace of the Hilbert space $W^0(E)$.

In the following, we want to prove that if L is elliptic, then $\mathcal{K}_L$ is finite dimensional. To do that, we need a little bit of functional analysis.

Definition 1.127. Let A, B be Banach spaces, a linear operator $T : A \to B$ is compact, if it sends bounded sets to precompact sets, i.e. if $\{x_n\}$ is a bounded sequence of elements in A, then there exists a subsequence x_{n_k} such that Tx_{n_k} converges to some $y \in B$.

Let's denote the space of compact operators from A to B by $\mathrm{Com}(A, B)$. It is easy to see, that compact operators are automatically bounded. Indeed, suppose that T is not bounded. Then there exists a sequence of unit vectors $\{x_n\}$ such that $\|Tx_n\| > n$. Now $\{x_n\}$ is a bounded sequence and T is compact, hence we have a subsequence x_{n_k} such that Tx_{n_k} converges, which is absurd.

Lemma 1.128. *Let H and K be Hilbert spaces. Then* 1) *$Com(H, K)$ is a closed linear subspace of $B(H, K)$ which is closed under composing with elements of $B(H)$ or $B(K)$, and* 2) *$C \in Com(H, K)$ if and only if $C^* \in Com(K, H)$.*

Proof. 1) It is trivial that $\mathrm{Com}(H, K)$ is a linear subspace. Suppose that $C \in \mathrm{Com}(H, K)$ and $L \in \mathrm{B}(K)$. Let $\{x_n\}$ be a bounded sequence in H, then there exists a subsequence $\{x_{n_k}\}$ such that $Cx_{n_k} \to y$, then since L is continuous, we see that $LCx_{n_k} \to Ly$, i.e. LC is a compact operator. Now let $T \in \mathrm{End}(H)$, then $\{Tx_n\}$ is a bounded sequence in H so it has a subsequence Tx_{n_k} such that $CTx_{n_k} \to y$, so CT is compact.

To prove that $\mathrm{Com}(H, K)$ is closed, choose a sequence $\{C_n\}$ in $\mathrm{Com}(H, K)$, such that $C_n \to C$ in the operator norm. Let $\{x_{n,0}\}$ be a bounded sequence in H. Since C_1 is compact we have a subsequence of $\{x_{n,0}\}$ denoted by $\{x_{n,1}\}$ such that $\{C_1 x_{n,1}\}$ converges. Now $\{x_{n,1}\}$ is a bounded sequence and C_2 is compact, so we have a subsequence $\{x_{n,2}\}$ such that $\{C_2 x_{n,2}\}$ converges. Since $\{x_{n,2}\}$ is a subsequence of $\{x_{n,1}\}$ we see that $\{C_1 x_{n,2}\}$ still converges. Repeating this procces inductively, we see that for each $j \in \mathbb{N}$ the sequence $\{C_k x_{n,j}\}$ converges for all $k \leq j$. Let $\tilde{x}_r = x_{r,r}$, we see that for this sequence $\{C_k \tilde{x}_r\}$ converges for all k. We show that C is compact, by showing that $\{C\tilde{x}_j\}$ is Cauchy. Note that $\tilde{x}_n$ is a bounded sequence, so there exists a $M > 0$, such that $\|\tilde{x}_n\| \leq M$ for all n, and by that we get the following

$$\|C\tilde{x}_n - C_k\tilde{x}_n\| \leq \|C - C_k\| \|\tilde{x}_n\| \leq \|C - C_k\| M.$$

Let ε be arbitrary, and choose $k > 0$ such that $\|C - C_k\| \leq \varepsilon$. Choose $n_0 > 0$ such that $\|C_k\tilde{x}_n - C_k\tilde{x}_m\| \leq \varepsilon$ for all $m, n > n_0$. Now we compute as follows

$$\begin{aligned}\|C\tilde{x}_n - C\tilde{x}_m\| \leq& \|C\tilde{x}_n - C_k\tilde{x}_n\| + \|C_k\tilde{x}_n - C_k\tilde{x}_m\| + \|C_k\tilde{x}_m - C\tilde{x}_m\| \\ \leq& \varepsilon M + \varepsilon + \varepsilon M = \varepsilon(2M + 1)\end{aligned}$$

This proves that $\mathrm{Com}(H, K)$ is closed.

2) Pick $C \in \mathrm{Com}(H, K)$ and suppose that $C^* \notin \mathrm{Com}(K, H)$. Therefore, we can choose a sequence of unit vectors $\{x_n\}$ in K, such that

$$\|C^*x_j - C^*x_i\| \geq \delta > 0,$$

if $i \neq j$. Since C^* is continuous, we see that $y_n = C^* x_n$ is a bounded sequence, and we also see that

$$\begin{aligned} 2\|Cy_n - Cy_m\| \geq & |(Cy_n - Cy_m, x_n - x_m)| \\ & = |(y_n - y_m, y_n - y_m)| = \|C^* x_n - C^* x_m\|^2 \geq \delta^2 > 0, \end{aligned}$$

if $m \neq n$. Thus, there is no convergent subsequence of $\{Cy_m\}$ which is a contradiction. □

Definition 1.129. Let H, K be Hilbert spaces. We say that the $T : H \to K$ continuous operator is Fredholm, if it is invertible modulo compact operators, i.e. if there exists $S_1, S_2 \in \mathrm{Hom}(K, H)$ such that

$$S_1 T - \mathrm{id}_H \in \mathrm{Com}(H) \text{ and } TS_2 - \mathrm{id}_K \in \mathrm{Com}(K).$$

We denote the Fredholm operators from H to K by $\mathrm{Fred}(H, K)$.

Remark 1.130. If such S_1 and S_2 exists, then $S_1 - S_2 \in \mathrm{Com}(K, H)$, thus we can assume, that $S_1 = S_2$. Also it is easy to see that T is Fredholm if and only if T^* is Fredholm.

Lemma 1.131. *Let H, K be Hilbert spaces, and $T \in Hom(H, K)$, then the following are equivalent*

a) *T is Fredholm.*

b) $\dim(\mathrm{Ker}(T)) < \infty$*,* $\dim(\mathrm{Ker}(T^*)) < \infty$*,* $\mathrm{Im}(T)$ *is closed, and* $\mathrm{Im}(T^*)$ *is closed.*

c) $\dim(\mathrm{Ker}(T)) < \infty$*,* $\dim(\mathrm{Ker}(T^*)) < \infty$ *and* $\mathrm{Im}(T)$ *is closed.*

Proof. We only prove that a) implies c). Let $T \in \mathrm{Fred}(H, K)$, and $\{x_n\} \in \mathrm{Ker}(T)$, $\|x_n\| = 1$. Then one has

$$x_n = (\mathrm{id} - S_1 T)x_n = Cx_n,$$

with $C \in \mathrm{Com}(H)$. Thus $\{x_n\}$ has a convergent subsequence. We just showed that the unit sphere in $\mathrm{Ker}(T)$ is compact thus $\mathrm{Ker}(T)$ is finite dimensional. Since $T^* \in \mathrm{Fred}(K, H)$ we also have that $\mathrm{Ker}(T^*)$ is finite dimensional.

To prove that $\mathrm{Im}(T)$ is closed pick a sequence $\{y_n\}$ in $\mathrm{Im}(T)$, such that $y_n \to y$. We want to prove that $y \in \mathrm{Im}(T)$. Since $y_n \in \mathrm{Im}(T)$ we can choose a sequence $\{x_n\} \subset H$, such that $Tx_n = y_n$ for all n. We can assume that $\{x_n\} \subset \mathrm{Ker}(T)^\perp$ and first we assume that there exists some $M > 0$ such that $\|x_n\| \leq M$ for all n i.e. $\{x_n\}$ is bounded. Then one has that

$$x_n = S_1 y_n + (\mathrm{id} - S_1 T)x_n.$$

Now $\mathrm{id} - S_1T$ is compact so we have a convergent subsequence of $\{(\mathrm{id} - S_1T)x_n\}$, moreover $\{S_1y_n\}$ is Cauchy. This means that we have a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to x$, and we have that $Tx = y$.

Now assume that $\{x_n\}$ is not bounded, i.e. $|x_n| \to \infty$. Then let $\tilde{x}_n = \frac{x_n}{|x_n|}$, and with this we get that $T(x_n) = \frac{y_n}{|x_n|} \to 0$. By the previous argument we get a subsequence $\{\tilde{x}_{n_k}\}$ with $\tilde{x}_{n_k} \to \tilde{x}$ and $T(\tilde{x}) = 0$. This means that $\tilde{x} \in \mathrm{Ker}(T)$, but $\tilde{x}_n \subset \mathrm{Ker}(T)^{\perp}$ which is closed, so we get that $\tilde{x} = 0$, but we also have that $\|\tilde{x}\| = 1$, which is clearly a contradiction. □

Remark 1.132. Since H, K are Hilbert spaces, for any $T \in \mathrm{Hom}(H, K)$ one has

$$\mathrm{Im}(T)^{\perp} = \mathrm{Ker}(T^*).$$

If we also know that $\mathrm{Im}(T)$ is a closed, then

$$\mathrm{Im}(T) = \mathrm{Ker}(T^*)^{\perp}.$$

Thus if $\mathrm{Im}(T)$ is closed one has $K = \mathrm{Im}(T) \oplus \mathrm{Im}(T)^{\perp}$, and $K/\mathrm{Im}(T) \simeq \mathrm{Im}(T)^{\perp}$. This means that T is Fredholm if and only if $\dim(\mathrm{Ker}(T)) < \infty$ and $\dim(\mathrm{coker}(T)) \leq \infty$.

Now we can go back to differential operators.

Corollary 1.133. *If $L \in \mathit{Diff}_k(E, F)$ is an elliptic operator of order k, then*

$$L_s : W^s(E) \to W^{s-k}(F)$$

is Fredholm for all s.

Proof. By Theorem 1.123 there exists $\tilde{L} \in \mathrm{OP}_{-k}(F, E)$ such that

$$\begin{aligned} \tilde{L} \circ L - \mathrm{id}_E &\in \mathrm{OP}_{-1}(E) \\ L \circ \tilde{L} - \mathrm{id}_F &\in \mathrm{OP}_{-1}(F). \end{aligned}$$

By Proposition 1.125 we know that $\tilde{L} \circ L - \mathrm{id}_E$ and $L \circ \tilde{L} - \mathrm{id}_F$ are compact operators of order 0, i.e. they are in $\mathrm{Com}(W^s(E))$ and $\mathrm{Com}(W^s(F))$ respectively for all s. □

Now we can prove the elliptic regularity theorem which is the key step towards the Hodge decomposition.

Theorem 1.134 (Elliptic regularity)**.** *Let $L \in \mathit{Diff}_k(E, F)$ be an elliptic operator and $\xi \in W^s(E)$. Suppose that $L_s\xi = \eta \in \Gamma(F)$, then $\xi \in \Gamma(E)$, i.e. ξ is smooth if its image is smooth.*

Proof. Since L is elliptic we have $\tilde{L}$ such that $\tilde{L} \circ L - \mathrm{id} = S \in \mathrm{OP}_{-1}(E)$. We know that $L_s\xi \in \Gamma(F)$ thus $\tilde{L}_{s-k} \circ L_s\xi = (\tilde{L} \circ L)_s\xi \in \Gamma(E)$. Now one has

$$\xi = (\tilde{L} \circ L - S)_s\xi$$

with $(\tilde{L} \circ L)_s\xi \in \Gamma(E)$ and $S_s\xi \in W^{s+1}$. Thus $\xi \in W^{s+1}(E)$, and by induction we see that $\xi \in W^{s+n}(E)$ for all n. By Theorem 1.111. *b)* we see that $\xi \in \Gamma(E)$. $\square$

Theorem 1.135. *Let $L \in Diff_k(E, F)$ be an elliptic operator, and denote by $\mathcal{K}_{L_s}$ the kernel of $L_s : W^s(E) \to W^{s-k}(F)$. Then we have the following*

a) $\mathcal{K}_{L_s} \subset \Gamma(E)$ *and hence* $\mathcal{K}_{L_s} = \mathcal{K}_L$ *for all* s.

b) $\dim(\mathcal{K}_{L_s}) = \dim(\mathcal{K}_L) < \infty$.

Proof. First we show that $\dim(\mathcal{K}_{L_s}) < \infty$, which is trivial since we know, that L_s is Fredholm, thus $\dim(\mathcal{K}_{L_s}) = \dim(\mathrm{Ker}(L_s)) < \infty$. Note that by the elliptic regularity theorem if $\xi \in \mathrm{Ker}(L_s)$ then ξ is smooth thus $\mathrm{Ker}(L_s) = \mathrm{Ker}(L)$. $\square$

Theorem 1.136. *Let $L \in Diff_k(E, F)$ be an elliptic operator. and suppose that $\tau \in \mathcal{K}_{L^*}^{\perp} \cap \Gamma(F)$. Then there exists a unique $\eta \in \Gamma(E)$ such that $L\eta = \tau$ and such that η is orthogonal to $\mathcal{K}_L$ in $W^0(E)$.*

Proof. First we show that $L\eta = \tau$ can be solved in $W^k(E)$, then it will follow by the elliptic regularity theorem that η is smooth and we will have our desired solution. To prove that we can solve $L_k\eta = \tau$ in $W^k(E)$ consider the following diagram

$$\begin{tikzcd} W^k(E) \arrow[r, "L_k"] \arrow[d, leftrightarrow] & W^0(F) \arrow[d, leftrightarrow] \\ W^{-k}(E) & W^0(F) \arrow[l, "(L_k)^*"'] \end{tikzcd}$$

The vertical arrows indicate the duality between the Banach spaces, and by 1.114 we know that $(L_k)^* = (L^*)_0$. We know from functional analysis that the closure of the image is the orthogonal complement of the kernel of the transpose, i.e. $\overline{\mathrm{Im}(L_k)} = \mathrm{Ker}((L_k)^*)$. In our case since L_k is Fredholm we know that $\mathrm{Im}(L_k)$ is closed thus $\mathrm{Im}(L_k) = \overline{\mathrm{Im}(L_k)}$. It follows that there exists $\eta \in W^k(E)$ such that $L_k\eta = \tau$. Since τ is smooth we know that by the elliptic regularity theorem that η is smooth. To get the unique solution we just have to project orthogonally η onto the closed subspace $\mathcal{K}_L^{\perp}$. $\square$

1.5.3 Elliptic complexes

Definition 1.137. Let $E_0, \ldots, E_N$ be complex vector bundles over the compact manifold X. Let's fix $k \in \mathbb{Z}_{>0}$, and let $L_i \in \mathrm{Diff}_k(L_i, L_{i+1})$ for $i = 0, 1, \ldots, N-1$.

We say that $(E, L) = (\{E_i\}, \{L_i\})$ is a complex if the following sequence is half exact

$$\Gamma(E_0) \xrightarrow{L_0} \Gamma(E_1) \xrightarrow{L_1} \Gamma(E_2) \xrightarrow{L_2} \dots \xrightarrow{L_{N-1}} \Gamma(E_N) \ ,$$

i.e. $L_{i+1}L_i = 0$ for $i = 0, 1, \dots, N-1$.

If we have a complex (E, L), then we can define it's cohomology groups as follows, let $E_{-1} = E_{N+1} = 0$, and let $L_{-1} = L_N = 0$, then the qth cohomology group of the complex is

$$H^q(E) = \frac{\mathrm{Ker}(L_q : \Gamma(E_q) \to \Gamma(E_{q+1}))}{\mathrm{Im}(L_{q-1} : \Gamma(E_{q-1}) \to \Gamma(E_q))},$$

with $q = 0, 1, \dots, N$.

Definition 1.138. Let (E, L) be a complex as above, then we say that (E, L) is elliptic if the following sequence of symbols is exact

$$0 \longrightarrow \pi^*(E_0) \xrightarrow{\sigma(L_0)} \pi^*(E_1) \xrightarrow{\sigma(L_1)} \dots \xrightarrow{\sigma(L_{N-1})} \pi^*(E_N) \longrightarrow 0 \ ,$$

i.e. for all $(x, v) \in T'(X)$ we have $\mathrm{Im}(\sigma(L_i)(x, v)) = \mathrm{Ker}(\sigma(L_{i+1})(x, v))$, where $\sigma(L_{-1}) = \sigma(L_N) = 0$.

Let (E, L) be an elliptic complex. Then we can equip each E_j with a Hermitian metric and thus we can define the formal adjoints $L_j^* \in \mathrm{Diff}_k(E_{j+1}, E_j)$. With respect to these Hermitian metric the Laplace of operators of (E, L) are

$$\Delta_j = L_j^* L_j + L_{j-1}L_{j-1}^* \in \mathrm{Diff}_{2k}(E_j, E_j).$$

with symbols

$$\begin{aligned}\sigma_2 k(\Delta_j) =& \sigma_k(L_j^*)\sigma_k(L_j) + \sigma_k(L_{j-1})\sigma_k(L_{j-1}^*)\\ =& \sigma_k(L_j)^*\sigma_k(L_j) + \sigma_k(L_{j-1})\sigma_k(L_{j-1})^*.\end{aligned}$$

Proposition 1.139. *Let (E, L) be a complex, then the following are equivalent*

a) (E, L) is an elliptic complex.

b) The Laplace operator Δ_j is an elliptic operator for $j = 0, 1, \dots, N$.

Proof. We have to prove the following linear algebraic fact. Let V', V, V'' be finite dimensional complex vector spaces equipped with Hermitian metric, and consider the following commutative diagram

$$\begin{tikzcd} V' \arrow[r, "A"] \arrow[d, leftrightarrow] & V \arrow[r, "B"] \arrow[d, leftrightarrow] & V'' \arrow[d, leftrightarrow] \\ V' & V \arrow[l, "A^*"'] & V'' \arrow[l, "B^*"'] \end{tikzcd}$$

Suppose that $B \circ A = 0$, then we have to prove that $\mathrm{Im}(A) = \mathrm{Ker}(B)$ if and only if $C = B^*B + AA^*$ is invertible. It is clear that $\mathrm{Ker}(C) \supseteq \mathrm{Ker}(B) \cap \mathrm{Ker}(A^*)$. Now suppose that $C(x) = 0$, then

$$0 = \langle Cx, x\rangle = \langle Bx, Bx\rangle + \langle A^*x, A^*x\rangle,$$

so we also see that $\mathrm{Ker}(C) \subseteq \mathrm{Ker}(B) \cap \mathrm{Ker}(A^*)$. Now suppose that $\mathrm{Im}(A) = \mathrm{Ker}(B)$ and suppose that $Cv = 0$, we want to show that in this case $v = 0$. By the argument above we have that $A^*v = 0$ and $Bv = 0$. Since $Bv = 0$ and we assumed that $\mathrm{Ker}(B) = \mathrm{Im}(A)$ we have $w \in V'$ with $Aw = v$, and we have the following

$$0 = \langle A^*v, w\rangle = \langle A^*Aw, w\rangle = \langle Aw, Aw\rangle,$$

thus $0 = Aw = v$.

Now suppose that C is invertible and let $v \in V$ with $Bv = 0$. We want to show, that $v = Aw'$ for some $w' \in V'$. Since C is invertible we have $w \in V$ with $Cw = v$, thus

$$\begin{aligned} 0 =& \langle Bv, Bw\rangle = \langle BCw, Bw\rangle \\ =& \langle BB^*Bw + BAA^*w, Bw\rangle = \langle B^*Bw, B^*Bw\rangle, \end{aligned}$$

where we used that $BA = 0$. The computation above shows that $B^*Bw = 0$, thus $v = Cw = B^*Bw + AA^*w = AA^*w$. □

We now have the following fundamental theorem concerning elliptic complexes.

Theorem 1.140 (Hodge decomposition)**.** *Let (E, L) be an elliptic complex. Then we have that*

a) $\dim(\mathrm{Ker}(\Delta_j)) < \infty$*, for* $j = 0, 1, \ldots, N$.

b) The following direct sum decompositions are orthogonal in the L^2 inner product:

$$\begin{aligned} \Gamma(E_j) &= \mathrm{Ker}(\Delta_j) \oplus L_{j-1}\Gamma(E_{j-1}) \oplus L_j^*\Gamma(E_{j+1}) \\ W^0(E_j) &= \mathrm{Ker}(\Delta_j) \oplus L_{j-1}W^k(E_{j-1}) \oplus L_j^*W^k(E_{j+1}) \end{aligned}$$

c) $\mathrm{Ker}(\Delta_j) = \mathrm{Ker}(L_j) \cap \mathrm{Ker}(L_{j-1}^*)$*, and the natural map*

$$\begin{aligned} \mathrm{Ker}(\Delta_j) &\to H^j(E) \\ \alpha &\mapsto [\alpha] \end{aligned}$$

is an isomorphism.

Proof. In this proof the extensions of Δ_j, L_j, L_j^* to $W^s(E)$ will be denoted by Δ_j, L_j, L_j^*, since $(\Delta_j)_s, (L_j)_s$ and $(L_j^*)_s$ are just way too wonky.

Part *a*) immediately follows from Proposition 1.139 and Theorem 1.135. We know from the proof of Theorem 1.136, that we have the following orthogonal decomposition:

$$W^0(E_j) = \mathrm{Im}(\Delta_j) \oplus \mathrm{Ker}(\Delta_j^*) = \mathrm{Im}(\Delta_j) \oplus \mathrm{Ker}(\Delta_j),$$

where we used the fact that the Laplace operator is (formally) self-adjoint, i.e. $\Delta_j^* = \Delta_j$.Thus if $\xi \in W^0(E)$, then

$$\xi = \xi_0 + (L_j^* L_j + L_{j-1} L_{j-1}^*)\eta,$$

with $\xi_0 \in \mathrm{Ker}(\Delta_j)$ and $\eta \in W^{2k}(E_j)$. Let's denote $L_{j-1}^*\eta \in W^k(E_{j-1})$ and $L_j\eta \in W^k(E_{j+1})$ by η_1 and η_2 respectively. We get that:

$$\xi = \xi_0 + L_{j-1}\eta_1 + L_j^*\eta_2.$$

Since $L_j L_{j-1} = 0$ we get that

$$(L_j^*\eta_2, L_{j-1}\eta_1) = (\eta_2, L_j L_{j-1}\eta_1) = 0,$$

thus $L_j^*\eta_2 \perp L_{j-1}\eta_1$. To show, that ξ_0 is orthogonal to $L_j^*\eta_2$ and $L_{j-1}\eta_1$ we show that $\mathrm{Ker}(\Delta_j) = \mathrm{Ker}(L_j) \cap \mathrm{Ker}(L_{j-1}^*)$. Let $\alpha \in \mathrm{Ker}(\Delta_j)$, then we have the following:

$$0 = (\Delta_j\alpha, \alpha) = ((L_j^* L_j + L_{j-1} L_{j-1}^*)\alpha, \alpha) = (L_j\alpha, L_j\alpha) + (L_{j-1}^*\alpha, L_{j-1}^*\alpha),$$

thus $\Delta_j\alpha = 0$ if and only if $\alpha \in \mathrm{Ker}(L_j) \cap \mathrm{Ker}(L_{j-1}^*)$. This shows, that ξ_0 is orthogonal to $L_j^*\eta_2$ and $L_{j-1}\eta_1$, and with that we proved the first half of *b*). For the second part assume that $\xi \in \Gamma(E_j)$, i.e. ξ is smooth. Then $\xi - \xi_0 = \Delta_j(\eta)$ is smooth and by elliptic regularity theorem we get that η is smooth thus η_1 and η_2 are smooth.

Lastly, from $\mathrm{Ker}(\Delta_j) \perp \mathrm{Im}(L_{j-1})$ it follows that $\alpha \mapsto [\alpha]$ is injective. Let $\alpha \in \mathrm{Ker}(L_j)$. To prove that the map above is surjective we have to show $[\alpha]$ can be represented by an element $\beta \in \mathrm{Ker}(\Delta_j)$. We know that $\alpha = \alpha_0 + L_{j-1}\alpha_1 + L_j^*\alpha_2$. If $L_j^*\alpha_2 = 0$, then we are done, since $[\alpha_0] = [\alpha]$, with $\alpha_0 \in \mathrm{Ker}(\Delta_j)$. To see that $L_j^*\alpha_2 = 0$ we compute as follows:

$$0 = (L_j\alpha, \alpha_2) = (L_j L_j^*\alpha_2, \alpha_2) = (L_j^*\alpha_2, L_j^*\alpha_2),$$

thus $L_j^*\alpha_2 = 0$ and we proved the Hodge decomposition theorem. $\square$

Example 1.141. 1) Let X be a compact oriented m-dimensional manifold with volume element $dVol$. We have seen that the complex $(A_X^*(X), d)$ have exact symbol sequence, hence it is an elliptic complex. If we introduce Hermitian metrics on the bundles $\bigwedge^* T_{\mathbb{C}}^* X$, then we can define the Laplace operators $\Delta_j = dd^* + d^*d$, where $j = 0, \ldots, m$. Let $\mathcal{K}_j = \ker(\Delta_j)$, then by the Hodge decomposition theorem we get that

$$\mathcal{K}_j \simeq H^j(A_X^*(X)).$$

If we denote by $\mathcal{A}_{\mathbb{C}}$ the sheaf of locally constant functions on X, then the complex (A_X^*, d) is clearly a soft resolution of $\mathcal{A}_X$. Let $H^*(X, \mathbb{C})$ denote the cohomology groups of $\mathcal{A}_{\mathbb{C}}$, then by lemma 1.78 we get that

$$H^j(X, \mathbb{C}) \simeq H^j(A_X^*(X)).$$

Let $b_i(X) = \dim(H^i(X, \mathbb{C}))$, then as a corollary of the Hodge decomposition theorem, we get thet $b_i(X) < \infty$ for all i, and $b_i(X) = 0$ for $i > m$. The numbers $b_i(X)$ are called the Betti numbers of X.

2) If we also assume that X is a compact complex manifold, then we have the Dolbeault complex $(A_X^{p,*}(X), \overline{\partial})$. We have seen that this complex also has exact symbol sequence, hence it is elliptic. If we introduce Hermitian metrics on the bundles $\bigwedge^{p,q} T^*X$, then we can define the Laplace operators $\overline{\Box}_q = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$. If $\mathcal{K}^{p,q}$ denotes $\ker(\overline{\Box}_q)$, then we get that

$$\mathcal{K}^{p,q} \simeq H^q(A_X^{p,*}(X)) = H^{p,q}(X) \simeq H^q(X, \Omega_X^p).$$

The latter isomorphism is the Dolbeault isomorphism from Theorem 1.106. Let's denote $\dim(H^q(X, \Omega_p))$ by $h^{p,q}(X)$, then $h^{p,q}(X) < \infty$ for all (p, q). The numbers $h^{p,q}(X)$ are called the Hodge numbers of X.

3) Let E be a holomorphic vector bundle over X (compact, complex). We have the complex $(A_E^{p,*}(X), \overline{\partial}_E)$, which is elliptic since it has exact symbol sequence. If we equip E with a Hermitian metric, then the tensor product of metrics clearly give a Hermitian metric on $\bigwedge^{p,q} T^*X \otimes E$ for all q, and we can define the Laplace operators $\overline{\Box}_{E,q} = \overline{\partial}_E\overline{\partial}_E^* + \overline{\partial}_E^*\overline{\partial}_E$. If we denote the kernel of $\overline{\Box}_{E,q}$ by $\mathcal{K}_E^{p,q}$, then we have the following

$$\mathcal{K}^{p,q} \simeq H^q(A_E^{p,*}(X)) = H^{p,q}(X, E) \simeq H^q(X, \Omega_E^p)$$

The latter isomorphism is from Theorem 1.109. Since $\mathcal{O}_E(X) \simeq H^{0,0}(X, E)$, we get as a corollary, that the global holomorphic sections of a holomorphic vector bundle E over a compact complex manifold is always finite dimensional. If we pick $E = T^{1,0}X$, then we see that the vector space of holomorphic vector fields are finite dimensional, which implies that the biholomorphism group of a compact complex manifold X is a finite dimensional Lie group. For more details we recommend the book [4] Kobayashi, S. - *Transformation Groups in Differential Geometry.*

2 Harmonic theory on compact complex manifolds

In the section Pincaré duality we follow the books of [3] Huybrechts, D. - *Complex Geometry: An Introduction.* and [7] Wells, R. O. - *Differential Analysis on Complex Manifolds.* The section Comparison of the Laplace operators follows the lecture notes of Róbert Szőke on Kähler manifolds.

2.1 Poincaré and Serre duality

For the beginning let's just assume that X is a compact oriented Riemannian manifold of dimension m with Riemannian metric g. Let $dVol$ be the induced volume form of the metric g. If $v_1, \ldots, v_m$ is a local frame of TX, and $v^1, \ldots, v^m$ is the dual frame, then locally $dVol$ is of the following form:

$$dVol = \sqrt{|\det(g_{ij})|}\, v^1 \wedge \cdots \wedge v^m,$$

where $g_{ij} = g(v_i, v_j)$.

The orientation and the Riemannian metric induces a $*_p : \bigwedge^* T_p^*X \to \bigwedge^* T_p^*X$ operator for all $p \in X$. This defines a $* : \bigwedge^* T^*X \to \bigwedge^* T^*X$ vector bundles homomorphism. It is smooth, since if we fix an positively oriented local orthonormal frame $e_1, \ldots, e_m$, then

$$*(e^{i_1} \wedge e^{i_2} \wedge \cdots \wedge e^{i_k}) = \operatorname{sgn}(I, J) e^{j_1} \wedge e^{j_2} \wedge \ldots e^{j_{m-k}},$$

with $I \cup J = \{1, \ldots, m\}$. We know by Proposition 1.21 that $* : \bigwedge^k T^*X \to \bigwedge^{m-k} T^*X$ is a bundle isomorphism. Notice that if we denote by 1 the constant 1 function, then $*(1) = dVol$. We can extend $*$ complex linearly to a $\bigwedge^* T_{\mathbb{C}}^*M \to \bigwedge^* T_{\mathbb{C}}^*M$ homomorphism, we also denote this map by $*$. Since $*$ restricted to $\bigwedge^k T^*X$ is an isomorphism it induces a $* : A^k(X) \to A^{m-k}(X)$ isomorphism. Now we can define a Hermitian metric on $A^*(X)$ the following way, let $\varphi, \psi \in A^k(X)$, then

$$(\varphi, \psi) = \int_X \varphi \wedge *\overline{\psi},$$

if $\varphi \in \bigwedge^p T^*X$ and $\psi \in \bigwedge^q T^*X$ with $p \neq q$, then $(\varphi, \psi) = 0$.

Proposition 2.1. *The form defined above is a positive definite Hermitian form on $A^*(X)$.*

Proof. The Riemannian metric g also induces a Hermitian metric on $\bigwedge^* T_{\mathbb{C}}^*X$, lets denote it by $\langle\ ,\ \rangle_{\mathbb{C}}$. By the definition of the Hodge $*$-operator we know that if $\varphi, \psi \in A^k(X)$, then

$$\varphi \wedge *\overline{\psi} = \langle \varphi, \psi \rangle_{\mathbb{C}} dVol,$$

thus $(\ ,\ )$ is Hermitian and positive semidefinite. To prove that it is definite let $\varphi \in A^k(X)$ be nonzero, i.e. there exists a $z_0 \in X$ such that $\varphi(z_0) \neq 0$. This means that $\langle \varphi, \varphi \rangle_{\mathbb{C}}$ is a non-negative function on X and it is positive in z_0, thus

$$(\varphi, \varphi) = \int_X \langle \varphi, \varphi \rangle_{\mathbb{C}} dVol > 0.$$

□

This means that on the elliptic complex $(A^*(X), d)$ we have a Hermitian product depending only on the orientation of X and the Riemannian metric g.

Suppose now also that X is also a complex manifold, with almost complex structure $I : TX \to TX$.

Definition 2.2. We say that a Riemannian metric g on a complex manifolds X with almost complex structure I is compatible with the complex structure, if

$$g(Iu, Iv) = g(u, v),$$

for all $u, v \in \Gamma(TX)$. We call g a compatible Riemannian metric if it is compatible with the complex structure.

Proposition 2.3. *If X is a complex manifold, then there exists at least one compatible Riemannian metric on X.*

Remark 2.4. The proof depends on the following, let V be a finite dimensional vector space with almost complex structure I. Suppose that we have a finite number compatible scalar product $\langle\ ,\ \rangle_i$, where $i = 1, \ldots, N$. Suppose that we have $\alpha_1, \ldots, \alpha_N \in \mathbb{R}_{>0}$ such that $\sum_j^N \alpha_j = 1$, then

$$\sum_j^N \alpha_j \langle\ ,\ \rangle_j = \langle\ ,\ \rangle$$

is a compatible scalar product on V, i.e. the compatible scalar products over (V, I) forms a convex set.

Proof. Let $\{(U_i, \varphi_i)\}_i$ a holomorphic atlas, with $\{U_i\}_i$ being a locally finite cover, and $\{\psi_i\}_i$ is a partition of unity subordonite to the cover. Over an open set $U_j \in \{U_i\}_i$ we have holomorphic coordinates z_i, where $z_i = x_i + iy_j$. With these notations we have, that $I\partial_{x_i} = \partial_{y_i}$ and $I\partial_{y_i} = -\partial_{x_i}$. Define a Riemannian metric over U_j as follows:

$$\begin{aligned} g_j(\partial_{x_i}, \partial_{x_j}) =& \delta_{ij} \\ g_j(\partial_{y_i}, \partial_{y_j}) =& \delta_{ij} \\ g_j(\partial_{x_i}, \partial_{y_j}) =& 0 \end{aligned}$$

It is easy to see that g_j over U_j is a compatible Riemannian metric. Thus, we have a compatible Riemannian metric g_j over each open set $U_j \in \{U_i\}_i$. Now define a compatible Riemannian metric on X as follows:

$$g = \sum_j \psi_j g_j.$$

It is clear that g is a Riemannian metric on X, so we only have to show that g is compatible with the complex structure. Let $u, v \in \Gamma(TX)$, clearly it is enough to chech the compatibility over each point of X, hence we just need to prove the that $g_p(Iu_p, Iv_p) = g_p(u_p, v_p)$ for all $p \in X$. Since $\{\psi_i\}$ is a partition of unity, we have a finite number ψ_i such that $\psi_i(p) \neq 0$, denote them by $\psi_{i_1} \dots, \psi_{i_N}$, and compute as follows:

$$\begin{aligned} g_p(Iu_p, Iv_p) &= \sum_j \psi_j(p) g_{j,p}(Iu_p, Iv_p) = \sum_j \psi_{i_j}(p) g_{i_j,p}(Iu_p, Iv_p) \\ &= \sum_j \psi_{i_j}(p) g_{i_j,p}(u_p, v_p) = \sum_j \psi_j(p) g_{j,p}(u_p, v_p) = g_p(u_p, v_p). \end{aligned}$$

□

Since a complex manifold has a natural orientation induced by the almost complex structure I, to define a Hodge $*$-operator on $\bigwedge^* T^*_{\mathbb{C}}X$ we only need a Riemannian metric on X.

Proposition 2.5. *Let X compact complex manifold with a compatible Riemannian metric g, then $A^k(X) = \oplus_{p+q=k} A^{p,q}(X)$ is an orthogonal decomposition with respect to the Hodge inner product.*

Proof. We have to show that if $\varphi \in A^{p,q}(X)$ and $\psi \in A^{p',q'}(X)$ with $p+q = p'+q' = k$, but $(p, q) \neq (p', q')$, then

$$\int_X \varphi \wedge *\overline{\psi} = \int_X \langle \varphi, \psi \rangle_{\mathbb{C}} dVol = 0,$$

and this follows from lemma 1.22. a). □

From now on if X is a complex manifold, and g is a Riemannian metric on X, then we will always assume that g is compatible with the complex structure.

By 1.22. b) we know that $* : \bigwedge^{p,q} T^*X \to \bigwedge^{n-q,n-p} T^*X$ is an isomorphism, but we want to modify $*$ a little bit, because we do not want this (p, q) swap. Let $\tilde{*} : \bigwedge^{p,q} T^*X \to \bigwedge^{n-q,n-p} T^*X$ be defined as follows:

$$\tilde{*}(\varphi) = *\overline{\varphi},$$

where $\varphi \in \bigwedge^{p,q} T^*X$. It follows that $\tilde{*}$ is an antilinear isomorphism of vector bundles, since it is a composition of two isomorphism.

Suppose now that E is an Hermitian vector bundle over X. The Hermitian product on E defines an antilinear isomorphism

$$\begin{aligned} \tau : E &\to E^* \\ x &\mapsto \langle -, x \rangle. \end{aligned}$$

Define the E valued Hodge $*$-operator as follows:

$$\begin{aligned} \tilde{*}_E : \bigwedge^k T^*_{\mathbb{C}}X \otimes E &\to \bigwedge^{2n-k} T^*_{\mathbb{C}}X \otimes E^* \\ \tilde{*}_E(\varphi \otimes e) &\mapsto \tilde{*}\varphi \otimes \tau(e). \end{aligned}$$

It is easy to see that the E valued Hodge $*$-operator $\tilde{*}_E$ is a conjugate linear isomorphism of vector bundles.

We want to define a non-degenerate pairing like the Hodge inner product, but to do this, we will need a wedge product between $A^*_E(X)$ and $A^*_{E^*}(X)$. Let $\alpha \in \bigwedge^r T^*_xX$, $\gamma \in \bigwedge^s T^*_xX$, $e \in E_x$ and $f \in E^*_x$, then we define the wedge product as follows:

$$(\alpha \otimes e) \wedge (\gamma \otimes f) = \alpha \wedge \gamma \cdot f(e).$$

One can prove that this gives a well defined smooth bundle morphism, :

$$\wedge : \bigwedge^r T^*X \otimes E \times \bigwedge^s T^*X \otimes E^* \to \bigwedge^{r+s} T^*X,$$

and this induces a wedge product on the sections:

$$\wedge : A^r_E(X) \times A^s_E(X) \to A^{r+s}(X).$$

Now we have a Hodge inner product on the E-valued forms defined as follows, let $\varphi, \psi \in A^k_E(X)$, then

$$(\varphi, \psi) = \int_X \varphi \wedge \tilde{*}_E(\psi).$$

Proposition 2.6. *The Hodge inner product on $A^k_E(X)$ is positive definite, and $A^k_E(X) = \sum_{p+q=k} A^{p,q}_E(X)$ is an orthogonal decomposition with respect to this inner product.*

Proof. Every section of $A^k_E(X)$ can be written as $\varphi \otimes e$, where $\varphi \in A^k(X)$ and $e \in \Gamma(E)$, and for sections $\varphi \otimes e, \alpha \otimes f \in A^k_E(X)$, the Hodge inner product is the following

$$(\varphi \otimes e, \alpha \otimes f) = \varphi \otimes e \wedge \tilde{*}_E \alpha \otimes f = \varphi \wedge \tilde{*}_E \alpha \cdot \langle e, f \rangle.$$

By that everything follows. $\square$

Proposition 2.7. *Let X be a compact m-dimensional oriented Riemannian manifold. Let d^* be the formal adjoint of d with respect to the Hodge inner product, and let $\Delta = d^*d + dd^*$ be the Laplace operators of the elliptic complex $(A^*(X), d)$. With these notations we have the following:*

a) $d^* = (-1)^{m+mp+1}\tilde{*}d\tilde{*} = (-1)^{m+mp+1} * d*$.

b) $*\Delta = \Delta*, \tilde{*}\Delta = \Delta\tilde{*}$.

Remark 2.8. Notice, that unlike in Example 1.141 the metrics on $\bigwedge^* T^*_{\mathbb{C}}X$ are related to each other. We will see later, that the better we choose metric on a manifold, the more informiation we get out of the harmonic forms.

Proof. Let $W = \sum(-1)^{mp+p}\pi^p$, by Proposition 1.21 c) we know that, $W = **$, also notice, that

$$\tilde{*}\tilde{*}(\varphi) = \overline{**(\overline{\varphi})} = *\overline{*}(\overline{\overline{\varphi}}) = * * \varphi = W\varphi,$$

where $\varphi \in A^*(X)$, and $\overline{*} = *$ since $*$ is the complexification of a real operator. Now let $\varphi \in A^{k-1}$ and $\psi \in A^k$, and compute as follows:

$$(d\varphi, \psi) = \int_X d\varphi \wedge \tilde{*}\psi = \int_X d(\varphi \wedge \tilde{*}\psi) - (-1)^{k-1} \int_X \varphi \wedge d\tilde{*}\psi.$$

On the second equality we used the Leibniz rule for d. Notice, that by Stokes theorem $\int_X d(\varphi \wedge \tilde{\psi} = 0$, and we have $(-1)^k \int_X \varphi \wedge d\tilde{*}\psi$ left only, which is almost what we want, we just have to put in the conjugate Hodge *-operator,

$$\begin{aligned}(-1)^k \int_X \varphi \wedge d\tilde{*}\psi =&(-1)^k \int_X \varphi \wedge \tilde{*}\tilde{*}^{-1}d\tilde{*}\psi = (-1)^k \int_X \varphi \wedge \tilde{*}(\tilde{*}W d\tilde{*}\psi) \\ =&(-1)^k(-1)^{m(m-k+1)+m-k+1} \int_X \varphi \wedge \tilde{*}(\tilde{*}d\tilde{*}\psi) \\ =&(-1)^{mk+m+1}(\varphi, \tilde{*}d\tilde{*}\psi).\end{aligned}$$

To finish a), we have to show that, $\tilde{*}d\tilde{*} = *d*$. Let $\alpha \in A^*(X)$ arbitry, then:

$$\tilde{*}d\tilde{*}(\alpha) = *\overline{d * (\overline{\alpha})} = *\overline{d}\overline{*}(\overline{\alpha}) = *d * (\alpha).$$

Here we used the fact that $\overline{d} = d$, since d is the complexification of a real operator.

We now want to prove b). Let $\varphi \in A^k(X)$, then

$$\begin{aligned} * \Delta\varphi &= (-1)^{m+mk+1}(*d * d * +(-1)^m * *d * d)\varphi \\ \Delta * \varphi &= (-1)^{m+m(m-k)+1}(d * d * * + (-1)^m * d * d*)\varphi.\end{aligned}$$

The equations above shows, that we only have to check the coefficients of $*d*d*\varphi$ and $d*d\varphi$. The easiest is $*d*d*\varphi$, in the first equation its coefficient is $(-1)^{m+mk+1}$, in the second it is $(-1)^{m+m(m-k)+1+m} = (-1)^{m+mk+1}$. Now we go to $d*d\varphi$, since $\varphi \in A^k(X)$ we have that $d*d\varphi \in A^{m-k}(X)$, thus $Wd*d\varphi = (-1)^{m(m-k)+m-k}d*d\varphi$, also $d*dW\varphi = (-1)^{mk+k}d*d\varphi$. The coefficient of $d*d\varphi$ in the first equation is the following:

$$(-1)^{m+mk+1}(-1)^m(-1)^{m(m-k)+m-k} = (-1)^{k+1},$$

and in the second equation:

$$(-1)^{m+m(m-k)+1}(-1)^{mk+k} = (-1)^{k+1}.$$

Thus $*\Delta = \Delta*$. □

We want to prove a similar statement for $\overline{\partial}_E$, but first note, that we have $\tilde{*}_{E^*}$ on $A^*_{E^*}(X)$, we just have to use τ^{-1}.

Theorem 2.9. *Let X be a compact complex manifold equipped with a compatible Riemannian metric, and let E be a holomorphic vector bundle over X. Then, we have the following:*

a) The operator $\overline{\partial}_E : A^{p,q}_E(X) \to A^{p,q+1}_E(X)$ has a formal adjoint with respect to the Hodge inner product, and it is given by the following:

$$\overline{\partial}^*_E = -\tilde{*}_{E^*}\overline{\partial}_{E^*}\tilde{*}_E.$$

b) If $\overline{\Box}_E = \overline{\partial}_E\overline{\partial}^*_E + \overline{\partial}^*_E\overline{\partial}_E$, and $\overline{\Box}_{E^*} = \overline{\partial}_{E^*}\overline{\partial}^*_{E^*} + \overline{\partial}^*_{E^*}\overline{\partial}_{E^*}$ are the Laplace operators on $A^{*,*}_E(X)$ and $A^{*,*}_{E^*}$ respectively, then we have the following relation:

$$\overline{\Box}_{E^*}\tilde{*}_E = \tilde{*}_E\overline{\Box}_E.$$

Remark 2.10. On E^*, we have a Hermitian metric induced by $\tau : E \to E^*$.

Proof. To prove part a), let $\varphi \in A^{p,q-1}_E(X)$, and $\psi \in A^{p,q}_E(X)$, then $\varphi \wedge \tilde{*}_E\psi \in A^{n,n-1}(X)$, thus

$$\overline{\partial}(\varphi \wedge \tilde{*}_E\psi) = d(\varphi \wedge \tilde{*}_E\psi),$$

also note that we have a product rule for the $\overline{\partial}$, i.e. we have

$$\overline{\partial}(\varphi \wedge \tilde{*}_E\psi) = \overline{\partial}_E\varphi \wedge \tilde{*}_E\psi + (-1)^{p+q-1}\varphi \wedge \overline{\partial}_{E^*}\tilde{*}_E\psi.$$

With these in our hands we compute as follows:

$$
\begin{aligned}
(\overline{\partial}_E\varphi,\psi) =& \int_X d(\varphi\wedge\tilde{*}_E\psi) - (-1)^{p+q-1}\int_X \varphi\wedge\overline{\partial}_{E^*}\tilde{*}_E\psi \\
=&(-1)^{p+q}\int_X \varphi\wedge\tilde{*}_E(W\tilde{*}_{E^*}\overline{\partial}_{E^*}\tilde{*}_E\psi) \\
=&-\int_X \varphi\wedge\tilde{*}_E(\tilde{*}_{E^*}\overline{\partial}_{E^*}\tilde{*}_E\psi) \\
=&(\varphi, -\tilde{*}_{E^*}\overline{\partial}_E\tilde{*}_E\psi).
\end{aligned}
$$

Here we also used, that if $\alpha\in A^*(X)$, and $f\in\Gamma(E)$, then

$$
\tilde{*}_{E^*}\tilde{*}_E(\alpha\otimes f) = \tilde{*}_{E^*}(\tilde{*}\alpha\otimes\tau f) = \tilde{*}\tilde{*}\alpha\otimes f = W(\alpha)\otimes f = W(\alpha\otimes f).
$$

The proof of part *b*) is exactly the same as the proof of part *b*) of proposition 2.7. □

Theorem 2.11 (Poincaré duality)**.** *Let X be a compact oriented m-dimensional manifold. Then there exists*

$$
\sigma : H^r(X;\mathbb{C}) \to H^{m-r}(X;\mathbb{C})
$$

conjugate linear isomorphism.

Proof. Let's introduce a Riemannian metric on X, then we have have Harmonic forms with respect to the induced Laplace operator, let's denote them by $\mathcal{K}^*(X)$. Then by proposition 2.7 and theorem 1.140 we have the following commutative diagram:

$$
\begin{array}{ccc}
A^r(X) & \xrightarrow{\tilde{*}} & A^{m-r}(X) \\
{\scriptstyle H_\Delta}\big\downarrow & & \big\downarrow{\scriptstyle H_\Delta} \\
\mathcal{K}^r(X) & \xrightarrow{\tilde{*}} & \mathcal{K}^{m-r}(X) \\
\big\updownarrow & & \big\updownarrow \\
H^r(X;\mathbb{C}) & \xrightarrow{\sigma} & H^{m-r}(X;\mathbb{C})
\end{array}
$$

□

Remark 2.12. We have a pairing between $H^m(X;\mathbb{C})$ and $H^{m-r}(X;\mathbb{C})$ given by the following, let $[\alpha]\in H^r(X;\mathbb{C})$ and $[\beta]\in H^{m-r}(X;\mathbb{C})$, then

$$
([\alpha],[\beta]) = \int_X \alpha\wedge\beta
$$

This is well defined by Stokes' theorem and we claim that it is non-degenerate. To show that this is non-degenerate pick $0\neq[\alpha]\in H^r(X;\mathbb{C})$. We need to find

$[\beta] \in H^{m-r}(X;\mathbb{C})$ such that $([\alpha],[\beta]) \neq 0$. Represent $[\alpha]$ with a harmonic form α and let $[\beta] = [\tilde{*}\alpha]$, where $\tilde{*}\alpha$ is a harmonic $m-r$-form, then

$$([\alpha],[\tilde{*}\alpha]) = \int_X \alpha \wedge \tilde{*}\alpha > 0.$$

Thus $H^{m-r}(X;\mathbb{C}) \simeq H^m(X;\mathbb{C})^*$. This is also called Poincaré duality.

Corollary 2.13. *If X is a compact oriented manifold, then*

$$b_i(X) = b_{m-i}(X)$$

Where $b_i(X)$ are the Betti numbers of X defined in Example 1.141 1).

Theorem 2.14 (Serre duality)**.** *Let X be a compact complex manifold, with complex dimension n, and let E be a holomorphic vector bundle over X. Then there exists a conjugate linear isomorphism*

$$\sigma : H^r(X;\Omega^p_E) \to H^{n-r}(X;\Omega^{n-r}_{E^*})$$

Proof. Let's introduce a compatible Riemannian metric on X, and a Hermitian metric on E, with these fixed we have E- and E^*-valued Harmonic (p,q)-forms, let's denote them by $\mathcal{K}^{*,*}_E(X)$ and $\mathcal{K}^{*,*}_{E^*}(X)$. By theorem 2.9 and theorem 1.140 and the Dolbeault isomoprhism we have the following commutative diagram:

$$\begin{tikzcd}
A^{p,q}_E(X) \arrow[r, "\tilde{*}"] \arrow[d, "H_\Delta"'] & A^{n-p,n-q}_{E^*}(X) \arrow[d, "H_\Delta"] \\
\mathcal{K}^{p,q}_E(X) \arrow[r, "\tilde{*}"] \arrow[d, leftrightarrow] & \mathcal{K}^{n-p,n-q}_{E^*}(X) \arrow[d, leftrightarrow] \\
H^{p,q}(X;E) \arrow[r, "\tilde{\sigma}"] \arrow[d, leftrightarrow] & H^{n-p,n-q}(X;E^*) \arrow[d, leftrightarrow] \\
H^q(X,\Omega^p_E) \arrow[r, "\sigma"] & H^{n-q}(X;\Omega^{n-q}_{E^*})
\end{tikzcd}$$

□

Remark 2.15. Just like with Poincaré duality, we have a natural pairing between $H^{p,q}(X;E)$ and $H^{n-p,n-q}(X;E^*)$, given by the following; let $[\alpha] \in H^{p,q}(X;E)$ and $[\beta] \in H^{n-p,n-q}(X;E^*)$, then

$$([\alpha],[\beta]) = \int_X \alpha \wedge \beta.$$

To show that this is well defined we need the following: If $\alpha \in A_E^{p,q-1}(X)$ and $\beta \in A_{E^*}^{n-p,n-q}(X)$, then $d(\alpha\wedge\beta) = \overline{\partial}(\alpha\wedge\beta)$., and $\overline{\partial}(\alpha\wedge\beta) = \overline{\partial}_E\alpha\wedge\beta+(-1)^{p+q-1}\alpha\wedge\overline{\partial}_{E^*}\beta$. Now also suppose that β is $\overline{\partial}_{E^*}$-closed, then

$$\int_X \overline{\partial}_E\alpha\wedge\beta = \int_X d(\alpha\wedge\beta) - (-1)^{p+q-1}\int_X \alpha\wedge\overline{\partial}_{E^*}\beta = 0$$

The first integral is 0 because of Stokes' theorem the second is zero, because we assumed that β is $\overline{\partial}_{E^*}$-closed. It is also non-degenerate, since if $0 \neq \alpha \in \mathcal{K}_E^{p,q}(X)$, then $\tilde{*}_E\alpha \in \mathcal{K}_{E^*}^{n-p,n-q}(X)$, and

$$([\alpha],[\tilde{*}_E\alpha]) = \int_X \alpha\wedge\tilde{*}_E\alpha > 0.$$

Thus, $H^{p,q}(X;E) \simeq H^{n-p,n-q}(X;E^*)^*$.

Corollary 2.16. *If E is the trivial line bundle, then $H^{p,q}(X,E) = H^{p,q}(X)$ and we get that there exists a conjugate linear isomorphism*

$$H^{p,q}(X) \simeq H^{n-p,n-q}(X).$$

Hence $h^{p,q}(X) = h^{n-p,n-q}(X)$, where $h^{p,q}$ are the Hodge numbers defined in Examples 1.141 2).

2.2 Comparison of the Laplace operators

Suppose that X is a compact complex manifold with a compatible Riemannian metric. We can define three Laplace operators on X with respect to the metric

1. $\Delta = d^*d + dd^* : A^k(X) \to A^k(X)$.
2. $\overline{\Box} = \overline{\partial}^*\overline{\partial} + \overline{\partial}\overline{\partial}^* : A^{p,q}(X) \to A^{p,q}(X)$.
3. $\Box = \partial^*\partial + \partial\partial^* : A^{p,q}(X) \to A^{p,q}(X)$.

By proposition 2.7 and theorem 2.9 we know that $d^* = -*d*$, $\overline{\partial}^* = -*\partial*$ and it is easy to see that $\partial^* = -*\overline{\partial}*$

When $p+q = k$, it is a natural question, whether these Laplace operators are related, i.e. is it true, that Δ maps (p,q) forms to (p,q) forms. We can also ask whether a form β being $\overline{\Box}$-closed implies that it is Δ-closed. For a general compact complex manifold X with compatible Riemannian metric, none of these hold. First we will give a counter example for the second case.

If β is a $(1,0)$-form, then $\overline{\partial}^*\beta = 0$ by definition. If we also assume that β is holomorphic, i.e. $\overline{\partial}\beta = 0$, then $\overline{\Box}\beta = 0$. The idea is to find a holomorphic $(1,0)$-form β which is not d-closed, since such a β can not be harmonic.

The next proposition shows, that if we want to find a holomorphic $(1,0)$-form which is not d-closed, the dimension of our space must be at least three.

Proposition 2.17. *Suppose that X is a compact 2-dimensional complex manifold. If $\alpha \in \Omega^1(X)$, then $d\alpha = 0$.*

Proof. Let $\alpha \in \Omega^1(X)$, then

$$d\alpha = (\partial + \overline{\partial})\alpha = \partial\alpha.$$

It is easy to see, that if $\partial\alpha \neq 0$, then:

$$\int_X \partial\alpha \wedge \overline{\partial\alpha} \neq 0,$$

also note that, $\overline{\partial\alpha} = \overline{\partial}\overline{\alpha}$, thus

$$\partial(\overline{\partial\alpha}) = \partial(\overline{\partial}\overline{\alpha}) = -\overline{\partial}(\partial\overline{\alpha}) = -\overline{\partial}(\overline{\overline{\partial}\alpha}) = 0,$$

and by that, we get the following:

$$d(\alpha \wedge \overline{\partial\alpha}) = d\alpha \wedge \overline{\partial\alpha} - \alpha \wedge d\overline{\partial\alpha} = \partial\alpha \wedge \overline{\partial\alpha}.$$

Suppose, that $d\alpha \neq 0$, then

$$0 \neq \int_X \partial\alpha \wedge \overline{\partial\alpha} = \int_X d(\alpha \wedge \overline{\partial\alpha}) = 0,$$

by Stokes' theorem, which is clearly a contradiction. □

Let H be the complex Heisenberg group:

$$H = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \middle| \, x, y, z \in \mathbb{C} \right\}$$

It is clear that $H \simeq \mathbb{C}^3$ as a complex manifold.

Proposition 2.18. *$\beta = dz - ydx$ is a right-invariant holomorphic $(1,0)$-form, and $d\beta \neq 0$.*

Proof. It is clear that β is holomorphic $(1,0)$-form, and $d\beta = -dy \wedge dx \neq 0$. Let

$$\gamma = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \in H,$$

then

$$R_\gamma \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+x & c+xb+z \\ 0 & 1 & b+y \\ 0 & 0 & 1 \end{pmatrix},$$

thus

$$R_\gamma^* \beta = d(c+xb+z) - (b+y)d(a+x) = bdx + dz - bdx - ydx = dz - ydx.$$

$\square$

The complex Heisenberg group is not compact, but let

$$\Gamma = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \middle| \, a, b, c \in \mathbb{Z}[i] \right\} < H.$$

Γ is a discrete subgroup of H which acts on H by right multiplication. Since Γ is a closed subgroup of H this action is proper, i.e. this is a properly discontinuous action. Thus H/Γ, called the Iwasawa manifold, is a complex manifold. Clearly the projection $\mathbb{C}^3 \to \mathbb{C}^2$ mapping (x, y, z) to (x, y) descends to a holomorphic submersion $H/\Gamma \to T \times T$, where T is a torus. It can be shown that this is a locally trivial bundle with torus fibers, hence the Iwasawa manifold is compact. Since β is invariant under the action of Γ it descends to a β' holomorphich $(1, 0)$-form which is not d-closed.

Now we want to show a complex manifold X and an $\alpha = \sum \alpha^{p,q}$ form, such that $\Delta\alpha = 0$, but $\Delta\alpha^{p,q} \neq 0$ for some (p, q).

Definition 2.19. Let X be a complex manifold. We call a smooth map $\rho : X \to \mathbb{C}$ pluriharmonic, if $\partial\overline{\partial}\rho = 0$.

Proposition 2.20. *Let X be a compact complex connected manifold. If $f : X \to \mathbb{C}$ is a pluriharmonic function, then f is constant.*

Before we prove this we need the following:

Lemma 2.21. *If $f : X \to \mathbb{R}$ is a pluriharmonic function, then locally f is the real part of a holomorphic function.*

Proof. Let $\overline{\partial} f = \alpha$, then $\partial\alpha = \overline{\partial}\alpha = 0$, thus $d\alpha = 0$. This means that locally there exists a function β with $d\beta = \alpha$. Since α is a $(0, 1)$-form we know that $\partial\beta = 0$. Let $h = f - \beta$, then

$$\begin{aligned} \overline{\partial} h &= \overline{\partial} f - \overline{\partial}\beta = \alpha - \alpha = 0 \\ \partial h &= \partial f - \partial\beta = \partial f, \end{aligned}$$

Write h as $u + iv$, where u, v are smooth real valued functions. By the equations above we get that

$$\partial_{x_j} u - i\partial_{y_j} u = \partial_{z_j} h = \partial_{z_j} f = \frac{1}{2}(\partial_{x_j} f - i\partial_{y_j} f),$$

thus locally $u/2 - f \equiv c$. Let $F = h/2 - c$, then F is holomorphic, since $\overline{\partial} F = \overline{\partial}(h/2 - c) = 0$, and $\mathrm{Re}(F) = f$. □

Now we can prove the previous proposition.

Proof. Let $f : X \to \mathbb{C}$ be a pluriharmonic function. Since $\partial\overline{\partial} f = 0$, we have that

$$0 = \overline{\partial\overline{\partial} f} = \overline{\partial}\partial\overline{f} = -\partial\overline{\partial}\overline{f},$$

thus if f is pluriharmonic, then $\overline{f}$ is also pluriharmonic, so we can assume that f is a real valued function. Since X is compact f reaches its maximum at $p \in X$. Let $U = f^{-1}(f(p))$, it is clear that U is non-empty and closed. Let $q \in U$, by the previous lemma we know that in a small neighbourhood of q $f = \mathrm{Re}(F)$, where F is a holomorphic function. Now e^F is a holomorphic function around q, and $|e^F|$ reaches its maximum at q, thus $e^F = c$ and we get that $f = \mathrm{Re}(F)$ is constant around q. This means that if $q \in U$, then a small neighbourhood of q is in U, hence U is also open, thus $U = X$. □

Let $\lambda \in \mathbb{C}^*$, $|\lambda| \neq 1$. Then $\mathbb{Z} \times \mathbb{C}^2 \setminus \{0\} \to \mathbb{C}^2 \setminus \{0\}$, $(k, x) \mapsto \lambda^k x$ defines a $\mathbb{Z}$ action on $\mathbb{C}^2 \setminus \{0\}$. Clearly this a properly discontinuous action, and $M = \mathbb{C}^2 \setminus \{0\}/\mathbb{Z}$ is a complex manifold called the Hopf surface. It is clear that topologically $M \simeq S^3 \times S^1$, thus by the Künneth formula $H^1(X; \mathbb{C}) = \mathbb{C}$.

Proposition 2.22. $H^0(M, \Omega^1_M) = 0$*, i.e. there are no non-trivial global holomorphic* $(1,0)$*-forms on* M*.*

Proof. Suppose that there is an $0 \neq \alpha$ global holomorphic $(1,0)$-form. Since $\dim(M) = 2$ we know that $d\alpha = 0$. We claim that $\overline{\alpha}$ is not $\overline{\partial}$ exact. Suppose on the contrary, that $\overline{\alpha} = \overline{\partial}\beta$. Then

$$\partial\overline{\partial}\beta = d\overline{\partial}\beta = d\overline{\alpha} = 0,$$

hence β is a pluriharmonic function, but M is compact, so by the previous proposition β is constant and $\alpha = \overline{\partial}\beta = 0$ which is a contradiction. We also have the following:

$$\overline{\partial}\overline{\alpha} = \overline{\partial\alpha} = \overline{d\alpha} = 0.$$

This means, that $\overline{\alpha}$ defines a non-trivial element in $H^{0,1}(M)$, and by Dolbeault's theorem we know that $H^{0,1}(M) = H^1(M, \mathcal{O}_M)$. Let's look at the following sequence of sheaves:

$$0 \longrightarrow \mathbb{C} \xrightarrow{\iota} \mathcal{O}_M \xrightarrow{d} \Omega_M \longrightarrow 0$$

The first map is the natural embedding of the locally constants sheaf to the sheaf of holomorphic functions, the second is the exterior derivative. It is clear that this sequence is exact, the only nontrivial part is that it is surjective, but if $\alpha \in \Omega^1(M)$, then locally $\alpha = d\beta$, but if we check the bi-grading, we get that $\overline{\partial}\beta = 0$, i.e. $\alpha = d\beta$, where β is a holomorphic function. By the short exact sequence of sheaves we get the following exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & H^0(M;\mathbb{C}) \arrow[r] & H^0(M;\mathcal{O}_M) \arrow[r] & H^0(M;\Omega^1_M) \arrow[dll, out=-10, in=170] \\ & H^1(M;\mathbb{C}) \arrow[r] & H^1(M;\mathcal{O}_M) \arrow[r] & H^1(M;\Omega^1_M) \arrow[r] & \dots \end{tikzcd}$$

We know that ι induces isomorphism between $H^0(M;\mathbb{C})$ and $H^0(M;\mathcal{O}_M)$, since both are one dimensional vector spaces and the induced map is injective. Thus the map $H^0(M;\mathcal{O}_M) \to H^0(M;\Omega^1_M)$ is the zero-map, hence we have to following exact sequence:

$$0 \longrightarrow H^0(M;\Omega^1_M) \longrightarrow H^1(M;\mathbb{C}) \longrightarrow H^1(M;\mathcal{O}_M) \longrightarrow \dots$$

Now by the assumption that we have a global non-trivial holomorphic $(1,0)$-form, i.e. $H^0(M;\Omega^1_M) \neq 0$, we got that $H^1(M;\mathcal{O}_M) \neq 0$, but this means that $\dim(H^1(M;\mathbb{C})) > 1$, which is a contradiction. $\square$

Corollary 2.23. *We have the following corollaries:*

a) $H^{1,0}(M) = 0$.

b) $H^{0,1}(M) \neq 0$.

Proof. By the Doulbeault theorem $H^{1,0}(M) = H^0(M, \Omega^1_M) = 0$. Since $H^0(M;\Omega^1_M) = 0$, we get that the map $\mathbb{C} = H^1(M;\mathbb{C}) \to H^1(M;\mathcal{O}_M) = H^{0,1}(M)$ is injective. $\square$

Corollary 2.24. *If g is a Riemannian metric on the Hopf surface M, and $0 \neq \alpha = \alpha^{1,0} + \alpha^{0,1}$ is a harmonic 1-form, i.e. $\Delta\alpha = 0$, then $\Delta\alpha^{1,0} \neq 0$ and $\Delta\alpha^{0,1} \neq 0$.*

Proof. We will only show that $\alpha^{0,1}$ can not be harmonic. Since $\Delta\alpha = 0$ we get that $d\alpha = 0$, but

$$0 = d\alpha = (\partial + \overline{\partial})(\alpha^{1,0} + \alpha^{0,1}) = \partial\alpha^{1,0} + \overline{\partial}\alpha^{1,0} + \partial\alpha^{0,1} + \overline{\partial}\alpha^{0,1},$$

hence $\partial\alpha^{1,0} = \overline{\partial}\alpha^{0,1} = 0$. Now suppose that $\Delta\alpha^{0,1} = 0$, then $d\alpha^{0,1} = 0$, thus $\partial\alpha^{0,1} = 0$, and this means that

$$\overline{\partial}\overline{\alpha}^{0,1} = \overline{\partial\alpha^{0,1}} = 0,$$

i.e. $\overline{\alpha}^{0,1}$ is a holomorphic $(1,0)$-form, hence $\alpha^{0,1} = 0$. This means that $\alpha = \alpha^{1,0}$, and $0 = d\alpha^{1,0} = \partial\alpha^{1,0} + \overline{\partial}\alpha^{1,0}$. Looking at the bigrades we get that $\overline{\partial}\alpha^{1,0} = 0$, but this means $\alpha^{1,0}$ is holomorphic, hence it is zero. Thus $\alpha = 0$ which is a contradiction. $\square$

The key point of this section was to show that in general the complex and real Laplace operators are not related. The reason behind this is that the compatibility of the complex structure and the metric, we used so far, is too weak. One needs a stronger compatibility condition between the Riemannian structure and complex structure.

3 Kähler manifolds

This section is devoted to introduce a special class of complex manifolds, the Kähler manifolds, and prove the Hodge decomposition theorem on their cohomologies. Kähler manifolds are in some sense a generalisation of projective manifolds and share really interesting properties with them.

The section Definitions and examples follows the books of [3] Huybrechts, D. - *Complex Geometry: An Introduction.*, [6] Voisin, C. - *Hodge Theory and Complex Algebraic Geometry I: Volume 1* and the section Harmonic theory on compact Kähler manifolds follows the book [7] Wells, R. O. - *Differential Analysis on Complex Manifolds.*.

3.1 Definitions and examples

Definition 3.1. Let X be a complex manifold with almost complex structure I and compatible Riemannian metric g. Let $v, w \in \Gamma(TX)$, then

$$\omega(v, w) = g(Iv, w)$$

is the fundamental form associated to g.

By lemma 1.14 we know that ω is a real $(1,1)$-form.

Definition 3.2. Let X be a complex manifold with compatible Riemannian metric g and fundamental form ω, then we have the Lefschetz operator

$$L : \bigwedge^{*} T^*X \to \bigwedge^{*+2} T^*X$$
$$\alpha \mapsto \omega \wedge \alpha$$

We can extend L complex linearly to $\bigwedge^* T^*_{\mathbb{C}}X$ and this induces a linear map between the sections of $\bigwedge^* T^*_{\mathbb{C}}X$ also denoted by L, i.e. $L : A^*(X) \to A^{*+2}(X)$.

Define the following operators

a) $\Lambda = *^{-1}L* : \bigwedge^* T^*X \to \bigwedge^{*-2} T^*$.

b) $H = \sum_{k=0}^{2n}(n-k)\pi^k : \bigwedge^* T^*X \to T^*X$, where $n = \dim(X)$.

Clearly these are vector bundle morphisms, hence at every point $x \in X$, the maps L_x, Λ_x and H_x give a representation of $\mathfrak{sl}(2,\mathbb{C})$ on $\bigwedge^* T^*_{\mathbb{C},x}X$. Thus we can use the results of section 1.2 while studying complex manifolds manifolds.

Corollary 3.3. *Let $P^k(X) = \ker\{L^{n-k+1} : \bigwedge^k T^*_{\mathbb{C}}X \to \bigwedge^{2n-k+2} T^*_{\mathbb{C}}X\}$ for $0 \leq k \leq n$ and $P^k(X) = 0$ for $k \geq n+1$. Then by Corollary 1.40 we get that* *a)* $\bigwedge^k T^*_{\mathbb{C}}X = \bigoplus_{i\geq 0} L^i(P^{k-2i}(X))$.

b) $L^{n-k} : \bigwedge^k T^*_{\mathbb{C}}X \to \bigwedge^{2n-k} T^*_{\mathbb{C}}X$ *is a bundle isomorphism.*

Let $P^{p,q}(X) = P^{p+q}(X) \cap \bigwedge^{p,q} T^*X$. *Then*

a) $P^k(X) = \bigoplus_{p+q=k} P^{p,q}(X)$

b) If $p+q=k$*, then* $L^{n-k} : P^{p,q}(X) \to P^{n-q,n-p}(X)$ *is a bundle ismomorphism.*

Clearly these impy that:

a) $L^{n-k} : A^k(X) \to A^{2n-k}(X)$ *is a linear isomorphism.*

b) $A^k(X) = \bigoplus_{i\geq 0} L^i(\Gamma(P^{k-2i}(X)))$ *is an orthogonal decomposition with respect to the Hodge inner product.*

To see *b)* let $\alpha_i \in \Gamma(P^{k-2i}(X))$ and $\beta_j \in \Gamma(P^{k-2j})$, where $i \neq j$. Then

$$(L^i\alpha_i, L^j\beta_j) = \int_X \langle L^i\alpha_i, L^j\beta_j\rangle dVol = 0,$$

since by Corollary 1.40 we are integrating 0 on X.

Proposition 3.4. *If* X *is compact, then the Lefschetz operator* L *has a formal adjoint* L^* *with respect to the Hodge inner product, and it is equal to* Λ.

Proof. Let $\alpha \in A^k(X)$ and $\beta \in A^{k+2}(X)$, then

$$(L\alpha, \beta) = \int_X g(L\alpha, \beta)dVol = \int_X g(\alpha, \Lambda\beta)dVol = (\alpha, \Lambda\beta)$$

Here we used the fact, that at every point $x \in X$, the adjoint of L_x with respect to g_x is Λ_x. □

Definition 3.5. Let X be a complex manifold and g a compatible Riemannian metric. We call (X, g) a Kähler manifold, if the fundamental form associated to g is d-closed, in which case g is called a Kähler metric on X. The complex manifold X is of Kähler type if there exists a Kähler metric on X.

Remark 3.6. The fundamental form associated to a Kähler metric g is also called the Kähler form of g.

At first this definition might seem strange, but as we will see, being Kähler type has many non-trivial consequence.

Proposition 3.7. *Let* X *be a compact Kähler manifold of dimension* n*. Then the forms* $\omega, \omega^2, \ldots, \omega^n$ *are* d*-closed, but not exact.*

Proof. It is trivial that ω^i is closed, since it is the product of d-closed forms. To see that they are not exact, first notice, that by Corollary 1.18 $\omega^n = n!dVol$. Now suppose that $\omega^n = d\alpha$, then by Stokes' theorem

$$0 = \int_X d\alpha = n! \int_X 1 \; dVol = n!Vol(X) > 0$$

which is clearly a contradiction. Now suppose that $\omega^i = d\alpha$ for some $i < n$, then

$$d(\omega^{n-i} \wedge \alpha) = d\omega^{n-i} \wedge \alpha + (-1)^{n-i}\omega^{n-i} \wedge d\alpha = (-1)^{n-i}\omega^n,$$

which is a contradiction since we just showed, that ω^n is not exact. □

Corollary 3.8. *If X is a compact Kähler manifold, then $H^{2i}(X;\mathbb{R}) \neq 0$, for $i = 0, \ldots, n$.*

As a corollary, we see that the Hopf surface is not Kähler, since $H^2(S^1 \times S^3) = 0$ by the Künneth formula. More generally we can define a $\mathbb{Z}$ action on $\mathbb{C}^n \setminus \{0\}$ by mapping k to multiplication with λ^k, where $\lambda \neq 0$, and $|\lambda| \neq 1$. This action is properly discontinuous, hence the manifold $M = \mathbb{C}^n/\mathbb{Z}$ admits a complex structure. The manifold M is called Hopf manifold. It's not hard to show, that $M \simeq S^1 \times S^{2n-1}$, hence the Hopf manifolds can not be Kähler manifolds for $n > 1$, since for $n > 1$ we have $H^2(S^1 \times S^{2n-1}, \mathbb{R}) = 0$.

Let (X, g) be a Kähler manifold with Kähler form ω_X. Suppose that $Y \subset X$ is complex submanifold. and denote the natural inclusion of Y by ι. Then $\iota^* g$ is clearly a compatible Riemannian metric on Y. If we denote the fundamental form of $\iota^* g$ by ω_Y, then it is trivial that $\omega_Y = \iota^*\omega_X$, hence $\iota^* g$ is a Kähler metric on Y. As a corollary, we get the following.

Corollary 3.9. *If Y is a compact complex submanifold of the Kähler manifold X. Then Y is not a boundary in X*

Proof. Suppose that $Y = \psi(\partial M)$ for a differentiable map $\psi : M \to X$ of a manifold with boundary M. Let $k = \dim(Y)$, then by Stokes' theorem we have that

$$k!Vol(Y) = \int_Y \omega_Y^k = \int_Y \iota^*(\omega_X^k) = \int_M d\psi^*(\omega_X^k) = 0.$$

which is clearly a contradiction. □

As we saw, being Kähler has some restriction on the topology of the underlying manifold. Before we give some examples we need the following.

Definition 3.10. Let's call a real $(1,1)$-form ω positive if locally in holomorphic coordinates it is of the form $\omega = \frac{i}{2}\sum_{r,s} h_{r,s}(x) dz_r \wedge d\overline{z}_s$, such that the matrix $(h_{r,s}(x))$ is positive definite Hermitian for all x where it is defined.

Proposition 3.11. *There is a one-one correspondence between d-closed real $(1,1)$-forms and Kähler metrics on X.*

Proof. Easy consequence of Proposition 1.19. $\square$

Hence to give examples of Kähler manifolds, it will be enough to show complex manifolds with d-closed real $(1,1)$-forms, which is sometimes easier, then giving the metric g and proving that it is Kähler.

Lemma 3.12 (local $\partial\overline{\partial}$-lemma)**.** *Let ω be a real $(1,1)$-form on the complex manifold X. Then $d\omega = 0$ if and only if for $x \in X$ there exists an U open neighborhood of x and a smooth function $\varphi : U \to \mathbb{R}$, such that $\omega|_U = i\partial\overline{\partial}\varphi$.*

Proof. For the "if" part, let $\varphi : U \to \mathbb{R}$ be a smooth function, then $i\partial\overline{\partial}\varphi$ is clearly a $(1,1)$-form, to see that it is real we compute as follows:

$$\overline{i\partial\overline{\partial}\varphi} = -i\overline{\partial}\partial\varphi = i\partial\overline{\partial}\varphi.$$

To compute that it is also d-closed, we use that $\partial^2 = \overline{\partial}^2 = 0$,

$$d(-i\partial\overline{\partial}\varphi) = i(\partial + \overline{\partial})(\partial\overline{\partial}\varphi) = 0$$

Now for the only if part let ω be a real d-closed $(1,1)$-form and let $x \in X$. By the real Poincaré lemma there exists an U open neighborhood of x, and τ real form on U such that $\omega = d\tau$. Let $\tau = \tau^{1,0} + \tau^{0,1}$, since τ is real 1-form we get that $\overline{\tau^{1,0}} = \tau^{0,1}$. Let's look at the following:

$$\omega = d\tau = (\partial + \overline{\partial})(\tau^{1,0} + \tau^{0,1}).$$

Since ω is $(1,1)$-form, $\partial\tau^{1,0}$ and $\overline{\partial}\tau^{0,1}$ has to be zero, and $\omega = \partial\tau^{0,1} + \overline{\partial}\tau^{1,0}$. By the $\overline{\partial}$-Poincaré lemma, on an even smaller open set U there exists a smooth function f such that $\overline{\partial}(f) = \tau^{0,1}$. Hence if we summarise everything we get that

$$\omega = \partial\tau^{0,1} + \overline{\partial}\tau^{1,0} = \partial\tau^{0,1} + \overline{\partial}\overline{\tau}^{0,1} = \partial\overline{\partial}f + \overline{\partial\overline{\partial}f} = \partial\overline{\partial}(f - \overline{f}) = \partial\overline{\partial}(2i\mathrm{Im}(f)).$$

To finish the proof let $\varphi = 2i\mathrm{Im}(f)$. $\square$

As a collorary of the local $\partial\overline{\partial}$-lemma, we see that locally a Kähler form of a Kähler manifold is of the form $\frac{i}{2}\partial\overline{\partial}\varphi$. We call this function the local potential function of the Kähler form.

Example 3.13. Now we will give some examples of Kähler manifolds.

1) Let $X = \mathbb{C}^n$, and $\varphi(z) = \|z\|^2 = \sum_j^n z_j \overline{z}_j$. Then

$$\frac{i}{2}\partial\overline{\partial}\Big(\sum_j^n z_j\overline{z}_j\Big) = \frac{i}{2}\sum_{j=1}^n \partial(z_j)d\overline{z}_j = \frac{i}{2}\sum_{j=1}^n dz_j \wedge d\overline{z}_j$$

The matrix $(h_{r,s})$ is the identity matrix, which is clearly positive definite. Hence there exists a Kähler metric on $\mathbb{C}^n$ such that $\frac{i}{2}\sum_j dz_j \wedge d\overline{z}_j$ is its Kähler form. One can check easily that the Kähler form comes from the standard Euclidean metric on R^{2n}.

2) Let $X = B^n = \{z \in \mathbb{C}^n \mid |z| < 1\}$, and $\varphi(z) = -\log\big(1 - \|z\|^2\big)$. Then

$$\begin{aligned} -\partial_{\overline{z}_j}\log\big(1-\|z\|^2\big) &= \frac{z_j}{(1-\|z\|^2)} \\ \partial_{z_i}\frac{z_j}{(1-\|z\|^2)} &= \frac{z_j\overline{z_i} + \delta_{i,j}(1-\|z\|^2)}{\big(1-\|z\|^2\big)^2} \end{aligned}$$

Hence we see that the $(1,1)$-form is

$$\frac{i}{2}\partial\overline{\partial}\varphi = \frac{i}{2}h_{i,j}dz_i \wedge d\overline{z}_j = \frac{i}{2}\sum_{i,j}\frac{\overline{z_i}z_j + \delta_{i,j}(1-\|z\|^2)}{\big(1-\|z\|^2\big)^2}dz^i \wedge d\overline{z}^j.$$

Clearly the matrix $(h_{i,j})$ is Hermitian matrix, and to see that it is positive definite, we only have to show, that the matrix $\tilde{h}_{i,j} = \big(1-\|z\|^2\big)^2 h_{i,j}$ is positive definite. Let $\xi = (\xi^1, \ldots, \xi^n) \in \mathbb{C}^n$, then we compute as follows:

$$\begin{aligned} \sum_{i,j}\tilde{h}_{i,j}\xi_i\overline{\xi}_j &= \sum_j(1-\|z\|^2)\xi_j\overline{\xi}_j + \sum_{i,j} z_j\overline{z_i}\xi_j\overline{\xi}_i = (1-\|z\|^2)\|\xi\|^2 + \langle z,\xi\rangle\langle\xi,z\rangle \\ &= (1-\|z\|^2)\|\xi\|^2 + |\langle z,\xi\rangle| > 0, \end{aligned}$$

hence the matrix is positive definite. B^n with the corresponding Kähler metric is called the complex hyperbolic space.

3) Let $X = \mathbb{C}P^n$, and let $U_i = \{z_i \neq 0\}$ for $i = 0, 1, \ldots, z_n$. On U_j we define the function $\rho_j([z]) = \log\left(\sum_{k=0}^n \left|\frac{z_k}{z_j}\right|^2\right)$. The functions ρ_j and ρ_i will not agree on $U_i \cap U_j$, but the forms $\partial\overline{\partial}\rho_i$ and $\partial\overline{\partial}\rho_j$ will. Indeed on $U_i \cap U_j$ we compute as follows

$$\begin{aligned} \rho_i([z]) = \log\left(\sum_{k=0}^n\left|\frac{z_k}{z_i}\right|^2\right) &= \log\left(\left|\frac{z_j}{z_i}\right|^2\sum_{k=0}^n\left|\frac{z_k}{z_j}\right|^2\right) \\ = \log\left(\left|\frac{z_j}{z_i}\right|^2\right) + \log\left(\sum_k\left|\frac{z_k}{z_j}\right|^2\right) &= \log\left(\left|\frac{z_j}{z_i}\right|^2\right) + \rho_j([z]) \end{aligned}$$

Hence, we only have to show that $\partial\overline{\partial}\log\left(\left|\frac{z_j}{z_i}\right|^2\right) = 0$. Notice $\frac{z_j}{z_i} \in \mathcal{O}^*_{\mathbb{C}P^n}(U_i \cap U_j)$. We will show something stronger. Suppose that $U \subset \mathbb{C}^n$ is a small polydisc, and $f \in \mathcal{O}^*(U)$, then $\partial\overline{\partial}(\log(|f|^2) = 0$. Indeed, locally there exists $\log(f) \in \mathcal{O}(U)$. It follows, that

$$\log(|f|^2) = \log(f\overline{f}) = \log(f) + \log(\overline{f}),$$

where $\log(f)$ is holomorphic and $\log(\overline{f})$ is anti-holomorphic, hence

$$\partial\overline{\partial}\log(\|f\|^2) = \partial\overline{\partial}\log(f) - \overline{\partial}\partial\log(\overline{f}) = 0.$$

Hence the forms $\partial\overline{\partial}\rho_i = \partial\overline{\partial}\rho_j$ on $U_i \cap U_j$. To show that it is a positive form first use the standard charts on U_i, i.e. $\varphi_i : U_i \to \mathbb{C}^n$, $[z] \mapsto (\frac{z_1}{z_j}, \ldots, \frac{\hat{z_i}}{z_i}, \ldots, \frac{z_n}{z_i}) = (w_1, \ldots, w_n)$. Under this map $\rho_i([z])$ corrseponds to $\rho_i(w) = \log(1 + \sum_{k=1}^{1} |w_k|^2)$. A similar calculation like in the previous example shows, that

$$\partial\overline{\partial}\rho_i = \frac{i}{2}\sum_{i,j} h_{i,j} dz_i \wedge d\overline{z}_j = \sum_{i,j} \frac{\delta_{i,j}(1 + \|w\|^2) - w_i\overline{w}_j}{\left(1 + \|w\|^2\right)^2} dw_i \wedge d\overline{w}_j,$$

It is clear again that the matrix $(h_{i,j})$ is Hermitian, and just like before, we show that $\left(1 + \|w\|^2\right)^2 h_{i,j} = \tilde{h}_{i,j}$ is positive definite. To do that, let $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n$, and compute as follows

$$\begin{aligned}\sum_{i,j} \tilde{h}_{i,j}\xi_i\overline{\xi}_j &= (1 + \|w\|^2)\sum_j \xi_j\overline{\xi}_j - \sum_{i,j} w_j\overline{w}_i\xi_i\overline{\xi}_j = (1 + \|w\|^2)\|\xi\|^2 - \langle w, \xi\rangle\langle \xi, w\rangle \\ &= \|\xi\|^2 + \|w\|^2\|\xi\|^2 - |\langle w, \xi\rangle| > 0\end{aligned}$$

For the inequality, we used the Cauchy-Schwarz inequality. Thus we see that the forms $\frac{i}{2}\partial\overline{\partial}\rho_i$ are positive. One should normalise this form by multiplying it with $\frac{1}{\pi}$, which corresponds to the potential functions $\frac{1}{\pi}\rho_i$. The Kähler metric corresponding to this normalised $(1,1)$-form is called the Fubini-Study metric, and the normalised form is denoted by ω_{FS}. We conclude this example by showing that $\int_{CP^1} \omega_{FS} = 1$. The compution is the following:

$$\begin{aligned}\int_{CP^1} \omega_{FS} &= \int_{\mathbb{C}} \frac{i}{2\pi} \frac{1}{\left(1 + |w|^2\right)^2} dw \wedge d\overline{w} \\ &= \frac{1}{\pi}\int_{\mathbb{R}^2} \frac{1}{\left(1 + |(x,y)|^2\right)^2} dx \wedge dy \\ &= 2\int_0^\infty \frac{1}{(1 + r^2)^2} dr = 1\end{aligned}$$

4) If X is a Kähler manifold, and G is a discrete group which acts on X properly discontinuously, and every $g : X \to X$ is a holomorphic isometry of X, then the factor manifold X/G is Kähler. Hence if we look at $\mathbb{C}^n$ and choose $z_1, \dots, z_{2n} \in \mathbb{C}^n$ such that these are linearly independent over $\mathbb{R}$, then we have $\mathbb{Z}^{2n}$ action on $\mathbb{C}^n$, where $(k_1, \dots, k_{2n})$ maps to the translations with $k_1 z_1 + \dots + k_{2n} z_{2n}$. This clearly satisfies the properties above, hence $\mathbb{C}^n/\mathbb{Z}^{2n} \simeq S^1 \times S^1 \dots \times S^1$ is a Kähler manifold. In particular every smooth elliptic curve is a Kähler manifold.

5) If X is a Rieamann surface, then any compatible Riemannian metric on X will be Kähler, since there are no non-trivial 3-form on X.

Corollary 3.14. *Every smooth projective manifold is Kähler manifold with the restriction of the Fubiny-Study metric.*

Hence we see, that there are actually a lot of Kähler manifolds.

3.2 Harmonic theory on compact Kähler manifolds

As we saw in the previous section, on a general compact complex manifold, there are no connections between the operators $\Delta, \overline{\Box}$ and $\Box$. Surprisingly on Kähler manifolds this is not case as we will see later in this section.

Theorem 3.15 (Kähler identities)**.** *Let X be a Kähler manifold, then we have the following commutation relations:*

$$\begin{aligned} &a)\ [L,\partial] = [L,\overline{\partial}] = [L^*,\partial^*] = [L^*,\overline{\partial}^*] = 0 \\ &b)\ [L,\partial^*] = i\overline{\partial} \quad [L,\overline{\partial}^*] = -i\partial \\ &\quad\ [L^*,\partial] = i\overline{\partial}^* \quad [L^*,\overline{\partial}] = i\partial^* \end{aligned}$$

Remark 3.16. One can easily check that all four in $a)$ and $b)$ are equivalent, so it is enough to prove only one in both of them.

Definition 3.17. Let $d_c = \mathbb{I}^{-1} d \mathbb{I}$ and let $d_c^* = \mathbb{I}^{-1} d^* \mathbb{I}$.

It is easy to see that d_c and d_c^* are real operators since they are compositions of real operators. Also, since $\mathbb{I}^{-1} = \mathbb{I}^*$ we also see that d_c^* is the formal adjoint of d_c. We claim that $d_c = -i(\partial - \overline{\partial})$. Indeed, let $\varphi \in A^{p,q}(X)$, then:

$$\begin{aligned} d_c(\varphi) =& \mathbb{I}^{-1}(\partial + \overline{\partial})\mathbb{I}(\varphi) = i^{p-q}\mathbb{I}^{-1}(\partial\varphi + \overline{\partial}\varphi) = i^{p-q}(i^{q-p-1}\partial\varphi + i^{q+1-p}\overline{\partial}\varphi) \\ =& -i\partial\varphi + i\overline{\partial}\varphi = -i(\partial - \overline{\partial})\varphi. \end{aligned}$$

Proposition 3.18. *Let X be a Kähler manifold. Then*

$$a_1)\ [L,d]=0 \quad a_2)\ [L^*,d^*]=0$$
$$b_1)\ [L,d^*]=d_c\ b_2)\ [L^*,d]=-d_c^*.$$

Proof. First notice that $a_1^*=a_2$ and $b_1^*=b_2$, so we only have to prove one in a_1 and b_2. To see a_1) let $\varphi\in A^k(X)$, then:

$$[L,d]\varphi=\omega\wedge d\varphi-d(\omega\wedge\varphi)=\omega\wedge d\varphi-(\underbrace{d\omega}_{0}\wedge\varphi+\omega\wedge d\varphi)=0.$$

We will prove b_2). Using the Lefschetz decomposition it is enough to prove for $L^j\alpha$, where $\alpha\in\Gamma(P^k(X))$. One can use the Lefschetz decomposition and write $d\alpha$ in the following form:

$$d\alpha=\alpha_0+L\alpha_1+L^2\alpha_2+\dots$$

where $\alpha_j\in\Gamma(P^{k+1-2j})$. Since L and d commute and $L^{n-k+1}\alpha=0$ we get that:

$$0=L^{n-k+1}\alpha_0+L^{n-k+2}\alpha_1+L^{n-k+3}\alpha_2+\dots$$

Since the Lefschetz decomposition is a direct sum decomposition we get that $L^{n-k+1+j}\alpha_j=0$ for all j. On the other hand we know that L^l restricted to $A^i(X)$ is injective if $l\le n-i$. Thus $\alpha_j\in A^{k+1-2j}$ is zero for all $j\ge 2$, and $d\alpha=\alpha_0+L\alpha_1$. First we will compute $[L^*,d](L^j\alpha)$. To compute we will use that L and d commute, $\Lambda\alpha_i=L^*\alpha_i=0$ and Theorem 1.33.

$$\begin{aligned}L^*dL^j\alpha&=L^*L^jd\alpha=L^*L^j\alpha_0+L^*L^{j+1}\alpha_1\\&=j(n-k-j)L^{j-1}\alpha_0+(j+1)(n-k-j+1)L^j\alpha_1.\\dL^*L^j\alpha&=j(n-k-j+1)L^{j-1}d\alpha\\&=j(n-k-j+1)(L^jj-1\alpha_0+L^j\alpha_1)\end{aligned}$$

Hence we get that:

$$[L^*,d](L^j\alpha)=-jL^{j-1}\alpha_0+(n-k-j+1)L^j\alpha_1.$$

Now we compute $-d_c^*(L^j\alpha)$, using Theorem 1.49 and that $\mathbb{I}^2$ on k-forms is $(-1)^k$.

$$\begin{aligned}
-d_c^*(L^j\alpha) &= *\,\mathbb{I}^{-1}d\mathbb{I}*L^j\alpha\\
&= *\,\mathbb{I}^{-1}d\mathbb{I}\big((-1)^{k(k+1)/2}\frac{j!}{(n-k-j)!}L^{n-k-j}\mathbb{I}\alpha\big)\\
&=(-1)^{k(k+1)/2+k}\frac{j!}{(n-k-j)!}\mathbb{I}^{-1}*L^{n-k-j}d\alpha\\
&=(-1)^{k(k+1)/2+k}\frac{j!}{(n-k-j)!}\mathbb{I}^{-1}(*L^{n-k-j}\alpha_0+*L^{n-k-j+1}\alpha_1)\\
&=-jL^{j-1}\alpha_0+(n-k-j+1)L^j\alpha_1
\end{aligned}$$

Hence we see that $[L^*,d]=-d_c^*$. □

Corollary 3.19. *a)* $[L,d_c]=0$, *b)* $[L^*,d_c^*]=0$, *c)* $[L,d_c^*]=-d$, *d)* $[L^*,d_c]=-d^*$.

Proof. It is clear that $a)*=b)$ and $c)^*=d)$, so we only have to prove $a)$ and $d)$. To prove $a)$ we compute as follows:

$$[L,d_c]=L\mathbb{I}^{-1}d\mathbb{I}-\mathbb{I}^{-1}d\mathbb{I}L=0.$$

since L commutes with $\mathbb{I},\mathbb{I}^{-1}$ and d. To see $d)$ we have to notice first, that L^* commutes with $\mathbb{I}$ and $\mathbb{I}^{-1}$, which is clear from the fact that $\mathbb{I}^*=\mathbb{I}^{-1}$. With this in mind we compute as follows:

$$[L^*,d_c]=\mathbb{I}^{-1}[L^*,d]\mathbb{I}=\mathbb{I}^{-1}-d_c^*\mathbb{I}=d^*.$$

□

With these we can finaly prove the Kähler identities.

Proof of the Kähler identities. First we will show $a)$.

$$0=[L,d_c]=[L,-i(\partial-\overline{\partial})]=i[L,\overline{\partial}]-i[L,\partial]$$

Since $[L,\overline{\partial}]$ is of bidegree $(1,2)$ and $[L,\partial]$ is of bidegree $(2,1)$ this can be zero if and only if $[L,\overline{\partial}]=0$ and $[L,\partial]=0$. To see $b)$ we compute as follows:

$$\partial^*+\overline{\partial}^*=[L^*,d_c]=i[L^*,\overline{\partial}]-i[L^*,\partial].$$

Checking bidegrees again we see that $[L^*,\partial]=-i\overline{\partial}^*$ and $[L^*\overline{\partial}]=-i\partial^*$. □

Corollary 3.20. *Let X be a compact Kähler manifold. Then*

$$\frac{1}{2}\Delta=\overline{\Box}=\Box.$$

Proof. First we show that $\overline{\Box} = \Box$.

$$\begin{aligned}\overline{\Box} =&\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial} = -i(\overline{\partial}[L^*, \partial] + [L^*, \partial]\overline{\partial}) = i(-\overline{\partial}L^*\partial - \partial\overline{\partial}L^* + L^*\overline{\partial}\partial + \partial L^*\overline{\partial})\\ =&i([L^*, \overline{\partial}]\partial + \partial[L^*, \overline{\partial}]) = \partial^*\partial + \partial\partial^* = \Box.\end{aligned}$$

Now we show that $\Delta = \overline{\Box} + \Box$.

$$\begin{aligned}\Delta =&dd^* + d^*d = (\partial + \overline{\partial})(\partial + \overline{\partial})^* + (\partial + \overline{\partial})^*(\partial + \overline{\partial})\\ =&\partial\partial^* + \partial\overline{\partial}^* + \overline{\partial}\partial^* + \overline{\partial}\overline{\partial}^* + \partial^*\partial + \partial^*\overline{\partial} + \overline{\partial}^*\partial + \overline{\partial}^*\overline{\partial}\\ =&\Box + \overline{\Box} + \underbrace{\partial\overline{\partial}^* + \overline{\partial}\partial^* + \partial^*\overline{\partial} + \overline{\partial}^*\partial}_{D}\end{aligned}$$

To finish the proof we only have to show that $D = 0$.

$$\begin{aligned}D =& - i\partial(L^*\partial - \partial L^*) + i\overline{\partial}(L^*\overline{\partial} - \overline{\partial}L^*) - i(L^*\partial - \partial L^*)\partial + i(L^*\overline{\partial} - \overline{\partial}L^*)\overline{\partial}\\ =& - i\partial L^*\partial + i\overline{\partial}L^*\overline{\partial} + i\partial L^*\partial - i\overline{\partial}L^*\overline{\partial} = 0\end{aligned}$$

Here we used that $\partial\partial = \overline{\partial}\overline{\partial} = 0$. $\Box$

Corollary 3.21. *The Laplacian Δ commutes with $L, \mathbb{I}, L^*, d, \partial, \overline{\partial}, \partial^*, \overline{\partial}^*$ and d^*.*

Proof. Since Δ is formally self adjoint we only have to show that Δ commutes with $\mathbb{I}, L, d, \partial$ and $\overline{\partial}$. Since $1/2\Delta = \Box$ it is of bidegree $(0, 0)$ hence $\Delta\mathbb{I} = \mathbb{I}\Delta$. It is clear that Δ commutes with d, and since $\Box$ commutes with ∂ and $\overline{\Box}$ commutes with $\overline{\partial}$ we also have that for Δ. To see that $[\Delta, L] = 0$ we compute as follows:

$$\begin{aligned}\Delta L - L\Delta =&dd^*L + d^*dL - Ldd^* - Ld^*d = dd^*L + d^*Ld - dLd^* - Ld^*d\\ =& - d[L, d^*] - [L, d^*]d = -dd_c - d_cd = -2i\partial\overline{\partial} + 2i\partial\overline{\partial} = 0\end{aligned}$$

$\Box$

Corollary 3.22. *Let X be a compact Kähler manifold. Then ω is a harmonic form.*

Proof. $\omega = L(1)$, and $\Delta\omega = \Delta L(1) = L\Delta(1) = L(0) = 0$. $\Box$

Corollary 3.23. *Let X be a compact complex manifold, with compatible Riemannian metric g and fundamental form ω. Then (X, g) is a Kähler manifold if and only if ω is harmonic.*

Proof. If ω is harmonic, then it is also d-closed. If ω is d-closed, then by the previous corollary it is also harmonic. $\Box$

Corollary 3.24. *Let $\alpha \in A^k(X)$ and suppose that $\alpha = \sum_{p+q=k} \alpha^{p,q}$. Then $\Delta\alpha = 0$ if and only if $\Delta\alpha^{p,q} = 0$ for all (p, q).*

Proof. Suppose that $\Delta\alpha = \sum_{p+q=k} \Delta\alpha^{p,q} = 0$. We have seen in the proof of Corollary 3.21 that Δ is of bidegree $(0,0)$, thus $\Delta\alpha^{p,q} \in A^{p,q}(X)$, hence $\sum_{p+q=k} \Delta\alpha^{p,q} = 0$ if and only if $\Delta\alpha^{p,q} = 0$ for all (p,q). $\square$

Corollary 3.25. *Let $\mathcal{H}^k = \ker(\Delta : A^k(X) \to A^k(X))$ and $\mathcal{H}^{p,q} = \mathcal{H}^k \cap A^{p,q}(X)$, where $p+q=k$, and lastly $\mathcal{H}^{p,q}_{\overline{\square}} = \ker(\overline{\square} : A^{p,q}(X) \to A^{p,q}(X))$. Then*

a) If $p+q=k$, then $\pi^{p,q}(\mathcal{H}^k) = \mathcal{H}^{p,q}_{\Delta} = \mathcal{H}^{p,q}_{\overline{\square}}$.

b) $\mathcal{H}^k = \bigoplus_{p+q=k} \mathcal{H}^{p,q}$.

c) $\overline{\mathcal{H}^{p,q}_{\Delta}} = \mathcal{H}^{q,p}_{\Delta}$.

Proof. Part $a)$ and $b)$ trivially follow from Corollary 3.24 and Corollary 3.20. To see $c)$ let $\alpha \in \mathcal{H}^{p,q}_{\Delta}$. Since Δ is a real operator we get that

$$\Delta\overline{\alpha} = \overline{\Delta}\overline{\alpha} = \overline{\Delta\alpha} = 0.$$

Hence $\overline{\alpha} \in \mathcal{H}^{q,p}$. $\square$

Theorem 3.26. *Let X be a compact complex manifold of Kähler type. Then there exists a direct sum decomposition*

$$H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X).$$

Moreover with respect to the conjugation on $H^*(X,\mathbb{C}) = H^*(X,\mathbb{R}) \otimes \mathbb{C}$ one has $\overline{H^{p,q}(X)} = H^{q,p}(X)$ and $H^{p,q}(X) \simeq H^q(X,\Omega^p)$.

Proof. Let g be a Kähler metric on X, then with respect to this metric we have $\mathcal{H}^k$, $\mathcal{H}^{p,q}_{\Delta}$ and $\mathcal{H}^{p,q}_{\overline{\square}}$. Then by Theorem 1.140 we know that $H^k(X,\mathbb{C}) \simeq \mathcal{H}^k$ and $H^q(X,\Omega^p) \simeq \mathcal{H}^{p,q}_{\overline{\square}} = \mathcal{H}^{p,q}_{\Delta}$. Hence by Corollary 3.25 $b)$ we get that

$$H^k(X,\mathbb{C}) \simeq \bigoplus_{p+q=k} H^{p,q}(X),$$

where $H^{p,q}(X)$ is the image of $K^{p,q}_{\Delta}$ under the canonical isomorphism. The conjugation in $H^k(X,\mathbb{C})$ is computed as follows, let $[\alpha] \in H^k(X,\mathbb{C})$, then $\overline{[\alpha]} = [\overline{\alpha}]$, and we can use Corollary 3.25 $c)$ te see that $\overline{H^{p,q}} = H^{q,p}$. $\square$

Proposition 3.27. *The decomposition above does not depend on the chosen Kähler metric.*

Proof. Suppose we chose a Kähler metric g and defined $H^{p,q}(X)$ as above. Define the following:

$$\mathcal{K}^{p,q} = \{[\alpha] \in H^k(X,\mathbb{C}) \mid \text{there exists } \beta \in A^{p,q}(X), \text{ such that } d\beta = 0 \text{ and } [\beta] = [\alpha]\}$$

It is clear that $\mathcal{K}^{p,q}$ does not depend on the metric. We claim that $H^{p,q}(X) = \mathcal{K}^{p,q}$. It is clear that $H^{p,q}(X) \subset \mathcal{K}^{p,q}$, since an element in $H^{p,q}(X)$ can be represented with a harmonic (p,q)-form, and we know that harmonic forms are d-closed. Now suppose that we have an element in $\mathcal{K}^{p,q}$. Then there exist a d-closed (p,q)-form ψ such that $[\psi]$ represents that element. In the proof of Theorem 1.140 we saw that $\Gamma(E_j) = \ker(\Delta_j) \oplus \operatorname{Im}(\Delta_j)$. In our case we get that $\psi = \alpha + \Delta\beta$, where α is a harmonic form. Since g is a Kähler metric we know that Δ is of bidegree $(0,0)$, hence $\psi = \psi^{p,q} = \alpha^{p,q} + \Delta\beta^{p,q}$, where $\alpha^{p,q}, \beta^{p,q}$ are (p,q)-forms and $\alpha^{p,q}$ is harmonic. Applying d to ψ we get that:

$$0 = d\psi = d\alpha^{p,q} + d(dd^* + d^*d)\beta^{p,q} = dd^*d\beta^{p,q}.$$

It follows that $d^*d\beta^{p,q} \in \ker(\Delta) \cap \operatorname{Im}(d^*) = 0$. Hence $\psi = \alpha^{p,q} + dd^*\beta^{p,q}$, and $[\psi] = [\alpha^{p,q}] \in H^{p,q}(X)$. □

Corollary 3.28. *The cup product on $H^*(X,\mathbb{C})$ respects the (p,q)-decomposition, i.e. if $[\alpha] \in H^{p,q}(X)$ and $[\beta] \in H^{p',q'}(X)$, then $[\alpha] \cup [\beta] \in H^{p+p',q+q'}(X)$.*

Proof. Since $[\alpha]\cup[\beta] = [\alpha\wedge\beta]$ and we know that $\alpha\wedge\beta$ is a d-closed $(p+p', q+q')$-form we get that $[\alpha] \cup [\beta] \in \mathcal{H}^{p,q} = H^{p+p',q+q'}(X)$. □

Remark 3.29. This does not follow from the definition of $H^{p,q}(X)$ since wedge product of harmonic forms does not have to be harmonic.

Corollary 3.30. *Let X be a compact Kähler manifold. Then $b_{2k+1}(X)$ is even for all k.*

Proof. By Theorem 3.26 *b*) we get that

$$b_{2k+1}(X) = h^{2k+1,0}(X) + h^{2k,1}(X) + \cdots + h^{1,2k}(X) + h^{0,2k+1}(X)$$

Also by Theorem 3.26 *c*) we know that $h^{i,j} = h^{j,i}$, hence

$$b_{2k+1}(X) = 2(h^{2k+1,0}(X) + h^{2k,1}(X) + \cdots + h^{k+1,k}(X)).$$

□

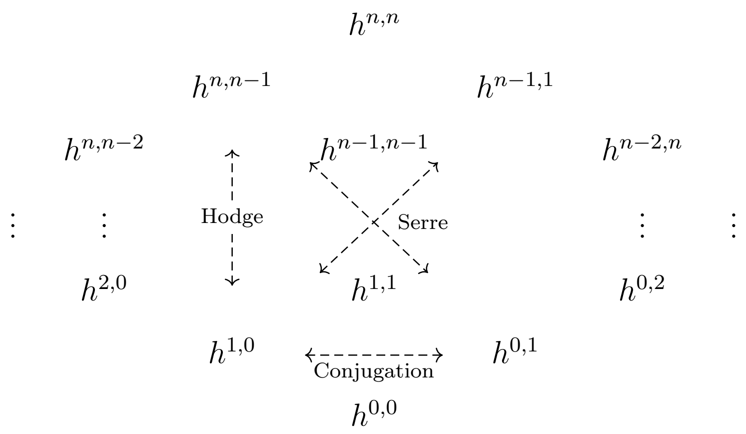
Corollary 3.31. *Let X be a compact Kähler manifold. If $\alpha \in H^0(X,\Omega^p(X))$, then $d\alpha = 0$.*

Proof. We know that $\overline{\partial}\alpha = 0$ since α is holomorphic. We also know that $\overline{\partial}^*\alpha = 0$ by definition, hence $\overline{\Box}\alpha = 0$. Since X is Kähler it follows that $\Delta\alpha = 0$, thus $d\alpha = 0$. □

Corollary 3.32. *There exists an injective map $H^0(X,\Omega^p(X)) \to H^p(X,\mathbb{C})$.*

Proof. Let $\alpha \in H^0(X, \Omega^p(X))$, then by the above $d(\alpha) = 0$, hence it defines an element in $H^p(X, \mathbb{C})$. Since α is harmonic $[\alpha]$ is zero if and only if $\alpha = 0 \in H^0(X, \Omega^p(X))$ hence the map $\alpha \to [\alpha]$ is injective. $\square$

Let X be a compact Kähler manifold of dimension n, then its hodge numbers can be visualised by the Hodge diamond:



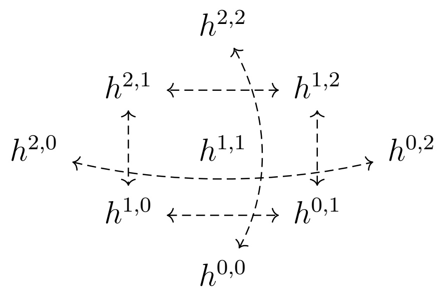
We know by Serre duality that $h^{p,q} = h^{n-p,n-q}$, i.e. the hodge diamond is stable under rotation by π around it's center. We also know that the Hodge $*$-operator induces an isomorphism between $H^{p,q}(X)$ and $H^{n-q,n-p}(X)$, hence we have that $h^{p,q} = h^{n-q,n-p}$, i.e. the hodge diamond is stable under reflection in the horizontal line passing through $h^{n,0}$ and $h^{0,n}$. We also know that the conjugation induces a conjugate linear isomorphism between $H^{p,q}(X)$ and $H^{q,p}(X)$, hence $h^{p,q} = h^{q,p}$, and the Hodge diamond remains unchanged after reflecting in the horizontal line which crosses $h^{0,0}$ and $h^{n,n}$.

Suppose that the dimension of X is one. Then by Theorem 3.26 we get that

$$H^1(X, \mathbb{C}) = H^{1,0}(X) + H^{0,1}(X),$$

where $H^{1,0}(X) \simeq H^0(X, \Omega(X))$, $H^{0,1}(X) \simeq H^1(X, \mathcal{O}_X)$, and $\overline{H^{1,0}(X)} = H^{0,1}(X)$. Hence $\dim(H^0(X, \Omega(X))) = \dim(H^1(X, \mathcal{O}_X)) = \frac{1}{2} b_1$ which is a topological invariant of X called the genus. It is just the amount of handles in X.

Now suppose that the dimension of X is two. Then we have the following Hodge diamond:



Hence $h^{2,1} = h^{1,2} = h^{1,0} = h^{0,1} = \frac{1}{2}b_1$ and $h^{2,0} = h^{0,2}$. We also know that $h^{1,1} > 0$, since $[\omega] \in H^{1,1}(X)$.

Corollary 3.21 has another important consequence. It says that Δ commutes with L and L^*, hence they map harmonic forms to harmonic forms. Denote the primitive harmonic r-forms by $\mathcal{H}_0^r = \Gamma(P^r(X)) \cap \mathcal{H}^r$ and the primitive harmonic (p,q)-forms by $\mathcal{H}_0^{p,q} = \Gamma(P^{p,q}(X)) \cap \mathcal{H}_\Delta^{p,q}$.

Theorem 3.33. *Let X be a compact Kähler manifold. Then*

a) $\mathcal{H}^r = \bigoplus_{s\geq 0} L^s(\mathcal{H}_0^{r-2s})$, and $\mathcal{H}_\Delta^{p,q} = \bigoplus_{s\geq 0} L^s(\mathcal{H}_0^{p-s,q-s})$ are both orthogonal decompositions with respect to the Hodge inner product.

b) $\mathcal{H}_0^r = \bigoplus_{p+q=r} \mathcal{H}_0^{p,q}$ is an orthogonal decomposition with respect to the Hodge inner product.

c) $L^{n-k} : \mathcal{H}^k \to \mathcal{H}^{2n-k}$ is an isomorphism. and restricting it to $\mathcal{K}_\Delta^{p,q}$ we get that $L^{n-k} : \mathcal{H}_\Delta^{p,q} \to \mathcal{H}_\Delta^{n-q,n-p}$ is an isomorphism.

Proof. Trivially follows from Remark 3.3 and that Δ commutes with λ and λ^*. □

If X is a Kähler manifold, then the map $L : A^*(X) \to A^*(X)$ induces a homorphism on the cohomology groups of X also denoted by L as follows: let $[\alpha] \in H^k(X;\mathbb{C})$, then $L([\alpha]) = [L\alpha] = [\omega \wedge \alpha]$. Define the primitive cohomology of X as $H_0^k(X;\mathbb{C}) = \ker(L^{n-k+1} : H^k(X;\mathbb{C}) \to H^{2n-k+2}(X;\mathbb{C}))$ and denote the (p,q)-primitive cohomology by $H_0^{p,q}(X) = H_0^{p+q}(X;\mathbb{C}) \cap H^{p,q}(X)$. With these notations we have the following theorem.

Theorem 3.34. *Let X be a compact Kähler manifold. Then*

a) $H^k(X;\mathbb{C}) = \bigoplus_{i\geq 0} L^i(H_0^{k-2i}(X;\mathbb{C}))$, and $H^{p,q}(X) = \bigoplus_{i\geq 0} L^i(H_0^{p-i,q-i}(X))$. We call this the Lefschetz decomposition of X.

b) $L^{n-k} : H^k(X;\mathbb{C}) \to H^{2n-k}(X;\mathbb{C})$ is an isomorphism. If $p+q=k$, then $L^{n-k} : H^{p,q}(X) \to H^{n-q,n-p}(X)$ is an isomorphism.

Proof. Notice that we have the following commutative diagram:

$$\begin{tikzcd} \mathcal{H}^k \arrow[r, "L"] \arrow[d, "\simeq"'] & \mathcal{H}^{k+2} \arrow[d, "\simeq"] \\ H^k(X;\mathbb{C}) \arrow[r, "L"] & H^{k+2}(X;\mathbb{C}) \end{tikzcd}$$

Hence if we show that the image of $\mathcal{H}_0^k$ is $H_0^k(X;\mathbb{C})$ and $\mathcal{H}_0^{p,q}$ is $H_0^{p,q}(X)$, then everyting in Theorem 3.33 is also true in the cohomology level if it makes sense. By Corollary 1.39 we see that $\alpha \in \mathcal{H}_0^k$ if and only if α is harmonic and $L^{n-k+1}\alpha = 0$, thus the image of $\mathcal{H}_0^k$ is indeed $H_0^k(X;\mathbb{C})$. One can show similarly that $\mathcal{H}_0^{p,q}$ maps to $H_0^{p,q}(X)$. $\square$

Part b) of Theorem 3.34 is called the hard Lefschetz theorem. It was originally proved by Lefschetz over integer coefficients but his proof turned out to be incorrect.

Corollary 3.35. *Let X be a compact complex manifold. Then $b_{i-2} \leq b_i$ for $i \leq n$, and $h^{p-1,q-1} \leq h^{p,q}$, for $p+q \leq n$.*

Proof. Follows from the fact that the Lefschetz operator $L : H^{p-1,q-1}(X) \to H^{p,q}(X)$ is injectice for $p+q \leq n$. $\square$

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NYILATKOZAT

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Diplomamunka címe:
Introduction to Hodge theory on compact Kähler manifolds

A **diplomamunka** szerzőjeként fegyelmi felelősségem tudatában kijelentem, hogy a dolgozatom önálló szellemi alkotásom, abban a hivatkozások és idézések standard szabályait következetesen alkalmaztam, mások által írt részeket a megfelelő idézés nélkül nem használtam fel.

Budapest, 2021.05.31

Koltai László

a hallgató aláírása