# CREDIT DEFAULT SWAPS

Diplomamunka

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Alkalmazott matematikus szak

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# Chapter 1

# Preface

A swap is a special contractual agreement between two parties, where they agree to make periodic payments to each other. Depending on the type of the swap, the agreement contains specifications regarding the currencies to be exchanged, the rate of interest applicable to each or the amount of commodities to be exchanged.

Risky corporate and sovereign bonds are among the most recent securities to benefit from the trading of associated derivative contracts. Credit derivatives are financial instruments that can be used to transfer credit risk from the investor exposed to the risk (the protection buyer) to an investor willing to assume that risk (the protection seller). Credit default swaps are the most liquid of the several credit derivatives currently traded and form the basic building blocks for more complex structured credit products.

This work provides methodologies for valuing credit default swaps. We explain how a plain vanilla CDS and basket credit default swap can be valued in different cases.

The paper is structured as follows:

First of all we say some words generally about the swap contracts.

In Chapter 2 we discuss the main risks connected to the swaps.

In the next chapter we write on the credit default swap (CDS), show a simple example, introduce the basic concepts and build up a methodology for valuing credit default swaps when the payoff is contingent on default by a single reference entity and there is no counterparty default risk.

We extend the analysis in Chapter 4. We focus on the CDS that takes account of counterparty default risk and allows the payoff to be contingent on default by multiple reference entities. A model of default correlations between different corporate or sovereign entities will be presented.

Finally we summarize our work and announce the data and valuation results.

# Chapter 2

# About Swap Contracts

## 2.1 What is a Swap?

A swap is a contractual agreement between two parties, called counterparties, where the counterparties agree to make periodic payments to each other. Depending on the type of the swap, the agreement contains specifications regarding the currencies to be exchanged (which may or may not be the same), the rate of interest applicable to each (which may be fixed or floating) or the amount of commodities to be exchanged. Moreover, the agreement contains specifications regarding the timetable by which the payments are to be made and any other provisions bearing on the relationship between the parties.

Since their inception in 1981, swaps have grown to a market of well over $6 trillion. Interest rate swaps have become common tools for interest rate risk management.

Swaps are by nature a series of forward contracts. This means that every payment is seen as an individual payment with a present value; this is though solemnly used in the process of pricing swaps (along the lines of pricing bonds).
The main characteristics of swap are,

- usually tailor-made (standardized in the future) and can be connected to almost any notional principal,
- brokers (matches the needs of clients) and dealers exist (puts himself 'on the line'),
- cheap and easy to use, versatile and keeps the amounts of money used small,

One can use swaps for hedging, speculation, arbitrage, cash flow management, trading and to enter new markets.

## 2.2 Risks in Swaps

Fullér [1] has collected the main risks connected to the swaps. The risks discussed below are mainly types of risk that arise in situations of speculation or by entering into several swaps in many different markets and thus taking several different type of position (i.e. holding entire portfolios of swaps). If we consider only one swap through which we fix our payments over time, the only risk will encounter is the opportunity risk i.e. what we could have done if we would have invested in another way. The main type of risks are the followings:

- *Interest rate risk as spread risk and market risk*
  Generally speaking the interest rate risk emerges as a result of the inverse relationship between the yield and the price of fixed-rate interest bearing debt, which consequently affects the debt management instrument. A change in the interest rate of a certain maturity will hence affect the instruments used to manage that debt; a decrease in the interest rate on debt might prove disastrous for a dealer whose strategy rests upon a steady swap spread over the underlying bond (used as the source for risk free yield in hedging operations). A no-reaction strategy and a reaction-but-not-enough-fast strategy puts the dealer in a position where he/she loses money as a result of the fluctuating interest rates. This specific interest rate risk resulting from yield curve movements is referred to as a spread risk.
  The interest rate risk referred as market risk is a form of risk that evolves during the lifetime of the swap. At the start of the swap the market value of the swap is zero; no-one has yet profited in any way of interest rate movements since and no-one has been exposed to any kind of risk against the counterparty. But at the instant the agreement is signed the contract becomes sensitive to interest rate movements. Then at the end of the time specified in the contract, for instance six months, the interest rates have moved in some direction or the other. Since the mere existence of a swap contract is dependent upon the counterparties having opposite views on market and interest rate movements, the end of the six-month period sees one counterparty owing the other one the difference between the contract specific fixed interest rate payment and the current contract specific floating interest rate index.

- *Currency/exchange rate risk*
  Currency risk is something that is a consequence of differences in the nominal currencies of the underlying interest bearing debt. In such a case the interest rate risk is accompanied by the currency risk. International transactions might easily concern

many different counterparties in many different countries whereby the currency risk becomes even more tangible and relevant to the actors involved. By using interest rate swaps or similar instruments to create hedges, one might lessen or totally eliminate the possible impacts of different currencies and different interest rate movements.

- *Credit default risk*

  Credit risk is defined as the probability of the counterparty in the swap agreement ceasing to exist, incorporating all the possible financial effects of the elimination. The reason for a counterparty becoming insolvent might be consequences of everything from bankruptcy to changes in the macroeconomic environment.

  When the total credit risk is asserted one has to take the current credit risk as well as the dynamics of this credit risk into account. This way one can create an understanding of the probability of default over an extended period of time as well as a notion of single and combinations of factors in the economic environment that might give birth to defaults and economic hardship.

- *Liquidity risk*

  This type of risk is characterized by the easiness by which one can transform the swap into liquid assets like cash. Consequently, this is highly dependent upon the structure of the secondary market for swaps as well as the structure of the swap itself and the independence of these two factors; the less developed secondary market, the harder it is to find a new counterparty in need of a contract under these specified conditions. The more tailor-made the contract is, again, the harder it is to find a new counterparty. This type of risk can be measured as a cost per time i.e.the longer it takes to find a new counterparty or in other ways dispose of the contract, the more costly the agreement becomes. The liquidity risk is thus consequently concerned with the difficulties of leaving a position without a price reduction.

- *Mismatch risk*

  Mismatch risk is a type of risk which evolves around differences in notional principal, maturity, the swap coupon, the floating index, the reset dates for the floating index and the payment frequencies between parallel agreements. As the number of agreements increase and they become more and more complex a treasurer faces an ever increasing risk that he might not be able to hedge every position taken by way of an identical agreement with opposite interest cash flows. This type of risk becomes even more eminent in context with credit risk; if mismatch has occurred as a result of differing payment dates and the treasurer pays a certain sum of money in six-month intervals while the counterparty pay on a yearly basis and the counterparty defaults, the dealer pays without receiving and loses thus the interest rate cash flow.

- *Basis risk*

  The basis is the difference between two prices; in the case of interest rate swaps it is the difference between two different floating-rate indexes. The basis risk arises in two different ways: first, suppose that a treasurer and a counterparty agree on a floating-floating interest rate swap in which the parties pay floating interest rate according to different floating-rate indexes like LIBOR and BUBOR. Second, in a matching pair of swaps a treasurer might pay according to one floating-rate index (for instance LIBOR) and receive according to another (for instance BUBOR). The risk arises as a result of the different characters of the two indexes; they fluctuate according to different economic environments.

- *Sovereign risk*

  The sovereign risk arises in cross-border interest rate swaps, i.e. in swaps that are concerned with parties in two different countries, and that thus reflect the countries financial standings in the world community and, to some degree, it is a function of the countries political stability. In general one could regard the sovereign risk as another aspect of credit risk with the exception that credit risk is specific for the counterparty while the sovereign risk is specific for the country in which the counterparty is operating. The sovereign risk might also be considered a political risk; while viewed in this manner the risk is given a more concrete size and shape in that it is not only international events that affect the investment climate but also national taxes, restrictions and other national policies. All these factors affect the price of the swap in that the higher the risk involved, the higher the price of the swap.

- *Delivery/settlement risk*

  The delivery risk, also called the settlement risk, exists when payments are made between counterparties who must effect their payments to each other at different times of the day owing to different settlement hours between the capital markets of the two parties. This most often occurs when payments are made between counterparties in two different countries.

- *Systematic risk*

  The systematic risk considers the probability that extensive disturbances that might affect other segments and institutions occur to the extent that the entire financial system crashes. This type of risk has its foundations in panic reactions and extensive loss of confidence in the current status quo as a consequence of a quickly changing reality. The construction of a method to comprise this kind of risk into the price of a swap is only relevant on a theoretical and philosophical level. In reality, if such a crisis would occur, the pricing strategy is most or less irrelevant to the outcome; this type of risk is always present in one way or another.

# Chapter 3

# Credit Default Swaps

The financial markets have recently developed securities that price directly the credit event called credit default swaps. The first such security was issued by Deutsche Bank in 2000. These securities payout only on the occurrence of a credit event, like a bond default and they have prices quoted in basis points as a percentage of a notional. In the USA trading in credit default swaps was facilitated by standard documentation produced by the International Swaps and Derivatives Association. Credit default swaps have become increasingly popular in recent years. Their purpose is to allow credit risks to be traded and managed in much the same way as market risks.

A CDS is a contract that provides protection against the risk of a credit event by a particular company or country. The company is known as the **reference entity** and a default by the company is known as **a credit event**. The buyer of the insurance obtains the right to sell a particular bond issued by the company for its par value when a credit event occurs. The bond is known as the **reference obligation** and the total par value of the bond that can be sold is known as the swap's **notional principal**. The buyer of protection makes periodic payments to the protection seller until the occurrence of a credit event or the maturity date of the contract, whichever is first. If a credit event occurs the buyer is compensated for the loss (possibly hypothetically) incurred as a result of the credit event.

A credit event usually requires a final accrual payment by the buyer. The swap is then settled by either physical delivery or in cash. If the terms of the swap require physical delivery, the swap buyer delivers the bonds to the seller in exchange for their par value. When there is cash settlement, the calculation agent polls dealers to determine the mid-market price, $Q$, of the reference obligation some specified number of days after the credit event. The cash settlement is then $\$(100 - Q)\%$ of the notional principal.

An example may help to illustrate how a typical deal is structured. Suppose that two parties enter into a five-year credit default swap on May 1, 2005. Assume that the notional principal is \$10 million and the buyer agrees to pay 90 basis points annually for protection against default by the reference entity. If the reference entity does not default (that is, there is no credit event), the buyer receives no payoff and pays \$90,000 on May 1 of each of the years 2006, 2007, 2008, 2009, and 2010. If there is a credit event a substantial payoff is likely. Suppose that the buyer notifies the seller of a credit event on November 1, 2008 (half way through the fourth year). If the contract specifies physical settlement, the buyer has the right to sell \$10 million par value of the reference obligation for \$10 million. If the contract requires cash settlement, the calculation agent would poll dealers to determine the mid-market value of the reference obligation a predesignated number of days after the credit event. If the value of the reference obligation proved to be \$35 per \$100 of par value, the cash payoff would be \$6.5 million. In the case of either physical or cash settlement, the buyer would be required to pay to the seller the amount of the annual payment accrued between May 1, 2008 and November 1, 2008 (approximately \$45,000), but no further payments would be required.

There are a number of variations on the standard credit default swap. In a **binary credit default swap**, the payoff in the event of a default is a specific dollar amount. In a **basket credit default swap**, a group of reference entities are specified and there is a payoff when the first of these reference entities defaults. In a **contingent credit default swap**, the payoff requires both a credit event and an additional trigger. The additional trigger might be a credit event with respect to another reference entity or a specified movement in some market variable. In a **dynamic credit default swap**, the notional amount determining the payoff is linked to the mark-to-market value of a portfolio of swaps.

In this chapter we explain how a plain vanilla and binary credit default swap can be valued assuming no counterparty default risk. Like most other approaches, we assumes that default probabilities, interest rates, and recovery rates are independent. Unfortunately, it does not seem to be possible to relax these assumptions without a considerably more complex model. The independence assumption will be discussed at the end of the chapter.

We test the sensitivity of the valuations to assumptions about the amount claimed in the event of a default and the expected recovery rate. We also test whether approximate no-arbitrage arguments give accurate valuations. In the next chapter, we will explain how the analysis can be extended to cover situations where the payoff is contingent on default by multiple reference entities and situations where there is counterparty default risk.

## 3.1 Estimation of Default Probabilities

The valuation of a credit default swap requires estimates of the risk-neutral probability that the reference entity will default at different future times. The prices of bonds issued by the reference entity provide the main source of data for the estimation. If we assume that the only reason a corporate bond sells for less than a similar Treasury bond is the possibility of default, it follows that:

$$\text{Value of Treasury Bond - Value of Corporate Bond} = \text{Present Value of Cost of Defaults}$$

By using this relationship to calculate the present value of the cost of defaults on a range of different bonds issued by the reference entity, and making an assumption about recovery rates, we can estimate the probability of the corporation defaulting at different future times. If the reference entity has issued relatively few actively traded bonds, we can use bonds issued by another corporation that is considered to have the same risk of default as the reference entity. This is likely to be a corporation whose bonds have the same credit rating as those of the reference entity – and ideally a corporation in the same industry as the reference entity.

We start with a simple example. Suppose that a five-year zero-coupon Treasury bond with a face value of 100 yields 5% and a similar five-year zero-coupon bond issued by a corporation yields 5.5%. Both rates are expressed with continuous compounding. The value of the Treasury bond is $100e^{-0.05\times 5} = 77.8801$ and the value of the corporate bond is $100e^{-0.055\times 5} = 75.9572$. The present value of the cost of defaults is, therefore

$$77.8801 - 75.9572 = 1.9229$$

Define the risk-neutral probability of default during the five-year life of the bond as $p$. If we make the simplifying assumption that there are no recoveries in the event of a default, the impact of a default is to create a loss of 100 at the end of the five years. The expected loss from defaults in a risk-neutral world is, therefore, $100p$ and the present value of the expected loss is

$$100pe^{-0.05\times 5}$$

It follows that:

$$100pe^{-0.05\times 5} = 1.9229$$

so that $p = 0.0247$ or 2.47%.

There are two reasons why the calculations for extracting default probabilities from bond prices are, in practice, usually more complicated than this. First, the recovery rate is usually non-zero. Second, most corporate bonds are not zero-coupon bonds. When the recovery rate is non-zero, it is necessary to make an assumption about the claim made by bondholders in the event of default. Jarrow and Turnbull [2] and Hull and White [3] assume that the claim equals the no-default of the bond. Sometimes it is assumed that the claim is equal to the value of the bond immediately prior to default. These assumptions do not correspond to the way bankruptcy laws work in most countries [4]. The best assumption is that the claim made in the event of a default equals the face value of the bond plus accrued interest.

As mentioned earlier, the payoff from a CDS in the event of a default at time $t$ is usually the face value of the reference obligation minus its market value just after time $t$. Using the best claim amount assumption just mentioned, the market value of the reference obligation just after default is the recovery rate times the sum of its face value and accrued interest. This means that the payoff from a typical CDS is

$$L - RL[1 + A(t)] = L[1 - R - A(t)] \tag{3.1}$$

where $L$ is the notional principal, $R$ is the recovery rate, and $A(t)$ is the accrued interest on the reference obligation at time $t$ as a percent of its face value.

### 3.1.1 A General Analysis Assuming Defaults at Discrete Times

We now present a general analysis that can be used in conjunction with alternative assumptions about the claim amount. We assume that we have chosen a set of $N$ bonds that are either issued by the reference entity or issued by another corporation that is considered to have the same risk of default as the reference entity. By the same risk of default we mean that the probability of default in any future time interval, as seen today, is the same. It is similar than in [5]. We assume that defaults can happen on any of the bond maturity dates. Later we will show a generalized analysis that allow defaults to occur on any date. Suppose that the maturity of the $i$th bond is $t_i$ with $t_1 < t_2 < t_3 ... < t_N$. Define:

- $B_j$: Price of the $j$th bond today.
- $G_j$: Price of the $j$th bond today if there were no probability of default (that is, the price of a Treasury bond promising the same cash flows as the $j$th bond).
- $F_j(t)$: Forward price of the $j$th bond for a forward contract maturing at time $t$ assuming the bond is default-free ($t < t_j$).
- $v(t)$: Present value of \$1 received at time $t$ with certainty.
- $C_j(t)$: Claim made by holders of the $j$th bond if there is a default at time $t$ ($t < t_j$).
- $R_j(t)$: Recovery rate for holders of the $j$th bond in the event of a default at time $t$ ($t < t_j$).

$\alpha_{ij}$: Present value of the loss, relative to the value the bond would have if there were no possibility of default, from a default on the $j$th bond at time $t_i$.

$p_i$: The risk-neutral probability of default at time $t_i$.

For ease of exposition, we first assume that interest rates are deterministic and that both recovery rates and claim amounts are known with certainty. We then explain how these assumptions can be relaxed.

Because interest rates are deterministic, the price at time $t$ of the no-default value of the $j$th bond is $F_j(t)$. If there is a default at time $t$, the bondholder makes a recovery at rate $R_j(t)$ on a claim of $C_j(t)$. It follows that

$$\alpha_{ij} = v(t_i)[F_j(t_i) - R_j(t_i)C_j(t_i)] \tag{3.2}$$

There is a probability, $p_i$ of the loss $\alpha_{ij}$ being incurred. The total present value of the losses on the jth bond is, therefore, given by:

$$G_j - B_j = \sum_{i=1}^{j} p_i \alpha_{ij} \tag{3.3}$$

This equation allows the $p$'s to be determined inductively:

$$p_j = \frac{G_j - B_j - \sum_{i=1}^{j-1} p_i \alpha_{ij}}{\alpha_{jj}} \tag{3.4}$$

### 3.1.2 Recovery Rate Assumption

These results have been produced on the assumption that interest rates are constant, recovery rates are known, and claim amounts are known. In what follows we will consider two assumptions about the claim amount. The first is that it equals the no-default value of the bond at the time of the default; the second is that it equals the face value plus accrued interest at the time of the default. It can be shown that, for either of these two assumptions, if a) default events, b) Treasury interest rates, and c) recovery rates are mutually independent, equations (3.2) and (3.3) are still true for stochastic interest rates, uncertain recovery rates, and uncertain default probabilities providing the recovery rate is set equal to its expected value in a risk-neutral world.

It is probably reasonable to assume that there is no systematic risk in recovery rates so that expected recovery rates observed in the real world are also expected recovery rates in the risk-neutral world. This allows the expected recovery rate to be estimated from historical data.

As mentioned earlier, the $N$ bonds used in the analysis are issued either by the reference entity or by another company that is considered to have the same risk of default as the reference entity. This means that the $p_i$ should be the same for all bonds. The recovery rates can in theory vary according to the bond and the default time. We will assume, for ease of exposition, that all the bonds have the same seniority in the event of default by the reference obligation and that the expected recovery rate is independent of time. The expected value of $R_j(t)$ is then independent of both $j$ and $t$. We will denote this expected value by $\hat{R}$.

### 3.1.3 Extension to Situation Where Defaults Can Happen at Any Time

The analysis used to derive equation (3.4) assumes that default can take place only on bond maturity dates. We now extend it to allow defaults at any time. Define $q(t)\triangle t$ as the probability of default between times $t$ and $t + \triangle t$ as seen at time zero. The variable $q(t)$ is not the same as the hazard (default intensity) rate. The hazard rate, $h(t)$, is defined so that $h(t)\triangle t$ is the probability of default between times $t$ and $t + \triangle t$ as seen at time $t$ assuming no default between time zero and time $t$. The variables $q(t)$ and $h(t)$ are related by

$$q(t) = h(t)e^{\int_0^t h(\tau)d\tau} \tag{3.5}$$

Some credit risk models (e.g. [2]) are formulated in terms of $h(t)$. However, we will follow the way of Hull and White [5] to express the results in terms of $q(t)$ rather than $h(t)$. We will refer to $q(t)$ as the **default probability density**.

We assume that $q(t)$ is constant and equal to $q_i$ for $t_{i-1} < t < t_i$. Setting

$$\beta_{ij} = \int_{t_{i-1}}^{t_i} v(t)[F_j(t) - \hat{R}C_j(t)]dt \tag{3.6}$$

a similar analysis to that used in deriving equation (3.4) gives:

$$q_j = \frac{G_j - B_j - \sum_{i=1}^{j-1} q_i\beta_{ij}}{\beta_{jj}} \tag{3.7}$$

The parameters $\beta_{ij}$ can be estimated using standard procedures, such as Simpson's rule, for evaluating a definite integral.

### 3.1.4 Claim Amounts and Value Additivity

We now present a numerical example and investigate the impact of different assumptions about the claim amount. As mentioned earlier, Jarrow and Turnbull [2] and Hull and White [3] assume that, in the event of a default, the bondholder claims the no-default value of the bond. This is an attractive assumption. It implies that $C_j(t) = F_j(t)$. The parameter, $\beta_{ij}$, is then proportional to $1 - \hat{R}$ so that equation (3.7) can be used to estimate $q_i(1 - \hat{R})$ directly from observable market variables. Furthermore, an analysis of equation (3.7) shows that, in this case, the value of the coupon-bearing bond $B_j$ is the sum of the values of the underlying zero-coupon bonds. This property is referred to as **value additivity**. It implies that it is theoretically correct to calculate zero curves for different rating categories (AAA, AA, A, BBB, etc) from actively traded bonds and use them for pricing less actively traded bonds.

As mentioned earlier, the best assumption is that $C_j(t)$ equals the face value of bond $j$ plus accrued interest at time $t$. As pointed out by Jarrow and Turnbull [4], value additivity does not apply when this assumption is made (except in the special case where the recovery rate is zero). This means that there is no zero-coupon yield curve that can be used to price corporate bonds exactly for a given set of assumptions about default probabilities and expected recovery rates.

Table 5.1 provides hypothetical data on six bonds issued by a reference entity. The bonds have maturities ranging from one to ten years and the spreads of their yields over Treasury yields are typical of those for BBB-rated bonds. The coupons are assumed to be paid semiannually, the Treasury zero curve is assumed to be flat at 5% (semiannually compounded), and the expected recovery rate is assumed to be 30%. Table 5.2 calculates the default probability densities for the two alternative assumptions about the claim amount. It can be seen that the two assumptions give similar results. This is usually the case. For the default probability densities to be markedly different, it would be necessary for the coupons on the bonds to be either very much greater or very much less than the risk-free rate.

### 3.1.5 Expected Recovery Rates and Bond Yields

Default probability densities must be greater than zero. From equation (3.7) this means that

$$B_j \leq G_j - \sum_{i=1}^{j-1} q_i \beta_{ij} \tag{3.8}$$

It is also true that the cumulative probability of default must be less than 1. This means that

$$\sum_{i=1}^{j} q_i (t_i - t_{i-1}) \leq 1$$

or

$$q_j (t_j - t_{j-1}) \leq 1 - \sum_{i=1}^{j-1} q_i (t_i - t_{i-1})$$

so that from equation (3.7)

$$B_j \geq G_j - \sum_{i=1}^{j-1} q_i \beta_{ij} - \frac{\beta_{jj}}{t_j - t_{j-1}} \Big[ 1 - \sum_{i=1}^{j-1} q_i (t_i - t_{i-1}) \Big] \tag{3.9}$$

Equations (3.8) and (3.9) impose both an upper and lower bound on the yield on the bond maturing at time $t_j$ once expected recovery rates and the yields on bonds maturing at earlier times have been specified. In the example in Table 5.1, when the expected recovery rate is 30%, a 20-year bond with a coupon of 7% must have a yield between 6.50% and 9.57% when the claim amount equals the face value plus accrued interest.

In general, one can use equations (3.8) and (3.9) to test whether a set of bond yields are consistent with the recovery rate assumption. Inconsistencies indicate that either the expected recovery rate assumption is wrong or bonds are mispriced.

## 3.2 The Valuation

We now move on to consider the valuation of a plain vanilla credit default swap with a $1 notional principal. We assume that default events, Treasury interest rates, and recovery rates are mutually independent. We also assume that the claim in the event of default is the face value plus accrued interest. Define

$T$: Life of credit default swap.

$q(t)$: Risk-neutral default probability density at time $t$.

$\hat{R}$: Expected recovery rate on the reference obligation in a risk-neutral world. As indicated in the previous section, this is assumed to be independent of the time of the default and the same as the recovery rate on the bonds used to calculate $q(t)$.

$u(t)$: Present value of payments at the rate of $1 per year on payment dates between time zero and time $t$.

$e(t)$: Present value of an accrual payment at time $t$ equal to $t - t^\star$ where $t^\star$ is the payment date immediately preceding time $t$.

$v(t)$: Present value of $1 received at time $t$.

$w$: Total payments per year made by credit default swap buyer.

$s$: Value of $w$ that causes the credit default swap to have a value of zero.

$\pi$: The risk-neutral probability of no credit event during the life of the swap.

$A(t)$: Accrued interest on the reference obligation at time $t$ as a percent of face value.

The value of $\pi$ is one minus the probability that a credit event will occur by time $T$. It can be calculated from $q(t)$:

$$\pi = 1 - \int_0^T q(t)dt$$

The payments last until a credit event or until time $T$, whichever is sooner. If a default occurs at time $t$ ($t < T$), the present value of the payments is $w[u(t) + e(t)]$. If there is no default prior to time $T$, the present value of the payments is $wu(T)$. The expected present value of the payments is, therefore:

$$w \int_0^T q(t)[u(t) + e(t)]dt + w\pi u(T)$$

Given our assumption about the claim amount, the risk-neutral expected payoff from the CDS is

$$1 - [1 + A(t)]\hat{R} = 1 - \hat{R} - A(t)\hat{R}$$

The present value of the expected payoff from the CDS is

$$\int_0^T [1 - \hat{R} - A(t)\hat{R}]q(t)v(t)dt$$

and the value of the credit default swap to the buyer is the present value of the expected payoff minus the present value of the payments made by the buyer or

$$\int_0^T [1 - \hat{R} - A(t)\hat{R}]q(t)v(t)dt - w \int_0^T q(t)[u(t) + e(t)]dt - w\pi u(T)$$

The CDS spread, $s$, is the value of $w$ that makes this expression zero:

$$s = \frac{\int_0^T [1 - \hat{R} - A(t)\hat{R}]q(t)v(t)dt}{\int_0^T q(t)[u(t) + e(t)]dt + \pi u(T)} \tag{3.10}$$

The variable $s$ is referred to as the **credit default swap spread** or **CDS spread**. It is the total of the payments per year, as a percent of the notional principal, for a newly issued credit default swap. Consider the data in Table 5.1 and suppose that the reference obligation is a five-year bond paying a semiannual coupon of 10% per annum with $\hat{R} = 0.3$. Equation (3.10) gives the value of $s$ for a five-year credit default swap with semiannual payments to be 1.944%. This is an annualized spread because of the way $w$ is defined. Payments equal to 0.972% of the CDS notional principal would be required every six months.

### 3.2.1 Approximate No-Arbitrage Arguments

There is an approximate no-arbitrage argument that can be used to understand the determinants of $s$. If an investor forms a portfolio of a $T$-year par yield bond issued by the reference entity and the credit default swap, the investor has eliminated most of the risks associated with default on the bond. If $y$ is the yield to maturity on the bond, the investor's net annual return is (at least, approximately) $y - s$. In the absence of arbitrage opportunities this should be (again, approximately) the $T$-year Treasury par yield, which we will denote by $x$. If $y - s$ is significantly higher than $x$, an arbitrageur will find it profitable to buy a $T$-year par yield bond issued by the reference entity, buy the credit default swap, and short a $T$-year par yield Treasury bond. If $y - s$ is significantly less than $x$, an arbitrageur will find it profitable to short a $T$-year par yield bond issued by the reference entity, sell the credit default swap, and buy a $T$-year Treasury par yield bond.

The argument just given suggests that $s$ should equal $y - x$. However, a close analysis of it shows that the arbitrage is less than perfect. Define:

$s^\star$: $y - x$

$L$: CDS notional principal.

$A^\star(t)$: The accrued interest as a percent of the face value at time $t$ on a $T$-year par yield bond that is issued at time zero by the reference entity with the same payment dates as the swap. We will refer to this bond as the **underlying par yield corporate bond**.

$R$: Realized recovery rate when a default happens.

We first consider the situation where the Treasury curve is flat and interest rates are constant. In this case the CDS spread is exactly $s^\star$ for a credit default swap where the payoff in the event of a credit event at time $t$ is $L[1 + A^\star(t)](1 - R)$. To see this, consider the position of an investor who buys both the credit default swap and an amount of the underlying corporate par yield bond with a face value of $L$ when the spread is $s^\star$. Using the notation above, $s^\star$ is the corporate par yield, $y$, minus the Treasury rate, $x$. The investor receives exactly the same cash flows as those from a Treasury par yield bond until either time $T$ or a credit event, whichever is earlier. If a credit event occurs at time $t$ $(0 < t < T)$, the investor has to make an accrual payment at time $t$ so that the net payoff from the CDS is

$$L[1 + A^\star(t)](1 - R) - L(y - x)(t - t^\star)$$

where as before $t^\star$ is the payment date immediately prior to time $t$. Because $A^\star(t) = y(t - t^\star)$, this reduces to

$$L[1 + x(t - t^\star)] - LR[1 + A^\star(t)]$$

The corporate bond holding is worth $LR[1 + A^\star(t)]$ so that the net value of the holding is

$$L[1 + x(t - t^\star)].$$

This is exactly what is required to buy a par yield Treasury bond with a face value of $L$ at time $t$. It follows that in all circumstances, the investor's portfolio exactly replicates the cash flows from the par yield Treasury bond showing that $s^\star$ must be the correct CDS spread. A spread greater than or less than $s^\star$ would give rise to an arbitrage opportunity.

We will refer to a CDS that provides a payoff of $[1 + A^\star(t)](1 - R)$ as an **idealized credit default swap**. Our analysis shows that the spread on such a CDS is exactly $s^\star$. In practice [5], the payoff from a credit default swap is usually $1 - R - A(t)R$ rather than $[1 + A^\star(t)](1 - R)$. This leads to $s^\star$ overestimating the true spread, $s$.

Continuing for a moment with the assumption that the Treasury curve is flat and interest rates are constant, we can correct for the difference between the payoff on the idealized CDS and the actual CDS. An analysis similar to that leading up to equation (3.10) shows that the spread for an idealized credit default swap is given by

$$s^\star = \frac{(1 - \hat{R}) \int_0^T [1 + A^\star(t)] q(t) v(t) dt}{\int_0^T q(t)[u(t) + e(t)]dt + \pi u(T)}$$

An approximation to this is

$$s^\star = \frac{(1 - \hat{R})(1 + a^\star) \int_0^T q(t) v(t) dt}{\int_0^T q(t)[u(t) + e(t)]dt + \pi u(T)} \tag{3.11}$$

where $a^\star$ is the average value of $A^\star(t)$ for $0 \le t \le T$. Similarly, from equation (3.10), an approximation to the actual CDS spread is

$$s = \frac{(1 - \hat{R} - a\hat{R}) \int_0^T q(t) v(t) dt}{\int_0^T q(t)[u(t) + e(t)]dt + \pi u(T)} \tag{3.12}$$

where $a$ is the average value of $A(t)$ for $0 \le t \le T$.

From equations (3.10) and (3.11)

$$s = \frac{s^\star(1 - \hat{R} - a\hat{R})}{(1 - \hat{R})(1 + a^\star)} \tag{3.13}$$

As an illustration of equation (3.13), consider the data in Table 5.1 and assume, as before, that the coupon on the reference obligation is 10%. (We will refer to this as Case A; see Table 5.3.) Suppose coupons are paid semiannually and all rates and yields are expressed with semiannual compounding in the whole Table 5.3. The five-year par yield for bonds issued by the reference entity is 7%. The five-year Treasury par yield is 5%. It follows that, for a five-year credit default swap with semiannual payments, $s^\star$ is 2.00%. The coupon paid every six months on a par yield bond issued by the reference entity is 3.5 per 100 of principal so that $a^\star = 0.0175$. Also $a = 0.025$ and $\hat{R} = 0.3$ so that equation (3.13) gives $s = 1.945\%$. This is very close to the 1.944% estimate reported earlier from using equation (3.10).

Equation (3.13) assumes a flat Treasury yield curve and constant interest rates.

Stochastic interest rates [6] make the no-arbitrage argument for the idealized CDS less than perfect, but do not affect valuations given our assumption that interest rates, default probabilities, and recovery rates are independent. However, the no-arbitrage argument for the idealized CDS swap requires a flat yield curve so that a par yield Treasury bond is always worth its face value plus accrued interest at the time of a default. An upward sloping yield curve will lead to the par yield Treasury bond being worth less than the face value plus accrued interest on average. As a result $s^\star$ underestimates the spread for the idealized CDS. Similarly a downward sloping yield curve leads to $s^\star$ overestimating the spread on the idealized CDS.

As pointed out by Duffie [7], we can deal with non-flat Treasury curves by considering par yield floating-rate bonds rather than par yield fixed-rate bonds. Define a Treasury par floater as a floating-rate bond where the interest rate is reset on each payment date of the credit default swap, and a par floater issued by the reference entity as a similar floating-rate bond that promises a prespecified spread above the Treasury par floater for the life of the credit default swap. If the payoff from the credit default swap is $[1 + A^\star(t)](1 - R)$ where $A^\star(t)$ is here defined as the accrual on the par floater issued by the reference entity, the arbitrage arguments are watertight and the CDS spread should exactly equal the spread of the reference entity floater over the Treasury floater.

In practice we rarely get the opportunity to observe the spreads on corporate par yield floaters. Credit default swaps must be evaluated from the yields on fixed rate bonds issued by the reference entity. The difference between the spread on par yield floaters and par yield fixed rate instruments is very small for flat term structures, but noticeable for non-flat term structures. As an extreme test of the effect of a non-flat term structure we changed the flat Treasury curve in Case A to a Treasury curve where the 1-, 2-, 3-, 4-, and 5-year par yields were 1%, 2%, 3%, 4%, and 5%, respectively. (We will refer to this as Case B; see Table 5.3.) Everything else, including the spreads between par yields on Treasuries and yields on bonds issued by the reference entity was maintained as in Case A. As a result, the five-year par yield for bonds issued by the reference was still 7% and $s^\star$ was still 2.00%. However, the value of $s$ given by equation (3.10) increased from 1.944% to 2.071%.

We also rarely get the chance to observe corporate bonds that are selling for exactly their par value. Assuming that the yield on a non-par-yield bond is the same as the yield on a par yield bond introduces some error. We tested this by changing the coupons on all bonds used to calculate default probabilities in Case A from 7% to 4% while keeping everything else (including the yield on the bonds) the same as in Case A. (We will refer to this as Case C; see Table 5.3.) The value of $s$ increased from 1.944 to 1.990. This change results entirely from the correct five year par yield being 7.048% rather than 7%. [1] For less creditworthy reference entities, the error from basing calculations on non-par-yield bonds can be much greater. Suppose Case A is changed so that the recovery rate is zero and the 1-, 2-, 3-, 4-, and 5-year yields on bonds issued by the reference entity are 10%, 20% 30%, 40% and 50%, respectively. (We will refer to this as Case D; see Table 5.3.) Assuming that the par yield is 50% and using equation (3.13) leads to an estimate of 40% for the value of $s$ ($s^\star = 45$, $a^\star = 0.125$, $a = 0.025$ and $\hat{R} = 0$). The correct value of $s$ given by equation (3.10) is 29.98%. This difference largely results from the correct par yield being about 38% rather than 50%.

### 3.2.2 Binary Credit Default Swaps

A binary credit default swap is structured similarly to a regular credit default swap except that the payoff is a fixed dollar amount. A similar analysis to that given earlier shows that the value a binary CDS spread that provides a payoff of $1 in the event of a default is

$$\frac{\int_0^T q(t)v(t)dt}{\int_0^T q(t)[u(t)+e(t)]dt + \pi u(T)}$$

### 3.2.3 The Independence Assumptions

The valuation approaches we have presented are based on the assumption that interest rates, default probabilities, and recovery rates are independent. These assumptions are unlikely to be perfectly true in practice. For example, it can be argued that high interest rates cause companies to experience financial difficulties and, as a result, default probabilities increase. Such a positive relation between interest rates and default probabilities has two effects. First, high default probabilities tend to be associated with high discount rates for the payoffs. This reduces the CDS spread. Second high default probabilities tend to be associated with relatively low market values for bonds issued by the reference entity. This increases the CDS spread (because it increases the value of the buyer's right to sell the reference bond for its face value). It is reassuring that these effects act in opposite directions so there is a partial offset. Note that the relevant correlation for the first effect is between default rates at time $t$ and the average short term interest rates between time zero and time $t$; the relevant correlation for the second effect is between interest rates at time $t$ and medium to long rates at time $t$. As far as the second effect is concerned, the

[1] Equation (3.13) provides an accurate estimate of $s$ when the correct par yield is used so that $s^\star = 2.048$, $a^\star = 0.01762$, $a = 0.025$ and $\hat{R} = 0.3$.

correlation is less than might be supposed because there are often significant time lags between the occurrence of high interest rates and the resultant defaults.

Moody's Investor's Service provides some statistics (see in [5]) which suggest that the correlations are small and provides a reasonable comfort level for the independence assumptions. These studies show that default rates are only weakly correlated with macroeconomic variables so we can reasonably hypothesize that the effect is small.

# Chapter 4

# Modelling Default Correlations

Now we will extend the model that was built up and explained in the previous chapter. The equation (3.10) is ofter used in reverse. Instead of CDS spreads being estimated from risk-neutral default probabilities and expected recovery rates, risk-neutral default probabilities are estimated from CDS spreads and expected recovery rates. These risk neutral probabilities are then used to value non-standard instruments.

In this chapter we will show the approach of Hull and White [8] for modelling default correlations. This allows us to reflect counterparty default risk in credit default swap valuations. It also allows us to handle instruments where the payoff is dependent on defaults by multiple reference entities.

There are two types of models of default risk in the literature: structural models and reduced form models.

The inspiration for structural models is provided by Merton [9]. He shows that the company's equity can be regarded as a European call option on its assets with a strike price of $D$ and maturity $T$. A default occurs at time $T$ if the option is not exercised. Merton's model has been extended later. The main drawback of traditional structural models is that they are not consistent with the risk-neutral probabilities of default backed out from corporate bond prices or CDS spreads.

Reduced form models focus on the risk-neutral hazard rate, $h(t)$, defined in the previous chapter. These models can incorporate correlations between defaults by allowing hazard rates to be stochastic and correlated with macroeconomic variables (see Duffie and Singleton [10]). Reduced form models are mathematically attractive. They can be made consistent with the risk-neutral probabilities of default backed out from corporate bond prices or CDS spreads. Their main disadvantage is that the range of default correlations that can be achieved is limited. Even when there is a perfect correlation between two hazard rates, the corresponding correlation between defaults in any chosen period of time is usually very low. This is liable to be a problem in some circumstances. For example, when two companies operate in the same industry and the same country or when the financial health of one company is for some reason heavily dependent on the financial health of another company, a relatively high default correlation may be warranted.

Hull and White's alternative approach is a natural development of the structural models of Merton. The model is exactly consistent with the risk-neutral default probabilities backed out from bond prices or CDS spreads. The default experience of large numbers of companies can be jointly simulated by sampling from multivariate normal distributions.

We first describe the model and then provide two applications. The first application is to vanilla swaps when there is counterparty default risk. The second is to basket credit default swaps.

## 4.1 The Model

We use the definitions of the **default probability density**, $q(t)$, and the **hazard default intensity rate**, $h(t)$ from the previous chapter. The two measures related by equation (3.5). They provide the same information about the default probability environment.

We assume that the risk-neutral default probability densities for $N$ companies have been estimated either from bond prices or CDS spreads. The key feature of this model is that there is a variable $X_j(t)$ describing the creditworthiness of company $j$ at time $t$ $(1 \leq j \leq N)$. We will refer to this variable as the **credit index** for company $j$. We can think of $X_j(t)$ in a number of ways. In the context of structural models, it can be regarded as some function of the value of the assets of the company $j$. Alternatively, we can imagine that the usual discrete credit ratings, produced by rating agencies, are replaced by continuous measures and that $X_j$ is some function of the measure for bonds issued by company $j$.

The objective is to select correlated diffusion processes for the credit indices of the $N$ companies and to determine a default barrier for each company such that the company defaults at time $t$ if its credit index first hits the default barrier at this time. We assume that $X_j(0) = 0$ and that the risk-neutral process for $X_j(t)$ is a Wiener process with zero drift and a variance rate of 1.0 per year. The usual measures of a firm's "credit quality" are of course conditionally non-normal. However, there is always some function of these measures that follows a Wiener process. The assumption that the credit indices follow Wiener processes can therefore be made without loss of generality.

The barrier must be chosen so that the first passage time probability distribution is the same as the default probability density, $q(t)$. As a first step, we discretise the default probability density so that defaults can happen only at times $t_i$ ($1 \leq i \leq n$). We define

$t_0 = 0$

$\delta_i = t_i - t_{i-1}$ $\quad (1 \leq i \leq n)$

$q_{ij}$: The risk-neutral probability of default by company $j$ at time $t_i$ ($1 \leq i \leq n$; $1 \leq j \leq N$).

$K_{ij}$: The value of the default barrier for company $j$ at time $t_i$ ($1 \leq i \leq n$; $1 \leq j \leq N$).

$f_{ij}(x)\triangle x$: The probability that $X_j(t_i)$ lies between $x$ and $x+\triangle x$ and there has been no default prior to time $t_i$ ($1 \leq i \leq n$; $1 \leq j \leq N$).

These definitions imply that the cumulative probability of company $j$ defaulting by time $t_i$ is

$$1 - \int_{K_{ij}}^{\infty} f_{ij}(x)dx$$

Both $K_{ij}$ and $f_{ij}(x)$ can be determined inductively from the risk-neutral default probabilities, $q_{ij}$. Based on the process for $X_j$, $X_j(t_1)$ is normally distributed with a mean of zero and a variance of $\delta_1$. As a result

$$f_{1j}(x) = \frac{1}{\sqrt{2\pi\delta_1}} exp\Big[-\frac{x^2}{2\delta_1}\Big] \tag{4.1}$$

and

$$q_{ij} = N\Big(\frac{K_{1j}}{\sqrt{\delta_1}}\Big) \tag{4.2}$$

where $N$ is the cumulative standard normal distribution function. This implies

$$K_{1j} = \sqrt{\delta_1}N^{-1}(q_{1j})$$

For $2 \leq i \leq n$, we first calculate $K_{ij}$. The relationship between $q_{ij}$ and $K_{ij}$ is

$$q_{ij} = \int_{K_{i-1,j}}^{\infty} f_{i-1,j}(u)N\Big(\frac{K_{ij}-u}{\sqrt{\delta_i}}\Big)du \tag{4.3}$$

Standard numerical methods can be used to set up a procedure for evaluating this equation for a given value of $K_{ij}$. An iterative procedure can then be used to find the value of $K_{ij}$ that solves the equation.

The value of $f_{ij}(x)$ for $x > K_{ij}$ is

$$f_{ij}(x) = \int_{K_{i-1,j}}^{\infty} f_{i-1,j}(u)\frac{1}{\sqrt{2\pi\delta_i}} exp\Big[-\frac{(x-u^2)}{2\delta_i}\Big]du \tag{4.4}$$

We solve equations (4.3) and (4.4) numerically. For each $i$ we consider $M$ values of $X_j(t_i)$ between $K_{ij}$ and $5\sqrt{t_i}$ (where $M$ is several hundred). We define $x_{ijm}$ as the $m$th value of $X_j(t_i)$ $(1 \leq m \leq M)$ and $\pi_{ijm}$ as the probability that $X_j(t_i) = x_{ijm}$ with no earlier default. The discrete versions of equations (4.3) and (4.4) are

$$q_{ij} = \sum_{m=1}^{M} (\pi_{i-1,j,m} N\Big(\frac{K_{ij} - x_{i-1,j,m}}{\sqrt{\delta_i}}\Big)$$

and

$$\pi_{i,j,n} = \sum_{m=1}^{M} \pi_{i-1,j,m} p_{ijmn}$$

where $p_{ijmn}$ is the probability that $X_j$ moves from $x_{i-1,j,m}$ at time $t_{i-1}$ to $x_{ijn}$ at time $t_i$. We set

$$p_{ijmn} = N\left[\frac{0.5(x_{i,j,n} + x_{i,j,n+1}) - x_{i-1,j,m}}{\sqrt{\delta_i}}\right] - N\left[\frac{0.5(x_{i,j,n} + x_{i,j,n-1}) - x_{i-1,j,m}}{\sqrt{\delta_i}}\right]$$

when $1 < n < M$. When $n = M$ we use the same equation with the first term on the right hand side equal to 1. When $n = 1$ we use the same equation with $0.5(x_{ijn} + x_{i,j,n-1})$ set equal to $K_{ij}$.

By increasing the number of default times, this model can be made arbitrarily close to a model where defaults can happen at any time. The default barrier is in general nonhorizontal; that is, in general, $K_{ij}$ is not the same for all $i$. This introduces some nonstationarity into the default process and is a price that must be paid to make the model consistent with the risk-neutral default probabilities backed out from bond prices or CDS spreads.

### 4.1.1 Data

The results in the rest of this chapter are based on the data in Table 5.4. This data shows credit spreads for AAA-, AA-, A-, and BBB-rated bonds. We assume that the recovery rates on all bonds is 30%, the risk-free zero curve is flat at 5% (with semiannual compounding), and that all the bonds pay a 7% coupon semiannually. Although credit ratings are attributes of bonds rather than companies, it will be convenient to refer to the companies issuing the bonds as AAA-, AA-, A-, and BBB-rated companies, respectively. Credit spreads vary through time. The spreads in Table 5.4 are designed to be representative of those encountered in practice. A few studies (e.g. [8]) in the literature are based on the same data.

The BBB data is the same as that we used in the previous chapter and leads to the default probability density in Table 5.5.

### 4.1.2 Default Correlations

Define $\rho_{jk}$ as the instantaneous correlation between the credit indices for companies $j$ and $k$. When $j$ and $k$ are public companies, we can assume that $\rho_{jk}$ is the correlation between their equity returns. When this is not the case, we can use other proxies. For example, when $j$ is a private company we can replace it by a public company that is in the same industry and geographical region for the purposes of calculating $\rho_{jk}$. When $j$ is a sovereign entity, we can use the exchange rate for the currency issued by the sovereign entity as a substitute for equity price when the $\rho_{jk}$'s are calculated. These proxies are less than ideal, but are used in practice. If reliable empirical estimates of default correlations were available the model could be calibrated to them with the correlation being perhaps a function of time. Alternatively, when the market for credit default swaps becomes sufficiently liquid, the correlations could be implied from the prices of the credit default swaps.

The default correlation between company $j$ and $k$ for the period between times $T_1$ and $T_2$ is usually defined as the correlation between the following two variables:

- A variable that equals 1 if company $j$ defaults between times $T_1$ and $T_2$ and zero otherwise;

- A variable that equals 1 if company $k$ defaults between times $T_1$ and $T_2$ and zero otherwise.

Define

$Q_j(T)$: The cumulative probability of a default by company $j$ between times 0 and $T$.

$P_{jk}(T)$: The probability that both company $j$ and $k$ will default between times 0 and $T$.

$\beta_{jk}(T)$: The default correlation between company $j$ and company $k$ for the period between times 0 and $T$.

It follows that

$$\beta_{jk}(T) = \frac{P_{jk}(T) - Q_j(T)Q_k(T)}{\sqrt{[Q_j(T) - Q_j^2(T)][Q_k(T) - Q_k^2(T)]}} \tag{4.5}$$

To calculate the default correlation, $\beta_{jk}(T)$, from the credit index correlation, $\rho_{jk}$, we can simulate the credit indices for companies $j$ and $k$ to calculate $P_{jk}(T)$ and equation (4.5) can then be used to obtain $\beta_{jk}(T)$. Table 5.6 shows the results of doing this for AAA, AA, A, and BBB companies. It illustrates that $\beta_{jk}(T)$ depends on $T$ and is less than $\rho_{jk}$. For a given value of $\rho_{jk}$, $\beta_{jk}$ increases as the credit quality of $j$ and $k$ decrease.

## 4.2 Calculation of CDS Spreads with Counterparty Credit Risk

In Chapter 3 we explained a method of Hull and White [5] how to value a CDS with a notional principal of \$1 when there is no counterparty default risk. Now we will show the method of [8] and we will explain the analysis that takes the possibility of a counterparty default into consideration. As before, we assume that default events, risk-free interest rates, and recovery rates are mutually independent. We also assume that a bondholder's claim in the event of a default equals the face value of the bond plus accrued interest. Define

- $T$: Life of credit default swap.
- $\hat{R}$: Expected recovery rate on reference obligation in the event of a default.
- $\theta(t)\triangle t$: Risk-neutral probability of default by reference entity between times $t$ and $t+\triangle t$ and no earlier default by counterparty.
- $\phi(t)\triangle t$: Risk-neutral probability of default by counterparty between times $t$ and $t+\triangle t$ and no earlier default by reference entity.
- $u(t)$: Present value of payments at the rate of \$1 per year on the CDS payment dates between time zero and time $t$.
- $e(t)$: Present value of an accrual payment on the CDS at time $t$ equal to $t-t^{\star}$ dollars where $t^{\star}$ is the CDS payment date immediately preceding time $t$.
- $v(t)$: Present value of \$1 received at time $t$.
- $w$: Total payments per year made by CDS buyer per \$1 of notional principal.
- $s$: Value of w that causes the credit default swap to have a value of zero. This is referred to as the CDS spread.
- $\pi$: The risk-neutral probability of no default by either counterparty or reference entity during the life of the credit default swap.
- $A(t)$: Accrued interest on the reference obligation at time $t$ as a percent of face value.

The CDS payments cease if there is a default by the reference entity or a default by the counterparty. If the reference entity defaults at time $t$ with no earlier default by the counterparty, there is a final accrual payment on the CDS so that the present value of all payments made is $w[u(t)+e(t)]$. If the counterparty defaults at time $t$ with no earlier default by the reference entity, we assume there is no final accrual payment so that the present value of all the payments made is $wu(t)$. If there is no default prior to time $T$ by either the counterparty or the reference entity, the present value of the payments is $wu(T)$. The expected present value of the payments is therefore

$$w\int_0^T [\theta(t)u(t)+\theta(t)e(t)+\phi(t)u(t)]dt + w\pi u(T)$$

If a credit event occurs at time $t$, the expected value of the reference obligation as a percent of its face value is $[1+A(t)]\hat{R}$. The expected payoff from the CDS is therefore

$$1-[1+A(t)]\hat{R} = 1-\hat{R}-A(t)\hat{R}$$

The present value of the expected payoff is

$$\int_0^T [1 - \hat{R} - A(t)\hat{R}]\theta(t)v(t)dt$$

and the value of the credit default swap to the buyer is the present value of the expected payoffs minus the present value of the payments the buyer will make, or

$$\int_0^T [1 - \hat{R} - A(t)\hat{R}]\theta(t)v(t)dt - w\int_0^T \big[\theta(t)u(t) + \theta(t)e(t) + \phi(t)u(t)\big]dt + w\pi u(T) \quad (4.6)$$

The CDS spread, $s$, is the value of $w$ that makes the expression (4.6) zero:

$$s = \frac{\int_0^T [1 - \hat{R} - A(t)\hat{R}]\theta(t)v(t)dt}{\int_0^T [\theta(t)u(t) + \theta(t)e(t) + \phi(t)u(t)]dt + w\pi u(T)} \quad (4.7)$$

The CDS spread can be calculated by evaluating both the numerator and denominator in equation (4.7) using simulation.

The credit index for both the reference entity and the counterparty must be simulated. If the reference entity defaults first (that is, the credit index for the reference entity falls below its default barrier before the credit index for the counterparty does so), payments continue up to the time of default with a final accrual payment and there is a payoff. If the counterparty defaults first (that is, the credit index for the counterparty falls below its default barrier before the credit index for the reference entity does so), payments continue up to the time of the default with no final accrual payment and no payoff. If neither the counterparty nor the reference entity default (that is, neither credit index reaches its barrier), payments continue for the life of the credit default swap and there is no payoff. If both sides default during the $i$th time period, we assume a 50% probability that the counterparty defaults first and a 50% probability that the reference entity defaults first.

In Chapter 3 we considered a CDS swap where

a) The life of the contract is five years

b) The buyer makers semiannual payments

c) The reference entity is rated BBB, as defined by Tables 5.4 and 5.5

d) The reference obligation lasts five years, pays a 10% coupon, and has a 30% recovery rate.

We showed that in the absence of counterparty default risk the CDS spread is 1.944%. Table 5.7 shows the spread for the same CDS when entered into with AAA, AA, A, and BBB counterparties. When the credit index correlation between the counterparty and the reference entity is zero, the impact of counterparty default risk is very small. But as the correlation increases and the credit quality of the counterparty declines, counterparty default risk has a bigger effect.

When the counterparty defaults, one option open to the purchaser of the CDS is to enter into new contract with a new counterparty to reinstate the default protection for the rest of the life of the original contract. If there is no correlation between the reference entity and the counterparty, the expected value of the reference entity credit index at the time of the counterparty default is its current value. If forward credit spreads are similar to spot credit spreads, the previous analysis shows that the CDS spread for the new contract should be similar to that for the original contract. This explains why the impact of counterparty default is small in the zero correlation case in Table 5.7.

### 4.2.1 Analytic Approximation When Default Correlations Are Known

Counterparty default risk reduces both the present value of the expected payoffs from a CDS and the present value of the purchaser's expected payments. To provide some insights into this, we now present a very simple analytic approximation for the change in the CDS spread when there is counterparty default risk. The approximation can be used when the default correlation between the reference entity and the counterparty has already been estimated – either directly from default data or in some other way. Define:

- $Q_r$: The probability of default by the reference entity between time 0 and $T$.
- $Q_c$: The probability of default by the counterparty between time 0 and $T$.
- $P_{rc}$: The joint probability of default by the counterparty and the reference entity between time 0 and $T$. (This can be calculated from $Q_r$, $Q_c$, and the default correlation using equation (4.5).)
- $g$: The proportional reduction in the present value of the expected payoff on the CDS arising from counterparty defaults.
- $h$: The proportional reduction in the present value of expected payments on the CDS arising from counterparty defaults.
- $\hat{s}$: The CDS spread assuming no counterparty default risk.

Counterparty default risk changes the CDS spread from $\hat{s}$ to $s$ where

$$s = \hat{s}\frac{1-g}{1-h} \tag{4.8}$$

The probability of a counterparty default during the life of the CDS conditional on the reference entity defaulting during the life of the CDS is $P_{rc}/Q_r$. We assume that there is a 0.5 chance that the counterparty default occurs before the reference entity defaults and a 0.5 chance that it occurs after the reference entity defaults. Ignoring discounting effects this implies

$$g = 0.5\frac{P_{rc}}{Q_r} \tag{4.9}$$

When the counterparty defaults, the payments made by the purchaser of the CDS may be less than they would be in the no-counterparty-default case. There is a probability of $Q_c - P_{rc}$ that the counterparty defaults and the reference entity does not default. Assume that, when this happens, the payments made by the CDS purchaser are half the average payments in the no-counterparty-default case. There is a probability of $P_{rc}$ that both the counterparty and the reference entity will default. As before we assume that there is a 50% chance that the counterparty default occurs first. We also assume that, when both default with the counterparty defaulting first, the payments made by the purchaser are one third less than in the no-counterparty-default case.[1] This leads to

$$h = \frac{Q_c - P_{rc}}{2} - \frac{P_{rc}}{6} = \frac{Q_c}{2} - \frac{P_{rc}}{3} \tag{4.10}$$

Equations (4.8), (4.9), and (4.10) suggest the following analytic approximation for the CDS spread is

$$s = \hat{s}\frac{1 - 0.5P_{rc}/Q_r}{1 - Q_c/2 + P_{rc}/3} \tag{4.11}$$

This result incorporates many courageous assumptions and approximations. It assumes that the probability of default by the reference entity is constant through the life of the CDS; it assumes that the probability of default by the counterparty is constant throughout the life of the CDS; it does not consider the impact of discounting effects on $g$ and $h$; it does not consider the impact of correlation on the relative timing of defaults by the reference entity and the counterparty; it ignores payment accrual issues; and so on. A much more complex analytic approximation would be required to deal with some of these points.

As a test of equation (4.11) we used the default correlations in Table 5.6 to estimate the numbers in Table 5.7. We considered situations where the company is BBB-rated. We tested the effect of the credit index correlation, the second company's credit rating and the length of the time period on the default correlation between the companies. The results are shown in Table 5.8. For the range of situations considered, the approximation appears to work reasonably well when the correlation is not too large. For example, when the credit index correlation is 0.4 or less, the analytic approximation is accurate to within 1.5 basis points. However, we stress that equation (4.11), and similar more complicated analytic approximations, can be used only when default correlations have already been estimated in some way.

[1] This assumption comes from the observation that, when defaults are equally likely at all times for both the counterparty and the reference entity and there is no default correlation, the average time between the two defaults is one third of the life of the CDS.

## 4.3 Basket Credit Default Swaps

In a basket credit default swap (sometimes called a first to default swap) a number of different reference entities and reference obligations are specified. The buyer makes payments in the usual way. The first reference entity to default triggers a payoff, either in cash or by physical delivery. As in the case of a regular CDS, the payoff typically equals $1 - R - A(t)R$ per dollar of principal where $R$ and $A(t)$ are the recovery rate and the accrued interest on the reference obligation for the defaulting reference entity. There are then no further payments or payoffs. As in the case of a vanilla credit default swap, a final accrual payment is usually required when there is a default.

When there is zero correlation between the reference entities and no counterparty default risk, a similar approach to [5] can be used to value a CDS or calculate the CDS spread. If $Q_{r,j}(t)$ $(1 \leq j \leq N)$ is the cumulative probability of the $j$th reference entity defaulting by time $t$, the probability of the first default happening between times $t_1$ and $t_2$ is

$$\prod_{j=1}^{N}[1 - Q_{r,j}(t_1)] - \prod_{j=1}^{N}[1 - Q_{r,j}(t_2)]$$

When the correlation between reference entities is non-zero, it is necessary to use a model such as the one we have explained in this chapter to value a basket credit default swap. Let us redefine variables as follows:

- $\theta(t)\triangle t$: Risk-neutral probability of the **first** default by a reference entity happening between $t$ and $t + \triangle t$ and no earlier default by the counterparty.
- $\phi(t)\triangle t$: Risk-neutral probability of the counterparty defaulting between times $t$ and $t + \triangle t$ and no earlier default by **any** of the reference entities.
- $\pi$: The risk-neutral probability of no default by the counterparty or **any** of the reference entities during the life of the CDS swap.
- $\hat{R}$: The expected recovery rate on the relevant reference obligation after first default.
- $A(t)$: Expected accrued interest as a percent of notional principal on the relevant reference obligation, conditional on the first default happening at time $t$.

Equations (4.6) and (4.7) then apply.

A basket CDS spread is calculated by evaluating both the numerator and denominator in equation (4.7) using simulation. The credit index for all reference entities and the counterparty must be simulated. If a reference entity defaults first (that is, the credit index for a reference entity reaches its default barrier before the credit index for the counterparty does), CDS payments continue up to the time of default with a final accrual payment and there is a payoff. If the counterparty defaults first (that is, the credit index for the counterparty falls below its default barrier before the credit index of any of the reference entities does), payments continue up to the time of the default with no final accrual payment and no payoff. If the credit indices for the counterparty and all reference entities stay above their respective default boundaries, payments continue for the life of the basket credit default swap and there is no payoff.

Table 5.9 shows results for a five-year basket credit default swap with semiannual payments where the counterparty is default-free. All reference entities are BBB-rated companies and the correlations between all pairs of reference entities are assumed to be the same. All reference obligations are assumed to be five-year bonds with 10% coupons and a 30% expected recovery rate. The table shows that the basket CDS spread increases as the number of reference entities increases and decreases as the correlation increases. The spread also decreases slightly as the expected recovery rate decreases.

# Chapter 5

# Tables, Figures

Table 5.1: Hypothetical Example of Bonds Issued by Reference Entity

| Bond Life (years) | Coupon (%) | Bond Yield (Spread Over Treasury Par Yield in bps) |
|---|---|---|
| 1 | 7.0 | 160 |
| 2 | 7.0 | 170 |
| 3 | 7.0 | 180 |
| 4 | 7.0 | 190 |
| 5 | 7.0 | 200 |
| 10 | 7.0 | 220 |

Table 5.2: Implied Probabilities of Default for Data in Table 5.1

| Time (years) | Default Probability Density (Claim = No-default Value) | Default Probability Density (Claim = Face Value + Accured Interest) |
|---|---|---|
| 0-1 | 0.0220 | 0.0219 |
| 1-2 | 0.0245 | 0.0242 |
| 2-3 | 0.0269 | 0.0264 |
| 3-4 | 0.0292 | 0.0285 |
| 4-5 | 0.0315 | 0.0305 |
| 5-10 | 0.0295 | 0.0279 |

Table 5.3: Cases Considered

| | Data Used | CDS Spread equation (3.10) | CDS Spread equation (3.13) |
|---|---|---|---|
| Case A | All Treasury rates are 5%; spreads and coupons on corporate bonds are as in Table 2; recovery rate is 30% | 1.944 | 1.945 |
| Case B | 1-, 2-, 3-, 4-, and 5-year Treasury par yields are 1%, 2%, 3%, 4%, and 5% respectively; spreads on corporate bonds are as in Table 5.1; recovery rate is 30% | 2.071 | 1.945 |
| Case C | All Treasury rates are 5%; spreads on corporate bonds are as in Table 5.1; coupons on corporate bonds are 4%; recovery rate is 30% | 1.990 | 1.945 |
| Case D | All Treasury rates are 5%; coupons on corporate bonds are as in Table 5.1; yields on 1-, 2-, 3-, 4-, and 5-year corporate bonds are 10%, 20%, 30%, 40%, and 50%; recovery rate is 0% | 29.98 | 40.00 |

Table 5.4: Spreads in Basis Points Between Corporate Bond Yields and Risk-free Bond Yields

| | Credit Rating | | | |
|---|---|---|---|---|
| Maturity | AAA | AA | A | BBB |
| 1 | 50 | 70 | 100 | 160 |
| 2 | 52 | 72 | 105 | 170 |
| 3 | 54 | 74 | 110 | 180 |
| 4 | 56 | 76 | 115 | 190 |
| 5 | 58 | 78 | 120 | 200 |
| 10 | 62 | 82 | 130 | 220 |

Table 5.5: Default Probability Density for a BBB-rated Company

| Time of Defalut (yrs) | Default Probability Density |
|---|---|
| 0-1 | 0.0219 |
| 1-2 | 0.0242 |
| 2-3 | 0.0264 |
| 3-4 | 0.0285 |
| 4-5 | 0.0305 |
| 5-10 | 0.0279 |

Table 5.6: Default Correlation of a BBB-Rated Company with a Second Company

| Time | Credit Index | Second | Company's | Credit | Rating |
|---|---|---|---|---|---|
| Period (yrs) | Correlation | AAA | AA | A | BBB |
| 2 | 0.0 | 0.00 | 0.00 | 0.00 | 0.00 |
| | 0.2 | 0.03 | 0.04 | 0.04 | 0.05 |
| | 0.4 | 0.09 | 0.10 | 0.11 | 0.12 |
| | 0.6 | 0.19 | 0.21 | 0.22 | 0.24 |
| | 0.8 | 0.35 | 0.37 | 0.40 | 0.43 |
| 5 | 0.0 | 0.00 | 0.00 | 0.00 | 0.00 |
| | 0.2 | 0.06 | 0.06 | 0.07 | 0.08 |
| | 0.4 | 0.14 | 0.15 | 0.16 | 0.18 |
| | 0.6 | 0.24 | 0.26 | 0.29 | 0.31 |
| | 0.8 | 0.39 | 0.42 | 0.47 | 0.50 |
| 10 | 0.0 | 0.00 | 0.00 | 0.00 | 0.00 |
| | 0.2 | 0.08 | 0.08 | 0.10 | 0.10 |
| | 0.4 | 0.17 | 0.18 | 0.21 | 0.22 |
| | 0.6 | 0.28 | 0.30 | 0.34 | 0.36 |
| | 0.8 | 0.41 | 0.45 | 0.51 | 0.55 |

Table 5.7: CDS Spreads in Basis Points for Different Counterparties and Different Correlations Between the Credit Indexes

| Credit Index | | Counterparty | Credit | Rating |
|---|---|---|---|---|
| Correlation | AAA | AA | A | BBB |
| 0.0 | 194.4 | 194.4 | 194.4 | 194.4 |
| 0.2 | 191.6 | 190.7 | 189.3 | 186.6 |
| 0.4 | 188.1 | 186.2 | 182.7 | 176.7 |
| 0.6 | 184.2 | 180.8 | 174.5 | 163.5 |
| 0.8 | 181.3 | 176.0 | 164.7 | 145.2 |

Table 5.8: Estimates of the CDS Spreads Using Equation 4.11

| Credit Index | | Counterparty | Credit | Rating |
|---|---|---|---|---|
| Correlation | AAA | AA | A | BBB |
| 0.0 | 194.0 | 193.9 | 193.7 | 193.2 |
| 0.2 | 191.0 | 190.2 | 190.2 | 185.6 |
| 0.4 | 186.7 | 184.8 | 184.8 | 175.8 |
| 0.6 | 181.0 | 177.7 | 171.7 | 163.2 |
| 0.8 | 173.5 | 168.1 | 158.5 | 145.3 |

Table 5.9: Basket CDS Spreads When Reference Entities ar all BBBs

| Expected | Credit Index | | Number of | Reference | Entities |
|---|---|---|---|---|---|
| Rec.Rate | Correlation | 1 | 2 | 5 | 10 |
| 0.1 | 0.0 | 196 | 390 | 959 | 1877 |
| | 0.2 | 196 | 376 | 848 | 1492 |
| | 0.4 | 196 | 357 | 730 | 1174 |
| | 0.6 | 196 | 332 | 604 | 888 |
| | 0.8 | 196 | 296 | 460 | 608 |
| 0.3 | 0.0 | 194 | 386 | 946 | 1842 |
| | 0.2 | 194 | 371 | 826 | 1441 |
| | 0.4 | 194 | 351 | 707 | 1122 |
| | 0.6 | 194 | 325 | 582 | 844 |
| | 0.8 | 194 | 289 | 444 | 580 |
| 0.5 | 0.0 | 192 | 380 | 925 | 1779 |
| | 0.2 | 192 | 363 | 794 | 1366 |
| | 0.4 | 192 | 342 | 672 | 1050 |
| | 0.6 | 192 | 315 | 551 | 786 |
| | 0.8 | 192 | 280 | 420 | 542 |

# Chapter 6

# Summary

Our aim was to give methodology for valuing credit default swaps and test the results on data. First we tried to show swaps. We wrote on the general properties of them and the most common risks connected to these contracts.

We explained the the function of credit default swap, introduce the basic concepts and build up a methodology for valuing credit default swaps when the payoff is contingent on default by a single reference entity and there is no counterparty default risk.

Then we extend the analysis. We focused on the CDS that takes account of counterparty default risk and allows the payoff to be contingent on default by multiple reference entities. A model of default correlations between different entities was presented.

We applied the mathematical formulas for valuing to hypothetical data and compared the results. Since it is hard to get real world data from the market directly we tried to use hypothetical data typical for data of real companies.

In our work we present different sources of literature. Since the CDS market is in dynamic growth the literature is growing as well. Later the analysis can be extended considering newly issued articles.

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