Unit 8. Topological and Differentiable Manifolds

The configuration space of a mechanical system, examples; the definition of topological and differentiable manifolds, smooth maps and diffeomorphisms; Lie groups, embedded submanifolds in $\mathbb{R}^n$, examples, Whitney's theorem, classification of closed 2-manifolds.

As the motion of a particle in $\mathbb{R}^3$ corresponds to a parameterized space curve, a motion of a system of n points can be described by n parameterized curves $\mathbf{x}_i: [a,b] \longrightarrow \mathbb{R}^3$ $i = 1,2,...,n$.

Putting these mappings together, we obtain a curve

$$(\mathbf{x}_1,...,\mathbf{x}_n): [a,b] \longrightarrow \mathbb{R}^3 x \mathbb{R}^3 x ... x \mathbb{R}^3 \text{ (n times)}$$

in the direct product of n copies of $\mathbb{R}^3$, the projections of which on the i-th factor of the product is just the curve $\mathbf{x}_i$.

For there is a one-to-one correspondence between points of the product $\mathbb{R}^3 x \mathbb{R}^3 x ... x \mathbb{R}^3$ (n times) and the possible configurations of n points in the space, we shall call the direct product of n copies of $\mathbb{R}^3$ the *configuration space* of the system of n points.

In general, the configuration space of a mechanical system is the set of all of its possible positions, equipped with some natural additional structures such as topology or the structure of a differentiable manifold (see later).

The advantage of introducing the configuration space is that the motion of the system can be interpreted as one single curve in the configuration space instead of a set of space curves.

Non-trivial examples can be obtained by putting some constraints on a system of n points. For example, some pairs of points can be connected by a rigid rod, some points can be fixed or forced to move along a line or a surface. Further constraints can be obtained by specifying the type of joint at the points where two or more rods meet.

The configuration space of a system of n points with constraints is a subspace of $\mathbb{R}^{3n}$ and it is quite natural to furnish it with the subspace topology inherited from $\mathbb{R}^{3n}$.

Examples.

i) The configuration space of the planar pendulum is the circle S^1.

ii) The configuration space of the spherical pendulum is the two-dimensional sphere S^2.

iii) The configuration space of a planar double pendulum is the direct product of two circles, i.e. the torus $T^2 = S^1 \times S^1$.

iv) The configuration space of a spherical double pendulum is the direct product of two spheres $S^2 \times S^2$.

v) A rigid segment in the plane has for its configuration space the direct product $\mathbb{R}^2 \times S^1$, which is homeomorphic to the open solid torus.

As we see in the above examples, the configuration space of a mechanical system is not necessarily homeomorphic to a linear space, but in each case the points of the configuration space have a neighborhood homeomorphic to an open ball.

In the following chain of definitions we *fix a positive integer* n.

Definition. Let X be an arbitrary set. A local parameterization of X is an injective mapping $\varphi : \Omega \longrightarrow X$ from an open subset Ω of $\mathbb{R}^n$ onto a subset of X.

The inverse $\varphi^{-1} : \varphi(\Omega) \longrightarrow \Omega$ of such a parameterization is called a chart because through φ^{-1} the region $\operatorname{im} \varphi \subset X$ is "charted" on $U \subset \mathbb{R}^n$, just as a region of the earth is charted on a topographic or a political map. φ^{-1} is also called a coordinate system because through φ^{-1} each point $p \in \operatorname{im} \varphi$ corresponds to an n-tuple of real numbers, the coordinates of p.

An atlas on X is a collection of charts $\mathcal{A} = \{ \varphi_i : i \in I \}$ such that every point is represented in at least one chart i.e. $\bigcup_{i \in I} \operatorname{dom} \varphi_i = X$.

Two charts $\varphi : X \hookrightarrow U$ and $\psi : X \hookrightarrow V$ are said to be $\mathcal{C}^r$-compatible if the domains $\varphi(\operatorname{dom} \psi \cap \operatorname{dom} \varphi)$ and $\psi(\operatorname{dom} \psi \cap \operatorname{dom} \varphi)$ of the "transit" mappings $\psi \circ \varphi^{-1}$ and $\varphi \circ \psi^{-1}$ are open subsets of $\mathbb{R}^n$, and $\psi \circ \varphi^{-1}$ and $\varphi \circ \psi^{-1}$ are r times continuously differentiable. (A mapping is 0 times continuously differentiable if it is continuous).

An atlas is $\mathcal{C}^r$-compatible if any two charts in the atlas are $\mathcal{C}^r$-compatible.

An atlas $\mathcal{A}$ on a set X defines a topology on X as follows

Let $U \subset X$ be open if and only if $\varphi(U \cap \operatorname{dom} \varphi)$ is open in $\mathbb{R}^n$ with respect to any chart φ from $\mathcal{A}$.

Proposition. The family of open sets yields a topology on X.

Proof. The statement follows directly from the following set theoretical identities.

i) $\varphi(\emptyset \cap \operatorname{dom} \varphi) = \emptyset$

ii) $\varphi(X \cap \operatorname{dom} \varphi) = \operatorname{im} \varphi$

iii) $\varphi(U \cap V \cap \operatorname{dom} \varphi) = \varphi(U \cap \operatorname{dom} \varphi) \cap \varphi(V \cap \operatorname{dom} \varphi)$

iv) $\varphi\left(\left(\bigcup_{i\in I} U_i\right)\cap \operatorname{dom}\varphi\right) = \bigcup_{i\in I} \varphi\,(U_i \cap \operatorname{dom}\varphi)$. ▮

Definition. An n-dimensional topological manifold is a pair $(X,\mathcal{A})$ consisting of a point set X and a $\mathcal{C}^0$-compatible atlas $\mathcal{A}$ on it, such that the topology induced by the atlas on X satisfies the following two conditions

i) for any two distinct points x, y $\in$ X, one can find two disjoint neighborhoods of x and y (i.e. X is a Hausdorff space);

ii) there exists a countable family of charts $\varphi_1,\varphi_2,\varphi_3,\ldots\in\mathcal{A}$, the domain of which cover X (X is a second countable topological space).

Remark. In physics, the dimension of the configuration space of a mechanical system (provided that it is a manifold) is called the number of degrees of freedom.

We say that a topological manifold is a $\mathcal{C}^r$-manifold if the atlas $\mathcal{A}$ of it is $\mathcal{C}^r$-compatible. Two atlases are *equivalent* or define the same $\mathcal{C}^r$-manifold structure on X if their union also consists of $\mathcal{C}^r$-compatible charts. It is clear that each equivalence class of atlases contains a unique *maximal atlas*.

We shall mainly be interested in $\mathcal{C}^\infty$-manifolds which will also be called smooth or differentiable manifolds.

Examples.

(i) $\mathbb{R}^n$ equipped with the atlas consisting of only one chart, the identity mapping of $\mathbb{R}^n$, is an n-dimensional differentiable manifold.

(ii) Open subsets $U \subset X$ of an n-dimensional manifold $(X,\mathcal{A})$ become n-dimensional manifolds with the atlas $\{\varphi|_{\operatorname{dom}(\varphi)\cap U} : \varphi \in \mathcal{A}\}$.

(iii) If $(X_1,\mathcal{A}_1)$ and $(X_2,\mathcal{A}_2)$ are two manifolds of dimension n and m respectively, then the product space $X_1 \times X_2$ has a natural (n+m)-dimensional manifold structure given by the atlas

$$\{(\varphi_1,\varphi_2):\operatorname{dom}(\varphi_1) \times \operatorname{dom}(\varphi_2)\longrightarrow \mathbb{R}^{n+m} : \varphi_1\in\mathcal{A}_1,\ \varphi_2\in\mathcal{A}_2\}.$$

(iv) We have introduced the topology on the Grassmann manifolds Gr(n,k) in Unit 2. The topology of these spaces comes from a k(n-k)-dimensional differentiable manifold structure. We construct a chart $\varphi_{\mathcal{B}}$ on Gr(n,k) to every ordered basis $\mathcal{B} = \{\mathbf{x}_1,\ldots,\mathbf{x}_n\}$ of $\mathbb{R}^n$. Let us denote by V the subspace spanned by the first k vectors of $\mathcal{B}$ and by W the subspace spanned by the last (n-k) vectors. It is clear that $\mathbb{R}^n = V \oplus W$. Denote by $\pi:\mathbb{R}^n\longrightarrow V$ the projection of $\mathbb{R}^n$ onto V along W. The chart $\varphi_{\mathcal{B}}$ will be defined on the set

$$\operatorname{dom}(\varphi_{\mathcal{B}}) = \{L \in Gr(n,k) : L \cap W = \{\mathbf{0}\}\}.$$

$\varphi_{\mathcal{B}}$ assigns to $L \in \operatorname{dom}(\varphi_{\mathcal{B}})$ a $k\times(n-k)$ matrix in the following way. The restriction of π onto L is an isomorphism between L and V. The preimages of the vectors $\mathbf{x}_1,\ldots,\mathbf{x}_k$ yield a basis of L which has the form

$$(\pi|_L)^{-1}(\mathbf{x}_i) = \mathbf{x}_i + \sum_{j=k+1}^{n} \alpha_{ij}\, \mathbf{x}_j.$$

It is clear that setting $\varphi_{\mathcal{B}}(L)$ to be equal to the matrix of coefficients (α_{ij}), $i=1,\dots,k$; $j=k+1,\dots,n$, we obtain a bijection between $\mathrm{dom}(\varphi_{\mathcal{B}})$ and the set of all $k\mathrm{x}(n-k)$ matrices. The family of all charts of the form $\varphi_{\mathcal{B}}$ is a $\mathcal{C}^{\infty}$-compatible atlas on $Gr(n,k)$.

$Gr(n,k)$ is a compact manifold, it can be covered by a finite number of charts. Indeed we get a finite atlas on $Gr(n,k)$ if we let $\mathcal{B}$ run through different permutations of the standard basis of $\mathbb{R}^n$.

The Grassmann manifold $Gr(n+1,1)$ is the n-dimensional projective space. The geometrical way to introduce projective spaces is the following. We take an n-dimensional Euclidean space and join to it a collection of extra points, called ideal points or points at infinity, in such a way, that we attach one point at infinity to each straight line and two straight line gets the same point at infinity if and only if they are parallel. If we put the n-dimensional space into the (n+1)-dimensional one and fix a point O outside it, then every straight line through O intersects the projective closure of the n-dimensional Euclidean space in a unique ordinary or ideal point and this is the natural correspondence between the two ways of introducing projective spaces.

A mapping $f:X\longrightarrow Y$ from a differentiable manifold $(X,\mathcal{A})$ into the differentiable manifold $(Y,\mathcal{B})$ is said to be smooth if for any two charts $\varphi\in\mathcal{A}$ and $\psi\in\mathcal{B}$, the mapping $\psi\circ f\circ\varphi^{-1}$ is smooth. The map f is a diffeomorphism if it is a bijection and both f and f^{-1} are smooth.

Two differentiable manifolds are diffeomorphic if there is a diffeomorphism between them.

Definition. A Lie group is a differentiable manifold G with a group operation such that the mapping

$$G \times G \longrightarrow G, \quad (x,y)\longmapsto x\, y^{-1}$$

is differentiable.

Example. $Gl(n,\mathbb{R})$ and $Gl(n,\mathbb{C})$ are open subset in the linear spaces of all nxn real/complex matrices, hence they have a differentiable manifold structure. They also have a group structure, which is smooth since the entries of the quotient of two matrices are rational functions of the entries of the original matrices and rational functions are smooth. This way, general linear groups are Lie groups.

The following theorem explains why the configuration space of a system of n points with constraints so often happens to be a manifold.

Theorem. Let $F:\mathbb{R}^n \longrightarrow \mathbb{R}^k$ be a smooth mapping, the image of which contains $\mathbf{0} \in \mathbb{R}^k$. Consider the preimage of the point $\mathbf{0}$

$$X = \{ \mathbf{x} \in \mathbb{R}^n : F(\mathbf{x}) = \mathbf{0} \}.$$

Let us suppose that the gradient vectors

$$\operatorname{grad} f_i(\mathbf{x}) = \left(\frac{\partial f_i}{\partial x_1}(\mathbf{x}), \frac{\partial f_i}{\partial x_2}(\mathbf{x}), \ldots, \frac{\partial f_i}{\partial x_n}(\mathbf{x}) \right)$$

of the coordinate functions of $F = (f_1, f_2, \ldots f_n)$ are linearly independent at each point $\mathbf{x}$ of X.

Then X is an (n-k)-dimensional topological manifold, furthermore, there is a well-defined differentiable manifold structure on X.

Remark. The condition on the independence of the gradient vectors of the coordinate functions is essential. By a theorem due to Whitney, for any closed set $C \subset \mathbb{R}^n$ there exists a smooth function f on $\mathbb{R}^n$ such that $C = \{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x})=0\}$.

Proof. Let us recall a fundamental result of multivariable calculus.

The Inverse Function Theorem. If $\mathbf{x}$ is a point in the domain of a smooth function $\tilde{F} = (f_1, f_2, \ldots, f_n): U \longrightarrow \mathbb{R}^n$, defined on an open subset of $\mathbb{R}^n$, and the gradient vectors of the coordinate functions $f_1, f_2, \ldots, f_n$ of $\tilde{F}$ are linearly independent at the point $\mathbf{x}$, then there exists an open neighborhood $V \subset U$ of $\mathbf{x}$ such that $\tilde{F}|_V$ is a diffeomorphism between V and $\tilde{F}(V)$. In addition, $\tilde{F}(V)$ is an open subset of $\mathbb{R}^n$.

The linear space $\mathbb{R}^{n-k}$ can be embedded into $\mathbb{R}^n$ through the mapping

$$\iota : (x_1, x_2, \ldots, x_{n-k}) \longmapsto (\underbrace{0, 0, \ldots, 0}_{k \text{ zeros}}, x_1, x_2, \ldots, x_{n-k}) .$$

Consider the set $\tilde{\mathcal{A}}$ of those diffeomorphisms $\tilde{\varphi} : V \longrightarrow U$ between open subsets of $\mathbb{R}^n$ through which the set $X \cap V$ is mapped onto the intersection $\iota(\mathbb{R}^{n-k}) \cap U$.

Put

$$\mathcal{A} = \{\varphi : \varphi \text{ has the form } \varphi = \iota^{-1} \circ \tilde{\varphi}|_{M \cap V} : X \cap V \to \mathbb{R}^{n-k}, \text{ where } \tilde{\varphi} \in \tilde{\mathcal{A}}, \operatorname{dom} \tilde{\varphi} = V \}.$$

Obviously, elements of $\mathcal{A}$ define a homeomorphism between open subsets of X and that of $\mathbb{R}^{n-k}$. It is also clear from the construction that the mappings $\varphi \circ \psi^{-1}$, defined on $\psi(\operatorname{dom} \varphi \cap \operatorname{dom} \psi)$, are smooth for any $\varphi, \psi \in \mathcal{A}$. In such a way, to prove that $\mathcal{A}$ is a $\mathcal{C}^{\infty}$-compatible atlas on X one has only to check that each point of X is represented in at least one chart belonging to $\mathcal{A}$.

For this purpose, consider an arbitrary point $\mathbf{x}$ of X and the gradient vectors of the coordinate functions of F at $\mathbf{x}$. Since they are linearly independent, we can obtain a basis by joining further n-k vectors $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_{n-k}$ to them.

The gradient vector of the linear function $g_i(\mathbf{x}) = \langle \mathbf{e}_i, \mathbf{x} \rangle$ is $\mathbf{e}_i$ at any point, consequently, the inverse function theorem can be applied to the mapping

$$\tilde{F} = (f_1, \dots, f_k, g_1, \dots, g_{n-k}) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

at the point $\mathbf{x}$. According to the theorem, $\tilde{F}$ is a diffeomorphism between a neighborhood V of $\mathbf{x}$ and an open neighborhood of $\tilde{F}(\mathbf{x})$. Denote by $\tilde{\varphi}$ the restriction of $\tilde{F}$ onto V. The mapping $\tilde{\varphi}$ belongs to $\tilde{\mathcal{A}}$ and the chart $\iota^{-1} \circ (\tilde{\varphi}|_{V \cap M})$ is defined in a neighborhood of $\mathbf{x}$, so the proof is finished. ▮

<u>Exercise</u>. Check that the topology of $(X, \mathcal{A})$ coincides with the subspace topology inherited from $\mathbb{R}^n$, consequently, it is Hausdorff and second countable.

<u>Definition</u>. We say that $X \subset \mathbb{R}^n$ is an <u>embedded (n - k)-dimensional submanifold in $\mathbb{R}^n$</u> if in a neighborhood U of every point $\mathbf{x} \in X$ there are functions $f_1, f_2, \dots, f_k : U \longrightarrow \mathbb{R}$ such that the intersection of U with X is given by the equations $f_1 = f_2 = \dots = f_k = 0$ and the vectors grad $f_1, \dots,$ grad f_k at $\mathbf{x}$ are linearly independent.

The study of higher dimensional manifolds was launched at the beginning of the 20-th century by H. Poincaré (1854 - 1912). At that time topology was in its cradle and the abstract definition of a topological space had not been created. Poincaré worked with embedded submanifolds in $\mathbb{R}^n$. This was not a real loss of generality since by *Whitney's theorem* every differentiable manifold of dimension n is diffeomorphic to an embedded submanifold in $\mathbb{R}^{2n+1}$.

<u>Examples</u>.

(i) As an application, consider the set S^{n-1} of points in $\mathbb{R}^n$ the distance of which from the origin is equal to one. They are characterized by the equality

$$f(x_1, \dots, x_n) = x_1^2 + \dots + x_n^2 - 1 = 0.$$

The gradient vector of f at the point $\mathbf{x} \in \mathbb{R}^n$ is just $2\mathbf{x}$. It is zero only at the origin, which does not belong to S^{n-1}, so S^{n-1} is a topological manifold with a natural differentiable structure on it. S^{n-1} is called the <u>(n-1)-dimensional sphere with the standard differentiable structure</u>.

(ii) Most important Lie groups are obtained as closed subgroups of $Gl(n,\mathbb{R})$, defined by some equalities on the matrix entries. For example, $Gl(n,\mathbb{C})$ is isomorphic to the subgroup of $Gl(2n,\mathbb{R})$ consisting of matrices of the form $\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$, where A, B are nxn matrices. The following example demonstrates how we can prove that a closed subgroup of $Gl(n,\mathbb{R})$ is a Lie group.

Consider the orthogonal group

$$O(n) = \{A \in Gl(n,\mathbb{R}) : A A^T = I\}.$$

We claim that it is an $\frac{n(n-1)}{2}$ - dimensional Lie group.

If $A = (a_{ij})$, then the equation $A A^T = I$ is equivalent to the system of equations

$$F_{ij}(A) = \sum_k a_{ik}a_{jk} = \delta_{ij}, \qquad (i,j = 1,\dots,n).$$

Since $F_{ij} = F_{ji}$, these equations are not independent. We get however an independent system of equations if we restrict ourselves to the equations with $1 \leqslant i \leqslant j \leqslant n$. The number of these equations is $\frac{n(n+1)}{2}$, so if we show the independence of the gradients then we obtain the required expression for the dimension of O(n).

The gradient of F_{ij} at $A \in O(n)$ is an nxn matrix with entries

$$\frac{\partial F_{ij}}{\partial a_{rs}} = \sum_k \left(\frac{\partial a_{ik}}{\partial a_{rs}} a_{jk} + a_{ik} \frac{\partial a_{jk}}{\partial a_{rs}} \right) = a_{js}\delta_{ir} + a_{is}\delta_{jr}$$

We show that these vectors, are orthogonal with respect to the usual scalar product on $\mathbb{R}^{n^2} = Mat(n,\mathbb{R})$. Indeed, taking the scalar product of the gradient vectors of F_{ij} and F_{kl} at A we obtain

$$\sum_{r,s} (a_{js}\delta_{ir} + a_{is}\delta_{jr})(a_{ls}\delta_{kr} + a_{ks}\delta_{lr}) =$$

$$= \sum_{r,s} a_{js}a_{ls}\delta_{ir}\delta_{kr}+a_{js}a_{ks}\delta_{ir}\delta_{lr}+a_{is}a_{ls}\delta_{jr}\delta_{kr}+a_{is}a_{ks}\delta_{jr}\delta_{lr} =$$

$$= \sum_{r} \delta_{jl}\delta_{ir}\delta_{kr}+\delta_{jk}\delta_{ir}\delta_{lr}+\delta_{il}\delta_{jr}\delta_{kr}+\delta_{ik}\delta_{jr}\delta_{lr} =$$

$$= \delta_{jl}\delta_{ik}+\delta_{jk}\delta_{il}+\delta_{il}\delta_{jk}+\delta_{ik}\delta_{jl} = 2\,\delta_{ik}\delta_{jl}(1+\delta_{ij}\delta_{kl}).$$

We know that the determinant of an orthogonal matrix is ±1. Thus O(n) has two components. The connected component on which determinant is 1, is the <u>special orthogonal group</u> SO(n), and it has also dimension $\frac{n(n-1)}{2}$. SO(n) is the configuration space of a rigid n-dimensional body with one fixed point. The configuration space of an n-dimensional body without a fix point is the space $SO(n) \times \mathbb{R}^n$.

For n = 2, SO(2) is the 1-dimensional group of rotations of the plane. For n = 3, SO(3) is a 3-dimensional manifold. With the help of quaternions we can show that this group is homeomorphic to the 3-dimensional projective space.

Let $\mathbb{H}$ denote the 4-dimensional space of quaternions x+yi+zj+wk, and let us identify $\mathbb{R}^3$ with the space of pure imaginary quaternions. For $0 \neq q \in \mathbb{H}$ let us denote by ρ_q the transformation $\rho_q(h)=q^{-1}hq$. Since $|q^{-1}hq|=|h|$, ρ_q is an orthogonal transformation. If h is a real number then $\rho_q(h)=h$, thus ρ_q maps $\mathbb{R} \subset \mathbb{H}$ into itself. Consequently, it maps $\mathbb{R}^3$, the orthogonal complement of $\mathbb{R}$ also into itself. The assignment $q \longmapsto \rho_q|_{\mathbb{R}^3}$ is a group homomorphism from the multiplicative group $\mathbb{H}\setminus\{\mathbf{0}\}$ to SO(3). The kernel of this homomorphism is the center of $\mathbb{H}\setminus\{\mathbf{0}\}$, i.e. $\mathbb{R}\setminus\{\mathbf{0}\}$. This homomorphism is also surjective as it follows from the following two exercises.

Exercise.

A) Show that every element of SO(3) is a rotation about a straight line.

B) Show that if $a \in \mathbb{R}^3$ is a pure imaginary quaternion, $\lambda \in \mathbb{R}$ and $q = 1+\lambda a$, then ρ_q is a rotation about a and varying λ we can obtain all rotations.

We conclude that SO(3) is isomorphic to the factorgroup $\mathbb{H}\setminus\{\mathbf{0}\}/\mathbb{R}\setminus\{\mathbf{0}\}$, but cosets of $\mathbb{R}\setminus\{\mathbf{0}\}$ in $\mathbb{H}\setminus\{\mathbf{0}\}$ are in one to one correspondence with straight lines through $\mathbf{0}$ in $\mathbb{H}$, i.e. there is a natural bijection between SO(3) and the projective space Gr(4,1).

The classification problem of n-dimensional manifolds can be formulated in different levels. For each r we may consider the category of $\mathcal{C}^r$-manifolds and r-times differentiable mappings. The following two theorems show that the classification problem of $\mathcal{C}^r$-manifolds is the same problem for all $1 \leqslant r \leqslant \infty$.

Theorem. For $r \geqslant 1$, every maximal $\mathcal{C}^r$-compatible atlas contains a $\mathcal{C}^\infty$-compatible atlas.

Theorem. It two $\mathcal{C}^\infty$-manifolds are $\mathcal{C}^1$-diffeomorphic, then they are $\mathcal{C}^\infty$-diffeomorphic.

None of the above theorems can be extended to $\mathcal{C}^0$-manifolds. There exist topological manifolds which have no $\mathcal{C}^1$-compatible atlas, and there exist homeomorphic but not diffeomorphic differentiable manifolds.

In 1956 J.W. Milnor constructed a differentiable manifold which is homeomorphic to the 7-dimensional sphere but not diffeomorphic to it. Such manifolds were given the name ”exotic spheres”. Later on an even more surprising result was published by Milnor and Kervaire. There are exactly 28 mutually non-diffeomorphic differentiable structures on a topological 7-sphere. Since then many examples of topological manifolds having many different differentiable structures have been obtained. One of the most interesting constructions is due to Donaldson, who invented exotic differentiable structures on $\mathbb{R}^4$.

We do not meet these problems in dimension two. The classification of compact surfaces is the same up to homeomorphism and diffeomorphism. The classification theorem of compact 2-dimensional manifolds gives a list of non-diffeomorphic compact surfaces and asserts that every compact surface is diffeomorphic to one of the surfaces in the list. The list of compact surfaces contains as a matter of fact two lists. The first list contains the orientable compact surfaces, the second contains the non-orientable ones.

Orientable compact surfaces. The simplest orientable closed surface is the sphere $S^2 = \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x}| = 1\}$. The next example is the torus $T^2=S^1 x S^1$. Cutting a small disc out of the torus, we get a surface with boundary, called a handle. A typical orientable compact surface is a sphere with g handles. We can obtain this surface if we cut g holes on the surface of the sphere and glue a handle to each of them.

Non-orientable compact surfaces. The first member of this list is the real projective plane. To understand the topological structure of the projective plane, let us cut the projective plane into two parts by a hyperbola. The interior of the hyperbola has two components in the Euclidean plane, but these components are glued together along a segment of the line at infinity , so the interior is a topological disc. The exterior of the hyperbola is a long infinite band in the Euclidean plane, the ”ends” of which are glued together along another segment of the line at infinity. One can see that the ends of the band are glued together by a half twist so what we get is a Möbius band. We conclude that the projective plane is the union of a disc and a Möbius band glued together along their boundaries. A typical non-orientable compact surface is a sphere with g Möbius bands. We obtain this surface cutting g discs out of the sphere and gluing to the boundary of each hole a Möbius band. Non-orientable compact surfaces can not be embedded into the 3-dimensional Euclidean space, so although one can easily construct a 3-dimensional model of a Möbius band and that of a sphere with g holes, it is impossible to glue the Möbius bands to the sphere in practice. If however we could try this in a 4-dimensional space there would be no difficulty.

Further Exercises

Exercise 8-1. The configuration space of the pentagon (closed chain of five rods in the plane) with one edge fixed is a compact surface (sometimes with singularities). What kind of surfaces can we obtain?

Exercise 8-2. Give an example of a set X with a $\mathcal{C}^0$-compatible atlas $\mathcal{A}$ on it such that the topology induced on X by $\mathcal{A}$ is (i) not Hausdorff; (ii) not second countable.

Exercise 8-3. Show that the special unitary group

$$SU(n) = \{ A \in Gl(n,\mathbb{C}) : A A^* = I, \det A = 1 \}$$

is a Lie group, determine its dimension. Prove that SU(2) is diffeomorphic to the 3-dimensional sphere S^3.

Exercise 8-4. Which surface shall we get from the classification list if we glue to the sphere $k \geq 1$ Möbius bands and l handles?

Exercise 8-5. Let P be a complex polynomial of degree k having k different roots. Consider the subset of $\mathbb{C}^2$ defined by

$$M = \{(z,w) \in \mathbb{C}^2 : z^l = P(w)\}.$$

Show that M is diffeomorphic to a sphere with g handles with N points omitted. Express g and N in terms of k and l.