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On well-balanced orientations, counter-examples for related problems

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Abstract

In this note we consider problems related to Nash-Williams' well-balanced orientation theorem and we present counter-examples for some of them.

1 Introduction

Let $G := (V, E)$ be an undirected (or a directed) graph. For two vertices $u, v \in V$ of G the **local edge-connectivity** $\lambda_G(u, v)$ from u to v in G is defined to be the maximum number of pairwise edge (arc resp.) disjoint paths from u to v in G. G is k**-edge-connected** (k**-arc-connected** resp.) if $\lambda_G(u, v) \geq k \quad \forall (u, v) \in V \times V$. More generally, for $U \subseteq V$, G is k**-edge-connected** (k**-arc-connected** resp.) **in** $\boldsymbol{U}$ if $\lambda_G(u, v) \geq k \;\; \forall (u, v) \in U \times U$.

Nash-Williams' well-balanced orientation theorem [16] states that for any undirected graph G there exists a **well-balanced** orientation $\vec{G}$ of G for which $\lambda_{\vec{G}}(u, v) \geq \lfloor \frac{1}{2}\lambda_G(u, v)\rfloor \; \forall (u, v) \in V \times V$. For global edge-connectivity this specializes to: G has a k-arc-connected orientation if and only if G is $2k$-edge-connected.

Let $G := (V + s, E)$ be an undirected graph. The operation **splitting off** is defined as follows: two edges rs, st incident to s are replaced by a new edge rt. The splitting off theorem of Lovász [11] concerns global edge-connectivity: if G is k-edge-connected in V $(k \geq 2)$ and $d(s)$ is even then there exists a pair of edges rs, st incident to s whose splitting off maintains the k-edge-connectivity in V. Lovász [11] also showed that the global case of the well-balanced orientation theorem is an easy consequence

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of his splitting off theorem. Mader [12] generalized Lovász' result for local edge-connectivity: if $d(s) \geq 4$ and no cut edge of G is incident to s then there exists a pair of edges rs, st incident to s whose splitting off maintains the local edge-connectivities in V. A simple proof for Mader's theorem can be found in [4]. Mader [12] provided a new proof for the well-balanced orientation theorem by applying his splitting off theorem.

Let $\vec{G} := (V+s, E)$ be a directed graph. Splitting off can be naturally reformulated for directed graphs: two arcs rs, st are replaced by rt. Mader [13] proved a splitting off theorem preserving global arc-connectivity in directed graphs: if G is k-arc-connected in V and $\varrho(s) = \delta(s)$ then there exists a pair of arcs rs, st incident to s whose splitting off maintains the k-arc-connectivity in V. An example of Enni [1] shows that there is no splitting off theorem preserving local arc-connectivities in directed graphs. In Question 3 we provide a smaller example showing that even if $\vec{G}$ is a well-balanced orientation of G there is no splitting off that preserves local arc-connectivities in V.

Nash-Williams' odd vertex pairing theorem [16] (that is unrelated to orientations) states that every undirected graph G has a **pairing** M (a set of new edges on the set T_G of odd degree vertices of G such that $d_M(v) = 1\ \forall v \in T_G$) that is **feasible** ($d_M(X) \leq b_G(X)\ \forall X \subset V$, where $b_G(X) := d_G(X) - 2\lfloor \frac{R_G(X)}{2} \rfloor$ and $R_G(X) := \max\{\lambda_G(x,y) : x \in X,\ y \notin X\}$. A simpler proof of the odd vertex pairing theorem can be found in [5]. For the global case, the odd vertex pairing theorem can be proved easily by the global splitting off theorem as it was shown in [10] by Király and Szigeti.

The original proof of the odd vertex pairing theorem in [16] and Frank's proof [5] as well relies heavily on the skew-submodularity of the function b_G. We show (Question 6) that the existence of a feasible pairing cannot be generalized to arbitrary skew-submodular functions. Skew-submodular functions correspond to local edge-connectivity, while crossing submodular functions can be considered as generalizations of global edge-connectivity. For such a function it is an open problem whether there exists a feasible pairing. However the corresponding orientation theorem can be proved easily (see Theorem 6.1).

The well-balanced orientation theorem is trivial for Eulerian graphs (any Eulerian orientation will do) but this special case plays an important role in the theory. It was shown in [10] that for Eulerian graphs, an orientation is well-balanced if and only if it is Eulerian.

Nash-Williams [16] showed that if M is a feasible pairing of G then for every Eulerian orientation $\vec{G} + \vec{M}$ of $G + M$, $\vec{G}$ is well-balanced and furthermore it is **smooth**, that is the in-degree and the out-degree of every vertex differ by at most one. A smooth well-balanced orientation is called **best-balanced**. We show (Question 10) that not every best-balanced orientation can be defined by a feasible pairing. On the other hand we prove in Theorem 7.1 that for the global case it can be.

Nash-Williams [17] claimed the following extension of the well-balanced orientation theorem for subgraph chain of length two: if H is a subgraph of G, then there exists an orientation of H that can be extended to an orientation of G both being best-balanced. A simple proof is given in [10]. It is shown there that the odd vertex pairing theorem easily implies it. It was also showed that the global case of this extension has a simple proof. We show that the *general subgraph chain property* is not valid, that is this extension cannot be generalized for subgraph chain of length three, neither for the global case (see Question 16).

In [10] Király and Szigeti generalized further the above extension by showing that the following *edge disjoint subgraphs property* is valid: if $\{G_1, G_2, ..., G_k\}$ is a partition of G into edge disjoint subgraphs then there is an orientation $\vec{G}$ of G such that each $\vec{G_i}$ and $\vec{G}$ are best-balanced orientations of G_i and of G. We show that deciding for two non-edge-disjoint graphs whether they have simultaneous best-balanced orientations is NP-complete, even for two Eulerian graphs (see Question 17).

Király and Szigeti [10] also showed that for every pairing M of G there exists an Eulerian orientation $\vec{G} + \vec{M}$ of $G + M$ so that $\vec{G}$ is best-balanced. We mention that, for an Eulerian subgraph H of G, any Eulerian orientation of H can be extended to a best-balanced orientation of G.

Frank [3] proved the following *reorientation property* for the k-arc-connected orientations: given two k-arc-connected orientations of G, there exists a series of k-arc-connected orientations of G (leading from the first to the second given orientation), such that in each step we reverse a directed path or a circuit. For well-balanced (or best-balanced) orientations it is not known whether the reorientation property is valid.

Frank [2] also proved that the *linkage property* is valid for the k-arc-connected orientation problem, namely there exists a k-arc-connected orientation whose in-degree function satisfies lower and upper bounds if and only if there is one satisfying the lower bound and one satisfying the upper bound. É. Tardos [18] showed that the linkage property is not valid for the well-balanced orientation problem. Here we present another example (see Question 13).

By the proof of Frank [3] it is easy to see that the following *matroid property* is valid for smooth k-arc-connected orientations: the family of sets, over smooth k-arc-connected orientations, consisting of vertices whose in-degree is larger than the out-degree, forms the basis of a matroid. We show that this is not true for best-balanced orientations (see Question 14).

The *polyhedron* of the fractional in-degree vectors of well-balanced orientations is not necessarily integral (see Question 12). On the other hand for the global case it is.

We summarize the above properties in the following table.

	GLOBAL edge-connectivity	LOCAL edge-connectivity
undirected splitting off	Lovász [11]	Mader [12] Frank [4]
directed splitting off	Mader [13]	Counter-examples Enni [1], IKKSz
feasible pairing	Nash-Williams [16] Király, Szigeti [10]	Nash-Williams [16] Frank [5]
well-balanced orientation	Nash-Williams [16] Lovász [11]	Nash-Williams [16] Mader [12]
feasible pairing for connectivity function	Open Problem	Counter-example IKKSz
feasible pairing defining a given best-balanced orientation	IKKSz	Counter-example IKKSz
well-balanced orientation with subgraph	Nash-Williams [17] Király, Szigeti [10]	Nash-Williams [17] Király, Szigeti [10]
general subgraph chain property	Counter-example IKKSz	Counter-example IKKSz
edge disjoint subgraphs property	Király, Szigeti [10]	Király, Szigeti [10]
reorientation between two well-balanced orientations	Frank [3]	Open Problem
linkage property	Frank [2]	Counter-examples Tardos [18], IKKSz
matroid property	Frank [3]	Counter-example IKKSz
polyhedron of in-degree vectors of best-balanced orientations	integral Frank	not integral IKKSz

The aim of this paper is to help to find a transparent proof for the well-balanced orientation theorem. A possible way could be to find a convenient generalization that has a simple inductive proof. Here we think of results like Theorems 3.3, 3.4 and 3.5. Unfortunately we do not have direct proofs for them, they follow easily from the odd vertex pairing theorem. This result (Theorem 3.10) is a miracle, it has no generalization, no application (except the well-balanced orientation theorem), no relation to any other result in Graph Theory.

2 Notation

A directed graph is denoted by $\vec{G} = (V, A)$ and an undirected graph by $G = (V, E)$. For a directed graph $\vec{G}$, a set $X \subseteq V$, a vector $z : A \to \mathbb{R}$ and $u, v \in V$, let $\delta_{\vec{G}}(X) := |\{uv \in A : u \in X,\ v \notin X\}|$, $\varrho_{\vec{G}}(X) := \delta_{\vec{G}}(V - X)$, $f_{\vec{G}}(X) := \varrho_{\vec{G}}(X) - \delta_{\vec{G}}(X)$,

$\delta^z_{\vec{G}}(X) := \sum_{\{uv \in A : u \in X,\, v \notin X\}} z(uv)$, $\varrho^z_{\vec{G}}(X) := \delta^z_{\vec{G}}(V - X)$, $\lambda_{\vec{G}}(u,v) := \min\{\delta_{\vec{G}}(Y) : u \in Y,\ v \notin Y\}$, and $\overleftarrow{G} := (V, \{vu : uv \in A\})$. For an undirected graph G, a set $X \subseteq V$ and $u, v \in V$, let $\Delta_G(X) := \{uv \in E : u \in X,\ v \notin X\}$, $d_G(X) := |\Delta_G(X)|$, $d_G(X,Y) := |\{uv \in E(G) : u \in X - Y, v \in Y - X\}|$, $\lambda_G(u,v) := \min\{d_G(X) : u \in X,\ v \notin X\}$, $R_G(X) := \max\{\lambda_G(x,y) : x \in X,\ y \notin X\}$, $\hat{R}_G(X) := 2\lfloor R_G(X)/2 \rfloor$, $b_G(X) := d_G(X) - \hat{R}_G(X)$ and $T_G := \{v \in V : d_G(v) \text{ is odd}\}$. Observe that $\forall X \subseteq V$,

$$f_{\vec{G}}(X) \;=\; \sum_{v \in X} f_{\vec{G}}(v). \tag{1}$$

Let $G = (V, E)$ be an undirected graph. G is **connected** if for every pair $u, v \in V$ of vertices there is a (u,v)-path in G. G is called **k-edge-connected** if $G - F$ is connected for $\forall F \subseteq E$ with $|F| \le k - 1$. For a function $\mathbf{r} : V \times V \to \mathbb{Z}^+_0$, we say that G is **r-edge-connected** if $\lambda_G(u,v) \ge \mathbf{r}(u,v)$ for every pair u, v of vertices.

Let $D = (V, A)$ be a directed graph. D is **strongly connected** if for every ordered pair $(u,v) \in V \times V$ of vertices there is a directed (u,v)-path in D. D is called **k-arc-connected** if $D - F$ is strongly connected for $\forall F \subseteq A$ with $|F| \le k - 1$. For a function $\mathbf{r} : V \times V \to \mathbb{Z}^+_0$, we say that D is **r-arc-connected** if $\lambda_D(u,v) \ge \mathbf{r}(u,v)$ for every ordered pair u, v of vertices.

An orientation $\vec{G}$ of G is called **well-balanced** if $\vec{G}$ satisfies (2), **smooth** if $\vec{G}$ satisfies (3) and **best-balanced** if it is smooth and well-balanced. Let us denote by $\mathcal{O}_w(G)$ and $\mathcal{O}_b(G)$ the set of well-balanced and best-balanced orientations of G. Note that if $\vec{G}$ is best-balanced then so is $\overleftarrow{G}$.

$$\lambda_{\vec{G}}(x,y) \;\ge\; \lfloor \lambda_G(x,y)/2 \rfloor \quad \forall\ (x,y) \in V \times V, \tag{2}$$
$$|f_{\vec{G}}(v)| \;\le\; 1 \quad \forall\ v \in V. \tag{3}$$

A **pairing** M of G is a new graph on vertex set T_G in which each vertex has degree one. Let M be a pairing of G. An orientation $\vec{M}$ of M that satisfies (4) is called **good**. Note that if $\vec{M}$ is good then every Eulerian orientation $\vec{G} + \vec{M}$ of $G + M$ that extends $\vec{M}$ defines a best-balanced orientation of G. Pairing M is **well-orientable** if there *exists* a good orientation of M, M is **strong** if *every* orientation of M is good and M is **feasible** if (5) is satisfied. Clearly an oriented pairing $\vec{M}$ is good iff $\overleftarrow{M}$ is good. Let us denote by $\mathcal{P}_f(G)$ the set of feasible pairings of G.

$$f_{\vec{M}}(X) \;\le\; b_G(X) \quad \forall X \subseteq V, \tag{4}$$
$$d_M(X) \;\le\; b_G(X) \quad \forall X \subseteq V. \tag{5}$$

We shall use that for subsets $X, Y, Z \subseteq V$ we have

$$d_G(X) + d_G(Y) \;=\; d_G(X \cap Y) + d_G(X \cup Y) + 2d_G(X,Y), \tag{6}$$
$$\begin{aligned} d_G(X) + d_G(Y) + d_G(Z) \;\ge\; & d_G(X \cap Y \cap Z) + d_G(X - (Y \cup Z)) + \\ & d_G(Y - (X \cup Z)) + d_G(Z - (X \cup Y)). \end{aligned} \tag{7}$$

3 Known results

The following were shown in [10] by Király and Szigeti.

Claim 3.1. *The following are equivalent:*

$$\vec{G} \in \mathcal{O}_w(G), \tag{8}$$

$$\delta_{\vec{G}}(X) \geq \lfloor \frac{R(X)}{2} \rfloor \qquad \forall\ X \subseteq V, \tag{9}$$

$$f_{\vec{G}}(X) \leq b_G(X) \qquad \forall\ X \subseteq V. \tag{10}$$

Claim 3.2. *A pairing is feasible if and only if it is strong.*

Theorem 3.3. *Every pairing is well-orientable.*

Theorem 3.4. *For every partition $\{E_1, E_2, ..., E_k\}$ of $E(G)$, if $G_i = (V, E_i)$ then G has a best-balanced orientation $\vec{G}$, such that the inherited orientation of each G_i is also best-balanced.*

Theorem 3.5. *For every partition $\{X_1, ..., X_l\}$ of $V = V(G)$, G has an orientation $\vec{G}$ such that $\vec{G}$, $((\vec{G}/X_1)...)/X_l$ and $\vec{G}/(V - X_i)$ $(1 \leq i \leq l)$ are best-balanced orientations of the corresponding graphs.*

Theorem 3.5 implies the following slight refinement of the best-balanced orientation theorem.

Theorem 3.6. *Let G be an undirected graph and M be a pairing of G. Then G has a well-balanced orientation $\vec{G}$ with $|f_{\vec{G}}(X)| \leq 1$ whenever $d_M(X) \leq 1$.*

The following theorems are due to Nash-Williams [16], [17].

Theorem 3.7. *A graph G has a k-arc-connected orientation if and only if G is $2k$-edge-connected.*

Theorem 3.8. *Every graph G has a best-balanced orientation.*

Theorem 3.9. *For every subgraph H of G, there exists a best-balanced orientation of H that can be extended to a best-balanced orientation of G.*

Theorem 3.10. *Every graph has a feasible pairing.*

4 Extensions of well-balanced orientation

4.1 r-arc-connected orientations

We can reformulate the well-balanced orientation theorem as follows.

Theorem 4.1. *Let $\mathbf{r} : V \times V \to \mathbb{Z}_0^+$ be a symmetric function. Then a graph $G = (V, E)$ has an $\mathbf{r}$-arc-connected orientation if and only if G is $2\mathbf{r}$-edge-connected.*

In light of this form the following problem is a natural generalization of the well-balanced orientation theorem.

Problem 1 Given an undirected graph $G = (V, E)$ and a function $\mathbf{r} : V \times V \to \mathbb{Z}_0^+$, decide if G has an $\mathbf{r}$-arc-connected orientation.

If the function $\mathbf{r}$ is symmetric then, by Theorem 4.1, there exists an $\mathbf{r}$-arc-connected orientation of G if and only if $\lfloor \frac{\lambda_G(x,y)}{2} \rfloor \geq \mathbf{r}(x, y) \;\; \forall\, x, y \in V$. If the function $\mathbf{r}$ is not symmetric then the problem is difficult.

Theorem 4.2. *[7] Problem 1 is NP-complete.*

4.2 Mixed graphs

A possible way to prove the well-balanced orientation theorem could be to characterize mixed graphs whose undirected edges can be oriented to have a well-balanced orientation of the underlying undirected graph.

Problem 2 Given a mixed graph G', decide whether the undirected edges can be oriented in such a way that the directed graph $\vec{G}$ obtained satisfies $\lambda_{\vec{G}}(x, y) \geq \lfloor \frac{\lambda_G(x,y)}{2} \rfloor \;\; \forall\, (x, y) \in V \times V$, where G is the undirected graph obtained from G' by deleting the orientation of the directed edges.

Question 1. *Is Problem 2 NP-complete?*

5 Splitting off

We have seen in the introduction that splitting off theorems are very useful in the proof of the global or local well-balanced orientation theorem. We also mention that Mader's proof [12] for the well-balanced orientation theorem as well as Frank's proof [5] for Theorem 3.10 uses Mader's splitting off theorem.

The odd vertex pairing theorem would be an easy task if the following was true.

Question 2. *For every 2-edge-connected graph G there exists a pair of adjacent edges rs, st such that for $G_{rt} := G - \{rs, st\} + rt$ we have:*

$$b_G(X) \geq b_{G_{rt}}(X) \quad \forall X \subseteq V. \tag{11}$$

Counter-example 2 Let $G = (U, V; E)$ be the complete bipartite graph $K_{3,4}$. Let us denote the vertices as follows: $U := \{a, b, c, d\}$ and $V := \{x, y, z\}$. By symmetry, $\{rs, st\}$ is either $\{xd, dy\}$ or $\{az, zb\}$. In the first case $b_G(z) = 0 < 2 = b_{G_{xy}}(z)$ and in the second case $b_G(\{a, x, y\}) = 3 < 5 = b_{G_{ab}}(\{a, x, y\})$. In both cases (11) is violated. □

Question 3. *If $\vec{G}$ is a best-balanced orientation of $G := (V + s, E)$ and $\varrho_{\vec{G}}(s) = \delta_{\vec{G}}(s)$ then there exist $rs, st \in A(\vec{G})$ so that for $\vec{G}_{rt} := \vec{G} - \{rs, st\} + rt$ we have*

$$\lambda_{\vec{G}_{rt}}(x, y) \geq \lambda_{\vec{G}}(x, y) \; \forall (x, y) \in V \times V. \tag{12}$$

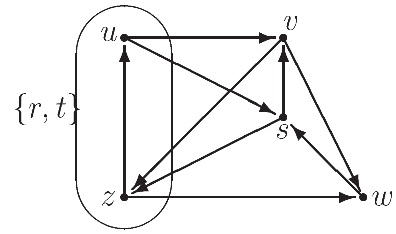

Figure 1

Counter-example 3 Let $G := (V + s, E)$ and $\vec{G} := (V + s, A)$ be defined as follows (see Figure 1): $V := \{u, v, w, z\}$, $E := \{uv, us, uz, vz, vs, vw, ws, wz, zs\}$, $A := \{uv, us, zu, vz, sv, vw, ws, zw, sz\}$. It is easy to check that $\vec{G} \in \mathcal{O}_b(G)$. In particular $\lambda_{\vec{G}}(v, z) = \lambda_{\vec{G}}(z, v) = 2$. Suppose that for some $(r, t) \in \{(u, z), (u, v), (w, z), (w, v)\}$, (12) is satisfied. Then, by (12), $3 = \varrho_{\vec{G}_{rt}}(\{r, t\}) + \delta_{\vec{G}_{rt}}(\{r, t\}) \geq \lambda_{\vec{G}_{rt}}(v, z) + \lambda_{\vec{G}_{rt}}(z, v) \geq \lambda_{\vec{G}}(v, z) + \lambda_{\vec{G}}(z, v) = 4$, a contradiction. □

We note that this is a smaller counter-example for a conjecture of Jackson than Enni's one [1].

Question 4. *If $\vec{G}$ is a best-balanced orientation of $G := (V + s, E)$ and $\varrho_{\vec{G}}(s) = \delta_{\vec{G}}(s)$ then there exist $rs, st \in A(\vec{G})$ so that $\vec{G}_{rt}$ is a best-balanced orientation of G_{rt}.*

Question 4 is an open problem. However the following is true.

Theorem 5.1. *For every pair rs, st of edges of a graph $G := (V + s, E)$ there exists a best-balanced orientation $\vec{G}$ of G so that $rs, st \in A(\vec{G})$ and $\vec{G}_{rt}$ is a best-balanced orientation of G_{rt}.*

Proof. By Theorem 3.10, there exists a feasible pairing M of G. Then M is a pairing of G_{rt} and hence, by Theorem 3.3, $G_{rt} + M$ has an Eulerian orientation $\vec{G}_{rt} + \vec{M}$ so that $\vec{G}_{rt} \in \mathcal{O}_b(G_{rt})$. Wlog. assume that rt is directed as $\vec{rt}$ in $\vec{G}_{rt}$. Then, for $\vec{G} := \vec{G}_{rt} - rt + rs + st$, $\vec{G} + \vec{M}$ is Eulerian, that is, since $M \in P_f(G)$, $\vec{G} \in \mathcal{O}_b(G)$. □

Question 5. *For every graph $G = (V + s, E)$ with $d(s) \geq 4$ there exist $rs, st \in E$ such that for every best-balanced orientation $\vec{G}_{rt}$ of G_{rt}, $\vec{G} := \vec{G}_{rt} - rt + rs + st$ is a best-balanced orientation of G.*

Question 5 is an open problem. If Question 5 had an affirmative answer, it would give us a possible way to prove the best-balanced orientation theorem.

6 Feasible pairing for connectivity functions

A set function $b : V \to \mathbb{R}$ is called **skew-submodular** if for every $X, Y \subseteq V$, at least one of the following two inequalities is satisfied:

$$b(X) + b(Y) \geq b(X \cap Y) + b(X \cup Y), \tag{13}$$
$$b(X) + b(Y) \geq b(X - Y) + b(Y - X). \tag{14}$$

A set function $p(X)$ is called **skew-supermodular** if $-p(X)$ is skew-submodular. We mention that, by [16], $\hat{R}_G(X)$ is skew-supermodular and hence $b_G(X)$ is skew-submodular. A set function b on V is called **crossing submodular** if (13) is satisfied for every $X, Y \subseteq V$ with $X \cap Y,\ X - Y,\ Y - X,\ V - (X \cup Y) \neq \emptyset$.

Question 6. *Let $b : V \to \mathbb{Z}_0^+$ be a symmetric, skew-submodular function with $b(\emptyset) = 0$ and $b(X) \equiv |X \cap T_b|$ mod 2, where $T_b = \{v : b(v)$ is odd$\}$. Then there exists a pairing M on T_b that satisfies*

$$d_M(X) \leq b(X) \qquad \forall\ X \subseteq V. \tag{15}$$

Counter-example 6 Let $b(X)$ be defined on V with $|V| = 6$ as follows $b(X) := 0$ if $X = \emptyset, V$; 1 if $|X|$ is odd and 2 otherwise. It is easy to see that b satisfies all the conditions. Note that $T_b = V$. For any pairing M on T_b, we may choose $X \subset V$ with $d_M(X) = 3$ but then X violates (15). □

Note that, by Theorem 3.10, the answer for Question 6 is affirmative for $b(X) = b_G(X)$.

The problem corresponding to the global case is the following open problem.

Question 7. *Let $b : V \to \mathbb{Z}_0^+$ be a symmetric crossing submodular function with $b(\emptyset) = 0$ and $b(X) \equiv |X \cap T_b|$ mod 2. Then there exists a pairing M on T_b that satisfies (15).*

If Question 7 was true then it would imply the following theorem that can be proved directly.

Theorem 6.1. *Let $G = (V, E)$ be an undirected graph. Let $b : V \to \mathbb{Z}_0^+$ be a crossing submodular function with $b(X) + d(X)$ even for every $X \subseteq V$. Then there exists an orientation $\vec{G}$ of G*

$$f_{\vec{G}}(X) \quad \leq \quad b(X) \quad \forall\ X \subseteq V. \tag{16}$$

Proof. Let $\vec{G} = (V, A)$ be an arbitrary orientation of G. Let $P := \{z \in \mathbb{R}^{|A|} : 0 \leq z_a \leq 1\ \forall a \in A,\ \delta^z_{\vec{G}}(X) - \varrho^z_{\vec{G}}(X) \leq (b(X) - f_{\vec{G}}(X))/2\ \ \forall X \subseteq V\}$. By the modularity of $f_{\vec{G}}(X)$ and by the assumptions, $(b(X) - f_{\vec{G}}(X))/2$ is integral and crossing submodular. Then, by the Edmonds-Giles theorem, P is an integral polyhedron. The vector $\frac{1}{2}\mathbb{1}$ belongs to P because b is non-negative. Then P contains an integral vector $\overline{z}$. Let $\vec{G}'$ be the orientation obtained from $\vec{G}$ by reversing the arcs $a \in A(\vec{G})$ for which $\overline{z}(a) = 1$. Then, since $\overline{z}$ is a $0-1$ vector in P, $f_{\vec{G}'}(X) = \varrho_{\vec{G}'}(X) - \delta_{\vec{G}'}(X) = (\varrho_{\vec{G}}(X) - \varrho^{\overline{z}}_{\vec{G}}(X) + \delta^{\overline{z}}_{\vec{G}}(X)) - (\delta_{\vec{G}}(X) - \delta^{\overline{z}}_{\vec{G}}(X) + \varrho^{\overline{z}}_{\vec{G}}(X)) = f_{\vec{G}}(X) + 2(\delta^{\overline{z}}_{\vec{G}}(X) - \varrho^{\overline{z}}_{\vec{G}}(X)) \leq b(X)\ \ \forall X \subseteq V$, and hence $\vec{G}'$ is the desired orientation. □

Note that if G is $2k$-edge-connected and $b(X) = d_G(X) - 2k\ \forall \emptyset \neq X \subset V$ and $b(\emptyset) = b(V) = 0$, then Theorem 6.1 is equivalent to Theorem 3.7. We remark that Theorem 6.1 also follows from a theorem of Frank [2] on orientations satisfying a G-supermodular function.

Question 8. *Let $d : V \to \mathbb{Z}_0^+$ be a symmetric function that satisfies $d(\emptyset) = 0$ and $\forall X, Y \subseteq V$*

$$d(X) + d(Y) + d(X \triangle Y) = d(X \cap Y) + d(X \cup Y) + d(X - Y) + d(Y - X) \quad (17)$$

$$d(X) + d(Y) - d(X \cup Y) \text{ is even if } X \cap Y = \emptyset. \quad (18)$$

Let $\hat{R} : V \to \mathbb{Z}_0^+$ be an even valued, symmetric, skew-supermodular function. Suppose that $\hat{R}(X) \leq d(X) \ \ \forall X \subseteq V$. Then there exists a pairing M on T_d that satisfies

$$d_M(X) \leq d(X) - \hat{R}(X) \qquad \forall\ X \subseteq V. \quad (19)$$

Counter-example 8 Let $V := \{u, v, w, z\}$, $G := (V, \{uw, uz, vw, vz, wz\})$, $H := (V, \{uv\})$, $d(X) := d_G(X) - d_H(X)$, $\hat{R}(X) := 2$ if $|X \cap \{w, z\}| = 1$ and 0 otherwise. Since for a proper subset X, $d_G(X) \geq 1$ and $d_H(X) \leq 1$, $d(X) \geq 0\ \forall X \subseteq V$. Clearly, d is integer valued and symmetric. d_G and d_H satisfy (17) and (18), consequently d also satisfies them. It is easy to see that $\hat{R}$ satisfies all the conditions. Note that $T_d = V$. Let M be an arbitrary pairing on T_d. Let e be the edge of M incident to w. Let $X := \{u, w\}$ and let $Y := \{v, w\}$. Then e leaves either X or Y but $d(X) - \hat{R}(X) = 0 = d(Y) - \hat{R}(Y)$ so either X or Y violates (19). □

Note that, by Theorem 3.10, the answer for Question 8 is affirmative for $d(X) = d_G(X)$ and $\hat{R}(X) = \hat{R}_G(X)$.

Question 9. *Let $G = (V, E)$ be a graph and $\hat{R} : V \to \mathbb{Z}_0^+$ an even valued, symmetric, skew-supermodular function. Suppose that $\hat{R}(X) \leq d_G(X) \ \ \forall X \subseteq V$. Then there exists a pairing M on T_G that satisfies*

$$d_M(X) \leq d_G(X) - \hat{R}(X) \qquad \forall\ X \subseteq V. \quad (20)$$

Question 9 is an open problem. If $\hat{R}$ satisfies $\hat{R}(X \cup Y) \leq \max\{\hat{R}(X), \hat{R}(Y)\}$ $\forall X, Y \subset V$ then $\hat{R}(X) = \max\{\mathbf{r}(x, y) : x \in X, y \in V - X\}$ for some $\mathbf{r} : V \times V \to \mathbb{Z}_0^+$ and hence, by Theorem 3.10, such a pairing exists.

7 Feasible pairing defining a best-balanced orientation

Nash-Williams' original idea was that every feasible pairing provides a best-balanced orientation. In fact Theorem 3.2 shows that every feasible pairing provides lots of best-balanced orientations. A natural question is whether every best-balanced orientation can be defined by a feasible pairing.

Question 10. *For every best-balanced orientation $\vec{G}$ of G there exists a feasible pairing M and an orientation $\vec{M}$ of M such that $\vec{G} + \vec{M}$ is Eulerian.*

Counter-example 10 Let $G := (V, E)$ and $\vec{G} := (V, A)$ be defined as follows (see Figure 2):

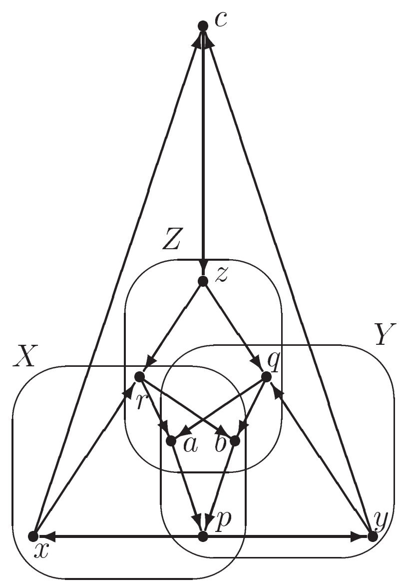

Figure 2

$V := \{a,b,c,p,q,r,x,y,z\}$, $E := \{ap,aq,ar,bp,bq,br,cx,cy,cz,xp,py,yq,qz,zr,rx\}$, $A := \{ap,qa,ra,\ bp,qb,rb,xc,yc,cz,px,py,yq,zq,zr,xr\}$. It is easy to check that $\vec{G} \in \mathcal{O}_b(G)$.

We show that if $M \in \mathcal{P}_f(G)$, then $ab \in M$. Note that $T_G = \{a,b,c,x,y,z\}$. Let $X := \{a,b,p,r,x\}$, $Y := \{a,b,p,q,y\}$, $Z := \{a,b,q,r,z\}$. Note that $d_G(W) = 5$ and $\hat{R}(W) = 4$ hence $b_G(W) = 1$ for $W \in \{X,Y,Z\}$. Then, by (5) and (7), $3{=}b_G(X) + b_G(Y) + b_G(Z) \geq d_M(X) + d_M(Y) + d_M(Z) \geq d_M(X \cap Y \cap Z) + d_M(X - (Y \cup Z)) + d_M(Y - (X \cup Z)) + d_M(Z - (X \cup Y)) = d_M(\{a,b\}) + d_M(x) + d_M(y) + d_M(z) \geq 0+1+1+1 = 3$, so $d_M(\{a,b\}) = 0$ that is $ab \in M$.

Then for every orientation $\vec{M}$ of any feasible pairing M of G either $\delta_{\vec{M}}(a) = 0$ or $\delta_{\vec{M}}(b) = 0$. Then, since $f_{\vec{G}}(a) = f_{\vec{G}}(b) = 1$, $\vec{G} + \vec{M}$ cannot be Eulerian. □

The following theorem shows that the answer for Question 10 is affirmative for global edge-connectivity.

Theorem 7.1. *Let $G := (V,E)$ be a $2k$-edge-connected graph and let $\vec{G} := (V,A)$ be a smooth k-arc-connected orientation of G. Then there is a pairing M of G and an orientation $\vec{M}$ of M so that*

$$d_M(X) \leq d_G(X) - 2k \quad \emptyset \neq \forall X \subset V \text{ and} \tag{21}$$

$$\vec{G} + \vec{M} \text{ is Eulerian.} \tag{22}$$

Proof. By induction on $|A|$. We shall apply the following stronger version of Mader's splitting off theorem [13] due to Frank [6].

Theorem 7.2. *Let $\vec{H} := (U + s, F)$ be k-arc-connected in U. If $\delta_{\vec{H}}(s) - \varrho_{\vec{H}}(s) < \varrho_{\vec{H}}(s) < 2\delta_{\vec{H}}(s)$ then there exist $rs, st \in F$ so that $\vec{H}_{rt} := \vec{H} - \{rs,st\} + rt$ is k-arc-connected in U.*

Case 1 If there is $s \in V$ with $d(s) \geq 2k+2$. Then, by (3) and Theorem 7.2, there exist $rs, st \in A$ so that $\vec{G}_{rt}$ is k-arc-connected in $V-s$. It follows, by the assumption of Case 1 and (3), that $\vec{G}_{rt}$ is k-arc-connected. Note that $T_{G_{rt}} = T_G$. $|A(\vec{G}_{rt})| < |A|$ so by induction there is a pairing M of G_{rt} and an orientation $\vec{M}$ of M so that (21) and (22) are satisfied for (G_{rt}, M) and for $(\vec{G}_{rt}, \vec{M})$ and hence for (G, M) and for $(\vec{G}, \vec{M})$ and we are done.

Case 2 If there is $s \in V$ with $d(s) = 2k$. This case can be handled in the same way as Case 1 but here we have to make a complete splitting off at s.

Case 3 Otherwise, $d(s) = 2k+1$ $\forall s \in V$. Then $T_G = V$. By a result of Mader [14], since there is no vertex v with $\varrho_{\vec{G}}(v) = \delta_{\vec{G}}(v)$, there exists $uv \in A$ such that $\vec{G}' := \vec{G} - uv$ is k-arc-connected. Note that, by the assumption of Case 3 and (3), $\vec{G}'$ satisfies (3). $|A(\vec{G}')| < |A|$ so by induction there is a pairing M' on $T_{G'} = T_G - \{u, v\}$ and an orientation $\vec{M}'$ of M' so that (21) and (22) are satisfied for (G', M') and for $(\vec{G}', \vec{M}')$. Let $M := M' + uv$ and $\vec{M} := \vec{M}' + vu$. Then $\vec{G} + \vec{M} = (\vec{G}' + \vec{M}') + uv + vu$ is Eulerian. Moreover, $\emptyset \neq \forall X \subset V$ either $d_M(X) = d_{M'}(X)$ and $d_G(X) = d_{G'}(X)$ or $d_M(X) = d_{M'}(X) + 1$ and $d_G(X) = d_{G'}(X) + 1$ so (21) is satisfied for G and M. □

8 The structure of feasible pairings

Question 11. *Let $a, b, c, d \in T_G$. If there exist $M_1, M_2 \in \mathcal{P}_f(G)$ such that $ab, cd \in M_1$ and $ad, bc \in M_2$ then there exists $M_3 \in \mathcal{P}_f(G)$ such that $ac, bd \in M_3$.*

Counter-example 11 Let $G := (V, E)$, M_1 and M_2 be defined as follows (see Figure 3): $V := \{a, b, c, d, e, f, g, h\}$, $E := \{ab, ad, ah, bc, bg, cd, cf, de, ef, eh, fg, gh\}$, $M_1 := \{ab, cd, eh, fg\}$, $M_2 := \{ad, bc, ef, gh\}$.

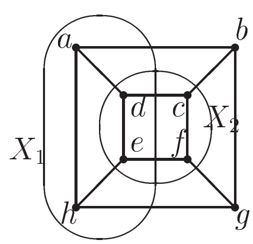

Figure 3

First we show that $M_1, M_2 \in \mathcal{P}_f(G)$. Note that $T_G = V$. For $i = 1, 2$, $G - M_i$ is a Hamilton cycle of G, so $G - M_i$ is 2-edge-connected, that is $2 \leq d_{G-M_i}(X)$ $\emptyset \neq \forall X \subset V$. Since G is 3-regular and 3-edge-connected, $\lambda_G(u, v) = 3$ $\forall u, v \in V$ and hence $\hat{R}_G(X) = 2$ $\emptyset \neq \forall X \subset V$. Then (5) is satisfied since $d_{M_i}(X) \leq d_G(X) - 2 = d_G(X) - \hat{R}_G(X) = b_G(X)$ $\emptyset \neq \forall X \subset V$.

Secondly we show that no feasible pairing of G contains ac and bd. Let M be a pairing of G that contains ac and bd. Let $X_1 := \{a, d, e, h\}$, $X_2 := \{c, d, e, f, \}$.

Note that for $i = 1, 2$, $d_G(X_i) = 4$ and $\hat{R}(X_i) = 2$ hence $b_G(X_i) = 2$. Then, by (6), $d_M(X_1) + d_M(X_2) = d_M(X_1 \cap X_2) + d_M(X_1 \cup X_2) + 2d_M(X_1, X_2) = d_M(\{d, e\}) + d_M(\{b, g\}) + 2d_M(\{a, h\}, \{c, f\}) \geq 2+2+2 = 6$, hence for some $i \in \{1, 2\}$, $d_M(X_i) \geq 3$ thus X_i violates (5) that is M is not feasible. □

9 The polyhedron

In this section we consider the polyhedron of the fractional in-degree vectors of all orientations, k-arc-connected orientations and best-balanced orientations. Though the first two polyhedra are integral we show that the third one is not necessarily integral.

Let $G = (V, E)$ be a graph. Let us introduce the following polyhedra:

$$
\begin{aligned}
P_0 &:= \{x \in \mathbb{R}^{|V|} : x(X) \geq i(X) \quad \forall X \subseteq V, \quad x(V) = |E|\}, \\
P_1 &:= \{x \in \mathbb{R}^{|V|} : x(X) \geq i(X) + k \quad \forall \emptyset \neq X \subset V, \ x(V) = |E|\}, \\
P_2 &:= \{x \in \mathbb{R}^{|V|} : x(X) \geq i(X) + \frac{\hat{R}(X)}{2} \quad \forall X \subseteq V, \ x(V) = |E|, \\
&\qquad \left\lfloor \frac{d(v)}{2} \right\rfloor \leq x(v) \leq \left\lceil \frac{d(v)}{2} \right\rceil \ \forall v \in V\}.
\end{aligned}
$$

It is known that $m \in \mathbb{Z}^{|V|}$ is the in-degree vector of an orientation, of a k-arc-connected orientation, of a best-balanced orientation of G if and only if m belongs to P_0, P_1, P_2. Since $i(X)$ is supermodular and $i(X) + k$ is crossing supermodular, P_0 and P_1 are base-polyhedra and hence they are integral polyhedra, that is P_0 and P_1 coincide with the convex hull of the indegree vectors of all orientations and of k-arc-connected orientations of G.

Question 12. *The polyhedron P_2 is integral.*

Counter-example 12 We show a graph G where the polyhedron P_2 is non-integral. This implies that the convex hull of in-degree vectors of best-balanced orientations is not described completely by these inequalities.

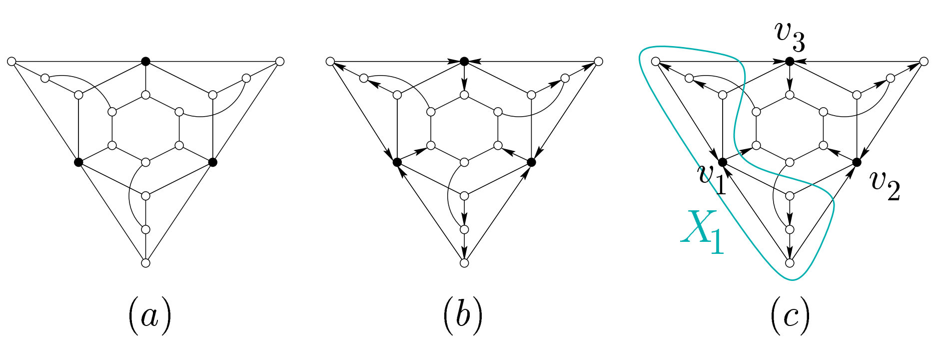

Figure 4

Let G be the graph on Figure 4 (a). It is easy to check that $\hat{R}(X) = 4$ if X separates two black nodes, and $\hat{R}(X) = 2$ otherwise. Figure 4 (b) represents a 'fractional orientation' x of G, where an undirected edge uv means that $x(uv) = \frac{1}{2}$ and $x(vu) = \frac{1}{2}$ (x can be seen as a capacity function on the underlying symmetric digraph). Let m_x denote the in-degree-vector of the fractional orientation x. This clearly satisfies $\lfloor \frac{d_G(v)}{2} \rfloor \le m_x(v) \le \lceil \frac{d_G(v)}{2} \rceil$ for every $v \in V$. It is also easy to verify that x contains a flow of size 1 from any node to any other, and it contains a flow of size 2 from any black node to any other black node. This implies that $\varrho_x(X) \ge \frac{\hat{R}(X)}{2}$ for every $X \subseteq V$, so m_x is in the polyhedron P_2.

In order to prove that P_2 is non-integral, it suffices to show a set $Z \subseteq V$ such that $m_x(Z)$ is non-integral and it minimizes $\{m(Z) : m \in P_2\}$. Let X_1 be the set indicated on Figure 4 (c), and let X_2, X_3 be the two similar sets obtained by symmetry, such that $v_2 \in X_2$ and $v_3 \in X_3$. We can observe that $\varrho_x(X_i) = 2 = \hat{R}(X_i)/2$, and $\varrho_x(V - v_i) = 2 = \hat{R}(V - v_i)/2$ $(i = 1, 2, 3)$. This means that $m_x(X_i)$ minimizes $\{m(X_i) : m \in P_2\}$, and $m_x(v_i)$ maximizes $\{m(v_i) : m \in P_2\}$, thus $m_x(X_i - v_i)$ minimizes $\{m(X_i - v_i) : m \in P_2\}$ $(i = 1, 2, 3)$.

Let $Z = X_1 \cup X_2 \cup X_3 - \{v_1, v_2, v_3\}$; then $\chi^Z \equiv \frac{1}{2}\sum_{i=1}^{3} \chi^{X_i - v_i}$, where χ^X is the characteristic function of the set X. It follows that $m_x(Z)$ minimizes $\{m(Z) : m \in P_2\}$. But $m_x(Z) = \frac{21}{2}$, hence the polyhedron P_2 cannot be integral. □

The cutting plane that cuts the non-integral vector m_x from the convex hull of the in-degree vectors of best-balanced orientations can be read out from the above proof.

10 Matroid property

Question 13. *Let $l, u : V \to \mathbb{Z}_0^+$ such that $l(v) \le u(v)$ for all $v \in V$. Then there exists $\vec{G} \in \mathcal{O}_w(G)$ such that $l(v) \le \varrho_{\vec{G}}(v) \le u(v)$ $\forall v \in V$ if and only if there exist $\vec{G}^1, \vec{G}^2 \in \mathcal{O}_w(G)$ such that $l(v) \le \varrho_{\vec{G}^1}(v)$ $\forall v \in V$ and $\varrho_{\vec{G}^2}(v) \le u(v)$ $\forall v \in V$.*

Counter-example 13 Let G, $\vec{G}^1 := \vec{G}, \vec{G}^2 := \overleftarrow{G} \in \mathcal{O}_w(G)$, X, Y and Z as in Figure 2. Let the functions l and u be defined as follows: $l(a) = l(b) = 2$ and $l(t) = \lfloor \frac{d_G(t)}{2} \rfloor$ $\forall t \in V - a - b$, $u(c) = 1$ and $u(t) = \lceil \frac{d_G(t)}{2} \rceil$ $\forall t \in V - c$. Then $l(v) \le \varrho_{\vec{G}^1}(v)$ $\forall v \in V$ and $\varrho_{\vec{G}^2}(v) \le u(v)$ $\forall v \in V$. Let $\vec{G}^3 \in \mathcal{O}_w(G)$ such that $l(v) \le \varrho_{\vec{G}^3}(v)$ $\forall v \in V$. Recall that $b_G(X) = b_G(Y) = b_G(Z) = 1$. Then, by Claim 3.1, $1 = b_G(X) \ge f_{\vec{G}^3}(X) = f_{\vec{G}^3}(x) + f_{\vec{G}^3}(p) + f_{\vec{G}^3}(a) + f_{\vec{G}^3}(b) + f_{\vec{G}^3}(r) = f_{\vec{G}^3}(x) + 0 + 1 + 1 + 0$, so $f_{\vec{G}^3}(x) \le -1$ and hence $f_{\vec{G}^3}(x) = -1$. Similarly, $f_{\vec{G}^3}(y) = f_{\vec{G}^3}(z) = -1$. Then, since $f_{\vec{G}^3}(V) = 0$, $f_{\vec{G}^3}(c) = 1$, that is $\varrho_{\vec{G}^3}(c) = 2 > 1 = u(c)$. Thus there is no well-balanced orientation of G whose in-degree function satisfies both the lower and upper bounds. □

Question 13 is valid for the global case by Frank [2]. This follows from the facts that the in-degree vectors of k-arc-connected orientations form a base-polyhedron and for such polyhedra the linkage property holds.

Let $\vec{G}$ be an orientation of G. Let $T^+_{\vec{G}} := \{v \in V(G) : \varrho_{\vec{G}}(v) > \delta_{\vec{G}}(v)\}$. Note that if $\vec{G}$ is smooth, then $|T^+_{\vec{G}}| = |T_G|/2$.

Question 14. *$\mathcal{T} := \{T^+_{\vec{G}} : \vec{G} \in \mathcal{O}_b(G)\}$ is the base family of a matroid.*

Counter-example 14 Let G, $\vec{G} \in \mathcal{O}_b(G)$, X, Y and Z be as in Figure 2. Then $\overleftarrow{G} \in \mathcal{O}_b(G)$ hence $B_1 := \{a, b, c\}$ and $B_2 := \{x, y, z\}$ are in $\mathcal{T}$. Suppose that $\mathcal{T}$ is the base family of a matroid. Then for $c \in B_1 - B_2$ there must exist $u \in B_2 - B_1$ such that $B_1 - \{c\} + \{u\} \in \mathcal{T}$, by symmetry we may suppose that $\{a, b, x\} \in \mathcal{T}$. Then there exists $\vec{G}' \in \mathcal{O}_b(G)$ so that $T^+_{\vec{G}'} = \{a, b, x\}$. Whence, by (10) and (1), $1{=}b_G(X){\geq}f_{\vec{G}'}(X){=}\sum_{u\in X} f_{\vec{G}'}(u){=}3$, contradiction. □

The answer for Question 14 is affirmative for the global case by the proof of Frank [3].

Question 15. *Let $\vec{G}^a, \vec{G}^b \in \mathcal{O}_w(G)$. Then there exist $\vec{G}^0 = \vec{G}^a, \vec{G}^1, ..., \vec{G}^l = \vec{G}^b \in \mathcal{O}_w(G)$ such that $\vec{G}^k$ is obtained from $\vec{G}^{k-1}$ by reversing a directed path or a directed cycle ($1 \leq k \leq l$).*

Question 15 is an open problem. We mention that, by Frank [3], the answer for Question 15 is affirmative for global edge-connectivity.

11 Simultaneous orientations

In this section we consider some possible generalizations of Theorem 3.9 and Theorem 3.4. Here we consider the statements of these theorems as assuring simultaneous (compatible) best-balanced orientations of some graphs.

The first two questions correspond to the global and local cases related to Theorem 3.9, i.e., the subgraph-chain-property.

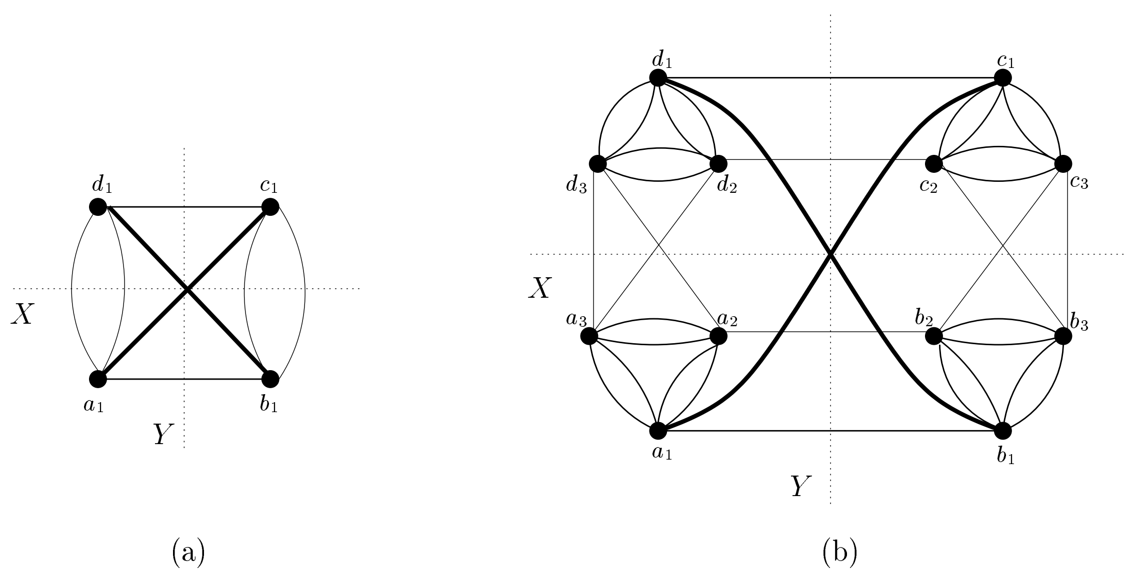

Figure 5

Question 16. *Let G_3 be a subgraph of G_2 and G_2 a subgraph of G_1. Then, for $i = 1,2,3$, there exists an orientation $\vec{G}_i$ of G_i such that $\vec{G}_j$ is a restriction of $\vec{G}_i$ if $1 \le i < j \le 3$ and*

(a) Local case: $\vec{G}_i \in \mathcal{O}_w(G_i)$.

(b) Global case: *$\vec{G}_i$ is a k_i-arc-connected orientation of G_i provided that G_i is $2k_i$-edge-connected.*

Counter-examples 16 Let $G_i := (V_i, E_i)$ $(i = 1,2,3)$ be defined as in Figure 5, that is

(a) $V_1 = V_2 = V_3 := \{a_1, b_1, c_1, d_1\}, E_3 := \{a_1d_1, a_1d_1, b_1c_1, b_1c_1\}, E_2 := E_3 \cup \{a_1b_1, c_1d_1\}, E_1 := E_2 \cup \{a_1c_1, b_1d_1\}$. Let $X := \{a_1, b_1\}, Y := \{a_1, d_1\}$.

(b) $V_3 := \{a_2, a_3, b_2, b_3, c_2, c_3, d_2, d_3\}$, $E_3 := \{a_2b_2, b_2c_3, c_3b_3, b_3c_2, c_2d_2, d_2a_3, a_3d_3, d_3a_2\}$, $V_2 := V_3 \cup \{a_1, b_1, c_1, d_1\}$, $E_2 := E_3 \cup \{a_1b_1, c_1d_1\} \cup \{x_1x_2, x_1x_2, x_2x_3, x_2x_3, x_3x_1, x_3x_1 : x \in \{a,b,c,d\}\}$ $V_1 := V_2, E_1 := E_2 \cup \{a_1c_1, b_1d_1\}$. Let $X := \{a_1, a_2, a_3, b_1, b_2, b_3\}, Y := \{a_1, a_2, a_3, d_1, d_2, d_3\}$.

We prove at the same time for (a) and (b) that the required orientations do not exist. Suppose that they do exist. First we observe that $(*)$ if $\vec{H}$ is an l-arc-connected orientation of a $2l$-edge-connected graph H and $d_H(X) = 2l$ for some $X \subset V$, then $f_{\vec{H}}(X) = 0$. Each connected component of G_1 and of G_3 is $2l$-edge-connected and $2l$-regular for some l, so by $(*)$, $\vec{G}_1$ and $\vec{G}_3$ are Eulerian orientations of G_1 and G_3, whence, by (1), $f_{\vec{G}_1}(X) = 0 = f_{\vec{G}_1}(Y)$ and $f_{\vec{G}_3}(X) = 0$. G_2 is $2k$-edge-connected and $d_{G_2}(Y) = 2k$, so by $(*)$, $f_{\vec{G}_2}(Y) = 0$. Then $f_{\vec{G}_1 - \vec{G}_2}(X) = f_{\vec{G}_1 - \vec{G}_3}(X) = f_{\vec{G}_1}(X) - f_{\vec{G}_3}(X) = 0$ and $f_{\vec{G}_1 - \vec{G}_2}(Y) = f_{\vec{G}_1}(Y) - f_{\vec{G}_2}(Y) = 0$. Note that $E(G_1 - G_2) = E_1 - E_2 = \{a_1c_1, b_1d_1\}$ and $a_1 \in X \cap Y, c_1 \in V - (X \cup Y), b_1 \in X - Y, d_1 \in Y - X$, contradiction. □

Regarding general simultaneous orientations, we may ask the following question:

Question 17. *Given two graphs (neither edge-disjoint nor containing each other), is there a good characterization for having simultaneous best-balanced orientations?*

The next theorem and corollary shows that this problem is NP-complete even for Eulerian graphs.

Theorem 11.1. *Deciding whether two Eulerian graphs, $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ have Eulerian orientations that agree on the common edges $E_1 \cap E_2$, is NP-complete.*

Proof. The problem is clearly in NP. For the completeness we show a reduction from ONE-IN-THREE 3SAT (see [9], Problem LO4). For a given 3SAT formula, n denotes the number of variables, the clauses are denoted by $C_1, \ldots, C_m$, and J_i denotes the set of indices of the clauses that contain the variable x_i.

Construct first the graph $G_1 = (V_1, E_1)$ as follows. Each connected component $G_1^i = (V_1^i, E_1^i)$ of G_1 corresponds to a clause C_i namely V_1^i contains the vertices

$\{C_i, C'_i\}$ and the vertices $\{x^i_j, \overline{x}^i_j : i \in J_j\}$ and E^i_1 contains the edge $C_iC'_i$, the edges $\{x^i_j\overline{x}^i_j : i \in J_j\}$, the edges $\{C_ix^i_j, C'_i\overline{x}^i_j\}$ if x_j occurs in C_i and the edges $\{C_i\overline{x}^i_j, C'_ix^i_j\}$ if $\overline{x}_j$ occurs in C_i. Note that vertices corresponding to literals are of degree two and vertices corresponding to clauses are of degree four.

The graph $G_2 = (V_2, E_2)$ is constructed in such a way that each connected component of G_2 is a cycle. One cycle has color classes $\{C_i : 1 \le i \le m\}$ and $\{C'_i : 1 \le i \le m\}$ and contains the edges $\{C_iC'_i : 1 \le i \le m\}$. We also have cycles for $1 \le i \le n$ with color classes $\{x^j_i : j \in J_i\}$ and $\{\overline{x}^j_i : j \in J_i\}$ containing the edges $\{x^j_i\overline{x}^j_i : j \in J_i\}$.

First we claim that if there is a truth assignment such that in each clause exactly one literal is TRUE then the required Eulerian orientations exist. Orient first G_2, it is enough to declare the orientation of one edge in each cycle. Let C'_1C_1 be oriented from C'_1 to C_1, and for each i let the edge $x^j_i\overline{x}^j_i$ (for any $j \in J_i$) be oriented from the FALSE value to the TRUE value. Now G_1 has a unique orientation that extends the orientation of the common edges and that makes each vertex of degree two Eulerian. Since each clause C_i contains exactly one literal of value TRUE, this orientation is Eulerian.

On the other hand suppose that we have Eulerian orientations $\vec{G}_1$ and $\vec{G}_2$ that agree on the common edges. If edge $C_1C'_1$ is oriented from C_1 to C'_1 then reverse both Eulerian orientations. The Eulerian orientation $\vec{G}_2$ first ensures that C'_iC_i is a directed edge for all i. Second, it also ensures that for all i either $x^j_i\overline{x}^j_i$ is a directed edge for all j, or $\overline{x}^j_ix^j_i$ is a directed edge for all j. Assign the value TRUE to variable x_i iff $\overline{x}^j_ix^j_i$ is a directed edge. We claim that this assignment makes TRUE exactly one literal in each clause. Indeed, from the three edges between C_i and the three literal-copies exactly one is directed towards C_i, and exactly the corresponding literal has value TRUE. □

We remark that another construction can be made by adding some extra vertices, in which both graphs are connected.

Corollary 1. *Deciding whether two graphs have simultaneous best-balanced orientations is NP-complete.*

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