On rearrangements of conditionally convergent series

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Balázs Gerencsér

Supervisor: **Gábor Tusnády**

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Eötvös Loránd University, Institute of Mathematics

Contents

Chapter 1

Introduction

Given a real series $a_1, a_2, \ldots$ it is meaningful to examine the "infinite commutativity" of addition. It is a well known fact that in general

$$\sum_{j=1}^{\infty} a_j \neq \sum_{j=1}^{\infty} a_{\pi(j)}$$

may happen for a permutation π of positive integers. In this paper we will focus on these conditionally convergent series where this can happen.

The starting point is Riemann's theorem about the possibility of obtaining any sum (more precisely any liminf and limsup) by an appropriate rearrangement. One may want to know more than existence, namely some characterization of rearrangements that preserve convergence or even the sum of conditionally convergent series.

A favorite object to analyze is the set of permutations of $\mathbb{N}$ which preserve convergence and sum for all conditionally convergent series. We will use a necessary and sufficient condition by R. P. Agnew [1], which states that a universally sum preserving rearrangement can break the intervals $[1, n]$ into only a limited number of intervals. There are a few different characterizations, a famous one is that of F. W. Levi [5], which examines how elements of the permutation jump over an integer n. Although these are exact conditions, they might be hard to check in some situation. Fortunately there are several simply formulated sufficient conditions. A few of them is presented by P. Schaefer, [8]. For example, it is enough that there is an integer B such that $\pi(j) \leq j + B$ for all j. Similar results can be found in U. Elias [2], J. R. Stefánsson [10], and U. C. Guha [4].

The original problem is simple enough and there is the possibility of countless modifications, generalizations or special cases.

One orientation of research is to require some extra properties of the permutation, and check whether there exist a permutation which preserves the summability of a given series or not. J. H. Smith proved in [9] existence of a permutation with the cycle structure fixed for any conditionally convergent series, only excluding trivial cases.

In [13], G. Tusnády estabilished connection between the relation of series and relation of their sets of convergence preserving permutations. This will be detailed later.

A possible extension is to change the range of the series from $\mathbb{R}$ to something else. This may increase the complexity of the problems because multiple types of convergence can coexist. For example, take real-valued measurable functions on $[0, 1]$. As proved by Nikishin in [6], if $\sum_{k=1}^{\infty} f_k \to F$ in measure, and $\sum_{k=1}^{\infty} f_k^2(t) < \infty$ almost everywhere, then the sum will converge to the same limit F almost everywhere for an appropriate rearrangement. G. Giorgobiani gave similar statements in [3] when the functions take their value from a normed space.

Starting from the fact that permutations of $\mathbb{N}$ form a group, one can meet new questions. Some algebraic problems are covered by G. S. Stoller in [11] and P. A. B. Pleasants in [7]. A detailed examination was done by Q. F. Stout in [12]. In this work, the "closure" of a set of permutation is defined as the set of permutations which preserve the sum of the same subset of series. The closure of a set of series is defined in a similar way. It is hard to produce the closure of a single permutation, but the closure of a single series is proved to be the linear span of the series and all absolutely convergent series.

In the present work, we will examine some structural questions about the set of convergence preserving permutations for a specific series. This will be developed in the next three chapters. After that, we will face the natural problem of reconstruction. The question is whether we can rebuild the series if only the set of convergence preserving permutations are given. In the last chapter we will have an outlook on a stochastic counterpart, where we use random permutations for rearrangement.

Chapter 2

Preliminaries, notations

We will work with real valued series that will be identified as the elements of $\mathbb{R}^{\mathbb{N}}$. Let us define the subset of series with convergent sums

$$S = \left\{ \mathbf{a} = \{a_i\}_{i=1}^{\infty} \in \mathbb{R}^{\mathbb{N}} \mid \exists A \in \mathbb{R} \lim_{n\to\infty} (A - \sum_{i=1}^{n} a_i) = 0 \right\},$$

and the subset of absolutely convergent series:

$$S_0 = \left\{ \mathbf{a} = \{a_i\}_{i=1}^{\infty} \in \mathbb{R}^{\mathbb{N}} \mid \exists A \in \mathbb{R} \lim_{n\to\infty} (A - \sum_{i=1}^{n} |a_i|) = 0 \right\}.$$

We would also like to define the set of series which might be convergent after an appropriate rearrangement. For an $\mathbf{a} \in \mathbb{R}^{\mathbb{N}} \setminus S_0$ it is clearly necessary that $\lim\limits_{n\to\infty} a_n = 0$ and that both the positive and the negative elements of the series must sum up to ∞. This is also sufficient, as it can be easily seen. We can start a rearrangement by the first positive element, then choose the first few negative elements until the sum goes below 0. Then continue with the first few unused nonnegative elements until the sum reaches 0, and so on. The condition that both the positive and the negative elements have infinite sums ensure that this procedure will never get stuck, and the condition about elements converging to 0 implies that the deviation from 0 will tend to zero, hence this rearrangement forces the partial sums to go to 0. Now we define:

$$\tilde{S} = S_0 \cup \left\{ \mathbf{a} = \{a_i\}_{i=1}^{\infty} \in \mathbb{R}^{\mathbb{N}} \mid \lim_{i\to\infty} a_i = 0, \sum_{i=1}^{\infty} |a_i|^+ = \infty, \sum_{i=1}^{\infty} |a_i|^- = \infty \right\}.$$

When investigating a series, we will often work with the subseries of nonnegative elements so let us introduce:

$$S^+ = \left\{ \mathbf{a} = \{a_i\}_{i=1}^{\infty} \in \mathbb{R}^{\mathbb{N}} \mid \forall i \in \mathbb{N}\ a_i \geq 0, \lim_{i\to\infty} a_i = 0 \right\}.$$

$$S_0^+ = \left\{ \mathbf{a} = \{a_i\}_{i=1}^{\infty} \in \mathbb{R}^{\mathbb{N}} \mid \forall i \in \mathbb{N}\ a_i \geq 0,\ \sum_{i=1}^{\infty} a_i < \infty \right\}.$$

Later we will need series with almost nonnegative elements, which means that we allow a subseries of negative elements with convergent sum.

$$\tilde{S}^+ = S^+ - S_0^+.$$

When we are interested in possible rearrangements, multiplying a series with a constant or adding an absolutely convergent series doesn't make sense. So we call two conditionally convergent series $\mathbf{a}, \mathbf{b} \in S \setminus S_0$ equivalent, $\mathbf{a} \equiv \mathbf{b}$, if $\mathbf{a} - \lambda \mathbf{b} \in S_0$ for some $\lambda \in \mathbb{R}$. Obviously $\lambda \neq 0$. This equivalency is also meaningful for elements of $\tilde{S} \setminus S_0$.

Let us move on to permutations. Let P be the set of all permutations of $\mathbb{N}$:

$$P = \{\pi : \mathbb{N} \to \mathbb{N} \mid \pi \textit{ is a bijection}\}.$$

A permutation describes a rearrangement of a series

$$\pi \mathbf{a} = \{a_{\pi_i}\}_{i=1}^{\infty}.$$

Note that for any $\pi, \rho \in P$ this definition has the property

$$\pi(\rho \mathbf{a}) = (\rho \pi) \mathbf{a}.$$

We introduce the notation:

$$s(\mathbf{a}, \pi) = \lim_{n \to \infty} \sum_{i=1}^{n} a_{\pi_i}.$$

This might be finite or infinite, or even might not exist. It is natural to use $|s(\mathbf{a}, \pi)| < \infty$ to indicate that a permutation assigns a convergent sum to $\mathbf{a}$. When we omit π, then we use the identity permutation.

Given a series $\mathbf{a}$, we are interested which permutations lead to a convergent sum. We denote the set of these permutations by $K(\mathbf{a})$ and call it the convergence set of $\mathbf{a}$:

$$K(\mathbf{a}) = \{\pi \in P \mid |s(\mathbf{a}, \pi)| < \infty\}.$$

It is obvious that if $\mathbf{a} \in S_0$, then $K(\mathbf{a}) = P$, if $\mathbf{a} \notin \tilde{S}$, then $K(\mathbf{a}) = \emptyset$. There are some permutations which do not ruin the convergence of any conditionally convergent series, these form the set P_0, that is,

$$P_0 = \bigcap_{\mathbf{a} \in S} K(\mathbf{a}).$$

Chapter 3

Small local deviations

Given a series $\mathbf{a}$, let us define the partial sums $s_n = \sum_{i=1}^{n} a_i$. A necessary condition for s_n to converge is that locally (whatever that means) the deviations should be small. An easy case to test this when several adjacent elements of $\mathbf{a}$ have the same sign. In this case one occurring deviation is the absolute sum of these adjacent elements.

To exploit this idea, we restrict the permutations so that the order within nonnegative and negative elements is fixed. This means that we split the series $\mathbf{a}$ into $\mathbf{b}$ and $\mathbf{c}$, where $\mathbf{b}$ contains the nonnegative and $\mathbf{c}$ contains the negative elements. Now a legal permutation looks like

$$b_1, b_2, \ldots, b_{n_1}, c_1, c_2, \ldots, c_{m_1}, b_{n_1+1}, \ldots, b_{n_2}, c_{m_1+1}, \ldots, c_{m_2}.$$

The necessary condition described above can be formulated as

$$\lim_{k\to\infty} \sum_{i=n_k+1}^{n_{k+1}} b_i = 0 \tag{3.1}$$

and the same for $\mathbf{c}$. Moreover, if this is true for $\mathbf{b}$ and $\mathbf{a} \in \tilde{S} \setminus S_0$, then one can build a convergent permutation where the elements of $\mathbf{b}$ are segmented this. Simply after each block, insert a few (maybe 0) elements of $\mathbf{c}$ until the partial sum decreases below 0. This method provides the $\liminf$ is 0 using $\lim_{i\to\infty} c_i = 0$, and (3.1) forces the $\limsup$ to be 0 by the same reason.

For a fixed $\mathbf{b}$, we try to find some structure of the sets $\{n_1, n_2, \ldots\} = N \subset \mathbb{N}$ which satisfy (3.1). Let $\mathcal{H}_b$ be the system of these sets. There are some trivial cases, we would like to avoid these.

If $\sum_{i=1}^{\infty} b_i < \infty$, this means $\sum_{k=1}^{\infty} \left(\sum_{i=n_k+1}^{n_{k+1}} b_i \right) < \infty$, but this involves (3.1), so $\mathcal{H}_b = 2^{\mathbb{N}}$. If $\limsup_{i\to\infty} b_i > 0$, then $\limsup_{k\to\infty} \sum_{i=n_k+1}^{n_{k+1}} b_i > \limsup_{i\to\infty} b_i > 0$, so $\mathcal{H}_b = \emptyset$.

To avoid these cases, we will assume that $\mathbf{a} \in \tilde{S} \setminus S_0$ or equivalently $\mathbf{b}, -\mathbf{c} \in S^+ \setminus S_0^+$ through the whole chapter. Obviously this means that every element of $\mathcal{H}_b$ must be infinite. These sets are somehow related to the speed of convergence to 0. It is clear that $\mathcal{H}_{b^1} \subset \mathcal{H}_{b^2}$ if $\lambda \mathbf{b}^1 > \mathbf{b}^2$ element-wise for some positive λ. A similar statement is true for $\mathbf{c}$.

We start the examination of the structure of $\mathcal{H}_b$ by showing a method to build a third element if two others are given.

Definition 1 *Let $N, K \in \mathcal{H}_b$, $N = \{n_1 < n_2 < \ldots\}$, $K = \{k_1 < k_2 < \ldots\}$ Let us define $H = N \wedge K = \{h_1 < h_2 < \ldots\}$, where*

$$h_1 = \max(n_1, k_1),$$

$$h_{i+1} = \max(\min(N \cap [h_i + 1, \infty)), \min(K \cap [h_i + 1, \infty))), \ i \geq 1.$$

We should note that $N \wedge K$ will also have infinite elements. This is true because N and K are infinite, thus we will take the minimum of non-empty sets, so the algorithm will not stop after finite number of steps.

Proposition 1 *We have $H \in \mathcal{H}_b$.*

Proof: We want to show that an interval $[h_i + 1, h_{i+1}]$ can be covered with at most two intervals of type $[n_{i'} + 1, n_{i'+1}]$ or $[k_{i''} + 1, k_{i''+1}]$. Without the loss of generality we may suppose $h_i = n_{i'}$. If $h_{i+1} = n_{i'+1}$, then

$$\sum_{j=h_i+1}^{h_{i+1}} b_j = \sum_{j=n_{i'}+1}^{n_{i'+1}} b_j.$$

If $h_{i+1} = k_{i''+1}$, then

$$\sum_{j=h_i+1}^{h_{i+1}} b_j \leq \sum_{j=n_{i'}+1}^{n_{i'+1}} b_j + \sum_{j=k_{i''}+1}^{k_{i''+1}} b_j.$$

Now for any $\varepsilon > 0$, let T be large enough such that for $n_{i'+1} > T$

$$\sum_{j=n_{i'}+1}^{n_{i'+1}} b_j < \varepsilon,$$

and the same be true for K, then choose $h_i > T$. From the previous observation,

$$\sum_{j=h_i+1}^{h_{i+1}} b_j < 2\varepsilon.$$

This means (3.1) for H, so $H \in \mathcal{H}_b$.

$\square$

If $N \subset K \subset \mathbb{N}$, then K gives a finer slicing than N, and it is clear that $N \in \mathcal{H}_b$ implies $K \in \mathcal{H}_b$. We use this latter property to define a partial ordering on $2^{\mathbb{N}}$:

Definition 2 *Let $H, K \subset \mathbb{N}$. We say $H \leq K$ if for any $\mathbf{b} \in S^+ \setminus S_0^+$ $H \in \mathcal{H}_b$ implies $K \in \mathcal{H}_b$. We say $H \cong K$ if $H \leq K$ and $K \leq H$. $H < K$ means $H \leq K$ but $H \ncong K$.*

Proposition 2 *$H \leq K$ if and only if there exists $C \in \mathbb{N}$ such that*

$$\forall i \geq 1 \ |[k_i, k_{i+1}] \cap H| \leq C.$$

Proof: First assume the inequality holds for all i, and choose $\mathbf{b}$ such that $H \in \mathcal{H}_b$. We shall prove $K \in \mathcal{H}_b$.

Fix $\varepsilon > 0$. $H \in \mathcal{H}_b$ means that there exists a T such that for $h_{j+1} > T$

$$\sum_{l=h_j+1}^{h_{j+1}} b_l < \varepsilon.$$

In the case $k_i > T$, choose the maximal j' and the minimal j'' for $[k_i, k_{i+1}] \subset [h_{j'}, h_{j''}]$ to hold. Our condition ensures $j'' - j' \leq C + 1$. Now

$$\sum_{l=k_i+1}^{k_{i+1}} b_l \leq \sum_{l=h_{j'}+1}^{h_{j''}} b_l = \sum_{l=h'_j+1}^{h_{j'+1}} b_l + \sum_{l=h_{j'+1}+1}^{h_{j'+2}} b_l + \ldots + \sum_{l=h_{j''-1}+1}^{h_{j''}} b_l < (C+1)\varepsilon.$$

So in the end, for $(C+1)\varepsilon$, T will be an adequate bound.

Conversely, suppose the inequality does not hold. This means that there exist a series $\{i_n\}_{n=1}^{\infty}$ such that for

$$A_n := [k_{i_n}, k_{i_n+1}] \cap H, \ |A_n| > n.$$

Let us define the series $\mathbf{b}$ as follows:

$$b_{j+1} = \begin{cases} \frac{1}{n}, & if\ j \in A_n \\ 0, & otherwise \end{cases}.$$

Clearly $\sum_{j=h_i+1}^{h_{i+1}} a_j$ is $\frac{1}{n}$, if $h_i \in A_n$, 0 otherwise, and using A_n is bounded, (3.1) holds. On the other hand, $\sum_{j=k_{i_n}+1}^{k_{i_n+1}} a_i > \frac{n-1}{n}$, so $K \notin \mathcal{H}_b$. This shows $H \not\leq K$.

□

Proposition 3 *Let $N, K \subset \mathbb{N}$, $H = N \wedge K$. In this case, $H \leq N$ and $H \leq K$.*

Proof: It is enough to prove the first, we will do this using Proposition 2. Let us choose $n_i \in N$. $[n_i, n_{i+1}] \cap H$ might be empty, but if not, let h_l be the first element of $[n_i, n_{i+1}] \cap H$. Looking at the definition of $\wedge$ it follows that $h_{l+1} \geq n_{i+1}$, so using $C = 2$ the condition of Proposition 2 will hold.

□

By Proposition 1 we may notice the operation $\wedge$ produces an element of $\mathcal{H}_b$ which is smaller than the elements we started with.

It is also true that $\mathcal{H}_b$ does not have a minimal element.

Proposition 4 *Suppose $K \in \mathcal{H}_b$ for some $\mathbf{b} \in S^+ \setminus S_0^+$. Then there exists $H < K$, $H \in \mathcal{H}_b$.*

Proof: We will delete some elements from K such that the condition of Proposition 2 does not hold. It is enough if we drop n consecutive elements of K somewhere for every n. Let us choose $i_1 = 1$, and then recursively choose $i_n \geq i_{n-1} + n$ to ensure

$$\sum_{j=k_i+1}^{k_{i+1}} b_j < \frac{1}{n^2}$$

for every $i \geq i_n$. Now delete the elements k_{i_n+j} for all $1 \geq n$, $0 \leq j \leq n-1$ to get H. $H \leq K$ is obvious, so is $K \not\leq H$. The series $\sum_{j=h_i+1}^{h_{i+1}} b_j$ is the combination of a subseries of $\sum_{j=k_i+1}^{k_{i+1}} b_j$ and of a series less than $\frac{1}{i}$, this occurs when we pass deleted elements of K. Both subseries converge to 0, consequently (3.1) holds, $H \in \mathcal{H}_b$.

□

We will show the previously defined partial ordering form a lattice structure on $2^{\mathbb{N}}/\cong$, with $\cup$ and $\wedge$ as the lattice operations. To see this, we need the following two propositions.

Proposition 5 *Let $N, K, H \subset \mathbb{N}$. If $H \leq N$ and $H \leq K$ then $H \leq N \wedge K$. If $H \geq N$ and $H \geq K$ then $H \geq N \cup K$.*

Proof: Let us start with the first statement. Suppose $H \in \mathcal{H}_b$ for some $\mathbf{b} \in S^+ \setminus S_0^+$, then our condition provides $N, K \in \mathcal{H}_b$. By Proposition 1 this implies $N \wedge K \in \mathcal{H}_b$. What we wrote here is the definition of $H \leq N \wedge K$. To prove the second statement, we will use Proposition 2. $H \geq N$ means that there is an appropriate C_N, and similarly a C_K, thus

$$|[h_i, h_{i+1}] \cap (N \cup K)| \leq |[h_i, h_{i+1}] \cap N| + |[h_i, h_{i+1}] \cap K| \leq C_N + C_K.$$

This is true for every i, therefore $H \geq N \cup K$.
□

Proposition 6 *For any $N, K \subset \mathbb{N}$, $(N \cup K) \wedge N = N$ and $(N \wedge K) \cup N \cong N$.*

Proof: The first statement is trivial from the definition of $\wedge$. Note that the max will always choose the element of the coarser partitioning. Concerning the second one, $(N \wedge K) \cup N \supset N$, so $(N \wedge K) \cup N \geq N$. The opposite direction is a consequence of the previous Proposition, $N \wedge K \leq N$, $N \leq N$, so $(N \wedge K) \cup N \leq N$, and we are ready.
□

Now we can sum up what we found until now about any $\mathcal{H}_b$:

Theorem 1 *For any $\mathbf{b} \in S^+ \setminus S_0^+$, $\mathcal{H}_b$ is a filter of $2^{\mathbb{N}}/\cong$ with no minimal elements.*

By the way, we can exclude some trivial cases. $\emptyset \in \mathcal{H}_b$ would mean $2^{\mathbb{N}} = \mathcal{H}_b$ but this is impossible for $\mathbf{b} \in S^+ \setminus S_0^+$. Clearly the same is true if $\mathcal{H}_b = 2^{\mathbb{N}} \setminus [\emptyset]$.

This is a well formed necessary condition. An open question is how far this is from being sufficient. We still have some work to do, as shown by the following statement.

Proposition 7 *Take $N \subset \mathbb{N}$, and $\mathcal{H} = \{H \subset \mathbb{N} \mid N < H\}$. This $\mathcal{H}$ will be a filter with no minimal elements, but $\mathcal{H} \neq \mathcal{H}_b$ for any $\mathbf{b} \in S^+ \setminus S_0^+$.*

Proof: It is clear that $\mathcal{H}$ is a filter. To prove it has no minimal elements, it is enough to show that for any $N < K$, one can find $N < H < K$. Using Proposition 2, $N < K$ implies we can find disjoint intervals $I_j = [n_{i_j}, n_{i_{j+1}}]$ such that $|I_j \cap K| \geq j$. Define

$$H = N \cup \bigcup_{j=1}^{\infty} (I_{2j} \cap K).$$

Hereby the intervals I_{2j} ensure $N < H$ and I_{2j+1} ensure $H < K$.

Suppose $\mathcal{H} = \mathcal{H}_b$. $N \notin \mathcal{H}_b$ means that there are infinitely many $I_j = [n_{i_j} + 1, n_{i_{j+1}}]$ for which $\sum_{k \in I_j} b_k > \varepsilon$ for some $\varepsilon > 0$. The fact that $\lim_{k \to \infty} b_k = 0$ causes these intervals to become longer and longer, so by thinning we may suppose $|I_j| \geq j + 1$. Let us define

$$H = N \cup \bigcup_{j=1}^{\infty} I_{2j}.$$

We filled up long empty intervals to achieve $N < H$, so $H \in \mathcal{H}$. We also retained infinitely many empty intervals where the sum is above ε, consequently $H \notin \mathcal{H}_b$.

□

Chapter 4

Incompatibility of series

When examining the slicing in the previous chapter, we have found a lattice-like structure on it. Now we move on to conditionally convergent series, where we will experience an opposite property.

It is meaningless to state that if $\mathbf{a} \in S_0$, $\mathbf{b} \in S \setminus S_0$, then $K(\mathbf{a}) \supsetneq K(\mathbf{b})$ holds for the convergence sets. The interesting point is that this inclusion can only happen in this situation.

Before proving this, let us have a closer look on P_0. In [1] we have a characterization on permutations which preserves not only convergence, but also the value of the sum:

Theorem 2 *(Agnew) Choose $\pi \in P$. Then π preserves the sum of all conditionally convergent series if and only if there is an integer M such that for all $n \in \mathbb{N}$, $\pi([1,n])$ is a union of M or fewer intervals of $\mathbb{N}$.*

We will show that exactly these are the universally convergence preserving permutations.

Theorem 3 *Choose $\pi \in P$. Then $\pi \in P_0$ if and only if there is an integer M such that for all n, $\pi([1,n])$ is a union of M or fewer intervals of $\mathbb{N}$.*

Proof: The condition is sufficient, this is obvious from the previous theorem. For the other implication, given a $\pi \in P$ which does not satisfy the condition, we will build an $\mathbf{a} \in S$ such that $\pi\mathbf{a}$ will not be convergent, $\pi \notin K(\mathbf{a})$.

By the properties of π, we can choose N_i as follows. Let N_1 be 1, this is the first number for which $\pi([1,N_1])$ consists of at least 1 intervals. If N_{i-1} is defined, let's choose N_i such that $\pi([1,N_i]) \supset [1,N_{i-1}]$, and $\pi([1,N_i])$ is the union of at least i intervals.

Fix an $i \geq 2$. $\pi([1, N_i]) = I \cup J_1 \cup \ldots \cup J_{i-1}$, where I is the interval containing $[1, N_{i-1}]$, J_k are the following intervals except the last one which is the union of the remaining (one or more) intervals. Let the first elements of J_k be j_k. Now we define some of the elements of $\mathbf{a}$:

$$a_{j_k - 1} = \frac{(-1)^{i-1}}{\sqrt{i-1}},$$

$$a_{j_k} = \frac{(-1)^{i}}{\sqrt{i-1}}.$$

We set all remaining elements of $[N_i + 1, N_{i+1}]$ to 0, then repeat this for all $i \geq 2$ to define all elements of $\mathbf{a}$.

In the end the series consists of a lot of 0-s, and when a nonzero element q appears, it is immediately followed by $-q$. This implies $\left|\sum_{k=1}^{n} a_k\right| \leq q$. Moreover, if $n > N_i$, then this holds for $q = \frac{1}{\sqrt{i-1}}$. So we got $\sum_{k=1}^{\infty} a_k = 0$.

Let's look at the permutated sum. Using $\pi([1, N_i]) = I \cup \bigcup_{k=1}^{i-1} J_k$ as above, and $\sum_{k \in I} a_k = 0$ and $\sum_{k \in J'_k} a_k = \frac{(-1)^i}{\sqrt{i-1}}$. Adding these up we get

$$\sum_{k=1}^{N_i} a_{\pi(k)} = (-1)^i \sqrt{i-1}.$$

So we got

$$\limsup_{n \to \infty} \left(\sum_{k=1}^{n} a_{\pi(k)} \right) = \infty.$$

In the end, the permutation ruined convergence for this specific $\mathbf{a}$, $\pi \notin K(\mathbf{a})$, $\pi \notin P_0$.

□

To quantify the relation between two series $\mathbf{a}$ and $\mathbf{b}$, let us define the following measures on $\overline{\mathbb{R}}$:

$$\mu_+ = \sum_{j: a_j > 0} a_j \delta_{\frac{b_j}{a_j}} + \sum_{j: a_j = 0,\ b_j \neq 0} a_j \delta_{sgn(b_j)\infty},$$

$$\mu_- = -\sum_{j:a_j<0} a_j \delta_{\frac{b_j}{a_j}}.$$

These measures give a sight on which ratios are more likely to appear. We will investigate the places where the measure is ∞, more precisely

Definition 3 *For a ν measure on $\overline{\mathbb{R}}$, $c \in \overline{\mathbb{R}}$ is a limit point if for all $c \in U$ neighborhoods, $\nu(U) = \infty$.*

Lemma 1 *μ_+ has at least one limit point (and similarly μ_-).*

Proof: We use only $\mu_+(\overline{\mathbb{R}}) = \infty$, which is a consequence of conditional convergence. If ∞ and $-\infty$ are not limit points, then $\mu_+(\overline{\mathbb{R}} \setminus [-K, K]) < \infty$ for an appropriate K.

Suppose there are no limit points in $[-K, K]$. This means for all $x \in [-K, K]$ there is $x \in U_x$, $\mu_+(U_x) < \infty$. Using compactness, finite of them will also cover, $[-K, K] \subset \bigcup_{i=1}^{n} U_{x_i}$.

This gives a finite upper bound $\sum_{i=1}^{n} \mu_+(U_{x_i})$ on $\mu_+([-K, K])$. Comparing with $\mu_+(\overline{\mathbb{R}}) = \infty$ ends up at a contradiction, so there must be at least one limit point.

□

We will have some interesting properties if special limit points appear.

Lemma 2 *Suppose 0 is a limit point of μ_+ (μ_-). Then for any $\varepsilon > 0, \delta > 0, T \geq 0$ ($T \leq 0$), and $J \subset \mathbb{N}$ finite set one can find a finite $I \subset \mathbb{N} \setminus J$ such that $\left|T - \sum_{i \in I} a_i\right| < \varepsilon$, $\left|\sum_{i \in I} b_i\right| < \delta$, and $\forall i \in I$ $a_i \geq 0$ ($a_i \leq 0$).*

Proof: The $T = 0$ case is trivial, otherwise without loss of generality, we may suppose $T > 0$. Discarding finite number of elements from the series does not change the limit point structure, so we may also suppose $J = \emptyset$. μ_+ has 0 as a limit point, so

$$\mu_+\left((-\frac{\delta}{T+\varepsilon}, \frac{\delta}{T+\varepsilon})\right) = \sum_{i:\left|\frac{b_i}{a_i}\right| < \frac{\delta}{T+\varepsilon}, 0 \leq a_i} a_i = \infty.$$

Using $a_i \to 0$, $\sum_{i:a_i \geq \varepsilon} a_i < \infty$, so it is also true that

$$\sum_{i:\left|\frac{b_i}{a_i}\right| < \frac{\delta}{T+\varepsilon}, 0 \leq a_i < \varepsilon} a_i = \infty.$$

Now let be

$$I' = \left\{ i \in \mathbb{N} | i : \left| \frac{b_i}{a_i} \right| < \frac{\delta}{T+\varepsilon}, 0 \leq a_i < \varepsilon \right\},$$

and choose minimal n sufficing $\sum_{i \in I' \cap [1,n]} a_i > T$. For this $I = I' \cap [1,n]$, automatically $\sum_{i \in I} a_i < T + \varepsilon$. Moreover,

$$\left| \sum_{i \in I} b_i \right| \leq \sum_{i \in I} a_i \left| \frac{b_i}{a_i} \right| < \frac{\delta}{T+\varepsilon} \sum_{i \in I} a_i < \delta.$$

□

Lemma 3 *Suppose ∞ is a limit point of μ_+ (μ_-). Then for any $\varepsilon > 0, \Delta > 0, T \geq 0$ ($T \leq 0$), and $J \subset \mathbb{N}$ finite set one can find a finite $I \subset \mathbb{N} \setminus J$ such that $\left| T - \sum_{i \in I} a_i \right| < \varepsilon$, $\sum_{i \in I} b_i > \Delta$, and $\forall i \in I$ $a_i \geq 0$ ($a_i \leq 0$).*

The proof is similar, the only difference is that instead of using an upper bound on $\left| \frac{b_i}{a_i} \right|$ in the definition of I', we need a lower bound on $\frac{b_i}{a_i}$. This is exactly what we can modify when changing 0 to ∞ as the limit point.

Now let's get back to the theorem mentioned at the beginning of the chapter.

Theorem 4 *Let $\mathbf{a}, \mathbf{b} \in S \setminus S_0$ be nonequivalent conditionally convergent series. In this case $K(\mathbf{a}) \setminus K(\mathbf{b}) \neq \emptyset$.*

Proof: Let us define $M = \max_{i \geq 1} |b_i|$.

We know μ_+ has a limit point c, and μ_- has a limit point d. It may happen that $c \neq d$. We can suppose $c < d$, and choose $c < T < U < d$. This means there are $I, J \subset \mathbb{N}$ such that

$$i \in I \Rightarrow a_i \geq 0,\ \frac{b_i}{a_i} < T,$$

$$j \in J \Rightarrow a_j < 0,\ \frac{b_j}{a_j} > U.$$

and $\sum_{i \in I} a_i = \infty$, $\sum_{j \in J} a_j = -\infty$.

Now we build the permutation needed for the theorem, we will define the values of the permutation one by one. There will be three types of steps.

In the first, we choose the first few unused elements of J such that the partial sum goes below 0.

In the second, we continue with some unused elements of I to increase the sum above 0.

In the third step, we insert the first unused element of $\mathbb{N} \setminus (I \cup J)$.

We will repeat the first two steps a lot, this will hold the partial sums of **a** near 0 while unboundedly decreasing that of **b**. We will will perform the third step relatively rarely not to ruin the effect of the first two steps, but to provide every element would occur.

To show the recursive method, suppose we already determined a few elements of the permutation, and the partial sum of **a** until now is K. Let us repeat the first two steps multiple times, starting with the one corresponding to the sign of K and ending with the first. Denote the indices chosen from I and J by I^* and J^*, respectively. If we do this long enough, we can achieve $\sum\limits_{i\in I^*} a_i > V + K$, $\sum\limits_{j\in J^*} -a_j > \sum\limits_{i\in I^*} a_i - K > V$ for any V, we will choose it later. These new elements have the following addition to the partial sum of the rearranged **b**:

$$\sum_{k\in I^*\cup J^*} b_k = \sum_{i\in I^*} b_i + \sum_{j\in J^*} b_j < T\sum_{i\in I^*} a_i + U\sum_{j\in J^*} a_j =$$

$$= T\left(\sum_{i\in I^*} a_i + \sum_{j\in J^*} a_j\right) + (U-T)\sum_{j\in J^*} a_j < TK - (U-T)V.$$

And the right hand side will be less than $-2M$ if we choose V large enough. If it is possible, we perform the third step once, in the end the partial sum of **b** decreased by at least M.

Now we start over. We can do this infinitely because I, J and the sums of **a** over I and J are infinite. The result must be a permutation as we use any element at most once and the possibility of third type steps provides we use them exactly once. The partial sum of $\pi\mathbf{b}$ decreases by M every time, finally it diverges to $-\infty$. It is clear that $\liminf\limits_{n\to\infty} \sum\limits_{i=1}^{n} a_{\pi(i)} \leq 0$ and $\limsup\limits_{n\to\infty} \sum\limits_{i=1}^{n} a_{\pi(i)} \geq 0$. If $\liminf\limits_{n\to\infty} \sum\limits_{i=1}^{n} a_{\pi(i)} < -3\varepsilon$, then we cross the -2ε barrier infinite times. Then there are infinitely many i such that $a_i < -\varepsilon$ since the barrier will be passed after a first and/or a third type step. This is impossible, so the lim inf is nonnegative, similarly the lim sup is nonpositive, which means. $s(\mathbf{a}, \pi) = 0$.

To sum up, this gives a legal permutation, one of $K(a) \setminus K(b)$ in the case when μ_+ and μ_- have different limit points.

The other possibility is that there is only one common limit point, this might be 0, a nonzero finite number, or $\pm\infty$.

Suppose this limit point is 0. We will describe an algorithm somewhat similar to the previous one to build an adequate permutation π.

We will describe step N, starting at $N = 1$. Choose $I \subset \mathbb{N}$ a finite subset belonging to negative elements of $\mathbf{b}$ satisfying

$$\sum_{i \in I} b_i < -4M.$$

This will help the sum of $\pi\mathbf{b}$ go to $-\infty$, but we need to compensate the effect of the sum of corresponding elements of $\mathbf{a}$. For each $i \in I$ we choose (a finite) $J_i \subset \mathbb{N}$ disjoint from each other and I such that

$$\left| \sum_{j \in J_i} a_j + a_i \right| < \frac{M}{|I|N},$$

$$\left| \sum_{j \in J_i} b_j \right| < \frac{M}{|I|}.$$

This is possible by using Lemma 2 multiple times. If all a_i are 0, let all J_i be empty.

$$\left| \sum_{i \in I} \left(a_i + \sum_{j \in J_i} a_j \right) \right| \le \sum_{i \in I} \left| a_i + \sum_{j \in J_i} a_j \right| \le |I| \frac{M}{N|I|} = \frac{M}{N},$$

$$\sum_{i \in I} \left(b_i + \sum_{j \in J_i} b_j \right) \le \sum_{i \in I} b_i + \sum_{i \in I} \left| \sum_{j \in J_i} b_j \right| < -4M + |I| \frac{M}{|I|} = -3M.$$

Using these relations we can tell the first part of the permutation. Insert the first element of I, then all of the corresponding J_i, then do this for all elements of I. After that, insert the first unused element $l \in \mathbb{N}$, to provide every index would appear in the permutation. Let A be the sum of elements of $\mathbf{a}$ used until now. In the end, we choose a $J' \subset \mathbb{N}$ finite subset of unused element such that $\left| A - \sum_{j \in J'} a_j \right| < \frac{M}{N}$, but $\left| \sum_{j \in J'} b_j \right| < M$. Put these elements to the end of the permutation.

Suppose we determined the $[K, L]$ elements of π during this step.

$$\sum_{i=K}^{L} b_{\pi(i)} = \sum_{i\in I} b_i + \sum_{i\in I}\sum_{j\in J_i} b_j + b_l + \sum_{j\in J'} b_j \leq -3M + M + M = -M,$$

$$\left|\sum_{i=1}^{L} a_{\pi(i)}\right| < \frac{M}{N}.$$

For any $K \leq L' \leq L$, $\pi([K, L'])$ consists of a few "blocks" - an $i \in I$ and the corresponding J_i or l and J', and of at most one partial block. The sum of $\mathbf{a}$ elements in a block is less then $\frac{M}{|I|N}$, or $\frac{M}{N}$ in the last block. In a partial block, there is an initial element, and the first few compensating terms. So the sum in a partial block is less than $\max(\max_{i\in I}|a_i|, |a_l|) \leq \max_{i\geq N}|a_i|$. This means

$$\left|\sum_{i=K}^{L'} a_{\pi(i)}\right| \leq |I|\frac{M}{|I|N} + \frac{M}{N} + \max_{i\geq N}|a_i| = \frac{2M}{N} + \max_{i\geq N}|a_i|.$$

After that we repeat this whole procedure on the remaining elements, using $N = 2, 3, \ldots$. We will end up at a real permutation, because it uses every element exactly once.

In each step, the sum of the permutated $\mathbf{b}$ series decreases by at least M, so it will diverge to $-\infty$. The sum of permutated $\mathbf{a}$ series is at most $\frac{M}{N}$ at the end of step N, which converges to 0, and even when looking at the middle of a step, we make at most $\frac{2M}{N} + \max_{i\geq N}|a_i|$ additional error, so in the end $s(\mathbf{a}, \pi) = 0$.

This means $\pi \in K(\mathbf{a}) \setminus (\mathbf{b})$.

Now suppose the common limit point is c. Define $\mathbf{b}' = \mathbf{b} - c\mathbf{a}$. $\frac{b'_i}{a_i} = \frac{b_i - ca_i}{a_i} = \frac{b_i}{a_i} - c$, so switching $\mathbf{b}$ to $\mathbf{b}'$ causes the limit point move to 0. From the previous case, there exists $\pi \in K(\mathbf{a}) \setminus K(\mathbf{b}')$. We only need $\pi \notin K(\mathbf{b})$, and this is true because $\pi\mathbf{b} = \pi\mathbf{b}' + c\pi\mathbf{a}$ is the sum of a series with convergent sum and one with divergent sum, so it must also have divergent sum.

The remaining case is when we have an infinite limit point, we may suppose this is ∞.

The construction will also be similar, only a bit simpler.

In step N, where N starts from 1, we choose the first unused element $l \in \mathbb{N}$. Let J be the indices used until now, and A be the sum of corresponding elements of $\mathbf{a}$. Then using Lemma 3 we can get $I \subset \mathbb{N} \setminus J$ such that

$$\left|A + \sum_{i\in I} a_i\right| < \frac{1}{N},$$

$$\sum_{i\in I} b_i > M.$$

Now put l and then I to the end of the permutation.

It is again clear that all elements will be used exactly once. No matter how far we go, we can find an $[N', N'']$ interval such that $\pi([K,L])$ is the I of a step. But

$$\left|\sum_{i=K}^{L} b_{\pi(i)}\right| = \left|\sum_{i\in I} b_i\right| > M.$$

Literally the oscillation will never go under M, so the permutated $\mathbf{b}$ series will diverge. The proof of convergence of the permutated $\mathbf{a}$ series is the same as in the previous case.

Finally, we could build permutations for all cases, so the proof is complete. □

This also gives an addition to the subject of [13]. There it is supposed that we have $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n \in S$ such that $\sum_{i=1}^{n} \lambda_i \mathbf{a}_i \in S_0$ can only happen with $\lambda_1 = \lambda_2 = \ldots = \lambda_n = 0$. As stated there, if $\mathbf{a}_{n+1} \in S$ and there exist $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that $\mathbf{a}_{n+1} - \sum_{i=1}^{n} \lambda_i \mathbf{a}_i \in S_0$, then $\bigcap_{i=1}^{n} K(\mathbf{a}_i) \subseteq K(\mathbf{a}_{n+1})$.

Now it turns out that $\bigcap_{i=1}^{n} K(\mathbf{a}_i) = K(\mathbf{a}_{n+1})$ can not happen, this would mean one nontrivial convergence set contains another, which is impossible.

We can prove an opposite statement telling that convergence sets cannot be completely different.

Theorem 5 *If* $\mathbf{a}^i \in S \setminus S_0$, $i \in \mathbb{N}$, *then*

$$\bigcap_{i=1}^{\infty} K(\mathbf{a}^i) \supsetneq P_0.$$

Proof:

$$\bigcap_{i=1}^{\infty} K(\mathbf{a}^i) \supseteq P_0.$$

is true because of the definition of P_0. We only have to find a $\pi \in \bigcap_{i=1}^{\infty} K(\mathbf{a}^i) \setminus P_0$. To satisfy $\pi \notin P_0$, we have to provide for every n that there is an N such that

$\pi([1,N])$ consists of at least n intervals. To achieve this, we will define π incrementally, fixing some elements for each n. We suppose after we are done with $2,3,\dots,n-1$, we defined π on $[1,N_{n-1}]$ such that $\pi([1,N_{n-1}])=[1,N_{n-1}]$. We need $\pi\in\bigcap_{i=1}^{\infty}K(\mathbf{a}^i)$, so in the next step, we will ensure the first n series to suffer only small changes regarding partial sums. We pick $K>N_{n-1}$ such that

$$|a_k^i|<\frac{1}{n^2},\ \forall 1\le i\le n,\ \forall k\ge K.$$

Now let's define

$$\pi(N_{n-1}+j)=K+2j,\ 1\le j\le n-1,$$

and N_n to be $K+2n-2$. Define π on $[N_{n-1}+n,N_n]$ to increasingly fill the remaining elements of $[N_{n-1},N_n]$. After that we can start over the iteration.

We constructed π to avoid P_0 as $\pi([1,N_{n-1}+n-1])$ consists of exactly n intervals. We still have to check $\pi\in K(\mathbf{a}^i)$.

Let $A^i=\sum_{j=1}^{\infty}a_j^i$, fix $\varepsilon>0$. It is enough to prove the partial sum of the rearranged series will approache A^i with at most ε error after a bound. Take K such that whenever $N>K$, $\left|A_i-\sum_{j=1}^{N}a_j^i\right|<\frac{\varepsilon}{2}$. Choose $n\ge i$ to $N_{n-1}>K$ and $n>\frac{2}{\varepsilon}$ to hold. This N_{n-1} will be the good bound for the rearranged series. To show this, suppose $M>N_{n-1}$. Then $M\in[N_{n'-1},N_{n'}]$ for some $n'\ge n$. From the construction $\pi([1,M])=[1,M']\cup H$ for an appropriate $M'\ge N_{n'}$ and $H\subset\{K+2,K+4,\dots,K+2n-2\}$. This allows us to estimate the sum of the rearranged series.

$$\left|\sum_{j=1}^{M}a_{\pi(j)}^i-A^i\right|\le\left|\sum_{j=1}^{M}a_{\pi(j)}^i-\sum_{j=1}^{M'}a_j^i\right|+\left|\sum_{j=1}^{M'}a_j^i-A^i\right|\le$$

$$\le\left|\sum_{j\in\pi([1,M])\setminus[1,M']}a_j^i\right|+\frac{\varepsilon}{2}<(n'-1)\cdot\frac{1}{n'^2}+\frac{\varepsilon}{2}<\varepsilon.$$

This works for all $i\in\mathbb{N}$ and $\varepsilon>0$, so $\pi\in K(\mathbf{a}^i)$, which means π suffices our needs.

□

A natural continuation is to extend the previous theorems on $\tilde{S}$. However, only the first will remain true.

Theorem 6 *Let* $\mathbf{a}, \mathbf{b} \in \tilde{S} \setminus S_0$ *nonequivalent series. In this case,* $K(\mathbf{a}) \setminus K(\mathbf{b}) \neq \emptyset$.

Proof: $\mathbf{a}$ can be rearranged to form a conditionally convergent series, choose such an adequate σ. This means $\sigma \in K(\mathbf{a})$. If $\sigma \notin K(\mathbf{b})$, we are ready. If not, then $\sigma\mathbf{a}, \sigma\mathbf{b} \in S \setminus S_0$ and are nonequivalent, so Theorem 4 can be used and provides a $\pi \in K(\sigma\mathbf{a}) \setminus K(\sigma\mathbf{b})$. Hence $\sigma\pi \in K(\mathbf{a}) \setminus K(\mathbf{b})$, we are ready.
□

Proposition 8 *There is an* $H \subset \tilde{S} \setminus S_0$, $|H| = c$ *such that* $K(\mathbf{a}) \cap K(\mathbf{b}) = \emptyset$ *for any* $\mathbf{a}, \mathbf{b} \in H$, $\mathbf{a} \neq \mathbf{b}$.

Proof: First we construct two series with disjoint convergence sets. An example is $a_n = (-1)^n \frac{1}{n}$, $b_{2n} = \frac{2}{2n}$, $b_{2n+1} = -\frac{1}{2n+1}$. In other words $a_n = b_n$ if n is odd, and $2a_n = b_n$ is n is even. $\mathbf{a}, \mathbf{b} \in \tilde{S} \setminus S_0$ is obvious.

The idea is that even if positive and negative terms are balanced in the rearranged $\mathbf{a}$ series, the rearranged $\mathbf{b}$ has the positive terms doubled, and because positive and negative parts both cumulate to ∞, the sum of the rearranged $\mathbf{b}$ will go out to ∞.

Formally, pick any $\pi \in P$. Choose an arbitrary T. If n is large enough, $\pi([1, n]) \supset [1, 2T]$.

$$\sum_{i=1}^{n} b_{\pi(i)} - \sum_{i=1}^{n} a_{\pi(i)} = \sum_{\substack{i \in [1,n] \\ \pi(i) \text{ is even}}} \frac{1}{\pi(i)} \geq \sum_{i=1}^{T} \frac{1}{2i} > \frac{\log T - 1}{2}.$$

But for two series with convergent sums, this difference would remain bounded. Consequently $\pi \in K(\mathbf{a})$ and $\pi \in K(\mathbf{b})$ cannot happen simultaneously, by other words, $K(\mathbf{a}) \cap K(\mathbf{b}) = \emptyset$.

We did use nothing special about the number 2 in the construction, only that it is different from 1. Therefore everything works for two series of the form $a^r_{2n} = \frac{r}{2n}$, $a^r_{2n+1} = -\frac{1}{2n+1}$ with two different positive r. Consequently $H = \{\mathbf{a}^r \mid r \in \mathbb{R}^+\}$ is the set we searched for.
□

The construction in the proof can be generalized, we don't need to start from the series $(-1)^n \frac{1}{n}$, any $\mathbf{a}^1 \in \tilde{S} \setminus S_0$ will suffice, we scale the positive elements the same way. This means we can even achieve $\mathbf{a}^1 \in H$ for some prescribed $\mathbf{a}^1$, in the end this shows $K(\mathbf{a}^1)$ is "small" in some sense.

An important consequence of the main theorem in this chapter is that nonequivalent series are really different when looking at possible convergence-preserving rearrangements. This will be the starting point of the next chapter.

Chapter 5

Reconstruction of series

We turn to a natural problem about conditionally convergent series and permutations. The question is whether we can reconstruct the conditionally convergent series $\mathbf{a}$ given the information $K(\mathbf{a})$.

Clearly we need to refine the problem. If $\mathbf{a} \equiv \mathbf{b}$, then $K(\mathbf{a}) = K(\mathbf{b})$, so it will be possible to reconstruct $\mathbf{a}$ only up to equivalency. On the other hand, Theorem 4 provides that nonequivalent series have different convergence sets, so this modified problem can be solved in theory.

Initially we make our work easier supposing we have some additional information.

We will need to combine multiple series. To describe a specific combination, we define indexing series. An acceptable indexing series to combine n series is an element of

$$\mathcal{K}_n = \left\{ \mathbf{k} = (k_i)_{i=1}^{\infty} \in [1,n]^{\mathbb{N}} \mid \forall i \in [1,n] \ \|\{j \in \mathbb{N} \mid k_j = i\}\| = \infty \right\}.$$

Definition 4 *Let us have series* $\mathbf{b}^1, \mathbf{b}^2 \ldots, \mathbf{b}^n$*, and an indexing series* $\mathbf{k} \in \mathcal{K}_n$*. We define the slicing of these series* $\mathbf{b} = \mathbf{k}(\mathbf{b}^1, \mathbf{b}^2, \ldots, \mathbf{b}^n)$ *recursively, let* b_i *be the first unused element of* $\mathbf{b}^{k_i}$*.*

Note that $s(b) = \sum_{i=1}^{n} s(b^i)$ if all $s(b^i)$ and their sum exists. If not, $s(b)$ may still exist. To be more precise,

$$\sum_{j=1}^{N} b_j = \sum_{i=1}^{n} \left(\sum_{j=1}^{|\{k \le N : n_k = i\}|} b_j^i \right).$$

This means we add up the partial sums of the subseries, but the length of these partial sums might grow with different speed. These speed can be controlled via $\mathbf{k}$. By fixing a slicing we mean fixing $\mathbf{k}$, allowing $\mathbf{b}^i$ to be changed or rearranged.

First we try to determine the signs of the elements of $\mathbf{a}$. For the same reason mentioned at the beginning of the chapter this is not possible, so instead we want to split $\mathbf{a}$ into two series $\mathbf{a}^1$ and $\mathbf{a}^2$ such that

$$\sum_{j=1}^{\infty} |a_j^1|^+ = \infty, \sum_{j=1}^{\infty} |a_j^2|^+ < \infty,$$

$$\sum_{j=1}^{\infty} |a_j^1|^- < \infty, \sum_{j=1}^{\infty} |a_j^2|^- = \infty,$$

or reversed. By other words, $\mathbf{a}^1, -\mathbf{a}^2 \in \tilde{S}^+ \setminus S_0$.

To do this, let us deduce some statements about splitting of a conditionally convergent series. In the following theorems, $\mathbf{a}^1, \mathbf{a}^2$ will play the role of a splitting of $\mathbf{a}$.

Proposition 9 $\mathbf{a}^1, -\mathbf{a}^2 \in \tilde{S}^+ \setminus S_0$ *or reversed if and only if for any* $\pi_1, \pi_2 \in P$ *there exist* $\mathbf{k} \in \mathcal{K}_2$ *slicing such that* $|s(\mathbf{k}(\pi_1 \mathbf{a}^1, \pi_2 \mathbf{a}^2))| < \infty$*, but we can find* $\rho_1, \rho_2 \in P$ *and* $\mathbf{k}' \in \mathcal{K}_2$ *slicing that* $s(\mathbf{k}'(\rho_1 \mathbf{a}^1, \rho_2 \mathbf{a}^2))$ *does not exist or is infinite.*

Proof:

Observe if $\mathbf{a}^1, \mathbf{a}^2$ is a splitting of $\mathbf{a}$, then this is a condition using only summability of some rearranged versions $\mathbf{a}$.

If we have any type of summability, $\mathbf{a}_j^1, \mathbf{a}_j^2$ will converge to 0, this is provided by both conditions, so we don't have to care about it.

The easier part is if we have $\mathbf{a}^1, -\mathbf{a}^2 \in \tilde{S}^+ \setminus S_0$. This means $\mathbf{a}^1 = \mathbf{b}^1 - \mathbf{c}^1$, $\mathbf{a}^2 = \mathbf{c}^2 - \mathbf{b}^2$, where $\mathbf{b}^1, \mathbf{b}^2 \in S^+$, $\mathbf{c}^1, \mathbf{c}^2 \in S_0^+$. We described in the first chapter how to slice a positive and a negative series to have convergent sum. We have to do this for $\mathbf{b}^1 = \mathbf{a}^1 + \mathbf{c}^1$ and $-\mathbf{b}^2 = \mathbf{a}^2 - \mathbf{c}^2$. We will get a slicing which is also good for $\mathbf{a}^1$ and $\mathbf{a}^2$ because the difference is only an absolutely convergent series. Of course the same thing works for $\pi_1 \mathbf{a}^1, -\pi_2 \mathbf{a}^2$.

If $-\mathbf{a}^1, \mathbf{a}^2 \in \tilde{S}^+ \setminus S_0$, we prove similarly.

For the reverse direction, let us suppose $\mathbf{a}^1 \in \tilde{S} \setminus S_0$. We will show that the summability conditions will not hold. Choose an arbitrary $\pi_2 \in P$. If $\liminf_{i \to \infty} \sum_{j=1}^{i} a_{\pi_2(j)}^2 > -\infty$, then find a $\pi_1 \in P$ such that $\sum_{j=1}^{i} a_{\pi_1(j)}^1 = \infty$. Obviously any slicing of these rearranged series will sum up to ∞. We do the same but negated if the lim sup is less than ∞. The remaining case is when the lim inf and the lim sup are $-\infty$ and ∞, respectively. We may then choose $\pi_1 \in K(\mathbf{a}^1)$, clearly slicing this with $\pi_2\mathbf{a}^2$ will not change these values.

Now let us prove the reverse direction, suppose our summability conditions hold. A basic consequence is that the combined series, e.g. $\mathbf{a} = \mathbf{k}(\mathbf{a}^1, \mathbf{a}^2)$ for some $\mathbf{k} \in \mathcal{K}_2$ both has rearrangements which have convergent and which have divergent sum. So $\mathbf{a} \in \tilde{S} \setminus S_0$. Adding the previous observation we have

$$\sum_{j=1}^{\infty} |a_j^1|^+ < \infty \textit{ or } \sum_{j=1}^{\infty} |a_j^1|^- < \infty,$$

$$\sum_{j=1}^{\infty} |a_j^2|^+ < \infty \textit{ or } \sum_{j=1}^{\infty} |a_j^2|^- < \infty,$$

$$\sum_{j=1}^{\infty} |a_j|^+ = \infty \textit{ and } \sum_{j=1}^{\infty} |a_j|^- = \infty.$$

This is true only in the cases which the proposition permits.

□

Proposition 10 $\mathbf{a}^1, \mathbf{a}^2 \in \tilde{S} \setminus S_0$ *if and only of there exists* $\pi_1, \pi_2, \rho_1, \rho_2 \in P$ *and* $\mathbf{k}', \mathbf{k}'' \in \mathcal{K}_2$ *slicing such that* $|s(\mathbf{k}(\pi_1\mathbf{a}^1, \pi_2\mathbf{a}^2))| < \infty$ *for any* $\mathbf{k} \in \mathcal{K}_2$ *slicing, but* $s(\mathbf{k}'(\rho_1\mathbf{a}^1, \pi_2\mathbf{a}^2))$ *and* $s(\mathbf{k}''(\pi_1\mathbf{a}^1, \rho_2\mathbf{a}^2))$ *are infinite or does not exist.*

Proof:

Suppose $\mathbf{a}^1, \mathbf{a}^2 \in \tilde{S} \setminus S_0$. Take $\pi_1 \in K(\mathbf{a}^1)$, $\pi_2 \in K(\mathbf{a}^2)$, $\rho_1 \notin K(\mathbf{a}^1)$, $\rho_2 \notin K(\mathbf{a}^2)$. In this case $s(\mathbf{k}(\pi_1\mathbf{a}^1, \pi_2\mathbf{a}^2)) = s(\mathbf{a}^1, \pi_1) + s(\mathbf{a}^2, \pi_2)$, so it is finite. On the other hand, the combination of a convergent and a divergent series is divergent, and $\rho_1\mathbf{a}^1$ and $\rho_2\mathbf{a}^2$ are divergent, so we are done using arbitrary k', k''.

Now suppose the conditions hold. We will show $|s(\mathbf{a}^1, \pi_1)|, |s(\mathbf{a}^2, \pi_2)| < \infty$. If this is not true, we will be able to construct a slicing so that the sum will not be finite. For $\pi_1\mathbf{a}^1$ there are four possibilities, the sum might be ∞, $-\infty$, the lim inf and lim sup might differ, or the sum exists and is finite. We will choose an $n_1^1 \leq n_2^1 \leq \ldots$, $\lim_{i\to\infty} n_i^1 = \infty$, similarly $(n_i^2)_{i=1}^{\infty}$ showing how long $\mathbf{k}$ should be 1, or 2.

In the first case, choose it to suffice $\sum_{j=1}^{n_i^1} (\pi_1\mathbf{a}^1)_j > 3i$.

In the second case to satisfy $-i > \sum_{j=1}^{n_i^1} (\pi_1\mathbf{a}^1)_j > -2i$.

In the third case, such that $\limsup_{i\to\infty} \sum_{j=1}^{n^1_{2i}} (\pi_1 \mathbf{a}^1)_j < \liminf_{i\to\infty} \sum_{j=1}^{n^1_{2i-1}} (\pi_1 \mathbf{a}^1)_j$.
In the fourth case, n^1_i can be arbitrary. Clearly all these n^1_i exist if i is large enough.

We generate n^2_i for $\mathbf{a}^2$ the same way. Now define $\mathbf{k}$ starting by n^1_1 elements 1, then n^2_1 elements 2, then $n^1_2 - n^1_1$ times 1 and so on. Now for the merged series $\mathbf{a} = \mathbf{k}(\pi_1\mathbf{a}^1, \pi_2\mathbf{a}^2)$,

$$\sum_{j=1}^{n^1_i+n^2_i} a_j = \sum_{j=1}^{n^1_i} (\pi_1\mathbf{a}^1)_j + \sum_{j=1}^{n^2_i} (\pi_2\mathbf{a}^2)_j.$$

It is easy to verify that this diverges to ∞ if the first case occur for one of the series. If not, it will diverge to $-\infty$ if the second case happen at least once. Otherwise if the third case is true for a series we will still end up at a series with divergent sum. In order to avoid contradiction with the condition, the fourth case has to happen for both series, so both $s(\mathbf{a}^1, \pi_1)$ and $s(\mathbf{a}^2, \pi_2)$ must be finite.

On the other hand, if $|s(\mathbf{a}^1, \rho_1)| < \infty$, then $s(\mathbf{k}'(\rho_1\mathbf{a}^1, \pi_2\mathbf{a}^2))$ must be finite for any $\mathbf{k}' \in \mathcal{K}_2$ slicing. This is not the case, consequently $s(\mathbf{a}^1, \rho_1)$ does not exist or is not finite. We got two rearrangements of $\mathbf{a}_1$, one with the sum converging, one without, this can only happen to a series in $\tilde{S} \setminus S_0$. We can say the same about $\mathbf{a}_2$.

□

Proposition 11 *Supposing* $\mathbf{k}'(\mathbf{a}^1, \mathbf{a}^2) \in \tilde{S}\setminus S_0$, $\mathbf{a}^1 \in S_0$ *if and only of there is* $\pi_2 \in P$ *and* $\mathbf{k} \in \mathcal{K}_2$ *slicing such that* $|s(\mathbf{k}(\pi_1\mathbf{a}^1, \pi_2\mathbf{a}^2))| < \infty$ *for any* $\pi_1 \in P$.

Proof: Suppose $\mathbf{a}^1 \in S_0$, this implies $\mathbf{a}^2 \in \tilde{S} \setminus S_0$, and we can choose $\pi_2 \in K(\mathbf{a}^2)$. Now for any $\pi_1 \in P = K(\mathbf{a}^1)$ and $\mathbf{k} \in \mathcal{K}_2$ $|s(k(\pi_1\mathbf{a}^1, \pi_2\mathbf{a}^2))| < \infty$ because we merged to series with convergent sums.

For the reverse direction, choose $\pi_1, \pi_1' \in P$.

$$s(\mathbf{k}(\pi_1\mathbf{a}^1, \pi_2\mathbf{a}^2)) - s(\mathbf{k}(\pi_1'\mathbf{a}^1, \pi_2\mathbf{a}^2)) = s(\mathbf{k}(\pi_1\mathbf{a}^1 - \pi_1'\mathbf{a}^1, \pi_2\mathbf{a}^2 - \pi_2\mathbf{a}^2)) =$$

$$= s(\pi_1\mathbf{a}^1 - \pi_1'\mathbf{a}^1),$$

and the condition tells this sum is finite. If $\mathbf{a}^1 \in \tilde{S} \setminus S_0$, then the choice $\pi_1 \in K(\mathbf{a}^1)$, $\pi_1' \in P\setminus K(\mathbf{a}^1)$ would lead to contradiction. If $\mathbf{a}^1 \in \tilde{S}^+ \setminus S_0$, then for n_1 large enough, $\sum_{i=1}^{n_1} a^1_i > 10$. If m_1 is large enough, $\sum_{i=m_1}^{m_1+n_1-1} a^1_i < 1$. Let us define the first elements of π_1 to be $1, 2, \ldots, n_1, m_1, m_1+1, \ldots, n_1+m_1-1$, and the start of π_1' to be $m_1, m_1+1, \ldots, n_1+m_1-1, 1, 2, \ldots, n_1$. Therefore

$$\sum_{i=1}^{n_1}(a^1_{\pi_1(i)} - a^1_{\pi_1'(i)}) > 9,$$

$$\sum_{i=1}^{2n_1}(a^1_{\pi_1(i)} - a^1_{\pi_1'(i)}) = 0.$$

By words, the sum oscillates at least 9 between 0 and $2n_1$. Now throw away used elements, and do this again to extend π_1 and π_1'. The oscillation will reach 9 from time to time, so the sum will not converge, and this is contradiction. To sum up $\mathbf{a}^1 \notin \tilde{S} \setminus S_0$, $\mathbf{a}^1 \notin \tilde{S}^+ \setminus S_0$, similarly $-\mathbf{a}^1 \notin \tilde{S}^+ \setminus S_0$, only $\mathbf{a}^1 \in S_0$ remained as a possibility.

□

The last three propositions made it possible to detect what type of series did we get at an arbitrary splitting. Take a decomposition of $\mathbf{a}$ to $\mathbf{a}^1$ and $\mathbf{a}^2$ for which $\mathbf{a}^1, -\mathbf{a}^2 \in \tilde{S}^+ \setminus S_0$. Adding an absolutely convergent series does not change anything, so we may suppose $\mathbf{a}^1, -\mathbf{a}^2 \in S^+ \setminus S_0$. By previous observations, for $N \subset \mathbb{N}$, $N \in \mathcal{H}_{a^1}$ exactly if there is a $\pi \in K(\mathbf{a})$ where $\mathbf{a}^1$ is segmented as described by N. Consequently we can determine $\mathcal{H}_{a^1}$ and $\mathcal{H}_{a^2}$, and this allows to get some bounds by comparing these with $\mathcal{H}_b$ for known $\mathbf{b} \in S^+ \setminus S_0$.

Although the problem of complete reconstruction remained unsolved, we could filter out some well formed numerical information by only knowing the convergence sets.

Chapter 6

Random permutations

We can go one step further and ask whether a random permutation preserves the convergence of a conditionally convergent series. Suppose we work on a probability space $(\Omega, \mathcal{M}, P)$.

The first problem we have to face is how to generate infinite random permutations. There might be several possibilities, however we will choose one which is natural in some sense.

We will start with a nontrivial distributions on $\mathbb{N}$, an element of:

$$\mathcal{Q} = \left\{ Q = \{q_i\}_{i=1}^{\infty} \in [0,1]^{\mathbb{N}} \mid \forall i \in \mathbb{N}\ q_i > 0,\ \sum_{i=1}^{\infty} q_i = 1 \right\}.$$

Definition 5 *Suppose there is a $Q \in \mathcal{Q}$ distribution given on $\mathbb{N}$. Take $X_1, X_2, \ldots$ IID random variables with distribution Q. For any $\omega \in \Omega$, define Π_Q permutation as follows:*

$$\Pi_Q(n) = \min\{X_k : |\{X_1, X_2, \ldots, X_l\}| = n\}.$$

By words, we write X_1 to the first position of Π_Q, then we examine $X_2, X_3, \ldots$, and write it to the next position of Π_Q if we find a new value. We need this is almost surely a permutation.

Proposition 12 *If $Q \in \mathcal{Q}$, then $P(\Pi_Q \in P) = 1$.*

Proof: When $\Pi_Q(n) = m$, then in the definition above, $X_k = m$, but $m \notin \{X_1, X_2, \ldots, X_{k-1}\}$ otherwise k would not be minimal. This also implies $m \notin \Pi_Q([1, n-1])$, so Π_Q is injective always. We miss an m value in the permutation only if none of X_k are m. Using $Q(m) > 0$, this happens with 0 probability. Thus the event that Π_Q is not surjective is the union of countably many zero-probability events, so we avoid it almost surely.

□

Our goal is to tell something about $\Pi_Q \in K(\mathbf{a})$ for a specific $\mathbf{a}$ and about $\Pi_Q \in P_0$. We must start with ensuring measurability. Let's fix the Q distribution for now.

Proposition 13 *Fix $\mathbf{a} \in \tilde{S}$. Then $\{\Pi_Q \in K(\mathbf{a})\}$ is measurable with respect to $\mathcal{F} = \sigma(X_1, X_2, \ldots)$.*

Proof: Let's define $S_{n,m} = \sum_{i=n}^{m} a_{\Pi_Q(i)}$, for $n \le m$. $\Pi_Q(i)$ is measurable, so is $a_{\Pi_Q(i)}$, and also a deterministic finite sum of these, namely $S_{n,m}$. Let's focus on Cauchy-convergence. The event "N is a good bound for ε" can be written as $A_{N,\varepsilon} = \bigcap_{n=N}^{\infty} \bigcap_{m=n}^{\infty} \{|S_{n,m}| < \varepsilon\}$, this does not take out from $\mathcal{F}$. In the end, the sum exist exactly on the set $\bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} A_{N,\frac{1}{k}} \in \mathcal{F}$.

□

Proposition 14 *$\{\Pi_Q \in P_0\}$ is measurable with respect to $\mathcal{F}$.*

Proof: We will check the property found at Theorem 3. $\{\Pi_Q([1,n]) = A\}$ for some $A \subset \mathbb{N}$ is measurable, taking the union of countable many of these gives the measurable set $A_{n,K} = \{\Pi_Q([1,n])$ *consists of at most K intervals*$\}$. Using the theorem $\{\Pi_Q \in P_0\} = \bigcup_{K=1}^{\infty} \bigcap_{n=1}^{\infty} A_{n,K} \in \mathcal{F}$.

□

Proposition 15 *For any $\mathbf{a} \in \tilde{S}$, $P(\Pi_Q \in K(\mathbf{a}))$, $P(\Pi_Q \in P_0)$ are all 0 or 1.*

Proof: This is an application of the Kolmogorov 0-1 rule. To use it, we have to show that $\{\Pi_Q \in K(\mathbf{a})\}$, $\{\Pi_Q \in P_0\} \in \mathcal{G}_n$, where $\mathcal{G}_n = \sigma(X_{n+1}, X_{n+2}, \ldots)$. Generate the permutation Π'_Q the same way as Π_Q from $X_{n+1}, X_{n+2}, \ldots$ setting $X_1 = X_2 = \ldots = X_n = X_{n+1}$. $\{\Pi'_Q \in K(\mathbf{a})\}$, $\{\Pi'_Q \in P_0\}$ are clearly $\mathcal{G}_n$ measurable. The point is that

$$\Pi'_Q \in K(\mathbf{a}) \Leftrightarrow \Pi_Q \in K(\mathbf{a}),$$

$$\Pi'_Q \in P_0 \Leftrightarrow \Pi_Q \in P_0.$$

We have to check what does happen to Π'_Q when we put the real randomized values to $X_1, X_2, \ldots, X_n$. It can be seen that nothing will change above

$\max(\Pi_Q'^{-1}(\{X_1, X_2, \ldots, X_{n+1}\}))$. In the end we modified only finite elements of the permutation which does not effect convergence preserving in any sense. So $\{\Pi_Q \in K(\mathbf{a})\}$, $\{\Pi_Q \in P_0\}$ are identical to $\mathcal{G}_n$ measurable sets, and this is enough to conclude.

□

In the next few theorems, we will show that this construction of random permutation is meaningful, both 0 and 1 will appear in the previous proposition. Notice that when looking at one $\mathbf{a}$ and one $Q = \{q_1, q_2, \ldots\}$, we end up at exactly the same situation if we apply the same rearrangement on $\{a_1, a_2, \ldots\}$ and $\{q_1, q_2, \ldots\}$. In fact we don't need the order of the elements of $\mathbf{a}$, only that which probability and element belong together.

Theorem 7 *For any $Q \in \mathcal{Q}$ there exists an $\mathbf{a} \in \tilde{S}$ such that $P(\Pi_Q \in K(\mathbf{a})) = 0$.*

Moreover, we may prescribe the sign of elements of $\mathbf{a}$, and even fix the values of the positive elements.

Proof: Denote the probabilities of positive elements by $q_1', q_2', \ldots$, that of negative elements by $q_1'', q_2'', \ldots$, the values of positive elements by $a_1', a_2', \ldots$, their indices by $j_1', j_2', \ldots$. We may suppose the series a_i' is nonincreasing.

The idea is to choose negative elements to be very small. So small that if big positive elements appear often enough, they cannot compensate and the sum goes to ∞.

When want to estimate the probability of a_1' to be in the first n_1 elements of the rearranged series. This surely happens if j_1' appears within $X_1, X_2, \ldots X_{n_1}$. The probability of this is $1 - (1 - q_1')^{n_1}$. This can be done for multiple terms: a_2' will be among the first $n_1 + n_2$ elements, if j_2' appears within $X_{n_1+1}, X_{n_1+2}, \ldots, X_{n_1+n_2}$. Even this is independent from what happens on $X_1, X_2, \ldots, X_{n_1}$.

Therefore for any $n_1, n_2, \ldots$ the probability of $j_i' \in \Pi_Q([1, n_1+n_2+\ldots+n_i])$ for all i is at least

$$\prod_{i=1}^{\infty}(1 - (1 - q_i')^{n_i}).$$

Let's call this event A.

Let's choose $n_1 \leq n_2 \leq \ldots$ such that this infinite product is positive, this can be easily achieved. This is what we call positive elements appear "often enough".

To assign the values of negative elements, let's choose a series $b_1, b_2, \ldots$ such that

$$\forall i \in \mathbb{N}\ a_i' > b_i \geq b_{i+1} > 0,$$

$$\sum_{i=1}^{\infty} b_i = \sum_{i=1}^{\infty} (a'_i - b_i) = \infty.$$

This implies $\forall i \in \mathbb{N}\ \frac{b_i}{n_i-1} \geq \frac{b_{i+1}}{n_{i+1}-1}$.

Now we can define the negative elements of $\mathbf{a}$, for every i let's have $n_i - 1$ pieces of $-\frac{b_i}{n_i-1}$.

When A holds, we want a lower bound on $\sum_{i=1}^{n_1+\ldots+n_k} a_{\Pi_Q(i)}$. A means we can find $a'_1, a'_2, \ldots, a'_k$ among the summands. In the worst case, the remaining terms are the possible smallest negative elements. These are $n_i - 1$ times $-\frac{b_i}{n_i-1}$ for $1 \leq i \leq k$. The resulting lower bound is

$$\sum_{i=1}^{k} a'_i - \sum_{i=1}^{k} (n_i - 1)\frac{b_i}{n_i - 1} = \sum_{i=1}^{k} (a'_i - b_i) \to \infty$$

if $k \to \infty$, so on A, $\sum_{i=1}^{\infty} a_{\Pi_Q(i)}$ will diverge.

Using $P(A) > 0$, we can say $P(\Pi_Q \in K(\mathbf{a})) < 1$, but using Proposition 15 $P(\Pi_Q \in K(\mathbf{a}))$ can only be 0.

□

Corollary 1 *There exists $Q \in \mathcal{Q}$ so that $P(\Pi_Q \in P_0) = 0$. Moreover, any Q' can be transformed to such a Q by permutating the appearing probabilities.*

Proof: Fix any $Q' \in \mathcal{Q}$, and get an $\mathbf{a} \in \tilde{S}$ from the theorem. Choose a $\pi \in K(\mathbf{a})$, and define $Q = \pi Q'$. The meaning of $\pi Q'$ is obvious, we can look at Q' as an element of S_0.

$P(\Pi_{Q'} \in K(a)) = P(\Pi_Q \in K(\pi\mathbf{a})) = 0$, but $P_0 \subset K(\pi\mathbf{a})$, so our claim holds.

□

To deal with the opposite question, we will have to work with the information when we first take an element of a fixed set.

Definition 6 *Given an $A \subset \mathbb{N}$, define the following stopping times:*

$$\tau_A^* = \min\{n : X_n \in A\},$$

$$\tau_A = \min\{n : \Pi_Q(n) \in A\}.$$

The fact that $P(\Pi_Q \in P) = 1$ implies that τ_A^*, τ_A will be finite for any nonempty A with probability one.

Lemma 4 *Take a* $Q \in \mathcal{Q}$ *and nonempty* $A_1, A_2 \subset \mathbb{N}$, $A_1 \cap A_2 = \emptyset$. *Then*

$$P(\tau_{A_1} < \tau_{A_2}) = \frac{Q(A_1)}{Q(A_1) + Q(A_2)}.$$

Moreover, a similar version is true for multiple terms. Take disjoint nonempty $A_1, A_2, \ldots A_n \subset \mathbb{N}$. *In this case,*

$$P(\tau_{A_1} < \tau_{A_2} < \ldots < \tau_{A_n}) = \prod_{i=1}^{n-1} \frac{Q(A_i)}{\sum_{j=i}^{n} Q(A_j)}.$$

Proof: The second statement contains the first, so it is enough to prove the second one. It is equivalent to compare $\tau^*_{A_i}$ instead of τ_{A_i}, and it will be easier to handle. Let us define $B_i = \mathbb{N} \setminus \bigcup_{j=i}^{n} A_j$ The event $\{\tau^*_{A_1} < \tau^*_{A_2} < \ldots < \tau_{A^*_n}\}$ means that there are $k_1, k_2, \ldots, k_{n-1}$ positive integers, $l_1 = 0$, $l_i = \sum_{j=1}^{i-1} k_j$ for $1 \leq i \leq n-1$, which may depend on $\omega \in \Omega$ of course, such that

$$\begin{array}{rcl} X_{l_1+1}, X_{l_1+2}, \ldots, X_{l_1+k_1-1} & \in & B_1,\ X_{l_1+k_1} \in A_1 \\ X_{l_2+1}, X_{l_2+2}, \ldots, X_{l_2+k_2-1} & \in & B_2,\ X_{l_2+k_2} \in A_2 \\ X_{l_{n-1}+1}, X_{l_{n-1}+2}, \ldots, X_{l_{n-1}+k_{n-1}-1} & \in & B_{n-1},\ X_{l_{n-1}+k_{n-1}} \in A_{n-1} \end{array}$$

Clearly these are disjoint events for different $(k_1, k_2, \ldots, k_{n-1})$, and they cover $\{\tau^*_{A_1} < \tau^*_{A_2} < \ldots < \tau_{A^*_n}\}$. The probability of this event for fixed $(k_1, k_2, \ldots, k_{n-1})$

$$\prod_{i=1}^{n-1} Q(B_i)^{k_i - 1} Q(A_i).$$

To get $P(\tau^*_{A_1} < \tau^*_{A_2} < \ldots < \tau^*_{A_n})$ we have to sum up the previous expression for every $(k_1, k_2, \ldots, k_{n-1}) \in \mathbb{N}^{n-1}$. This will be

$$\prod_{i=1}^{n-1} \left(\sum_{k_i=1}^{\infty} Q(B_i)^{k_i-1} Q(A_i) \right) = \prod_{i=1}^{n-1} \left(Q(A_i) \frac{1}{1 - Q(B_i)} \right).$$

By the definition of B_i, $1 - Q(B_i) = Q\left(\bigcup_{j=i}^{n} A_j \right) = \sum_{j=i}^{n} Q(A_j)$. Writing this back to the previous expression results in the claim of the lemma.

□

Theorem 8 *For any $Q \in \mathcal{Q}$ there exists an $\mathbf{a} \in S \setminus S_0$ such that $P(\Pi_Q \in K(\mathbf{a})) = 1$.*

Moreover, we can find $\mathbf{a}$ in the form $\mathbf{a} = \pi\mathbf{b}$, where $\mathbf{b} \in \tilde{S} \setminus S_0$ is fixed, and we require $\pi \in K(\mathbf{b})$.

Proof: Let's choose an arbitrary $\mathbf{c} \in S \setminus S_0$. First we will create a $\mathbf{c}'$ series by inserting a lot of zeros into $\mathbf{c}$. The idea is to do this in such a way that Π_Q retains the order of original nonzero elements with positive probability, therefore convergence remains.

Define $n_1 = 1$, and choose n_2 to meet $\dfrac{q_1}{q_1 + \sum_{i=n_2}^{\infty} q_i} > \dfrac{1}{2}$. By Lemma 4, 1 will appear before any $n_2, n_2+1, \ldots$ in Π_Q with probability at least $\frac{1}{2}$. To continue, choose n_3 to satisfy $\dfrac{q_1}{q_1 + \sum_{i=n_2}^{\infty} q_i} \cdot \dfrac{q_{n_2}}{q_{n_2} + \sum_{i=n_3}^{\infty} q_i} > \dfrac{1}{2}$. Using Lemma 4,

$$P(\tau_{\{n_1\}} < \tau_{\{n_2\}} < \tau_{[n_3,\infty)}) = \frac{q_1}{q_1 + q_{n_2} + \sum_{i=n_3}^{\infty} q_i} \cdot \frac{q_{n_2}}{q_{n_2} + \sum_{i=n_3}^{\infty} q_i} >$$

$$> \frac{q_1}{q_1 + \sum_{i=n_2}^{\infty} q_i} \cdot \frac{q_{n_2}}{q_{n_2} + \sum_{i=n_3}^{\infty} q_i} > \frac{1}{2}.$$

We can repeat this infinitely which means we end up at $n_1 < n_2 < \ldots$ such that

$$P(\tau_{\{n_1\}} < \tau_{\{n_2\}} < \ldots < \tau_{\{n_{k-1}\}} < \tau_{[n_k,\infty)}) > \frac{1}{2}$$

for any $k \geq 2$. These are monotonically decreasing events in k, so for their intersection

$$P(\tau_{\{n_1\}} < \tau_{\{n_2\}} < \ldots) \geq \frac{1}{2}.$$

Let $c'_{n_i} = c_i$, and set all other elements of $\mathbf{c}'$ to 0. With probability at least $\frac{1}{2}$, $\Pi_Q \mathbf{c}'$ will contain the elements c_i in the original order, only the zeros will be mixed. This implies the sum will converge, therefore $P(\Pi_Q \in K(\mathbf{c}')) \geq \frac{1}{2}$, in fact it is 1 by Proposition 15.

This is enough to prove the first part of the theorem, however we have to slightly modify it to get the second part. Take a $\mathbf{b} \in \tilde{S} \setminus S_0$. $\lim_{n\to\infty} b_n = 0$ means we can take a $\mathbf{b}'' \in S_0$ subseries of $\mathbf{b}$. Rearrange the remaining elements to form a conditionally convergent series $\mathbf{b}'$. Perform the previous construction with $\mathbf{b}'$ playing the role of $\mathbf{c}$, and insert the elements of $\mathbf{b}''$ instead of zeros. The construction still works because mixing an absolutely convergent series is irrelevant to the sum as much as mixing zeros in the original case. Moreover, now this is the rearrangement of $\mathbf{b}$, $\mathbf{a} = \pi\mathbf{b}$. $\mathbf{a}$ has convergent sum, so $\pi \in K(\mathbf{b})$ holds.

□

Corollary 2 *For any* $\mathbf{a} \in \tilde{S} \setminus S_0$ *there is a* $Q \in \mathcal{Q}$ *satisfying* $P(\Pi_Q \in K(\mathbf{a})) = 1$.

Moreover, we can find Q *as a rearrangement* $Q = \pi Q^*$*, where* $Q^* \in \mathcal{Q}$ *is fixed.*

Proof: Use the stronger statement of the previous theorem on Q^* and $\mathbf{a}$. We obtain π such that $P(\Pi_{Q^*} \in K(\pi\mathbf{a})) = 1$. But from a previous remark, $\Pi_{Q^*} \in K(\pi\mathbf{a}) \Leftrightarrow \Pi_{\pi^{-1}Q^*} \in K(\pi^{-1}\pi\mathbf{a})$, so for $Q = \pi^{-1}Q^*$ the claim will hold. $\square$

The same idea can give us even more:

Theorem 9 *There exists* $Q \in \mathcal{Q}$ *with the property* $P(\Pi_Q \in P_0) = 1$.

Proof: We will provide the identic permutation occurs with positive probability, that is

$$P(\tau_{\{1\}} < \tau_{\{2\}} < \ldots) > 0.$$

Using the same train of thought, it is enough to suffice

$$\frac{n}{2n-1} \cdot \frac{q_n}{\sum_{i=n}^{\infty} q_i} \geq \frac{n+1}{2n+1}$$

for all positive n. Equivalently

$$q_n \geq \frac{2n^2+n+1}{2n^2+n} \sum_{i=n}^{\infty} q_i = \left(1 - \frac{1}{2n^2+n}\right) \sum_{i=n}^{\infty} q_i,$$

$$\sum_{i=n+1}^{\infty} q_i \leq \frac{1}{2n^2+n} \sum_{i=n}^{\infty} q_i.$$

This clearly holds if we achieve $\sum_{i=n}^{\infty} q_i = \frac{1}{3^{n-1}(n-1)!^2}$, so let us define

$$q_n = \frac{1}{3^{n-1}(n-1)!^2} - \frac{1}{3^n n!^2} = \frac{3n^2-1}{3^n n!^2}.$$

$\square$

By a simple transformation, we can get more convergence-preserving random permutations for a specific series.

Proposition 16 *For any choice of* $Q \in \mathcal{Q}$ *and* $\pi \in P$, $\pi\Pi_Q$ *and* $\Pi_{\pi Q}$ *have the same distribution.*

Proof: Suppose we generate Π_Q using the independent random variables $X_1, X_2, \ldots$, which all have distribution Q. In this case, πX_i have distribution πQ, and are still independent, so they generate $\Pi'_{\pi Q}$ which has the distribution of $\Pi_{\pi Q}$.

On the other hand, if we look at the definition of Π_Q, and put πX_i instead of X_i, the resulting permutation will be $\pi \Pi_Q$. This finishes the proof.
□

If $P(\Pi_Q \in P_0) = 1$ for some $Q \in Q$, and $\pi \in K(\mathbf{a})$ for some $\mathbf{a} \in \tilde{S} \setminus S_0$, then obviously $\Pi_Q(\pi\mathbf{a}) = (\pi\Pi_Q)\mathbf{a}$ has convergent sum with probability 1. According to the previous Proposition, we may rephrase this as $P(\Pi_{\pi Q} \in K(\mathbf{a})) = 1$.

Although we generated a family of distributions which give convergence-preserving permutation for an $\mathbf{a} \in \tilde{S} \setminus S_0$, it is still not clear how to determine $P(\Pi_Q \in K(\mathbf{a}))$ for general Q and $\mathbf{a}$. Even if this is complex to solve, it would be interesting to know the answer for special distributions, eg. geometric distribution (which will not be in P_0 almost surely) or Poisson distribution.

Another direction for going forward is to introduce other methods of generating random permutations.

An interesting method generates the same distribution class, therefore it gives a new sight on it. Let X_n be a q_n-exponential random variable, choose them to be independent. The permutation will be the ordering of X_n.

A direct generalization of the one we investigated until now is when we have a $Q_H : 2^H \to [0, 1]$ distribution on every infinite subset of $\mathbb{N}$, and we choose one element after another, always independently from previous choices by the distribution of the remaining elements. The situation we worked with until now fits this scheme with $Q_H(i) = \dfrac{q_i}{\sum_{j \in H} q_j}$ if $i \in H$.

A different possibility is to fix a series $\sigma_1^2, \sigma_2^2, \ldots$, and take independent normal variables $X_n \sim N(n, \sigma_n^2)$. The permutation will be defined by the ordering of X_n.

Chapter 7

Final words

We observed the diversity of problems originating from Riemann's observation. Let us conclude with two more questions not covered here.

We can use randomness not only when choosing the permutation, but when taking a series. These two might be independent or not, one example on the latter one is the extension of the exponentially generated permutation defined in the previous chapter. The point is that we may choose the value a_n independently by a distribution F_x, this depends only on the value x taken by X_n.

To find new problems we can exchange summability to some other asymptotic property, which depends on the order of elements. An example for that is to require

$$\lim_{n\to\infty} \frac{\sum_{j=1}^{n} a_j}{n}$$

to exist. From this point we can do the similar analysis on permutations which does or does not preserve this property.

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