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Augmenting undirected node-connectivity by one

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Abstract

We present a min-max formula for the problem of augmenting the node-connectivity of a graph by one and give a polynomial time algorithm for finding an optimal solution. We also solve the minimum cost version for node-induced cost functions.

1 Introduction

An undirected graph $G = (V, E)$ is **k-node-connected**, or shortly, **k-connected** if $|V| \geq k + 1$ and after the deletion of any set of at most $k - 1$ nodes, the remaining graph is still connected. By Menger's well-known theorem, a graph is k-connected if and only if it contains k openly disjoint paths between any two nodes. The node connectivity augmentation problem consists of finding a minimum number of edges whose addition to a given graph G results in a k-connected graph. The complexity of this problem is a longstanding open question. In this paper we give a min-max formula and a polynomial time algorithm for augmenting connectivity by one, the special case when the input graph G is already $(k-1)$-connected.

Besides node-connectivity, one may study edge-connectivity as well, and both augmentation problems may also be asked for directed graphs. The other three among these four basic connectivity augmentation problems were solved beforehand: undirected edge-connectivity by Watanabe and Nakamura [16], directed edge-connectivity by Frank [5], and directed node-connectivity by Frank and Jordán [6].

For the undirected node-connectivity version, the best previously known result is due to Jackson and Jordán [10]. They gave a polynomial time algorithm for finding an optimal augmentation for any fixed k. The running time is bounded by $O(|V|^5 + f(k)|V|^3)$, where $f(k)$ is an exponential function of k. For some special classes of graphs they prove even stronger results: for example, the running time of the algorithm is a polynomial of $|V|$ if the minimum degree is at least $2k - 2$. Liberman and Nutov [13] gave a polynomial time algorithm for augmenting connectivity by one for the graphs

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[⋆] Dedicated to András Frank on the occasion of his 60th birthday.

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satisfying the following property: there exists a set $A \subseteq V$ with $|A| = k-1$ so that $G - A$ has at least k connected components.

Prior to these results, cases $k = 2, 3, 4$ were solved by Eswaran and Tarjan [3], Watanabe and Nakamura [17] and Hsu [8], respectively. For the case $k = |V|-2$, observe that for a graph G, connectivity augmentation for $k = |V| - 2$ is equivalent to finding a maximum matching in the complement graph of G. Similarly, the case $k = |V| - 3$ is equivalent to finding a maximum square-free 2-matching in the complement. This is still open, however, augmentation by one (or equivalently, finding a maximum square-free 2-matching in a subcubic graph) was recently solved by Bérczi and Kobayashi [1].

It is straightforward to give a 2-approximation for connectivity augmentation by bidirecting all edges and using that directed node-connectivity may be augmented optimally based on [6]. For augmenting connectivity by one, Jordán [11, 12] gave an algorithm finding an augmenting edge set larger than the optimum by at most $\left\lceil \frac{k-2}{2} \right\rceil$. Jackson and Jordán [9] extended this result for general connecitivity augmentation with an additive term of $\left\lceil \frac{k(k-1)+4}{2} \right\rceil$. (The running time of this algorithm is polynomial also in k.)

Let us now formulate our theorem, conjectured by Jordán [2] in 1994. In the $(k-1)$-connected graph $G = (V, E)$, a subpartition $X = (X_1, \ldots, X_t)$ of V with $t \geq 2$ is called a **clump** if $|V - \cup X_i| = k - 1$ and $d(X_i, X_j) = 0$ for any $i \neq j$. The sets X_i are called the **pieces** of X while $|X|$ is used to denote t, the number of pieces. If $t = 2$ then X is a **small clump**, while for $t \geq 3$ it is a **large clump**. An edge $uv \in V^2$ **connects** X if u and v lie in different pieces of X. Two clumps are said to be **independent** if there is no edge $uv \in V^2$ connecting both.

A **bush** $\mathcal{B}$ is a set of pairwise different small clumps, so that each edge in V^2 connects at most two of them. A **shrub** is a set consisting of pairwise independent (possibly large) clumps. For a bush $\mathcal{B}$ let $def(\mathcal{B}) = \left\lceil \frac{|\mathcal{B}|}{2} \right\rceil$, and for a shrub $\mathcal{S}$ let $def(\mathcal{S}) = \sum_{K \in \mathcal{S}} (|K| - 1)$.

A **grove** is a set consisting of some (possibly zero) bushes and one (possibly empty) shrub, so that the clumps belonging to different bushes are independent, and a clump belonging to a bush is independent from all clumps belonging to the shrub. For a grove Π consisting of the shrub $\mathcal{B}_0$ and bushes $\mathcal{B}_1, \ldots, \mathcal{B}_\ell$, let $def(\Pi) = \sum_i def(\mathcal{B}_i)$. For a $(k-1)$-connected graph $G = (V, E)$, let $\tau(G)$ denote the minimum number of edges whose addition makes G k-connected, and let $\nu(G)$ denote the maximum value of $def(\Pi)$ over all groves Π.

Theorem 1.1. *For a $(k-1)$-connected graph $G = (V, E)$ with $|V| \geq k+1$, $\nu(G) = \tau(G)$.*

The theorem is illustrated on Figure 1. Both the proof and the algorithm are motivated by the algorithm given by Frank and the author [7] for augmenting directed node-connectivity by one. Let us now state the min-max formula for this problem. In a digraph $D = (V, A)$, an ordered pair (X^-, X^+) of disjoint non-empty subsets of V is called a **one-way pair** if $|V - (X^- \cup X^+)| = k - 1$ and there is no arc in A from X^- to X^+. $uv \in V^2$ **covers** (X^-, X^+) if $u \in X^-$, $v \in X^+$ and two pairs are **independent** if they cannot be covered by the same arc.

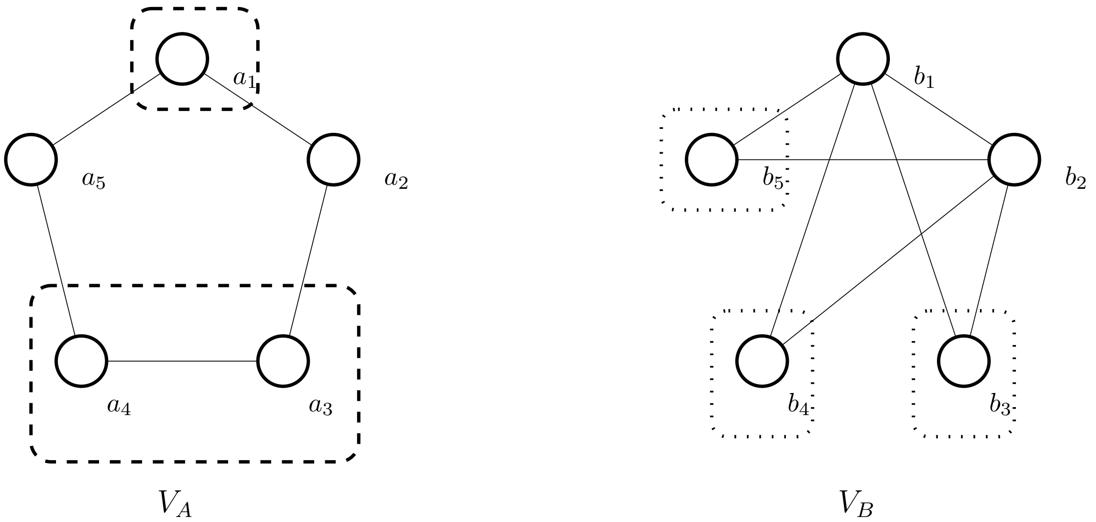

Figure 1: Let G be the graph on the figure with the addition of a complete bipartite graph between V_A and V_B and let $k = 8$. G is 7-connected, and it can be made 8-connected by the addition of the edge set $\{a_1a_3, a_2a_4, a_3a_5, b_3b_4, b_4b_5\}$. Two clumps $(\{a_1\}, \{a_3, a_4\})$ and $(\{b_3\}, \{b_4\}, \{b_5\})$ are shown on the figure. A grove Π with $def(\Pi) = 5$ consists of the shrub $\mathcal{B}_0$ and the bush $\mathcal{B}_1$ with $\mathcal{B}_0 = \{(\{b_3\}, \{b_4\}, \{b_5\})\}$, and $\mathcal{B}_1 = \{(\{a_1\}, \{a_3, a_4\}), (\{a_2\}, \{a_4, a_5\}), (\{a_3\}, \{a_5, a_1\}), (\{a_4\}, \{a_1, a_2\}), (\{a_5\}, \{a_2, a_3\})\}$.

Theorem 1.2 ([6]). *The minimum number of arcs whose addition to the* $(k-1)$*-connected digraph* $D = (V, A)$ *with* $|V| \geq k+1$ *results in a* k*-connected digraph equals the maximum number of pairwise independent one-way pairs.*

We give a brief outline of the argument of [7]. A natural partial order $\preceq$ may be defined on the set of one-way pairs. A subset $\mathcal{K}$ of one-way pairs is called **cross-free** if any two non-independent pairs in $\mathcal{K}$ are comparable with respect to $\preceq$; such a $\mathcal{K}$ maximal for inclusion is called a **skeleton**. The two main ingredients of the proof are as follows: (i) for a cross-free $\mathcal{K}$, the maximum number of pairwise independent one-way pairs in $\mathcal{K}$ and an arc set F of the same cardinality covering all pairs in $\mathcal{K}$ may be determined using Dilworth's theorem on finding a maximum antichain and a minimum chain cover of a poset; (ii) an arc set F covering all one-way pairs in a skeleton $\mathcal{K}$ may be transformed to an arc set F' of the same cardinality covering every one-way pair in D.

Our proof for Theorem 1.1 will roughly follow the same lines. Although no natural partial order can be defined on the set of clumps, nestedness may be defined as a natural notion analogous to comparability: a cross-free system will be a set of clumps so that any two non-independent clumps are nested. For a cross-free $\mathcal{K}$ we will be able to determine an edge set F covering all clumps in $\mathcal{K}$ and a grove consisting of a shrub and bushes of clumps in $\mathcal{K}$ with deficiency $|F|$. Instead of Dilworth's theorem, we apply a reduction to Fleiner's theorem [4] on covering a symmetric poset by symmetric chains. For part (ii), the argument of [7] may be adapted with minor modifications.

While Dilworth's theorem can be derived from the Kőnig-Hall theorem on finding a maximum matching in bipartite graphs, Fleiner's theorem may be deduced from the Berge-Tutte theorem on the size of a maximum matching in general graphs. The relation between the directed and undirected connectivity augmentation problems is somewhat analogous: for example, the formula in Theorem 1.1 involves parity. This is the reason why the strikingly simple proof of Frank and Jordán for Theorem 1.2 cannot be adapted for the undirected case.

Another difficulty is that in contrast to one-way pairs, clumps may have more than two pieces. Fortunately, it turns out that the clumps of size at least three are nested with every other clump they are dependent with. Therefore, although such clumps will cause certain difficulties in the first part of the proof, they play only little role in the second part.

For the algorithm, we are going to construct a subroutine determining the dual optimum value $\nu(G)$ for a $(k-1)$-connected graph G. This gives rise to the following simple algorithm for finding an optimal augmenting edge set based on Theorem 1.1. First compute $\nu(G)$, and let $J = V^2 - E$ be the complement of E. In each step choose an edge $e \in J$, compute $\nu(G+e)$, and remove e from J. If $\nu(G+e) = \nu(G) - 1$ then add e to E, otherwise keep the same G. Note that Theorem 1.1 ensures the existence of an edge e with $\nu(G+e) = \nu(G) - 1$.

The paper is organized as follows. We introduce the necessary concepts and prove some basic claims in Section 2. Section 3 contains the proof of Theorem 1.1, while the algorithm is given in Section 4. Finally, Section 5 describes the minimum cost version for node-induced cost functions, and Section 6 deals with possible extensions for general connectivity-augmentation. We remark that the rooted connectivity augmentation problem can be solved using the same techniques; this will be contained in a later version of this paper.

2 Preliminaries

For the undirected graph $G = (V, E)$ and a subset $A \subseteq V$, $d(A) = d_G(A) = d_E(A)$ denotes the degree of A, and $N(A) = N_G(A)$ the set of neighbours of A, that is, $\{v \in V - A, \exists u \in A, uv \in E\}$. A^* is used to denote $V - (A \cup N(A))$. For subsets $A, B \subseteq V$, $d(A, B)$ is the number of edges between $A - B$ and $B - A$. For $u \in V$, u sometimes refers to the set $\{u\}$, for example, $A + v$ and $A - v$ denote the sets $A \cup \{v\}$ and $B - \{v\}$, respectively. Similar notation is used concerning edges. For two sets A and B, $A \subset B$ means that A is a proper subset of B. Let $n = |V|$, the number of nodes.

First we give a brief motivation for the concepts related to clumps. In a $(k-1)$-connected graph G, we may have sets $A \subset V$ with $|A| = k-1$, so that $V - A$ has $t \geq 2$ connected components. The components of $V - A$ form a clump, and any partition of these components to at least two sets forms a clump as well, since in the definition, the pieces are not required to be connected. In order to make G k-connected, we have to add at least $t-1$ edges between different components of $V - Z$. For $t = 2$, an arbitrary edge suffices between the two components, however for $t \geq 3$ not any such edge set of cardinality at least $t-1$ is suitable. For $t \geq 3$, such the set A is often called a shredder in the literature.

Now we list some definitions. For a clump $X = (X_1, X_2, \dots, X_t)$, let $N_X = V - \cup_i X_i$. X is called **basic** if all pieces X_i are connected. The clump Y is **derived** from the basic clump X if each piece of Y is the union of some pieces of X. By $D(X)$ we mean the set of all clumps derived from X, while $D_2(X)$ is used for the set of small clumps derived from X. Let $\mathcal{C}$ denote the set of all basic clumps. For a set $\mathcal{F} \subseteq \mathcal{C}$, $D(\mathcal{F})$ denotes the union of the sets $D(X)$ with $X \in \mathcal{F}$. The clumps being in the same $D(X)$ can easily be characterized (see e.g. [11, 12, 13]):

Claim 2.1. *(i) Two clumps X and Y are derived from the same basic clump if and only if $N_X = N_Y$. (ii) If two basic clumps X and Y have a piece in common, then $X = Y$.* ☐

For a clump X and an edge set F, let F/X be the graph obtained from (V, F) by deleting N_X and shrinking the components X_i to single nodes. Let $c_F(X)$ denote the number of connected components of F/X. F **covers** X if F/X is connected, that is, $c_F(X) = 1$. Observe that if F is an augmenting edge set if and only if F covers all clumps X, and we need at least $|X| - 1$ edges of F between different components of X in order to cover X. If X is a small clump then F covers X if and only if F connects X. F covers (connects) $\mathcal{H} \subseteq D(\mathcal{C})$ if it covers (connects) all clumps in $\mathcal{H}$. The following simple claim shows that in order to cover a set $\mathcal{F}$ of clumps, it suffices to connect every small clump derived from the members of $\mathcal{F}$.

Claim 2.2. *For an edge set $F \subseteq V^2$ and $\mathcal{F} \subseteq \mathcal{C}$, the following three statements are equivalent: (i) F covers $\mathcal{F}$; (ii) F covers $D(\mathcal{F})$; and (iii) F connects $D_2(\mathcal{F})$.* ☐

We have already defined when two clumps are independent: if no edge in V^2 connects both. Two clumps are **dependent**, if they are not independent. In the rest of the section we introduce the concept of nestedness of clumps and uncrossing for dependent clumps, and furthermore we define crossing and cross-free subsets of clumps. The reader may find useful to compare these to the concepts related to one-way pairs in case of directed connectivity augmentation as in [6]. These will also be defined later in this section as we will also use them directly. A major difference between the undirected and directed setting is that in the directed case, a natural partial order can be defined for the one-way pairs, which cannot be done for clumps. Nestedness will be the natural analogue of comparability for clumps.

We say that two clumps $X = (X_1, \dots, X_t)$ and $Y = (Y_1, \dots, Y_h)$ are **nested** if $X = Y$ or for some $1 \le a \le t$ and $1 \le b \le h$, $Y_i \subset X_a$ for all $i \neq b$ and $X_j \subset Y_b$ for all $j \neq a$. We call X_a the **dominant piece** of X with respect to Y, and Y_b the dominant piece of Y w.r.t X. The following important lemma shows that a large basic clump is automatically nested with any other basic clump (see also in [13]).

Lemma 2.3. *Assume X is a large basic clump, and Y is an arbitrary basic clump. If X and Y are dependent then X and Y are nested.*

To prove this, first we need two simple claims.

Claim 2.4. *For the basic clumps $X = (X_1, \dots, X_t)$ and $Y = (Y_1, \dots, Y_h)$, $X_i \cap N_Y = \emptyset$ implies $X_i \subseteq Y_j$ for some $1 \le j \le h$.* ☐

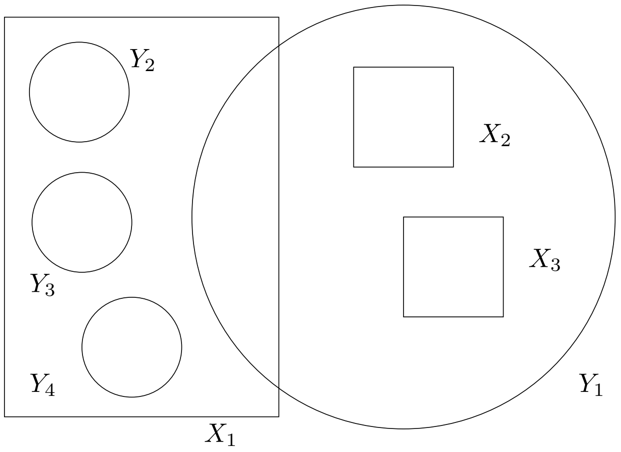

Figure 2: The nested clumps $X = (X_1, X_2, X_3)$ and $Y = (Y_1, Y_2, Y_3, Y_4)$ with dominant pieces X_1 and Y_1.

Claim 2.5. *Let* $X = (X_1, \dots, X_t)$ *and* $Y = (Y_1, \dots, Y_h)$ *be two different clumps both basic or both small. If* $X_s \subset Y_z$ *for some* $1 \le s \le t$, $1 \le z \le h$, *then* X *and* Y *are nested with* Y_z *being the dominant piece of* Y *w.r.t* X.

Proof. Consider an $\ell \neq z$. $X_s \subseteq Y_z$ implies $d(X_s, Y_\ell) = 0$ for $\ell \neq z$, thus $Y_\ell \cap N_X = \emptyset$. Hence $Y_\ell \subseteq X_w$ for some $w \neq s$ follows either by Claim 2.4 or by $t = 2$. We claim that this w is always the same independently from the choice of ℓ. Indeed, assume that for some $\ell' \notin \{z, \ell\}$, $Y_{\ell'} \subseteq X_{w'}$ with $w' \neq w$.

The same argument applied with changing the role of X and Y (by making use of $Y_\ell \subseteq X_w$) shows that $X_{w'} \subseteq Y_j$ for some j, giving $Y_{\ell'} \subseteq Y_j$, a contradiction. $X_i \subseteq Y_z$ for $i \neq w$ may be proved by changing the role of X and Y again. Thus X and Y are nested with dominant pieces X_w and Y_z. ☐

Proof of Lemma 2.3. The dependence implies $X_1 \cap Y_1 \neq \emptyset$, $X_2 \cap Y_2 \neq \emptyset$ by possibly changing the indices. Let $x_i = |N_Y \cap X_i|$, $y_i = |N_X \cap Y_i|$, $n_0 = |N_X \cap N_Y|$. Then $k - 1 \le |N(X_1 \cap Y_1)| \le n_0 + x_1 + y_1$. Since $k - 1 = |N_Y| = n_0 + \sum_i y_i$ this implies $\sum_{i \neq 1} y_i \le x_1$ and similarly $\sum_{i \neq 1} x_i \le y_1$. The same argument for $X_2 \cap Y_2$ gives $\sum_{i \neq 2} y_i \le x_2$ and $\sum_{i \neq 2} x_i \le y_2$.

Thus we have $x_i = y_i = 0$ for $i \ge 3$. This gives $X_3 \cap N_Y = \emptyset$ hence $X_3 \subseteq Y_i$ for some i by Claim 2.4. The nestedness of X and Y follows by the previous claim. ☐

The notion of one-way pairs from the directed connectivity augmentation setting will also be used. A **one-way pair** $K = (K^-, K^+)$ is an ordered pair of disjoint sets with $|V - (K^- \cup K^+)| = k - 1$ with $d(K^-, K^+) = 0$, or equivalently, the subpartition consisting of K^- and K^+ forms a (small) clump. K^- is called the **tail**, while K^+ the **head** of K. For each small clump X, there are two corresponding one-way pairs, called the orientations of X. For a large clump X, we mean by the orientations of X the **orientations** of the small clumps in $D_2(X)$.

For a one-way pair K, $\underline{K}$ denotes the corresponding small clump. An arc uv **covers** the one-way pair $K = (K^-, K^+)$, if $u \in K^-$, $v \in K^+$. Note that if uv covers K, then

vu does not cover it. If an edge uv connects a small clump X, then uv covers exactly one of its two orientations (in the directed sense). For the one-way pair $K = (K^-, K^+)$, its reverse is $\overleftarrow{K} = (K^+, K^-)$.

Two one-way pairs are **independent**, if no (directed) edge covers both or equivalently, if either their tails or their heads are disjoint. Two non independent set pairs are called **dependent**. Let us define a partial order $\preceq$ on the one-way pairs as follows. For the one-way pairs $K = (K^-, K^+)$ and $L = (L^-, L^+)$, $K \preceq L$ if $K^- \subseteq L^-$, $K^+ \supseteq L^+$. For dependent one-way pairs K and L, let $K \wedge L = (K^- \cap L^-, K^+ \cup L^+)$ and $K \vee L = (K^- \cup L^-, K^+ \cap L^+)$. A simple argument (e.g. in [6]) shows that these are also one-way pairs.

Take two dependent small clumps $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$. We say that their orientations L_X and L_Y are **compatible** if they are dependent one-way pairs. Clearly, any two dependent one-way pairs admit compatible orientations, and if L_X and L_Y are compatible, then so are $\overleftarrow{L_X}$ and $\overleftarrow{L_Y}$. X and Y are said to be **simply dependent** if for an orientation L_X of X, there is exactly one compatible orientation L_Y of Y and **strongly dependent** if both possible choices of L_Y are compatible with L_X. (Note that the definition does not depend on the choice of L_X). X and Y are strongly dependent if and only if $X_i \cap Y_j \neq \emptyset$ for every $i = 1, 2$, $j = 1, 2$. The following claim is easy to see.

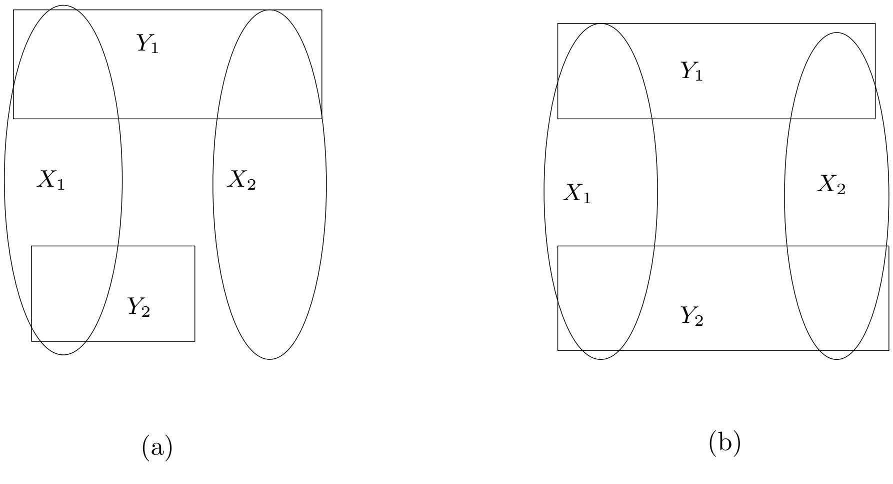

Figure 3: Simply dependent one-way pairs (a), and strongly dependent ones (b).

Claim 2.6. *Two small clumps X and Y are nested if and only if for some orientations K_X and K_Y, $K_X \preceq K_Y$.* □

We are ready to define uncrossing of clumps. By uncrossing the dependent one-way pairs K and L we mean replacing them by $K \wedge L$ and $K \vee L$ (which coincide with K and L if K and L are comparable). For dependent basic clumps X and Y, we will define a set $\chi(X, Y)$ consisting of two or four pairwise nested clumps in the analogous sense. If X and Y are nested, then let $\chi(X, Y) = \{X, Y\}$. By Lemma 2.3, this is always the case if one of X and Y is large. For the small basic clumps X and Y, consider some compatible orientations L_X and L_Y. If X and Y are simply dependent then let $\chi(X,Y) = \{\underline{L_X \wedge L_Y}, \underline{L_X \vee L_Y}\}$. (Altough there are two possible choices for L_X and L_Y, the set $\chi(X,Y)$ will be the same.) If they are strongly dependent, then L_X is also compatible $\overleftarrow{L_Y}$. In this case let $\chi(X,Y) = \{\underline{L_X \wedge L_Y}, \underline{L_X \vee L_Y}, \underline{L_X \wedge \overleftarrow{L_Y}}, \underline{L_X \vee \overleftarrow{L_Y}}\}$. It is easy to see that the clumps in $\chi(X,Y)$ are nested with X and Y and each other in both cases. We will need the following submodular-type property:

Claim 2.7. *For dependent basic clumps X, Y, if an edge uv connects a clump in $\chi(X,Y)$ then it connects at least one of X and Y.* □

We say that two clumps are **crossing** if they are dependent but not nested. Again by Lemma 2.3, two basic clumps may be crossing only if both are small. A subset $\mathcal{F} \subseteq \mathcal{C}$ is called **crossing**, if for any two dependent clumps $X, Y \in \mathcal{F}$, $\chi(X,Y) \subseteq D(\mathcal{F})$. The reason for assuming containment in $D(\mathcal{F})$ instead of $\mathcal{F}$ is that there might be non basic members of $\chi(X,Y)$. Note that $\mathcal{C}$ itself is crossing. For a crossing system $\mathcal{F}$ and a clump $K \in \mathcal{F}$, let $\mathcal{F} \div K$ denote the set of clumps in $\mathcal{F}$ independent or nested with K. Similarly, for a subset $\mathcal{K} \subseteq \mathcal{F}$, $\mathcal{F} \div \mathcal{K}$ denotes the set of clumps in $\mathcal{F}$ not crossing any clump in $\mathcal{K}$. An $\mathcal{F} \subseteq \mathcal{C}$ is **cross-free** if it contains no crossing clumps, that is, any two dependent clumps in $\mathcal{F}$ are nested. (Note that a cross-free system is crossing as well.) A cross-free $\mathcal{K}$ is called a **skeleton** of $\mathcal{F}$ if it is maximal cross-free in $\mathcal{F}$, that is, $\mathcal{F} \div \mathcal{K} = \mathcal{K}$. By Lemma 2.3, a skeleton of $\mathcal{C}$ should contain every large clump.

Lemma 2.8. *For a crossing system $\mathcal{F} \subseteq \mathcal{C}$ and $K \in \mathcal{F}$, $\mathcal{F} \div K$ is a crossing system as well.*

Proof. Let $\mathcal{F}' = \mathcal{F} \div K$. If K is large then $\mathcal{F}' = \mathcal{F}$ by Lemma 2.3, therefore K is assumed being small in the sequel. Let us fix an orientation L_K of K. Take crossing basic clumps $X, Y \in \mathcal{F}'$. Again by Lemma 2.3, if a clump in $\chi(X,Y)$ is not basic, then it is automatically in $D(\mathcal{F}')$. We consider all possible cases as follows.

(I) Both are nested with K. Choose orientations L_X and L_Y compatible with L_K (but not necessarly with each other). (a) If $L_X \preceq L_K \preceq L_Y$ or $L_Y \preceq L_K \preceq L_X$, then X and Y are nested by Claim 2.6. (b) Let $L_X, L_Y \preceq L_K$. If L_X and L_Y are dependent, then $L_X \wedge L_Y, L_X \vee L_Y \preceq L_K$. Otherwise, L_X and $\overleftarrow{L_Y}$ are dependent. Then $L_X \wedge \overleftarrow{L_Y} \preceq L_K$ and $\overleftarrow{L_K} \preceq L_X \vee \overleftarrow{L_Y}$. These arguments show $\chi(X,Y) \subseteq D(\mathcal{F}')$. (c) In case of $L_X, L_Y \succeq L_K$, the claim follows analogously.

(II) Both X and Y are independent from K. By Claim 2.7, all clumps in $\chi(X,Y)$ are independent from K.

(III) One of them, say X is nested with K and Y is independent from K. Let L_X be an orientation of X compatible with L_K and L_Y an orientation of Y compatible with L_X. By symmetry, we may assume $L_X \preceq L_K$. Now $L_X \wedge L_Y \preceq L_K$, and we show that $\underline{L_X \vee L_Y}$ is independent from K. As L_Y is an arbitrary orientation compatible with L_X, these again imply $\chi(X,Y) \subseteq D(\mathcal{F}')$. L_Y and L_K are independent, but $L_K^- \cap L_Y^- \neq \emptyset$, thus $L_K^+ \cap L_Y^+ = \emptyset$, so the one-way pairs $L_X \vee L_Y$ and L_K are independent. We also need to show that $\overleftarrow{L_X \vee L_Y}$ and L_K are independent. Indeed, their dependence would imply $L_Y^+ \cap L_K^- \neq \emptyset$, $L_Y^- \cap L_K^+ \neq \emptyset$, contradicting the independence of K and Y. □

Finally, the sequence $K_1, K_2, \ldots, K_\ell$ of clumps is called a **chain** if they admit orientations $L_1, L_2, \ldots, L_\ell$ with $L_1 \preceq L_2 \preceq \ldots \preceq L_\ell$. If $u \in L_1^-$, $v \in L_\ell^+$ then the edge uv connects all members of the chain.

3 The proof of Theorem 1.1

For a crossing system $\mathcal{F} \subseteq \mathcal{C}$, let $\tau(\mathcal{F})$ denote the minimum cardinality edge set covering $\mathcal{F}$. Let $\nu(\mathcal{F})$ denote the maximum of $def(\Pi)$ over groves consisting of a shrub and bushes of clumps in $D(\mathcal{F})$. First, we give the proof of the following slight generalization of Theorem 1.1 based on two lemmas proved in the following subsections.

Theorem 3.1. *For a crossing system $\mathcal{F} \subseteq \mathcal{C}$, $\nu(\mathcal{F}) = \tau(\mathcal{F})$.*

The two lemmas are these:

Lemma 3.2. *For a cross-free system $\mathcal{F}$, $\nu(\mathcal{F}) = \tau(\mathcal{F})$.*

Lemma 3.3. *If an edge set F covers $\mathcal{F} \div K$, then there exists an F' covering $\mathcal{F}$ with $|F'| = |F|$, and furthermore $d_{F'}(v) = d_F(v)$ for every $v \in V$.*

Proof of Theorem 3.1. $\nu \le \tau$ is straightforward. The proof of $\nu \ge \tau$ is by induction on $|\mathcal{F}|$. If $\mathcal{F}$ is cross-free, we are done by Lemma 3.2. Otherwise, consider two crossing clumps $K, K' \in \mathcal{F}$ and let $\mathcal{F}' = \mathcal{F} \div K$, a crossing system by Lemma 2.8. As $K' \notin \mathcal{F}'$, we may apply the inductive statement for $\mathcal{F}'$ giving a grove Π and an edge set F covering $\mathcal{F}'$ with $def(\Pi) = |F|$. The proof is finished using Lemma 3.3. □

The following theorem may be seen as a reformulation of this proof, however, it will be more convenient for the aim of the algorithm and to handle the minimum cost version for node induced cost functions.

Theorem 3.4. *For a crossing system $\mathcal{F} \subseteq \mathcal{C}$ and a skeleton $\mathcal{K}$ of $\mathcal{F}$, $\nu(\mathcal{K}) = \nu(\mathcal{F})$. Furthermore, if an edge set F covers the skeleton $\mathcal{K}$ of $\mathcal{F}$, then there exists an F' covering $\mathcal{F}$ with $|F'| = |F|$ and $d_{F'}(v) = d_F(v)$ for every $v \in V$.*

Proof. Let $\mathcal{K} = \{K_1, \ldots, K_\ell\}$. For $i = 1, \ldots, \ell$, let $\mathcal{F}_i = \mathcal{F} \div \{K_1, \ldots, K_i\}$. Lemma 2.8 implies that $\mathcal{F}_i$ is a crossing system as well. $\mathcal{F}_\ell = \mathcal{K}$ since $\mathcal{K}$ is a skeleton. By Lemma 3.2, $\mathcal{K}$ admits a cover F_ℓ with $|F_\ell| = \tau(\mathcal{K}) = \nu(\mathcal{K})$. Applying Lemma 3.3 inductively for $\mathcal{F}_{i-1}$ and K_i for $i = \ell, \ell-1, \ldots, 1$, we get a cover F_{i-1} of $\mathcal{F}_{i-1}$ with $|F_{i-1}| = |F_\ell|$. Finally, F_0 is a cover of $\mathcal{F} = \mathcal{F}_0$, hence $\nu(\mathcal{F}) \le |F_0| = |F_\ell| = \nu(\mathcal{K})$, implying the first part of the theorem. The identity of the degree sequences follows by the second part of Lemma 3.3. □

3.1 Covering cross-free systems

In this section we prove Lemma 3.2. The analogous statement in case of directed connectivity augmentation simply follows by Dilworth' theorem. Dilworth' theorem is well-known to be a consequence of Kőnig's theorem on the size of a maximum bipartite matching. In contrast, Lemma 3.2 follows by a theorem of Fleiner which may be deduced from the Berge-Tutte theorem on maximum matching in general graphs.

To state Fleiner's theorem, we need the following notion. A triple $P = (U, \preceq, M)$ is called a **symmetric poset** if $(U, \preceq)$ is a finite poset and M a perfect matching on U with the property that $u \preceq v$, $uu', vv' \in M$ implies $u' \succeq v'$. A subset $\{u_1v_1, \ldots, u_kv_k\} \subseteq M$ is called a **symmetric chain** if $u_1 \preceq u_2 \preceq \ldots \preceq u_k$. The symmetric chains $S_1, S_2, \ldots, S_t$ **cover** P if $M = \cup S_i$. A set $\mathcal{L} = \{L_1, L_2 \ldots, L_\ell\}$ of disjoint subsets of M forms a **legal subpartition** if $uv \in L_i$, $u'v' \in L_j$, $u \preceq u'$ yields $i = j$ and there is no symmetric chain of length three contained in any L_i. The value of $\mathcal{L}$ is $val(\mathcal{L}) = \sum_i \left\lceil \frac{|L_i|}{2} \right\rceil$.

Theorem 3.5 (Fleiner, [4]). *Let $P = (U, \preceq, M)$ be a symmetric poset. The minimum number of symmetric chains covering P is equal to the maximum value of a legal subpartition of P.*

Note that the max $\leq$ min direction follows easily since a symmetric chain may contain at most two edges of M belonging to at most one class of a legal subpartition. This theorem gives a common generalization of Dilworth's theorem and the well-known min-max formula for the minimum size edge cover of a graph (a theorem equivalent to the Berge-Tutte formula).

First we show that Lemma 3.2 is a straigthforward consequence if $\mathcal{F}$ contains only small clumps. Consider the nested family $\mathcal{F}$ of clumps, and let U be the set of all orientations of the pairs in $\mathcal{F}$. The edges of the matching M contain the two orientations of the same clump, while $\preceq$ is the usual partial ordering of the one-way pairs. A symmetric chain corresponds to a chain of clumps. Since all clumps in a chain can be connected by a single edge, a symmetric chain cover gives a cover of $\mathcal{F}$ of the same size. On the other hand, a legal subpartition corresponds to a grove with a shrub and bushes consisting of the clumps corresponding to the one-way pairs in L_i.

Let us now turn to the general case when $\mathcal{F}$ may contain large clumps as well. An edge set F **semi-covers** the clump X if F contains at least $|X| - 1$ edges connecting X and each clump (X_i, X_i^*) is covered. F semi-covers $\mathcal{F}$ if it semi-covers every $X \in \mathcal{F}$. A semi-cover of $\mathcal{F}$ is not necessary a cover, nevertheless, the following lemma shows that it is yet enough to find a semi-cover.

Lemma 3.6. *If F is a semi-cover of $\mathcal{F}$, then there exists an edge set H covering $\mathcal{F}$ with $|F| = |H|$ with $d_H(v) = d_F(v)$ for every $v \in V$.*

Proof. We are done if F covers all clumps in $\mathcal{F}$. Otherwise, consider a clump $X \in \mathcal{F}$ semi-covered but not covered. A semi-covered small clump is covered as well, hence X is large. Since X is connected by at least $|X| - 1$ edges of F, there is an edge $e = x_1y_1 \in F$ so that $c_F(X) = c_{F-e}(X)$. Each (X_i, X_i^*) is connected, thus we may consider an edge $x_2y_2 \in F$ connecting X with x_2y_2 in a component of F/X different from the one containing x_1y_1. Let $F' = F - \{x_1y_1, x_2y_2\} + \{x_1y_2, x_2y_1\}$. Clearly, $c_{F'}(X) = c_F(X) - 1$. We show that $c_{F'}(Y) \leq c_F(Y)$ for every $Y \in \mathcal{F} - X$, hence by a sequence of such steps we finally arrive at an H covering $\mathcal{F}$.

Indeed, assume $c_{F'}(Y) > c_F(Y)$ for some $Y \in \mathcal{F}$. X and Y are dependent as at least one of x_1y_1 and x_2y_2 connects Y. By Lemma 2.3, X and Y are nested with dominant pieces X_a and Y_b. At least one of x_1y_1 and x_2y_2 is not incident to X_a (as they are in different connected components of F/X), assume this is x_2y_2. This means that $x_2, y_2 \in Y_b$, and it is also clear that x_1y_1 connects (Y_b, Y_b^*) (as otherwise $x_1, y_1 \in X_a$ contradicting the fact that x_1y_1 connects X). In such a configuration, F'/Y may not have less components than F/Y, a contradiction. □

Now we show how a semi-cover F of $\mathcal{F}$ may be found based on a reduction to Fleiner's theorem. For a basic clump $X = (X_1, \dots, X_t)$, let $u_i^X = (X_i, X_i^*)$, $v_i^X = (X_i^*, X_i)$ and $U^X = \{u_i^X, v_i^X : i = 1, \dots, t\}$. Let $U = \cup_{X \in \mathcal{F}} U^X$. A one-way pair is of type X if it is in U^X. Let the matching M consist of the edges $u_i^X v_i^X$.

If X is small ($t = 2$), then $u_1^X = v_2^X$ and $v_1^X = u_2^X$, thus $|U^X| = 2$. If X is large, then there $|U^X| = 2t$. In this case, let u_1^X and v_1^X be called the **special one-way pairs** w.r.t X. Note that here it is important, which piece of X is called X_1 (though arbitrarily chosen). Let the partial order $\preceq'$ on U be defined as follows. If x and y are one-way pairs of different type, then let $x \preceq' y$ if and only if $x \preceq y$ for the standard partial order $\preceq$ on one-way pairs. If x and y are both of type X, then let $x \preceq y$ if either $x = u_1^X$, $y = v_i^X$, or $x = u_i^X$, $y = v_1^X$, for some $i > 1$. In other words, $\preceq'$ is the same as $\preceq$ except that x and y be uncomparable whenever x and y are of the same type X and neither of them is a special one-way pair w.r.t. X.

Claim 3.7. *$P = (U, \preceq', M)$ is a symmetric poset.*

Proof. The only nontrivial property to prove is the transitivity of $\preceq'$: $x \preceq' y$ and $y \preceq' z$ implies $x \preceq' z$. This follows from the transitivity of $\preceq$ unless x and z are different one-way pairs of the same type X and neither of them is special. X is a large clump and by possibly changing the indices, $x = u_2^X$, $z = v_3^X$. y could be of type X only if it were special, excluded by $x = u_2^X \not\preceq u_1^X$ and $v_1^X \not\preceq v_3^X = z$. Hence y is of a different type Y.

Assume first $y = u_i^Y$ for some i. Now $X_2 \subseteq Y_i \subseteq X_3^*$ thus $N_X \cap Y_i = \emptyset$, giving by Claim 2.4 $Y_i \subseteq X_j$ for some $j \neq 3$. Consequently, $X_2 = Y_i$, a contradiction as $X = Y$ by Claim 2.1. Next, assume $y = v_i^Y$. $X_3 \subseteq Y_i \subseteq X_2^*$ gives a contradiction the same way. □

The following claim points out the connection between dependency of clumps and comparability in P.

Claim 3.8. *In a cross-free system $\mathcal{F}$, the clumps X and Y are dependent if and only if for arbitrary i, j, u_i^X is comparable with either u_j^Y or v_j^Y.* □

Consider a symmetric chain cover $S_1, \dots, S_t$ and a legal subpartition $\mathcal{L} = \{L_1, L_2, \dots, L_\ell\}$ with $val(\mathcal{L}) = t$. Let us choose $\mathcal{L}$ so that ℓ is maximal, and subject to this, $\cup_{i=1}^{\ell} L_i$ contains the maximum number of edges in M consisting of special one-way pairs. A symmetric chain S_i naturally corresponds to a chain of the clumps (X_j, X_j^*) for $u_j^X v_j^X \in S_i$. These can be covered by a single edge; hence the symmetric chain cover corresponds to an edge set F of the same size. A symmetric chain may contain both $u_j^X v_j^X$ and $u_{j'}^X v_{j'}^X$ for $j \neq j'$ only if $j = 1$ or $j' = 1$. Consequently, F is a semi-cover as there are at least $|X| - 1$ different edges in F connecting X, and all (X_j, X_j^*)s are connected.

It is left to show that $\mathcal{L}$ gives a grove Π with $def(\Pi) = val(\mathcal{L})$. For a clump X, let $B(X)$ denote the set of indices j with $u_j^X v_j^X \in \cup_i L_i$.

Claim 3.9. *For any clump X, the edges in M corresponding to $B(X)$ are either all contained in the same L_i or all are singleton L_is. $1 \in B(X)$ always gives the first case.*

Proof. There is nothing to prove for $|X| = 2$, so let us assume $|X| \geq 3$. As $\mathcal{L}$ is chosen with ℓ maximal, if $u_j^X v_j^X \in L_i$ with $|L_i| > 1$, then there is an $u_h^Y v_h^Y \in L_i$ with u_h^Y comparable with either u_j^X or v_j^X. Claim 3.8 gives that u_h^Y is also comparable with $u_{j'}^X$ or $v_{j'}^X$ for any $j' \in B(X)$. $1 \in B(X)$ gives the first case as $u_1^X \preceq v_j^X$ for every $j \geq 2$. □

In the first case of this claim let $\beta(X) = i$ while in the second case, let $\beta(X) = 0$. Let $\mathcal{I}$ denote the set of indices for which L_i is a singleton. Take a clump X with $\beta(X) = i \notin \mathcal{I}$. Let us say that a piece X_j is a **dominant piece** of X, if for some Y with $\beta(Y) = i$, X_j is the dominant piece of X w.r.t. Y. Let $U(X)$ denote the set of the indices of the dominant pieces of X.

Claim 3.10. *If $|B(X)| \geq 2$, then $|B(X) \cap U(X)| = \emptyset$.*

Proof. First assume $B(X) \cap U(X) \neq \emptyset$ and $|U(X)| \geq 2$. Consider a $j \in B(X) \cap U(X)$ and a $j' \in U(X) - \{j\}$, say X_j is the dominant piece of X w.r.t. Y and $X_{j'}$ the dominant piece of X w.r.t. Z with $\beta(Y) = \beta(Z) = i$. It is easy to see that L_i contains a symmetric chain of lenght three consisting of $u_j^X v_j^X$ and two edges in M corresponding to certain pieces of Y and Z.

Thus if $B(X) \cap U(X) \neq \emptyset$ then $|U(X)| = 1$; let $U(X) = \{j\}$. We claim that $1 \notin B(X)$. Indeed, if $1 \in B(X)$ and $j \neq 1$ then L_i contains a symmetric chain of length three containing $u_j^X v_j^X$, $u_1^X v_1^X$. If $j = 1$, then we also have a symmetric chain of length three containing $u_1^X v_1^X$ and $u_h^X v_h^X$ for arbitrary $h \in B(X) - \{1\}$.

Let us replace L_i by $L_i' = L_i - \{u_j^X v_j^X\} + \{u_1^X v_1^X\}$. By Claim 3.8, any element of L_i' is independent from any element of L_h, $h \neq i$. L_i' does not contain any symmetric chain of length three, as it contains no edge corresponding to a dominant piece of X, thus a symmetric chain containing edges corresponding to pieces of X may have length at most two. This is a contradiction as $\mathcal{L}$ was chosen so that the number of edges consisting of special one-way pairs is maximal. □

Let us construct the grove Π as follows. For any X with $\beta(X) = 0$, $B(X) \neq \emptyset$, let $\tilde{X} \in D(X)$ denote the clump consisting of the pieces X_i with $i \in B(X)$ and the piece $\cup_{j \notin B(X)} X_j$. The latter set is nonempty since $1 \notin B(X)$, thus $|\tilde{X}| - 1 = |B(X)|$. Define the shrub as $\mathcal{B}_0 = \{\tilde{X} : \beta(X) = 0\}$. For $i \notin \mathcal{I}$, let $\mathcal{B}_i = \{(X_j, X_j^*) : u_j^X v_j^X \in L_i\}$. The following easy claim completes the proof.

Claim 3.11. *Π is a grove with $def(\mathcal{B}_0) = |\mathcal{I}|$ and $def(\mathcal{B}_i) = \left\lceil \frac{|L_i|}{2} \right\rceil$ if $i \notin \mathcal{I}$.*

Proof. Claim 3.8 implies that clumps in different bushes are independent from each other and from those in $\mathcal{B}_0$. Assume an edge $uv \in V^2$ covers three clumps in some $\mathcal{B}_i$. If these three clumps are derived from three different clumps, then L_i would contain a symmetric chain of length three. Thus we need to have two clumps derived from the same basic clump X: uv covers (X_j, X_j^*), $(X_{j'}, X_{j'}^*)$ and (Y_h, Y_h^*) for $\beta(X) = \beta(Y) = i$. This is also impossible since either X_j or $X_{j'}$ would need to be a dominant piece of X, a contradiction to Claim 3.10. □

3.2 The proof of Lemma 3.3.

First we need the following lemmas.

Lemma 3.12. *Assume for three clumps* $X = (X_1, X_2)$, $Y = (Y_1, Y_2)$, $Z = (Z_1, Z_2)$, *all four sets* $X_1 \cap Y_1 \cap Z_1$, $X_1 \cap Y_2 \cap Z_2$, $X_2 \cap Y_1 \cap Z_2$, $X_2 \cap Y_2 \cap Z_1$ *are nonempty. Then all of* X, Y *and* Z *are derived from the same basic clump (and thus none of them is basic itself).*

Proof. Let $X_c = N_X$, $Y_c = N_Y$, $Z_c = N_Z$. By A_s for a sequence s of three literals each 1,2 or c, we mean the intersection of the corresponding sets. For example, $A_{12c} = X_1 \cap Y_2 \cap Z_c$.

The conditions mean that the sets A_{111}, A_{122}, A_{212}, A_{221} are nonempty. $V - (A_{111} \cup N(A_{111})) \neq \emptyset$ as there is no edge between A_{111} and X_2, thus $|N(A_{111})| \geq k - 1$ as G is $(k-1)$-connected. This implies

$$k - 1 \leq |A_{c11} \cup A_{1c1} \cup A_{11c} \cup A_{1cc} \cup A_{c1c} \cup A_{cc1} \cup A_{ccc}| \tag{1}$$

as $N(A_{111})$ is a subset of the set on the RHS. Let us take the sum of these types of inequalities for all A_{111}, A_{122}, A_{212}, A_{221}. This gives $4(k-1) \leq S_1 + 2S_2 + 4|A_{ccc}|$, where S_1 is the sum of the cardinalities of the sets having exactly one c in their indices, while S_2 is the same for two cs.

On the other hand, $|X_c| = |Y_c| = |Z_c| = k-1$. This gives $3(k-1) = S_1 + 2S_2 + 3|A_{ccc}|$. These together imply $S_1 = S_2 = 0$, $|A_{ccc}| = k - 1$. We are done by Claim 2.1 since $N_X = N_Y = N_Z = A_{ccc}$. □

Lemma 3.13. *[7] (i) Let* L_1, L_2, L_3 *be one-way pairs with* L_1 *and* L_2 *dependent,* $L_1 \wedge L_2$ *and* L_3 *also dependent, but* L_2 *and* L_3 *independent. Then* $L_1^- - L_2^- \subseteq L_3^-$. *(ii) Let* L_1, L_2, L_3 *be one-way pairs with* L_1 *and* L_2 *dependent,* $L_1 \vee L_2$ *and* L_3 *also dependent, but* L_2 *and* L_3 *independent. Then* $L_1^+ - L_2^+ \subseteq L_3^+$.

Proof. (i) The dependence of $L_1 \wedge L_2$ and L_3 implies $L_2^- \cap L_3^- \neq \emptyset$, so L_2 and L_3 can only be independent if $L_2^+ \cap L_3^+ = \emptyset$. Consider now the pair $N = (L_1 \wedge L_2) \vee L_3$. $N^+ = (L_1^+ \cup L_2^+) \cap L_3^+ = L_1^+ \cap L_3^+$, hence $N^+ \subseteq L_1^+$. By Claim 2.5, $N^- \supseteq L_1^-$, implying the claim. (ii) follows from (i) by reverting the orientations of all pairs. □

Proof of Lemma 3.3. Let $\mathcal{F}' = \mathcal{F} \div K$. If K is large then $\mathcal{F}' = \mathcal{F}$ by Lemma 2.3, therefore K will be assumed to be small with an orientation L_K.

If F covers $\mathcal{F}'$ but not $\mathcal{F}$, then by Claim 2.2 there exists a small clump $X \in D_2(\mathcal{F}) - D_2(\mathcal{F}')$ not connected by F, thus X and K are crossing. Choose X with the orientation L_X compatible with L_K so that L_X is minimal to these properties w.r.t. $\preceq$ (that is, there is no other uncovered X' with orientation $L_{X'}$ compatible with L_K so that $L_{X'} \prec L_X$.) Choose Y not connected by F with $L_X \preceq L_Y$, and L_Y maximal in the analogous sense ($X = Y$ is allowed).

$\underline{L_X \wedge L_K}$ and $\underline{L_Y \vee L_K}$ are nested with L_K and thus connected by edges $x_1y_1, x_2y_2 \in F$ with $x_1 \in L_X^- \cap L_K^-$, $y_2 \in L_Y^+ \cap L_K^+$. As X and Y are not connected, $y_1 \in L_K^+ - L_X^+$, $x_2 \in L_K^- - L_Y^-$ follows. Let $F' = F - \{x_1y_1, x_2y_2\} + \{x_1y_2, x_2y_1\}$. F' connects X and Y and we prove that F' covers all small clumps in $D_2(\mathcal{F})$ connected by F. Hence after a finite number of such operations all small clumps in $D_2(\mathcal{F})$ will be connected, so by Claim 2.2, $\mathcal{F}$ will be covered.

For a contradiction, assume there is a small clump S connected by F but not by F'. (S is not necessary basic.) No edge in $F \cap F'$ may connect S, hence either exactly one of x_1y_1 and x_2y_2 connect it, or if both then x_1 and y_2 are in the same piece and y_1 and x_2 in the other piece of S. In this latter case, K and S are strongly dependent.

(I) First, assume that only x_1y_1 connects S, and choose the orientation L_S with $x_1 \in L_S^-$, $y_1 \in L_S^+$. We claim that L_S and L_Y are also dependent. Indeed, if they are independent, then Lemma 3.13(i) is applicable for $L_1 = L_K$, $L_2 = L_Y$, $L_3 = L_S$, since $L_K \wedge L_Y$ and L_S are dependent because x_1y_1 covers both. This gives $x_2 \in L_K^- - L_Y^- \subseteq L_S^-$, that is, x_2y_1 connects S, a contradiction.

Hence we may consider the one-way pair $L_S \vee L_Y$. $L_S \vee L_Y$ is strictly larger than L_Y, as if $L_S \preceq L_Y$, then S were connected by x_1y_2. By the maximal choice of L_Y, $\underline{L_S \vee L_Y}$ is connected by some edge $f \in F$. By Claim 2.7, f also connects one of S and Y, implying $f = x_1y_1$. This is a contradiction as $x_1 \in L_S^- \cup L_Y^-$ and $y_1 \notin L_S^+ \cap L_Y^+$.

(II) If x_2y_2 is the only edge connecting S, we may use the same argument by exchanging $\vee$ and $\wedge$, X and Y, "minimal" and "maximal" everywhere and applying Lemma 3.13(ii) instead of (i).

(III) Finally, if both x_1y_1 and x_2y_2 cover S, let L_S be chosen with $x_1 \in L_S^-$, $y_1 \in L_S^+$. The argument in (I) may be applied with the only difference that at the end $f = x_2y_2$ is possible. This gives $x_2 \in L_Y^+ \cap L_S^+$, thus $x_2 \in L_X^+$. Analogously, the argument in (II) applies for $\overleftarrow{L_S}$, and we get $y_1 \in L_X^- \cap L_S^+$, thus $y_1 \in L_X^-$.

Now the clumps K, S and X satisfy the condition in Lemma 3.12 as shown by the nodes x_1, y_1, x_2, y_2. This contradicts the assumption that K was a basic clump. $\square$

4 The Algorithm

As outlined in the Introduction, our algorithm is a simple iterative application of a subroutine determining the dual optimum $\nu(G)$. Theorem 3.4 shows that $\nu(G) = \nu(\mathcal{K})$ for an arbitrary skeleton $\mathcal{K}$. Given a skeleton $\mathcal{K}$, $\nu(\mathcal{K})$ can be determined based on Fleiner's theorem: [4] gives a proof of Theorem 3.5 based on a (linear time) reduction to maximum matching in general graphs. Hence the only nontrivial question is how a skeleton can be found. A naive approach is simply choosing clumps greedily so that they do not cross the previously selected ones. The difficulty arises from the fact that the number of the clumps might be exponentially large, hence we cannot try all clumps one-by-one. In fact, it is not even clear how to decide whether a given cross-free system is a skeleton. To overcome these difficulties, we restrict ourselves to a special class of cross-free systems as described in the next subsection.

4.1 Constructing a skeleton

Let us first introduce some new notation concerning pieces. If the set $A \subseteq V$ is a piece of the basic clump X, then let $A^\sharp$ denote X. Let $\mathcal{P}$ be the set of all connected pieces of all basic clumps, and $\mathcal{P}_1$ the set of all (not necessarily connected) pieces of all clumps. For a subset $\mathcal{A} \subseteq \mathcal{P}$, $\mathcal{A}^\sharp$ is the set of corresponding clumps (e.g. $\mathcal{P}^\sharp = \mathcal{C}$).

Next we define the special class of cross-free systems we wish to use. A cross-free set of $\mathcal{H} \subseteq \mathcal{C}$ is **stable** if it fulfills the following: U crosses some element of $\mathcal{H}$ whenever $U \in \mathcal{C} - \mathcal{H}$; $K, K' \in \mathcal{H}$, and K, U, K' forms a chain. The following simple claim will be used for handling chains of length three.

Claim 4.1. *For pieces A, B, C, if (i) $A \subseteq B \subseteq C$ or (ii) $A \subseteq B$ and $C \subseteq B^*$ then the corresponding clumps $A^\sharp, B^\sharp, C^\sharp$ form a chain.* ☐

Clearly, $\mathcal{H} = \emptyset$ is stable, and every skeleton is stable as well. Let $\mathcal{M} \subseteq \mathcal{P}$ denote the set of the pieces minimal for inclusion. Based on the following claim, we will be able to determine when a stable cross-free system is a skeleton. The subroutine for finding the elements of $\mathcal{M}$ will be given in the Appendix among other technical details of the algorithm.

Claim 4.2. *The stable cross-free system $\mathcal{H} \subseteq \mathcal{C}$ is a skeleton if and only if $\mathcal{M}^\sharp \subseteq \mathcal{H}$.*

Proof. On the one hand, every skeleton should contain $\mathcal{M}^\sharp$. Indeed, consider an $M \in \mathcal{M}$. $M^\sharp$ cannot cross any $X \in \mathcal{C}$, as $\chi(X, M^\sharp)$ would contain a clump with a piece being a proper subset of M.

On the other hand, assume $\mathcal{H}$ is not a skeleton even though $\mathcal{M}^\sharp \subseteq \mathcal{H}$. Hence there exists a $U = (U_1, \dots U_t) \in \mathcal{C} - \mathcal{H}$, not crossing any element of $\mathcal{H}$. Consider minimal pieces $M_1 \subseteq U_1$, $M_2 \subseteq U_2$. Then $M_1^\sharp, U, M_2^\sharp$ forms a chain by Claim 4.1, contradicting the stability. ☐

Assume $\mathcal{H}$ is a stable cross-free system, but not a skeleton. In the following, we show how $\mathcal{H}$ may be extended to a stable cross-free system larger by one. By the above claim, there is an $M \in \mathcal{M}$ with $M^\sharp \in \mathcal{C} - \mathcal{H}$. Let

$$\mathcal{L}_1 := \{X \in \mathcal{H} : X \text{ and } M^\sharp \text{ are nested}\}, \quad \mathcal{L}_2 := \{X \in \mathcal{H} : X \text{ and } M^\sharp \text{ are independent}\} \quad (2)$$

Claim 4.3. *If $\mathcal{L}_1 = \emptyset$, then $\mathcal{H} + M^\sharp$ is a stable cross-free system.*

Proof. Indeed, assume that for some $U \in \mathcal{C} - \mathcal{H}$ and $K \in \mathcal{H}$, $\mathcal{H} + U$ is cross-free, although $K, U, M^\sharp$ form a chain. Now K and M are dependent and thus nested, a contradiction. ☐

In the sequel we assume $\mathcal{L}_1 \neq \emptyset$. The key concept of the algorithm will be "fitting": it will be defined when a piece $Z \in \mathcal{P}$ **fits** the pair $(\mathcal{H}, M)$. Since this definition is fairly technical, we formulate the main lemma in advance:

Lemma 4.4. *Let Z be a minimal member of $\mathcal{P} - \cup\mathcal{H}$ fitting $(\mathcal{H}, M)$. Then $\mathcal{H} + Z^\sharp$ is a stable cross-free system.*

The set of pieces fitting $(\mathcal{H}, M)$ will be nonempty as according to the definition, the components of M^* fit $(\mathcal{H}, M)$. Thus there exists a Z satisfying the conditions in Lemma 4.4. Such a Z may be found using standard bipartite matching theory similarly as in [7]; the technical details are left to the Appendix.

The minimality of M implies that for any $X \in \mathcal{L}_1$, the dominant piece of $M^\sharp$ w.r.t. X is a connected component of M^*. One simple notion before giving the definition of fitting is the following. For $A, B \in \mathcal{P}$, we say that A **supports** B if $A \subseteq B \subseteq M^*$. $A \in \mathcal{P}$ supports $Y \in \mathcal{C}$ if A supports some piece of Y; $X \in \mathcal{C}$ supports $B \in \mathcal{P}$, and $X \in \mathcal{C}$ supports $Y \in \mathcal{C}$ are defined in the analogous sense.

Definition 4.5. We say that the piece $Z \in \mathcal{P}$ **fits** the pair $(\mathcal{H}, M)$ if

(a) $Z^\sharp \in \mathcal{C} - \mathcal{H}, Z \subseteq M^*$;

(b) there exists a $W \in \mathcal{L}_1$ supporting Z;

(c) for every $X = (X_1, X_2, \ldots, X_h) \in \mathcal{L}_1$ with $X_i \subseteq M^*$, $X_j \supseteq M$ for some $1 \leq i, j \leq h$, either $X_i \subset Z$ or $X_i \cap Z = \emptyset$, and if $X_j \cap Z \neq \emptyset$ then $X_i \cap Z^* = \emptyset$;

(d) $Z^\sharp$ is independent from every $X \in \mathcal{L}_2$.

The proof of Lemma 4.4 is based on the following claim:

Claim 4.6. *Let $Z \in \mathcal{P} - \cup\mathcal{H}$, $Z \subseteq M^*$, supported by some $W \in \mathcal{L}_1$, so that Z does not support any $X \in \mathcal{L}_1$. The following two properties are equivalent: (i) Z fits $(\mathcal{H}, M)$; (ii) $\mathcal{H} + Z^\sharp$ is cross-free.*

Proof. First we show that (i) implies (ii). $Z^\sharp$ is independent from all pairs in $\mathcal{L}_2$. Consider an $X \in \mathcal{L}_1$. $Z^\sharp$ and X cannot cross by Lemma 2.3 whenever X or $Z^\sharp$ is large, thus let us assume they both are small basic clumps, $X = (X_1, X_2)$ with $X_1 \subset M^*$. If X and Z are dependent, then $X_1 \cap Z \neq \emptyset$ or $X_2 \cap Z \neq \emptyset$. In the first case, (c) implies $X_1 \subset Z$ hence nestedness follows by Claim 2.5. So let us assume $X_1 \cap Z = \emptyset$. By the dependency, $X_1 \cap Z^* \neq \emptyset$, contradicting $X_2 \cap Z \neq \emptyset$ by the second part of (c).

Next, we show how (ii) implies (i). (a) and (b) are given among the conditions. For (c), consider an $X \in \mathcal{L}_1$ with pieces $X_i \subseteq M^*$ and $X_j \supseteq M$. If X and $Z^\sharp$ are independent, then $X_i \cap Z = \emptyset$ as otherwise an edge between $X_i \cap Z$ and M would connect both. If they are dependent so that the dominant side of X w.r.t. $Z^\sharp$ is different from X_i, then $X_i \subseteq Z$ or $X_i \cap Z = \emptyset$ follows. Finally, if the dominant side is X_i, then Z cannot be the dominant side of $Z^\sharp$ (as it would imply $M \subseteq X_j \subseteq Z$), thus $Z \subset X_i$, that is, Z supports X, a contradiction.

Assume next $X_j \cap Z \neq \emptyset$ and $X_i \cap Z^* \neq \emptyset$. X and $Z^\sharp$ are again dependent and thus nested, and as above, the dominant side of X cannot be X_i. Z cannot be the dominant side of $Z^\sharp$ as $X_i \subseteq Z$ would contradict $X_i \cap Z^* \neq \emptyset$. Hence $Z \subseteq X_i^*$. By Claim 4.1, W, Z, X_i form a chain, a contradiction to stability.

Finally for (d), assume $Z^\sharp$ and $X \in \mathcal{L}_2$ are dependent. Z cannot be the dominant piece of $Z^\sharp$ w.r.t. X as it would give $X \in \mathcal{L}_1$. Thus $W, Z^\sharp, X$ form a chain, a contradiction to stability. □

Proof of Lemma 4.4. Using Claim 4.6, it is left to show that no chain $Z^\sharp, U, K$ may exist with $K \in \mathcal{H}$, $U \in \mathcal{C} - \mathcal{H} - Z^\sharp$ so that $\mathcal{H} + Z^\sharp + U$ is stable. Indeed, if such a chain exists, then $Z^\sharp$ and K are nested. Let Z' be the dominant piece of $Z^\sharp$ w.r.t. K. If $Z' \neq Z$ then the chain $W, Z^\sharp, K$ contradicts the stability of $\mathcal{H}$, where W is the clump supporting Z ensured by (b).

If $Z' = Z$, then for some pieces U_1 of U and K_1 of K, $K_1 \subset U_1 \subset Z$. $\mathcal{H} + U$ is stable, K supports U_1, and U_1 does not support any $X \in \mathcal{L}_1$, as otherwise K, U, X would form a chain. Hence we may apply Claim 4.6, thus U_1 fits $(\mathcal{H}, M)$, a contradiction to the minimal choice of Z. □

4.2 Description of the Dual Oracle

To determine the value of $\nu(G)$, first we construct a skeleton $\mathcal{K}$ as described above. For $\mathcal{K}$ we apply the reduction to Theorem 3.5 as in Section 3.1. As already mentioned, a minimal chain decomposition and a maximal legal subpartition of a symmetric poset $P = (U, \preceq, M)$ may be found via a reduction to the Berge-Tutte theorem. For the sake of completeness and because it will be needed in the minimum cost version, we include this reduction. Define the graph $C = (U, H)$ by

$$H := \{uv' : \exists v \text{ such that } vv' \in M \text{ and } u \prec v\}.$$

It is easy to see that the set $\{m_1, m_2, \ldots, m_\ell\} \subseteq M$ is a symmetric chain if and only if there exists a edges $e_1, \ldots, e_{\ell-1} \in H$ such that $m_1 e_1 m_2 e_2 \ldots m_{k-1} e_{k-1} m_k$ is a path, called an M-alternating path. The transitivity of $\preceq$ ensures that $M \cup H$ contains no M-alternating cycles. Let $N \subseteq H$ be a matching in C. Then the components of $M \cup N$ are M-alternating paths, each containing exactly two nodes not covered by N. Hence finding a maximum matching in H is equivalent to finding a minimum chain cover, in P. The running time of the most efficient maximum matching algorithm for a graph on n_1 nodes with m_1 edges is $O(\sqrt{n_1} m_1)$ [14, Vol I, p. 423].

Let us now give an upper bound on $|\mathcal{K}|$. Jordán [11, 12] has shown that the size of the optimal augmenting edge set is at most $\max(b(G) - 1, \left\lceil \frac{t(G)}{2} \right\rceil) + \left\lceil \frac{k-2}{2} \right\rceil$. Here $b(G)$ is the maximum size of a clump, while $t(G)$ is the number of pairwise disjoint sets in $\mathcal{P}$. Since $b(G) \leq n - (k-1)$, $t(G) \leq n$, we get an upper bound n on the size of an augmenting edge set. In a skeleton $\mathcal{K}$, an edge connects the elements of a chain of clumps. The size of a chain can also be bounded by n, hence the size of a skeleton is at most n^2. Using the running time estimation in the Appendix, this gives a bound $O(kn^5)$ for finding $\mathcal{K}$.

In Section 3.1 the minimum semi-cover of $\mathcal{K}$ is reduced to a minimum symmetric chain cover of a poset $P = (U, \preceq, M)$ with $|U| = O(n^2)$ since there are $2|X|$ nodes in U corresponding the clump $|X|$. Hence the running time of the matching algorithm may be bounded by $O(n^5)$. As indicated in the introduction, at most n^2 calls of the Dual Oracle enable us to compute an optimal augmentation. This gives a total running time $O(kn^7)$.

As in [7], another algorithm may be constructed which calls the dual oracle only once. First, let us find a skeleton $\mathcal{K} = \{K_1, \ldots, K_\ell\}$ with a cover F and a grove Π of

$\mathcal{K}$ with $def(\Pi) = |F|$. Then we iteratively apply sequences of flipping operations as in Lemma 3.3 for $\mathcal{F}_{i-1} = \mathcal{C} \div \{K_1, \dots, K_{i-1}\}$ and K_i for $i = \ell, \ell-1, \dots, 1$ resulting finally in a cover F' of $\mathcal{C}$ with $|F| = |F'|$. For each i it can be easily seen that after $O(n^2)$ flippings we get a cover of $\mathcal{F}_{i-1}$, thus $O(n^4)$ improving flipping suffice. The realization of a flipping step can be done using similar techniques as in the Appendix. We omit this analysis as it is highly technical and we cannot get a better running time estimation as for the previous algorithm.

5 Node-induced cost function

The problem of finding a minimum cost edge set whose addition makes a $(k-1)$-connected graph k-connected is NP-complete as already making the graph $G = (V, \emptyset)$ connected by adding a minimum cost edge set generalizes the TSP problem, even for 0-1-valued cost functions. However, there is a special type of cost-functions for which directed connectivity augmentation and also both directed and undirected edge-connectivity augmentation are solvable: the node-induced cost functions. We show that augmenting undirected connectivity by one is also tractable for such cost functions.

$c' : E \to \mathbb{R}$ is a **node-induced cost function**, if there exists a $c : V \to \mathbb{R}$ so that $c'(uv) = c(u) + c(v)$ for every $uv \in E$. By the second part of Theorem 3.4, for a skeleton $\mathcal{K}$ and a node-induced cost function c', the minimum cost of an edge set covering $\mathcal{C}$ is the same as the minimum cost cover of $\mathcal{K}$. Hence it is enough to construct an oracle for determining the minimum cost $\nu_c(\mathcal{K})$ of a cover of $\mathcal{K}$. A minimum cost edge set whose addition makes G k-connected can be found by iteratively calling this oracle.

Furthermore, by Lemma 3.6, $\nu_c(\mathcal{K})$ equals the minimum cost of a semi-cover of $\mathcal{K}$. Finding a minimum-cost semi-cover can be easily done based on the following weighted version of Fleiner's theorem, which reduces to maximum cost matching in general graphs.

Given a symmetric poset $P = (U, \preceq, M)$ and a cost function $w : U \to \mathbb{R}$, let us define the cost of the symmetric chain $S = \{u_1v_1, \dots, u_\ell v_\ell\} \subseteq M$ with $u_1 \preceq \dots \preceq u_\ell$, $v_1 \succeq \dots \succeq v_\ell$ by $w(S) = w(u_\ell) + w(v_1)$. Our aim is now to find a chain cover of minimum total cost.

Consider the reduction to the matching problem in Section 4.2. For a matching $N \subseteq H$ of C, the components of $M \cup N$ are M-alternating paths each corresponding to a symmetric chain. The alternating path corresponding to the chain S is $v_1u_1v_2u_2 \dots v_\ell u_\ell$, hence the cost of the two nodes not covered by N equals the cost of the chain. Consequently, the cost of a symmetric chain cover equals the total cost of the nodes not covered by N. Hence minimizing the cost of a symmetric chain cover is equivalent to finding a maximum cost matching. Note that here we need a maximum cost matching only for node induced cost functions, although this can be done for arbitrary cost functions.

To find a minimum cost semi-cover of $\mathcal{K}$ we construct the symmetric poset $P = (U, \preceq', M)$ as in Section 3.1. For a one-way pair $u = (u^-, u^+) \in U$, let $w(u) = \min_{x \in u^+} c(x)$. We claim that finding a minimum cost symmetric chain cover for this w is equivalent to finding a minimum cost semi-cover of $\mathcal{K}$.

Indeed, there is a one-to-one correspondence between chains consisting of clumps of the form (X_i, X_i^*) and the symmetric chains of U (with the restriction that a chain may not contain both (X_i, X_i^*), (X_j, X_j^*) for $i, j > 1$). A chain $K_1, K_2, \dots, K_\ell$ of clumps with orientations $L_1 \preceq L_2 \preceq \dots \preceq L_\ell$ can be covered by any edge between L_1^- and L_ℓ^+, thus the minimum cost of an edge covering it is $w(L_\ell) + w(\overleftarrow{L_1})$ with w defined as above. Hence a minimum c-cost of a semi-cover in $\mathcal{K}$ equals the minimum w-cost of a symmetric chain cover of P.

5.1 Degree sequences

What can we say about the degree sequences of the augmenting edge sets? It is well-known that in a graph G with some cost function on the edges, the sets of nodes covered by a minimum cost matching form the bases of a matroid. A natural generalization of matroid bases are base polyhedra, and the analogous concept for supermodular functions are supermodular base polyhedra. For a supermodular function $p : V \to \mathbb{R}$ the corresponding supermodular base polyhedron is (see e.g. [14, Vol II, p. 840]):

$$B(p) = \{x \in \mathbb{R}^V : x(Z) \geq p(Z) \ \forall Z \subseteq V, x(V) = p(V)\}.$$

For undirected edge connectivity augmentation, the degree sequences of the augmenting edge sets form a supermodular base polyhedron, and the same holds for the in- and out-degree sequences for directed connectivity augmentation (see e.g. [5]). This is also true in case of directed connectivity augmentation [6]. Moreover, all these results can be generalized for node-induced cost functions: the degree (resp. in- and out-degree) sequences of the minimum cost augmenting edge sets form a base polyhedron. Hence a natural conjecture is the following:

Conjecture 5.1. *Given a $(k-1)$-connected graph G and a node-induced cost function, the degree sequences of the minimum cost augmenting edge sets form supermodular base polyhedron.*

This was essentially proved by Szabó proved in his master's thesis [15] for $k = n - 2$. His result holds even without the assumption that the graph is $(k - 1)$-connected, indicating that the conjecture might hold for arbitrary graphs as well.

6 Towards general connectivity augmentation

In this section, we point out why there is no straigthforward way of generalizing Theorem 1.1 for general connectivity augmentation. For this, let us discuss directed connecitvity augmentation first. In case of augmentation by one, Theorem 1.2 states that the minimum size of an augmenting arc set equals the maximum number of pairwise independent one-way pairs. The min-max formula will be quite similar if $(k - 1)$-connectivity is not assumed. In this case, we need to consider a broader class of one-way pairs: for a digraph $D = (V, A)$ and nonempty disjoint $X^-, X^+ \subseteq V$, $X = (X^-, X^+)$ is a one-way pair if there is no arc in A from X^- to X^+ ($|V - (X^- \cup X^+)| = k - 1$

is not assumed). Let us define $p(X) = \max(0, k - |V - (X^- \cup X^+)|)$. Clearly, an augmenting arc set should contain at least $p(X)$ arcs covering X. Then the minimum size augmenting edge set equals the maximum of $\sum_i p(X_i)$ over pairwise independent one-way pairs X_i. Actually, this is still only a special case of the theorem of Frank and Jordán in [6] where minimum coverings of positively crossing bisupermodular functions are considered.

A possible analogous approach for general undirected connectivity augmentation would be the following. Let a clump be a subpartition $X = (X_1, \ldots, X_\ell)$ of V with $d(X_i, X_j) = 0$ (we do not assume $|N_X| = k - 1$), and let us $p(X)$ be a lower bound on the number of edges needed to cover X. There are multiple possible candidates for $p(X)$ and we do not commit us to any of them, but work only with the natural assumption that $(\star)$ $p(X) = \max(0, k - |N_X|)$ whenever $|X| = 2$; and $p(X) = 0$ whenever $|N_X| \geq k$. Hence a natural conjecture is the following: the minimum size of an augmenting edge set equals the maximum deficiency of a grove, where in the definition of deficiency, each term $|X| - 1$ is replaced by $p(X)$.

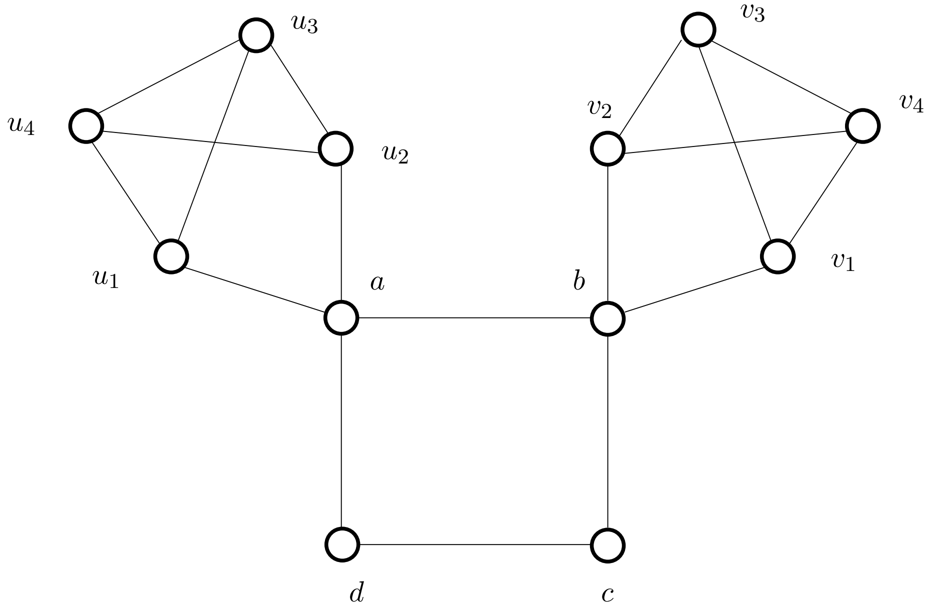

Figure 4: Example concerning general connectivity augmentation.

We show by an example that this conjecture fails even if $(\star)$ is the only assumption on $p(X)$. Let $G = (V, E)$ be the complement of the graph on Figure 4 and let $k = 9$. For a node $z \in V$, let $Z_z = (\{z\}, \{z\}^*)$. The only basic clumps in G with $|N_X| < 9$ are Z_a, Z_b, Z_{u_1}, Z_{u_2}, Z_{v_1}, Z_{v_2}, $(\{u_1, u_2\}, \{u_3\}, \{u_4\})$, $(\{v_1, v_2\}, \{v_3\}, \{v_4\})$ and $(\{a, c\}, \{b, d\})$. $\{u_1u_4, u_2u_3, v_1v_4, v_2v_3, ab, ad, bc\}$ is an augmenting edge set of size 7, while a grove of value 6 is the one consisting of two bushes $\mathcal{B}_1 = \{Z_{u_1}, Z_{u_2}, Z_{u_3}, Z_{u_4}, (\{a\}, \{u_1, u_2, d\}\}$ and $\mathcal{B}_2 = \{Z_{v_1}, Z_{v_2}, Z_{v_3}, Z_{v_4}, (\{b\}, \{v_1, v_2, c\}\}$.

We show that neither an augmenting edge set of size 6, nor a grove of value 7 exists. On the one hand, assume an augmenting edge set F exists with $|F| = 6$. Then F can be partitioned into $F = F_1 \cup F_2$ with $|F_1| = |F_2| = 3$, F_1 covering $\mathcal{B}_1$ and F_2 covering $\mathcal{B}_2$. However, we need at least two edges to cover Z_a and two to cover Z_b, and these can only be contained in F_1 and F_2, respectively. If $ad \in F_1$, then F_1 cannot contain any of au_1 and au_2 as otherwise at least one of Z_{u_3} and Z_{u_4} remain uncovered. Hence $ad \notin F_1$, and similarly $bc \notin F_2$. $ab, cd \notin F$ as they do not cover any of $\mathcal{B}_1$ and $\mathcal{B}_2$, thus $(\{a, c\}, \{b, d\})$ remains uncovered.

On the other hand, assume a grove of value 7 exists. We claim that it should contain $(\{a, c\}, \{b, d\})$, and two clumps $\{\{a\}, A\}$ and $\{\{b\}, B\}$ with $b \in A$ and $a \in B$. This is clearly a contradiction as they cannot be simultaneously contained in a grove since the edge ab connects all three of them. It can easily be checked that if we do not require that $(\{a, c\}, \{b, d\})$ or at least one clump of the form $\{\{a\}, A\}$ and one of the form $\{\{b\}, B\}$ should be covered then the remaining clumps may all be covered by six edges. This is the reason why all of them should be included in every grove.

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7 Appendix

In this Appendix we present how the subroutine for constructing a skeleton may be implemented using bipartite matching theory. The argument follows the same lines as the argument in the Appendix of [7]. Let us start with a simple claim concerning pieces.

Claim 7.1. *For a piece $Y \in \mathcal{P}_1$ and an arbitrary set $X \subseteq V$, if $X^* \supseteq Y^*$, then $X \subseteq Y$.*

Proof. Indeed, assume X is not a subset of Y, thus $|X \cup Y| > |Y|$. The condition gives $(X \cup Y)^* = Y^*$ hence $|N(X \cup Y)| < |N(Y)| = k-1$, contradicting that G is $(k-1)$-connected. □

Given the $(k-1)$-connected graph $G = (V, E)$, let us construct the bipartite graph $B = (V', V''; H)$ as follows. With each node $v \in V$ associate nodes $v' \in V'$ and $v'' \in V''$ and an edge $v'v'' \in H$. With each edge $uv \in V$ associate two edges $v'u'', u'v'' \in H$. For a set $X \subseteq V$, we denote by X' and X'' its images in V' and V'', respectively. The $(k-1)$-connectivity of G implies that B is $(k-1)$-**elementary bipartite**, that is, for each $\emptyset \neq X' \subseteq V'$, either $\Gamma(X') = V''$ or $|\Gamma(X')| \geq |X'| + k - 1$, where $\Gamma(X')$ denotes the set of neighbours of X'. We say that $X' \subseteq V'$ is **tight** if $|\Gamma(X')| = |X'| + k - 1$ and $\Gamma(X') \neq V''$. Observe that X' is tight if and only if $X \in \mathcal{P}_1$.

Given a function $f : V' \cup V'' \to \mathbb{N}$ we call the set $F \subseteq H$ an **f-factor** if $d_F(x) = f(x)$ for every $x \in V' \cup V''$. Let $f(Z) = \sum_{x \in Z} f(x)$ for $Z \subseteq V' \cup V''$.

Claim 7.2. *Consider a bipartite graph $G = (V', V''; H)$ and a function $f : V' \cup V'' \to \mathbb{N}$ so that $f(V') = f(V'')$ and $f(x) = 1$ or $f(y) = 1$ for every $xy \in H$. An f-factor exists if and only if $f(X) \leq f(\Gamma(X))$ for every $X \subseteq V'$.*

Proof. An easy consequence of Hall's theorem, replacing each $x \in V' \cup V''$ by $f(x)$ copies. Note that by the condition $f(x) = 1$ or $f(y) = 1$ for every $xy \in H$, at most one copy of the same edge may be used. $\square$

First we show how the set of minimal pieces $\mathcal{M}$ can be found. Let us consider nodes $u, v \in V$ with $uv \notin E$. A piece $X \in \mathcal{P}_1$ is called an ***uv*-piece**, if $u \in X$ and $v \in X^*$. For a $uv \notin E$, consider the following f. Let $f(u') = f(v'') = k + 1$ and for $z \in (V' - u') \cup (V'' - v'')$, let $f(z) = 1$. An f-factor for this f is called a **k-*uv*-factor**. If G is $(k-1)$-connected and thus B a $(k-1)$-elementary bipartite graph, then Claim 7.2 implies the existence of a $(k-1)$-uv-factor. Let F_{uv} denote one of them.

Claim 7.3. *If there is a k-uv-factor, then there exists no uv-piece.*

Proof. Assume X is a uv-piece. As $X \in \mathcal{P}_1$, $|\Gamma(X')| = |X'| + k - 1$. Since $u' \in X'$, $v'' \notin \Gamma(X')$, we have $f(X') = |X'| + k$, $f(\Gamma(X')) = |X'| + k - 1$, thus by Claim 7.2, no k-uv-factor exists. $\square$

It is easy to see that any two uv-pieces are dependent and the union and intersection of two uv-pieces are uv-pieces as well. Thus if the set of uv-pieces is nonempty, then it contains a unique minimal element. In what follows we show how this can be found algorithmically. For an edge set $F \subseteq H$, we say that the path $U = x_0 y_0 x_1 y_1 \ldots x_t y_t$ is an **alternating path** for F from x_0 to y_t, if $x_i \in V'$, $y_i \in V''$, $x_i y_i \in H - F$ for $i = 0, \ldots, t$, and $y_i x_{i+1} \in F$ for $i = 0, \ldots, t-1$. Under the same conditions we also say that $x_0 y_0 x_1 y_1 \ldots x_t$ is an alternating path for F from x_0 to x_t.

Claim 7.4. *(a) If there exists an alternating path for F_{uv} between u' and v'', then there exists no uv-piece. (b) Assume there is no alternating path for F_{uv} from u' to v''; let S denote the set of nodes $z \in V$ having an alternating path for F_{uv} from u' to z'. Then S is the unique minimal uv-piece and S is connected.*

Proof. (a) Let U be an alternating path for F_{uv} from u' to v''. Then $F_{uv} \Delta U$ is a k-uv-factor so by Claim 7.3 no uv-piece exists. (b) Let Z be an arbitrary uv-piece. For every $x \in Z - u$, $\Gamma(Z')$ contains a unique y'' with $x'y'' \in F_{uv}$. The number of $y \in V$ with $u'y'' \in F_{uv}$ is exactly k, and all of them are contained in $\Gamma(Z')$. These are $|Z'| + k - 1$ different elements of $\Gamma(Z')$, and since $Z \in \mathcal{P}_1$, $\Gamma(Z')$ has no elements other than these. This easily implies that Z' contains every $x' \in V$ for which there is an alternating path for F_{uv} from u' to x', showing $S \subseteq Z$. It is left to prove that $S \in \mathcal{P}_1$. From the definition of S, it follows that for every $y'' \in \Gamma(S')$, there exists an $x \in S$ with $x'y'' \in F_{uv}$, proving $\Gamma(S') = |S'| + k - 1$. The connectivity of S follows since otherwise the connected component containing u would be a smaller uv-piece. $\square$

For the initialization of the algorithm, we determine the edge sets F_{uv} by a single max-flow computation for every $u, v \in V$, $uv \notin E$. By Claim 7.4 the minimal uv-pieces may be found by a breadth-first search. The minimal ones among these will give the elements of $\mathcal{M}$ (note that the minimal $u_i v_i$-set might be contained in some other $u_j v_j$-set). We will use the sets F_{uv} also later steps on.

To implement the basic step of the algorithm, consider a stable cross-free $\mathcal{H}$ which is not complete, a minimal element $M \in \mathcal{P} - \cup\mathcal{H}$ and $\mathcal{L}_1$, $\mathcal{L}_2$ as defined by (2). By Lemma 4.4, our task is to find a Z fitting $(\mathcal{H}, M)$ minimal subject to this property. Let $\mathcal{T}$ be the set of the maximal ones among those pieces of the clumps in $\mathcal{L}_1$ which are subsets of M^*.

Claim 7.5. *$\mathcal{T}$ consits of pairwise disjoint sets.*

Proof. Consider clumps $X, Y \in \mathcal{L}_1$ with pieces $X_1, Y_1 \in \mathcal{T}$. If X and Y are independent then $X_1 \cap Y_1 = \emptyset$ as otherwise an edge between $X_1 \cap Y_1$ and M would connect both. If they are dependent, then we show that the dominant side X_i of X w.r.t Y is different from X_1. Indeed, if $X_i = X_1$, then the dominant side of Y w.r.t. X shall be $Y_j \neq Y_1$ as otherwise $M \subseteq Y_1$ would follow. Hence $Y_1 \subset X_1$, a contradiction to the maximality of Y_1. Similarly, the dominant side of Y w.r.t. X may not be Y_1. Hence $Y_1 \subseteq X^*$, thus $X_1 \cap Y_1 = \emptyset$. $\square$

Let us construct the bipartite graph $B_1 = (V', V''; H_1)$ from B by adding some new edges as follows. (1) For each $X \in \mathcal{L}_2$, let $x'y'', y'x'' \in H_1$ for every xy connecting X. (2) Let $x'y'' \in H_1$ whenever $T \in \mathcal{T}$, $x \in T$ and $y \in T \cup N(T)$. (3) For each $X \in \mathcal{L}_1$ with a piece $X_j \supseteq M$, let $x'y'' \in H_1$ for every $x \in X_j$, $y \in X_j^*$.

Claim 7.6. *Let $Z \in \mathcal{P} - \cup\mathcal{H}$, $Z \subseteq M^*$, supported by some $W \in \mathcal{H}$. Z fits $(\mathcal{H}, M)$ if and only if Z' is tight in B_1.*

Proof. The tight sets of B_1 are those tight sets Z' of B for which there is no edge in $x'y'' \in H_1 - H$ with $x' \in Z'$ and $y'' \in V'' - \Gamma(Z')$. For such edges we say that $x'y''$ **blocks** Z'. (This is equivalent to that xy connects Z.)

Assume Z fits $(\mathcal{H}, M)$. Property (d) forbids that any $x'y'' \in H_1 - H$ of the first type block Z', while (c) forbids any $x'y''$ of the second or third type to block Z'. For the other direction, properties (a) and (b) follow by the conditions. For (d), if Z were dependent with some $X \in \mathcal{L}_2$ then a new edge of the first type would block Z'. For (c), if $Z \cap X_i \neq \emptyset$, $X_i - Z \neq \emptyset$ for some $X \in \mathcal{L}_1$ with piece $X_i \subset M^*$, then consider a $T \in \mathcal{T}$ with $X_i \subseteq T$. By Claim 7.1, $Z^* \cap (T \cup N(T)) \neq \emptyset$, hence a new edge of the second type blocks Z'. Finally, if $X_j \cap Z \neq \emptyset$, $X_i \cap Z^* \neq \emptyset$, then there is a new edge of the third type blocking Z'. $\square$

To find a Z as in Lemma 4.4, we need to add some further edges to B_1. Indeed, we need to ensure that $Z \in \mathcal{P} - \cup\mathcal{H}$ and furthermore that Z is supported by some $W \in \mathcal{L}_1$. Consider now a $W \in \mathcal{L}_1$ with a piece $W_1 \in \mathcal{T}$ and a connected set Q with $W_1 \subset Q \subseteq M^*$. Let $Z(Q)$ denote the unique minimal X satisfying the following property:

$$X \in \mathcal{P},\ Q \subseteq X, \text{ and } X \text{ fits } (\mathcal{H}, M). \tag{3}$$

We will determine $Z(Q)$ for different sets Q in order to find K. $Z(Q)$ is well-defined since it is easy to see the following: (i) the connected component of M^* containing Q itself satisfies (3); (ii) if X and X' satisfy (3), then X and X' are dependent and $X \cap X'$ also satisfies (3); (iii) $Z(Q)$ is connected. The next claim gives an easy algorithm for finding $Z(Q)$ for a given Q.

Claim 7.7. *Fix some $u \in Q$, $v \in M$. Let B_2 denote the graph obtained from B_2 by adding all edges $u'y''$ with $y \in Q \cup N(Q)$. Let S denote the set of nodes z for which there exists an alternating path for F_{uv} from u' to z'. Then $Z(Q) = S$.*

Proof. As M^* is an uv-piece in B_2, applying Claim 7.4(a) for B_2 instead of B, we get that B_2 contains no alternating path for F_{uv} between u' and v''. By Claim 7.4(b), S is the unique minimal uv-piece in B_2. $\Gamma(S' \cup Q') = \Gamma(S')$ thus $Q \cup N(Q) = S \cup N(S)$, hence by Claim 7.1, $Q \subseteq X$. By making use of Claim 7.6, X is the unique minimal set satisfying (3), thus $Z(Q) = X$. □

Consider now a $W = (W_1, W_2, \ldots, W_h) \in \mathcal{L}_1$ with $W_1 \in \mathcal{T}$. We want to find a Z_W fitting $(\mathcal{H}, M)$ supported by W_1. For each $q \in N_W \cap M^*$, let us compute $Z(Q)$ for $Q = W + q$. Let Z_W denote a minimal set among these. A $Z(Q)$ can be found by a single breadth-first search, thus we need at most $k - 1$ breadth-first searches. We may compute such a Z_W for all possible choices of W, and a minimal among these gives a minimal Z fitting $(\mathcal{H}, M)$. Therefore the running time may be bounded by $(k-1)n$ breadth-first searches since by Claim 7.5, $|\mathcal{T}| \leq n$.

7.1 Complexity

To find a skeleton system first we need n^2 Max Flow computations for computing the minimal pieces and the auxiliary graphs. The running time for determining a member of the skeleton is dominated by $(k-1)n$ breadth first searches. Thus if s is an upper bound for the size of a skeleton then we can determine a complete down-closed system in $O(n^5 + skn^3)$ running time by using an $O(n^3)$ maximum flow algorithm and an $O(n^2)$ breadth first search algorithm.