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Theory and Numerical Analysis of Multi-factor Term Structure Models

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Introduction

Tools for modelling the behaviour of interest rates are essential for all participants of financial markets. When aiming to construct a *realistic* model the most important question is what features of bond markets we aim to capture. Some models accurately evaluate the bonds, others provide a good approximation of the prices of interest rate derivatives such as options on bonds, caps and floors, some produces yield curve shapes which are close to the observable ones etc. Many objectives can be worked out to assure the model reflects the real-life features of the market, but unfortunately, none of the models can perform well in respect to all these features. Thus in case of applying an interest rate model our choice depends on the task we would like to accomplish.

Not surprisingly, the more accurate we would like to be, the more complex and the less tractable the model becomes. In case of yield curve dynamics, one has to describe the special joint behaviour of infinite number of points of the curve. Some approaches assume that the problem can be handled by introducing finite number of factors.

In the present study, firstly I discuss a general mathematical framework of bond markets modelled by random factors, namely the use of *Brownian motions*, *Itô calculus* and the *PDE method* for pricing interest rate derivatives. Secondly, focusing on practical issues and historical data, I introduce a three-factor model of the Hungarian government bond market. Finally, I examine some applications of the derived pricing PDEs.

Summarising the main results of the study, firstly, I refer to the derivation of the relation between the pricing PDEs based on two different approaches, and some mathematical consequences of the economic relations between the factors. I also point out that as a result of having access to historical bond data, the three-factor model can be calibrated numerically. Finally, I present a method to approximate the risk-adjusted drift terms of the factors.

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Lajos Gergely Gyurkó

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Chapter 1

Mathematical framework of bond markets

In the present study, I examine the theory of government bond markets. *General bonds* are simple financial products[1]: the issuer of a bond has the obligation to pay pre-specified coupons[2] and principal on pre-specified dates to the owner. A *zero-coupon* bond pays no coupons but principal, and theoretically all bonds can be broken down into series of zero-coupon bonds. Pricing bonds is based on calculating the *present value* of future cash-flows.

In a simple approach, if the future cash-flow is fixed and guaranteed and the time value of money, that is the *interest rates* are deterministic and known, the present value is calculated as the sum of the discounted payments.

Fixed-rate government bonds are assumed to be risk-less in the sense that there is no default risk on the payment of coupons and the repayment of principal[3], thus the first assumption holds.

Indeed the interest rates are not known. Bonds are issued and traded on the market, the demand and supply create the prices which are observable. Discount rates and interest rates can be estimated based upon bond prices. [Anderson 1996] chapter 2 reviews the main approaches and methods of estimation of interest rates based on market prices.

On the other hand interest rates *are* stochastic even if there is no default risk on the repayments because demand and supply appreciate or devalue the price of guaranteed future cash-flows.

In this chapter, the stochastic framework of bond markets is discussed.

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[1] *"A bond is securitized form of a loan."* as in [Cairns 2004].

[2] Usually fix coupons are specified at issue (*fix-rate bonds*), or coupons are linked to a benchmark, index or floating rate such as inter-bank rate (*floating-rate bonds*).

[3] One of my colleagues pointed on the fact that default on Russian government bonds has already been recorded thus the *no default on government bonds* assumption does not hold universally .

1.1 Fundamentals of interest rates

The elementary notions of interest rate models are defined below. These definitions are more-or-less identical in the referred literature:[4], although the notation may differ.

When considering term structures and interest rates, the elemental product is the *bank account.*

Definition 1. Bank account *is a continuously interest paying financial product. If one unit of money invested at time zero into a bank account its value at time t is denoted by $\beta(t)$. The evolution of β is as follows:*

$$\beta(0) = 1, \tag{1.1}$$
$$d\beta(t) = r(t)\beta(t)\,dt, \tag{1.2}$$

thus

$$\beta(t) = \exp\left\{\int_0^t r(s)\,ds\right\}, \tag{1.3}$$

where $r(t)$ is the instantaneous rate of interest*.*

□

The instantaneous rate is assumed to be non-negative. If non-negativity holds, the value of a bank account is non-decreasing. As the instantaneous rate is guaranteed it can be viewed as the minimum expected rate of return on investments, that is the *risk-free rate* of return, and the bank account is *the risk-free asset.* All the other financial products whose value can either increase or decrease at any time are called *risky assets.*

As we will see later, the interest rate derivatives, including bonds, can be valued using the process $r(t)$. Factor models of interest rates mainly differ in the way they describe this process.

In the factor models, usually the instantaneous rate is the factor (or one of the factors) or can be derived from the factors.

Example 1. One-factor models[5]

- *In the* Vasicek *model $r(t)$ is given by the following stochastic differential equation (further on SDE):*
$$dr(t) = \alpha(\mu - r(t))\,dt + \sigma\,dW(t),$$
- *In the* Black, Derman and Toy *model $r(t)$ is given as:*
$$r(t) = e^{X(t)},$$

[4] [Cairns 2004], [Elliott-Kopp 2000], [James-Webber 2004], [Shreve 1998] etc.
[5] See [Cairns 2004], Chapter 4 for details.

where $X(t)$ is the factor:

$$dX = \alpha(\mu - X(t))dt + \sigma dW(t).$$

□

Example 2. Multi-factor models[6]

- *In the* Longstaff and Schwartz *model there are two factors:*

$$dX(t) = \alpha_X(\mu_X - X(t))dt + \sqrt{X(t)}dW_X(t), \tag{1.4}$$
$$dY(t) = \alpha_Y(\mu_Y - Y(t))dt + \sqrt{Y(t)}dW_Y(t), \tag{1.5}$$

  and $r(t) = X(t) + Y(t)$.

- *The model of* Duffie and Kan *also has two factors, but the processes are given differently:*

$$dX(t) = (\mu_{11}X(t) + \mu_{12}Y(t))dt + \sigma_1\sqrt{\alpha + \beta_1 X(t) + \beta_2 Y(t)}dW_X(t), \tag{1.6}$$
$$dY(t) = (\mu_{21}X(t) + \mu_{22}Y(t))dt + \sigma_2\sqrt{\alpha + \beta_1 X(t) + \beta_2 Y(t)}dW_{XY}(t), \tag{1.7}$$

  where

$$dW_{XY}(t) = \rho dW_X(t) + \sqrt{1-\rho^2}dW_Y(t).$$

  In this model $r(t) \stackrel{\circ}{=} X(t)$ and $Y(t)$ denotes the long rate.

□

In the above examples the $\mathrm{d}W(t)$, $\mathrm{d}W_X(t)$ and $\mathrm{d}W_Y(t)$ terms denote the increments of Brownian motions. The accurate mathematical definition of financial markets is given in the next section.

Definition 2. *A* simple zero-coupon bond *(further on bond) is a product, which has one unit of money payoff at maturity. The* price of a bond *with maturity T at time t $(0 \leq t \leq T)$ is denoted by $P(t,T)$. Obviously $P(T,T) = 1$.*

□

Definition 3. *The* discount factor *of maturity T at time t is defined as:*

$$D(t,T) \stackrel{\circ}{=} P(t,T)^{-1}. \tag{1.8}$$

□

[6]See [Cairns 2004] chapter 6.2.3 and [Shreve 1998] chapter 28 for details. Further examples are classified in [James-Webber 2004] chapter 7.6.

Definition 4. *The* spot rate *at time t for maturity T is defined as the (logarithmic) yield to maturity of a $P(t,T)$ bond, and denoted by $R(t,T)$. Given the economic relation between bond prices and yield-to-maturity*

$$P(t,T) = \exp\left\{-(T-t)R(t,T)\right\},$$

we get

$$R(t,T) \mathrel{\hat{=}} -\frac{\log P(t,T)}{T-t}. \tag{1.9}$$

□

Remark 1. *We will see later, that $R(t,t) \mathrel{\hat{=}} \lim_{T\to t^+} R(t,T) = r(t)$ also holds. That is why instantaneous rate is sometimes referred to as* short rate*.*

Definition 5. *The* yield curve *at time t is defined as a $[0,T^*] \to \mathbb{R}^+$ mapping:*

$$Y_t(T) \mathrel{\hat{=}} R(t,T), \tag{1.10}$$

where T^* denotes the time horizon ($T \ll T^*$).

□

Trivially, there is a one-to-one relation between the bond prices and discount factors. As the logarithm function is strictly monotone, equation (1.9) also indicates a one-to-one relation between $R(t,T)$ and $P(t,T)$. Although in reality there is finite number of bonds traded on the market, the yield curve is defined on the whole $[0,T^*]$ interval[7]. Because of economic argumentation, the yield curve is assumed to be continuous and *smooth*[8]. Hereafter, let assume that at time t the $T \mapsto P(t,T)$ curve is also continuous and smooth[9], that is theoretically there exists a *bond price curve* as well.

The next definition requires the introduction of *forward rate agreements* (hereinafter FRA). Let $0 \leq t \leq T \leq T+\varepsilon \leq T^*$ be given. Under a FRA contract at time t one participant agrees to borrow one unit of money at time T with repayment (principle and interest) at time $T+\varepsilon$. Assuming that there is no arbitrage[10] opportunity on the market, the interest to be payed under that agreement can be given without the

[7] Indeed, the Hungarian zero-coupon government bonds are issued with maturity less then one year. Bonds maturing over one year are coupon-paying bonds. These coupon-paying bonds can be broken down into series of theoretical zero-coupon bonds. Although the individual prices of these theoretical *zeros* are not known, the market values of coupon-paying bonds provide us enough information for fitting the theoretical continuous yield curve. See [Anderson 1996] for methods.

[8] Smoothness means that the discount curve indicated by the yield curve is monotonically decreasing without oscillations. See [Anderson 1996] chapter 1 for the economic reasoning.

[9] As that is needed for the (1.12) formula.

[10] At this level, arbitrage defined as a portfolio valued zero at time t and having guaranteed positive payoff at time T. The definition is given in the next subsection.

specification of a precise mathematical framework. Suppose that at time t one buys one unit of bond maturing at time T and short sell $\frac{P(t,T)}{P(t,T+\varepsilon)}$ units of a bond maturing at time $T+\varepsilon$. Since

$$P(t,T) - \frac{P(t,T)}{P(t,T+\varepsilon)} P(t,T+\varepsilon) = 0$$

at time t the value of the portfolio is zero. As time goes by, the portfolio has a double payoff, firstly at time T:

$$P(T,T) = 1,$$

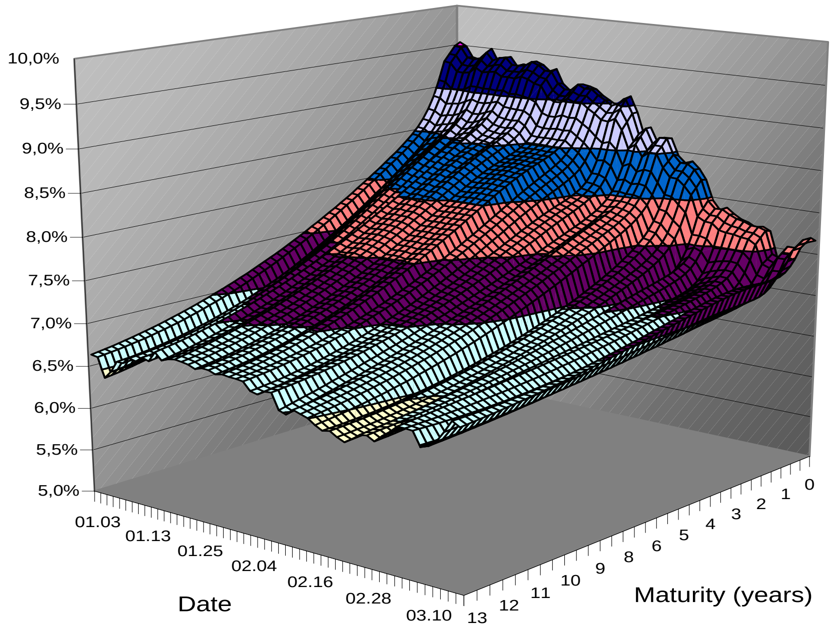

Figure 1.1: Estimated yield curves of the Hungarian bond market (1st quarter, 2005)

and secondly at time $T+\varepsilon$:

$$-\frac{P(t,T)}{P(t,T+\varepsilon)}P(T+\varepsilon,T+\varepsilon) = -\frac{P(t,T)}{P(t,T+\varepsilon)}.$$

The *effective interest rate* of the contract on the time interval $[T, T+\varepsilon]$ is derived from the following formula:

$$\frac{P(t,T)}{P(t,T+\varepsilon)} = \exp\left\{\varepsilon F(t,T,T+\varepsilon)\right\},$$

hence:

Definition 6. *i) The* forward rate *at time t for the time interval* $[T, T+\varepsilon]$ *is defined as:*

$$F(t,T,T+\varepsilon) \overset{\circ}{=} -\frac{\log(P(t,T+\varepsilon)) - \log(P(t,T))}{\varepsilon}. \tag{1.11}$$

ii) The instantaneous forward rate *at time t for the maturity T is defined as*

$$f(t,T) \overset{\circ}{=} \lim_{\varepsilon\to+0} F(t,T,T+\varepsilon) = -\frac{\partial \log P(t,T)}{\partial T}. \tag{1.12}$$

□

The above portfolio determines the hedge strategy for the contract. If the interest rate of the FRA differs from $F(t,T,T+\varepsilon)$, simple consideration shows that arbitrage opportunity exists. Thus the agreed rate of an FRA is given by the corresponding forward rate.

Equation (1.12) indicates that given the instantaneous forward curve the bond prices can be determined and vice versa:

$$P(t,T) = \exp\left\{-\int_t^T f(t,u)\mathrm{d}u\right\}. \tag{1.13}$$

Remark 2. *As we will see later* $r(t) = f(t,t)$ *also holds.*

□

The consequence of the relationship between the bond price curve and the instantaneous forward curve indicates that by modelling the dynamics of the latter one the dynamics of the former one are described as well. The approach of *Heath, Jarrow and Morton*[11] models the processes $f(t,T)$ for each $T \in [t, T^*]$. The drawback of this approach for my purpose is that complex versions lead to such non-Markovian cases when

[11] For example, see [Cairns 2004] chapter 5.3, [Shreve 1998] chapter 34.1, [Elliott-Kopp 2000] chapter 9.7 etc. for details.

the *PDE* method cannot be applied. That is the reason why I insist on multi-factor Markovian models in this study.

Finally, the formal definition of *term-structure models*[12] is given as follows.

Definition 7. *Any mathematical model which determines any of the following stochastic processes:*

i) $P(t,T)$, *or*

ii) $D(t,T)$, *or*

iii) $R(t,T)$

for all $t \in [0,T]$ *and all* $T \in [0,T^*]$, *is called* term-structure model.

□

1.2 Mathematical markets

In the present study, I assume that the price processes of financial products (such as bonds, shares and derivatives) are deduced from random factors. The mathematical formulation of this approach is analogous to the definition of *markets* in [Øksendal 2003] chapter 12:

Definition 8. *Suppose there exists a probability space given by the triplet* $(\Omega, \mathcal{F}^{(m)}, \mathbf{P})$ *with a filtration* $\left(\mathcal{F}_t^{(m)}\right)_{t\geq 0}$ *and the* σ*-algebra* $\mathcal{F}_t^{(m)}$ *is generated by* $\{W^{(m)}(s); s \leq t\}$ *for all* $t \geq 0$ *where* $W^{(m)}(s)$ *denotes an* m*-dimensional* $\mathbf{P}$*-Brownian motion.*

The random factors of the mathematical market *are (or the* market *is) given by an* $\mathcal{F}_t^{(m)}$*-adapted* n*-dimensional Itô process* $X(t) = (X_1(t), \ldots, X_n(t))^T$; $0 \leq t \leq T^*$, *where the following SDEs hold:*

$$\begin{aligned} dX_i(t) &= M_i(t,\omega)\,dt + \sum_{j=1}^{m} \Sigma_{ij}(t,\omega)\,dW_j(t) \qquad (1.14) \\ &= M_i(t,\omega)\,dt + \Sigma_i(t,\omega)\,dW^{(m)}(t);\ i = 1,2,\ldots,n. \end{aligned}$$

□

In the above reference, the first factor is the bank account:

$$X_1(t) = \beta(t)$$

[12] Analogously as in [Shreve 1998].

and the other factors are themselves risky assets, which are traded on the market. I present the fundamental results of derivative pricing in that special case based on [Medvegyev 2004] chapter 4.3.1 and [Øksendal 2003] chapter 12. The following definitions are essential.

Definition 9. *i) A* portfolio *of the assets (or* investment strategy*) on the time interval* $[0,T]$ *is an* n*-dimensional* (t,ω)*-measurable and* $\mathcal{F}_t^{(m)}$*-adapted stochastic process:*

$$\theta(t,\omega) = (\theta_1(t,\omega),\ldots,\theta_n(t,\omega))^T;\ 0 \le\ t \le T. \tag{1.15}$$

ii) The value *at time* t *of a portfolio* $\theta(t)$ *is defined by*

$$V^\theta(t,\omega) \mathrel{\hat{=}} \theta(t)^T X(t) = \sum_{i=1}^{n} \theta_i(t) X_i(t). \tag{1.16}$$

iii) The portfolio $\theta(t)$ *is called* self-financing *if the inequality*

$$\int_0^T \left\{ \left| \theta_1(s) r(s) X_1(s) + \sum_{i=2}^{n} \theta_i(s) M_i(s) \right| + \sum_{j=1}^{m} \left[\sum_{i=2}^{n} \theta_i(s) \Sigma_{ij}(s) \right]^2 \right\} ds < \infty \tag{1.17}$$

a.s. holds and

$$V(t) - V(0) = \sum_{i=1}^{n} \int_0^t \theta_i(s) dX_i(s), \tag{1.18}$$

i.e.

$$dV(t) = \sum_{i=1}^{n} \theta_i(t) dX_i(t). \tag{1.19}$$

iv) A self-financing portfolio is called admissible *if the corresponding value process* $V^\theta(t)$ *is* (t,ω) *almost surely (a.s) lower bounded, i.e. there exists* $K = K(\theta) < \infty$ *such that:*

$$V^\theta(t,\omega) \ge -K \tag{1.20}$$

for almost all $(t,\omega) \in [0,T] \times \Omega$. □

The condition (1.20) is necessary for the sake of reality of finance: that gives a limit for the level of debt the creditors can tolerate.

Definition 10. *An admissible investment strategy* $\theta(t)$ *is* arbitrage *on the time interval* $[0,T]$ *if the following conditions hold:*

$$\begin{aligned} V^\theta(0) &= 0, \\ V^\theta(T) &\ge 0 \end{aligned}$$

a.s. and

$$\mathbf{P}\left[V^{\theta}(T) > 0\right] > 0.$$

□

Because of content limits, the scope of the study is restricted to *European type derivatives.*

Definition 11. *i) A* European T-claim *(or* claim*) is given by a (lower bounded) $\mathcal{F}_T^{(m)}$-measurable random variable*[13] *$F(\omega) \in L^2(\mathbf{Q})$. That is the derivative*[14] *has a payoff F at time T.*

ii) The value of a claim *at time t is denoted by $c_F(t)$.*

iii) The claim is called attainable *if there exist an admissible portfolio $\theta(t)$ and a real number z such as:*

$$F(\omega) = V_z^{\theta}(T) = z + \int_0^T \theta(s) dX(t) \tag{1.21}$$

a.s. holds.

iv) The strategy $\theta(t)$ is called replicating *or* hedging *strategy for F.*

v) The market is complete *if every T-claim is attainable.*

□

The main idea behind derivative pricing is *hedging*[15]. If there exists a unique self-financing strategy with a.s. the same payoff as the given T-claim, then the price of the claim at time t ($0 \leq t \leq T$) must be equal to the value of the hedging portfolio otherwise an arbitrage portfolio could be created. The question is *how to find the strategy.*

In this study the bank account is used as the *numeraire*, that is we express the discounted prices of assets in units of bank account. The discounted price of the asset X_i is denoted by

$$\overline{X}_i(t) = \frac{X_i(t)}{\beta(t)},$$

[13]The measure $\mathbf{Q}$ in the expression $F \in L^2(\mathbf{Q})$ will be introduced later on.

[14]Basically, in this study only claims with F payoff are considered, where F is someway the function of the factors. In that sense claims are derivatives.

[15]Practically, a hedging strategy is realizable if the following assumptions hold. On one hand, markets are frictionless, that is the assets and derivatives can be freely bought and sold, without restrictions, there are no tax consequences associated with trading, and there are no transaction cost. On the other hand, trading is continuous with no gaps in the price change of the assets. See [Natenberg 1994] for details.

thus the discounted value of a portfolio equals:

$$\overline{V}^{\theta}(t) = \frac{V^{\theta}(t)}{\beta(t)} = \sum_{i=1}^{n} \theta_i(t)\overline{X}_i(t). \tag{1.22}$$

Definition 12. *Let the* $(\Omega, \mathcal{F}^{(m)}, \mathbf{P})$ *probability space and the* $\left(\mathcal{F}_t^{(m)}\right)_t$ *filtration be given. The probability measure* $\mathbf{Q}$ *on* $\left(\Omega, \mathcal{F}_T^{(m)}\right)$ *equivalent to the measure* $\mathbf{P}$ *is called* equivalent martingale measure *(up to time* T*) if the discounted value processes of assets are local martingales.*

□

The equivalent martingale measure has two important features. The first one is indicated by the following proposition[16].

Proposition 1. *Suppose that an equivalent martingale measure exists. Then there is no arbitrage on the market.*

See [Øksendal 2003] *Lemma* 12.1.6 for proof.

□

Further on, we assume that there exists an equivalent martingale measure.

Proposition 2. *Suppose a process* $u(t,\omega)$ *satisfies the following conditions*

i) $(t,\omega) \to u(t,\omega)$ *is* $\mathcal{B} \times \mathcal{F}^{(m)}$*-measurable, where* $\mathcal{B}$ *denotes the Borel* σ*-algebra on* $[0,\infty)$,

ii) $u(t,\omega)$ *is* $\mathcal{F}_t^{(m)}$*-adapted, and*[17]

iii) $\mathbf{E_P}\left[\exp\left\{\frac{1}{2}\int_0^T u^2(s,\omega)\,ds\right\}\right] < \infty.$

Define the measure $\mathbf{Q} = \mathbf{Q}_u$ *on* $\mathcal{F}_T^{(m)}$ *by*

$$d\mathbf{Q}(\omega) \mathrel{\hat{=}} \exp\left\{-\int_0^T u(s,\omega)\,dW(s) - \frac{1}{2}\int_0^T u(s,\omega)^2\,ds\right\} d\mathbf{P}(\omega). \tag{1.23}$$

Consequently

$$\widetilde{W}(t) \mathrel{\hat{=}} \int_0^t u(s,\omega)\,ds + W(t) \tag{1.24}$$

[16] The second one is discussed later in this subsection

[17] The last condition is referred to as the *Novikov* condition

is an $\mathcal{F}_t^{(m)}$-Brownian motion with respect to $\mathbf{Q}$ and any random variable $G \in L^2\left(\mathcal{F}_T^{(m)}, \mathbf{Q}\right)$ has a unique representation

$$G(\omega) = \mathbf{E_Q}[G] + \int_0^T \phi(s,\omega) d\widetilde{W}(s), \tag{1.25}$$

where $\phi(t,\omega)$ is an $\mathcal{F}_t^{(m)}$-adapted, (t,ω)-measurable $\mathbb{R}^m$-valued process such that

$$\mathbf{E_Q}\left[\int_0^T \phi(s,\omega)^2 ds\right] < \infty \tag{1.26}$$

holds.

See [Øksendal 2003] chapter 12 for proof.

□

Proposition 3. *Suppose there exist an m-dimensional process $u(t,\omega)$ as in the previous proposition and for the $X(t,\omega) = (X_2(t), \ldots, X_n(t))$ factors*

$$\Sigma(t,\omega)u(t,\omega) = M(t,\omega) - r(t,\omega)X(t,\omega) \tag{1.27}$$

a.a. (t,ω) holds. Define the measure $\mathbf{Q}_u$ and the process $\widetilde{W}(t)$ as above.

Then in terms of $\widetilde{W}(t)$, the discounted assets have the following representation:

$$d\overline{X}_1(t) = 0 \tag{1.28}$$

$$d\overline{X}_i(t) = \beta(t)^{-1}\Sigma_i(t)d\widetilde{W}(t);\ 2 \leq i \leq n. \tag{1.29}$$

The discounted value process of the portfolio θ^* is local $\mathbf{Q}$-martingale given by

$$d\overline{V}^{\theta^*}(t) = \beta(t)^{-1}\sum_{i=2}^{n}\theta_i(t)\Sigma_i(t)d\widetilde{W}(t). \tag{1.30}$$

Here only the strategy of risky assets is defined. The term θ^* refers to a strategy: $\theta^* = (\theta_2, \ldots, \theta_n)^T$. See [Øksendal 2003] chapter 12 for proof.

□

Remark 3. *As a consequence, the discounted value process*

$$\overline{V}^{\theta}(t) = \overline{V}^{\theta}(0) + \int_0^t \theta^*(s)d\overline{X}^*(s) \tag{1.31}$$

can be rewritten as

$$\overline{V}^{\theta^*}(t) = z + \int_0^t \beta(s)^{-1}\sum_{i=2}^{n}\theta_i(s)\Sigma_i(s)d\widetilde{W}(s). \tag{1.32}$$

with $z = \overline{V}^{\theta^*}(0)$.

□

Given a T-claim with a payoff F, the proposition 3 indicates the process $u(t,\omega)$ and a strategy can be derived from the process $\phi(t,\omega)$ of the proposition 2 on $G \mathrel{\mathring{=}} \overline{F}$ combined with the representation of the remark 3.

In this representation the coefficient of the numeraire is arbitrary. That indicates the following proposition.

Proposition 4. *Suppose the $X(t)$ and $\overline{V}^{\theta^*}(t)$ processes are a.s. continuous. Then the strategy* (1.32) *on X^* can be completed with a strategy on the bank account β, such as the complete strategy on $X = \{\beta, X^*\}$ is* self-financing.

Proof[18] *of Proposition 4:*

For the sake of simplicity, let us use the following notation: $N(t) \mathrel{\mathring{=}} \overline{V}^{\theta^*}(t)$. It is clear form the representation (1.32) that:

$$N(T) = \overline{F},$$

and

$$N(t) = \mathbf{E}_{\mathbf{Q}}\left[\overline{F}\middle|\,\mathcal{F}_t\right] = N(0) + \int_0^t \sum_{i=2}^n \theta_i \mathrm{d}\overline{X}_i = N(0) + \int_0^t \theta^* \mathrm{d}\overline{X}^*, \tag{1.33}$$

where $\overline{X}^* \mathrel{\mathring{=}} (\overline{X}_2, \ldots, \overline{X}_n)$. The process X^* is defined analogously.

Let θ_1 be defined as follows

$$\theta_1(t) \mathrel{\mathring{=}} \frac{N(t)\beta(t) - \theta^*(t)^T X^*}{\beta(t)}. \tag{1.34}$$

Thus the value of the portfolio at time t is given by:

$$V(t) = \beta(t)N(t) = \theta_1(t)\beta(t) + \theta^*(t)^T X^*(t). \tag{1.35}$$

The left hand side of the formula (1.18) can be rewritten as

$$\begin{aligned} \sum_{i=1}^n \int_0^t \theta_i(s)\mathrm{d}X_i(s) &= \qquad\qquad (1.36) \\ &= \int_0^t \theta_1(s)\mathrm{d}\beta(s) + \int_0^t \theta^*(s)\mathrm{d}X^*(s) \\ &= \int_0^t \frac{N(s)\beta(s) - \theta^*(s)^T X^*(s)}{\beta(s)}\mathrm{d}\beta(s) + \int_0^t \theta^*(s)\mathrm{d}X^*(s) \\ &= \int_0^t N(s)\mathrm{d}\beta(s) - \int_0^t \theta^*(s)\mathrm{d}Y(s) + \int_0^t \theta^*(s)\mathrm{d}X^*(s), \end{aligned}$$

[18] The proof of [Medvegyev 2004] for a market with one risky asset is generalised below.

where $Y(t) \mathrel{\hat{=}} \int_0^t \overline{X}^*(s)\mathrm{d}\beta(s)$.

Since $X^*(t) = \overline{X}^*(t)\beta(t)$, we have

$$\mathrm{d}X^*(t) = \overline{X}^*\mathrm{d}\beta(t) + \beta(t)\mathrm{d}\overline{X}^* + \mathrm{d}\langle\beta(t), \overline{X}^*(t)\rangle. \tag{1.37}$$

Here $\mathrm{d}\langle\beta(t), \overline{X}^*(t)\rangle = 0$ is indicated by the equation (1.2) of $\beta(t)$.

Applying the result of (1.37) on the $\int_0^t \theta^*(s)\mathrm{d}X^*(S)$ term of (1.36), we get

$$\begin{aligned}\int_0^t \theta^*(s)\mathrm{d}X^*(S) &= \qquad\qquad (1.38)\\ &= \int_0^t \theta^*(s)\mathrm{d}Y(s) + \int_0^t \beta(s)\theta^*(s)\mathrm{d}\overline{X}^*\\ &= \int_0^t \theta^*(s)\mathrm{d}Y(s) + \int_0^t \beta(s)\mathrm{d}Z(s),\end{aligned}$$

where $Z(t) \mathrel{\hat{=}} \int_0^t \theta^*(s)\mathrm{d}\overline{X}^*(s) = N(t) - N(0)$, that is $\mathrm{d}Z(s) = \mathrm{d}N(s)$. Thus

$$\int_0^t \theta^*(s)\mathrm{d}X^*(S) = \int_0^t \theta^*(s)\mathrm{d}Y(s) + \int_0 \beta(s)\mathrm{d}N(s). \tag{1.39}$$

Analogously to (1.37)

$$\mathrm{d}V(t) = \overline{V}\mathrm{d}\beta(t) + \beta(t)\mathrm{d}\overline{V}. \tag{1.40}$$

Finally, (1.36), (1.39) and (1.40) indicate:

$$\begin{aligned}\sum_{i=1}^n \int_0^t \theta_i(s)\mathrm{d}X_i(s) &= \qquad\qquad (1.41)\\ &= \int_0^t \beta(s)\mathrm{d}N(s) + \int_0^t N(s)\mathrm{d}\beta(s)\\ &= \beta(t)N(t) - \beta(0)N(0) = V(t) - V(0),\end{aligned}$$

that is the strategy is self-financing.

□

Since the self-financing feature of the strategy is ensured by the previous proposition, equation 1.33 determines the discounted value process of the claim by a conditional expected value formula. Thus the real price[19] of the claim is given as follows.

Corollary 1. (The fundamental theorem of pricing T-claims)

[19]That is price without discounting.

Let a T-claim with a payoff F at time T be given. The price of the T-claim at time t equals to the following formula:

$$c_F(t) = \beta(t)N(t) = \beta(t)\mathbf{E_Q}\left[\left.\frac{F}{\beta(T)}\right|\mathcal{F}_t\right]. \tag{1.42}$$

□

For the pricing formula, knowing the particular hedging strategy is not necessary but the existence of the strategy is. The expected value in the price is calculated with respect to $\mathbf{Q}$ not with respect to the real world measure $\mathbf{P}$. This is the *other* important feature of the equivalent martingale measure.

Corollary 2. *i) As $\beta(t)$ is defined as in Definition 1, the value of a derivative at time t is given by*

$$c_F(t) = \mathbf{E_Q}\left[\left.F\exp\left\{-\int_t^T r(s)\,ds\right\}\right|\mathcal{F}_t\right]. \tag{1.43}$$

ii) In particular, a bond maturing at time T has a payoff 1, then its' value at time t is as follows:

$$P(t,T) = c_1(t) = \mathbf{E_Q}\left[\left.\exp\left\{-\int_t^T r(s)\,ds\right\}\right|\mathcal{F}_t\right]. \tag{1.44}$$

□

Remark 4. *Although the above pricing argumentation is based on the case where the factors are traded assets, the result* (1.43) *may hold when the factors themselves are not traded. The necessary condition is that enough derivatives of the factors should be traded thus a portfolio hedging the randomness in the claim could be created. The exact proof will be derived from the* PDE *method.*

□

A term-structure model is well *tractable* if formulae like (1.43) can be given by analytical forms[20] or can be estimated by fast and easy methods. Unfortunately, as we make our model more complex for the sake of fitting the reality we loose the tractability even for bond prices given by the formula (1.44). In the next sections such model will be introduced.

I close the section with the proof of the remarks 2 and 1.

Remark 5. *The propositions of remark 2 and 1 are derived from the formula 1.44:*

[20] See the *Vasicek*, *CIR* etc. models in [Cairns 2004], [Elliott-Kopp 2000], [Shreve 1998], or [James-Webber 2004]

On one hand, based on 1.44 we have

$$\frac{\partial P(t,T)}{\partial T} = \mathbf{E}_{\mathbf{Q}}\left[-r(T)\exp\left\{ -\int_t^T r(s)\mathrm{d}s \right\} \middle| \mathcal{F}_t \right],$$

thus the following holds

$$\left.\frac{\partial P(t,T)}{\partial T}\right|_{T=t} = \mathbf{E}_{\mathbf{Q}}\left[-r(t) \middle| \mathcal{F}_t \right] = -r(t). \tag{1.45}$$

On the other hand, the formula (1.13) indicates

$$\frac{\partial P(t,T)}{\partial T} = -f(t,T)\exp\left\{ -\int_t^T f(t,s)\mathrm{d}s \right\},$$

thus

$$\left.\frac{\partial P(t,T)}{\partial T}\right|_{T=t} = -f(t,t). \tag{1.46}$$

The equations (1.45) and (1.46) result in $r(t) = f(t,t)$.

According to the formula (1.9), we have:

$$\begin{aligned} \lim_{T\to t^+} R(t,T) &= \qquad (1.47) \\ &= \lim_{T\to t^+} -\frac{\log P(t,T)}{T-t} \\ &= -\left.\frac{\partial \log P(t,T)}{\partial T}\right|_{T=t} \\ &= -\left.\frac{1}{P(t,T)}\right|_{T=t} \left.\frac{\partial P(t,T)}{\partial T}\right|_{T=t}. \end{aligned}$$

Thus, $R(t,t) = r(t)$ is indicated by (1.45).

□

Chapter 2

PDE approach

In this chapter, the pricing method, based on partial differential equations derived from hedging argument and referred as PDE-approach is presented.

The theorem called *Itô-lemma* is necessary for the argumentation.

2.1 Itô-lemma

Before stating the theorem, some notions need to be defined[1].

Definition 13. *Let space* $(\Omega, \mathcal{F})$ *be given with the filtration* $(\mathcal{F}_t)_{t\geq 0}$*. The mapping* $\tau : \Omega \to [0, \infty)$ *is called* stopping time *if*

$$\{\tau < t\} \in \mathcal{F}_t \tag{2.1}$$

holds for all $t \in [0, \infty)$*.*

□

Definition 14. *Let* $(\Omega, \mathcal{F}, \mathbf{P})$ *probability space be given with the filtration* $(\mathcal{F}_t)_{t\geq 0}$*. The process* X *is* local martingale *if there exists a series of stopping times* $(\tau_n)_n$*, where* $\tau_n \to \infty$ *monotonically a.s. holds, and the stopped process* X^{τ_n} *is uniformly integrable martingale for all* n*.*

□

Definition 15. *Let* $(\Omega, \mathcal{F}, \mathbf{P})$ *probability space be given with the filtration* $(\mathcal{F}_t)_{t\geq 0}$*. The adapted process* S *is called* semi-martingale *if there exist a representation*

$$S = L + V, \tag{2.2}$$

where L *is local martingale and* V *is an* $(\mathcal{F}_t)$*-adapted process with finite variance.*

□

[1] As in [Medvegyev 2004].

Theorem 1. (Itô-lemma)
Let the vector $X = (X_1, \ldots, X_n)$ *with semi-martingale components and the function* $F \in C^2(\mathbb{R}^n)$ *be given. Then*

$$\begin{aligned} F(t) - F(0) &= \\ &= \sum_{k=1}^{n} \int_0^t \frac{\partial F}{\partial X_k}(X)\, dX_k + \frac{1}{2} \sum_{i,j} \int_0^t \frac{\partial^2 F}{\partial X_i \partial X_j}(X)\, d\langle X_i, X_j \rangle \end{aligned} \tag{2.3}$$

holds.

See [Medvegyev 2004] chapter 3.1 for proof.

□

2.2 Deriving the PDE - first version

The main idea behind the PDE-approach is applied on a one factor share-price model in [Hull 1997] and on two factor market models in [Willmott 1998]. The slight generalisation of the idea is presented below.

Suppose the market is given by n factors $X = (X_1, \ldots, X_n)^T$:

$$\mathrm{d}X_i = M_i(X(t))\mathrm{d}t + \Sigma_i(X(t))\mathrm{d}W_i(t), \ i = 1, \ldots n. \tag{2.4}$$

where the correlation between the terms $\mathrm{d}W_i(t)$ and $\mathrm{d}W_j(t)$ is denoted by $\rho_{i,j}$, for $i, j = 1, \ldots n$. In this study, the correlation coefficients assumed to be constant. Let the first factor $X_1 = r$ be the instantaneous rate of return as in the previous chapter.

Remark 6. *An equivalent form*[2] *of definition* (2.4) *can be given as below*

$$dX = M\,dt + \Sigma dW(t) \tag{2.5}$$

where $M = (M_1, \ldots, M_n)^T$,

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 & \cdots & 0 \\ 0 & \Sigma_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & \Sigma_n \end{bmatrix} \begin{bmatrix} 1 & \rho_{1,2} & \cdots & \rho_{1,n} \\ \rho_{2,1} & 1 & \cdots & \rho_{2,n} \\ \vdots & & \ddots & \vdots \\ \rho_{n,1} & \cdots & \rho_{n,n-1} & 1 \end{bmatrix}^{\frac{1}{2}} \tag{2.6}$$

and $W = (W_1, \ldots, W_n)^T$ *denotes an* n *dimensional Brownian motion*[3].

□

[2] The mathematical markets were defined in the first chapter by this representation.
[3] The increments of the components are independent by definition

Suppose there is a T-claim C_0 to be valued. Suppose there are n derivative products traded on the market with prices depending on the factors[4]. Let these prices be denoted by $C_1, \dots, C_n$. The idea is to construct a portfolio V at time t, which includes one contract of C_0 and some of the other derivatives.

$$V(t) = C_0(t) - \sum_{i=1}^{n} \Delta_i(t) C_i(t) \tag{2.7}$$

where the Δ_i weights of the elements are chosen at every moment to eliminate the *risk*. As it was underlined earlier, the bank account is risk-less in the sense, that its' value over time is non-decreasing. Any product with a non-decreasing value must produce the same return as the bank account, otherwise arbitrage opportunity would exist on the market. The increment of the portfolio is derived by the Itô-lemma:

$$\mathrm{d}C_i = \frac{\partial C_i}{\partial t} + \sum_{j=1}^{n} \frac{\partial C_i}{\partial X_j} \mathrm{d}X_j + \frac{1}{2} \sum_{j,k} \frac{\partial^2 C_i}{\partial X_j \partial X_k} \mathrm{d}\langle X_j, X_k \rangle \tag{2.8}$$

for $i = 0, 1, \dots, n$. The $\mathrm{d}\langle . \rangle$ terms can be derived from (2.4):

$$\mathrm{d}\langle X_j, X_k \rangle = \rho_{j,k} \Sigma_j \Sigma_k \mathrm{d}t. \tag{2.9}$$

where $\rho_{j,j} = 1$. Therefore

$$\begin{aligned} \mathrm{d}V &= \mathrm{d}C_0 - \sum_{i=1}^{n} \Delta_i \mathrm{d}C_i & (2.10) \\ &= \left(\Lambda_0 - \sum_{i=1}^{n} \Delta_i \Lambda_i \right) \mathrm{d}t + \sum_{i=1}^{n} \Psi_i \mathrm{d}X_i & (2.11) \end{aligned}$$

where

$$\Lambda_i = \frac{\partial C_i}{\partial t} + \frac{1}{2} \sum_{j,k} \rho_{j,k} \Sigma_j \Sigma_k \frac{\partial^2 C_i}{\partial X_j \partial X_k}, \tag{2.12}$$

and

$$\Psi_i = \frac{\partial C_0}{\partial X_i} - \sum_{j=1}^{n} \Delta_j \frac{\partial C_j}{\partial X_i}. \tag{2.13}$$

The risk is eliminated, if the increment of the portfolio includes only the $\mathrm{d}t$ term, thus $\Psi_i = 0$ for all $i = 1; \dots, n$.

If the coefficients $\Delta_1, \dots, \Delta_n$ can be found then the portfolio produces the risk-free rate of return

$$\mathrm{d}V(t) = r(t) V(t) \mathrm{d}t. \tag{2.14}$$

[4]The necessary characteristics of the dependence is described later.

The equations (2.10) and (2.14) indicate

$$r\left(P_0 - \sum_{i=1}^{n} \Delta_i P_i\right) = rV = \Lambda_0 - \sum_{i=1}^{n} \Delta_i \Lambda_i. \tag{2.15}$$

From the formulas (2.13) and (2.15) there are $n+1$ equations given on n variables, thus the system is over-prescribed. The solution exists if and only if the $n+1 \times n+1$ matrix A is singular for all t, where

$$A = \begin{pmatrix} \Lambda_0 - rC_0 & \cdots & \Lambda_n - rP \\ \partial C_0/\partial X_1 & \cdots & \partial C_n/\partial X_1 \\ \partial C_0/\partial X_2 & \cdots & \partial C_n/\partial X_2 \\ \vdots & \ddots & \vdots \\ \partial C_0/\partial X_n & \cdots & \partial C_n/\partial X_n \end{pmatrix}. \tag{2.16}$$

The dependence of the hedging products on the factors is required to ensure that the first row is linear combination of the second to nth rows. In that case the above matrix is singular, and there exist n adapted functions $\hat{\lambda}_1, \ldots, \hat{\lambda}_n$ such that

$$\Lambda_i - rC_i = -\sum_{j=1}^{n} \hat{\lambda}_j \frac{\partial C_i}{\partial X_j} \tag{2.17}$$

holds for $i = 1, \ldots, n$. Since the equation (2.17) holds for $i = 0$ and the hedging elements of the portfolio were not specified but C_0, the functions $\hat{\lambda}_j$ do not depend on the choice of the elements. These coefficients are referred to as the *risk adjusted drift rates*, and thus often written in the following form

$$\hat{\lambda}_j = \lambda_j \Sigma_j - M_j \; j = 1, \ldots, n. \tag{2.18}$$

The functions λ_j are called the *market price of risk*.

Proposition 5. *If the factors are time-homogeneous, that is M and Σ do not depend on t but X, the $\hat{\lambda}_j$ can be chosen to be functions of X only.*

Proof. The same argumentation as above can be derived at two different points of time t_1 and t_2 firstly for the bond $P_1(t_1, t_1+t)$ and secondly for the bond $P_2(t_2, t_2+t)$. Assuming that $X(t_1) = X(t_2)$, since P_1 relatively is the same product at time t_1 as the bond P_2 at time t_2, their values and derivatives must be equal, and therefore $\hat{\lambda}_j(t_2, X) \circeq \hat{\lambda}_j(t_1, X)$ is a right choice for $j = 1, \ldots, n$.

□

Example 3. *In a classic one factor share-model*[5]*, the instantaneous rate is constant, and the factor S is given by a geometric Brownian motion*

$$dS = \mu S\,dt + \sigma S\,dW \tag{2.19}$$

with constant coefficients. Suppose one is about to price a European call option C *with maturity T and exercise price K. The hedge portfolio is $V = C - \Delta S$. From Itô-lemma*

$$dC = \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial S}dS + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}dt. \tag{2.20}$$

Thus the increment of V is

$$dV = \left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}\right)dt + \left(\frac{\partial C}{\partial S} - \Delta\right)dS. \tag{2.21}$$

Therefore the choice $\Delta \doteq \partial C/\partial S$ eliminates the dS term, and from $dV = rV\,dt$ we get

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = r(C - \Delta S) = r\left(C - S\frac{\partial C}{\partial S}\right), \tag{2.22}$$

thus for the price of the derivative the equation

$$0 = -rC + \frac{\partial C}{\partial t} + \hat{\lambda}\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \tag{2.23}$$

holds, with

$$\hat{\lambda} = rS = \left(\mu - \frac{\mu - r}{\sigma}\sigma\right)S = (\mu - \lambda\sigma)S. \tag{2.24}$$

In this case, r is called the risk adjusted rate of return[6]*, and $(\mu - r)/\sigma$ is referred to as the market price of risk.*

□

We can summarise the results as follows:

Proposition 6. *Expanding the equation* (2.17) *for $i = 0$, we get the pricing PDE for the T-claim with a payoff F*

$$0 = -rC + \frac{\partial C}{\partial t} + \sum_{j=1}^{n} \hat{\lambda}_j \frac{\partial C}{\partial X_j} + \sum_{j,k} \rho_{j,k}\Sigma_j\Sigma_k \frac{\partial^2 C}{\partial X_j \partial X_k} \tag{2.25}$$

with the terminal condition

$$C(T) = F. \tag{2.26}$$

□

[5] See [Hull 1997] or [Elliott-Kopp 2000] for details

[6] Although precisely, rS would be the risk adjusted rate of return and $S(\mu - r)/\sigma$ the market price of risk.

Remark 7. *The equation* (2.25) *is referred to as the* Black-Scholes equation.

□

Example 4. *i) When the claim to be priced is a bond, then* $F = 1$.

ii) When the claim is an European call option on a factor[7] *S with exercise price K, then*

$$F = \max(S - K, 0). \tag{2.27}$$

iii) In case, when the option is written on a bond maturing at T_2, $T_2 > T_1$ *where* T_1 *is the maturity of the option, then*

$$F = \max(P(T_1, T_2) - K, 0). \tag{2.28}$$

□

2.3 Deriving the PDE - second version

In some of the references, the starting point of the PDE approach is the representation of the factors with respect to the equivalent martingale measure $\mathbf{Q}$:

$$\mathrm{d}X_i = \widetilde{M}_i(X(t))\mathrm{d}t + \Sigma_i(X(t))\mathrm{d}\widetilde{W}_i(t) \tag{2.29}$$

with correlation coefficients $\widetilde{\rho}_{i,j}$ between the increments $\mathrm{d}\widetilde{W}_i$ and $\mathrm{d}\widetilde{W}_j$ for $i, j = 1, \ldots, n$.

Referring to the claim-pricing formula derived in the first chapter, the value of a T-claim with a payoff F at time t is given as

$$C_F(t) = \mathbf{E_Q}\left[F \exp\left\{-\int_t^T r(s)\mathrm{d}s\right\}\middle|\mathcal{F}_t\right] \tag{2.30}$$

which is equivalent to the following equation

$$C_F(t)\exp\left\{-\int_0^t r(s)\mathrm{d}s\right\} = \mathbf{E_Q}\left[F \exp\left\{-\int_0^T r(s)\mathrm{d}s\right\}\middle|\mathcal{F}_t\right]. \tag{2.31}$$

The increment of the left hand side is derived by the Itô-lemma:

$$\mathrm{d}\left(\exp\left\{-\int_0^t r(s)\mathrm{d}s\right\}C\right) = \tag{2.32}$$

$$\exp\left\{-\int_0^t r(s)\mathrm{d}s\right\}\left[-r(t)C\mathrm{d}t + \frac{\partial C}{\partial t}\mathrm{d}t + \sum_{i=1}^n \frac{\partial C}{\partial X_i}\mathrm{d}X_i + \frac{1}{2}\sum_{i,j}\frac{\partial^2 C}{\partial X_i \partial X_j}\mathrm{d}\langle X_i, X_j\rangle\right]. \tag{2.33}$$

[7] E.g. a share.

The terms $\mathrm{d}\langle . \rangle$ are derived as in the previous section, but for the representation (2.29). Expanding the $[\dots]$ term of the above expression, we get

$$\left\{ -rC + \frac{\partial C}{\partial t} + \sum_{i=1} \widetilde{M}_i \frac{\partial C}{\partial X_i} + \frac{1}{2} \sum_{i,j} \widetilde{\rho}_{i,j} \Sigma_i \Sigma_j \frac{\partial^2 C}{\partial X_i \partial X_j} \right\} \mathrm{d}t + \sum_{i=1}^{n} \widetilde{M}_i \frac{\partial C}{\partial X_i} \mathrm{d}\widetilde{W}_i(t) \quad (2.34)$$

The right hand side of the equation (2.31) is trivially martingale. Hence the left hand side is also martingale, therefore the drift is zero. The drift term is the expanded $\mathrm{d}t$ term of (2.34).

On the whole, this approach can be summarised in the following proposition.

Proposition 7. *Given the representation* (2.29) *of the market factors with respect to the equivalent martingale measure* $\mathbf{Q}$*, the value function of a* T*-claim with payoff* F *satisfies the PDE*

$$0 = -rC + \frac{\partial C}{\partial t} + \sum_{i=1}^{n} \widetilde{M}_i \frac{\partial C}{\partial X_i} + \frac{1}{2} \sum_{i,j} \widetilde{\rho}_{i,j} \Sigma_i \Sigma_j \frac{\partial^2 C}{\partial X_i \partial X_j} \quad (2.35)$$

with the terminal condition

$$C(T) = F. \quad (2.36)$$

□

2.4 The relation between the approaches

In the previous sections two PDEs, (2.25) and (2.35) were derived. The first one requires the market price of risk functions. The second one is derived from the risk-adjusted representation of the factors, thus this PDE requires the transformation. In this section, the relation between these representation is presented.

Before stating the relationship, some preparation[8] is necessary.

Given the triplet $(\Omega, \mathcal{F}, \mathbf{P})$ and the filtration $(\mathcal{F}_t)_{t \geq 0}$, let L be a continuous martingale, starting from zero. Let $(\tau_n)_{n>0}$ be a series of bounded stopping times. Suppose that

$$\exp\left\{ -L - \frac{1}{2} \langle L \rangle \right\}^{\tau_n} \quad (2.37)$$

is a bounded martingale. Let the elements of $\cup_n \mathcal{F}_{\tau_n}$ be the domain of definition of the mapping $\mathbf{Q}_L$, which is given as

$$\mathbf{Q}_L(A) = \int_A \exp\left\{ -L - \frac{1}{2} \langle L \rangle \right\}_{\tau_n} \mathrm{d}\mathbf{P} \quad (2.38)$$

[8]The preparation is based on [Prokaj 2004] chapter 5.

for all $A \in \cup_n \mathcal{F}_{\tau_n}$, that is

$$\mathrm{d}\mathbf{Q}_L = \exp\left\{-L - \frac{1}{2}\langle L\rangle\right\}\mathrm{d}\mathbf{P}. \tag{2.39}$$

Theorem 2. (Girsanov theorem)[9]

Let N and L be continuous martingales such as $L(0) = 0$. Suppose that $\mathbf{Q}_L$ is probability measure on $\sigma(\cup_n \mathcal{F}_{\tau_n})$. Then

$$(N + \langle L, N\rangle, \mathcal{F}) \tag{2.40}$$

is local martingale with respect to $\mathbf{Q}_L$.

Furthermore,

$$\langle N + \langle L, N\rangle\rangle = \langle N\rangle \tag{2.41}$$

holds.

See [Prokaj 2004] chapter 5 for proof.

□

Let suppose that the market is given by the representation of the form (2.5) with respect to the objective measure $\mathbf{P}$.

Proposition 8. *Let the market $X = (X_1, \ldots, X_k)$ represented by*

$$dX(t) = M(X)dt + \Sigma(X)dW(t), \tag{2.42}$$

where $W(t)$ is an m-dimensional Brownian motion, $M(t,\omega) \in \mathbb{R}^k$ and $\Sigma(t,\omega) \in \mathbb{R}^{k\times m}$. Suppose there exists an m-dimensional local martingale L such that the equivalent martingale measure is given as $\mathbf{Q} = \mathbf{Q}_L$ and conditions of the Girsanov theorem hold. Then the representation with respect to $\mathbf{Q}$ is

$$dX(t) = \widetilde{M}(t)dt + \Sigma(t)d\widetilde{W}(t), \tag{2.43}$$

where

$$\widetilde{W}(t) = W(t) + \langle L, W\rangle \tag{2.44}$$

$$\widetilde{M}(t) = M(t) - \int_0^t \Sigma(t)d\langle L, W\rangle \tag{2.45}$$

and

$$\langle \Sigma W\rangle = \langle \Sigma\widetilde{W}\rangle. \tag{2.46}$$

[9]The general version of the theorem is presented in [Medvegyev 2004] and [Revuz-Yor 1999].

In particular, if u is given as in the Proposition 2 and

$$L(t) = \int_0^t u(s,\omega)dW(s) \tag{2.47}$$

then

$$\widetilde{M} = M - \Sigma u. \tag{2.48}$$

Proof. The proposition is the simple application of the Girsanov theorem[10].

□

Taken the representation (2.5), the previous proposition indicates the followings. From (2.46) comes

$$\rho_{i,j} = \widetilde{\rho}_{i,j} \tag{2.49}$$

for $i, j = 1, \dots k$.

Remark 8. *Since* (2.49) *holds, defining* $\hat{\lambda}_i \mathring{=} \widetilde{M}_i$*, the equation* (2.35) *indicates* (2.17) *and with* (2.18) *we get that the PDE* (2.25) *is equivalent to* (2.35)*.*

□

The equation (2.45) indicates

$$\hat{\lambda}_i = \widetilde{M}_i(t) = M_i(t) - \sum_{j=1}^{m} \int_0^t \Sigma_{i,j} \mathrm{d}\langle L, W_j \rangle \tag{2.50}$$

for $i = 1, \dots k$.

Example 5. *In case when* $\rho_{i,j} = 0$ *from* (2.50) *we get*

$$\widetilde{M}_i(t) = M_i(t) - \int_0^t \Sigma_i d\langle L, W_i \rangle \tag{2.51}$$

for $i = 1, \dots k$.

□

2.5 Ensuring the economic relations

Up to the present section, the pricing PDEs were derived in a general framework. However, there might be conceptual economic relations between the factors, and the PDE must ensure these relations. In the present section two examples with conceptual relationship are presented.

[10] The particular case is directly proved in [Øksendal 2003] Theorem 8.6.6.

2.5.1 Involvement of the long rate

Suppose there is an n-factor representation of the market where the long rate $X_2 = l(t) \mathrel{\mathring{=}} R(t, t+\tau)$ with fixed $\tau \gg 1$ is the second factor. Let the instantaneous rate $X_1(t) = r(t)$ and the long rate be denoted by

$$\begin{aligned} \mathrm{d}r(t) &= M_r(X(t))\mathrm{d}t + \Sigma_r(X(t))\mathrm{d}W_r(t) \\ \mathrm{d}l(t) &= M_l(X(t))\mathrm{d}t + \Sigma_l(X(t))\mathrm{d}W_l(t) \end{aligned} \tag{2.52}$$

with respect to the objective measure $\mathbf{P}$ and with a correlation coefficient $\rho \mathrel{\mathring{=}} \rho_{r,l}$ between the terms $\mathrm{d}W_r$ and $\mathrm{d}W_l$.

Since there is one-to-one relation between the yield curve and the bond curve[11], the following holds for all $t > 0$

$$P(t, t+\tau) = \exp\{-\tau R(t, t+\tau)\} = \exp\{-\tau l(t)\}. \tag{2.53}$$

Proposition 9. *Suppose the second factor is the long rate, then the pricing PDE* (2.25) *and the equation* (2.53) *implies*

$$\hat{\lambda}_l = \widetilde{M_l} = -\frac{r}{\tau} + \frac{1}{2}\tau\Sigma_l^2 + \frac{1}{\tau P}\frac{\partial P}{\partial t}. \tag{2.54}$$

Proof. From (2.53) the partial differentials of the bond price at time t can be derived:

$$\begin{aligned} \frac{\partial P}{\partial l} &= -\tau P \\ \frac{\partial^2 P}{\partial l^2} &= \tau^2 P \end{aligned} \tag{2.55}$$

and

$$\frac{\partial P}{\partial X_i} = \frac{\partial^2 P}{\partial X_i^2} = \frac{\partial^2 P}{\partial X_i \partial X_j} = 0 \tag{2.56}$$

for all $i = 1, 3, \ldots n$ and $j = 1, 2, \ldots n$.

Deriving the PDE (2.25) on the bond $P(t, t+\tau)$ with the consideration of the equations (2.55) and (2.56), we get

$$0 = -rP - \tau P\widetilde{M_l} + \frac{1}{2}\tau^2 P\Sigma_l^2 + \frac{\partial P}{\partial t}. \tag{2.57}$$

Since τ is fixed and positive, furthermore $P > 0$ holds for any $l > 0$, therefore (2.54) holds.

□

[11] See chapter 1.

2.5.2 Involvement of the spread

Suppose that the second factor is the spread $X_2(t) = s(t) \hat{=} r(t) - l(t)$ with the following representation

$$\begin{aligned} \mathrm{d}r(t) &= M_r(X(t))\mathrm{d}t + \Sigma_r(X(t))\mathrm{d}W_r(t) \\ \mathrm{d}s(t) &= M_s(X(t))\mathrm{d}t + \Sigma_s(X(t))\mathrm{d}W_s(t) \end{aligned} \tag{2.58}$$

and a correlation coefficient $\rho = \rho_{r,s}$.

In that case, the equation (2.53) has the form

$$P(t, t+\tau) = \exp\left\{-\tau\left[r(t) - s(t)\right]\right\}. \tag{2.59}$$

Proposition 10. *Suppose the second factor is the spread, then the pricing PDE* (2.25) *and the equation* (2.59) *implies*

$$\hat{\lambda}_s = \widetilde{M}_s = \widetilde{M}_r + \frac{r}{\tau} - \tau\frac{\Sigma_r^2 + \Sigma_s^2 - 2\rho\Sigma_r\Sigma_s}{2} + \frac{1}{\tau P}\frac{\partial P}{\partial t}. \tag{2.60}$$

From (2.59) the derivatives of P are

$$\begin{aligned} \frac{\partial P}{\partial s} = -\frac{\partial P}{\partial r} &= \tau P \\ \frac{\partial^2 P}{\partial s^2} = \frac{\partial^2 P}{\partial r^2} = -\frac{\partial^2 P}{\partial s \partial r} &= \tau^2 P \end{aligned} \tag{2.61}$$

and

$$\frac{\partial P}{\partial X_i} = \frac{\partial^2 P}{\partial X_i^2} = \frac{\partial^2 P}{\partial X_i \partial X_j} = 0 \tag{2.62}$$

for $i = 3, \ldots n$ and $j = 1, \ldots n$.

Deriving the PDE (2.25) with (2.61) and (2.62) we get

$$0 = -rP - \tau P\widetilde{M}_r + \tau P\widetilde{M}_s + \frac{1}{2}\tau^2 P(\Sigma_r^2 + \Sigma_s^2) - \rho\tau^2 P\Sigma_r\Sigma_s + \frac{\partial P}{\partial t}. \tag{2.63}$$

The same argumentation as in the previous subsection indicates (2.60).

□

Chapter 3

Multi-factor term-structure models

In this chapter, the main characteristics of factor term-structure models are presented. Further on, a three-factor model of the Hungarian market is constructed based on the slight generalisation of the method described in [Willmott 1998].

3.1 Model formulating considerations

As it was emphasised earlier, when formulating a term-structure model, the necessary complexity is up to the modelling objectives. Some *time homogeneous* one-factor models approximate bond prices with small error. [Hull 1997] emphasises that small error in bond prices may lead to significant misprizing of bond derivatives. *Time-inhomogeneous* extensions of one-factor models usually result in better valuation of bond derivatives. For *risk management* purposes the well approximation of dynamics is needed. That can be handled by capturing as many sources of randomness as many is reasonable and tractable for explaining the randomness. [Cairns 2004] and [James-Webber 2004] refer to principal component analysis (PCA) methods on yield curve data for determining the number of factors reasonable in the model. However, the examined yield curve series here were estimated by fitting the logarithm of the discount curve by six base functions. Thus, no more then six factors can be found by PCA. Therefore I approach the formulation more intuitively.

Beyond the modelling objectives, the specification of the market factors is depend on the *characteristics of the modelled yield curve dynamics* as well. From economic point of view, the following characteristics are common:

i) Instantaneous rates are non-negative.

ii) Instantaneous rates tend to be mean reverting.

iii) The mean reversion level may vary in time.

iv) The volatility of the interest rates is not constant.

v) The movements in the different points of the yield curve are not perfectly correlated. For example, the changes in the short rate and in the spread, the difference between the short and the long rate, are weakly related.

Fulfilling these requirements, I have chosen a model, with the following factors intuitively:

i) mean reverting instantaneous rate,

ii) mean reverting spread,

iii) volatility of instantaneous rate.

The Hungarian market has one further and important speciality rising from the *convergence mechanism to the European Monetary Union*. As Hungary is going to join the EMU sometime in 2010-2012, the level of interest rates is expected to gradually reach the level of EMU rates. There are several solutions for modelling this convergence.

Firstly, one could let the mean reversion level be non-constant:

- it could be deterministic function of time, or
- the mean reversion level itself could be stochastic.

The former solution requires professional estimates of the deterministic trend and the latter indicates the introduction of an additional random factor.

Secondly, one could let the mean reversion level be constant and equal to the mean reversion level of the Euro instantaneous rate and let the speed of reversion be small enough for the slow long-term convergence. This solution only requires the estimation of two parameters.

The main difficulty in choosing solution for the convergence phenomenon is the fact that Hungary is about half way to the destination, only less then the first half of the process can be observed. Anyway, some kind of economic expectations and intuitions are necessary.

For tractability reasons, I have chosen the second solution[1]. The long-term level of mean reversion with slow reversion speed is applied and calibrated with the use of historical Euro yield curve data.

The Hungarian government bond yield curve is currently downward sloping contrary to the EMU yield curve which is upward sloping. If the mean reversion level of the spread is chosen to be negative, the long term equilibrium yield curve becomes upward sloping. Hence the long term equilibrium shape of yield curve can be adjusted by calibration.

[1]In the present study, I focus on numerical tractability and only briefly consider calibration issues. Comparing the model types of mean reversion could be the scope of an other study.

3.2 Model construction

After detailed analysis of the Hungarian bond yield curve dynamics, a three factor model, including the instantaneous rate, the volatility coefficient of the short rate and the spread as factors, proved to be appropriate. Considering the general characteristics of term structures presented in the previous section, the market is defined as below.

Definition 16. *i) Let the* instantaneous rate *be given by the following SDE*

$$dr(t) = M_r(t,\omega)\,dt + \Sigma_r(t,\omega)\,dW_r(t) \tag{3.1}$$

where the drift term assumed to be

$$M_r(t,\omega) = \alpha_r(\mu_r - r(t)) \tag{3.2}$$

and the volatility term has the form

$$\Sigma_r(t,\omega) = \Sigma_r(r(t),s(t)) = \sigma_r(t,\omega)f_r(r(t),s(t)). \tag{3.3}$$

ii) Let the spread $s(t)\mathring{=}r(t) - R(t,t+\tau)$ *(with a τ fixed at a high level) be given by*

$$ds(t) = M_s(t,\omega)\,dt + \Sigma_s(t,\omega)\,dW_s(t) \tag{3.4}$$

with

$$M_s(t,\omega) = \alpha_s(\mu_s - s(t)) \tag{3.5}$$

and

$$\Sigma_s(t,\omega) = \Sigma_s(r(s),s(s)) = \sigma_s f_s(r(s),s(t)). \tag{3.6}$$

iii) The third factor is the volatility coefficient of the instantaneous rate *(further on the* volatility coefficient*):*

$$d\sigma_r(t) = M_\sigma(t,\omega)\,dt + \Sigma_\sigma\,dW_\sigma(t). \tag{3.7}$$

The parameters α_r, α_s and σ_s are time-homogeneous constants. The terms dW_r, dW_s and dW_σ are increments of Brownian motions with respect to the objective measure $\mathbf{P}$*, and have constant correlation coefficients $\rho_{r,s}$, $\rho_{r,\sigma}$ and $\rho_{s,\sigma}$ respectively.*

The functions f_r and f_s are assumed to be continuous. The non-constant elements of the drift and volatility of the process σ_r, namely $M_\sigma(t,\omega)$ and $\Sigma_\sigma(t,\omega)$, are assumed to be adapted to the filtration of the market, so that

$$\begin{aligned}
P\left[\int_0^t |M_\sigma(t,\omega)|\,ds < \infty \;\forall t \geq 0\right] &= 1 \\
P\left[\int_0^t \Sigma_\sigma(t,\omega)^2\,ds < \infty \;\forall t \geq 0\right] &= 1
\end{aligned}$$

□

This formulation mainly differs from the *Balduzzi*, *Das*, *Foresi* and *Sundaram* (BDFS)[2] three factor model in that above the spread is introduced as a factor instead of the stochastic mean reversion level approach.

The specification of the function f_r, Σ_l, M_σ and Σ_σ is to be done.

3.3 Specification of the coefficients

The slight generalisation of the method introduced in [Willmott 1998] chapter 23 and 38 is applied below.

The factor-specification method of [Willmott 1998] is a two step process. In the first step the volatility term is specified based on the observation of day-to-day increments in the factors. In the second step the drift term is calibrated with the use of steady-state distribution of the factors. Because of the referred convergence phenomenon, up to the current date only non-stationary path of the instantaneous rate[3] can be observed. Therefore the steady-state distribution cannot be estimated. The same problem holds for the spread. In this study, it is handled by making assumptions on the drift terms of the first two factors. The mean reversion levels μ_r and μ_s are estimated from the long term average of the short rate and the spread of EURO yield curve data[4] respectively. The mean reversion speed parameters are calculated to ensure, that the level of factors reach the level of their correspondings in 6 years in expected value.

In case of the volatility coefficient factor stationarity is assumed, thus the drift-specification method of [Willmott 1998] is applicable. The difficulty with the application is that the method requires the factor to be observable. Below, the volatility process is estimated based on the information given by the instantaneous rate specification.

3.3.1 The instantaneous rate and the spread

Let start the theoretical calibration with the volatility term of the spread process $\Sigma_l(t,\omega)$. The process $s(t)$ and its' day-to-day increments δs are observable.

[Willmott 1998] approximates the square of the increment as follows:

$$(\delta s)^2 \approx \Sigma_s^2(s,r)\phi_s^2\delta t, \tag{3.8}$$

[2][James-Webber 2004] chapter 7.5.2

[3]In this study the calibration is based on the Hungarian government bond data of the period 01/01/2004-01/05/2005 provided by the Government Debt Management Agency Ltd. (ÁKK).

[4]The EURO yield curve data were provided by *Reuters*. The short end, up to one year, is estimated using EURO zeros, the long end is estimated based on swap yields. The data of the period between 01/01/1999-01/05/2005 were included.

where $\phi_s\sqrt{\delta t} = W_s(t+\delta) - W(t) \sim N(0, \sqrt{\delta t})$. Using the information given by the drift term of definition 16, better approximation can be derived:

$$\Delta_s(t) \mathrel{\mathring{=}} (\delta s - \alpha_s(\mu_s - s(t))\delta t)^2 \approx \Sigma_s^2(s,r)\phi_s^2\delta t. \tag{3.9}$$

The expected value of ϕ_s is 1, thus

$$\mathbf{E}\left[\Delta_s | r, s\right] = \mathbf{E}\left[\Sigma_s^2(r,s)\phi_s^2\delta t | r, s\right] = \Sigma_s^2(r,s)\delta t. \tag{3.10}$$

Our aim is to specify the function $\Sigma_s(r,s)$ from (3.10), that is the relationship between the drift eliminated increments of s and the level of r and s.

In [Willmott 1998] only one-dimensional relationship is analysed by approximating the left hand side of (3.10) by splitting the factor into k buckets and calculating the bucket averages of the argument of the expectation for each bucket. [Willmott 1998]

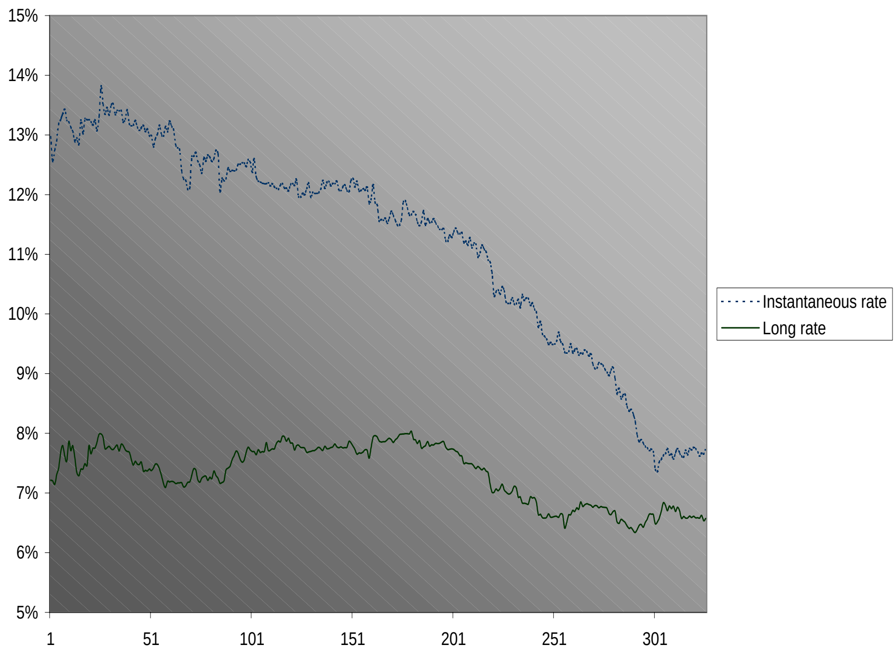

Figure 3.1: The instantaneous rate and the long rate

plotted the logarithm of the estimated conditional averages against the logarithmic level of the factor in each bucket and found linear relationship. Repeating similar analysis on the spread of the Hungarian bond market, no clear relationship was found. This experience and some technical considerations indicated the introduction of the instantaneous factor into the calculations.

I split $l = r - s$ into $k = 10$ buckets and approximated $\mathbf{E}[\Delta_s | l_i]$ by calculating the average of Δ_s for all $i = 1, \ldots, k$. The following linear relationship was found:

$$\log(\mathbf{E}[\Delta_s | l]) = 2 \log(\Sigma_s(r - s)) + \log(\delta t) \approx A_s + B_s \log(r - s), \tag{3.11}$$

which indicates the following function form

$$\Sigma_s(r - s) \stackrel{\circ}{=} \sigma_s (r - s)^\gamma \tag{3.12}$$

where the parameters σ_s and γ are estimated from (3.11).

I also examined if the increments of the spread better explained by the combinations of r and s, but statistically that solution was much less significant. Therefore the spread is described by the following SDE

$$\mathrm{d}s(t) = \alpha_s(\mu_s - s(t))\mathrm{d}t + \sigma_s(r(t) - s(t))^\gamma \mathrm{d}W_s(t). \tag{3.13}$$

Assuming the volatility coefficient σ_r to be stationary and independent from both the instantaneous rate and the spread, the modification of (3.10) for the instantaneous rate has the form

$$\mathbf{E}[\Delta_r | r, s] = \mathbf{E}\left[\Sigma_r^2(r, s)\phi_r^2 \delta t | r, s\right] = \mathbf{E}\left[\sigma_r^2 f_r^2(r, s)\phi_r^2 \delta t | r, s\right] = \mathbf{E}\left[\sigma_r^2\right] f_r^2(r, s)\delta t. \tag{3.14}$$

Therefore in the case of the instantaneous rate, the same analysis can be done as above and $f_r(r, s) = f_r(r) = r^\kappa$ type function is found to be significant.

Hence, the instantaneous rate factor is specified as

$$\mathrm{d}r(t) = \alpha_r(\mu_r - r(t))\mathrm{d}t + \sigma_r(t, \omega) r^\kappa \mathrm{d}W_r(t). \tag{3.15}$$

3.3.2 The volatility process of the instantaneous rate

After specifying the structure of the instantaneous rate process, the σ_r process can be estimated from the increments of $r(t)$. In this study, the process σ_r was estimated by calculating the exponentially weighted variance of the drift eliminated and r^γ standardised increments of the instantaneous rate. Providing the observations of the daily σ_r, the method of [Willmott 1998] can be applied to specify the $\Sigma_\sigma(t, \omega) = \Sigma_\sigma(\sigma_r)$ coefficient.

Analogously, the square of the day-to-day changes in the volatility factor is given as[5]:

$$(\delta\sigma_r)^2 \approx \Sigma_\sigma^2 \phi_\sigma^2 \delta t \tag{3.16}$$

where ϕ_σ also denotes a standardised Normal variable. Similarly

$$\mathbf{E}\left[(\delta\sigma_r)^2|\sigma_r\right] = \mathbf{E}\left[\Sigma_\sigma(\sigma_r)^2\phi^2\delta t|\sigma_r\right] = \Sigma_\sigma(\sigma_r)^2\delta t \tag{3.17}$$

holds. And the functional form estimation is based on

$$\log(\mathbf{E}\left[(\delta\sigma_r)^2|\sigma_r\right]) = 2\log(\Sigma_\sigma(\sigma_r)) + \log(\delta t) \approx A_\sigma + B_\sigma \log(\sigma_r), \tag{3.18}$$

[5]Since there are no assumption on the drift term of $d\sigma_r$, the drift is not eliminated form the increments as in [Willmott 1998]. The approximation is reasonable as the order of neglected terms is $O((\delta t)^2)$ or $O((\delta t)^{\frac{3}{2}})$.

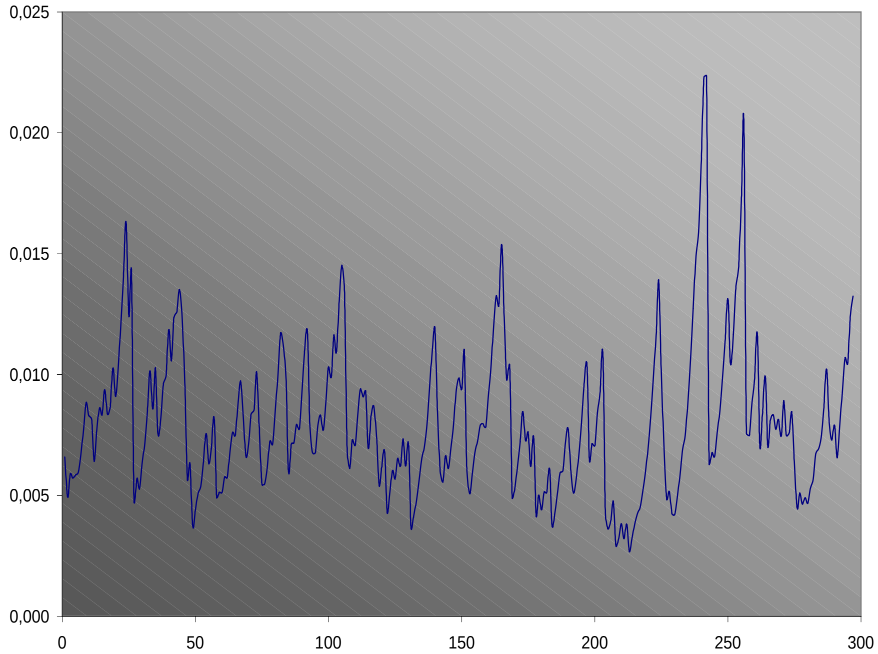

Figure 3.2: The estimated volatility process

which indicates

$$\Sigma_\sigma(\sigma_r) \mathrel{\hat{=}} \nu\sigma_r^\eta, \tag{3.19}$$

where the ν and η parameters can be estimated from (3.18).

Finally, we have

$$\mathrm{d}\sigma_r = M_\sigma(\sigma_r)\mathrm{d}t + \nu\sigma_r^\eta \mathrm{d}W_\sigma(t). \tag{3.20}$$

For the estimation of the drift term $M_\sigma(\sigma_r)$ [Willmott 1998] assumes the existence of the *steady-state probability density function* for σ_r. To apply this approach to the Hungarian market we further assume that the distribution of σ_r is independent of the convergence, that is we can estimate the steady-state distribution based on the available data.

The referred approach applies the *Fokker-Planck* equation:

Proposition 11. *(Fokker-Planck or forward Kolmogorov equation)*
Consider a process governed by the SDE

$$dX(t) = a(X(t))dt + b(X(t))dW(t). \tag{3.21}$$

Let the random density function of $X(t)$ denoted by

$$p(t,y) \mathrel{\hat{=}} \mathbf{P}\{X(t) = y|X(0) = x\}.$$

Then the following holds

$$\frac{\partial p(t,y)}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(b^2(y)p(t,y)\right) - \frac{\partial}{\partial y}\left(a(y)p(t,y)\right). \tag{3.22}$$

If the process $X(t)$ has an equilibrium density, it will be

$$p(y) = \lim_{t\to\infty} p(t,y). \tag{3.23}$$

For $p(y)$ the equilibrium *equation holds:*

$$\frac{1}{2}\frac{\partial^2}{\partial y^2}\left(b^2(y)p(y)\right) - \frac{\partial}{\partial y}\left(a(y)p(y)\right) = 0. \tag{3.24}$$

See [Shreve 1998] chapter 31.2 for proof.

□

In case of the volatility process $\sigma_r(t)$ the equilibrium Fokker-Planck equation has the form

$$\frac{1}{2}\frac{\partial^2}{\partial \sigma_r^2}\left(\Sigma_\sigma^2(\sigma_r)p(\sigma_r)\right) - \frac{\partial}{\partial \sigma_r}\left(M_\sigma(\sigma_r)p(\sigma_r)\right) = 0. \tag{3.25}$$

thus the drift process is given as

$$M_\sigma(\sigma_r) = \frac{1}{2p(\sigma_r)}\frac{\partial}{\partial \sigma_r}\left(\Sigma_\sigma^2(\sigma_r)p(\sigma_r)\right). \tag{3.26}$$

Remark 9. *This specification of the drift term assumes that the steady-state distribution is observable. From an economic point of view one can assume that the process σ_r is stationary. It was emphasised earlier that due to the convergence phenomenon the steady-state density of the instantaneous rate and the spread processes are not observable, thus the above method cannot be applied for them. That is why we rely on intuitions in their case.*

□

Since $\Sigma_\sigma(\sigma_r)$ is given in (3.19), with the estimation of the steady-state density $p(\sigma_r)$ of σ_r the function form of $M_\sigma(\sigma_r)$ can be approximated. Calibrating the volatility process of Dow Jones Industrial Average based on thirty day volatility of daily returns, [Willmott 1998] could fit a lognormal curve to the empirical distribution.

In our case, the figure 3.3.2 suggests the lognormal distribution to be suitable,

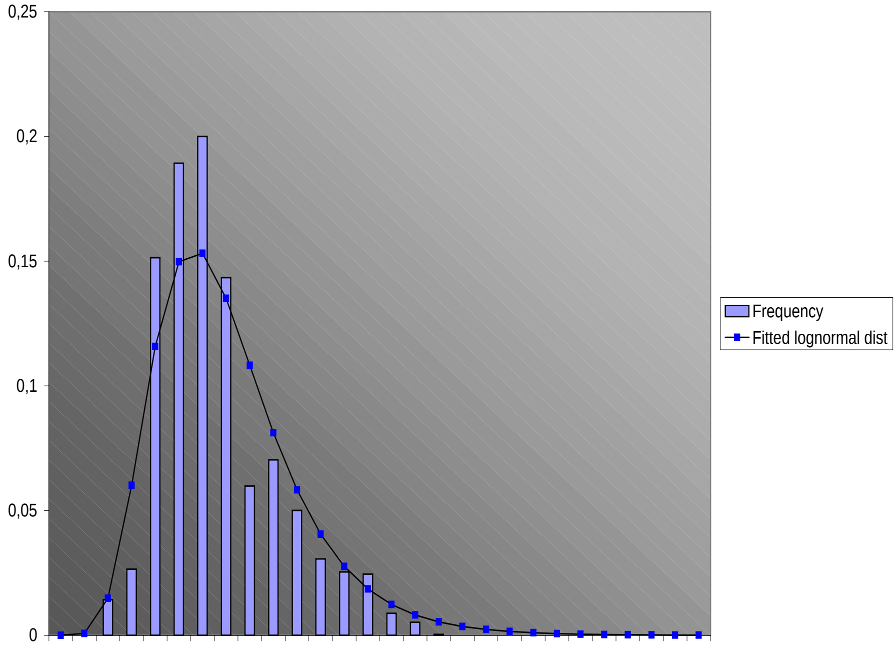

Figure 3.3: Fitting the empirical steady-state distribution of σ_r

hence

$$p(\sigma_r) = \frac{1}{a\sigma_r\sqrt{2\pi}} \exp\left\{-\frac{1}{2a^2}\left(\log(\sigma_r) - \log(\sigma^*)\right)^2\right\}, \tag{3.27}$$

and therefore M_σ has the form

$$M_\sigma(\sigma_r) = \nu^2\sigma_r^{2\eta-1}\left(\left[\eta - \frac{1}{2} + \frac{1}{2a^2}\log(\sigma^*)\right] - \frac{1}{2a^2}\log(\sigma_r)\right). \tag{3.28}$$

After all, for the increment of the volatility factor is given:

$$\mathrm{d}\sigma_r(t) = \nu^2\sigma_r^{2\eta-1}\left(M_\sigma^* - \frac{1}{2a^2}\log(\sigma_r(t))\right)\mathrm{d}t + \nu\sigma_r^\eta(t)\mathrm{d}W_\sigma(t). \tag{3.29}$$

Where

$$M_\sigma^* \mathrel{\hat{=}} \left[\eta - \frac{1}{2} + \frac{1}{2a^2}\log(\sigma^*)\right].$$

The $\mathrm{d}t$ term of the above SDE indicates the σ_r to be mean reverting to the level $\exp M_\sigma^*$.

3.3.3 Statistical computations

The process and results of the statistical analysis of the Hungarian bond market can be summarised as follows.

The parameters of the drift terms:

r	s	σ_r
$\alpha_r = 0.7755$	$\alpha_s = 0.6904$	$a = 36.64\%$
$\mu_r = 2.504\%$	$\mu_s = -1.9709\%$	$\sigma^* = 0.754\%$

and of the volatility terms, including the R^2 statistics of the linear regression:

s	r	σ_r
$\sigma_s = 2.6919$		$\nu = 3.2267$
$\gamma = 2.6949$	$\kappa = 0.92145$	$\eta = 1.3593$
$R^2 = 0.7141$	$R^2 = 0.8764$	$R^2 = 0.8755$

The correlation coefficients are

$\rho_{r,s} = 0.8044$
$\rho_{r,\sigma} = 0.0427$
$\rho_{s,\sigma} = 0.005$

Remark 10. *Since the coefficients $\rho_{r,\sigma}$ and $\rho_{s,\sigma}$ statistically do not differ from zero, further on these coefficients assumed to be zero.*

□

The calibration process is summarised as below.

Process of calibration

Daily observations of $P(t,T)$ $\downarrow$ Fitting the daily yield curves $\downarrow$ Estimates of $R(t,T)$ $\downarrow$ Daily observations of $r(t)$ and $s(t)$	Euro yield data observations

$\downarrow$

Estimation of the parameters α_r, μ_r, α_s, μ_s $\longrightarrow M_r$ and M_s

$\downarrow$

Calculation of the series Δ_r and Δ_s

$\downarrow$

Estimation of the parameters κ, γ and σ_s $\longrightarrow \Sigma_r$ and Σ_s

$\downarrow$

Implied estimation of the process σ_r

$\downarrow$

Calibrating the parameters ν and η $\longrightarrow \Sigma_\sigma$

$\downarrow$

Calibrating the drift term of σ_r $\longrightarrow M_\sigma$

3.4 Remarks on model-construction

The spread $s(t) = r(t) - R(t, t+\tau)$ as a factor is introduced into to the model to ensure the stochastic relation between the short and long end of the yield curve. The choice of τ depends on the modelling purposes. If the purpose is the valuation of a group of interest rate derivatives or the risk management of an interest rate dependent portfolio, then the most typical maturity or the longest maturity could be a reasonable choice.

If the maturities of the portfolio elements disperse around two typical levels, then other spread factors could be introduced. When increasing the number of factors tractability issues should be considered.

[Willmott 1998] refers to some models with different structures:

i) In the *Hull-White* volatility model:

$$\mathrm{d}(\sigma)^2 = a(b - \sigma^2)\mathrm{d}t + c\sigma^2\mathrm{d}W_\sigma. \tag{3.30}$$

ii) The *Heston* model describes the volatility as

$$\mathrm{d}\sigma = -a\sigma\mathrm{d}t + b\mathrm{d}W_\sigma. \tag{3.31}$$

Chapter 4

Numerical analysis of the three-factor model

So far, the PDE for pricing derivative products of the market factors has been derived.

Corollary 3. *In the special case of the introduced three-factor model, the PDE for a T-claim C_F is given as*

$$0 = -rC + \frac{\partial C}{\partial t} + \hat{\lambda}_r(r,s,\sigma_r)\frac{\partial C}{\partial r} + \hat{\lambda}_s(r,s,\sigma_r)\frac{\partial C}{\partial s} + \hat{\lambda}_\sigma(r,s,\sigma_r)\frac{\partial C}{\partial \sigma_r} + \mathcal{L}_{r,s} + \mathcal{L}_\sigma, \quad (4.1)$$

where

$$\begin{aligned} \mathcal{L}_{r,s} &= \frac{1}{2}\Sigma_r^2(r,\sigma)\frac{\partial^2 C}{\partial r^2} + \frac{1}{2}\Sigma_s^2(r-s)\frac{\partial^2 C}{\partial s^2} + \rho\Sigma_r(r,\sigma)\Sigma_s(r-s)\frac{\partial^2 C}{\partial r \partial s} \quad (4.2)\\ \mathcal{L}_\sigma &= \frac{1}{2}\Sigma_\sigma(\sigma)\frac{\partial^2 C}{\partial \sigma_r^2}. \end{aligned}$$

for $r \geq 0$, $s \geq r$ and $\sigma_r \geq 0$ with the terminal condition

$$C(T) = F, \quad (4.3)$$

and coefficients $\hat{\lambda}_r$, $\hat{\lambda}_s$, $\hat{\lambda}_\sigma$, Σ_r, Σ_s and Σ_σ specified in the previous chapter.

□

Later in this chapter we will show how the information provided by the historical bond prices, and thus the information of the historical yield curves, can be applied to estimate the functions $\hat{\lambda}_r$, $\hat{\lambda}_s$, and $\hat{\lambda}_\sigma$. Knowing these coefficient functions, the current prices of interest rate derivatives or the future values of bonds and derivatives under different scenarios of the factors can be established. Also knowing the (empirical) distribution of the factor's future value, the distribution of the derivative's price can be estimated. This latter possibility has useful applications in risk management.

The coefficients might be so complex that no closed form solution of the pricing PDE can be derived. In such cases numerical methods are required. In this chapter the pricing PDE is analysed from a numerical point of view.

4.1 Boundary conditions

When using finite difference methods the domain of definition must be bounded. There are two ways to make the domain bounded. The first one is the transformation of variables. The second one is *cutting the edge* of the domain. In the present study, the latter version is examined.

Before cutting, the behaviour of the PDE on the edge is to be analysed. This behaviour might depend on the specification of the derivative product to be valued. Some special cases are discussed below.

Proposition 12. *Suppose that C is bounded around $r = 0$. Then in case, when $r \to 0$ the PDE* (4.1) *has the form*

$$0 = \frac{\partial C}{\partial t} + \hat{\lambda}_r \frac{\partial C}{\partial r} + \hat{\lambda}_s \frac{\partial C}{\partial s} + \hat{\lambda}_\sigma \frac{\partial C}{\partial \sigma_r} + \frac{1}{2}\Sigma_s \frac{\partial^2 C}{\partial s^2} + \mathcal{L}_\sigma \tag{4.4}$$

Proof. As in [Seydel 2003], due to the boundedness around $r = 0$, the term

$$\Sigma_r^2 \frac{\partial^2 C}{\partial r^2} = \sigma_r^2 r^{2\kappa} \frac{\partial^2 C}{\partial r^2} \tag{4.5}$$

vanishes for $r \to 0$. By contradiction, assuming a nonzero value leads to

$$\left| \frac{\partial^2 C}{\partial r^2} \right| \geq O\left(\frac{1}{r^{2\kappa}} \right). \tag{4.6}$$

Integrating twice the term (4.6), we get

$$|C| \geq O\left(\frac{1}{r^{2\kappa - 2}} \right) + O\left(\frac{1}{r^{2\kappa - 1}} \right) + c_1 \tag{4.7}$$

which contradicts the boundedness for $0 < \gamma < 2$ and $r \to 0$.

The same argumentation as above indicates that the other the terms

$$\Sigma_r \Sigma_s \frac{\partial^2 C}{\partial r \partial s} \tag{4.8}$$

vanish for $r \to 0$.

□

Proposition 13. *Suppose that C is bounded around each point of the subspace $r = s$. Then in case when $s \to r$, the PDE* (4.1) *has the form*

$$0 = -rC + \frac{\partial C}{\partial t} + \hat{\lambda}_r \frac{\partial C}{\partial r} + \hat{\lambda}_s \frac{\partial C}{\partial s} + \hat{\lambda}_\sigma \frac{\partial C}{\partial \sigma_r} + \frac{1}{2}\Sigma_r \frac{\partial^2 C}{\partial r^2} + \frac{1}{2}\Sigma_\sigma \frac{\partial^2 C}{\partial \sigma_r^2}. \tag{4.9}$$

Proof. The vanishing of the term

$$\Sigma_s \frac{\partial^2 C}{\partial s^2} = \sigma_s (r-s)^\gamma \frac{\partial^2 C}{\partial \sigma_r^2} \tag{4.10}$$

can be proved analogously to the previous proposition.

Proposition 14. *Suppose C is bounded around $\sigma_r = 0$. Then in case when $\sigma_r \to 0$ the PDE* (4.1) *has the form*

$$0 = -rC + \frac{\partial C}{\partial t} + \hat{\lambda}_r \frac{\partial C}{\partial r} + \hat{\lambda}_s \frac{\partial C}{\partial s} + \hat{\lambda}_\sigma \frac{\partial P}{\partial \sigma_r} + \frac{1}{2}\Sigma_s^2 \frac{\partial^2 C}{\partial s^2}. \tag{4.11}$$

Proof.

i) The equation

$$\Sigma_r = \sigma_r r^\gamma = 0 \tag{4.12}$$

is indicated by $\sigma_r = 0$.

ii) The vanishing of the term

$$\Sigma_\sigma^2 \frac{\partial^2 C}{\partial \sigma_r^2} = \nu \sigma_r^\eta \frac{\partial^2 C}{\partial \sigma_r^2} \tag{4.13}$$

can be proved analogously to the proposition (12) for $0 < \eta < 2$ and $\sigma_r \to 0$.

□

Remark 11. *The key point in the above propositions is the assumption on the boundedness of C around $r = 0$ and around $\sigma_r = 0$. The proof of the boundedness depends on the type of derivative.*

i) When the derivative is a simple bond maturing at T, the equation

$$P(t,T) = \mathbf{E}_\mathbf{Q}\left[\exp\left\{-\int_t^T r(s)\,ds\right\}\middle|\mathcal{F}_t\right] \tag{4.14}$$

and $r \geq 0$ indicates $P \in [0,1]$.

ii) In case of a T-claim C with a payoff F at time T, if there exist $K > 0$ and $\varepsilon > 0$, such as

$$\mathbf{E}_\mathbf{Q}\left[|F|\middle|\mathcal{F}_t\right] < K \tag{4.15}$$

(a.s.) for $0 \le r \le \varepsilon$, then the following (a.s.) holds

$$\begin{aligned} |C(t)| &= \left| \mathbf{E_Q} \left[\exp\left\{ -\int_t^T r(s)\,ds \right\} F \middle| \mathcal{F}_t \right] \right| \\ &\le \mathbf{E_Q} \left[\exp\left\{ -\int_t^t r(s)\,ds \right\} |F| \middle| \mathcal{F}_t \right] \\ &\le \mathbf{E_Q} \left[|F| \middle| \mathcal{F}_t \right] < K \end{aligned} \tag{4.16}$$

thus the boundedness (a.s.) holds.

□

Proposition 15. *In case when $s(0) = r(0)$, the bond price for all maturities $T \in [0, \tau]$ equals to 1:*

$$P(0, T) = 1, \text{ for all } T \in [0, \tau]. \tag{4.17}$$

Proof. If $s(0) = r(0)$ then $R(0, \tau) = r(0) - s(0) = 0$ and therefore

$$P(0, \tau) = \exp\{-\tau R(0, \tau)\} = 1. \tag{4.18}$$

Since

$$P(0, \tau) = \mathbf{E_Q} \left[\exp\left\{ -\int_0^\tau r(u)\mathrm{d}u \right\} \middle| \mathcal{F}_0 \right] = \mathbf{E} \left[\exp\left\{ -\int_0^\tau r(s)\mathrm{d}s \right\} \right] = 1 \tag{4.19}$$

also holds, and $r \ge 0$

$$\exp\left\{ -\int_0^\tau r(u)\mathrm{d}u \right\} \le 1 \tag{4.20}$$

for almost all $\omega \in \Omega$, therefore (4.19) only holds if $r(u, \omega) = 0$ for almost all $(u, \omega) \in [0, \tau] \times \Omega$.

Consequently

$$\exp\left\{ -\int_0^T r(u)\mathrm{d}u \right\} = 1, \tag{4.21}$$

for almost all and $\omega \in \Omega$, and therefore

$$P(0, T) = 1 \tag{4.22}$$

for almost all $T \in [t, t + \tau]$.

□

Remark 12. *The extension of the above proposition for the case $r(t) = s(t)$ can be proved analogously.*

□

Corollary 4. *When applying the pricing PDE on a bond maturing at time T, on the bound where $s = r$ the boundary condition is*

$$P(t, T) = 1. \tag{4.23}$$

□

Remark 13. *Deriving accurate boundary conditions for a given interest rate derivative, there are different PDE numerical methods available for approximating the solution. The main classes of method are the* finite difference *method and the* finite element *method. See [Stoyan 1997] and [Seydel 2003] for details.*

□

4.2 Application of the pricing PDE

As it was derived in the chapter 1, the price of an interest rate derivative C at time t is given by the formula

$$C = \mathbf{E}_{\mathbf{Q}} \left[F \exp \left\{ - \int_t^T r(u) \mathrm{d}u \right\} \middle| \mathcal{F}_t \right] \tag{4.24}$$

where the expectation is calculated with respect to the equivalent martingale measure $\mathbf{Q}$. The expectation can be estimated by simulation of the factors[1]. The representation of the factors with respect to $\mathbf{Q}$ is applied for the simulation, therefore the estimation $\widetilde{M}_i$ terms are required. In this section, a method for the approximation of the *risk-adjusted drift* terms based on historical bond prices is presented in the special case of two-factor r, l term structure model, where r denotes the instantaneous rate and $l \mathrel{\hat{=}} R(t, t + \tau)$ denotes the long spot rate.

4.2.1 Pointwise estimation of the risk-adjusted drift

The representation to be applied is given as below

$$\mathrm{d}r = \widetilde{M}_r(r, l)\mathrm{d}t + \Sigma_r(r, l)\mathrm{d}\widetilde{W}_r(t) \tag{4.25}$$
$$\mathrm{d}l = \widetilde{M}_l(r, l)\mathrm{d}t + \Sigma_l(r, l)\mathrm{d}\widetilde{W}_l(t) \tag{4.26}$$

with a correlation coefficient ρ between $\mathrm{d}\widetilde{W}_r$ and $\mathrm{d}\widetilde{W}_l$. Suppose the functions Σ_r and Σ_l to be known (calibrated after observed market data).

The pricing PDE written on a bond price P in the two factor case is given as:

$$0 = -rP + \frac{\partial P}{\partial t} + \widetilde{M}_r \frac{\partial P}{\partial r} + \widetilde{M}_l \frac{\partial P}{\partial l} + \frac{1}{2}\Sigma_r^2 \frac{\partial^2 P}{\partial r^2} + \frac{1}{2}\Sigma_l^2 \frac{\partial^2 P}{\partial l^2} + \rho \Sigma_r \Sigma_l \frac{\partial^2 P}{\partial r \partial l}. \tag{4.27}$$

The above PDE is an important equation for calibrating the risk-adjusted drift.

[1] See [Kloeden-Platen 1999] for details.

Remark 14. *As the pricing PDE for bonds does not depend directly on t and T, but depends on $\hat{t} \circeq T - t$, r and l the model is called* time homogeneous. *Hereafter the notation*

$$P^*(\hat{t}) \circeq P(t, t + \hat{t}) \tag{4.28}$$

is also applied. This also indicates the following

$$P(t + \Delta_t, T, r, l) = P(t, T - \Delta_t, r, l) = P^*(\hat{t} - \Delta_t, r, l). \tag{4.29}$$

□

The other important equation for the calibration of $\widetilde{M_r}$ derived from the *economic relation* used for bonds[2]:

$$\widetilde{M_l} = -\frac{r}{\tau} + \frac{1}{2}\tau\Sigma_l^2 + \frac{1}{\tau P^*(\tau)}\frac{\partial P^*(\tau)}{\partial t}. \tag{4.30}$$

Considering the equation (4.27) applied on $P \circeq P^*(\hat{t})$ for an arbitrary chosen $\hat{t} \in (0, \tau)$, the coefficients $\widetilde{M_r}$ for a given pair (r, l) can be estimated if all the other terms of the PDE (4.27) are known at least approximately. Suppose the price of the bond is observable. Since the coefficient $\widetilde{M_l}$ is given by (4.30) and the coefficients of the second order derivatives are known, for the estimation of $\widetilde{M_r}$ only the approximation of the partial derivative terms are required. A method for this purpose is presented below.

Applying the Taylor expansion, the following holds[3]:

$$\frac{\partial P^*(\tau)}{\partial t} = \frac{-P^*(\tau - 2\Delta_t) + 4P^*(\tau - \Delta_t) - 3P^*(\tau)}{2\Delta_t} + O(\Delta_t^2). \tag{4.31}$$

Corollary 5. *i) Given a pair r, l and the bond curve $P^*(\hat{t}, r, l)$ for all $\hat{t} \in [0, \tau]$, the term*

$$\frac{\partial P^*(\tau)}{\partial t}$$

is approximated by the formula (4.31) in order $O(\Delta_t^2)$.

ii) The order of approximation also holds for $\widetilde{M_l}$.

□

Remark 15. *If the bond curve is given analytically, then the partial derivative of P with respect to t can be derived without approximation.*

□

[2]See chapter 2 for derivation.

[3]The bond curve $P^*(\hat{t})$ at time t is defined on the interval $[0, \tau]$ that is why the difference scheme is given in the asymmetrical form.

Suppose there are six pairs of $(r+\Delta_{i,r}, l+\Delta_{i,l})$ observed for $i=0,1,\ldots,5$, where $\Delta_{0,r}=\Delta_{0,l}=0$. Hence the bond prices $P_i(\hat{t}, r+\Delta_{i,r}, l+\Delta_{i,l})$ are also known for all $\hat{t}\in[0,\tau]$. Expanding P_i for $i=1,\ldots,5$ into Taylor series around P_0, the following is implied

$$P_i = P_0+\Delta_{i,r}\frac{\partial P_0}{\partial r}+\Delta_{i,l}\frac{\partial P_0}{\partial l}+\frac{1}{2}\Delta_{i,r}^2\frac{\partial^2 P_0}{\partial r^2}+\frac{1}{2}\Delta_{i,l}^2\frac{\partial^2 P_0}{\partial l^2}+\Delta_{i,r}\Delta_{i,l}\frac{\partial^2 P_0}{\partial r\partial l}+O(|\Delta_{i,r}^3|+|\Delta_{i,l}^3|). \tag{4.32}$$

introducing the following notation:

$$A \mathrel{\hat{=}} \begin{bmatrix} \Delta_{1,r} & \Delta_{1,l} & \Delta_{1,r}^2 & \Delta_{1,l}^2 & \Delta_{1,r}\Delta_{1,l} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \Delta_{5,r} & \Delta_{5,l} & \Delta_{5,r}^2 & \Delta_{5,l}^2 & \Delta_{5,r}\Delta_{5,l} \end{bmatrix}, \tag{4.33}$$

$$b \mathrel{\hat{=}} \begin{bmatrix} P_1-P_0 \\ \vdots \\ P_5-P_0 \end{bmatrix} \tag{4.34}$$

$$\delta b \mathrel{\hat{=}} \begin{bmatrix} O(|\Delta_{1,r}^3|+|\Delta_{1,l}^3|) \\ \vdots \\ O(|\Delta_{5,r}^3|+|\Delta_{5,l}^3|) \end{bmatrix}, \tag{4.35}$$

and

$$x \mathrel{\hat{=}} \begin{bmatrix} \frac{\partial P_0}{\partial r} & \frac{\partial P_0}{\partial l} & \frac{\partial^2 P_o}{\partial r^2} & \frac{\partial^2 P_0}{\partial l^2} & \frac{\partial^2 P_0}{\partial r\partial l} \end{bmatrix}^T. \tag{4.36}$$

Assuming non-singularity of A, the problem is reformulated into a linear system

$$Ax = b+\delta b. \tag{4.37}$$

Since δb is not known, the vector x is approximated by the solution $\hat{x}$ of the following system:

$$A\hat{x} = b, \tag{4.38}$$

and the error $\|x-\hat{x}\|$ in an arbitrary norm $\|.\|$ is given by

$$\|x-\hat{x}\| = \left\|A^{-1}\delta b\right\| \leq \left\|A^{-1}\right\| \|\delta b\|. \tag{4.39}$$

Thus the order of the error depends on the placement of the six pairs. The order relative error equals to

$$\frac{\|x-\hat{x}\|}{\|x\|} = \mathrm{cond}(A)O(\max(|\Delta_{i,r}^3|)+|\max(|\Delta_{i,l}^3|)) \tag{4.40}$$

where $\mathrm{cond}(A)$ denotes the conditional number of A with respect to the chosen norm $\|.\|$.

The method introduced above is summarised as follows.

Proposition 16. *Suppose that there exist five observable pairs* $(r + \Delta_{i,r}, l + \Delta_{i,l})$ *for* $i = 1, \ldots, 5$ *close around* (r, l) *such that the matrix* A *defined as in* (4.33) *is non-singular and the relative error* (4.40) *is small enough. Then by solving the equation* (4.37), *and substituting the estimates of the partial derivatives into* (4.27), *the value of the function* $\widetilde{M_r}$ *in the point* (r, l) *is well approximated.*

□

Remark 16. *The accurate approximation feature of the method derived above depends on the choice of points around the base point* (r, l) *to be involved in the calculations. In practice, one can assume, that there are great number of observations of market states* (r_i, l_i) *around the base point, and therefore the accuracy of approximation is ensured.*

□

Remark 17. *If the market is represented by* n *factors, and hence there are* n *risk-adjusted terms to be approximated, similar method can be constructed.*

The method. Suppose there are k economic relation type equations defined, given by k independent linear equations ($k \geq 0$). Choosing five different points $X_1, \ldots, X_5$ close around a base point X such the corresponding relative error defined by (4.40) is small enough[4], and choosing $n - k$ different time values $\hat{t}_i$ for $i = 1, 2, \ldots, n - k$, then the first and second order partial derivatives of the functions $P(\hat{t}_i)$ in the base point X is approximated by solving (4.37) type equations for all $P(\hat{t}_i)$.

Writing the pricing PDE on $P(\hat{t}_i)$ for $i = 1, 2, \ldots, n - k$, and substituting the approximation of the partial derivatives, we get $n-k$ linear equations. The k equations on the economic relations and the recently derived $n - k$ equations all together gives n linear equations on the n risk-adjusted drift terms. If the linear system has solution, then an approximation of the risk-adjusted drift term is the base point X is implied.

□

4.2.2 Interpolation of the pointwise risk-adjusted drift

Suppose that the error of the approximation method introduced above is kept under a common small limit in the points (r_i, l_i) for $i = 1, 2, \ldots, n$. Thus the risk-adjusted drift is well estimated in the specified points. The following methods are available for extension:

- linear interpolation (e.g. by the triangularisation of the domain),
- polynomial interpolation,
- spline interpolation.

[4]Without any specification of the economic relation type equations and the replacement of the chosen points, no precise order of error can be derived.

The method is summarised as below.

<table>
<tr><th colspan="2">Process of estimating the risk-adjusted drift terms</th></tr>
<tr><td colspan="2">Daily observation of $P^*(\hat{t})$ for all $\hat{t} \in [0, \tau]$.
in points (r_i, l_i) for $i = 0, 1, \ldots, 5$.</td></tr>
<tr><td>$\downarrow$
Calculating/ Estimating
the partial derivatives $\frac{\partial P^*(\tau)}{\partial t}$</td><td>$\downarrow$
Calibrating the factor processes
r and l</td></tr>
<tr><td>$\swarrow$
The economic type relation implies
a linear equation on $\widetilde{M}_l(r_0, l_0)$</td><td>$\searrow$
Given the Taylor expansion
in (r_i, l_i) $(i = 1, \ldots, 5)$, around (r_0, l_0)
for $P^*(\hat{t})$ for a fixed $\hat{t} \in [0, \tau]$
$\downarrow$
Five linear equations implied
on the partial derivatives
$\frac{\partial P^*}{\partial r}$, $\frac{\partial P^*}{\partial l}$,
$\frac{\partial^2 P^*}{\partial r^2}$, $\frac{\partial^2 P^*}{\partial l^2}$ and $\frac{\partial^2 P^*}{\partial r \partial l}$
$\downarrow$
Writing the estimates of the partial derivatives into the pricing PDEs of $P^*(\hat{t}_j)$
$\downarrow$
We get a linear equation
on $\widetilde{M}_r(r_0, l_0)$ and $\widetilde{M}_l(r_0, l_0)$</td></tr>
<tr><td colspan="2">$\downarrow$
Estimation of $\widetilde{M}_r(r_0, l_0)$ for all observed (r_0, l_0)
$\downarrow$
Extending $\widetilde{M}(r, l)$</td></tr>
</table>

Chapter 5

Closing thoughts and conclusions

In the first two chapters the mathematical introduction of financial markets and a general summary of the theory for pricing derivative products were presented. I discussed the relation between the pricing PDEs derived from two different argumentations. Additionally, the necessary conditions were given for the economic relation between the factors in special cases.

In the third chapter the slight generalisation of the calibration method presented by [Willmott 1998] was applied on a three-factor model of the Hungarian government bond market. The numerical calculations were based on economic considerations about the referred convergence phenomenon and on the historical daily yield curves estimated by the Raiffeisen Bank Rt.

In the last chapter, some practical applications of the pricing PDE were sketched. Firstly, in the case of an accurate definition of boundary conditions, the interest rate derivatives can be valued using PDE numerical methods, Secondly, a method for the pointwise calibration of the risk-adjusted drift terms based on the pricing PDE was introduced.

In spite of my original intentions, some unsolved problems, such as giving sufficient conditions for the economic relations in case of the pricing PDE, the general derivation of accurate boundary conditions for the interest rate derivatives pricing PDE, and precise development of extension of the pointwise estimation of the risk-adjusted drift terms remained. These might be the topic of a following study.

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