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Tree-compositions and submodular flows

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Abstract

Tree-composition of a set was defined in [3]. We prove a theorem that makes the construction of a tree-composition rather easy. Using this result we give an algorithmic proof of the main theorem of [3] on submodular flows. The result can be applied to decide if a mixed graph admits a k-edge-connected orientation.

1 Introduction

Submodular flows were introduced and investigated by Edmonds and Giles in [1], while the paper [3] described a theorem on existence of feasible submodular flows with a crossing submodular border function. This theorem can be used to prove some orientation theorems, so it is useful to make it algorithmic. To this end we will rely on an algorithm that computes a feasible submodular flow given by a submodular border function ([8], [5]). The main task is to compute the tree-composition that tells us that no feasible submodular flow exists if the submodular flow polyhedra is empty.

1.1 Definitions

Two sets A and $B \subseteq V$ are **crossing** if $A - B \neq \emptyset$, $B - A \neq \emptyset$, $A \cap B \neq \emptyset$, $V - (A \cup B) \neq \emptyset$. We say that a family $\mathcal{F}$ on a ground set V is a **crossing family** if $A \cap B \in \mathcal{F}$ and $A \cup B \in \mathcal{F}$ for any two crossing members $A, B \in \mathcal{F}$. We call a family $\mathcal{F}$ on a ground set V **cross-free** if there are not any crossing pairs in $\mathcal{F}$.

A $t\overline{s}$**-set** is a set that contains t but does not contain s. For a given family $\mathcal{F}$ we usually use the term $t\overline{s}$-set only for the members of $\mathcal{F}$. Let S and T be disjoint sets. We call a family $\mathcal{F}$ on the ground set $S \cup T$ $\overline{S}$**-**T**-separating** if for any $s \in S$ and $t \in T$ there is an $t\overline{s}$-set in $\mathcal{F}$. We say that an S-T-separating family $\mathcal{F}$ is **minimal** if no proper subfamily of $\mathcal{F}$ is $\overline{S}$-T-separating.

Let $\{Z_1, Z_2, ..., Z_t\}$ be a partition of $Z \subseteq S$, and let $\{Z_i^1, Z_i^2, ..., Z_i^{t_i}$ be a partition of $S - Z_i$. Then the set-system $\mathcal{D} := \{S - Z_i^j : 1 \leq i \leq t, q \leq j \leq t_i\}$ is called the **double partition of** Z.

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Let $\varphi : V \to U$ be a map, and $F = (U, A)$ an oriented tree. For $e \in A$ let V_e be $\varphi^{-1}(U_e)$, where U_e is the component of $F - e$, that contains the head of e. Let S and T be disjoint non-empty sets. We call a family $\mathcal{F}$ on a ground set $V = S \cup T$, **S-T-tree-composition** if there is an oriented tree $F = (U = U_S \cup U_T, A)$ and a surjective map $\varphi : V \to U$ for which for any $u \in U_S$ $\varphi^{-1}(u) \subseteq S$ for any $u \in U_T$ $\varphi^{-1}(u) \subseteq T$, every edge of F has a tail in U_S and a head in U_T, and $\mathcal{F} = \{V_e : e \in A\}$. By convention a family on a ground set V is a **tree-composition on a set $\emptyset \neq Z \subseteq V$** if it is a $(V - Z)$-Z-tree-composition. The **ground-degree** of a double partition or tree-composition $\mathcal{F}$ is $\min\{d_{\mathcal{F}}(v) : v \in V\}$.

Let $G = (S, T, E)$ be a bipartite graph with color classes S and T. We say that an **S-T-tree-composition $\mathcal{F}$ belongs to the graph** if for any $s \in S, t \in T, st \in E$ the image $\varphi(s)\varphi(t) \in A$, where $F = (U, A)$ is the oriented tree belonging to $\mathcal{F}$ with the surjective map $\varphi : V \to U$.

Let $G = (V, \mathcal{E})$ be a graph. For $v \in V$ we denote by $d(v)$ or $d_G(v)$ the **degree** of v. For $X \subseteq V$ we denote by $i(X)$ the **number of edges induced by** X. For an oriented graph $D = (V, A)$, we denote by $\rho(v)$ or $\rho_D(v)$ the **in-degree** of the node v, and we denote by $\delta(v)$ or $\delta_D(v)$ the **out-degree** of the node v. We use $d(X), \rho(X), \delta(X)$ or $d_G(X), \rho_D(X), \delta_D(X)$ to denote the **degree, in-degree or out-degree of a subset** $X \subseteq V$. For $X, Y \subseteq V$ we denote by $d(X, Y)$ or $d_G(X, Y)$ the number of edges, that have got an endpoint in both of $X - Y$ and $Y - X$. For a partition $\mathcal{P}$ of the node set V we denote by $e(\mathcal{P})$ the number of edges that endpoints belong to two elements of $\mathcal{P}$.

A digraph is **k-edge-connected** if for any two node u and v there is k edge disjoint oriented path from u to v. Let r be a node of a digraph D, and let $0 \leq \ell \leq k$. We say that D is **r-rooted (k, ℓ)-edge-connected** if there is k edge disjoint path from r to any node of D, and there is ℓ edge disjoint path from any node of D to r.

We call a set-function $b : 2^S \to \mathbb{R} \cup \{\infty\}$ **(crossing) submodular** if $b(X) + b(Y) \geq b(X \cap Y) + b(X \cup Y)$ for every (crossing) $X, Y \subseteq S$. We call a set-function $p : 2^S \to \mathbb{R} \cup \{-\infty\}$ **(crossing) supermodular** if $p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y)$ for every (crossing) $X, Y \subseteq S$.

For a vector $x \in \mathbb{R}^S$ and $X \subseteq S$ $\tilde{x}(X) := \sum_{v \in X} x(v)$.

For a (crossing) submodular function b we define $B(b)$, as $B(b) := \{x \in \mathbb{R}^S : \tilde{x}(X) \leq b(X)(\forall X \subseteq S), b(S) = \tilde{x}(S)\}$. If b is submodular, then we say that $B(b)$ is the **base-polyhedron of** b. For a (crossing) supermodular function p we define $B'(p)$, as $B'(p) := \{x \in \mathbb{R}^S : \tilde{x}(X) \geq p(X)(\forall X \subseteq S), p(S) = \tilde{x}(S)\}$. If p is supermodular, then we say that $B'(p)$ is the **base-polyhedron of** p.

For a graph $G = (V, E)$ a set-function $h : 2^V \to \mathbb{R}$ is **(crossing) G-supermodular** if $h(X) + h(Y) \leq h(X \cap Y) + h(X \cup Y) + d_G(X, Y)$ for every (crossing) $X, Y \subseteq V$.

Let $b : 2^S \to \mathbb{R} \cup \{\infty\}$ be an arbitrary set function. Then the **lower truncation of** b is a set function $b^{\vee} : 2^S \to \mathbb{R} \cup \{\infty\}$, for which

$$b^{\vee}(X) := \min\{b(\mathcal{P}) : \mathcal{P} \text{ is a partition of } X\} \tag{1}$$

holds for any $X \subseteq S$. Let $p : 2^S \to \mathbb{R} \cup \{-\infty\}$ be an arbitrary set function. Then

the upper truncation of p is a set function $p^\wedge : 2^S \to \mathbb{R} \cup \{-\infty\}$, for which

$$p^\wedge(X) := \max\{p(\mathcal{P}) : \mathcal{P} \text{ is a partition of } X\} \tag{2}$$

holds for any $X \subseteq S$.

Let $b : 2^S \to \mathbb{R} \cup \{\infty\}$ be a crossing submodular function. It is well known that if $B(b) \neq \emptyset$, then it is a base-polyhedron whit unique submodular border function. We call this submodular function the **full (lower) truncation or bi-truncation** of b, and we denote it by $b^\downarrow$. It is also known, that if $b(S) = 0$ and $B(b) \neq \emptyset$, then $b^\downarrow$ can be computed by the following formula.

$$b^\downarrow(Z) = \min\left\{\sum_{X\in\mathcal{D}} b(X) : \mathcal{D} \text{ is a double-partition of } Z\right\}. \tag{3}$$

Let $p : 2^S \to \mathbb{R} \cup \{-\infty\}$ be a crossing supermodular function. It is well known, that if $B'(p) \neq \emptyset$, then it is a base-polyhedron, whit unique submodular border function. We call this submodular function the **full (upper) truncation or bi-truncation** of p, and we denote it by $p^\uparrow$. It is also known, that if $B'(p) \neq \emptyset$, then $p^\uparrow$ can be computed, by the following formula.

$$p^\uparrow(Z) = \tag{4}$$

$$= \max\left\{\sum_{X\in\mathcal{D}} p(X) - \Delta p(S) : \mathcal{D} \text{ is a double-partition of } Z, \text{ with ground-degree } \Delta\right\}$$

(see [6], [3]). These formulas were simplified by [3]. We will give a new proof to these simplified versions.

Let $D = (V, A)$ be a directed graph, $f : A \to \mathbb{Z} \cup \{-\infty\}$, $g : A \to \mathbb{Z} \cup \{\infty\}$ two integer-valued bounding functions for which $f \leq g$. Moreover, we are given a crossing submodular set-function $b : 2^V \to \mathbb{Z} \cup \{\infty\}$ for which $b(\emptyset) = 0$ and $b(V)$ is finite. A function (or vector) $x : A \to \mathbb{R}$ is called a **submodular flow or subflow** confined by b if $\Psi_x(Z) := \rho_x(Z) - \delta_x(Z) \leq b(Z)$ for every $Z \subseteq V$, where $\rho_x(Z) := \sum\{x(uv) : u \in V - Z, v \in Z\}, \delta_x(Z) := \sum\{x(uv) : u \in Z, v \in V - Z\}$. Hence $\Psi_x(V) = 0$, and we can modify b on V, we can assume, that $b(V) = 0$. A subflow x is feasible if $f \leq x \leq g$. The set $Q(f, g; b)$ of feasible subflows is called a **submodular flow polyhedron**.

2 The main result

Theorem 2.1. *Let $\mathcal{F}$ be an $\overline{S}$-T-separating crossing family. Then there is an S-T-tree-composition $\mathcal{T}$ such that $\mathcal{T} \subseteq \mathcal{F}$. Moreover a family $\mathcal{T}$ is an S-T-tree-composition if and only if it is a minimal cross-free $\overline{S}$-T-separating family.*

Proof: First we will show that a crossing $\overline{S}$-T-separating family always contains a cross-free $\overline{S}$-T-separating family. Hence the minimal (or the maximal) $t\overline{s}$-sets do not form always a cross-free family, we need to work a bit more.

Lemma 2.2. *Let $\mathcal{F}$ be an $\overline{S}$-T-separating crossing family. For $s \in S$ let the elements of $\mathcal{F}_s$ be the connected components of the hypergraph $\mathcal{H}_s = (S \cup^* T - \{s\}, \mathcal{E})$, where $\mathcal{E} = \{M_{t\overline{s}} : t \in T\}$, where $M_{t\overline{s}}$ denotes the minimum $t\overline{s}$-set in $\mathcal{F}$. Then the subfamily*

$$\mathcal{G} := \bigcup_{s \in S} \mathcal{F}_s$$

of $\mathcal{F}$ is cross-free $\overline{S}$-T-separating.

Note that hence $\mathcal{F}$ is crossing, the intersection of $t\overline{s}$-sets in $\mathcal{F}$ is also in $\mathcal{F}$, so $M_{t\overline{s}}$ really exists.

Proof: By the definition of $\mathcal{G}$ it is easy to see that $\mathcal{G}$ is $\overline{S}$-T-separating, because $\mathcal{F}_s$ contains an $t\overline{s}$-set for all $t \in T$, that is the component of t in $\mathcal{H}_s$. We get the components of $\mathcal{H}_s$ by the union of crossing sets of $\mathcal{F}$, so $\mathcal{G} \subseteq \mathcal{F}$ is also clear.

Hence the elements of $\mathcal{F}_s$ are disjoint, we only need to prove that for two $s_1 \neq s_2$ elements of S a set $F_1 \in \mathcal{F}_{s_1}$ will not cross a set $F_2 \in \mathcal{F}_{s_2}$. For a contradiction assume that F_1 and F_2 are crossing.

Case 1: If $s_2 \notin F_1$, then for $t \in F_1 \cap T$ $M_{t\overline{s_1}} = M_{t\overline{s_2}}$, because $M_{t\overline{s_1}} \subseteq F_1 \not\ni s_2$, so $M_{t\overline{s_1}}$ is a $t\overline{s_2}$-set also, but if $M_{t\overline{s_2}}$ is smaller than $M_{t\overline{s_1}}$, then it would be a smallest $t\overline{s_1}$-set. That means there is a component of H_{s_2} that contains F_1, so F_1 and F_2 are disjoint or $F_1 \subseteq F_2$, that is a contradiction.

Case 2: If $s_2 \in F_1$, then there is a $t \in F_1 \cap T$, for which $s_2 \in M_{t\overline{s_1}}$. $M_{t\overline{s_1}} \cup M_{t\overline{s_2}} \subseteq F_1 \cup F_2 \neq S \cup^* T$, because F_1 and F_2 are crossing sets by our assumption. We also know that $t \in M_{t\overline{s_1}} \cap M_{t\overline{s_2}} \neq \emptyset$, that means $M_{t\overline{s_1}}$ and $M_{t\overline{s_2}}$ are crossing sets or $s_2 \notin M_{t\overline{s_2}} \subset M_{t\overline{s_1}} \ni s_2$. So we got that $s_2 \notin M_{t\overline{s_1}} \cap M_{t\overline{s_2}} \in \mathcal{F}$ is a smaller $t\overline{s_1}$-set than $s_2 \in M_{t\overline{s_1}}$, that is a contradiction. □

The following claim is obvious by the definition of S-T-tree-composition.

Claim 2.3. *If $\mathcal{T}$ is an S-T-tree-composition, then it is a minimal cross-free $\overline{S}$-T-separating family.*

First we show the other direction for laminar families.

Claim 2.4. *If $\mathcal{F}$ is a minimal laminar $\overline{S}$-T-separating family, then it is an S-T-tree-composition, and the tree representing $\mathcal{F}$ is a star, with the center node S.*

Proof: A minimal laminar $\overline{S}$-T-separating family is just a partition of T, because if it has an element F for which $s \in F \cap S$, then hence for all $t \in F \cap T$ there must be an $t\overline{s}$, that must be the subset of F by the laminarity, we can delete F from $\mathcal{F}$ without violating that $\mathcal{F}$ is $\overline{S}$-T-separating, that contradicts to the minimality. A partition of T is obviously represented by a star, with the center node S. □

Now we are ready to prove the main lemma.

Lemma 2.5. *If $\mathcal{F}$ is a minimal cross-free $\overline{S}$-T-separating family, then it is an S-T-tree-composition.*

Proof: We will give the representing tree. Let the vertices of the tree be the **atoms**, that are the maximal disjoint subsets of $S \cup T$, from which every $F \in \mathcal{F}$ can be built up. If there is a subset of $S \cup T$, that is not contained in any element of $\mathcal{F}$, then $\mathcal{F}$ is laminar, and we are done by Claim 2.4. So we can assume that the atoms cover $S \cup T$. Hence $\mathcal{F}$ is an $\overline{S}$-T-separating family, there is not any atom, that intersects both of S and T, because if it contains an $s \in S$ and a $t \in T$, then there is not any $t\overline{s}$-set in $\mathcal{F}$. Let $\varphi : V \to U$ be a surjective map which maps every atoms to a single node. To define the edges we need the following claims:

Claim 2.6. *For any $F \in \mathcal{F}$ there uniquely exists an atom $A_F \subseteq T \cap F$, such that for all $F' \in \mathcal{F}$, that is a subset of F, $A_F \nsubseteq F'$.*

Proof: There must exist such an atom by minimality, otherwise we could omit F from $\mathcal{F}$, so we only need to prove the uniqueness.

Case 1: If $F \cap S = \emptyset$, then we can prove that F is an atom. For $F' \subseteq F$ $F' \notin \mathcal{F}$ always holds, otherwise F' would be omittable from $\mathcal{F}$. If $F' \in \mathcal{F}$, then $S \nsubseteq F'$, because otherwise F' is omittable. So if F' intersects F, then it must contain F, because we have seen that F could not contain F', and they cannot cross. So in this case the atom $A_F = F$ is unique.

Case 2: If $F \cap S \neq \emptyset$, then assume for a contradiction that there are two atoms A_1 and A_2 with the property described in the claim. There must be a set $F_1 \in \mathcal{F}$, that separates A_1 and A_2, because A_1 and A_2 are different atoms. We can assume, by changing the indices that $A_2 \in F_1$, and $A_1 \cap F_1 = \emptyset$. Hence A_1 and A_2 hold the property described in the claim, this set could not be contained by F. Moreover $A_2 \subseteq F_1 \cap F \neq \emptyset$, and $A_1 \subseteq F \nsubseteq F_1$. $S - F_1 \neq \emptyset$, because otherwise F_1 is omittable. So $(S \cup T) - F \subseteq F_1$ and $(S \cup T) - F_1 \subseteq F$, because $\mathcal{F}$ is cross-free. For all $s \in S - F_1$ there is a set $F_s \in \mathcal{F}$ that avoids s and contains A_1. We can prove that $(S \cup T) - F \subseteq F_s$ as for F_1. So $F_1 \cap F_s \subseteq (S \cup T) - F$. $s \notin F_1 \cup F_2$, and $A_1 \in F_s - F_1$, so, hence $\mathcal{F}$ is cross-free, $F_1 \subseteq F_s$. Hence this hols for any $s \in S - F_1$, for any $t \in T \cap F_1$ F_s is also a $t\overline{s}$-set, so we can omit F_1 from $\mathcal{F}$ that is a contradiction. $\square$

Applying the above claim to the set system, that we get by complementing every element of $\mathcal{F}$ we get the following:

Claim 2.7. *For any $F \in \mathcal{F}$ there uniquely exists an atom $B_F \subseteq S - F$, such that for all $F' \in \mathcal{F}$, that contains F, $B_F \subseteq F'$.*

Let us define the edge set to be

$$A := \{e_F = \overrightarrow{\varphi(B_F)\varphi(A_F)} : F \in \mathcal{F}\},$$

so we get the oriented graph $D = (U, A)$.

Claim 2.8. *The only edge of D that enters to a set $F \in \mathcal{F}$ is e_F.*

Proof: Assume that $e_{F'}$ also enters $\varphi(F)$. It means that $B_{F'} \subseteq S - (F \cup F')$ and $A_{F'} \subseteq F \cap F' \cap T$. Hence F and F' do not cross, $F \subset F'$ or $F' \subset F$ holds. If $F \subset F'$, then $A_{F'} \nsubseteq F$ by definition, so $e_{F'}$ does not enters $\varphi(F)$, a contradiction. And if $F' \subset F$, then $B_{F'} \subseteq F$ by definition, so $e_{F'}$ does not enters $\varphi(F)$, a contradiction. $\square$

Claim 2.9. *No edge of D leaves a set $F \in \mathcal{F}$.*

Proof: Assume that $e_{F'}$ leaves $\varphi(F)$ for an $F' \in \mathcal{F}$. First we show that $B_F \subset F'$. To do this we show that if $B_F \not\subset F'$, then $B_{F'} \not\subset F$, so $e_{F'}$ does not leave $\varphi(F)$. By the definition of $B_{F'}$ every set $F'' \in \mathcal{F}$, that contains F', contains $B_{F'}$, so it intersects F. If B_F would not be a subset of F', then we could choose F'' not to contain B_F, hence $B_F \neq B_{F'}$. So $F \cup F'' \neq S \cup T$. Hence F and F' intersect each other and they do not cross, $F \subseteq F''$ or $F'' \subseteq F$. But hence $e_{F'}$ leaves $\varphi(F)$, $A_{F'} \subseteq F' - F \subseteq F'' - F$, so the second case could not hold. In the first case F'' would be an element of $\mathcal{F}$, that contains F, but does not contain B_F, that contradicts to the definition of B_F. So $B_F \subseteq F'$. For any $s \in B_F$ there is a $G \in \mathcal{F}$ that contains $A_{F'}$, but avoids s, because $\mathcal{F}$ is $\overline{S}$-T-separating. $G \nsubseteq F'$ by the definition of $A_{F'}$. Hence $\mathcal{F}$ is cross-free G does not cross F'. So $(S \cup T) - F' \subset G$. G does not cross F also. Hence $B_{F'} \subseteq G \cap F$ and they does not contain s, $F \subset G$. But in this case G would be an element of $\mathcal{F}$, that contains F, but does not contain B_F, that contradicts to the definition of B_F. $\square$

The above two claims show that the underlying unoriented graph of D is a forest, hence we gave a set for every edge e of D, that has only e as in-edge, and no out-edge. So to show that D is an oriented tree, we only need to show that its vertex number is at least one more than its edge number. We know that the edge number is $|\mathcal{F}|$ by the definition of the edges.

Claim 2.10. *A cross-free family $\mathcal{F}$ on the ground set V has at most one more atoms than elements.*

Proof: We do induction on $|\mathcal{F}|$. The claim is obvious for $|\mathcal{F}| = 1$. Let $\mathcal{F}' := \mathcal{F} \cup \{F\}$, where $\mathcal{F}$ is cross free. We will show that $\mathcal{F}'$ has at most one more atoms, or $\mathcal{F}'$ is not cross free. $\mathcal{F}'$ could have at least two more atoms than $\mathcal{F}$ only if F intersects two atoms of $\mathcal{F}$, and does not contain them, and they do not contain F. Let these atoms be A and B. $\mathcal{F}$ has an element F' that contains exactly one of them, we can assume that it contains A. But then F and F' are crossing, because $\emptyset \neq F \cap A \subseteq F \cap F'$, $\emptyset \neq A - F \subseteq F' - F$, $\emptyset \neq B - F \subseteq V - (F \cup F')$. $\emptyset \neq F \cap B \subseteq F - F'$. $\square$

Now we are done with the proof of the lemma, because D is an oriented tree with edges from S to T, and we proved that $V_{e_F} = F$, hence F has one in-edge, that is e_F, and its out-degree is 0. $\square$

Minimizing the family $\mathcal{G}$ of Lemma 2.2 we get an S-T-tree-composition according to the Lemma 2.5. $\square$

The proof above give rise to a polynomial algorithm to compute a subfamily of an (almost) crossing $\overline{S}$-T-separating family, that is an S-T-tree-composition.

2.1 Another proof

In this section we will give another proof to the main statement of Theorem 2.1, because the second part of the previous proof could be simplified using the following result of Edmonds and Giles [1]:

Lemma 2.11. *For every cross-free family $\mathcal{F}$ on a ground set V, there exists a directed tree $T = (U, A)$, along with a map $\varphi : V \to U$, so that the sets in $\mathcal{F}$ and the edges of T are in a one-to-one correspondence, as follows. For every edge $e \in A$, the corresponding set of $\mathcal{F}$ is $\varphi^{-1}(V_e)$.*

We will modify the cross free family $\mathcal{G}$ given in Lemma 2.2 until its tree-representation is such as wanted. We will also use the following observation:

Remark 2.12. It can be proved by induction, that a directed tree $T = (U, A)$ admits a **level function** $\pi : U \to \{0, 1, 2, ...\}$ so that $\pi(v) - \pi(u) = 1$ for every $uv \in A$. One can see that if (T, φ) is the tree representation of the cross-free family $\mathcal{F}$ on the ground set V, and π is a level function of T, then $\pi(\varphi(v))-\pi(\varphi(v')) = d_{\mathcal{F}}(v)-d_{\mathcal{F}}(v')$.

Proof: Using Lemma 2.2 we get a cross-free subfamily $\mathcal{G} \subseteq \mathcal{F}$, that is $\overline{S}$-T-separating, moreover all the elements of T are in $\Delta = |S|$ elements of $\mathcal{G}$, and all the elements of S are in less than $\Delta = |S|$ elements of $\mathcal{G}$. By Lemma 2.11 and Remark 2.12 $\mathcal{G}$ has a tree representation $(F, \varphi) = (U, A, \varphi)$, with a level function π. Let L^* and L_* denote the highest and lowest level's nodes, respectively.

If $\varphi^{-1}(u)$ is empty for a node $u \in L^*$, then $\{V_e : e \in A, e = vu\}$ is a co-partition of V. Revise $\mathcal{G}$ by removing the members of this co-partition. Since a co-partition is a regular hypergraph, the revised $\mathcal{G}$ continues to cover the elements of T Δ times, and the elements of S less then Δ times for some Δ, and by this it remains to be an $\overline{S}$-T-separating family.

If $\varphi^{-1}(u)$ is empty for a node $u \in L^*$, then $\{V_e : e \in A, e = uv\}$ is a partition of V. Revise $\mathcal{G}$ by removing the members of this partition. Since a partition is a regular hypergraph, the revised $\mathcal{G}$ continues to cover the elements of T Δ times, and the elements of S less then Δ times for some Δ, and by this it remains to be an $\overline{S}$-T-separating family.

Consider now the case when $\varphi^{-1}(u)$ is non-empty and F has more than two levels. Then $d_{\mathcal{G}}(s) \leq \Delta - 2$ for every $s \in \varphi^{-1}(u) \subseteq S$ since the difference of the highest and the lowest levels is at least two. Now the members of $\mathcal{G}$ corresponding to the tree edges leaving u form a partition of $V - \varphi^{-1}(u)$. Revise again $\mathcal{G}$ by removing the members of this subpartition. The revised $\mathcal{G}$ continues to cover the elements of T Δ' times, and the elements of S less then Δ' times for $\Delta' = \Delta - 1$, because the elements of $\varphi^{-1}(u)$ were in at most $\Delta - 2$ elements of the original $\mathcal{G}$, and the other elements will be in one less element of the revised $\mathcal{G}$, then they were in the original.

In this way we arrive at a family $\mathcal{G} \subseteq \mathcal{F}$ for which the representing directed tree has two levels and no empty nodes, which means that $\mathcal{G}$ is a subfamily of $\mathcal{F}$ that is an S-T-tree-composition. $\square$

Hence the proof of Lemma 2.11 is algorithmic, the above proof also gives us a polynomial algorithm.

3 Computing the full truncation

Here we will give an algorithm to compute the full truncation of a crossing supermodular function b. We will use the following well known theorem, see for example in [6].

Theorem 3.1. *Let b be a submodular function. Then $b(Z) = \max\{\tilde{m}(Z) : m \in B(b)\}$. Let p be a supermodular function. Then $p(Z) = \min\{\tilde{m}(Z) : m \in B'(p)\}$. If b or p is integer valued, then the maximum or minimum could be taken by an integer vector.*

We give an algorithmic proof to the following theorem of [3].

Theorem 3.2. *Let b be a crossing submodular function for which $b(S) = 0$ and $B(b) \neq \emptyset$. Then*

$$b^{\downarrow}(Z) = \min\left\{\sum_{F \in \mathcal{F}} b(F) : \mathcal{F} \text{ is a tree-composition on } Z\right\}.$$

Proof: Hence a tree-composition is a special double-partition, (3) implies that $b^{\downarrow} \leq \min\{\sum_{F \in \mathcal{F}} b(F) : \mathcal{F}$ is a tree-composition on $Z\}$, so we need to show a tree-composition, for which equality holds.

$B(b) = B(b^{\downarrow})$ by definition. Theorem 3.1 implies that there is an element m of $B(b)$, for which $\tilde{m}(Z) = \sum_{z \in Z} m(z) = b^{\downarrow}(Z)$. Call a subset $X \subset S$ tight if $\tilde{m}(X) = b(X)$, and let $\mathcal{F}$ be the family of tight sets. Then $\mathcal{F}$ is a crossing system by submodularity: $\tilde{m}(X) + \tilde{m}(Y) = b(X) + b(Y) \geq b(X \cap Y) + b(X \cup Y) \geq \tilde{m}(X \cap Y) + \tilde{m}(X \cup Y) = \tilde{m}(X) + \tilde{m}(Y)$, whenever $X, Y \subseteq S$ are crossing.

Claim 3.3. *There exists a tight $t\overline{s}$-set for every $s \in S - Z,\ t \in Z$, so $\mathcal{F}$ is $(S-Z)$-Z-separating.*

Proof: If there is an $s \in S - Z,\ t \in Z$, for which no tight $t\overline{s}$-set exists, then for $\varepsilon := \min\{b(X) - \tilde{m}(X) : X$ is an $t\overline{s}$-set$\}$ the vector m', for which $m'(s) = m(s) - \varepsilon,\ m'(t) = m(t) + \epsilon,\ m'(v) = m(v)\ (v \in S - \{s,t\})$, would belong to $B(b)$, but would not belong to $B(b^{\downarrow})$, that is a contradiction. □

By Theorem 2.1, there is an $(S-Z)$-Z-tree-composition $\mathcal{T}$ on tight sets. Let Δ be the degree of $\mathcal{T}$ on the elements of $S - Z$. (Then the degree of $\mathcal{T}$ on the elements of Z is $\Delta + 1$.) Then $\sum_{X \in \mathcal{T}} b(X) = \sum_{X \in \mathcal{T}} \tilde{m}(X) = \Delta\tilde{m}(S-Z) + (\Delta+1)\tilde{m}(Z) = \Delta\tilde{m}(S) + \tilde{m}(Z) = \Delta b(S) + \tilde{m}(Z) = 0 + b^{\downarrow}(Z)$. □

With a similar proof one can get the following.

Theorem 3.4. *Let p be a crossing submodular function for which $b(S) = 0$ and $B'(p) \neq \emptyset$. Then*

$$p^{\uparrow}(Z) =$$

$$= \max\left\{\sum_{F \in \mathcal{F}} p(F) - \Delta p(S) : \mathcal{F} \text{ is a tree-composition on } Z, \text{ with ground-degree } \Delta\right\}.$$

4 Feasible submodular flows

In this section we give an algorithmic proof to the following theorem of [3] on existence of submodular flows. Now we will define the tree-composition of the ground set, and the empty set also, as follows. We say that $\mathcal{F}$ is a tree-composition of V or $\emptyset$ if $\mathcal{F}$ is a partition or a co-partition of V.

Theorem 4.1. *Let b be a crossing submodular function for which $b(V) = 0$. There is an integer feasible submodular flow if and only if*

$$\rho_f(Z) - \delta_g(Z) \leq \sum_{X \in \mathcal{F}} b(X) \tag{5}$$

for every subset $Z \subseteq V$ and every tree-composition $\mathcal{F}$ of Z.

Proof: First we check whether $B(b)$ is empty or not. This can be done, for example, by the algorithm of [6] (see also [12] for correctness). If $B(b) = \emptyset$, then the algorithm give us a partition or a co-partition $\mathcal{F}$ of V for which $0 > \sum[b(X) : X \in \mathcal{F}]$. In this case, the algorithm for testing $Q = Q(f, g; b)$ for emptiness concludes that Q is empty and returns $\mathcal{F}$ as a tree-composition of V which violates (5) since $\rho_f(Z) - \delta_g(Z) = 0$.

If $B(b)$ is non-empty, then the algorithm gives us an element $m \in B(b)$. In this case $b^{\downarrow}$ exists and by definition $B(b) = B(b^{\downarrow})$. Obviously $Q(f, g; b) = Q(f, g; b^{\downarrow})$, so we need an algorithm for finding a feasible flow for a fully submodular functions. Fujishige and Zhang [8] developed such an algorithm, for a simplified version see [5]. There are two issues to solve.

First, the algorithm require a subroutine for computing $\Delta_{b^{\downarrow}-\tilde{m}}(u, v) := \min\{b^{\downarrow}(X) - \tilde{m}(X) : u \in X \subseteq V - v\}$. The next lemma shows that it suffices to have a subroutine for computing $\Delta_{b-\tilde{m}}(u, v) := \min\{b(X) - \tilde{m}(X) : u \in X \subseteq V - v\}$. This is simpler since $\Delta_{b-\tilde{m}}(u, v)$ is a function depending on b and not on its full truncation $b^{\downarrow}$. (In applications to orientation problems, the above subroutine is typically available via a max-flow min-cut computation.)

Lemma 4.2. *Let b be a crossing submodular function for which $b(V) = 0$ and m a member of $B(b)$. Then $\Delta_{b^{\downarrow}-\tilde{m}}(u, v) = \Delta_{b-\tilde{m}}(u, v)$.*

Proof: First we prove a bit different claim:

Claim 4.3. *Let h be a non-negative crossing submodular function on ground set V for which $h(V) = 0$. Then $\Delta_h = \Delta_{h^{\downarrow}}$.*

Note that $B(h)$ is non-empty since the non-negativity of h implies that the origin is in $B(h)$. Hence the full truncation of h is submodular.

Proof: Hence $h^{\downarrow} \leq h$, we have $\Delta_{h^{\downarrow}} \leq \Delta_h$. Let $u, v \in V$ and let Z be a $u\bar{v}$-set for which $\Delta_{h^{\downarrow}}(u, v) = h^{\downarrow}(Z)$. By Theorem 3.2 there exists a tree-composition $\mathcal{F}$ of Z such that $h^{\downarrow}(Z) = \sum_{X \in \mathcal{F}} h(X)$. By Claim 2.3 there is a $u\bar{v}$-set X in $\mathcal{F}$ and by the non-negativity $h(X) \leq h^{\downarrow}(X)$, from which $\Delta_h(u, v) \leq \Delta_{h^{\downarrow}}(u, v)$. $\square$

We can use the above claim for $h = b - \tilde{m}$, hence it holds the conditions required, because $m \in B(b)$. We need to prove that $(b - \tilde{m})^{\downarrow} = b^{\downarrow} - \tilde{m}$. First we prove that $B(b - \tilde{m}) = B(b^{\downarrow} - \tilde{m})$. To this let x be a vector with $\tilde{x}(V) = 0$. We have $x \in B(b - \tilde{m}) \Leftrightarrow x \leq b - \tilde{m} \Leftrightarrow \tilde{x} + \tilde{m} \leq b \Leftrightarrow x + m \in B(b) \Leftrightarrow x + m \in B(b^{\downarrow}) \Leftrightarrow \tilde{x} + \tilde{m} \leq b^{\downarrow} \Leftrightarrow \tilde{x} \leq b^{\downarrow} - \tilde{m} \Leftrightarrow x \in B(b^{\downarrow} - \tilde{m})$, as we wanted. As noted above $B(b - \tilde{m})$ is non-empty, so the full truncation $(b - \tilde{m})^{\downarrow}$ of $(b - \tilde{m})$ exists, it is submodular and $B(b - \tilde{m}) = B((b - \tilde{m})^{\downarrow})$ by definition. By this and the first statement we get $B((b - \tilde{m})^{\downarrow}) = B(b^{\downarrow} - \tilde{m})$. Hence both function are submodular, we get by Theorem 3.1 our statement. So the lemma follows by the claim. $\square$

The second problem arises when the subflow polyhedron is empty and the algorithm terminates by returning a subset Z for which $\rho_f(Z) - \delta_g(Z) > b^{\downarrow}(Z)$. Then we can use the algorithm outlined in the proof of Theorem 3.2 for computing a tree-composition $\mathcal{F}$ of Z for which $b^{\downarrow}(Z) = \sum_{X \in \mathcal{F}} b(X)$. $\square$

5 Orientations

The above theorem implies the following (see in [3]), so we also got an algorithm to orient a given graph, as follows.

Theorem 5.1. *Let $G = (V, E)$ be a graph, and let $h : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be a crossing G-supermodular function, for which $h(\emptyset) = 0$. G has an orientation covering h, that is an orientation for which $\rho(X) \geq h(X)$ for all $X \subseteq V$, if and only if*

$$e(\mathcal{F}) \geq \sum_{i=1}^{q} h(V_i) = h(\mathcal{F}), \tag{6}$$

holds for all tree composition $\mathcal{F} = \{V_1, V_2, ..., V_q\}$ of V.

The strength of the above theorem is shown in [3], so here we just recall that by the above algorithm we got a simpler algorithm for the following problems. Given a mixed graph $M = (V, E, A)$. Orient E such that the resulting digraph $\overrightarrow{M} = (V, \overrightarrow{E} \cup A)$ is k-edge-connected. In the second problem let r be a fixed node of M. Then we want to orient E such that the resulting digraph is r-rooted (k, ℓ)-edge-connected.

For an other application of our new algorithm to the full truncation recall the following theorem of [2]:

Theorem 5.2. *Let $G = (V, E)$ be a graph, and let $h : 2^V \to \mathbb{Z}_+$ be a crossing G-supermodular function, for which $h(\emptyset) = 0$. G has an orientation covering h, that is an orientation for which $\rho(X) \geq h(X)$ for all $X \subseteq V$, if and only if*

$$e(\mathcal{P}) \geq \sum_{i=1}^{q} h(V_i), \tag{7}$$

and

$$e(\mathcal{P}) \geq \sum_{i=1}^{q} h(V - V_i) \tag{8}$$

hold for all partition $\mathcal{P} = \{V_1, V_2, ..., V_q\}$ of V.

Theorem 5.2 can be proved using Fujishige's theorem [7] on non-emptiness of the base-polyhedron and the Orientation lemma of Hakimi [9] (see in [4]). Let $p(X) = h(X) + i(X)$. It can be shown that p is crossing supermodular and that an integer vector in the base-polyhedron $B'(p)$ belongs to the in-degree vector of an orientation covering h. (With a little calculation one can see that from (7) and (8) the conditions of Fujishige's theorem follow, that gives the proof.)

Let $X \subseteq V$. Observe that an integer vector $x \in B'(p)$, for which $\tilde{x}(X) = \min\{\tilde{m}(X) : m \in B'(p)\}$, belongs to an in-degree vector of an orientation covering h for which $\rho(X)$ is minimal. So by Theorem 3.1 and Theorem 3.4 $\min\{\rho_{\overrightarrow{G}}(X) : \overrightarrow{G}$ covers $h\} = p^{\uparrow}(X) - i(X) = \max\Big\{\sum_{F\in\mathcal{F}} p(F) - \Delta p(V) : \mathcal{F}$ is a tree-composition on X, with ground-degree $\Delta\Big\} - i(X) = \max\Big\{\sum_{F\in\mathcal{F}} (h(F) + i(F)) - \Delta i(V) - i(X) : \mathcal{F}$ is a tree-composition on X, with ground-degree $\Delta\Big\}$. We got the following theorem:

Theorem 5.3. *Let $G = (V,E)$ be a graph, and let $h : 2^V \to \mathbb{Z}_+$ be a crossing G-supermodular function, for which $h(\emptyset) = 0$, and G has an orientation covering h, so h holds the conditions of Theorem 5.2. Let $X \subseteq V$. Then*

$$\min\Big\{\rho_{\overrightarrow{G}}(X) : \overrightarrow{G} \text{ covers } h\Big\} = \max\Bigg\{\sum_{F\in\mathcal{F}} (h(F) + i(F)) - \Delta i(V) - i(X) :$$
$$\mathcal{F} \text{ is a tree-composition on } X, \text{ with ground-degree } \Delta\Bigg\}.$$

Hence the proof was algorithmic, we have got an algorithm to find an orientation of G, that covers h, and in which the in-degree of X is minimal. Note that previously this orientation could be found with the minimum cost submodular flow algorithm [10] but with our algorithm it is much simpler. The only difficult part is to find an orientation covering h that uses generally the algorithm described in [6, 12]. Note that this is simpler for r-rooted $(k,1)$-edge-connected orientations by the algorithm of [11].

We can use the above theorem to the most important corollaries of Theorem 5.2. So for example we can find that k-edge-connected orientation of a $2k$-edge-connected graph, in which the in-degree of a given subset of nodes $X \subseteq V$ is minimal. We get the following theorem.

Theorem 5.4. *Let $G = (V,E)$ be a $2k$-edge connected graph, and let $X \subseteq V$. Then*

$$\min\Big\{\rho_{\overrightarrow{G}}(X) : \overrightarrow{G} \text{ is } k\text{-edge-connected}\Big\} = \max\Bigg\{\sum_{F\in\mathcal{F}} i(F) + k|\mathcal{F}| - \Delta|E| - i(X) :$$
$$\mathcal{F} \text{ is a tree-composition on } X \text{ with ground-degree } \Delta\Bigg\}.$$

Consider the case when $G = (S, T, E)$ is a bipartite graph, $k = 1$ and $X = T$. In this case we can show that there is a minimizing S-T-tree-composition, that belongs to the graph. Note that one edge of G can be induced in at most Δ elements of an S-T-tree-composition $\mathcal{F}$, and if $\mathcal{F}$ is a tree-composition, that belongs to the graph, then every edge of G is induced in exactly Δ elements of $\mathcal{F}$, where Δ is the ground-degree of $\mathcal{F}$. So $\sum_{F\in\mathcal{F}} i(F) - \Delta|E|$ is the sum of the deficits. Note also that $i(T) = 0$.

Assume that for the minimizing S-T-tree-composition $\mathcal{F}$ and for $s \in S, t \in T, st \in E$ the image to the representating oriented tree $F = (U = U_S \cup U_T, A)$ $\varphi(s)\varphi(t) \notin A$, and the deficit of $\mathcal{F}$ is minimal. Let P be the unoriented path of length $2t+1$ between $\varphi(s)$ and $\varphi(t)$. Contract the nodes of $V(P) \cap U_T$ in F, and let F' be the resulting tree and $\mathcal{F}'$ the S-T-tree-composition that belongs to F' with ground-degree Δ'. It is easy to see that $|\mathcal{F}'| = |\mathcal{F}| - t$, and $\sum_{F\in\mathcal{F}'} i(F) - \Delta'|E| \leq \sum_{F\in\mathcal{F}} i(F) - \Delta|E| - t$, hence the deficit of the edge st becomes 0 from t, and the deficit of the other edges does not increase. So $\sum_{F\in\mathcal{F}} i(F) + |\mathcal{F}| - \Delta|E| \geq \sum_{F\in\mathcal{F}'} i(F) + |\mathcal{F}'| - \Delta'|E|$, so $\mathcal{F}'$ is also a minimizing S-T-tree-compositionwith less deficit, a contradiction. We get the following theorem.

Theorem 5.5. *Let $G = (S, T, E)$ be a 2-edge connected bipartite graph. Then*

$$\min\left\{\rho_{\overrightarrow{G}}(T) : \overrightarrow{G} \text{ is strongly edge-connected}\right\} = \max\left\{\sum_{F\in\mathcal{F}} i(F) + |\mathcal{F}| - \Delta|E| : \right.$$

$$\left. \mathcal{F} \text{ is a tree-composition that belongs to } G, \text{ and its ground-degree is } \Delta\right\}.$$

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