Unit 9. The Tangent Bundle

The tangent space of a submanifold of $\mathbb{R}^n$, identification of tangent vectors with derivations at a point, the abstract definition of tangent vectors, the tangent bundle; the derivative of a smooth map.

The aim of this chapter is to give and reconcile different commonly used definitions of a tangent vector to a manifold. Before passing over to the abstract situation, we shall deal with submanifolds of $\mathbb{R}^n$.

Definition. Smooth mappings $\gamma : [a,b] \longrightarrow M$ of an interval into a differentiable manifold $(M,\mathcal{A})$ are called smooth curves in the manifold.

Definition **A**. Let M be a differentiable manifold embedded in $\mathbb{R}^n$, $\mathbf{x}_0 \in M$. A vector $\mathbf{v}$ is called a tangent vector to M at $\mathbf{x}_0$ if there is a smooth curve $\mathbf{x}: [-\varepsilon,\varepsilon] \longrightarrow M$ passing through $\mathbf{x}_0 = \mathbf{x}(0)$ such that $\mathbf{v} = \mathbf{x}'(0)$. The tangent space $T_{\mathbf{x}_0}M$ of M at $\mathbf{x}_0$ is the set of all tangent vectors to M at $\mathbf{x}_0$.

Theorem. Let us suppose that a k-dimensional manifold M embedded in $\mathbb{R}^n$ is given in a neighborhood U of $\mathbf{x}_0 \in M$ by a system of equalities $f_1 = \ldots = f_{n-k} = 0$, where $f_1,\ldots,f_{n-k}$ are smooth functions on U such that the vectors grad $f_1(\mathbf{x}_0),\ldots,$grad $f_{n-k}(\mathbf{x}_0)$ are linearly independent at $\mathbf{x}_0$. Then the tangent space of M at $\mathbf{x}_0$ consists of the vectors orthogonal to grad $f_1(\mathbf{x}_0),\ldots,$ grad $f_{n-k}(\mathbf{x}_0)$.

Corollary. The tangent space of a k-dimensional submanifold of $\mathbb{R}^n$ is a k-dimensional linear subspace of $\mathbb{R}^n$.

Proof. If $\mathbf{x}: [-\varepsilon,\varepsilon] \longrightarrow \mathbb{R}^n$ is a smooth curve having coordinate functions $x_1,\ldots,x_n$ and lying on M, then we have

$$f_i(x_1(t),\ldots,x_n(t)) = 0 \qquad i = 1,\ldots,n-k$$

for each $t \in [-\varepsilon,\varepsilon]$. Differentiating by t we get

$$\frac{\partial f_i}{\partial x_1}(\mathbf{x}(0))x_1'(0) + \ldots + \frac{\partial f_i}{\partial x_n}(\mathbf{x}(0))x_n'(0) = 0$$

which means that the vectors grad $f_i(\mathbf{x}(0))$ and $\mathbf{x}'(0)$ are orthogonal.

Now let us prove that if a vector $\mathbf{v}$ is orthogonal to the vectors grad $f_i(\mathbf{x}_0)$, $1 \leqslant i \leqslant n-k$, then $\mathbf{v}$ is a tangent vector.

Let us take a smooth local parameterization $F:\mathbb{R}^k \longrightarrow M \subset \mathbb{R}^n$ of M around the point $\mathbf{x}_0$. The curve $t \longmapsto F(F^{-1}(\mathbf{x}_0) + t\mathbf{y})$, where $\mathbf{y} \in \mathbb{R}^k$ fixed, is a curve on M passing through $\mathbf{x}_0$. The speed vector of this curve for t = 0 is

$$\frac{\partial F}{\partial x_1}(F^{-1}(\mathbf{x}_0))y_1 + \ldots + \frac{\partial F}{\partial x_k}(F^{-1}(\mathbf{x}_0))y_k,$$

where $y_1,\ldots,y_k$ are the coordinates of $\mathbf{y}$. By the construction of local parameterizations of embedded manifolds, F is a restriction of a diffeomorphism between open subsets of $\mathbb{R}^n$ onto $\mathbb{R}^k$, consequently, the vectors $\frac{\partial F}{\partial x_i}(F^{-1}(\mathbf{x}_0))$ are linearly independent. We conclude, that the tangent space is contained in the k-dimensional linear subspace orthogonal to grad $f_1(\mathbf{x}_0),\ldots,$grad $f_{n-k}(\mathbf{x}_0)$ and contains the k-dimensional linear subspace spanned by the vectors $\frac{\partial F}{\partial x_i}(F^{-1}(\mathbf{x}_0))$ $1 \leqslant i \leqslant k$, which means that both linear subspaces coincide with the tangent space.▮

The definition of tangent vectors can also be given in intrinsic terms, independent of the embedding of M into $\mathbb{R}^n$.

Let us define an equivalence relation on the set

$$\mathcal{C}urve(M,p) = \{\gamma\colon[-\varepsilon,\varepsilon]\longrightarrow M : \gamma(0) = p\},$$

consisting of curves passing through p ∈ M, by calling two curves $\gamma_1,\gamma_2 \in \mathcal{C}urve(M,p)$ equivalent if $(\mathbf{x}\circ\gamma_1)'(0) = (\mathbf{x}\circ\gamma_2)'(0)$ for some chart $\mathbf{x}$ around p. Then this condition is true for any chart (prove this!).

Definition **B**. A tangent vector to a manifold M at the point p ∈ M is an equivalence class of curves belonging to $\mathcal{C}urve(M,p)$. The set of equivalence classes is called the tangent space of M at p and denoted by T_pM.

Given a chart $\mathbf{x}$ around p, we can establish a one-to-one correspondence between the equivalence classes and points of $\mathbb{R}^m$, (m = dim M), assigning to the equivalence class of a curve $\gamma \in \mathcal{C}urve(M,p)$ the vector $(\mathbf{x}\circ\gamma)'(0) \in \mathbb{R}^m$. With the help of this identification, we can introduce a vector space structure on the tangent space, not depending on the choice of the chart.

For embedded manifolds definition **B** agrees with definition **A**. The advantage of definition **B** lies in the fact that it is applicable also for abstract manifolds, not embedded anywhere.

Definition. If $\mathbf{x} = (x_1,\ldots,x_m)$ is a chart on the manifold M around the point p, $\gamma \in \mathcal{C}urve(M,p)$, then the numbers $(x_1\circ\gamma)'(0)$,..., $(x_m\circ\gamma)'(0)$ are called the components of the tangent vector represented by γ with respect to the chart $\mathbf{x}$.

The main difficulty of defining tangent vectors to a manifold is due to the fact that an abstract manifold might not be embedded into a fixed finite dimensional linear space. Nevertheless, there is a universal embedding of each differentiable manifold into an infinite dimensional linear space.

Let us denote by $\mathcal{F}$ (M) the linear vector space of smooth functions on M, and by $\mathcal{F}^*$ (M) the dual space of $\mathcal{F}$ (M) that is the space of linear functions on $\mathcal{F}$ (M), and consider the embedding ι of M into $\mathcal{F}^*$ (M) defined by the formula

$$[\iota(p)](f) = f(p), \text{ where } p \in M, \; f \in \mathcal{F}(M).$$

Having embedded the manifold M into $\mathcal{F}^*$ (M), we can define tangent vectors to M to be elements of the linear space $\mathcal{F}^*$ (M).

<u>Definition</u>. Let M be a differentiable manifold, $p \in M$. We say that a linear function $D \in \mathcal{F}^*$ (M) defined on smooth functions on M is a <u>derivation at the point p</u> if the equality

$$D(fg) = D(f)g(p) + f(p)D(g)$$

holds for every $f,g \in \mathcal{F}$ (M).

Each curve $\gamma \in \mathcal{C}urve(M,p)$ defines a derivation at the point p by the formula $D_{\gamma'(0)}(f) = (f\circ\gamma)'(0)$, where $f \in \mathcal{F}$ (M). $D_{\gamma'(0)}$ is the speed vector of the curve $\iota\circ\gamma$ in $\mathcal{F}^*(M)$. Since two curves define the same derivation iff they are equivalent, there is a one-to-one correspondence between the equivalence classes of curves and the derivations obtained as $D_{\gamma'(0)}$ for some γ.

<u>Definition</u> **C**. A <u>tangent vector</u> to a manifold M at the point $p \in M$ is a derivation of the form $D_{\gamma'(0)}$, where $\gamma \in \mathcal{C}urve(M,p)$.
The <u>tangent space</u> T_pM of M at the point p is the set of derivations $D_{\gamma'(0)}$ along curves in M passing through $p = \gamma(0)$.

<u>Theorem</u>. The tangent space to a differentiable manifold M at the point $p \in M$ coincides with the space of derivations on $\mathcal{F}$ (m) at p, which is a linear space having the same dimension as M has.

<u>Lemma</u> 1. If $f \in \mathcal{F}$ (M) is a constant function and D is a derivation at a point $p \in M$, then $D(f) = 0$.

<u>Proof</u>. Because of linearity, it is enough to show that $D(\mathbf{1})=0$, where $\mathbf{1}$ is the constant 1 function on M. But we have

$$D(\mathbf{1}) = D(\mathbf{1}\ \mathbf{1}) = D(\mathbf{1})\ \mathbf{1}(p) + \mathbf{1}(p)\ D(\mathbf{1}) = 2\ D(\mathbf{1}). \blacksquare$$

Lemma 2. If two functions $f, g \in M$ coincide on a neighborhood U of $p \in M$ and D is a derivation at p then $D(f) = D(g)$.

Proof.

Sublemma. If $\mathbf{x} \in \mathbb{R}^n$ and $B(\mathbf{x}, \varepsilon)$ is a fixed open ball about it, then there exists a smooth function $h: \mathbb{R}^n \longrightarrow \mathbb{R}$ such that $h(\mathbf{y})$ is equal to 1 if $\mathbf{y} \in B(\mathbf{x}, \varepsilon/2)$, positive if $\mathbf{y} \in B(\mathbf{x}, \varepsilon)$ and zero if $\mathbf{y} \notin B(\mathbf{x}, \varepsilon)$.

Define the function h_0 of the real variable t by the formula

$$h_0(t) = \begin{cases} e^{-(1-t^2)^{-1}} & \text{if } t \in (-1,1) \\ 0 & \text{otherwise.} \end{cases}$$

It is a good exercise to prove that h_0 is a smooth function on $\mathbb{R}$. Set $h_1(\mathbf{y}) := h_0(4\|\mathbf{y}\| / \varepsilon)$, let χ denote the characteristic function of the ball $B(\mathbf{x}, 3\varepsilon/4)$, and define the function h_2 as follows

$$h_2(\mathbf{y}) := \int_{\mathbb{R}^n} \chi(\mathbf{z}) h_1(\mathbf{y}-\mathbf{z}) d\mathbf{z}.$$

If we put $h(\mathbf{y}) = h_2(\mathbf{y})/h_2(\mathbf{x})$ then we get a desirable function.

Now let us prove the lemma. Using the construction above we can define a smooth function h on M which is zero outside U and such that $h(p) = 1$. In this case $h(f-g)$ is the constant 0 function on M. Thus we have

$$0 = D(\mathbf{0}) = D(h(f-g)) = D(h)\ (f(p)-g(p)) + h(p)\ D(f-g) = D(f) - D(g). \blacksquare$$

Remarks.

i) The sublemma shows that the mapping $\iota: M \longrightarrow \mathcal{F}^*(M)$ above is indeed an inclusion. If $p \neq q$ are distinct points of M, then there is a smooth function h on M such that

$$[\iota(p)](h) = h(p) = 1 \neq [\iota(q)](h) = h(q) = 0.$$

ii) We can extend a derivation D at a point p on functions f defined only in a neighborhood U of p by taking a smooth function h on M such that h is zero outside U and constant 1 in a neighborhood of p and putting $D(f) := D(\tilde{f})$, where

$$\tilde{f}(x) = \begin{cases} f(x)h(x) & \text{for } x \in U \\ 0 & \text{for } x \notin U. \end{cases}$$

By lemma 2 this extension of D is correctly defined.

Lemma 3. Let $f: B \longrightarrow \mathbb{R}$ be a smooth function defined on an open ball $B \subset \mathbb{R}^n$ around the origin. Then there exist smooth functions g_i $1 \leqslant i \leqslant n$ on B such that

$$f(\mathbf{x}) = f(\mathbf{0}) + \sum_{i=1}^{n} x_i g_i(\mathbf{x}) \qquad \text{for } \mathbf{x} = (x_1, \ldots, x_n) \in B$$

and

$$g_i(\mathbf{0}) = \frac{\partial f}{\partial x_i}(\mathbf{0}).$$

Proof. Since

$$f(\mathbf{x}) - f(\mathbf{0}) = \int_0^1 \frac{d\, f(t\mathbf{x})}{dt} dt = \int_0^1 \sum_{i=1}^n x_i \frac{\partial f}{\partial x_i}(t\mathbf{x})\, dt =$$

$$= \sum_{i=1}^n x_i \int_0^1 \frac{\partial f}{\partial x_i}(t\mathbf{x})\, dt, \quad \text{we may take } g_i(\mathbf{x}) = \int_0^1 \frac{\partial f}{\partial x_i}(t\mathbf{x})\, dt. \blacksquare$$

Now we are ready to prove the theorem.

Let us take a differentiable manifold $(M,\mathcal{A})$ and a chart $\mathbf{x} = (x_1,...,x_n) \in \mathcal{A}$ defined in a neighborhood of $p \in M$.

Define the derivations $\partial_i(p)$ as follows

$$\Big[\partial_i(p)\Big](f) := \frac{\partial\, f\circ\mathbf{x}^{-1}}{\partial x_i}(\mathbf{x}(p)).$$

We prove that the derivations $\partial_i(p)$ form a basis in the space of derivations at p. They are linearly independent since if we have

$$\sum_{i=1}^n \alpha_i \partial_i(p) = 0,$$

then applying this derivation to the j-th coordinate function x_j we get

$$\sum_{i=1}^n \alpha_i \frac{\partial x_j}{\partial x_i}(\mathbf{x}(p)) = \alpha_j = 0.$$

On the other hand, if D is an arbitrary derivation at p, then we have

$$D = \sum_{i=1}^n D(x_i)\partial_i(p).$$

Indeed, let $f \in \mathcal{F}(M)$ be an arbitrary smooth function on M and apply lemma 3 to $f\circ\mathbf{x}^{-1}$ around $\mathbf{x}(p)$. We obtain functions g_i defined around $\mathbf{x}(p)$ such that

$$f = f(p) + \sum_{i=1}^n (x_i - x_i(p)) g_i\circ\mathbf{x} \quad \text{and} \quad g_i(\mathbf{x}(p)) = \frac{\partial\, f\circ\mathbf{x}^{-1}}{\partial x_i}(\mathbf{x}(p)).$$

In this case however we have

$$D(f) = D(f(p)) + \sum_{i=1}^n D((x_i - x_i(p)))\, g_i(\mathbf{x}(p)) + (x_i(p) - x_i(p))\, D(g_i\circ\mathbf{x}) =$$

$$\sum_{i=1}^n D(x_i) \frac{\partial\, f\circ\mathbf{x}^{-1}}{\partial x_i}(\mathbf{x}(p)) = \sum_{i=1}^n D(x_i)\Big[\partial_i(p)\Big](f).$$

To finish the proof, we only have to show that every derivation at the point p can be obtained as a speed vector of a curve passing through p.

Define the curve $\gamma : [-\varepsilon,\varepsilon] \longrightarrow M$ by the formula

$$\gamma(t) := \mathbf{x}^{-1}(\mathbf{x}(p) + (t\alpha_1,...,t\alpha_n)).$$

Then obviously the speed vector $\gamma'(0)$ is just $\sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i}(p)$. $\blacksquare$

The tangent bundle

The union of the tangent spaces of M at the various points, $\bigcup_{p\in M} T_pM$, has a natural differentiable manifold structure, the dimension of which is twice the dimension of M.

This manifold is called the tangent bundle of M and is denoted by TM. A point of this manifold is a vector D, tangent to M at some point p. Local coordinates on TM are constructed as follows. Let $\mathbf{x} = (x_1,\ldots,x_n)$ be a chart on M the domain U of which contains p, and $D(x_1),\ldots,D(x_n)$ be the components of D in the basis $\partial_i(p)$. Then the mapping

$$D\longmapsto (x_1(p),\ldots,x_n(p),D(x_1),\ldots,D(x_n))$$

give a local coordinate system on $\bigcup_{p\in U} T_pM \subset TM$. The set of all local coordinate systems constructed this way forms a $\mathscr{C}^{\infty}$-compatible atlas on TM, that turns TM into a differentiable manifold.

Exercise. Check the last statement.

The mapping $\pi : TM\longrightarrow M$ which takes a tangent vector D to the point $p \in M$ at which the vector is tangent to M is called the natural projection. The inverse image of a point $p \in M$ under the natural projection is the tangent space T_pM. This space is called the fiber of the tangent bundle over the point p.

The derivative of a map

Definition. Let $f:M\longrightarrow N$ be a smooth mapping between the differentiable manifolds $(M,\mathscr{A})$, $(N,\mathscr{B})$, and let $p \in M$. The derivative of f at the point p is the linear map of the tangent spaces $f'_p : T_pM \longrightarrow T_{f(p)}N$, which is given in the following way.

Let $D \in T_pM$ and consider a curve $\gamma : [-\varepsilon,\varepsilon]\longrightarrow M$ with $\gamma(0)= p$, and speed vector D. Then $f'_p(D)$ is the tangent vector represented by the curve $f\circ\gamma$.

Proposition. The derivative f'_p is correctly defined (does not depend on the choice of γ) and is linear.

Proof. We derive a formula for f'_p using local coordinates which will show both parts of the proposition clearly.

Let $\mathbf{x} = (x_1,\ldots,x_m)$ and $\mathbf{y} = (y_1,\ldots,y_n)$ be local coordinates in a neighborhood of $p \in M$ and $f(p) \in N$ respectively.

If the components of D in the basis $\partial_i(p)$ corresponding to the chart $\mathbf{x}$ are $\{ \alpha_i : 1 \leqslant i \leqslant m \}$ then we have $(x_i \circ \gamma)'(0) = \alpha_i$. Observe, that α_i depends only on D. The components $\{ \beta_j : 1 \leqslant j \leqslant n \}$ of $f'_p(D)$ in the basis $\tilde{\partial}_j(f(p))$ generated by the chart $\mathbf{y}$ can be computed by the formula $\beta_j = (y_j \circ f \circ \gamma)'(0)$. Denote by $\tilde{f}_j$ the j-th coordinate function of the mapping $\mathbf{y} \circ f \circ \mathbf{x}^{-1}$, i.e.

$$\tilde{f}_j = y_j \circ f \circ \mathbf{x}^{-1}.$$

Then we have

$$\beta_j = (y_j \circ f \circ \gamma)'(0) = [(y_j \circ f \circ \mathbf{x}^{-1}) \circ \mathbf{x} \circ \gamma]'(0) = [\tilde{f}_j \circ (\mathbf{x} \circ \gamma)]'(0) =$$

$$= \sum_{i=1}^{m} \frac{\partial \tilde{f}_j}{\partial x_i} (\mathbf{x}(p))(x_i \circ \gamma)'(0) = \sum_{i=1}^{m} \frac{\partial \tilde{f}_j}{\partial x_i} (\mathbf{x}(p))\alpha_i,$$

which shows that $f'_p(D)$ depends only on D and that f'_p is a linear mapping the matrix of which in the bases $\left\{\partial_i(p)\right\}$ and $\left\{\tilde{\partial}_j(f(p))\right\}$ is

$$\left(\frac{\partial \tilde{f}_j}{\partial x_i} (\mathbf{x}(p))\right)_{\substack{1 \leqslant i \leqslant m \\ 1 \leqslant j \leqslant n}}. \blacksquare$$