Kneser-Poulsen conjecture for large radii

Igors Gorbovickis

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Abstract

In this paper we prove the Kneser-Poulsen conjecture for large radii. Namely, if a finite number of points in Euclidean space E^n is rearranged so that the distance between each pair of points does not decrease, then there exists a positive number r_0 that depends on the rearrangement of the points, such that if we consider n-dimensional balls of radius $r > r_0$ with centers at these points, then the volume of the union (intersection) of the balls before the rearrangement is not less (no more) than the volume of the union (intersection) after the rearrangement. Also under the same conditions we prove a similar result about perimeters instead of volumes.

1 Introduction

Let $|\dots|$ be the Euclidean norm. Let $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_N)$ and $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_N)$ be two configurations of N points, where each $\mathbf{p}_i \in \mathbb{E}^n$ and each $\mathbf{q}_i \in \mathbb{E}^n$. If for all $1 \le i < j \le N$, $|\mathbf{p}_i - \mathbf{p}_j| \le |\mathbf{q}_i - \mathbf{q}_j|$, we say that $\mathbf{q}$ is an *expansion* of $\mathbf{p}$ and $\mathbf{p}$ is a *contraction* of $\mathbf{q}$. We denote by $B_n(\mathbf{p}_i, r_i)$ the closed n-dimensional ball of radius $r_i \ge 0$ in $\mathbb{E}^n$ about the point $\mathbf{p}_i$, and let Vol_n represent the n-dimensional volume.

The following is a longstanding conjecture independently stated by Kneser [Kne55] and Poulsen [Pou54] for the case when $r_1 = \dots = r_N$:

Conjecture 1.1. *If* $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_N)$ *is an expansion of* $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_N)$ *in* $\mathbb{E}^n$*, then*

$$\mathrm{Vol}_n \left[\bigcup_{i=1}^{N} B_n(\mathbf{p}_i, r_i) \right] \le \mathrm{Vol}_n \left[\bigcup_{i=1}^{N} B_n(\mathbf{q}_i, r_i) \right].$$

A similar conjecture was mentioned in [KW91] by Klee and Wagon:

Conjecture 1.2. *If* $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_N)$ *is an expansion of* $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_N)$ *in* $\mathbb{E}^n$*, then*

$$\mathrm{Vol}_n \left[\bigcap_{i=1}^{N} B_n(\mathbf{p}_i, r_i) \right] \ge \mathrm{Vol}_n \left[\bigcap_{i=1}^{N} B_n(\mathbf{q}_i, r_i) \right].$$

Even though in full generality these conjectures still remain a mystery, there are results that provide support for them with an additional assumption that there exists a *continuous expansion* from $\mathbf{p}$ to $\mathbf{q}$ — a continuous motion $\mathbf{p}(t) = (\mathbf{p}_1(t), \ldots, \mathbf{p}_N(t))$, with $\mathbf{p}_i(t) \in \mathbb{E}^n$ for all $t \in [0, 1]$ and $i = 1, \ldots, N$ such that $\mathbf{p}(0) = \mathbf{p}$ and $\mathbf{p}(1) = \mathbf{q}$, and $|\mathbf{p}_i(t) - \mathbf{p}_j(t)|$ is non-decreasing for all $1 \le i < j \le N$. Assuming that there exists a continuous expansion from $\mathbf{p}$ to $\mathbf{q}$, Csikós in [Csi97], [Csi98] and [Csi01] proves Conjectures 1.1 and 1.2 and similar conjectures on n-dimensional sphere and in n-dimensional hyperbolic space.

Finally, in [BC02] Bezdek and Connelly prove Conjectures 1.1 and 1.2 for $n = 2$ by finding a continuous expansion from $\mathbf{p}$ to $\mathbf{q}$ in a 4-dimensional Euclidean space and relating higher dimensional volumes with the 2-dimensional ones. More precisely, they claim that if there exists a smooth continuous expansion from $\mathbf{p}$ to $\mathbf{q}$ in $(n+2)$-dimensional Euclidean space, then Conjectures 1.1 and 1.2 hold. Also in their paper they mention other results related to these conjectures.

In this paper we prove the following theorem:

Theorem 1.3. *If* $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_N)$ *is an expansion of* $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_N)$ *in* $\mathbb{E}^n$*, then there exists* $r_0 > 0$ *such that for any* $r \ge r_0$

$$\mathrm{Vol}_n \left[\bigcup_{i=1}^{N} B_n(\mathbf{p}_i, r) \right] \le \mathrm{Vol}_n \left[\bigcup_{i=1}^{N} B_n(\mathbf{q}_i, r) \right], \tag{1}$$

and

$$\mathrm{Vol}_n \left[\bigcap_{i=1}^{N} B_n(\mathbf{p}_i, r) \right] \ge \mathrm{Vol}_n \left[\bigcap_{i=1}^{N} B_n(\mathbf{q}_i, r) \right], \tag{2}$$

and if the point configurations $\mathbf{q}$ *and* $\mathbf{p}$ *are not congruent, then the inequalities are strict.*

More specifically in Lemma 3.2 and Theorem 3.3 we prove that starting from sufficiently large r the volumes in (1) and (2) can be represented as Laurent series in variable r, and

$$\mathrm{Vol}_n \left[\bigcup_{i=1}^{N} B_n(\mathbf{q}_i, r) \right] - \mathrm{Vol}_n \left[\bigcup_{i=1}^{N} B_n(\mathbf{p}_i, r) \right] = Cr^{n-1} + o(r^{n-1}), \tag{3}$$

$$\mathrm{Vol}_n \left[\bigcap_{i=1}^{N} B_n(\mathbf{q}_i, r) \right] - \mathrm{Vol}_n \left[\bigcap_{i=1}^{N} B_n(\mathbf{p}_i, r) \right] = -cr^{n-1} + o(r^{n-1}), \tag{4}$$

as r tends to infinity, where c and C are non-negative constants that can be zero only if $\mathbf{p}$ and $\mathbf{q}$ are congruent. Theorem 1.3 is an easy consequence of this result.

Let us by $\mathrm{Bdy}[X]$ denote the boundary of a set $X \subset \mathbb{E}^n$. By differentiating (3) and (4) with respect to r we get another corollary:

Corollary 1.4. *If $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_N)$ is an expansion of $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_N)$ in $\mathbb{E}^n$, for $n \geq 2$, then there exists $r_0 > 0$ such that for any $r \geq r_0$*

$$\mathrm{Vol}_{n-1}\left[\mathrm{Bdy}\left[\bigcup_{i=1}^{N} B_n(\mathbf{p}_i, r)\right]\right] \leq \mathrm{Vol}_{n-1}\left[\mathrm{Bdy}\left[\bigcup_{i=1}^{N} B_n(\mathbf{q}_i, r)\right]\right], \tag{5}$$

and

$$\mathrm{Vol}_{n-1}\left[\mathrm{Bdy}\left[\bigcap_{i=1}^{N} B_n(\mathbf{p}_i, r)\right]\right] \geq \mathrm{Vol}_{n-1}\left[\mathrm{Bdy}\left[\bigcap_{i=1}^{N} B_n(\mathbf{q}_i, r)\right]\right], \tag{6}$$

and if the point configurations $\mathbf{q}$ and $\mathbf{p}$ are not congruent, then the inequalities are strict.

The question, whether inequality (6) holds for all values of r was asked by R. Alexander and remains open even for $n = 2$. In [BCC06] Bezdek, Connelly and Csikós give a positive answer to this question for $n = 2$ and $N \leq 4$. On the other hand, in the case of unions of balls Habicht and Kneser provide an example of an expansion, described in [KW91], for which inequality (5) does not hold for a particular value of r. However, Corollary 1.4 implies that by increasing r we can make inequality (5) hold.

2 Properties of the volume functions

Definition 2.1. *We call a nonempty set $P \subset \mathbb{E}^n$ a (convex) polyhedral set, if it is an intersection of $\mathbb{E}^n$ with finitely many (may be zero) closed halfspaces.*

Definition 2.2. *A nonempty set $F \subset P$ is called a face of a polyhedral set P, if for some positive integer k*

$$F = P \bigcap \left(\bigcap_{i=1}^{k} H_i\right),$$

where $H_1, \ldots H_k$ are the boundary hyperplanes of some k halfspaces that form the polyhedral set P.

Note that a face of a convex polyhedral set is also a convex polyhedral set. We also say that a convex polyhedral set P is *k-dimensional*, if the dimension of the affine span of P equals k.

Let $\mathbf{p}_0$ be a point in $\mathbb{E}^n$, $n \geq 2$, and let $P \subset \mathbb{E}^n$ be a convex polyhedral set.

Definition 2.3. *We will call the set $\hat{P}(r) = P \cap B_n(\mathbf{p}_0, r)$ a polytope truncated by a ball of radius r centered at $\mathbf{p}_0$, or just a truncated polytope.*

Now the focus of our interest is how the volume of a truncated polytope depends on the radius, so we define the following function which is the n-dimensional volume of a truncated polytope $\hat{P}(r)$ depending on r:

$$V_{P,\mathbf{p}_0,n}(r) = \begin{cases} 0, & \text{if } r < 0; \\ \mathrm{Vol}_n\left[\hat{P}(r)\right], & \text{if } r \geq 0. \end{cases}$$

Let $F_1, \dots, F_m$ be $(n-1)$-dimensional faces of the polyhedral set P. For each face F_i we denote by h_i the distance from the point $\mathbf{p}_0$ to the hyperplane containing that face, and we put the sign of the face $\epsilon_i = 1$, if the point $\mathbf{p}_0$ is contained in the halfspace related to that face, or we put $\epsilon_i = -1$ otherwise.

Lemma 2.4. *The function $V_{P,\mathbf{p}_0,n}(r)$ satisfies the following differential equation on the interval $(0, +\infty)$:*

$$V_{P,\mathbf{p}_0,n}(r) = \frac{1}{n}\sum_{i=1}^{m} \epsilon_i h_i \mathrm{Vol}_{n-1}[F_i \cap B_n(\mathbf{p}_0, r)] + \frac{r}{n}\frac{d}{dr}V_{P,\mathbf{p}_0,n}(r). \tag{7}$$

Proof. One can see that the volume of a truncated polytope $\hat{P}(r)$ is the volume of the cone with the vertex at $\mathbf{p}_0$ about the spherical part of the boundary $\mathrm{Bdy}[B_n(\mathbf{p}_0, r)] \cap P$ of $\hat{P}(r)$ plus the sum of the volumes of cones with the common vertex at the same point $\mathbf{p}_0$ about the planar faces of $\hat{P}(r)$ multiplied by the sign of the corresponding face of the polyhedral set P (see an example in Figure 1):

$$V_{P,\mathbf{p}_0,n}(r) = \frac{1}{n}\sum_{i=1}^{m} \epsilon_i h_i \mathrm{Vol}_{n-1}[F_i \cap B_n(\mathbf{p}_0, r)] + \frac{r}{n}\mathrm{Vol}_{n-1}[\mathrm{Bdy}[B_n(\mathbf{p}_0, r)] \cap P].$$

Finally we note that $\mathrm{Vol}_{n-1}[\mathrm{Bdy}[B_n(\mathbf{p}_0, r)] \cap P] = \frac{d}{dr}V_{P,\mathbf{p}_0,n}(r)$, which completes the proof of the lemma. □

Differential equation (7) is linear, so one can easily find its solution:

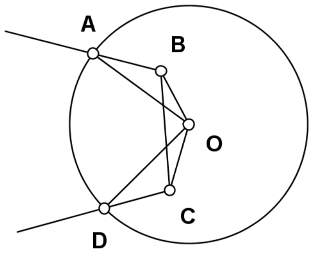
Corollary 2.5. *There exists a real number c that depends on the polyhedral set P and on position of the point $\mathbf{p}_0$, such that for $r \geq 0$*

$$V_{P,\mathbf{p}_0,n}(r) = cr^n - r^n \sum_{i=1}^{m} h_i \epsilon_i \int_0^r \frac{\mathrm{Vol}_{n-1}[F_i \cap B_n(\mathbf{p}_0, s)]}{s^{n+1}} ds. \tag{8}$$

In case if r is small enough, the truncated polytope $\hat{P}(r)$ is a cone, so $V_{P,\mathbf{p}_0,n}$ is proportional to r^n. Here is a more precise statement:

Lemma 2.6. *Let h be equal to the minimal nonzero h_i. Then $V_{P,\mathbf{p}_0,n}(r) = cr^n$ on the interval $[0, h]$.*

Figure 1:



2-dimensional volume of the truncated polytope ABCD is the sum of the volumes of the cone AOD about the spherical part of the boundary of ABCD, plus the the volumes of the cones AOB and COD, minus the volume of the cone BOC.

Proof. When $r \leq h$, for any $i \in \{1, \ldots, m\}$ either $\mathrm{Vol}_{n-1}[F_i \cap B_n(\mathbf{p}_0, r)] = 0$, or $h_i = 0$, so

$$h_i \epsilon_i \int_0^r \frac{\mathrm{Vol}_{n-1}[F_i \cap B_n(\mathbf{p}_0, s)]}{s^{n+1}} = 0.$$

Thus, from formula (8), when $r \in [0, h]$, we have $V_{P,\mathbf{p}_0,n}(r) = cr^n$. □

Definition 2.7. *Let $\Omega(P, \mathbf{p}_0)$ be a subset of non-negative real numbers, such that*

(i) $0 \in \Omega(P, \mathbf{p}_0)$, if and only if $\mathbf{p}_0 \in P$,

(ii) $r \in \Omega(P, \mathbf{p}_0)$ if $r > 0$ and the ball centered at $\mathbf{p}_0$ with radius r is tangent to some k-dimensional face of a polyhedral set P, where $k \in \{0, \ldots, n-1\}$.

We call the set $\Omega(P, \mathbf{p}_0)$ a *singular set* of the truncated polytope $\hat{P}(r)$ for the reasons explained in Proposition 2.8. Since a polyhedral set can have only finitely many faces, a singular set is also finite. On the other hand, the set $\Omega(P, \mathbf{p}_0)$ is non-empty, because if $P = \mathbb{E}^n$, then $0 \in \Omega(P, \mathbf{p}_0)$, but if $P \neq \mathbb{E}^n$, then at least one of the faces of P is a k-dimensional plane for some $k \in \{0, \ldots, n-1\}$, hence there exists r, such that the ball centered at $\mathbf{p}_0$ with radius r is tangent to this plane.

The next proposition is not used further in this paper, but it is of independent interest.

Proposition 2.8. *The function $V_{P,\mathbf{p}_0,n}(r)$ is real analytic on $\mathbb{R} \backslash \Omega$ and at least $[n/2]$ times continuosly differentiable in any point of $\Omega(P, \mathbf{p}_0)$.*

Proof. For the point $r = 0$ the statement follows from Lemma 2.6. For positive values of r the proof goes by induction on n. Note that when $n = 1$ the polyhedral set P is just an interval. Then one can see that the function $V_{P,\mathbf{p}_0,n}(r)$ is piecewise linear (hence piecewise analytic) and continuous. Moreover, the function can be non-differentiable only for those non-zero values of r, for which the ball touches an endpoint of the interval. Thus, for $n = 1$ the statement is true.

We assume that the statement is true for $(n-1)$-dimensional euclidean space. Let the points $\bar{\mathbf{p}}_i$ be the orthogonal projections of the point $\mathbf{p}_0$ onto the hyperplanes containing the faces F_i. On each of these hyperplanes there is a Euclidian metric induced from the enclosing space, so one can define the functions

$$V_{F_i,\bar{\mathbf{p}}_i,n-1}(r) = \begin{cases} 0, & \text{if } r < 0; \\ \mathrm{Vol}_{n-1}\left[F_i \cap B_{n-1}(\bar{\mathbf{p}}_i, r)\right], & \text{if } r \geq 0. \end{cases} \tag{9}$$

Now we can rewrite formula (8) in the following way:

$$V_{P,\mathbf{p}_0,n}(r) = cr^n - r^n \sum_{i=1}^{m} h_i \epsilon_i \int_0^r \frac{V_{F_i,\bar{\mathbf{p}}_i,n-1}(r_i(s))}{s^{n+1}} ds, \tag{10}$$

where

$$r_i(s) = \begin{cases} 0, & \text{if } s < h_i; \\ \sqrt{s^2 - h_i^2}, & \text{if } s \geq h_i. \end{cases} \tag{11}$$

We note that each function $r_i(s)$ is real analytic everywhere except the point $s = h_i$. Consider the radius $r > 0$, such that the ball $B_n(\mathbf{p}_0, r)$ is not tangent to any of the faces of P. Then each ball $B_{n-1}(\bar{\mathbf{p}}_i, r_i(r))$ in the hyperplane containing the face F_i is not tangent to any of the faces of F_i. Now for all $i \in \{1, \ldots, m\}$, if $r \neq h_i$, then by the inductive assumption the function $V_{F_i,\bar{\mathbf{p}}_i,n-1}(r_i(s))$ is real analytic at the point $s = r$. On the other hand, if $r = h_i$, then $0 \notin \Omega(F_i, \bar{\mathbf{p}}_i)$, so the function $V_{F_i,\bar{\mathbf{p}}_i,n-1}(r_i(s))$ is identically zero in some neighborhood of the point $s = r$, hence it is also real analytic at this point. Finally, from formula (10) it follows that the function $V_{P,\mathbf{p}_0,n}(s)$ is also real analytic at the point $s = r$. Therefore we proved that the function $V_{P,\mathbf{p}_0,n}(r)$ is real analytic on the set $\mathbb{R} \setminus \Omega(P, \mathbf{p}_0)$.

Now we prove that the function $V_{P,\mathbf{p}_0,n}(r)$ is at least $[n/2]$ times continuously differentiable when $r > 0$. For each i consider the function

$$G_i(r) = h_i \epsilon_i \int_0^r \frac{V_{F_i,\bar{\mathbf{p}}_i,n-1}(r_i(s))}{s^{n+1}} ds. \tag{12}$$

If $h_i = 0$, then $G_i(r) \equiv 0$, so $G_i(r)$ is infinitely differentiable, hence we can assume that $h_i > 0$. By the inductive assumption the function $V_{F_i,\bar{\mathbf{p}}_i,n-1}(r)$ is $[(n-1)/2]$ times continuously differentiable. Since $r_i(s)$ is infinitely smooth everywhere except $s = h_i$, from formula (12) we can conclude that the function $G_i(r)$ is at least $[(n-1)/2] + 1 \geq [n/2]$ times continuously differentiable everywhere except $r = h_i$, but according to Lemma 2.6

there exists a neighborhood of the point $r = h_i$, such that in this neighborhood

$$G_i(r) = \begin{cases} 0, & \text{if } r < h_i; \\ h_i \epsilon_i \int_0^r \frac{c_i (s^2 - h_i^2)^{(n-1)/2}}{s^{n+1}} ds, & \text{if } r \geq h_i, \end{cases}$$

for some real c_i. It is easy to verify that this function is $[n/2]$ times continuously differentiable at $r = h_i$, which finishes the proof. $\square$

The set $\Omega(P, \mathbf{p}_0)$ is finite, so it has a maximal element which we denote by b. As in the proof of Proposition 2.8 we define the points $\bar{\mathbf{p}}_i$ to be the orthogonal projections of the point $\mathbf{p}_0$ onto the hyperplanes containing the faces F_i. We also define $b_i = \max(\Omega(F_i, \bar{\mathbf{p}}_i))$, for all $i = 1, \ldots, m$.

Lemma 2.9. *The following formula holds:* $b = \max\{\sqrt{b_i^2 + h_i^2} \mid i = 1, \ldots, m\}$.

Proof. This formula is an easy concequence of the relationship

$$\Omega(P, \mathbf{p}_0) = \bigcup_{i=1}^{m} \{\sqrt{r^2 + h_i^2} \mid r \in \Omega(F_i, \bar{\mathbf{p}}_i)\}.$$

$\square$

Proposition 2.10. *There exists a Laurent series with real coefficients* $L(r) = \sum_{j=-n}^{+\infty} a_j r^{-j}$, *such that* $L(r)$ *converges to* $V_{P,\mathbf{p}_0,n}(r)$ *on the interval* $(b, +\infty)$, *and* $|a_j| \leq 2^{n^2}(m+1)^n(b+1)^{n+j}$.

Proof. Here we also use induction on n. For $n = 1$, $L(r)$ is just a linear function that coincides with $V_{P,\mathbf{p}_0,n}(r)$ on some interval $(b_0, +\infty)$. One can check that the inequalities on the coefficients a_j also hold.

We define the functions $V_{F_i,\bar{\mathbf{p}}_i,n-1}(r)$ as in (9). If the statement is true for $(n-1)$-dimensional Euclidean space, then there exist series $L_i(r) = \sum_{j=-n+1}^{+\infty} a_{i,j} r^{-j}$ that converge to $V_{F_i,\bar{\mathbf{p}}_i,n-1}(r)$ on the intervals $(b_i, +\infty)$ respectively. Since the number of $(n-2)$-dimensional faces of each polyhedral set F_i is less than m, then by the inductive hypothesis, for each $a_{i,j}$ we have inequalities $|a_{i,j}| \leq 2^{(n-1)^2}(m+1)^{n-1}(b_i+1)^{n+j-1}$.

Let us consider one of the faces F_i, where $i \in \{1, \ldots, m\}$. For all $j \geq 1-n$ and for all $k \geq 0$ we define

$$a_{i,j,k} = (-1)^k \binom{-j/2}{k} h_i^{2k} \tag{13}$$

The series $\sum_{k=0}^{+\infty} a_{i,j,k} s^{-j-2k}$ is a binomial series that converges absolutely to the function $(r_i(s))^{-j}$ for all $s > h_i$ and all $j \in \mathbb{Z}$, where $r_i(s)$ is defined as in (11). Note that when $j > 0$, all the coefficients $a_{i,j,k}$ are positive, hence the following equality holds:

$$\sum_{j=-n+1}^{+\infty} |a_{i,j}| \sum_{k=0}^{+\infty} |a_{i,j,k} s^{-j-2k}| = \sum_{j=-n+1}^{0} |a_{i,j}| \sum_{k=0}^{+\infty} |a_{i,j,k} s^{-j-2k}| + \sum_{j=1}^{+\infty} |a_{i,j}| (r_i(s))^{-j}. \tag{14}$$

Since the series $L_i(r)$ converges absolutely for all $r > b_i$, the sum in (14) is defined and finite for all $s > \sqrt{b_i^2 + h_i^2}$. Thus, when $s > \sqrt{b_i^2 + h_i^2}$, the result of the sum $\sum_{j=-n+1}^{+\infty} a_{i,j} \sum_{k=0}^{+\infty} a_{i,j,k} s^{-j-2k}$ does not change after reordering the summands. Hence there exists series

$$\tilde{L}_i(s) = \sum_{j=-n+1}^{+\infty} \tilde{a}_{i,j} s^{-j}, \quad \text{where } \tilde{a}_{i,j} = \sum_{k=0}^{[(n+j-1)/2]} a_{i,j-2k} a_{i,j-2k,k} \tag{15}$$

that converges to $V_{F_i,\bar{\mathbf{p}}_i,n-1}(r_i(s))$ on the interval $(\sqrt{b_i^2 + h_i^2}, +\infty)$.

Having upper estimates for $|a_{i,j}|$ and formula (13) one can find upper estimates for $|\tilde{a}_{i,j}|$:

$$|\tilde{a}_{i,j}| \le \sum_{k=0}^{[(n+j-1)/2]} |a_{i,j-2k}||a_{i,j-2k,k}| \le 2^{(n-1)^2}(m+1)^{n-1} \sum_{k=0}^{[(n+j-1)/2]} (b_i+1)^{n+j-2k-1} \left| \binom{k-j/2}{k} \right| h_i^{2k}.$$

When $j \ge 0$, we have

$$\sum_{k=0}^{[(n+j-1)/2]} (b_i+1)^{n+j-2k-1} \left| \binom{k-j/2}{k} \right| h_i^{2k} \le \sum_{k=0}^{[(n+j-1)/2]} \binom{[(n+j-1)/2]}{k} (b_i+1)^{n+j-2k-1} h_i^{2k} \le ((b_i+1)^2 + h_i^2)^{(n+j-1)/2} \le (b+1)^{n+j-1}. \tag{16}$$

When $j < 0$, we have

$$\sum_{k=0}^{[(n+j-1)/2]} (b_i+1)^{n+j-2k-1} \left| \binom{k-j/2}{k} \right| h_i^{2k} \le \sum_{k=0}^{[(n+j-1)/2]} \binom{[(n-1)/2]}{k} (b+1)^{n-j-1} \le 2^{(n-1)/2}(b+1)^{n-j-1}. \tag{17}$$

We used Lemma 2.9 in the last inequality of (16) and in the first inequality of (17).

Combining (16) and (17) together we get that

$$|\tilde{a}_{i,j}| \le 2^{(n-1)^2+(n-1)/2}(m+1)^{n-1}(b+1)^{n-j-1} \le 2^{n^2}(m+1)^{n-1}(b+1)^{n-j-1}. \tag{18}$$

Finally we can rewrite formula (10) in the following way:

$$V_{P,\mathbf{p}_0,n}(r) = cr^n + r^n \sum_{i=1}^{m} h_i \epsilon_i \int_0^r \frac{V_{F_i,\bar{\mathbf{p}}_i,n-1}(r_i(s))}{s^{n-1}} d\left(\frac{1}{s}\right),$$

The series for the functions $\frac{V_{F_i,\bar{\mathbf{p}}_i,n-1}(r_i(s))}{s^{n-1}}$ do not have s to a positive degree, so they are power series in variable $1/s$, which means that the integrals of the series with respect to $d(1/s)$ agree with the integrals of the corresponding functions on the intervals $(\sqrt{b_i^2+h_i^2}, +\infty)$ respectively. Thus, the series for the function $V_{P,\mathbf{p}_0,n}(r)$ are of the form

$$L(r) = cr^n + r^n \sum_{i=1}^{m} h_i \epsilon_i \left(\int_0^{\sqrt{b_i^2+h_i^2}} \frac{V_{F_i,\bar{\mathbf{p}}_i,n-1}(r_i(s))}{s^{n-1}} d\left(\frac{1}{s}\right) + \int_{\sqrt{b_i^2+h_i^2}}^{r} \frac{\tilde{L}_i(s)}{s^{n-1}} d\left(\frac{1}{s}\right) \right) = \sum_{j=-n}^{+\infty} a_j r^{-j} \quad (19)$$

and converges for all $r > \max\{\sqrt{b_i^2+h_i^2} \mid i = 1, \dots, m\} = b$. The last equality is due to Lemma 2.9.

For $j > -n$ from (18) and (19) we have

$$|a_j| \le \sum_{i=1}^{m} 2^{n^2} \frac{h_i}{n+j} (m+1)^{n-1} (b+1)^{n-j-1} \le 2^{n^2} (m+1)^n (b+1)^{n-j}.$$

For $j = -n$ we have $|a_{-n}| \le \delta_n = \mathrm{Vol}_n[B(\mathbf{0},1)]) \le 2^{n^2}(m+1)^n(b+1)^{n-j}$, so the proposition is proven. □

Remark 2.11. *Since the statement of Proposition 2.10 is true, the series* $\tilde{L}_i(r) = \sum_{j=-n+1}^{+\infty} \tilde{a}_{i,j} r^{-j}$ *defined in (15) actually converge to the functions* $V_{F_i,\bar{\mathbf{p}}_i,n-1}(r_i(r)) = \mathrm{Vol}_{n-1}[F_i \cap B_n(\mathbf{p}_0, r)]$ *on the interval* $(b, +\infty)$*, and the inequality (18) on the coefficients* $\tilde{a}_{i,j}$ *holds.*

Note that all the statements in this section can be easily generalized for the case of non-convex and not connected polyhedral sets that can be defined as a finite union of intersections of finitely many halfspaces.

3 Voronoi decomposition

Let $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_N)$ be a configuration of N distinct points in $\mathbb{E}^n$. Consider the n-dimensional nearest point and farthest point Voronoi regions for the point configuration $\mathbf{p}$:

$$C_{i,n} = \{\mathbf{p}_0 \in \mathbb{E}^n \mid \text{for all } j, |\mathbf{p}_0 - \mathbf{p}_i| \le |\mathbf{p}_0 - \mathbf{p}_j|\},$$

$$C^{i,n} = \{\mathbf{p}_0 \in \mathbb{E}^n \mid \text{for all } j, |\mathbf{p}_0 - \mathbf{p}_i| \geq |\mathbf{p}_0 - \mathbf{p}_j|\}.$$

For each pair $C_{i,n}$ and $C_{j,n}$ of distinct nearest point Voronoi regions we can consider the set $W_{ij} = C_{i,n} \cap C_{j,n}$ called the wall between two nearest point Voronoi regions. In the similar way one can define the wall between two farthest point Voronoi regions: $W^{ij} = C^{i,n} \cap C^{j,n}$. Note that the Voronoi regions are convex polyhedral sets, and each nonempty wall is a common $(n-1)$-dimensional faces of a corresponding pair of Voronoi regions. This allows us to define the truncated nearest point and farthest point Voronoi regions $C_{i,n}(\mathbf{p}, r) = C_{i,n} \cap B_n(\mathbf{p}_i, r)$ and $C^{i,n}(\mathbf{p}, r) = C^{i,n} \cap B_n(\mathbf{p}_i, r)$ as corresponding truncated polytopes. Similarly $W_{ij}(\mathbf{p}, r) = W_{ij} \cap B_n(\mathbf{p}_i, r)$ and $W^{ij}(\mathbf{p}, r) = W^{ij} \cap B_n(\mathbf{p}_i, r)$ are truncated walls between two nearest point and farthest point Voronoi regions.

Further we will be interested in the behavior of the following functions:

$$V_n(\mathbf{p}, r) = \mathrm{Vol}_n \left[\bigcup_{i=1}^{N} B_n(\mathbf{p}_i, r) \right],$$

$$V^n(\mathbf{p}, r) = \mathrm{Vol}_n \left[\bigcap_{i=1}^{N} B_n(\mathbf{p}_i, r) \right].$$

Consider a smooth (infinitely many times differentiable) motion $\mathbf{p}(t) = (\mathbf{p}_1(t), \ldots, \mathbf{p}_N(t))$ of some configuration of N points in $\mathbb{E}^n$. Let $d_{ij} = |\mathbf{p}_i(t) - \mathbf{p}_j(t)|$, and let d'_{ij} be the t-derivative of d_{ij}. The following is Csikós's formula [Csi98] for the t-derivative of the functions $V_n(\mathbf{p}(t), r)$ and $V^n(\mathbf{p}(t), r)$.

Theorem 3.1. *Let $n \geq 2$ and let $\mathbf{p}(t)$ be a smooth motion of a configuration of points in $\mathbb{E}^n$ such that for each t, all the points are pairwise distinct. Then the functions $V_n(\mathbf{p}(t), r)$ and $V^n(\mathbf{p}(t), r)$ are differentiable with respect to t and,*

$$\frac{d}{dt} V_n(\mathbf{p}(t), r) = \sum_{1 \leq i < j \leq N} d'_{ij} \mathrm{Vol}_{n-1} \left[W_{ij}(\mathbf{p}(t), r) \right], \tag{20}$$

$$\frac{d}{dt} V^n(\mathbf{p}(t), r) = \sum_{1 \leq i < j \leq N} -d'_{ij} \mathrm{Vol}_{n-1} \left[W^{ij}(\mathbf{p}(t), r) \right]. \tag{21}$$

Note that Theorem 3.1 immediately provides a proof of Conjectures 1.1 and 1.2 in case there exists a smooth expansion from $\mathbf{p}$ to $\mathbf{q}$ in $\mathbb{E}^n$ (i.e. a smooth motion $\mathbf{p}(t) = (\mathbf{p}_1(t), \ldots, \mathbf{p}_N(t))$, such that $\mathbf{p}(0) = \mathbf{p}$, $\mathbf{p}(1) = \mathbf{q}$ and $|\mathbf{p}_i(t) - \mathbf{p}_j(t)|$ are non-decreasing functions for any pair of i and j).

Lemma 3.2. *Let $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_N)$ be a configuration of N points in $\mathbb{E}^n$. Then there exist a positive real number $\tilde{b}$ and two series $L_n(\mathbf{p}, r) = \sum_{j=-n}^{+\infty} \alpha_j(\mathbf{p}) r^{-j}$ and $L^n(\mathbf{p}, r) = \sum_{j=-n}^{+\infty} \beta_j(\mathbf{p}) r^{-j}$ with real coefficients, such that* *(i) The series $L_n(\mathbf{p}, r)$ and $L^n(\mathbf{p}, r)$ converge to the functions $V_n(\mathbf{p}, r)$ and $V^n(\mathbf{p}, r)$ respectively on the interval $(\tilde{b}, +\infty)$,*

(ii) $\alpha_{-n}(\mathbf{p}) = \beta_{-n}(\mathbf{p}) = \mathrm{Vol}_n[B(\mathbf{0}, 1)] = \delta_n$,

(iii) $|\alpha_j(\mathbf{p})|, |\beta_j(\mathbf{p})| \le 2^{n^2} N^{n+1} (\tilde{b} + 1)^{n+j}$, for all $j > -n$.

Proof. We note that $\bigcup_{i=1}^N B_n(\mathbf{p}_i, r) = \bigcup_{i=1}^N C_{i,n}(\mathbf{p}, r)$ and $\bigcap_{i=1}^N B_n(\mathbf{p}_i, r) = \bigcup_{i=1}^N C^{i,n}(\mathbf{p}, r)$. Since intersection of any two distinct truncated Voronoi regions is a set of dimension at most $n - 1$, we have

$$V_n(\mathbf{p}, r) = \sum_{i=1}^N \mathrm{Vol}_n \left[C_{i,n}(\mathbf{p}, r) \right],$$

$$V^n(\mathbf{p}, r) = \sum_{i=1}^N \mathrm{Vol}_n \left[C^{i,n}(\mathbf{p}, r) \right].$$

Truncated Voronoi regions are truncated polytopes, so by Proposition 2.10 there exist a real number $\tilde{b} > 0$ and two series $L_n(\mathbf{p}, r) = \sum_{j=-n}^{+\infty} \alpha_j(\mathbf{p}) r^{-j}$ and $L^n(\mathbf{p}, r) = \sum_{j=-n}^{+\infty} \beta_j(\mathbf{p}) r^{-j}$ that converge to the functions $V_n(\mathbf{p}, r)$ and $V^n(\mathbf{p}, r)$ respectively on the interval $(\tilde{b}, +\infty)$. Each Voronoi region has at most $N - 1$ faces, so plugging $N - 1$ instead of m into the estimates in Proposition 2.10 and multiplying the result by N — the number of Voronoi regions, we get the statement of part (iii).

Finally, part (ii) follows from the inequalities

$$\delta_n r^n = \mathrm{Vol}_n \left[B_n(\mathbf{p}_1, r) \right] \le V_n(\mathbf{p}, r) \le \mathrm{Vol}_n \left[B_n(\mathbf{p}_1, r + l) \right] = \delta_n r^n + o(r^n),$$

and

$$\delta_n r^n + o(r^n) = \mathrm{Vol}_n \left[B_n(\mathbf{p}_1, r - l) \right] \le V^n(\mathbf{p}, r) \le \mathrm{Vol}_n \left[B_n(\mathbf{p}_1, r) \right] = \delta_n r^n,$$

where $l = \max\{|\mathbf{p}_1 - \mathbf{p}_i| \mid i = 1, \ldots, N\}$. □

Now we formulate our main result which will be proven in section 6:

Theorem 3.3. *Let $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_N)$ be an expansion of $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_N)$ in $\mathbb{E}^n$. Then $\alpha_{1-n}(\mathbf{p}) \le \alpha_{1-n}(\mathbf{q})$ and $\beta_{1-n}(\mathbf{p}) \ge \beta_{1-n}(\mathbf{q})$, and the equality holds if and only if the point configurations $\mathbf{p}$ and $\mathbf{q}$ are congruent.*

4 Cone property

Consider an n-dimensional convex polyhedral set $P \subset \mathbb{E}^n$ together with a fixed point $\mathbf{p}_0 \in \mathbb{E}^n$.

Lemma 4.1. *If P contains an infinite n-dimensional circular cone, then $V_{P,\mathbf{p}_0,n}(r) = Cr^n + o(r^n)$, for some positive $C \in \mathbb{R}$, as r goes to infinity.*

Proof. Let $\mathbf{v} \in \mathbb{E}^n$ be the vertex of the cone and $l = |\mathbf{p}_0 - \mathbf{v}|$. In the conditions of the lemma the function $V_{P,\mathbf{p}_0,n}(r)$ is bounded from above by the volume of the ball $B_n(\mathbf{p}_0, r)$, and from below by the volume of the cone intersected with the ball $B(\mathbf{v}, r - l)$. Thus $V_{P,\mathbf{p}_0,n}(r) = Cr^n + o(r^n)$, for some positive C. □

Let $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_N)$ be a configuration of points in Eucledian space $\mathbb{E}^n$, and let $H(\mathbf{p})$ be a convex hull of the set of these points. $H(\mathbf{p})$ is a convex polytope with vertices from the set $\{\mathbf{p}_1, \ldots, \mathbf{p}_N\}$. If some point $\mathbf{p}_i$ appears to be the inner point of an edge of the polytope $H(\mathbf{p})$, then we split this edge into two with the common endpoint $\mathbf{p}_i$, making the point $\mathbf{p}_i$ one of the vertices of the polytope $H(\mathbf{p})$. Thus, we may think that the points from the set $\{\mathbf{p}_1, \ldots, \mathbf{p}_N\}$ cannot be the inner points of the edges of the polytope $H(\mathbf{p})$. On the other hand, if a point $\mathbf{p}_i$ is the inner point of some k-dimensional face of $H(\mathbf{p})$, where $k > 1$, then we do not consider it as a vertex of the polytope $H(\mathbf{p})$.

Definition 4.2. *We will say that a point $\mathbf{w} \in \mathbb{E}^n$ is in a strictly convex position with respect to a set $S \subset \mathbb{E}^n$ if it is not contained in the closed convex hull of $S \setminus \{\mathbf{w}\}$.*

Lemma 4.3. *If two points $\mathbf{p}_i$ and $\mathbf{p}_j$ are vertices of the polytope $H(\mathbf{p})$ connected by an edge, then the wall W_{ij} contains an $(n-1)$-dimensional infinite circular cone.*

Proof. Let A be the hyperplane that goes through the midpoint of the edge between $\mathbf{p}_i$ and $\mathbf{p}_j$ perpendicular to it. Then the wall W_{ij} can be obtained by intersecting A with the closed halfspaces $S_k = \{\mathbf{p}_0 \in \mathbb{E}^n \mid |\mathbf{p}_0 - \mathbf{p}_i| \leq |\mathbf{p}_0 - \mathbf{p}_k|\}$, for all $k \neq i$. Clearly, each of these halfspaces either contains A or intersects it by a non-empty set.

Let $\bar{\mathbf{p}} = (\bar{\mathbf{p}}_1, \ldots, \bar{\mathbf{p}}_N)$ be the orthogonal projection of $\mathbf{p}$ onto the hyperplane A. Since the points $\mathbf{p}_i$ and $\mathbf{p}_j$ are not in the interior of any more than one-dimensional face of $H(\mathbf{p})$, their projection $\bar{\mathbf{p}}_i = \bar{\mathbf{p}}_j$ is in a strictly convex position with respect to the point configuration $\bar{\mathbf{p}}$ considered being in the Euclidean space A.

Those halfspaces S_k that do not contain the hyperplane A, intersect it by $(n-1)$-dimensional halfspaces $S'_k \subset A$ with boundary planes perpendicular to the corresponding line segments $\bar{\mathbf{p}}_k\bar{\mathbf{p}}_i$. Moreover for each halfspace S'_k the intersection of it with the corresponding ray $\bar{\mathbf{p}}_k\bar{\mathbf{p}}_i$ is non-compact. Since $\bar{\mathbf{p}}_i$ is in a strictly convex position with respect to the point configuration $\bar{\mathbf{p}} \subset A$, this implies that the intersection of halfspaces S'_k contains an $(n-1)$-dimensional infinite circular cone. □

Lemma 4.4. *If two points $\mathbf{p}_i$ and $\mathbf{p}_j$ are in a strictly convex position with respect to the set $\{\mathbf{p}_1, \ldots, \mathbf{p}_N\}$ and are connected by the sequence of collinear edges of the polytope $H(\mathbf{p})$, then the wall W^{ij} contains an $(n-1)$-dimensional infinite circular cone.*

Proof. We define the hyperplane A and the point configuration $\bar{\mathbf{p}}$ in the same way as in the proof of Lemma 4.3. By the same reason as in the proof of Lemma 4.3, the point $\bar{\mathbf{p}}_i = \bar{\mathbf{p}}_j$ is in a strictly convex position with respect to the point configuration $\bar{\mathbf{p}}$ considered being in the Euclidean space A.

The wall W^{ij} can be obtained by intersecting A with the closed halfspaces $T_k = \{\mathbf{p}_0 \in \mathbb{E}^n \mid |\mathbf{p}_0 - \mathbf{p}_i| \geq |\mathbf{p}_0 - \mathbf{p}_k|\}$, for all $k \neq i$. One can check that each of these halfspaces either contains A or intersects it by a non-empty set. Those halfspaces T_k that do not contain the hyperplane A, intersect it by $(n-1)$-dimensional halfspaces $T'_k \subset A$ with boundary planes perpendicular to the corresponding line segments $\bar{\mathbf{p}}_i\bar{\mathbf{p}}_k$. Moreover for each halfspace T'_k the intersection of it with the corresponding ray $\bar{\mathbf{p}}_i\bar{\mathbf{p}}_k$ is non-compact. Since $\bar{\mathbf{p}}_i$ is in a strictly convex position with respect to the point configuration $\bar{\mathbf{p}} \subset A$, this implies that the intersection of halfspaces T'_k contains an $(n-1)$-dimensional infinite circular cone. □

5 Continuous case

Consider a smooth motion $\mathbf{p}(t) = (\mathbf{p}_1(t), \ldots, \mathbf{p}_N(t))$ of a configuration of N points in $\mathbb{E}^n$, where $t \in [0,1]$. As before, let $d_{ij}(t) = |\mathbf{p}_i(t) - \mathbf{p}_j(t)|$, and let $d'_{ij}(t)$ be the t-derivative of $d_{ij}(t)$.

Definition 5.1. *We call the motion $\mathbf{p}(t)$ properly expansive if for any $t \in (0,1)$ all derivatives $d'_{ij}(t)$ are non-negative, and $d'_{ij}(t_0) = 0$ for some $t_0 \in (0,1)$ implies that $d_{ij}(0) = d_{ij}(1)$.*

Proposition 5.2. *If there exists a smooth properly expansive motion $\mathbf{p}(t)$ of N points in $\mathbb{E}^n$ defined for $t \in [0,1]$, where $\mathbf{p} = \mathbf{p}(0)$, and $\mathbf{q} = \mathbf{p}(1)$, then Theorem 3.3 holds.*

Proof. First note that the coefficients $\alpha_{1-n}(\mathbf{p}(t))$ and $\beta_{1-n}(\mathbf{p}(t))$ are respectively non-decreasing and non-increasing as functions of t. Otherwise it would contradict Theorem 3.1 for sufficiently large value of r. We will show that if $\mathbf{p}$ and $\mathbf{q}$ are not congruent, then there is a subinterval $I \subset (0,1)$ on which the functions $\alpha_{1-n}(\mathbf{p}(t))$ and $\beta_{1-n}(\mathbf{p}(t))$ are strictly monotone.

Let $A_1, \ldots A_{2^N}$ be all subsets of the set $\{1, \ldots, N\}$. We define the sequence of intervals $\{I_0, \ldots I_{2^N}\}$ so that it would satisfy the following properties for $1 \leq i \leq 2^N$:

(i) $I_0 = (0,1)$.

(ii) The interval I_i is a closed subinterval of I_{i-1} consisting of more than one point.

(iii) For each $t \in I_i$ the dimension of the affine span of point set $B_i(t) = \{\mathbf{p}_j(t) \mid j \in A_i\}$ is constant and does not depend on t.

Since the maximality of the dimension of the affine span is an open condition, we can always satisfy properties (ii) and (iii) by chosing I_i as a subinterval of the set of such $t \in I_{i-1}$ for which the affine span of $B_i(t)$ has maximal dimension.

Finally, we chose $I = I_{2^N}$. Thus the interval I satisfies the property that for any $i = 1, \ldots, 2^N$ the dimension of the affine span of the set $B_i(t)$ is constant and does not depend on t.

Lemma 5.3. *For all $i = 1, \ldots, N$ the sets $\Omega(C_{i,n}(t), \mathbf{p}_i(t))$ and $\Omega(C^{i,n}(t), \mathbf{p}_i(t))$ are uniformly bounded on the interval I.*

Proof. From definition of Voronoi regions we know that each k-dimensional face of convex polyhedral sets $C_{i,n}(t)$ and $C^{i,n}(t)$ lies on a k-dimensional plane equidistant from some $N-k+1$ points $\mathbf{p}_{j_1}(t), \ldots, \mathbf{p}_{j_{N-k+1}}(t)$, where $1 \le j_1 < j2 < \cdots < j_{N-k+1} \le N$. Since the dimension of the affine span of the points $\mathbf{p}_{j_1}(t), \ldots, \mathbf{p}_{j_{N-k+1}}(t)$ is constant on the interval I, this k-dimensional plane either does not exist for all $t \in I$, or always exists and changes its position continuously. Thus the distance from the point $\mathbf{p}_i(t)$ to this plane is a continuous function on I, hence uniformly bounded. The lemma is proved. $\square$

Due to the last lemma we can assume that the number $\tilde{b}$ is such that the series $L_n(\mathbf{p}(t), r)$ and $L^n(\mathbf{p}(t), r)$ converge to the corresponding functions $V_n(\mathbf{p}(t), r)$ and $V^n(\mathbf{p}(t), r)$ on the interval $r \in (\tilde{b}, +\infty)$ for all $t \in I$. Moreover, one can see that when r is fixed, the volumes of truncated walls between Voronoi regions depend continuously on $t \in I$. Together with Theorem 3.1 it implies that the functions $\frac{d}{dt}V_n(\mathbf{p}(t), r)$ and $\frac{d}{dt}V_n(\mathbf{p}(t), r)$ are continuous in $t \in I$, and by Remark 2.11 these functions can be decomposed into Laurent series

$$\frac{d}{dt}V_n(\mathbf{p}(t), r) = \sum_{j=1-n}^{+\infty} \tilde{\alpha}_j(\mathbf{p}(t)) r^{-j} \tag{22}$$

and

$$\frac{d}{dt}V^n(\mathbf{p}(t), r) = \sum_{j=1-n}^{+\infty} \tilde{\beta}_j(\mathbf{p}(t)) r^{-j}$$

on the interval $r \in (\tilde{b}, +\infty)$.

Lemma 5.4. *For all $j > -n$ the functions $\alpha_j(\mathbf{p}(t))$ and $\beta_j(\mathbf{p}(t))$ are continuously differentiable on the interval $t \in I$, and $\frac{d}{dt}\alpha_j(\mathbf{p}(t)) = \tilde{\alpha}_j(\mathbf{p}(t))$, $\frac{d}{dt}\beta_j(\mathbf{p}(t)) = \tilde{\beta}_j(\mathbf{p}(t))$.*

Proof. We will prove the lemma only for functions $\tilde{\alpha}_j(\mathbf{p}(t))$, but exactly the same arguments work for functions $\tilde{\beta}_j(\mathbf{p}(t))$.

Assume, not all of the functions $\tilde{\alpha}_j(\mathbf{p}(t))$ are continuous. Let k be the smallest integer, such that $\tilde{\alpha}_k(\mathbf{p}(t))$ is discontinuous. Then since the series (22) is continuous in $t \in I$, the series $S(r,t) = \sum_{j=k}^{+\infty} \tilde{\alpha}_j(\mathbf{p}(t)) r^{k-j}$ is also continuous in $t \in I$ for all $r > \tilde{b}$.

According to Remark 2.11, there exists a real number M, such that

$$|\tilde{\alpha}_j(\mathbf{p}(t))| \le M(\tilde{b}+1)^j, \tag{23}$$

for all $j > -n$ and $t \in I$. This means that when r tends to infinity, the series $S(r,t)$ uniformly converges to the function $\tilde{\alpha}_j(\mathbf{p}(t))$ on the interval $t \in I$, hence $\tilde{\alpha}_j(\mathbf{p}(t))$ cannot be discontinuous.

Inequality (23) also implies uniform convergence of the series (22) in $t \in I$ for all $r > \tilde{b} + 1$, so if t_0 is an arbitrary point from interval I, then

$$V_n(\mathbf{p}(t), r) = V_n(\mathbf{p}(t_0), r) + \int_{t_0}^{t} \frac{d}{ds} V_n(\mathbf{p}(s), r) ds =$$
$$\alpha_{-n}(\mathbf{p}(t_0)) r^n + \sum_{j=1-n}^{+\infty} r^{-j} (\alpha_j(\mathbf{p}(t_0)) + \int_{t_0}^{t} \tilde{\alpha}_j(\mathbf{p}(s)) ds) = \sum_{j=-n}^{+\infty} \alpha_j(\mathbf{p}(t)) r^{-j},$$

for all $r > \tilde{b} + 1$ and $t \in I$. This finishes the proof of the lemma. □

Because of the last lemma, to finish the proof of Proposition 5.2, we only need to show that if $\mathbf{p}$ and $\mathbf{q}$ are not congruent, then the functions $\tilde{\alpha}_{1-n}(\mathbf{p}(t))$ and $-\tilde{\beta}_{1-n}(\mathbf{p}(t))$ are positive on interval I. We will give a proof only for $\tilde{\alpha}_{1-n}(\mathbf{p}(t))$, but in the case of the coefficient $\tilde{\beta}_{1-n}(\mathbf{p}(t))$ the proof goes in exactly the same way, except that instead of Lemma 4.3 one should use Lemma 4.4.

Before we continue with the proof, we would like to give a definition and formulate two theorems that will be used here. The first theorem is a simplified version of so called "Cauchy's arm lemma" (Lemma 5 from [Con82]) and the second is Theorem 8.6 from [Whi84].

Definition 5.5. *Let $\mathbf{q}(t) = (\mathbf{q}_1(t), \ldots, \mathbf{q}_n(t))$ be a continuous motion of some point configuration in $\mathbb{E}^d$. We say that the motion is rigid, if the distances between any pair of points in the configuration are constant throughout the motion.*

Theorem 5.6. *Let $\mathbf{q}(t) = (\mathbf{q}_1(t), \ldots, \mathbf{q}_n(t))$ be a continuous motion of some point configuration in $\mathbb{E}^d$, such that the distances between any two points are non-decreasing functions of t. Assume for some $t = t_0$ the points $\mathbf{q}_1(t_0), \ldots, \mathbf{q}_n(t_0)$ are the vertices of a 2-dimensional convex n-gon, and the distances between any two points connected by some edge of the n-gon are constant throughout the motion. Then the motion $\mathbf{q}(t)$ is rigid.*

Theorem 5.7. *Let $\mathbf{q}(t) = (\mathbf{q}_1(t), \ldots, \mathbf{q}_n(t))$ be a continuous motion of some point configuration in $\mathbb{E}^d$, such that the affine span of the point configuration $\mathbf{q}(t)$ has the same dimension troughout the motion. Assume for some $t = t_0$ the following conditions are satisfied:*

(i) there exists a convex polytope P such that all of its vertices are from the point set $\mathbf{q}(t_0)$ and are in a strictly convex position with respect to $\mathbf{q}(t_0)$;

(ii) if some point $\mathbf{q}_i(t_0)$ is not a vertex of P, then it is an interior point of some edge of P;

(iii) for any subset of points $\mathbf{q}_{i_1}(t_0), \ldots, \mathbf{q}_{i_k}(t_0)$ that belong to the same edge or 2-face of P, its motion $\mathbf{q}_{i_1}(t), \ldots, \mathbf{q}_{i_k}(t)$ is rigid.

Then the motion $\mathbf{q}(t)$ is rigid.

Since the walls between two Voronoi regions are $(n-1)$-dimensional truncated polytopes, there exist non-negative functions $c_{ij}(t)$, such that for each $t \in I$

$$\mathrm{Vol}_{n-1}[W_{ij}(\mathbf{p}(t), r)] = c_{ij}(t) r^{n-1} + o(r^{n-1}),$$

as r tends to infinity. According to Theorem 3.1 we have $\tilde{\alpha}_{1-n}(\mathbf{p}(t)) = \sum_{i<j} d'_{ij}(t) c_{ij}(t)$, which implies that $\tilde{\alpha}_{1-n}(\mathbf{p}(t)) \geq 0$.

Assume, for some $t_0 \in I$, we have $\tilde{\alpha}_{1-n}(\mathbf{p}(t_0)) = 0$. Because of Lemma 4.3 and Lemma 4.1 this means that for each pair of points $\mathbf{p}_i$ and $\mathbf{p}_j$ that satisfy the conditions of Lemma 4.3, we have $d'_{ij}(t_0) = 0$. Since the motion $\mathbf{p}(t)$ is properly expansive, it implies that $d'_{ij}(t) = 0$ for all $t \in I$. Thus, the edges of the polytope $H(\mathbf{p}(t_0))$ have the same length for all $t \in I$. Note that the other points cannot get closer throughout their motion, so by Theorem 5.6 the motion of any subconfiguration of points that belong to the same 2-face of the polytope $H(\mathbf{p}(t_0))$, is rigid, when $t \in I$. Now by applying Theorem 5.7 we get that the motion of the set of all vertices of the polytope $H(\mathbf{p}(t_0))$ is rigid for $t \in I$.

Finally assume the point $\mathbf{p}_i(t_0)$ is not a vertex of $H(\mathbf{p}(t_0))$. Then there is a subset of the set of all vertices of the polytope $H(\mathbf{p}(t_0))$, such that the point $\mathbf{p}_i(t_0)$ lies in the relative interior of the convex hull of these points. According to our choice of the interval I, the point $\mathbf{p}_i(t)$ stays in the affine span of these vertices for all $t \in I$. Since the distances between this point and the vertices cannot decrease as t increases, it means that the point $\mathbf{p}_i(t)$ does not move with respect to the polytope $H(\mathbf{p}(t_0))$ as t changes in I. Thus the motion $\mathbf{p}(t)$ is rigid on the interval I, and since it is a properly expansive motion, it is rigid on the whole interval $[0, 1]$, which means that the point configurations $\mathbf{p}$ and $\mathbf{q}$ are congruent. Thus, the Proposition 5.2 is proven. □

6 Discontinuous case

The following lemma is an easy consequence of Lemma 7 from [BC02].

Lemma 6.1. *Let $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_N)$ be a fixed configuration of points in $\mathbb{E}^n \subset \mathbb{E}^{n+2}$. Then*

$$\frac{d}{dr} V_{n+2}(\mathbf{p}, r) = 2\pi r V_n(\mathbf{p}, r),$$

and

$$\frac{d}{dr} V^{n+2}(\mathbf{p}, r) = 2\pi r V^n(\mathbf{p}, r).$$

Proof of Theorem 3.3. Let k be a positive integer, and we regard $\mathbb{E}^n$ as the subset $\mathbb{E}^n = \mathbb{E}^n \times \{0\} \subset \mathbb{E}^n \times \mathbb{E}^{2k} = \mathbb{E}^{n+2k}$. We chose k to be sufficiently large, so that there exists a smooth properly expansive motion from $\mathbf{p}$ to $\mathbf{q}$ in $\mathbb{E}^{n+2k}$. According to Lemma 1 from [BC02] such k always exists.

Now from Proposition 5.2 we know that Theorem 3.3 holds for the volume functions $V_{n+2k}(\mathbf{p},r)$, $V^{n+2k}(\mathbf{p},r)$, $V_{n+2k}(\mathbf{q},r)$ and $V^{n+2k}(\mathbf{q},r)$. According to Lemma 6.1 if we differentiate these functions with respect to r and divide them by $2\pi r$, we get the functions $V_{n+2k-2}(\mathbf{p},r)$, $V^{n+2k-2}(\mathbf{p},r)$, $V_{n+2k-2}(\mathbf{q},r)$ and $V^{n+2k-2}(\mathbf{q},r)$. After repeating this operation k times we get the n-dimensional volume functions $V_n(\mathbf{p},r)$, $V^n(\mathbf{p},r)$, $V_n(\mathbf{q},r)$ and $V^n(\mathbf{q},r)$. Clearly, this operation preserves the property of satisfying the result of Theorem 3.3, so since the result of Theorem 3.3 holds for the $(n+2k)$-dimensional volume functions, it also holds for the n-dimensional ones. □

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