Calculation of Thom polynomials for minimal orbits

Diploma Thesis

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Contents

1. Introduction

The theory of Thom polynomials is part of an enormous area of algebraic topology called theory of obstructions. The word obstruction means '"obstacle to the existence of an object". A great variety of problems can be translated into the following type of question: when has a fiber bundle a section? In this case obstructions are cohomology elements of the base space so that if they do not vanish, then this is an obstacle: the bundle will not have any section.

The Thom polynomials show up when our fiber bundle is a vector bundle, and $\eta \subset V$ is a closed subset of the fiber, which is invariant under the structure group of the bundle. In this case it make sense to speak about η in an arbitrary fiber, and to ask whether there exist a section avoiding η at every point. There is a so called avoiding ideal of the cohomology ring of the base space, whose elements must vanish if the bundle admits such section, so elements of this ideal are obstructions, this ideal is an obstruction ideal. When the avoiding ideal has a generator, it is called Thom polynomial.
Determination of the avoiding ideal is usually slightly easier problem than finding the generator of it, which is also called *first obstruction*
By now theory of Thom polynomials has become more than a homotopy theoretic question; it connects several branches of mathematics, like singularity theory, representation theory, algebraic geometry, etc. For example the *degeneracy loci* formulas in algebraic geometry turned out to be Thom polynomials for group actions.

Calculation of Thom polynomials is not an easy task, despite the quite simple definition. By now there are three known efficient methods for calculating them. The first, classical way is the method of resolutions – this is also effective at computations of other invariants of singular varieties. The second, which turned out to be the most effective method is based on an idea of R. Rimányi and called the method of restriction equations. A standard reference of this is [**?**], and a nice summary of this method and its applications is [**?**].
This thesis deals with the third method, the so-called Duistermaat- Heckman formula, which is a well-known formula in algebraic topology. A special form of this formula makes us able to compute Thom polynomials for smooth invariant projective varieties.

Organization of the paper. This thesis can be divided into four parts. The first part consist of section §2, where we present the background and the notions of the theory. The second part (§3, §4, §5) deals with Lie groups and Lie algebras and gives a short summary of their representations. In §4 we present the basic facts about representations of the classical Lie groups, which we will use later in computations. In section §5 we introduce the subjects of our later examinations, namely the flag manifolds and minimal orbits: the new result of this thesis is the method of applying the Duistermaat-Heckman formula for these varieties.
In the third part (§6) we prove the Duistermaat-Heckman formula from the integral formula of Atiyah and Bott, and we discuss the connection between the affine and projective Thom polynomial.
In the last part (§7) we present the method of calculations of Thom polynomials for

minimal orbits, and give some examples. The new result here is the Thom polynomials of the orthogonal grassmannians and the spinor varieties, see **??**, **??**.

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2. Obstruction Theory, Theory of Thom polynomials

Let us start with some motivation and some words about the historical background and the origin of the theory.

2.1. **Motivation.** The name Thom polynomial comes from singularity theory, where R. Thom proposed the following question ([**?**])

Problem 2.1. *Given a smooth map $f : M \to N$ consider the points of M where f has singularity of type η, this is denoted by $\eta(f)$. What is the cohomology class $[\overline{\eta(f)}] \in H^*(M)$ defined via the Poincaré duality by the closure of $\eta(f) \subset M$?*

The simplest case of this problem is the following: *We have a smooth map $f : M \to N$. What is the Poincaré dual of the closure of points in M where df, the Jacobian of f has corank k?*

As Thom observed, this class can be expressed as a polynomial of characteristic classes of the vector bundles TM and f^*TN. We can see easily the connection between this problem and that of finding a section of a vector bundle avoiding an invariant closed subset. Let us see the most simple case, the so called Σ_k singularities. The following problem is the same that of Thom, if we use it for $E = TM$, $F = f^*TN$, and for the section df:

Problem 2.2. *Let E and F be complex vector bundles of complex dimension n and p over the manifold M, and let $s : E \to F$ be a vector bundle homomorphism, that is, a section of the bundle $HOM(E, F)$. Let $\Sigma_k(s)$ denote the set of $m \in M$ where the linear map $s(m)$ has corank k. What is the cohomology class $[\overline{\Sigma_k(s)}] \in H^*(M)$, the Poincaré dual of the closure of $\Sigma_k(s)$?*

Note, that $\overline{\Sigma_k(s)}$ is just $\cup_{j=k}^{n} \Sigma_j(s)$, since the closure of the points where the corank is k contains the points where the corank is greater than k.
Actually, the class $[\overline{\Sigma_k(s)}]$ is an obstruction for the existence of a section of the following fiber bundle: let we look at the subspace $\Sigma_{<k} \subset \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^p)$ containing matrices of corank smaller than k. $\Sigma_{<k}$ is an invariant subset for the group $G = GL(n, \mathbb{C}) \times GL(p, \mathbb{C})$ acting on $\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^p)$. So in each fiber F^x, $(x \in M)$ of the bundle $\mathrm{HOM}(E, F)$ we can define $\Sigma_{<k}^x \subset F^x$. A section s with $s(x) \in \Sigma_{<k}^x$ for every $x \in M$ is the same as $s(x) \notin \overline{\Sigma}_k^x$ for every $x \in M$. If there is a section s avoiding $\cup_{x \in M} \overline{\Sigma}_k^x$, then—since s induces an isomorphism between the cohomology of the total space and base space—$[\overline{\Sigma_k(s)}] \in H^*(M)$ must be zero, so this Poincaré dual is an obstruction.

We give the precise definitions and background to these notions in the next section. The problem was solved by Porteous, here are the formulas ([**?**])

Theorem 2.3. ***(Porteous)***[**?**]

$$[\overline{\Sigma_k(s)}] = det(A)$$

where $A_{i,j} = c_{i-j+k}$ for $i, j = 1, \ldots, k + p - n$ and $c_l = c_l(F - E)$ are the Chern-classes of the difference bundle $F - E = f^*TN - TM$.

Our goal is to find cohomological obstructions of sections of fiber bundles, that is, cohomology classes of the base space which have to be zero if the fiber bundle admits a section.

2.2. **The Borel construction, equivariant cohomology.** Let F be an arbitrary topological space, G a Lie group, and $\rho : G \to \mathrm{Homeo}(F)$ a (left) action of G on F with homeomorphisms (with diffeomorphisms if F is differentiable manifold, and in general the group of morphism of the given category) Then we call $F \to E \ \to B$ a *ρ-bundle*, if the structure group is given by ρ. When $F = V$ a vector space and $\rho : G \to GL(V)$, this is a vector bundle. When $F = G$ is a Lie group, and $\rho : G \to \mathrm{Homeo}(G)$ is the left multiplication this is called principal G-bundle. The total space of a principal G-bundle admits a free right G-action, because G acts on itself by right multiplication as well, and this commutes with the left multiplication. The factor by this free right action is the projection map of the bundle.

For a given bundle $\xi : F_\xi \to E_\xi \to B_\xi$ and a map $f : B \to B_\xi$ we can define $f^*\xi$, the *pull back* of ξ, which is a F_ξ-fiber bundle over B. Pull backs of homotopic maps $f \sim f' : B \to B_\xi$ are isomorphic. In a similar construction we get the *associated bundle*: if $\xi : G \to P \to M$ is a principal G-bundle, and $\rho : G \to \mathrm{Homeo}(F)$ is a left action, then the associated F-fiber bundle is $\rho_*\xi : F \to E \to M$ where

$$E = P \times_\rho F = (P \times F)/(p, f) \sim (pg^{-1}, gf)$$

Note that here we used the fact, that G acts from left on the fiber but on the right on the total space. We can do from a right action of G on a space X a left action by $xg \rightsquigarrow g^{-1}x$, so we can say, that we factored by the left diagonal action.

With association we can construct universal bundles. We call a ρ-bundle *universal ρ-bundle*, if every ρ-bundle is the pull back of this. If we want to get a universal ρ-bundle, first we have to build a universal principal G-bundle, which was done by Milnor:

Theorem 2.4. ***(Milnor)*** *For every Lie group G there exist universal principal G-bundle $G \to EG \to BG$. The space EG is contractible with a free (right) G-action on it, and $BG = EG/G$. Every universal principal G-bundle can be obtained from a contractible space, which has a free right G-action on the total space, and the base space is the factor space. Moreover the space BG is unique up to homotopy equivalence if we require to be a CW-complex. This is called the classifying space of G.*

If $\xi : G \to P \to M$ is a principal G-bundle then by there is a map $f : M \to BG$ with $f^*(EG) = \xi$, this is called *classifying map*. For example $BGL(n)$ is the Grassmannians manifold.
Now we get a universal ρ bundle where $\rho : G \to \mathrm{Homeo}(F)$ with association: $F \to B_G F = EG \times_\rho F \to BG$ is a universal ρ-bundle.
Note that the classifying space mapping $G \to BG$ is a functor from the category of Lie groups to the category of classifying spaces: if $H \to G$ is a Lie group homomorphism then we can define a map $Bf : BH \to BG$ in the following way: G has a natural left H-action given by $hg = f(h)g$, and Bf is the classifying map of the principal G-bundle $EH \times_H G \to BH$.
Analogously, forming the universal ρ bundles is a functor too, introduced by Borel:

Borel-construction:
For any G-space X we can construct $B_G X = EG \times_G X$, which is the total space of a bundle with fiber X over the classifying space BG associated to the universal

bundle $EG \to BG$. Note that wee will sometimes denote this Borel construction by $B_G X = X_G$.
This is a *covariant* functor between the category of G-spaces with morphisms the G-equivariant maps, and the Borel-construction of the corresponding spaces, so a G-equivariant map $f : X \to Y$ induce a map $Bf : B_G X \to B_G Y$, which is not well defined just the homotopy class of it, but if we fix the space BG, then Bf will be well-defined: it is the natural bundle map between $B_G X \to BG$ and $B_G Y \to BG$ over the same base space which is "the map f in each fiber".
Note, that $BG = EG \times_G pt$ is the Borel construction of the point.
Now

$$H_G^*(X) = H^*(B_G X)$$

is a G-equivariant cohomology theory, i.e. it is a *contravariant* functor from the category of G-spaces to graded rings. This is an extraordinary cohomology theory, that is, all axioms of cohomology is satisfied except for the axiom for the cohomology of a point, which can be rather complicated, as a theorem of Borel says, see Theorem **??**.Observe, that the cohomology group $\oplus H_G^i(X)$ is also a module over the ring $H_G^* = H_G^*(pt) = H^*(BG)$ where the module structure is induced by the projection $\pi : BV \to BG$, so $\alpha\sigma := \pi^*(\alpha)\sigma$ for $\alpha \in H_G^*$. The usual properties of ordinary cohomology remain valid, so for a G-equivariant map $X \to Y$ we have $H_G^*(Y) \to H_G^*(X)$, we have exact sequences, excision, etc.

We have a natural map σ of $B_G M$ onto the quotient space M/G, which fits into the following diagram of Cartan and Borel:

$$\begin{tikzcd}
EG \arrow[d] & EG \times M \arrow[l] \arrow[r] \arrow[d] & M \arrow[d] \\
BG & (EG \times_G M) \arrow[l, "\pi"'] \arrow[r, "\sigma"] & M/G
\end{tikzcd}$$

In general σ is not a fibering. In fact, it is easy to see that the fiber $\sigma^{-1}(Gm)$ over the orbit Gm is the quotient of EG by the stabilizer G_m of m at the right action of G on EG. So $\sigma^{-1}(Gm)$ is the classifying space of G_m:

$$B_G\eta = \sigma^{-1}(Gm) = EG/G_m \simeq BG_m$$

where $\eta = Gm$ is the orbit of m in M, and trivially $\sigma^{-1}(\eta) = B_G\eta$. So we have $H_G^*(\eta) = H^*(BG_\eta)$ for any orbit of M. Note that this equation includes only the orbit η, so we have

Theorem 2.5. *If the Lie group G acts transitively from the left on the space η, then $H_G^*\eta \cong H_{G_x}^* pt = H^*(BG_x)$ where G_x is the stabilizer group of arbitrary point of η.*

If G is a compact Lie-group acting smoothly on M, the map σ induces homotopy equivalence whenever G acts freely, so that $B_G M \simeq M/G$ in that case. So the map

$$\sigma^* : H^*(M/G) \to H_G^*(M)$$

is an isomorphism if G acts freely. In general, $B_G M$ can be considered as the "homotopy theoretic quotient" of M by G.
Let H be a closed subgroup of G, then G/H is an orbit at the natural action of G with stabilizer H, so we get as above:

$$H_G^*(G/H) = H_H^* = H^*(BH).$$

There is a natural homomorphism

$$i^* : H^*_G(M) \to H^*(M)$$

from equivariant theory to ordinary theory which corresponds to the inclusion $i : M \hookrightarrow B_G M$ of M as the fiber over a point of BG.

Although the equivariant theory is largely analogous to the ordinary theory, its overtones are rather different, precisely because the equivariant theory of a "point" is so large. Observe, that in the equivariant theory the "points" of M/G really correspond to orbits of G on M, and hence are of various "sizes", according to the size of the stabilizer group. Thus the free orbits carry only zero-dimensional equivariant cohomology, while the point orbits stand at the other extreme and their cohomology is H^*_G.

2.3. **Obstruction ideals.** The standard reference of this subsection is [?], we go more or less along this reference.
Let us suppose that a Lie group G act on the fiber F, and $F \to E \to M$ be a fiber bundle associated to a principal G-bundle $G \to P \to M$ – i.e. $E = P \times_G F$. We are mainly interested in the case when G is a semisimple Lie group and F is a G-manifold.

Our starting point is the following well-known observation:

Observation 2.6. *Sections of the fiber bundle $E = P \times_G F$ are in one to one correspondence with G equivariant maps from P to F. (We have left action on F, right action on F, which can be transformed into a left action.)*

In other words in the category of left G-spaces a section s of E is a lift of the collapse map $pt_P : P \to pt$:

$$\begin{tikzcd} & F \arrow[d, "pt_F"] \\ P \arrow[ur, "s"] \arrow[r, "pt_P"'] & pt \end{tikzcd}$$

Suppose now that h is any contravariant functor from the category of (left) G-spaces into (graded) rings (the main example is G-equivariant cohomology). Then a section s induces a commutative diagram:

$$\begin{tikzcd} & h(F) \arrow[dl, "h(s)"'] \\ h(P) & h(pt) \arrow[u, "h(pt_F)"'] \arrow[l, "h(pt_P)"] \end{tikzcd}$$

So the existence of the section s implies that

$$\ker h(pt_F) \subset \ker h(pt_P). \qquad (1)$$

We will use the following notation: $\mathcal{O}_F = \ker h(pt_F)$ is the *obstruction ideal* of F. Now we can reformulate (??) in the following way:

Theorem 2.7. *If the fiber bundle $E = P \times_G F$ admits a section then all characteristic classes of P from the obstruction ideal $\mathcal{O}_F$ vanish.*

So the goal of a general theory would be the calculation of $\mathcal{O}_F$ for G-spaces F and G-equivariant cohomology theories, and from now on we deal with the introduced cohomology H_G.
Observe, that $B_G P = EG \times_G P \cong P/_r G = M$, since the right action of G on the total

space of a principal G-bundle is free. Moreover, $Bpt_P : M \to BG$ is the classifying map of the principal G-bundle $P \to M$. So after using the Borel construction we get from the diagram above:

$$\begin{tikzcd} & B_G F \arrow[d, "Bpt_F"] \\ B_G P = M \arrow[ur, "Bs"] \arrow[r, "Bpt_P"'] & Bpt = BG \end{tikzcd}$$

and

$$\begin{tikzcd} & H^*_G(F) \arrow[dl, "H_G(s)"'] \\ H^*_G(P) = H^*(M) & H^*_G(pt) = H^*(BG) \arrow[l, "H_G(pt_P)"] \arrow[u, "H_G(pt_F)"'] \end{tikzcd}$$

So in this case $\mathcal{O}_F \subset H^*(BG)$. Suppose now, that F is $d-2$ connected, and $\pi_{d-1}(F) \cong \mathbb{Z}$. Then if we have a triangulation of M with small simplices (so the bundle over every simplex is trivial) then we have a section over the $d-1$-skeleton, and the first obstruction sits in $H^d(M, \pi_{d-1}(F)) = H^d(M, \mathbb{Z})$. The vanishing of the first obstruction is a sufficient and necessary condition for the existence of a section over the d-skeleton of the bundle. For the universal F-bundle $EG \times_G F \to BG$, the first obstruction is an element of $H^d(BG, \mathbb{Z})$. Let we denote the first obstruction of the bundle E by $fo(E)$ and that of the universal bundle by $fo(F)$. If k is the classifying map of E, that is

$$\begin{tikzcd} E = P \times_G F \arrow[r, "K"] \arrow[d, "F"] & B_G F \arrow[d, "F"] \\ M \arrow[r, "k"] & BG \end{tikzcd}$$

then $k^* fo(F) = fo(E)$ (that is, the first obstruction is natural). So it is enough to define it for the universal case, and we do it after Fuchs, Fomenko and Gutenmacher (see [**?**]) with the aid of the spectral sequence of the bundle $B_G F \to BG$. In the E_2 table of this $E_2^{p,0} = H^p(BG)$, $E_2^{p,q} = 0$ for $1 \le q \le d-2$ (since F is $d-2$-connected), and $E_2^{0,d-1} = H^{d-1}(F) = \pi_{d-1}(F) = \mathbb{Z}$. Let us denote the boundary homomorphism from $E_*^{0,d-1}$ to $E_*^{d,0}$ by f. We have a natural homomorphism $e_p : E_2^{p,0} \to E_\infty^{p,0} \to H^p(B_G F)$ which is called *edge homomorphism.* Now the edge homomorphism e_d is quite simple: the kernel of it is just the image of f, so $\operatorname{Im} e_d = \ker f$. It can be seen easily that $e_d = \pi_d^*$ where $\pi_d^* : H^d(BG) \to H^d(B_G F)$ is induced by the projection. Let $\mathbf{1} \in \mathbb{Z} = E_2^{0,d-1}$ be the generator, then we define the first obstruction as $fo(F) := f(\mathbf{1}) \in H^d(BG)$. It can be proved, that it has the properties described above. Since $fo(F)$ is the generator of $\ker \pi_d^* = \mathcal{O}_F \cap H^d(BG)$, by definition we have

Lemma 2.8. *The first obstruction $fo(F)$ is the generator of the d-dimensional part of the obstruction ideal $\mathcal{O}_F^d = \mathcal{O}_F \cap H^d(BG)$. Moreover $\mathcal{O}_F^i = 0$ for $i < d$. (Similar statement holds for $fo(E)$ as well)*

The last statement of the lemma comes in the following way: if $\mathcal{O}_F^j \neq 0$ for some $j < d-1$ then E has no section, and it has no section over sk_jM either (because $i_j^* : H^j(M) \to H^j(sk_jM)$ is injective) which is a contradiction, since $\pi_{j-1}(F) = 0$, so every section can be extended over sk_jM.

Now let us suppose, that we have an open embedding of F into a contractible G-space V. (In most applications F is given as an open subset of a vector space.) Let s be a generic section, and

$$F^c(s) := \{m \in M : s(m) \notin F\}$$

We cannot say, that $F^c(s)$ represents a cohomology element in the usual way, because (for example) it is not smooth in general. So we have to extend the notion of Poincaré dual in the following way, see [**?**]

Definition 2.9. [**?**] *Let X be a closed subset of a topological space Y. X represents the cohomology class $[X] \in H^d(Y;\mathbb{Z})$ if for the inclusion $i : Y \setminus X \to Y$*

$$\ker H^j(i) \cong \begin{cases} 0 & \text{if } j < d \\ \mathbb{Z} & \text{if } j = d \end{cases}$$

Then $[X]$ is the generator of $\ker H^d(i)$.

The class $[X]$ is defined up to sign, we can call the two choices orientations. If $f : X \to Y$ is a proper map of smooth oriented manifolds, then Thom isomorphism for the normal bundle of X shows that X represents the ordinary Poincaré dual of X, so this is indeed a generalization. Moreover, if $A \subset X$ is a complex algebraic subvariety, then A will always represent a dual class in X, and we will use this case of the definition in every case.

If we use this definition for $X = P \times_G (V \setminus F) \subset P \times_G V$ and $Y = P \times_G V$ then since $H^*(P \times_G V) = H^*(M)$ we get that this generalized Poincaré dual is exactly the first obstruction: $[P \times_G (V \setminus F)] = fo(E)$.
Now by the naturality property of the Poincaré dual we have the following, see [**?**]

Theorem 2.10. *Suppose that the Lie group G acts on the vector space V and F is a G-invariant open subspace of V. Assume moreover that $\pi_i(F) = 0$ for $i < d-1$ and $\pi_{d-1}(F) \cong \mathbb{Z}$ and that for the principal G-bundle $P \to M$ the cohomology class $fo(P \times_G F)$ is not zero and $F^c(s)$ represents a cohomology class in $H^d(M;\mathbb{Z})$. Then $fo(P \times_G F) = \pm[F^c(s)]$.*

2.4. **The definition of Thom polynomials.** Thom polynomials are special first obstructions. Given a representation $\rho : G \to GL(V)$ we are looking for obstructions of having a section of a V-bundle associated to this representation *avoiding* a certain orbit η (or more generally a G-invariant subset of V). In effect we want to avoid the *closure* of η. In our examples G will be a complex Lie group acting on the complex vector space V, and $\eta \subset V$ is a G-invariant algebraic subvariety of V. Then the assumptions stand for $V \setminus \eta$, so it is highly connected by the following observation (see [**?**])

Observation 2.11. *Let $X \subset \mathbb{C}^N$ be a d (real) codimensional complex algebraic variety. Then*

$$\pi_i(\mathbb{C}^N \setminus X) \cong \begin{cases} 0 & \text{if } i < d-1 \\ \mathbb{Z} & \text{if } i = d-1. \end{cases}$$

(If $d = 2$ then π_1 should be replaced by H_1.)

Remark 2.12. Similar theorem holds for real varieties with some extra condition. However the closure of an orbit of a real representation is not necessarily a real algebraic variety, the algebraic closure can contain some other orbits with the same codimension.

Now we can define Thom polynomials:

Definition 2.13. *(Thom polynomial) For an invariant complex algebraic subvariety of the complex representation $\rho : G \to GL(V)$ we define*

$$\mathrm{Tp}(\eta) = fo\big(EG \times_G (V \setminus \overline{\eta})\big).$$

In other words $\mathrm{Tp}(\eta)$ is a generator of the d-dimensional part of the so called *avoiding ideal*, which is simply $\mathcal{O}_{V \setminus \overline{\eta}}$.

Now the assumptions of **??** hold, so we have

Lemma 2.14. *Let $\eta \subset V$ be an invariant complex algebraic subvariety of the complex representation $\rho : G \to GL(V)$, where G is a complex Lie group. Then*

$$Tp(\eta) = [\overline{B_G \eta}] \in H^*(BV) = H^*(BG).$$

Remark 2.15. Thom polynomials are generalizations of the Euler class, because by definition $Tp(0 \in V) = e_G(V)$, where $e_G(V)$ denotes the Euler class of $BV \to BG$.

3. Representations of Lie groups and Lie algebras

In this section we give a short summary of the representations of Lie groups and Lie algebras. We will focus on Lie algebras, which will use quite a lot later. In this section we follow more or less the structure of the work [?] We deal only with complex Lie groups and Lie algebras.
A representation of a Lie group G on the complex vector space V is simply a homomorphism of Lie groups $G \to GL(V)$. A representation of a Lie algebra $\mathfrak{g}$ on a complex vector space V is a Lie algebra (i.e Lie-bracket preserving) map $\mathfrak{g} \to \mathfrak{gl}(V) = End(V)$. A fundamental theorem says, that representations of a connected, simply connected Lie group are in one-to one correspondence with representations of its Lie algebra: by associating $\rho : G \to GL(V)$ its differential $d\rho : \mathfrak{g} \to \mathfrak{gl}(V)$. We can form tensor products, symmetric powers, exterior powers of representations in a natural way.

A Lie algebra is simple, if it contains no nontrivial ideals. The *lower central series* of subalgebras defined by $\mathfrak{D}_1\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, and $\mathfrak{D}_k\mathfrak{g} = [\mathfrak{g}, \mathfrak{D}_{k-1}\mathfrak{g}]$, and the *derived series* by $\mathfrak{D}^1\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{D}^k\mathfrak{g} = [\mathfrak{D}^{k-1}\mathfrak{g}, \mathfrak{D}^{k-1}\mathfrak{g}]$.

Definition 3.1. *A Lie algebra is*

- *(i) nilpotent, if $\mathfrak{D}_k\mathfrak{g} = 0$ for some k;*
- *(ii) solvable, if $\mathfrak{D}^k\mathfrak{g} = 0$ for some k;*
- *(iii) perfect, if $\mathfrak{D}\mathfrak{g} = \mathfrak{g}$;*
- *(iv) semisimple, if $\mathfrak{g}$ has no nonzero ideals;*
- *(v) the radical $Rad(\mathfrak{g})$ of $\mathfrak{g}$ is the maximal solvable ideal in $\mathfrak{g}$.*

  We say that a Lie group has one of this properties, if its Lie algebra has.

The standard example for nilpotent and solvable Lie algebra is the algebra of strictly upper-triangular $n \times n$ matrices and Lie algebra of upper-triangular matrices, respectively. Every Lie algebra fits into an exact sequence

$$0 \to Rad(\mathfrak{g}) \to \mathfrak{g} \to \mathfrak{g}/Rad(\mathfrak{g}) \to 0$$

where the first algebra is solvable, the last is semisimple. Now by Lie's theorem an irreducible representation of a solvable Lie algebra is one dimensional, and the following consequence of the exact sequence says, that it is enough to know the representations of semisimple Lie algebras:

Theorem 3.2. *If $\mathfrak{g}$ is a complex Lie algebra, then every irreducible representation of $\mathfrak{g}$ is of the form $V = V_0 \otimes L$ where V_0 is an irreducible representation of $\mathfrak{g}/Rad(\mathfrak{g})$ and L is a one-dimensional representation of $\mathfrak{g}$.*

Complete reducibility – that is, every representation decomposes into a sum of irreducible representations – is not true in general, but for semisimple Lie group it is true. One more note about this classification: we can describe all the simple Lie algebras. With five exceptions every simple complex Lie algebra is isomorphic to either ${}_n\mathbb{C}$, $\mathfrak{so}_n\mathbb{C}$, $\mathfrak{sp}_{2n}\mathbb{C}$, these are the *classical Lie algebras.* The five exceptions are denoted by $\mathfrak{g}_2, \mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7$ and $\mathfrak{e}_8$, these are the *exceptional Lie algebras.*

For every Lie algebra $\mathfrak{g}$ there is a unique simply connected Lie group $\tilde{G}$ with this Lie algebra (called *simply connected form*) and all other complex Lie groups with Lie algebra $\mathfrak{g}$ are quotients of G by a discrete subgroup of the center $Z(G)$. ($G/Z(G)$ is called the *adjoint form.*) For any representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ corresponds a representation $\tilde{\rho}$ of $\tilde{G}$ by $\tilde{\rho}(exp(X)) = exp(\rho(X))$. For any other group $G = \tilde{G}/C$ with $C < Z(G)$ the representations of G are the representations of $\tilde{G}$ trivial on C. We will turn back to the representations of Lie groups, but we first start with Lie algebras.

3.1. **Representations of Lie algebras.** (reference:[**?**])

Let $\mathfrak{g} = LieG$ be a semisimple Lie algebra, and let $\mathfrak{h} \subset \mathfrak{g}$ be its Cartan-subalgebra, that is, the maximal abelian subalgebra of $\mathfrak{g}$, then any action of $\mathfrak{h}$ on any representation of $\mathfrak{g}$ will be diagonalizable. Let $\mathfrak{h}$ act on $\mathfrak{g}$ by the adjoint representation, and decompose $\mathfrak{g}$ accordingly, this is the Cartan-decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus (\oplus \mathfrak{g}_\alpha)$$

where the action of $\mathfrak{h}$ preserves each $\mathfrak{g}_\alpha$ and acts on it by scalar multiplication by the linear functional $\alpha \in \mathfrak{h}^*$, that is, for any $H \in \mathfrak{h}$ and $X \in \mathfrak{g}_\alpha$ we will have

$$ad(H)(X) = \alpha(H)X$$

The second direct sum in the expression above is over a finite set of eigenvalues $\alpha \in \mathfrak{h}^*$, these are called the *roots* of the Lie-algebra, and the corresponding subspaces $\mathfrak{g}_\alpha$ are called the *root spaces.* $\mathfrak{h}$ is the eigenspace to the eigenvalue 0. R will denote the set of roots. The adjoint action of $\mathfrak{g}_\alpha$ carries the eigenspace $\mathfrak{g}_\beta$ into another eigenspace $\mathfrak{g}_{\alpha+\beta}$.

Proposition 3.3. *(i) Each root space $\mathfrak{g}_\alpha$ is one dimensional*
(ii) R generates a lattice $\Lambda_R \subset \mathfrak{h}^*$ of rank equal to the dimension of $\mathfrak{h}$.
(iii) R is symmetric about the origin, i.e., if α is a root then $-\alpha$ is a root as well.

Let V be an irreducible representation of $\mathfrak{g}$, then V has a direct sum decomposition
$$V = \oplus V_\alpha$$
where the direct sum runs over a finite set of $\alpha \in \mathfrak{h}^*$ and $\mathfrak{h}$ acts diagonally on each V_α by multiplication by the eigenvalue α, so for $H \in \mathfrak{h}$ and $v \in V_\alpha$
$$H(v) = \alpha(H)v.$$
The eigenvalues in the direct sum decomposition are called the *weights* of V, and V_α are the *weight spaces*, the dimension of V_α is the *multiplicity* of α. The action of the rest of $\mathfrak{g}$ on V can be described in these terms: for any root β
$$\mathfrak{g}_\beta : V_\alpha \to V_{\alpha+\beta}$$
so we can think of the action of $\mathfrak{g}_\beta$ on V as a translation in the weight diagram. The weights of an irreducible representation are congruent modulo the the root lattice Λ_R.

Now we can find distinguished subalgebras $s_\alpha \cong \mathfrak{sl}_2\mathbb{C} \subset \mathfrak{g}$ in the following way: Let $\mathfrak{g}_\alpha \subset \mathfrak{g}$ be a root space, then by proposition **??** there is another root space $\mathfrak{g}_{-\alpha} \subset \mathfrak{g}$ and $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is a subspace of $\mathfrak{h}$ of dimension one. Now the subalgebra
$$\mathfrak{s}_\alpha = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$$
is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. So we can pick a basis $X_\alpha \in \mathfrak{g}_\alpha$, $Y_\alpha \in \mathfrak{g}_{-\alpha}$ and $H_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$, satisfying the standard commutation relations
$$[H_\alpha, X_\alpha] = 2X_\alpha, [H_\alpha, Y_\alpha] = -2Y_\alpha, [X_\alpha, Y_\alpha] = H_\alpha$$
and H_α is uniquely determined by these relations. These $H_\alpha \in \mathfrak{h}$ elements are called the distinguished elements. All eigenvalues of the action of H_α must be integers, so if Λ_W denotes the following lattice
$$\Lambda_W = \{\beta \in \mathfrak{h}^* : \beta(H_\alpha) \in \mathbb{Z} \ for \ all \ \alpha\}$$
called the *weight lattice* then

Proposition 3.4. *All weights of all representations of $\mathfrak{g}$ will lie in Λ_W.*

Then $\Lambda_R \subset \Lambda_W$ is a sub-lattice of finite index. We have the following for the weights of tensor products, exterior and symmetric powers and duals of representations:

Proposition 3.5. *Let V, W be two representations of $\mathfrak{g}$, the weights of V and W are $v_1, \dots, v_n$, and $w_1, \dots, w_m$ respectively. Then we have*
(i) The weights of $V \otimes W$ are $\{v_i + w_j : \ 1 \le i \le n, \ 1 \le j \le m\}$.
(ii) The weights of $\wedge^k V$ are $\{v_{i_1} + v_{i_2} + \ldots + v_{i_k} : \ 1 \le i_1 < \ldots < i_k \le n\}$.
(iii) The weights of $Sym^k V$ are $\{v_{i_1} + v_{i_2} + \ldots + v_{i_k} : \ 1 \le i_1 \le \ldots \le i_k \le n\}$.
(iv) The weights of V^*, the dual representation of V are $\{-v_i : \ 1 \le i \le n\}$.

For every α we can introduce an involution W_α of $\mathfrak{h}^*$, which is the reflection in the hyperplane $\Omega_\alpha = \{\beta \in \mathfrak{h}^* : \beta(H_\alpha) = 0\}$, that is
$$W_\alpha(\beta) = \beta - \frac{2\beta(H_\alpha)}{\alpha(H_\alpha)}\alpha = \beta - \beta(H_\alpha)\alpha.$$
The group generated by these involutions is called the *Weyl-group* of the Lie-algebra.

Proposition 3.6.

(i) The set of weights of any representation of $\mathfrak{g}$ *is invariant under the Weyl-group.*
(ii) Every element of the Weyl-group is induced by an automorphism of the Lie-algebra $\mathfrak{g}$ *carrying* $\mathfrak{h}$ *to itself.*

Now we can define the *Killing form*, which is a symmetric bilinear form on $\mathfrak{g}$, it will be denoted by B. For $X, Y \in \mathfrak{g}$

$$B(X,Y) = Trace(ad(X) \circ ad(Y) : \mathfrak{g} \to \mathfrak{g})$$

Proposition 3.7.
(i) B *is positive definite on the real subspace of* $\mathfrak{h}$ *spanned by the vectors* $H_\alpha : \alpha \in R$.
(ii) The Killing form is nondegenerate on the semisimple Lie algebra $\mathfrak{g}$.

The Killing form on $\mathfrak{h}$ defines a symmetric bilinear form on $\mathfrak{h}^*$. The element T_α of $\mathfrak{h}$ which corresponds to $\alpha \in \mathfrak{h}^*$ by the Killing form satisfies by definition the condition

$$B(T_\alpha, H) = \alpha(H) \ for \ all \ H \in \mathfrak{h}$$

It can be seen easily, that $T_\alpha = (2/B(H_\alpha, H_\alpha)H_\alpha$, and so the Killing form on $\mathfrak{h}^*$ is defined by $B(\alpha, \beta) = B(T_\alpha, T_\beta)$

The dual of the previous proposition is the following.

Proposition 3.8. *The Killing form* B *is positive definite on the real vector space spanned by the root lattice* Γ_R.

Proposition 3.9. *The space* $\mathfrak{h}^*$ *is an irreducible representation of the Weyl group.*

We can characterize the Weyl group as up to scalars it is the unique inner product on $\mathfrak{h}^*$ preserved by the Weyl-group, i.e, in terms of which the the Weyl group acts as a group of orthogonal transformations. The Weyl-group is the group generated by the reflections in the hyperplanes perpendicular to the roots of the Lie algebra with respect to the Killing form.

Now choose a real linear functional l on the lattice Λ_R irrational with respect to this lattice. This gives us a decomposition of the set

$$R = R^+ \cup R^-$$

where $R^+ = \{\alpha : l(\alpha) > 0\}$, these are called positive roots, the remaining are the negative roots, this decomposition is called an ordering of the roots.

Definition 3.10. *Let* V *be a representation of* $\mathfrak{g}$. *A nonzero vector* $v \in V$ *that is both an eigenvector for the action of* $\mathfrak{h}$ *and in the kernel of* $\mathfrak{g}_\alpha$ *for all* $\alpha \in R^+$ *is called a* highest weight *vector of* V.

Proposition 3.11. *For any semisimple complex Lie algebra* $\mathfrak{g}$
(i) Every finite dimensional representation V *of* $\mathfrak{g}$ *possesses a highest weight vector.*
(ii) The subspace W *of* V *generated by the images of a highest weight vector* v *under successive applications of root spaces* $\mathfrak{g}_\beta$ *for* $\beta \in R^-$ *is an irreducible subrepresentation.*
(iii) An irreducible representation possesses a unique highest weight vector up to scalars.This is called the highest weight of the irreducible representation.

We say that a positive (resp. negative) root α *primitive* or *simple* if it cannot be expressed as a sum of two positive (resp. negative) roots.

Proposition 3.12.

(i) Any irreducible representation V is generated by the images of its highest weight vector v under successive applications of root spaces $\mathfrak{g}_\beta$ where β ranges over the primitive negative roots.

(ii) The set of weights of V will be exactly the weights that are congruent to α modulo the root lattice Λ_R and that lie in the convex hull of the images of α under the Weyl group.

The highest weight of any representation of V will be a weight α satisfying $\alpha(H_\gamma) \geq 0$ for every $\gamma \in R^+$, that is in terms of the Killing form, it makes an acute or right angle with each of the positive roots. The locus $\mathcal{W}$ in the real span of the roots of these points is called the *Weyl chamber* associated to the ordering of the roots. The Weyl group acts transitively on the set of Weyl chambers. Now we can classify all the irreducible representations of $\mathfrak{g}$:

Theorem 3.13. *For any α in the intersection of the Weyl chamber $\mathcal{W}$ with the weight lattice Λ_W there exist a unique irreducible, finite dimensional representation Γ_α of $\mathfrak{g}$ with highest weight α; this gives a bijection between $\mathcal{W} \cap \Lambda_W$ and the set of irreducible representations of $\mathfrak{g}$. The weights of Γ_α will consist of those elements of the weight lattice congruent to α modulo the root lattice Λ_R and lying in the convex hull of the set of points in $\mathfrak{h}^*$ conjugate to α under the Weyl group.*

There are *fundamental weights* $\omega_1, \dots, \omega_n$ such that any dominant weight can be expressed uniquely as a nonnegative integral linear combination of them. They can be characterized geometrically as the first weights met along the edges of the Weyl chamber, or algebraically as those elements ω_i in $\mathfrak{h}^*$ such that $\omega_i(H_{\alpha_j}) = \delta_{i,j}$ where $\alpha_1, \dots, \alpha_n$ are the simple roots.

3.2. **Representation of Lie groups.** Remember, that the representation of a semisimple Lie group $G = \tilde{G}/C$ where $C < Z(G)$ is simply a representation of $\tilde{G}$ trivial on C. The center C lies in the image of the Cartan subalgebra under the exponential map. This image, which is the maximal abelian subgroup of G, called *Cartan subgroup* of G, or *maximal torus* denoted by H. Then $\mathfrak{h} = LieH$. Now we can define the dual lattices of Λ_W and Λ_R, (the weight lattice and root lattice) namely:

$$\Gamma_R = \mathbb{C}\{H_\gamma : \gamma \in R\} \subset \mathfrak{h}, \quad \Gamma_W = \{X \in \mathfrak{h} : \alpha(X) \in \mathbb{Z} \; for \; all \; \alpha \in R\} \subset \mathfrak{h}$$

There is a natural pairing from Γ_W/Γ_R to the center of $\tilde{G}$, which is given by the exponential. More precisely, let $e_G : \mathfrak{h} \to H \subset G$ be the homomorphism defined by $e_G(X) = exp(2\pi i X)$. When $G = \tilde{G}$ is the simply-connected group, then $Ker(e_{\tilde{G}}) = \Gamma_R$ and $e_{\tilde{G}}(\Gamma_W)$ is the center of $\tilde{G}$, and $e_{\tilde{G}}$ induces an isomorphism $\Gamma_W/\Gamma_R \cong Z(\tilde{G})$. More generally, for any $G = \tilde{G}/C$, define a lattice

$$\Gamma(G) = Ker(e_G) \subset \mathfrak{h},$$

then e_G determines an isomorphism

$$\Gamma_W/\Gamma(G) \cong Z(G),$$

and we have

Theorem 3.14. *There is a one-to one correspondence between connected Lie groups G with Lie algebra $\mathfrak{g}$ and lattices $\Lambda(G) \subset \mathfrak{h}^*$ such that $\Lambda_R \subset \Lambda(G) \subset \Lambda_W$. The correspondence is given by associating to a group G the lattice dual to the kernel of the exponential map. In terms of this correspondence, the irreducible representation V_λ of $\mathfrak{g}$ with highest weight $\lambda \in \mathfrak{h}^*$ will lift to a representation of the group G corresponding to $\Lambda(G) \subset \mathfrak{h}^*$ iff $\lambda \in \Lambda(G)$.*

Note also, that

$$H \cong \mathfrak{h}/\Gamma(G) \cong \mathbb{C}^* \times \cdots \mathbb{C}^*$$

with $n = \dim_{\mathbb{C}} \mathfrak{h}$ copies of $\mathbb{C}^*$.

Now elements of the lattice $\Lambda(G) \subset \Lambda_W$ are in bijection with the set of characters of H. (a character is a Lie-group homomorphism $H \to U(1)$. The bijection is given by $w \in \Lambda(G) \to exp(w) \in U(1)$, and conversely, the differential of a homomorphism $H \to U(1)$ is a weight $w \in \Lambda(G) : \mathfrak{h} \to \mathbb{R}$. Given a representation $\rho : G \to GL(V)$ and $T \subset G$ is a maximal torus, $\mathfrak{t} = \mathrm{Lie}T$, then we have restriction $\rho|T : T \to GL(V)$, and V decomposes into the direct sum of one-dimensional eigenspaces, and this decomposition is the same as for the action of the Lie algebra $\mathfrak{t}$, in the weight spaces V_α T acts by multiplication by $exp(2\pi i\alpha) \in U(1)$, thus we can associate a set of weights $\mathcal{W} = \{w_1, \dots, w_n\} \subset \Lambda(G)$ to any representation ρ of G.
The Weyl group $\mathfrak{W}$, which we defined as a subgroup of $Aut(\mathfrak{h}^*)$ can be described for the Lie group G in the following way: let $N(H)$ be the normalizer of the Cartan subgroup, which is the greatest subgroup of G in which H is a normal subgroup. Then we have homomorphisms:

$$N(H) \to Aut(H) \to Aut(\mathfrak{h}) \to Aut(\mathfrak{h}^*)$$

the first defined by conjugation, the second by differentiation at the identity, and the third using the identification of $\mathfrak{h}$ and $\mathfrak{h}^*$ via the Killing form. This map determines an isomorphism $N(H)/N \cong \mathfrak{W}$.

3.3. **Calculation of equivariant cohomologies.** Here we present some methods of calculating equivariant cohomologies. First of all a theorem of Borel tell us how to compute the equivariant cohomology of the point. We work with coefficients $\mathbb{Q}$ or $\mathbb{R}$.
Let $T \subset G$ be a maximal torus. As we have seen, there is a bijection between the weight lattice Γ_W , and one dimensional irreducible representations, namely, if $\lambda \in \Gamma_W$ then the associated representation is $E_\lambda = ET \times_{\rho_\lambda} \mathbb{C}^1 \to BT$. (the action $\rho_\lambda : T \to Aut(\mathbb{C}^1)$ is multiplication by $exp(2\pi i\lambda)$). So we have a map

$$f_G : \Gamma_W \to H^2(BT)$$

$$\lambda \to c_1(E_\lambda).$$

This is a homomorphism: if $\lambda, \mu \in \Gamma_W$ then $\rho_{\lambda+\mu} = \rho_\lambda \rho_\mu$, and $E_{\rho_\lambda \rho_\mu} = E_{\rho_\lambda} \otimes E_{\rho_\mu}$, whose first Chern class is the sum of the Chern classes of the components. So $f_G(\lambda + \mu) = c_1(E_\lambda) + c_1(E_\mu)$, so f_G is indeed a homomorphism.
When V is a representation of G, and the weights of this representation are $\{w_1, \dots, w_m\}$ (which are the Chern classes of the one-dimensional representations of the action of

the maximal torus of G), then the Chern classes of the bundle $BV \to BG$—which are called equivariant Chern classes—are

$$c_i^G(V) = \sigma_i(w_1, \ldots, w_m)$$

where σ_i is the ith elementary symmetric polynomial. An important special case of this the equivariant Euler class $e_G(V) = w_1 \cdots w_m$.
If $T^n = U(1) \times U(1) \times \ldots \times U(1)$ is a complex torus, then we have $H_G^T pt = H^*(BT) = H^*(BU(1) \times \ldots \times BU(1)) = \mathbb{R}[\alpha_1, \ldots, \alpha_n]$ where α_i correspond to the first Chern class of the principal complex line bundle $\mathbb{C}^1 \to \gamma_{\mathbb{C}}^1 \to BU(1)$, these are the generators of the cohomology ring. For arbitrary G Lie group we have

Theorem 3.15. ***(Borel)***
Let $T \subset G$ be a maximal torus of the complex Lie group G, which is a maximal compact abelian subgroup. Then the map

$$i^* : H^*(BG, \mathbb{Q}) \to H^*(BT, \mathbb{Q})$$

is injective, and $\operatorname{Im} i^* = [H^*(BT)]^{\mathcal{W}}$, *so*

$$H^*(BG) = [H^*(BT)]^{\mathfrak{W}} = \mathbb{R}[\alpha_1, \ldots, \alpha_n]^{\mathfrak{W}}$$

where the upper index means the $\mathfrak{W}$-invariant part of the ring. (The Weyl group acts on the weight lattice, and α_i are elements of the weight lattice.)

4. Representations of the classical Lie algebras

The standard reference for this section is [**?**].

4.1. **Analysis of $\mathfrak{sl}_n\mathbb{C}$.** As we have seen, $\mathfrak{sl}_n$ is the subalgebra of $n \times n$ traceless matrices. The Cartan subalgebra is the subalgebra of diagonal matrices. Let $E_{i,i}$ denote the diagonal matrix with all 0-s in the diagonal except for the ith entry, which is 1, so that takes e_i to itself and kills e_j for $i \neq j$. Then

$$\mathfrak{h} = \{a_1 H_1 + a_2 H_2 + \ldots + a_n H_n : a_1 + \ldots + a_n = 0\}$$

$$\mathfrak{h}^* = \mathbb{C}\{L_1, L_2, \ldots, L_n\}/(L_1 + \ldots + L_n = 0)$$

where $L_i(H_j) = \delta_{i,j}$, and so $L_i \in \mathfrak{h}^*$. (The notation $\mathbb{C}\{\}$ means $\mathbb{C}\langle\rangle$, so the generated space). If $E_{i,j}$ is the endomorphism of $\mathbb{C}^n$ carrying e_j to e_i and killing e_k for all $k \neq j$, then

$$ad(a_1 H_1 + a_2 H_2 + \ldots + a_n H_n) = (a_i - a_j) E_{i,j}$$

so $E_{i,j}$ is an eigenvector for the action of $\mathfrak{h}$ with eigenvalue $L_i - L_j \in \mathfrak{h}^*$: the roots of $\mathfrak{sl}_n\mathbb{C}$ are just the pairwise differences of the L_i: $R = \{L_i - L_j : 1 \leq i, j \leq n\}$, and the root spaces are $\mathbb{C}E_{i,j}$. The root lattice $\Lambda_R \subset \mathfrak{h}^*$ can thus be described as

$$\Lambda_R = \{\sum a_i L_i : \ a_i \in \mathbb{Z}, \ \sum a_i = 0\}/(\sum L_i = 0).$$

We can now calculate the Killing form directly: by the definition we have $B(X, Y) = \sum_{\alpha \in R} \alpha(X)\alpha(Y)$ so

$$B(\sum a_i H_i, \sum b_i H_i) = \sum_{i \neq j} (a_i - a_j)(b_i - b_j) = \sum_i \sum_{j \neq i} (a_i b_i + a_j b_j - a_i b_j - a_j b_i)$$

Since $\sum_{i \neq j} a_j = -a_i$ and $\sum_{i \neq j} b_j = -b_i$ this simplifies to

$$B(\sum a_i H_i, \sum b_i H_i) = 2n \sum a_i b_i.$$

So for the dual form on $\mathfrak{h}^*$:

$$B(\sum a_i L_i, \sum b_i L_i) = (1/2n)(\sum_i a_i b_i - (1/n)\sum_{i,j} a_i b_j).$$

The Weyl group in this case is the symmetric group σ_n, and this Killing form is the unique one up to scalars invariant under this group. The L_i all have the same length, and the angles between all pairs are the same. To picture the roots in $\mathfrak{h}^*$ we can think of the points L_i situated at the vertices of a regular $(n-1)$-simplex with the origin located at the barycenter of that simplex.The next step, finding the distinguished subalgebras $\mathfrak{s}_\alpha$. The root space $\mathfrak{g}_{L_i - L_j}$ corresponding to the root $L_i - L_j$ is generated by $E_{i,j}$, so the subalgebra $\mathfrak{s}_{L_i - L_j}$ is generated by

$$E_{i,j},\ E_{j,i} \text{ and } [E_{i,j}, E_{j,i}] = H_i - H_j.$$

The eigenvalue of $H_i - H_j$ acting on $E_{i,j}$ is $(L_i - L_j)(H_i - H_j) = 2$, so that the corresponding distinguished element $H_{L_i - L_j}$ in $\mathfrak{h}$ is $H_i - H_j$. The annihilator is the hyperplane $\Omega_{L_i - L_j} = \{\sum a_i L_j : a_i = a_j\}$.
Knowing the H_α we can determine the weight lattice: the linear functional $\sum a_i L_i \in \mathfrak{h}^*$ to have integral values on all the distinguished elements, it is necessary and sufficient that all the a_i be congruent to one another modulo $\mathbb{Z}$. Since $\sum L_i = 0$ the weight lattice is the following:

$$\Lambda_W = \mathbb{Z}\{L_1, \dots, L_n\}/(\sum L_i = 0).$$

So the weight lattice of $\mathfrak{sl}_n\mathbb{C}$ may be realized as the lattice generated by the vertices of a regular $(n-1)$-simplex Δ centered at the origin, and the roots as the pairwise differences of these vertices.

Remark 4.1. Since the lattice Λ_W can be generated by Λ_R together with any of the vertices L_i of our simplex, Λ_W/Λ_R is cyclic, generated by an L_i, and it can be shown, that $\Lambda_W/\Lambda_R = \mathbb{Z}/n\mathbb{Z}$.

We can see, that the reflection in the hyperplane perpendicular to the root $L_i - L_j$ exchanges L_i and L_j, and fixes the other L_k, so this is a transposition on $\{L_i\}$, and the Weyl group is the symmetric group acting on the generators L_i of $\mathfrak{h}^*$. Let us choose a direction, that is, a linear functional l on $\mathfrak{h}^*$ which has the following form:

$$l(\sum a_i L_i) = \sum c_i a_i$$

with $\sum c_i = 0$, and $c_1 > \ldots > c_n$. This determines an ordering of $\mathfrak{h}^*$, and of the roots with

$$R^+ = \{L_i - L_j :\ i < j\},\ R^- = \{L_i - L_j :\ j < i\}.$$

The primitive negative roots are the roots $L_{i+1} - L_i$ $(i = 1, ...n-1)$, and the (closed) Weyl chamber associated to this ordering is

$$\mathfrak{W} = \{\sum a_i L_i : a_1 \geq a_2 \geq \ldots \geq a_n\}.$$

Geometrically, if we take the barycentric subdivision of the faces of the simplex Δ, the Weyl chamber will be the cone over one $(n-2)$-simplex of the barycentric subdivision. The edges of the Weyl chamber are the lines generated by the vectors

$$L_1,\ L_1 + L_2,\ L_1 + L_2 + L_3, \dots, L_1 + \ldots + L_{n-1} = -L_n,$$

these are the fundamental weights. The faces of $\mathfrak{W}$ will thus be the hyperplanes

$$\Omega_{L_i - L_{i+1}} = \{\sum a_j L_j : \ a_i = a_{i+1}\}$$

perpendicular to the primitive negative roots $L_{i+1} - L_i$. We can index the irreducible representations of $\mathfrak{sl}_n\mathbb{C}$ in the following way: $\Gamma_{a_1,\dots,a_{n-1}}$ with $a_1,\dots,a_{n-1} \in \mathbb{N}$ denotes the irreducible representation with highest weight $a_1L_1 + a_2(L_1 + L_2) + \dots + a_{n-1}(L_1 + \dots + L_{n-1})$. As a consequence of Proposition **??** $\Gamma_{a_1,\dots,a_{n-1}}$ will occur as a subrepresentation of the tensor product of symmetric powers

$$Sym^{a_1}V^{(1)} \otimes Sym^{a_2}V^{(2)} \otimes \cdots \otimes Sym^{a_{n-1}}V^{(n-1)}$$

where $V^{(i)} = \Gamma_{0,\dots,1,\dots 0}$ denotes the irreducible representation corresponding to the fundamental weight $\omega_i = L_1 + \dots + L_i$. Now it can be proved, that $\wedge^k V$ is irreducible, where $V = \mathbb{C}^n$ is the standard representation of $\mathfrak{sl}_n\mathbb{C}$, so $V^{(i)} = \Gamma_{0,\dots,1,\dots 0} = \wedge^k V$.

Proposition 4.2. *The exterior powers $\wedge^k V$ of the standard representations V are irreducible for $k = 1,\dots,n$, and these are the fundamental representations $\Gamma_{0,\dots,0,1,0,\dots,0} = \Gamma_{\omega_i} = \wedge^k V$.*

We will use moreover the symmetric powers of the standard representations, which are irreducible representations as well: $\Gamma_{k,0,\dots,0} = Sym^k V$.

4.2. **Analysis of $\mathfrak{sp}_n\mathbb{C}$.** Let V be a $2n$-dimensional complex vector space, $\mathcal{Q}$ a nondegenerate, skew-symmetric bilinear form on V. Then as we have seen

$$Sp_{2n}\mathbb{C} = \{A \in \text{Aut } V : \ \mathcal{Q}(Av, Aw) = \mathcal{Q}(v, w), \ v, w \in V\},$$

$$\mathfrak{sp}_n^{\mathbb{C}} = \{A \in \text{End } V : \ \mathcal{Q}(Av, w) + \mathcal{Q}(v, Aw) = 0, \ v, w \in V\}$$

Let $e_1,\dots,e_{2n}$ be a basis such that $\mathcal{Q}(e_i, e_{i+n}) = 1, \ \mathcal{Q}(e_{i+n}, e_i) = -1, \ \mathcal{Q}(e_i, e_j) = 0$ if $j \neq i \pm j$, so the matrix of $\mathcal{Q}$ in this basis is $M = \left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)$. Then

$$Sp_{2n}\mathbb{C} = \{A \in GL_{2n}\mathbb{C} : \ M = A^t M A\}, \ \mathfrak{sp}_{2n}\mathbb{C} = \{A \in Mat_{2n}\mathbb{C} : \ A^t M + M A = 0\},$$

and we get that

$$\mathfrak{sp}_{2n}\mathbb{C} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with B, C symmetric, $A = -D^t$. The Cartan subalgebra is again the subalgebra of diagonal matrices, so $\mathfrak{h} = \mathbb{C}\{H_i : 1 \leq i \leq n\}$ where $H_i = E_{i,i} - E_{n+i,n+i}$, and the dual basis elements of $\mathfrak{h}^*$ are L_i, $(i = 1,\dots,n)$ with $\langle L_i, H_j\rangle = \delta_{i,j}$. Now it can be easily checked that:

Proposition 4.3. *$X_{i,j} = E_{i,j} - E_{n+j,n+i} \in \mathfrak{sp}_{2n}\mathbb{C}$ is an eigenvector for the action of $\mathfrak{h}$ with eigenvalue $L_i - L_j$.*
$Y_{i,j} = E_{i,n+j} + E_{j,n+i}$ is an eigenvector for the action of $\mathfrak{h}$ with eigenvalue $L_i + L_j$.
$Z_{i,j} = E_{n+i,j} + E_{n+j,i}$ is an eigenvector for the action of $\mathfrak{h}$ with eigenvalue $-L_i - L_j$.
$U_i = E_{i,n+i}$ and $V_i = E_{n+i,i}$ are eigenvectors for the action of $\mathfrak{h}$ with eigenvalue $2L_i$ and $-2L_i$, respectively.

So the roots are $R = \{\pm L_i \pm L_j\}$. In similar way, as in the case of $\mathfrak{sl}_n\mathbb{C}$ we can compute the Killing form on $\mathfrak{h}$: for $H = \sum a_i H_i$ and $H^* = \sum b_i H_i$ we have

$$B(H, H^*) = (4n+4)(\sum a_i b_i)$$

and the corresponding form on $\mathfrak{h}^*$ is

$$B(\sum a_i L_i, \sum b_i L_i) = (1/(n+1)) \sum a_i b_i.$$

It can be easily shown, that for the roots $L_i - L_j$ and $L_j - L_i$ corresponding to the elements $X_{i,j}$ and $X_{j,i}$ we have $[X_{i,j}, X_{j,i}] = H_i - H_j$, so the distinguished element $H_{L_i - L_j}$ is a multiple of $H_i - H_j$ and $ad(H_i - H_j)(X_{i,j}) = 2X_{i,j}$, so $H_{L_i-L_j} = H_i - H_j$. In a similar way we get:

Proposition 4.4. *The distinguished elements of* $\mathfrak{sp}_{2n}\mathbb{C}$ *are*

$$H_{L_i - L_j} = H_i - H_j,\ H_{L_i + L_j} = H_i + H_j,$$

$$H_{-L_i - L_j} = -H_i - H_j,\ H_{2L_i} = H_i,\ H_{-2L_i} = -H_i.$$

As a corollary we have the weight lattice:

$$\Lambda_W = \{\sum a_i L_i :\ a_i \in \mathbb{Z}\}$$

Now the index $\Lambda_W : \Lambda_R$ is 2. The Weyl group action on $\mathfrak{h}^*$ is generated by reflections: the reflection in the plane Ω_{2L_i} given by $\langle H_i, L\rangle = 0$ reverse the sign of L_i and leaves the other L_j fixed; reflection in the plane $\langle H_i - H_j, L\rangle = 0$ exchange L_i and L_j and leave the remaining L_k alone, so the action can be characterized by a permutation of the L_i-s, and changing the signs some of them, so it fits into the sequence:

$$1 \to (\mathbb{Z}_2)^n \to \mathfrak{W} \to \sigma_n \to 1.$$

and this sequence splits: $\mathfrak{W}$ is the semi direct product of σ_n and $\mathbb{Z}_2^n$. Choosing an ordering $l(\sum a_i L_i) = \sum c_i a_i,\ c_1 > \ldots > c_n > 0$, the positive roots are

$$R^+ = \{L_i + L_j\}_{i \le j} \cup \{L_i - L_j\}_{i<j}$$

with primitive positive roots $\{L_i - L_{i+1}\}_{i=1,\ldots,n-1}$ and $2L_n$. The corresponding closed Weyl chamber is

$$\mathcal{W} = \{a_1 L_1 + \ldots + a_n L_n :\ a_1 \ge \ldots \ge a_n \ge 0\}.$$

The walls of this chamber- the maximal dimensional faces are

$$\{\sum a_i L_i : a_1 > \ldots > a_i = a_{i+1} > \ldots > a_n > 0\},\ \{\sum a_i L_i :\ a_1 > \ldots > a_n = 0\}$$

and the edges are

$$\{\sum a_i L_i :\ a_1 = \ldots = a_i > a_{i+1} = \ldots = a_n = 0\}.$$

The fundamental weights are $\omega_i = L_1 + \ldots + L_i$, and as before we denote the irreducible representation with highest weight $a_1\omega_1 + \ldots + a_n\omega_n = (a_1 + \ldots + a_n)L_1 + (a_2 + \cdots + a_n)L_2 + \ldots + a_n L_n$ with $\Gamma_{a_1,\ldots,a_n}$. In the simplectic case $\wedge^k V$ is not an irreducible representation. We have the natural contraction map

$$\varphi_k :\ \wedge^k V \to \wedge^{k-2} V$$

defined by

$$\varphi_k(v_1 \wedge \ldots \wedge v_k) = \sum_{i<j} (v_i, v_j)(-1)^{i+j-1} v_1 \wedge \ldots \wedge \hat{v}_i \wedge \ldots \wedge \hat{v}_j \wedge \ldots \wedge v_k$$

Since the representation $\wedge^{k-2}V$ does not have the weight $L_1+\ldots+L_k$, the irreducible representation with this highest weight will have to be contained in the kernel of this map.

Theorem 4.5. *For $1 \le k \le n$, the kernel of the map φ_k is exactly the irreducible representation $V^{(k)} = \Gamma_{0,\ldots,0,1,0,\ldots,0}$ with highest weight $L_1 + \ldots + L_k$.*

Now every irreducible representation occurs in a tensor product of symmetric powers of these representations.

4.3. **Analysis of $\mathfrak{so}_m\mathbb{C}$.** In the case of orthogonal Lie algebras there is a difference between the so called even orthogonal Lie algebras $\mathfrak{so}_{2n}\mathbb{C}$ and odd orthogonal Lie algebras $\mathfrak{so}_{2n+1}\mathbb{C}$.

Let V be a m-dimensional complex vector space, $\mathcal{Q}$ a nondegenerate, symmetric bilinear form on V. Then as we have seen

$$SO_m\mathbb{C} = \{A \in GL(V) : \ \mathcal{Q}(Av, Aw) = \mathcal{Q}(v,w) \ v,w \in V, \ detA = 1\},$$

$$\mathfrak{so}_n^{\mathbb{C}} = \{A \in End\ V : \ \mathcal{Q}(Av,w) + \mathcal{Q}(v,Aw) = 0 \ v,w \in V\}$$

In case $m = 2n$ let $e_1, \ldots, e_{2n}$ be a basis such that $\mathcal{Q}(e_i, e_{i+n}) = \mathcal{Q}(e_{i+n}, e_i) = 1$, $\mathcal{Q}(e_i, e_j) = 0$ if $j \neq i \pm n$, so the matrix of $\mathcal{Q}$ in this basis is $M = \left(\begin{smallmatrix} 0 & I_n \\ I_n & 0 \end{smallmatrix}\right)$. Then

$SO_{2n}\mathbb{C} = \{A \in GL_{2n}\mathbb{C} : \ M = A^tMA\}$, $\mathfrak{so}_{2n}\mathbb{C} = \{A \in Mat_{2n}\mathbb{C} : \ A^tM + MA = 0\}$,

and we get that

$$\mathfrak{so}_{2n}\mathbb{C} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with B, C skew-symmetric, and A, D negative transposes of each other.

When $m = 2n+1$ we choose a basis $e_1, \ldots, e_{2n+1}$ in which the bilinear form has matrix $M = \left(\begin{smallmatrix} 0 & I_n & 0 \\ I_n & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$, and then

$$\mathfrak{so}_{2n+1}\mathbb{C} = \begin{pmatrix} A & B & E \\ C & D & F \\ G & H & J \end{pmatrix}$$

with B, C skew-symmetric, and $A = -D^t$, $E = -H^t$, $F = -G^t$, $J = 0$. The Cartan subalgebra is generated by the $2n \times 2n$ diagonal matrices $H_i = E_{i,i} - E_{n+i,n+i}$ (this is the same in the odd and the even case), the dual basis of $\mathfrak{h}^*$ is denoted by L_i where $\langle L_j, H_i\rangle = \delta_{i,j}$. Description of the roots:

Proposition 4.6. ***m=2n:*** *$X_{i,j} = E_{i,j} - E_{n+j,n+i} \in \mathfrak{so}_{2n}\mathbb{C}$ is an eigenvector for the action of $\mathfrak{h}$ with eigenvalue $L_i - L_j$.*
$Y_{i,j} = E_{i,n+j} - E_{j,n+i}$ is an eigenvector for the action of $\mathfrak{h}$ with eigenvalue $L_i + L_j$.
$Z_{i,j} = E_{n+i,j} - E_{n+j,i}$ is an eigenvector for the action of $\mathfrak{h}$ with eigenvalue $-L_i - L_j$.
So the roots of $\mathfrak{so}_{2n}\mathbb{C}$ are $\{\pm L_i \pm L_j\}_{i \neq j} \subset \mathfrak{h}^*$.

Proposition 4.7. ***m=2n+1:*** *$X_{i,j} = E_{i,j} - E_{n+j,n+i}$ viewed as an element of the Lie algebra $\mathfrak{so}_{2n+1}\mathbb{C}$ is an eigenvector for the action of $\mathfrak{h}$ with eigenvalue $L_i - L_j$.*
$Y_{i,j} = E_{i,n+j} - E_{j,n+i}$ viewed as an element of $\mathfrak{so}_{2n+1}\mathbb{C}$ is an eigenvector for the action of $\mathfrak{h}$ with eigenvalue $L_i + L_j$.
$Z_{i,j} = E_{n+i,j} - E_{n+j,i}$ viewed as an element of $\mathfrak{so}_{2n+1}\mathbb{C}$ is an eigenvector for the action of $\mathfrak{h}$ with eigenvalue $-L_i - L_j$.

$U_i = E_{i,2n+1} - E_{2n+1,n+i}$ and $V_i = E_{n+i,2n+1} - E_{2n+1,i}$ are eigenvectors for the action of $\mathfrak{h}$ with eigenvalue L_i and $-L_i$, respectively.
So the roots of $\mathfrak{so}_{2n}\mathbb{C}$ are $\{\pm L_i \pm L_j\}_{i\neq j} \cup \{\pm L_i\} \subset \mathfrak{h}^*$.

So the root diagram of $\mathfrak{so}_{2n+1}\mathbb{C}$ is the same that of $\mathfrak{sp}_{2n}\mathbb{C}$ with the roots $\pm 2L_i$ replaced by $\pm L_i$, hence the Weyl group is the same as that of $\mathfrak{sp}_{2n}\mathbb{C}$:

$$1 \to (\mathbb{Z}_2)^n \to \mathfrak{W}_{\mathfrak{so}_{2n+1}\mathbb{C}} \to \sigma_n \to 1.$$

In the case of $\mathfrak{so}_{2n}\mathbb{C}$ the Weyl group is the subgroup of $\mathfrak{W}_{\mathfrak{sp}_{2n}\mathbb{C}}$ generated by reflection in the hyperplanes perpendicular to the roots $\pm L_i \pm L_j$, without the additional generator given by reflection in the roots $\pm L_i$. So the kernel of this action, instead of acting as $\pm I$ on each of the coordinate axes independently, will consist of transformations of determinant 1, i.e, will act as -1 on an even number of axes. And every such transformation is indeed in the Weyl group, because reflection in $\Omega_{L_i+L_j}$ followed by reflection in $\Omega_{L_i-L_j}$ will send L_i to $-L_i$, L_j to $-L_j$ and fix the other L_k. So the Weyl group of $\mathfrak{so}_{2n}\mathbb{C}$ consist of transformations whose determinant agrees with the sign of the induced permutation of coordinate axes, and so we have the following exact sequence:

$$1 \to (\mathbb{Z}_2)^{n-1} \to \mathfrak{W}_{\mathfrak{so}_{2n}\mathbb{C}} \to \sigma_n \to 1.$$

Choosing an ordering $l(\sum a_i L_i) = \sum c_i a_i,\ c_1 > \ldots > c_n > 0$, the positive roots in the case of $\mathfrak{so}_{2n+1}\mathbb{C}$ are

$$R^+ = \{L_i + L_j\}_{i<j} \cup \{L_i - L_j\}_{i<j} \cup \{L_i\}_i$$

whereas in the case of $\mathfrak{so}_{2n}\mathbb{C}$

$$R^+ = \{L_i + L_j\}_{i<j} \cup \{L_i - L_j\}_{i<j}.$$

The primitive positive roots are

$$\{\{L_i - L_{i+1}\}_{i=1,\ldots,n-1}, L_n\} \ for\ \mathfrak{so}_{2n+1}\mathbb{C}$$

$$\{\{L_i - L_{i+1}\}_{i=1,\ldots,n-1}, L_{n-1} + L_n\} \ for\ \mathfrak{so}_{2n}\mathbb{C}.$$

In the first case the Weyl chamber is exactly the same as for $\mathfrak{sp}_{2n}\mathbb{C}$, namely with $m = 2n+1$

$$\mathcal{W} = \{\sum a_i L_i : \ a_1 \geq \ldots \geq a_n \geq 0\}.$$

In the case of $\mathfrak{so}_{2n}\mathbb{C}$ since there is no root along the line spanned by the L_i, the equality $a_n = 0$ does not describe a face of the Weyl chamber, however, since $L_{n-1} + L_n$ is still a root, we still have the inequality $a_{n-1} + a_n \geq 0$, so

$$\mathcal{W} = \{\sum a_i L_i : \ a_1 \geq \ldots \geq a_{n-1} \geq |a_n|\}.$$

The Killing form on $\mathfrak{h}$ with the same computations

$$B(\sum a_i H_i, \sum b_i H_i) = \begin{cases} (4n-2)\sum a_i b_i & \text{if} \quad m = 2n+1 \\ (4n-4)\sum a_i b_i & \text{if} \quad m = 2n \end{cases}$$

The corresponding Killing form on $\mathfrak{h}^*$ is

$$B(\sum a_i L_i, \sum b_i L_i) = \begin{cases} 4/(4n-2)\sum a_i b_i & \text{if} \quad m = 2n+1 \\ 1/(n-1)\sum a_i b_i & \text{if} \quad m = 2n \end{cases}$$

The distinguished elements H_α $(\alpha \in R)$ are the following:

Proposition 4.8. *The distinguished elements of* $\mathfrak{so}_m\mathbb{C}$ *are*

$$H_{L_i-L_j} = H_i - H_j,\; H_{L_i+L_j} = H_i + H_j,$$

$$H_{-L_i-L_j} = -H_i - H_j,$$

and in the even case $m = 2n$

$$H_{L_i} = 2H_i,\; H_{-L_i} = -2H_i.$$

The weight lattice Λ_W is now for both even and odd orthogonal Lie algebras is the lattice generated by the L_i together with the element $(L_1 + \ldots + L_n)/2$:

$$\Lambda_W = \mathbb{Z}\{L_1, \ldots, L_n, (L_1 + \ldots + L_n)/2\}.$$

Now it can be checked, that Λ_W/Λ_R is $\mathbb{Z}_2$ if $m = 2n+1$, $\mathbb{Z}_4$ if $m = 2n$ and n is odd and $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ if $m = 2n$ and n is even.
The fundamental weights of $\mathfrak{so}_{2n}\mathbb{C}$ are $\omega_1 = L_1,\ \omega_2 = L_1 + L_2,\ \ldots, \omega_{n-2} = L_1 + \ldots + L_{n-2}$ and the vectors $\omega_{n-1} = \alpha = (L_1 + \ldots + L_n)/2$ and $\omega_n = \beta = (L_1+L_2+\ldots+L_{n-1}-L_n)/2$. The fundamental weights of $\mathfrak{so}_{2n+1}\mathbb{C}$ are $\omega_1 = L_1,\ \omega_2 = L_1 + L_2,\ \ldots, \omega_{n-1} = L_1 + \ldots + L_{n-1}$ and the vector $\omega_n = \alpha = (L_1 + \ldots + L_n)/2$.
The fundamental irreducible representations of the orthogonal groups can be described completely:

Proposition 4.9.

(i) ***representations of*** $\mathfrak{so}_{2n}\mathbb{C}$
The exterior powers $\wedge^k V$ *of the standard representation* V *are irreducible for* $k = 1, \ldots, n-1$ *and* $\Gamma_{\omega_i} = \wedge^k V$ *and* $\wedge^n V = \Gamma_\alpha \oplus \Gamma_\beta$. *The irreducible representations* Γ_α *and* Γ_β *are called spin representations.*

(ii) ***representations of*** $\mathfrak{so}_{2n+1}\mathbb{C}$
The exterior powers $\wedge^k V$ *of the standard representation* V *are irreducible for* $k = 1, \ldots, n$ *and* $\Gamma_{\omega_i} = \wedge^k V$ *for* $i = 1, \ldots, n-1$. Γ_α *is called the spin representation of* $\mathfrak{so}_{2n+1}\mathbb{C}$.

5. Homogeneous Spaces, Flag Manifolds, Minimal Orbits

(see [?], 23.3)

A *homogeneous space* of a compact connected group G is a space X with a transitive $G-action$. In this section we introduce and describe the compact homogeneous spaces associated to the classical Lie groups.
We begin by introducing the notion of Borel subalgebras and Borel subgroups. A choice of Cartan-subalgebra $\mathfrak{h}$ determines a decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus (\oplus_{\alpha \in R}\mathfrak{g}_\alpha).$$

If we choose an ordering of the root system $R = R^+ \cup R^-$, we can associate a subalgebra

$$\mathfrak{b} = \mathfrak{h} \oplus (\oplus_{\alpha \in R^+}\mathfrak{g}_\alpha)$$

called a *Borel subalgebra*. Then $\mathfrak{b}$ is solvable, in fact $\mathfrak{b}$ is a maximal solvable subalgebra.If G is a Lie group with semisimple Lie algebra $\mathfrak{g}$, the connected subgroup $B < G$ with Lie algebra $\mathfrak{b}$ is called a *Borel subgroup.*

Lemma 5.1. *B is a closed subgroup of G, and the quotient G/B is compact.*

Proof: Consider the adjoint representation of G on $\mathfrak{g}$. The action of the Borel subalgebra $\mathfrak{b}$ obviously preserves the subspace $\mathfrak{b} \subset \mathfrak{g}$, and, in fact, $\mathfrak{b}$ is just the inverse image of the subalgebra of $\mathfrak{gl}(\mathfrak{g})$ preserving this subspace: if $X = \sum X_\alpha$ is any element of $\mathfrak{g}$ with $X_\alpha \in \mathfrak{g}_\alpha$ and $X_\alpha \neq 0$ for some $\alpha \in R^-$, we could find an element H of $\mathfrak{h} \subset \mathfrak{g}$ with $ad(X)(H) \notin \mathfrak{b}$ – any H not in the annihilator of $\alpha \in \mathfrak{h}^*$ would do. B is thus (the connected component of the identity in) the inverse image in G of the subgroup of $GL(\mathfrak{g})$ carrying $\mathfrak{b}$ into itself. It follows that B is closed, and the quotient G/B is contained in a Grassmannian, and hence compact. (Alternatively, we could consider the action of G on the projective space $\mathbb{P}(\wedge^m \mathfrak{g})$ where m is the number of positive roots, and observe that B is the stabilizer of the point corresponding to the exterior product of the positive root spaces.) $\boxtimes$

In the case of the classical groups it is easy to describe the Borel subgroups and the corresponding quotients.
For $G = SL_{n+1}\mathbb{C}$, B is the group of all upper triangular matrices in G, i.e, the automorphisms preserving the standard flag. It follows that G/B is the usual complete flag manifold, i.e, the variety of all flags

$$G/B = \{0 \subset V_1 \subset \ldots \subset V_n \subset \mathbb{C}^{n+1}\}$$

of subspaces with $\dim(V_r) = r$.
For $G = SO_{2n+1}\mathbb{C}$ the orthogonal group of automorphisms of $\mathbb{C}^{2n+1}$ preserving a quadratic form $\mathcal{Q}$, B is the subgroup of automorphisms, which preserve a fixed flag $V_1 \subset \ldots \subset V_n$ of isotropic subspaces with $dim(V_r) = r$. All such flags being conjugate, G/B is the variety of all such flags, i.e

$$G/B = \{0 \subset V_1 \subset \ldots \subset V_n \subset \mathbb{C}^{n+1} : \mathcal{Q}(V_n, V_n) = 0\}.$$

Since B automatically preserves the flag of orthogonal subspaces, so we could also characterize G/B as the space of complete flags equal to their orthogonal complements, i.e

$$G/B = \{0 \subset V_1 \subset \ldots \subset V_n \subset \mathbb{C}^{n+1} : \mathcal{Q}(V_i, V_{2n+1-i}) = 0\}.$$

The same holds for $Sp_{2n}\mathbb{C}$: the Borel subgroups $B \subset Sp_{2n}\mathbb{C}$ are just the subgroups preserving a half-flag of isotropic subspaces, or equivalently a full flag of pairwise complementary subspaces; and the quotient G/B is the variety of all such flags.
For $G = SO_{2n}\mathbb{C}$, B fixes an isotropic flag $V_1 \subset \ldots \subset V_{n-1}$, and

$$G/B = \{0 \subset V_1 \subset \ldots \subset V_{n-1} \subset \mathbb{C}^{2n} : \mathcal{Q}(V_{n-1}, V_{n-1}) = 0\}.$$

We say, that a closed, connected subgroup P of G is *parabolic*, if the quotient G/P can be realized as the closed orbit of the action of G on $\mathbb{P}(V)$ for some representation V of G, so in particular G/P is a projective algebraic variety. As we will see many famous variety is of this kind, so they are orbits at representations of a classical Lie group, and for minimal orbits we will have a strong method (The Atiyah-Bott integral formula and the Duistermaat-Heckman formula) to determine the Thom-polynomial. It follows from the proof of the previous lemma that every Borel subgroup of G is parabolic, and parabolic subgroups are those subgroups containing a Borel subgroup, so the Borel subgroups are the minimal parabolic subgroups:

Lemma 5.2. *([?], 23.47, 23.48)*

(i) If B is a Borel, P is a parabolic subgroup of G, then there is an $x \in G$ with $B \subset xPx^{-1}$.
(ii) If a subgroup P of G contains a Borel subgroup B, then P is parabolic.

So we can completely classify the parabolic subgroups of a simple group up to conjugacy. We may assume that P contains a Borel subgroup B, so its Lie algebra $\mathfrak{p}$ is a subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}$ so it is of the form:

$$\mathfrak{p} = \mathfrak{h} \oplus (\oplus_{\alpha \in T} \mathfrak{g}_\alpha)$$

where $T \subset R$ is generated by R^+ and some negative simple roots. So if $\Sigma \subset$ {positive simple roots}, $T(\Sigma) = (\mathbb{Z}\{-\Sigma\} \cap R^-) \cup R^+$ then

$$\mathfrak{p}(\Sigma) = \mathfrak{h} \oplus (\oplus_{\alpha \in T(\Sigma)} \mathfrak{g}_\alpha)$$

is a parabolic subalgebra, the corresponding Lie group $P(\Sigma)$ is a parabolic subgroup, and we obtain in this way all the parabolic subgroups of G. We will use later the expression: a parabolic subgroup corresponding to a set of positive simple roots. (It would be better to talk about parabolic subgroup corresponding to a set of negative roots, but never mind)

Proposition 5.3. *Up to conjugacy, parabolic subgroups of the simple group G are in one-to-one correspondence with subsets of the nodes of the Dynkin diagram, i.e, with subsets of the set of simple roots.*

Parabolic subgroups of a classical Lie group G may be described as the subgroups preserving a partial flag in the standard representation. In particular, a maximal parabolic subgroup, corresponding to omitting one simple root (or one node in the Dynkin diagram) may be described as the subgroup of G preserving a single subspace, so G/P is the variety of these single subspaces.

Examples 5.4. *For $G = SL_m\mathbb{C}$, if we omit the k^{th} simple root (the kth node in the Dynkin diagram) we get the ordinary Grassmannian $G(k,m)$ of k-dimensional subspaces of $\mathbb{C}^m$.*
For $Sp_{2n}\mathbb{C}$, the k^{th} node of the Dynkin diagram corresponds to the Lagrangian Grassmannian of isotropic k-planes for $k = 1, \dots, n$.
Similarly for $G = SO_{2n+1}\mathbb{C}$ the kth node corresponds to the orthogonal Grassmannian of isotropic k-planes in $\mathbb{C}^{2n+1}$.
Finally, for $SO_{2n}\mathbb{C}$, for $k = 1, 2, \dots, n-2$ the kth node of the Dynkin diagram yields the orthogonal Grassmannian of isotropic k-planes in $\mathbb{C}^{2n}$, but there is one anomaly: either of the last two nodes gives one of the two connected components of the Grassmannian of isotropic n-planes, these are also called spinor varieties.

There is another way to realize the compact homogeneous spaces associated to a simple compact, connected group G. Let $V = \Gamma_\lambda$ be an irreducible representation of G with highest weight λ, and consider the action of G on the projective space $\mathbb{P}V$.(Every representation is a linear action, so it is an action on the corresponding projective space.) Let $p \in \mathbb{P}V$ be the point corresponding to the weight space of λ, i.e to the one-dimensional eigenspace, with eigenvalue λ. We have then the

Observation 5.5. *The orbit $G \circ p$ is the unique closed orbit of the action of G on $\mathbb{P}V$,so this is the minimal orbit.*

Proof: The point p is fixed under the Borel subgroup B, because p corresponds to a highest weight space. So the stabilizer of $p \in \mathbb{P}V$ is a parabolic subgroup P_λ. The orbit G/P_λ is thus compact and hence closed. Conversely, by the Borel fixed point theorem ([**?**]) any closed orbit of G contains a fixed point for the action of B, but p is the unique point in $\mathbb{P}V$ fixed by B. ⊠

In fact, it is not hard to say which parabolic subgroup P_λ is the stabilizer of this projective orbit in terms of the classification above: it is the parabolic subgroup corresponding to the subset of positive simple roots that are perpendicular to the weight λ.(**??**)
Now, sets Σ of simple roots correspond to faces of the Weyl chamber, namely, the face that is the intersection of all hyperplanes perpendicular to all roots in Σ.
We thus have a correspondence between *faces* of the Weyl chamber, and *parabolic subgroups* P, such that *if* $V = \Gamma_\lambda$ *is the irreducible representation with highest weight* λ*, then the unique closed orbit of the action of* G *on* $\mathbb{P}V$ *is of the form* G/P*, where* P *is the parabolic subgroup corresponding to the open face of the Weyl-chamber containing* λ. In particular, weights in the interior of the Weyl chamber correspond to $P_\lambda = B$, and so determine the full flag manifold G/B, whereas weights on the edges give rise to the quotients of G by maximal parabolics.

Remark 5.6. *We do obtain in this way all compact homogeneous spaces for* G*.*

6. The Duistermaat-Heckman formula

Let G be a compact, connected group, $T \subset G$ a maximal torus, and $\rho : G \curvearrowright V$ a linear action of G on the vector space V. It means that it induces an action $G \curvearrowright \mathbb{P}V$. Let $\mathbb{P}\eta \subset \mathbb{P}V$ be a G-invariant, smooth projective variety of complex codimension c, and $\eta = cone(\mathbb{P}\eta) \subset V$ its cone. Then η is a G-invariant affine variety, which is homogeneous, that is, it is union of lines through the origin. Let us suppose, that G has finite number of fixed points in $\mathbb{P}\eta$. The fixed points in $\mathbb{P}\eta$ correspond to one-dimensional eigenspaces, i.e one-dimensional representations of the action $G \curvearrowright V$. This proof of the Duistermaat Heckman formula from the Atiyah-Bott integral formula is due to L. Fehér.

6.1. Connection between affine and projective Thom polynomials.

Let $[\eta]_G \in H^{2c}_G(V) = H^{2c}_G(pt) = H^{2c}(BG)$ be the equivariant Poincaré dual of η where working over reals $\mathbb{R}$, $H^*(BG) = \mathbb{R}[\alpha_1, \dots, \alpha_r]^{\mathfrak{W}}$, $(r = dimT)$ as we have seen. We call $[\eta]_G$ the *affine Thom polynomial.*
Similarly $[\mathbb{P}\eta]_G \in H^{2c}_G(\mathbb{P}V)$ means the equivariant Poincaré dual of $\mathbb{P}\eta$, this is the *projective Thom polynomial.* Our next goal is to determine the connection between $[\eta]_G$ and $[\mathbb{P}\eta]_G$. We will use the notations introduced for equivariant characteristic classes: for example $e_G(V)$ denotes the Euler class of the bundle $BV \to BG$.
By the Leray-Hirsch theorem we know, that $H^*(B\mathbb{P}V)$ is a module over $H^*(BG)$ in the following way: The fiber of $B\mathbb{P}V$ is $\mathbb{P}V = \mathbb{C}P^n$, let $y \in H^2(\mathbb{C}P^n)$ denote the generator of the ring $H^*(\mathbb{C}P^n) = R[1, y, \dots, y^{n-1}]$ then there is $x \in H^2(B\mathbb{P}V)$ with $j^*(x^i) = y^i$ where $j : \mathbb{P}V \to B\mathbb{P}V$ is the embedding of an arbitrary fiber, and $H^*(B\mathbb{P}V)$ is a free $H^*(BG)$ module with generators x^i. Now we want to determine the ring structure on $H^*(B\mathbb{P}V)$:

Lemma 6.1. *Let $w_1(V), \dots, w_n(V) \in H^2(BG)$ denote the weights of the representation $G \curvearrowright V$, where $n = dimV$. Then*

$$H^*_G(\mathbb{P}V) = H^*(BG)[x]/(\Pi_1^n(x + w_j)).$$

We will use the notation $\Gamma(x)$ for $\Pi_1^n(x + w_j)$.

Note: Let $\pi : B\mathbb{P}V \to BG$ be the projection map, then to be correct we should write $\pi^* w_i$ instead of w_i in the previous formula, but we will omit it.

Proof: Let τ denote the tautological line bundle over $\mathbb{P}V$, then $B\tau \to B\mathbb{P}V$ is the "equivariant" line bundle over $B\mathbb{P}V$, so $c(B\tau) = 1 + c_1(B\tau)$.

Now let $\pi^* BV$ denote the pull-back of the principal V-bundle

$$\begin{tikzcd} \pi^*(BV) \arrow[r] \arrow[d] & BV . \arrow[d] \\ B\mathbb{P}V \arrow[r, "\pi"] & BG \end{tikzcd}$$

Then

$$c(\pi^* BV) = 1 + \pi^* c_1(BV) + \pi^* c_2(BV) + \ldots + \pi^* c_n(BV) = \prod_{i=1}^n (1 + \pi^*(w_i)).$$

Let $Q = \pi * BV \ominus B\tau$ be the formal difference of the two bundle over $B\mathbb{P}V$, then

$$c(Q) = \frac{c(\pi^* BV)}{c(B\tau)} = c(\pi^* BV)(1 + x + x^2 + \ldots)$$

where we introduced the notation $x = -c_1(B\tau)$. Now since Q is a $n-1$ dimensional bundle over $B\mathbb{P}V$, we have

$$0 = c_n(Q) = \sum_{i=0}^n c_{n-i}(\pi^* BV)x^i = \sum_{i=0}^n \sigma_{n-i}(\pi^* w_1, \dots, \pi^* w_n)x^i = \prod_{i=1}^n (x + \pi^* w_i).$$

Now let us consider the following map from the polynomial ring over H^* to $H^*(B\mathbb{P}V)$:

$$q : H^*(BG)[x] \to H^*(B\mathbb{P}V)$$

$$x \to -c_1(B\tau)$$

$$\alpha \to \pi^* \alpha$$

for $\alpha \in H^*(BG)$. Since restriction of $c_1(B\tau)$ to any fiber gives a generator of $H^*(fiber)$, the Leray-Hirsch theorem says that q is surjective.
We have just proved, that $q(\prod(x + w_j)) = 0$. So it remains to prove, that there are no other relations in the polynomial ring. Since we work over $\mathbb{Q}$, it is enough to prove, that the Betti-numbers of $H^*(BG)[x]/\prod(x + w_i)$ and $H^*(B\mathbb{P}V)$ are equal. From the spectral sequence of $\mathbb{C}P^(n-1) \to B\mathbb{P}V \to BG$ we get that $b^{2i}(B\mathbb{P}V) = \sum_{j=0}^{n-1} b^{2(i-j)}(BG)$. Since the coefficient of x^n is 1 in $\prod(x + w_i)$, we have the same Betti-numbers for the other side. ⊠

Corollary 6.2. *Since the main coefficient of $\Gamma(x)$ is 1, every element $\alpha \in H^k(B\mathbb{P}V)$ can be written in the form $\alpha = \sum_{i=0}^{n-1} a_i x^i$ where $a_i \in H^{k-2i}(BG)$.*

Corollary 6.3. $[\mathbb{P}\eta]_G = \sum_{i=0}^{c} g_i x^i$ *where $g_i \in H^{2c-2i}(BG)$.*

First we show how can we get the affine Thom polynomial and the degree of $\mathbb{P}\eta$ from the projective Thom polynomial. First here is the definition of degree:

Definition 6.4. *Let $X^k \subset \mathbb{P}^n$ be a closed projective variety of dimension k. Since $H^*(\mathbb{P}^n;\mathbb{Z}) = \mathbb{Z}[\xi]/\xi^{n+1}$, there is an integer d, such that the dual class $[X]_{\mathbb{P}V} = d\cdot\xi^{n-j}$. This number d is called the degree of the variety X, and is denoted by $deg(X)$.*

Remark 6.5. There is a more intuitive, but less precise definition of the degree as follows: let $H_1,\dots,H_{n-k}$ be generic hyperplanes in $\mathbb{P}V$. Then $X\cap H_1\cap H_2\cap\dots\cap H_{n-k}$ consist of finitely many points, and the cardinality of this set is the degree. The equivalence of the definitions can be shown easily: it follows from the fact that $\xi = [H_i]_{\mathbb{P}V}$ $(i=1,\dots,n-k)$ and $[A\cap B]_{\mathbb{P}V} = [A]_{\mathbb{P}V}\cup[B]_{\mathbb{P}V}$ for "generic" A and B.

Lemma 6.6. [?] *If $[\mathbb{P}\eta]_G = \sum_{i=0}^c g_i x^i = p(x)$ then $g_0 = [\eta]_G \in H^{2c}(BG)$, and $g_c = deg(\mathbb{P}\eta)\cdot 1$, where $1\in H^0(BG)$ is the generator.*

Proof: We don't prove the degree part of the assertion. Let us consider the following diagram:

$$\begin{tikzcd} & V \\ \mathbb{P}V & V\setminus 0 \arrow[l, "\pi"'] \arrow[u, "i"'] \end{tikzcd}$$

where i is the embedding, π is the natural projection. Now applying the functor H_G^* to this diagram we get

$$\begin{tikzcd} & H^*(BG) \arrow[dl] \arrow[d, "i_G^*"] \\ H_G^*(\mathbb{P}V) = H^*(BG)[x]/\Gamma(x) \arrow[r, "\pi_G^*"] & H_G^*(V\setminus 0) \end{tikzcd}$$

where $\pi_G^*([\mathbb{P}\eta]_G) = [\eta\setminus 0]_G = i_G^*([\eta])$.

Now $\pi_G^*(x) = \pi_G^*(c_1(\tau)) = 0$ since $x = -c_1(B\tau)$, where $B\tau \to B\mathbb{P}V$ is the tautological line bundle over $B\mathbb{P}V$, and $\pi^*(\tau)$ is trivial line bundle over $V\setminus 0$ hence $\pi^*(c_1(\tau) = 0$, so the equivariant Chern class goes to 0 as well, so $\pi_G^*(p(x)) = \pi_G^*(p(0))$. But i_G^* is isomorphism for $* < n$ from the Gysin sequence, so $\pi_G^*(p(0)) = p(0)$ under this isomorphism using the commutativity of the diagramm, and the lemma is proved.

After this lemma we can think that projective Thom polynomials contain more information than the affine ones, but this is not the truth. In fact a simple substitution into the affine Thom polynomial provides the projective Thom polynomial in the following way, see [?]

Let $m: U(1)^n \to G$ be a coordinated maximal torus of G and let α_i are the corresponding weights, i.e the generators of $H^*(BG)$. By the Borel theorem (see **??**) $[\eta]_G = Tp(\eta)$ is a polynomial in variables α_i and $[\mathbb{P}\eta]_G = Tp(\mathbb{P}\eta)$ is a polynomial in variables α_i and ξ. We assume that the image of ρ contains the scalars, i.e. there is a homomorphism $\varphi: GL(1)\to G$ and a non-zero integer q such that $\rho\circ\varphi(\lambda)v = \lambda^q v$ for $v\in V$, $\lambda\in GL(1)$.

We assume that $Im\varphi|_{U(1)} \subset Im\ m$, so φ factors through $U(1)^r$, that is, we have a homomorphism $\tilde{\varphi} : U(1) \to U(1)^n$ such that $\varphi|_{U(1)} = m \circ \tilde{\varphi}$, i.e. the following diagram is commutative:

$$\begin{tikzcd} GL(1) \arrow[r, "\varphi"] & G \\ U(1) \arrow[u, "i"] \arrow[r, "\tilde{\varphi}"] & U(1)^r \arrow[u, "m"'] \end{tikzcd}$$

The homomorphism $\tilde{\varphi}$ must be of the form $\tilde{\varphi}(z) = (z^{\omega_1}, \dots, z^{\omega_n})$ where $z \in U(1)$ and ω_i are integers. The choice of φ is not unique.

Theorem 6.7. *([?]) Let $G \to GL(V)$ be a representation of the Lie group G, such that the image of ρ contains the scalars. Let α_1, q, ω_i, η as above. Then*

$$Tp(\mathbb{P}\eta)(\alpha_1, \dots, \alpha_n, \xi) = Tp(\eta)(\alpha_1 + \frac{\omega_1}{q}\xi, \dots, \alpha_n + \frac{\omega_n}{q}\xi).$$

Proof The trick is that we relate the projective Thom polynomial to an affine Thom polynomial for a different group. Let us consider the bundle $\chi : Hom(\mathbb{C}, V) \to Hom(\tau, \pi^* BV) \to B_G\mathbb{P}V$. (here we use the notations of **??**) This bundle has $\tilde{G} = GL(1) \times G$ as its structure group, with the action on $Hom(\mathbb{C}, V)$ by $\tilde{\rho} = Hom(\rho^1, \rho)$ where ρ^1 is the standard representation of $GL(1)$ on $\mathbb{C}$. Since $Hom(\mathbb{C}, V)$ is isomorphic to V in a natural way, we can think as $\tilde{G}$ acting on V. Moreover the $\tilde{G}$-invariant subsets of $Hom(\mathbb{C}, V)$ are the same as the G-invariant subsets of V, if we use this isomorphism. The bundle χ has a canonical section σ, since a fiber of τ is a part of the corresponding fiber π^*V over the same point x of $B_G\mathbb{P}V$, and $\sigma(x)$ is this embedding. By definition, $\sigma^{-1}(\eta) = B_G\mathbb{P}\eta$. So if $f : B_G\mathbb{P}V \to B\tilde{G}$ denotes the classifying map of χ then $f^*Tp_{\tilde{\rho}}(\eta) = Tp_\rho(\mathbb{P}\eta)$. Now it is easy to see, that $H^*(B\tilde{G}) \cong H^*(BG)[\xi_1]$ where $\xi_1 = -c_1(\pi_1^*\tau_1)$, $\pi_1 : B\tilde{G} \to BGL(1)$ is the projection and τ_1 is the tautological line bundle over $BGL(1)$. Moreover $f^* : H^*_G(\xi_1) \to H^*_G(\mathbb{P}V)$ is a H^*_G-module homomorphism which maps ξ_1 to ξ, so it s simply the factorization with the relation $\Gamma(\xi) = \prod(\xi + \omega_i)$.
So we must concentrate on calculating $Tp_{\tilde{\rho}}(\eta)$. To do that we choose the maximal torus of $\tilde{G}$ of the form $\tilde{m}(t_0, t_1, \dots, t_n) = (t_0, m(t_1, \dots, t_n))$. The key observation is that restricted to these maximal tori the representation $\tilde{\rho}$ almost factors through ρ. We define a homomorphism $\kappa : U(1)^{n+1} \to U(1)^n$ in the following way:

$$\kappa(t_0, t_1, \dots, t_n) = (t_0^{\omega_1} t_1^q, \dots, t_0^{\omega_n} t_n^q) = \tilde{\varphi}(t_0)(t_1^q, \dots, t_n^q).$$

Then

$$\rho \circ \kappa(t_0, \dots, t_n) = \rho(\tilde{\varphi}(t_0)(t_1^q, \dots, t_n^q)) = (t_0 \cdot \rho(t_1, \dots, t_n))^q = (\tilde{\rho} \circ \tilde{m}(t_0, \dots, t_n))^q.$$

We will use the following proposition twice:

Proposition 6.8. *Let $h : K \to G$ be a homomorphism of Lie groups and $\sigma = \rho \circ h$. Then*

1. *if $\eta \subset V$ is ρ-invariant, then η is σ-invariant too.*
2. *$Tp_\sigma(\eta) = (Bh)^* Tp_\rho(\eta)$ where $Bh : BK \to BG$ is induced by h.*

This proposition is an obvious consequence of the fact, that the universal σ-bundle is the pull back of the universal ρ-bundle via Bh.
Applying this twice: first to $h = \kappa$, $(Bh)^*(\alpha_i) = q\alpha_i + \omega_i\xi$, then to $h(x) = x^q$, $(Bh)^*(\alpha_i) = q\alpha_i$ finishes the proof of **??**.

Corollary 6.9. *The degree of the projective variety $\mathbb{P}\eta$ comes from a simple substitution:*

$$deg(\mathbb{P}\eta) = Tp(\eta)\left(\frac{\omega_1}{q}, \dots, \frac{\omega_n}{q}\right).$$

We will see one application of this theorem for Grassmannians in **??**

6.2. **Push-forward and Thom-polynomials.** A good reference for this and the next subsection is [**?**]. Let N^n, M^m be compact, oriented manifolds of dimension n, m and $f : N \to M$ be a map. Then we can define the "push-forward", or Gysin-map of H^*

$$f_! : H^*(N) \to H^{*+m-n}(M)$$

with the aid of Poincaré duals:

$$\begin{tikzcd} H^*(N) \arrow[r, "f_!"] \arrow[d, "PD"] & H^{*+m-n}(M) \\ H_{n-*}(N) \arrow[r, "f_*"] & H_{n-*}(M) \arrow[u, "PD"'] \end{tikzcd}$$

so $f_!(\alpha) = PD(f_*PD(\alpha))$. More general we can define "push-forward" for proper maps of spaces. (proper means, that the pre-image of compact set is compact) This "push-forward" has the following properties:

(i) It is functorial: $(f \circ g)_! = f_! \circ g_!$

(ii) It is a homomorphism of $H^*(M)$-modules,i.e

$$f_!(\alpha f^*\beta) = (f_!\alpha)\beta \ for\ \alpha \in H^*(N),\ \beta \in H^*(M)$$

this is called adjunction formula.

(iii) If f is a fibering, then $f_!$ corresponds to integration over the fiber.

(iv) If $f : N \hookrightarrow M$ is embedding of a submanifold, then $f_!(1) = PD(N \hookrightarrow M)$, the Thom-class of the normal bundle of N in M, so $f^*(f_!(1)) = e(\nu_f)$, the Euler-class, and in general $f_! = j^* \circ \varphi$ where φ is the Thom-isomorphism:

$$\begin{tikzcd} \cdots \arrow[r] & H^{*-1}(M-N) \arrow[r, "\delta"] & H^*(M, M-N) \arrow[r, "j^*"] \arrow[d, "\varphi"] & H^*(M) \arrow[r] & \cdots \\ & & H^{*-(m-n)}(N) & & \end{tikzcd}$$

This "push-forward" can be defined almost word-for-word in the equivariant situation. Let G be a compact group, N, M be G-spaces, and $g : N \to M$ G-equivariant map. In this situation we can define the "equivariant push-forward" map

$$g_!^G : H_G^*(N) \to H_G^{*+m-n}(M).$$

The map g induces a map $Bg : B_G(N) \to B_G(M)$, but these spaces are already not manifolds, and not compact, so the definition with Poincaré dual does not work. Fortunately by the graph construction every map g can be factored into an inclusion followed by a product projection in the following way:

$$N \xrightarrow{i} N \times M \xrightarrow{\pi} M$$

$$n \longrightarrow (n, g(n)) \longrightarrow g(n)$$

So it is enough define the "push-forward" for inclusions and fibering.

(i) If $g : N \hookrightarrow M$ is inclusion, so N is G-equivariant submanifold of M, then since G is compact, there is tubular neighborhood U of N in M, which is G-equivariant, so $B_G U$ is tubular neighborhood of $B_G(N)$ in $B_G(M)$, so we can define $g_! = j^* \circ \varphi$ as above, with the Thom-isomorphism.

(ii) If $g : N \to M$ is a fibering, the induced map $Bg : B_G N \to B_G M$ is one also with compact oriented fiber, and integration over the fiber is well-defined in any fibering with an oriented compact manifold as fiber, so we can define $g_!^G$ in this way.

The "equivariant push-forward" satisfies moreover:

(iii) $g_!^*$ preserves the $H^*(BG)$-module structure.

(iv) The special case of (ii) is when $M = pt$: if $\pi : N \to pt$ then $\pi_!^G$ corresponds to integration over the fiber in the fibering $B_G M \to BG$.

Notation: We introduce a special notation for the "equivariant push-forward map" in the previous case (iv), when $M = pt$. Let

$$\int_N \alpha = \pi_!^G(\alpha) \in H^*(BG),\ \alpha \in H_G^*(N)$$

according to the fact, that $\pi_!^G$ is integration over the fiber. So integration is a map:

$$\int_N : H_G^*(N) \to H^{*-n}(BG)$$

(v) As we mentioned in **??** there is a natural map $i^* : H_G^*(X) \to H^*(X)$ from equivariant theory to ordinary theory which corresponds to the inclusion $i : X \to B_G X$ of X as the fiber over a point of BG. Let N, M be n and m-dimensional manifolds, respectively, and $f : N \to M$ be a proper map. Then we have the following commutative diagram:

$$\begin{tikzcd}
H_G^*(N) \arrow[r, "i_N^*"] \arrow[d, "f_!^G"] & H^*(N) \arrow[d, "f_!"] \\
H^{*+m-n}(BG) \arrow[r, "i_M^*"] & H^{*+m-n}(M)
\end{tikzcd}$$

Lemma 6.10. *Let $\sum_{i=0}^{n-1} a_i x^i$, ($a_i \in H^{k-2i}(BG)$) be an arbitrary element of $H^k(B\mathbb{P}V) = H^*_G[x]/\Gamma(x)$, see ??. Then*

$$\int_{\mathbb{P}V} \sum_{i=0}^{n-1} a_i x^i = a_{n-1}.$$

Proof: Since the integration $\int_{\mathbb{P}V} : H^*_G(\mathbb{P}V) \to H^{*-2n+2}(BG)$ lower the dimension by $2n-2$, it sends $1, x, \ldots, x^{n-2}$ to 0. The remaining x^{n-1} goes to $1 \in H^0(BG)$, which can be seen from the following commutative diagram, which is a special case of (v):

$$\begin{tikzcd} H^{2n-2}_G(\mathbb{P}V) \arrow[r, "i^*_{\mathbb{P}V}"] \arrow[d, "\pi^G_!"] & H^{2n-2}(N) \arrow[d, "\pi_!"] \\ H^0(BG) \arrow[r, "i^*_{pt}"] & H^0(pt) \end{tikzcd} .$$

Now $i^*_{\mathbb{P}V}(x^{n-1})$ is the generator of $H^{2n-2}(\mathbb{P}V)$, and by the definition of $\pi_!$ in the non-equivariant case $(i^*_{\mathbb{P}V} \circ \pi_!)(x^{n-1}) = 1 \in H^0(pt)$. The diagram is commutative, and $i^*_{pt} : H^0(BG) \to H^0(pt)$ is isomorphism, so $\pi^G_!(x^{n-1}) = 1 \in H^0(BG)$. ⊠

We have seen at the proof of Lemma ?? that $x = -c_1(B\tau)$ where τ is the tautological line bundle over $\mathbb{P}V$, and $\Gamma(x) = \prod(x + \pi^* w_i)$ where w_i are the weights of the representation V, so

$$\Gamma(x) = x^n + c_1(\pi^* BV) + \ldots + c_{n-1}(\pi^* BV) + e(\pi^* BV),$$

so we can define

$$q = -\frac{\pi^* BV}{x} = x^{n-1} + c_1(\pi^* BV) x^{n-2} + \ldots + c_{n-1}(\pi^* BV) \in H^{2n-2}(B\mathbb{P}V)$$

The following lemma gives the connection between the projective, and affine Thom-polynomial:

Lemma 6.11.

$$[\eta]_G = \int_{\mathbb{P}V} q \cdot [\mathbb{P}\eta]_G$$

Proof: Let $[\mathbb{P}\eta]_G = \sum_{i=1}^{c} g_i x^i$ where $g_i \in H^{2c-2i}(BG)$ be as above. Then

$$q \cdot [\mathbb{P}\eta]_G = q(\sum_{i=1}^{c} g_i x^i) = g_0 q + (\sum_{i=1}^{n-1} g_i x^{i-1}) e(\pi^* BV),$$

so using the previous theorem we get

$$\int_{\mathbb{P}V} q \cdot [\mathbb{P}\eta]_G = \int_{\mathbb{P}V} g_0 q + \int_{\mathbb{P}V} (\sum_{i=1}^{n-1} g_i x^{i-1}) e(\pi^* BV) = g_0 = [\eta]_G. ⊠$$

Lemma 6.12. *Let $Bi : B\mathbb{P}\eta \to B\mathbb{P}V$ denote the map induced by the inclusion $\mathbb{P}\eta \hookrightarrow \mathbb{P}V$. Then*

$$\int_{\mathbb{P}V} q \cdot [\mathbb{P}\eta]_G = \int_{\mathbb{P}\eta} (Bi)^* q.$$

Proof: We have the following diagram:

$$\begin{tikzcd} \mathbb{P}\eta \arrow[r, "i"] \arrow[dr, "\pi^{\mathbb{P}\eta}"'] & \mathbb{P}V \arrow[d, "\pi^{\mathbb{P}V}"] \\ & pt \end{tikzcd} \qquad \begin{tikzcd} B_G\mathbb{P}\eta \arrow[r, "Bi"] \arrow[dr, "B\pi^{\mathbb{P}\eta}"'] & B\mathbb{P}V \arrow[d, "B\pi^{\mathbb{P}V}"] \\ & BG \end{tikzcd}$$

Now using that the push-forward is module-homomorphism:

$$\int_{\mathbb{P}\eta}(Bi)^*q = \int_{\mathbb{P}V} i_!^G((Bi)^*q\cdot 1) = \int_{\mathbb{P}V} q\cdot i_! 1 = \int_{\mathbb{P}V} q\cdot[\mathbb{P}\eta]_G.\boxtimes$$

6.3. **The Atiyah-Bott localization theorem, and integral formula.** The standard reference is [**?**]. From now on, we will work with complex coefficients, so H_G^* means equivariant cohomology with complex coefficients. When $G = T^l$ is a torus of dimension l, then H_G^* is the polynomial ring $\mathbb{C}[u_1,\dots,u_l]$ and this is not a principal ideal domain. As a consequence, modules over this ring no longer have so simple structure as a sum of a free and a torsion module. However we can define the torsion sub-module and the rank. This can be done by passing to the field of fractions $\mathbb{C}(u_1,\dots,u_k)$.

We can define the support of a finitely generated module in the following way: Let H be a module over $\mathbb{C}[u_1,\dots,u_l]$ its *support* is the subset of $\mathbb{C}^l$ defined by:

$$\mathrm{Supp}H = \bigcap_f V_f \textit{ over all } f\in\mathbb{C}[u_1,\dots,u_l] \textit{ with } fH=0$$

where V_f is the hypersurface $f(u)=0$. Thus a free module has the whole space $\mathbb{C}^l$ as a support, while the support of a torsion module indicates which "primes" occur in the torsion.
A related notion is that of *localization.* If $f\in\mathbb{C}[u_1,\dots,u_l]$ is any non-zero polynomial, localizing to the open set $U_f=\mathbb{C}^l-V_F$ means that we allow ourselves to divide by powers of f. More formally we form the ring $\mathbb{C}[u_1,\dots,u_l]_f$ consisting of all rational functions with denominators power of f, and consider the corresponding module

$$H_f = H\otimes_{\mathbb{C}[u_1,\dots,u_l]}\mathbb{C}[u_1,\dots,u_l]_f$$

for any module H over $\mathbb{C}[u_1,\dots,u_l]$. This process kills torsion supported in V_f, i.e.

$$SuppH\subset V_f \Rightarrow H_f = 0.$$

Let T act smoothly on the compact manifold M with F as the set of fixed points. Since the action of T on F is trivial, we have $B_TF\simeq BT\times F$ so $H_T^*(F)=H_T^*\otimes H^*(F)$, so $H_T^*(F)$ is a module over H_T^*.
As we have seen for the inclusion $i: F\hookrightarrow M$ the push-forward map $i_!: H_T^*(F)\to H_T^*(M)$ is a $H_T^*(M)$-module homomorphism, and

$$i^*i_!1 = e_T(\nu_F)$$

so $i^*i_!$ is multiplication in $H_T^*(F)$ by the equivariant Euler-class of the normal bundle ν_F:

$$i^*i_!(i^*\alpha) = i^*i_!(i^*\alpha\cdot 1) = i^*(\alpha\cdot i_!1) = (i^*\alpha)e_T(\nu_F).$$

We will show, that $e_T(\nu_F)$ is be invertible in the localized module

$$H_T^*(F)_f = (H_T^*)_f \otimes H^*(F)$$

where $f \in \mathbb{C}[u_1, \dots, u_l] = H_T^*$ is a suitable polynomial. We can give this polynomial explicitly in the following way. Let $F = \cup P$ be the decomposition of F into its connected components, and we may consider separately each $e_T(\nu_P)$. Since terms of positive degree in $H^*(P)$ are nilpotent, an element in $H^*(P)$ is invertible provided its component in $H^0(P)$ is nonzero. Hence $e_T(\nu_P)$ will be invertible in some localization $H_T^*(P)_f$ provided its component in $H_T^* \otimes H^0(P)$ is a non-zero polynomial, and we choose this polynomial as f. But this component is determined by restricting to any point p of P.
The action of T on the fiber ν_p has no fixed vectors, because the only fixed directions are tangential to P, so ν_p decomposes as a direct sum of irreducible one-dimensional representations of T, and if $a_j \in H^*(BT)$ denotes the weights of the representation ν_p then $e_T(\nu_p) = \prod_j a_j \neq 0$, because if a_i was 0, it would mean, that this weight space is a fix one-dimensional subspace, a contradiction. If we denote this non-zero polynomial by f_P it follows that $i^* i_!$ becomes invertible after localizing with respect to

$$f_F = \prod_P f_P.$$

Working over such a localized ring (or over the full field of rational functions) we therefore see that

$$Q = \sum_P \frac{i_P^*}{e_T(\nu_P)}$$

is inverse to $i_!^F : H_T^*(F) \to H_T^*(M)$. (This is the same as $i_!$, but we need the upper index to distinguish this map from $i_!^P : H_T^*(P) \to H_T^*(M)$.) Thus for any $\varphi \in H_T^*(M)$ we have after localizing

$$\varphi = i_!^F Q \varphi = \sum_P \frac{i_!^P i_P^* \varphi}{e_T(\nu_P)}.$$

Applying the push-forward to point $\pi_!^M : H_T^*(M) \to H_T^*$ to both sides, we get

$$\pi_!^M \varphi = \sum_P \pi_!^M i_!^P \frac{i_P^* \varphi}{e_T(\nu_P)}$$

Now using the functoriality of push-forward, that is $\pi_!^M \circ i_!^P = (\pi^M \circ i^P)_! = \pi_!^P$ we get the *integration formula*:

Theorem 6.13. ***Integration formula***[?] *Let M be a compact oriented manifold with a T-action, F is the set of fixed points of the action and P are the connected components of F. Let $i_P : P \hookrightarrow M$ be the inclusion of P, $i_P^* : H_T(M) \to H_T(P)$, $\varphi \in H_T^*(M)$. Then $e_T(\nu_P) \neq 0$ and*

$$\int_M \varphi = \sum_P \int_P \frac{i_P^* \varphi}{e_T(\nu_P)}.$$

6.4. The Duistermaat-Heckman formula for Thom-polynomials. Here we present the proof of the Duistermaat-Heckman formula from the integration formula, this is due to L. Fehér. We will use the integration formula in the following situation. Let T be a torus, and $T \curvearrowright V$ a linear action of T on the vector space V. (In most cases T is the maximal torus of a compact, connected Lie-group G.) It means that it induces an action $G \curvearrowright \mathbb{P}V$. Let us suppose that the weights of V are different, then $\mathbb{P}V$ has finitely many fixed points, which correspond to weight spaces of V, so the set F of fixed points in this situation is a finite union of single points. Let η be an orbit of the action, such that $\mathbb{P}\eta$ is a smooth variety. Then using the notations of the following commutative diagram of inclusions

$$\begin{tikzcd} f \in F \cap \mathbb{P}\eta \arrow[r, "i_f"] \arrow[d, "i_{f\subset\mathbb{P}\eta}"'] & \mathbb{P}V \\ \mathbb{P}\eta \arrow[ur, "i_{\mathbb{P}\eta}"'] & \end{tikzcd}$$

we get by the integral formula

$$[\eta]_T = \int_{\mathbb{P}\eta} i^*_{\mathbb{P}\eta} q = \sum_{f\in\mathbb{P}\eta\cap F} \frac{i^*_{f\subset\mathbb{P}\eta} i^*_{\mathbb{P}\eta} q}{e_T(\nu_f)} = \sum_{f\in\mathbb{P}\eta\cap F} \frac{i^*_f q}{e_T(\nu_f)}.$$

where i^* always means the corresponding map between equivariant cohomologies. Since f is a point, $\nu_f = T_f\mathbb{P}\eta$, so $e_T(\nu_f) = e_T(T_f\mathbb{P}\eta)$. Recall, that

$$q = \frac{e(\pi^* BV)}{x} = \sum_{i=0}^{n-1} c_{n-i}(\pi^* BV) x^{i-1},$$

and $x = c_1(\tau)$, so $i^*_f x = c_1(i^*_f \tau) = w_f$, where w_f is the weight corresponding to f, which is a weight-space, and $i^*_f e(\pi^* BV) = e_T(V)$. So

$$i^*_f q = \frac{e_T(V)}{w_f}$$

and we get the desired formula

Theorem 6.14. ***Duistermaat-Heckman formula***
Let $T^k \curvearrowright V$ be a linear action of T on the vector space V, $\mathbb{P}\eta \subset \mathbb{P}V$ a T-invariant smooth variety, $\eta = cone\mathbb{P}\eta$,and $B_T\eta \subset B_TV$ represents a cohomology element of $H^*(BT) = \mathbb{Q}[\alpha_1, \dots, \alpha_n]$. Let us suppose that the action $T \curvearrowright \mathbb{P}V$ has a finite number of fixed points, F denote the set of these, and the weight corresponding to $f \in F$ is w_f. Then

$$Tp(\eta) = [\eta]_T = e_T(V) \sum_{f\in\mathbb{P}\eta\cap F} (e_T(T_f\mathbb{P}\eta) w_f)^{-1}$$

Remark 6.15. We don't know any appearance of this theorem in the literature. The formula was communicated to us by A. Knutson.
The right hand side of the Duistermaat Heckman formula is an element of the field of fractions of the cohomology ring H^*_T, but we know a priori, that this is simply a symmetric polynomial, so it is a polynomial of the basic symmetric polynomials which are the Chern classes. If $T \subset G$ is a maximal torus of a compact, connected group, then $[\eta]_T = [\eta]_G$.

7. Calculation of Thom polynomials for homogeneous spaces

7.1. **The method.** We can use the Duistermaat Heckman formula to compute Thom polynomials of homogeneous spaces, which appear when G is a Lie group, $T \subset G$ is a maximal torus, and V is an irreducible representation of G. As we have seen in Chapter **??** we can get all compact homogeneous spaces of G as the orbit of a highest weight vector at the action of G on $\mathbb{P}V$, where V is an irreducible representation of G, so every homogeneous G-space is of the form G/P_λ where P_λ is the parabolic subgroup corresponding to the highest weight λ.
So we will use The Duistermaat Heckman formula when $T \subset G$ is maximal torus, $\mathbb{P}\eta$ is an orbit of G, (then it is invariant subset of T) and this will be a smooth orbit, so it will define a cohomology element. The fixed points under the projective action of T are precisely the weight spaces, which will be different, so all the assumptions of the theorem stand.
So let V be an irreducible representation of G, λ is a highest weight corresponding to an ordering of the weight lattice, $\mathbb{P}\eta$ its orbit in $\mathbb{P}V$. Then $e_T(V)$ is simply the product of the weights of V, it remains two problems: (i) to determine which of the weights are in $\mathbb{P}\eta$; (ii) to compute $e_T(T_f\mathbb{P}\eta)$.

Theorem 7.1. *The weights of V which are in the orbit $\mathbb{P}\eta$ are the weights of the orbit of λ under the action of the Weyl group on the weight lattice, so*

$$F \cap \mathbb{P}\eta = \mathfrak{W} \circ \lambda.$$

These are exactly the highest weights of the representation.

Proof: From the definition and properties of the Weyl group.

The orbit of one highest weight under the action of the Weyl group consist of precisely all the highest weights of the representation. To solve the second problem, we observe that the action of a Lie group on $T_f(\mathbb{P}\eta)$, where $f \in F$ is a fix point of the action, is the infinitesimal action of the Lie group at f, so $T_f(\mathbb{P}\eta)$ is a representation of the corresponding Lie algebra, namely, the adjoint representation.
Precisely, let B_f be the stabilizer of f, which is a Borel subgroup of G, and $X = G/B_f$ is the compact homogeneous space. If $V_\lambda \in \mathbb{P}V$ is the highest weight space, then $B_{V_\lambda} = P_\lambda$, and for other fixed points, which are images of v_λ under the Weyl group, B_f will be conjugate to P_λ. Now the action of G on G/B_f is given by $b(gB_f) = bgB_f = bgb^{-1}B_f$, so it coincides with conjugation, therefore the action of G on $T_e(G/B_f)$ is the adjoint action. Now $T_e(G/B_f) = \mathfrak{g}/\mathfrak{b}_f$, so we have proved the

Theorem 7.2. *$T_f(G/B_f) = \mathfrak{g}/\mathfrak{b}_f$, where the equality means the isomorphism of the two representation, where on the right hand side there is the adjoint representation.*

The following well-known lemma helps us to determine the Lie algebra $\mathfrak{b}_f$ of the stabilizer group of the fix point f:

Lemma 7.3. *Let B_f denote the stabilizer group of the fix point f, which is a Borel subgroup. Then*

$$\mathrm{Lie}(B_f) = \{H \in \mathfrak{g} : \ H(f) = 0\} = Ann(f)$$

where $H(f)$ is the action of H on f. (V is a representation of $\mathfrak{g}$ too.)

So our exercise is to compute the annihilators of the highest weight spaces of the representation, which is now not a difficult problem, since for a highest weight space

V_λ we we have seen in Chapter **??** how can we describe the parabolic subgroup, which is the stabilizer:

Proposition 7.4. *$LieP_\lambda$ corresponds to $\Sigma = \{\mathfrak{g}_\alpha\}_{B(\alpha,\lambda)=0} \subset R^+$, so*

$$T_{V_\lambda}\mathbb{P}\eta = \mathfrak{g}/LieP_\lambda = \mathbb{C}\{\mathfrak{g}_\alpha \in R^- : \ B(\alpha,\lambda) < 0\}.$$

Since we know how to compute Killing form in most cases, we can write up the DH formula easily. Let we see some examples.

7.2. Thom polynomials of Grassmannians.

7.2.1. *The ordinary Grassmannians.*

We can embed the ordinary Grassmannian manifold $Gr(n,k)$, the k-dimensional complex subspaces of $\mathbb{C}^n$ into $\mathbb{P}\wedge^k\mathbb{C}^n$ by the usual Plücker embedding:

$$Gr(n,k) \hookrightarrow \mathbb{P}\wedge^k\mathbb{C}^n$$

$$\mathbb{C} < v_1, \dots, v_k > \to [v_1 \wedge \dots \wedge v_k]$$

Now as we have seen $\wedge^k\mathbb{C}^n$ is an irreducible representation of $SL_n\mathbb{C}$, and so it is an irreducible representation of the corresponding Lie algebra $\mathfrak{sl}_n\mathbb{C}$, with highest weight $\omega_k = L_1 + \dots + L_k$, the kth fundamental weight of the weight lattice, so this is the representation $\Gamma_{0,\dots,0,1,0,\dots,0}$ with 1 at the ith column. $Gr(n,k)$ is the orbit of the highest weight space in $\mathbb{P}\wedge^k\mathbb{C}^n$. Let l_{ω_i} denote the highest weight space. The negative roots are $R^- = \{L_i - L_j\}_{i>j}$, and by proposition **??** we should determine the negative roots with

$$B(L_1 + \dots + L_k, L_i - L_j) < 0.$$

Now as we have seen

$$B(L_i, L_j) = \begin{cases} \frac{-1}{(2n^2)} & i \neq j \\ \frac{n-1}{(2n^2)} & i = j \end{cases}$$

so we get that for $i > j$

$$B(L_1 + \dots + L_k, L_i - L_j) = \begin{cases} \frac{-1}{(2n)} & \text{if} \quad j \leq k < i \\ \frac{n-1}{(2n^2)} & \text{if} \quad j < i \leq k \text{ or } k < j < i \end{cases}.$$

Hence as we have seen, the tangent space of $Gr(n,k)$ at the fix point l_{ω_k} is a representation of $\mathfrak{sl}_n\mathbb{C}$ with weights $\{L_i - L_j\}_{j\leq k<i}$, so we get using the notation L_i for the Chern root $\alpha_i \in H^2(BT)$:

$$e_T(T_{l_{\omega_k}}Gr(n,k)) = \prod_{j\leq k<i}(L_i - L_j).$$

As a consequence we derive the dimension of $Gr(n,k)$, it is $k(n-k)$, hence the codimension of $Gr(n,k)$ in $\wedge^k\mathbb{C}^n$ (which is the degree of the Thom-polynomial) is $\binom{n}{k} - k(n-k) - 1$.
We know, that the the fix points are the images of the weight space l_{ω_k} under the Weyl group. The Weyl group simply permutes the roots, so in a general fix point we must write $L_{i_1} + \dots + L_{i_k}$ instead of $L_1 + \dots + L_k$, and we get the Thom polynomial of the cone of the Grassmannian, which is denoted by $cGr(n,k)$

Example 7.5.

$$Tp(cGr(n,k)) = \prod_{\substack{I\in\binom{n}{k}\\ I=\{i_1<\ldots<i_k\}}} (L_{i_1}+L_{i_2}+\ldots+L_{i_k})\cdot$$

$$\cdot \sum_{\substack{I\in\binom{n}{k}\\ I=\{i_1<\ldots i_k\}}} \frac{1}{(L_{i_1}+L_{i_2}+\ldots+L_{i_k})\prod_{\substack{i\notin I\\ j\in I}}(L_i-L_j)}$$

Here $I\subset\binom{n}{k}$ means that I runs over the k-element subsets of the index set.

Unfortunately we have not succeed to determine the polynomial form of the right side of the DH formula for $n>7$ even with the aid of computers. It is possible to give a not too simple formula for the degree of the Grassmannians using theorem **??**. The substitution in the formula does not work directly, since substituting $\alpha_i=1/k$ the denominator will be zero. But with a little trick we reach our goal: substitute $\alpha_i=\frac{(1+i\varepsilon)}{k}$ and then take the limit $\varepsilon\to 0$. After writing the formula in common denominator form we see, that both the denominator and nominator tends to 0 in the order of $\binom{n}{2}$, so applying the l'Hospital rule $\binom{n}{2}$ times, and then substituting $\varepsilon=0$ we get the following with the notation $\mathcal{I}$ for the set of k-element subsets of $[n]$, and $d=k(n-k)$:

$$deg(Gr(n,k) = \sum_{\binom{\mathcal{J}\in\mathcal{I}}{d}}\left(\prod_{J\in\mathcal{J}}\left(\sum_{j\in J} j\right)\sum_{I\notin\mathcal{J}}\left(\prod_{\substack{i\in I\\ j\notin I}}(j-i)^{-1}\right)\right)$$

There is a well-known formula for this degree, see [?]:

$$deg(Gr(n,k) = (k(n-k))!\prod_{i=0}^{k-1}\frac{i!}{(n-k+i)!}.$$

It is now a good exercise to prove that this is the same as the previous formula, unfortunately we have not succeeded to do it. These are the bad news. The good is that from the Duistermaat Heckman formula it is possible to compute degrees of other symmetric varieties. For example L. Fehér has generalized this technique to compute degrees of duals of Grassmannians (which is in fact the same as the Thom-polynomial itself).

The next example shows a bit more simple case, when we can explicitly determine the Thom polynomial.

7.2.2. *The Veronese variety.*

For any (complex) vector space V and any positive integer n we have a natural map called the Veronese embedding

$$\mathbb{P}V\hookrightarrow\mathbb{P}(Sym^k V)$$

$$[v]\to[v^k]$$

so if V is n-dimensional this is a smooth embedding $\mathbb{C}P^{n-1}\hookrightarrow\mathbb{C}P^{\binom{n+k-1}{k-1}}$. If V is the standard representation of $SL_n\mathbb{C}$ then we have seen that $Sym^kV=\Gamma_{k,0,\ldots,0}$

is an irreducible representation with highest weight $\lambda = kL_1$, and the orbit of a highest weight vector in $\mathbb{P}(Sym^k V)$ is the image of the Veronese embedding, called the Veronese variety, this will be denoted by $\mathcal{V}(n,k)$. Now

$$B(kL_1, L_i - L_j) = \begin{cases} 0 & 1 < j < i \\ -\frac{k}{2n} & 1 = j < i \end{cases}$$

and from Proposition **??** the Lie algebra of the stabilizer subgroup of λ

$$LieP_\lambda = \{R^+, L_k - L_1 \ (k = 2, \dots, n\}, \quad e(T_{kL_1}(\mathcal{V}(n,k)) = \prod_{i=2}^{n} (L_i - L_1).$$

Now by Proposition **??** and Theorem **??** and using that the Weyl group is simply σ_n we get the Thom polynomial of the Veronese embedding:

$$Tp(\mathcal{V}(n,k)) = \prod_{1 \le i_1 \le i_2 \dots \le i_k} (L_{i_1} + \dots + L_{i_k}) \cdot$$

$$\cdot \sum_{i=1}^{n} \frac{1}{kL_i(L_1 - L_i) \dots (L_{i-1} - L_i)(L_{i+1} - L_i) \dots (L_n - L_i)}$$

Now a little computation shows that

$$\sum_{i=1}^{n} \frac{1}{L_i(L_1 - L_i) \dots (L_{i-1} - L_i)(L_{i+1} - L_i) \dots (L_n - L_i)} = \frac{1}{L_1 \dots L_n}.$$

So the Thom polynomial of the cone is the following:

Example 7.6.

$$Tp(c\mathcal{V}(n,k)) = \prod_{\substack{1 \le i_1 \le i_2 \dots \le i_k \\ i_1 \ne i_k}} (L_{i_1} + \dots + L_{i_k}).$$

7.2.3. ***The orthogonal Grassmannian, and the spinor variety.***

Next consider the irreducible representation $\wedge^k V = \Gamma_{\omega_k}$ of $SO_{2n}\mathbb{C}$ where V is the standard representation.($1 \le k \le n-2$) The highest weight of this is the fundamental weight $\lambda = \omega_k = L_1 + \dots + L_k$, and the corresponding homogeneous space, which is the orbit of this highest weight space in $\mathbb{P} \wedge^k V$ is the smooth manifold of k-dimensional orthogonal Grassmannian, which is the set of k-dimensional isotropic subspaces of V for the given bilinear form on V (see **??**) we denote it by $Gr_o(n,k)$. These orthogonal Grassmannians are therefore minimal orbits, the stabilizer of the highest weight space is a parabolic subgroup of $SO_{2n}\mathbb{C}$. The negative roots are $R^- = \{-L_i - L_j\}_{i>j} \cup \{L_i - L_j\}_{i>j}$ and

$$B(L_1 + L_2 + \dots + L_k, -L_i - L_j)_{i>j} = \begin{cases} 0 & k < j < i \\ -1/(n-1) & j \le k < i \\ -2/(n-1) & j < i \le k \end{cases}$$

so $B(\lambda, -L_i - L_j) < 0$ iff $\{1, 2, \dots, k\} \cap \{i, j\} \ne 0$, and we get in the same way that $B(\lambda, L_i - L_j) < 0$ iff $j \le k < i$, so $\{1, 2, \dots, k\} \cap \{i, j\} = j$. Now again by Proposition **??**

$$e(T_\lambda Gr_o(n,k)) = \prod_{j < i \le k} (L_i + L_j) \prod_{j \le k < i} (L_j + L_i)(L_j - L_i).$$

The Weyl group in this case permutes the L_is, and changes the sign of even of them. So a general element of $\mathcal{W}$ takes L_s to $\varepsilon_{i_s} L_{i_s}$ where $\varepsilon_i = \pm 1$, $|\{i : \varepsilon_i = -1\}| = even$. The image of $\omega_k = L_1 + \ldots + L_k$ under this action is $\varepsilon_{i_1} L_{i_1} + \varepsilon_{i_2} L_{i_2} + \ldots + \varepsilon_{i_k} L_{i_k}$.

Remark 7.7. Note that $L_1 + \ldots + L_k$ can have the same image under different elements of the Weyl group, since we can permute the weights $L_{i_{k+1}}, \ldots, L_{i_n}$ arbitrarily after saying the images of $L_1, \ldots, L_k$. But $e(T_{L_{i_1}+\ldots+L_{i_k}} Gr_o(n,k))$ will be the same at every two such element of the Weyl group, despite of the fact, that in the expression $e(T_{L_{i_1}+\ldots+L_{i_k}} Gr_o(n,k))$ occur $L_{i_{k+1}}, \ldots, L_{i_n}$ as well. This can be seen from the expression we have got.

So the equivariant Euler class of the tangent space at a general fix point $\delta = L_{i_1} + \ldots + L_{i_k}$ of the action is

$$e(T_\delta Gr_o(n,k)) = \prod_{s<t\le k} (\varepsilon_{i_t} L_{i_t} + \varepsilon_{i_s} L_{i_s}) \prod_{s\le k<t} (\varepsilon_{i_s} L_{i_s} + L_{i_t})(\varepsilon_{i_s} L_{i_s} - L_{i_t}).$$

The Thom polynomial of the cone is the following:

Example 7.8.

$$Tp(cGr_o(n,k) =$$

$$= A \sum_{\substack{I \in \binom{n}{k} \\ \{e_1,\ldots,\varepsilon_k\} \in \{\pm 1\}^k}} \left((\varepsilon_1 L_{i_1} + \ldots + \varepsilon_k L_{i_k}) \prod_{s,t \in I} (\varepsilon_t L_{i_t} + \varepsilon_s L_{i_s}) \prod_{\substack{1 \le s \le k \\ t \notin I}} (\varepsilon_{i_s} L_{i_s} + L_t)(\varepsilon_{i_s} L_{i_s} - L_t) \right)^{-1},$$

where

$$A = \prod_{1 \le i_1 \le i_2 \le \ldots \le i_k \le n} (L_{i_1} + L_{i_2} + \ldots + L_{i_k})$$

Example **??** tells us the Thom polynomials of minimal orbits corresponding to highest weights Γ_{ω_k} for $k = 1, \ldots n-2$. It remains to determine the corresponding Thom polynomials for the remaining two spinor representations: the minimal orbits are called spinor varieties. Here are the results:

First we give a short description of the spin representations, since these can be not constructed from the standard representation with the usual methods like making tensor products, exterior powers, symmetric powers, kernels or co-kernels of homomorphisms.

Let $\mathcal{Q}$ be a nondegenerate symmetric bilinear form on a vector space V^m, the Clifford algebra $C = C(\mathcal{Q}) = \text{Cliff}(V, \mathcal{Q})$ is an associative algebra with unit 1, which contains and is generated by V, with $v \cdot v = \mathcal{Q}(v,v) \cdot 1$ for all $v \in V$. Equivalently, we have the equation $v \cdot w + w \cdot v = 2\mathcal{Q}(v,w)$ for all $v, w \in V$. If $e_1, e_2, \ldots, e_m$ form a basis for V, then the products $e_{i_1} e_{i_2} \cdots e_{i_k}$ with $e_\emptyset = 1$ form a basis for $C(\mathcal{Q}) = Cliff(V, \mathcal{Q})$, so $dim C(\mathcal{Q}) = 2^m$. Now the Clifford algebra has a natural Z_2 grading: $C = C^{even} \oplus C^{odd}$, where C^{even} is spanned by products of an even number of elements in V, and C^{odd} by products of an odd number, in particular C^{even} is a subalgebra of dimension 2^{m-1}. Since $C(Q)$ is an associative algebra, it determines a Lie algebra, with bracket $[a,b] = ab - ba$. The spinor representations of $\mathfrak{so}_m \mathbb{C}$ can be found in two steps:

(i) embedding $\mathfrak{so}_m\mathbb{C}$ inside the Lie algebra of the even part C^{even}. This can be done easily, see [?].

(ii) identifying the Clifford algebra with one or two copies of matrix algebras. To do this let $V = W + W'$ where W and W' are n-dimensional isotropic subspaces for $\mathcal{Q}$. With our standard $\mathcal{Q}$ on $V = \mathbb{C}^{2n}$ W can be the subspace spanned by the first n basis vectors. Now we have isomorphism of algebras $C(\mathcal{Q}) \cong End(\wedge^\bullet W)$, where $\wedge^\bullet W = \wedge^0 W \oplus \ldots \oplus \wedge^n W$.
Now we have a decomposition $\wedge^\bullet W = \wedge^{even} W \oplus \wedge^{odd} W$ into the sum of even and odd exterior powers, and $C(W)^{even}$ respects this splitting. By the isomorphism $C(\mathcal{Q}) \cong End(\wedge^\bullet W)$ we have

$$C(Q)^{even} \cong End(\wedge^{even} W) \oplus End(\wedge^{odd} W)$$

so

$$\mathfrak{so}(\mathcal{Q}) \subset C(\mathcal{Q})^{even} \cong \mathfrak{gl}(\wedge^{even} W) \oplus \mathfrak{gl}(\wedge^{odd} W),$$

and hence we have two representations of $\mathfrak{so}(\mathcal{Q}) = \mathfrak{so}_{2n}\mathbb{C}$, which we denote by

$$S^+ = \wedge^{even} W \text{ and } S^- = \wedge^{odd} W.$$

Proposition 7.9. *The representations $S^\pm$ are the irreducible representations of $\mathfrak{so}_{2n}\mathbb{C}$, with highest weights $\alpha = \frac{1}{2}(L_1 + \ldots + L_n)$ and $\beta = \frac{1}{2}(L_1 + \ldots + L_{n-1} - L_n$ (the half-spin representations), more precisely*

$$S^+ = \Gamma_\alpha \textit{ and } S^- = \Gamma_\beta \textit{ if } n \textit{ is even}$$

$$S^+ = \Gamma_\beta \textit{ and } S^- = \Gamma_\alpha \textit{ if } n \textit{ is odd.}$$

The representation $S = S^+ \oplus S^- = \wedge^\bullet W$ is called the spin representation.

It is can be shown easily that the natural basis vectors $e_I = e_{i_1} \wedge \ldots \wedge e_{i_k}$ for $\wedge^\cdot W$ span weight spaces with weight $\frac{1}{2}(\sum_{i\in I} L_i - \sum_{j\notin I} L_j)$. All such weights with given I mod 2 are congruent by the Weyl group, so each of S^+ and S^- must be an irreducible representation with highest weights trivially α and β.

For the odd case, write $V = W \oplus W' \oplus U$ where W and W' are n-dimensional isotropic subspaces, and U is a one-dimensional space perpendicular to them. For the standard $\mathcal{Q}$ these are spanned by the first and second n basis vectors, and the last one. Now this decomposition determines an isomorphism of algebras $C(\mathcal{Q}) \cong End(\wedge^\bullet W) \oplus End(\wedge^\bullet W')$. The subalgebra $C(\mathcal{Q})^{even}$ of $C(\mathcal{Q})$ is mapped onto either of the factors by this isomorphism, so we have

$$C(\mathcal{Q})^{even} \cong End(\wedge^\bullet W),$$

and we get

$$\mathfrak{so}_{2n+1}\mathbb{C} = \mathfrak{so}(\mathcal{Q}) \subset C(\mathcal{Q})^{even} \cong \mathfrak{gl}(\wedge^\bullet W).$$

Proposition 7.10. *The representation $S = \wedge^\bullet W$ is the irreducible representation of $\mathfrak{so}_{2n+1}\mathbb{C}$ with highest weight $\alpha = \frac{1}{2}(L_1 + \ldots + L_n)$, this is the spinor representation of $\mathfrak{so}_{2n+1}\mathbb{C}$.*

The simply-connected covering spaces of the orthogonal groups are called spin groups $Spin(\mathcal{Q})$. We have a two-sheeted covering homomorphism of $SO(\mathcal{Q})$

$$\rho : \ \mathrm{Spin}(\mathcal{Q}) \to SO(\mathcal{Q})$$

with kernel $\{1, -1\}$. The spin representations are representations of $Spin(\mathcal{Q})$. In the odd case the closed orbit of the spin representation S is the orthogonal Grassmannian of n-dimensional isotropic subspaces of $\mathbb{C}^{2n+1}$, the corresponding projective variety has dimension $\binom{n+1}{2}$ in $\mathbb{P}^N$, $N = 2^n - 1$, and called the spinor variety, denoted by $\mathcal{S}$ or the variety of pure spinors. For $Spin_{2n}\mathbb{C}$ the orthogonal Grassmannian of n-dimensional isotropic subspaces has two components, and the first is the minimal orbit of S^+, the other is of S^-. The corresponding spinor varieties $\mathcal{S}_n^+$ and $\mathcal{S}_n^-$ have dimension $\binom{n}{2}$ in $\mathbb{P}^{2^{n-1}-1}$.

Remark 7.11. The spinor variety for $Spin_{2n-1}\mathbb{C}$ is isomorphic to each of the spinor varieties for $Spin_{2n}\mathbb{C}$. In fact they are projectively equivalent as subvarieties of projective space $\mathbb{P}^{2^{n-1}-1}$. So it is enough to determine the Thom polynomial in the even case.

Since the Lie algebra of $Spin(\mathcal{Q})$ is the same as that of $SO(\mathcal{Q})$, we can use the method described above with $\mathfrak{so}(\mathcal{Q})$. This is an example, where this method, which uses only the structure of the Lie algebra is more effective then the other method we mentioned earlier: the method of restriction equations. For computing restriction equations it should be necessary to understand the action of the Lie group $Spin(\mathcal{Q})$ on S^+ and S^-, which is quite complicated, and we have not succeeded to calculate the Thom polynomial by this method.

The highest weights of the representation S^+ are

$$\left\{ \frac{1}{2}(\varepsilon_1 L_1 + \ldots + \varepsilon_n L_n \right\}_{\substack{\{\varepsilon_1, \ldots, \varepsilon_n\} \in \{-1,+1\}^n \\ \#\{-1\} = even}},$$

while those of S^- are

$$\left\{ \frac{1}{2}(\varepsilon_1 L_1 + \ldots + \varepsilon_n L_n \right\}_{\substack{\{\varepsilon_1, \ldots, \varepsilon_n\} \in \{-1,+1\}^n \\ \#\{-1\} = odd}}.$$

moreover, as we have seen before, these are all the weights of these representations. So we get

Example 7.12.

$$Tp(\mathcal{S}_n^+) = \prod_{\substack{\varepsilon_1, \ldots, \varepsilon_n \in \{-1,+1\}^n \\ \#\{-1\} = even}} (\varepsilon_1 L_1 + \ldots + \varepsilon_n L_n) \cdot$$

$$\cdot \sum_{\substack{\varepsilon_1, \ldots, \varepsilon_n \in \{-1,+1\}^n \\ \#\{-1\} = even}} \left((\varepsilon_1 L_1 + \ldots + \varepsilon_n L_n) \prod_{i<j} (-\varepsilon_i L_i - \varepsilon_j L_j) \right)^{-1}$$

$$Tp(\mathcal{S}_n^-) = \prod_{\substack{\varepsilon_1, \ldots, \varepsilon_n \in \{-1,+1\}^n \\ \#\{-1\} = odd}} (\varepsilon_1 L_1 + \ldots + \varepsilon_n L_n) \cdot$$

$$\cdot \sum_{\substack{\varepsilon_1,\dots,\varepsilon_n\in\{-1,+1\}^n \\ \#\{-1\}=odd}} \left((\varepsilon_1 L_1 + \ldots + \varepsilon_n L_n) \prod_{i<j} (-\varepsilon_i L_i - \varepsilon_j L_j) \right)^{-1}$$

Unfortunately even the fastest personal computer "dies" for $n > 5$. In the case $n = 5$ the spinor varieties are 10-dimensional complex varieties in the 15-dimensional spinor representations, so $Tp(\mathcal{S}_n^{\pm}) \in H^{10}(BSO_{10}\mathbb{C}) = \mathbb{Z}$, which is generated by the Euler-class. So the Thom polynomial is some integer multiple of the Euler class, and some computer computation gives that it is itself the Euler class, so $Tp(\mathcal{S}_5^{\pm}) = L_1 \cdot \ldots \cdot L_5$.
It would be necessary to simplify the expression on the right hand side above before we use computers. This is our next goal: to determine the polynomial form of the Thom polynomial for these minimal orbits, by simplifying the expression.

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