On the problems of edge disjoint cliques in graphs

Thesis

Nguyen Huu Hoi

Supervisor: **Győri Ervin**

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Eötvös Loránd University, Mathematics Department

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Preface

Many results have been published in extremal graph theory on the number of given size cliques in graphs. To some extent, there are three main topics:

- on the maximal number of cliques.
- on the maximal number of vertex disjoint cliques.
- on the maximal number of edge disjoint ones.

For the first topic, Moon and Moser [11] proved that a graph of order n and size e (i.e it has n vertices and e edges) contains at least $(e/3n)(4e-n^2)$ triangles, the analogous results can be proved for the general case. If e is bounded from above then better lower bounds can be proved, say for the number of triangles, Bollobás [2] proved that if $n^2/4 \leq e \leq n^2/3$ then the graph contains at least $(n/9)(4e-n^2)$ triangles.

From another viewpoint, Erdős conjectured that if G is a graph of order n and size $\lfloor n^2/4 \rfloor + m$ $(m < n/2)$ then G has at least $m\lfloor n/2 \rfloor$ triangles. Lovász and Simonovits [12] proved this conjecture whenever n is sufficiently large. More results can be found in [3], [4] and [14].

Dealing with the second one, Hajnal and Szemerédi [1] showed that if the minimum degree $\delta(G) \geq (1-1/r)n$ then G contains $\lfloor n/r \rfloor$ vertex-disjoint copies of K_r. One can replace K_r by a H graph of order r, and the extended version of the Hajnal-Szemerédi becomes a theorem of Alon and Yuster [15] concerning the vertex disjoint copies of H when $\delta(G)$ is greater than $(1-1/\chi(H))n$. These problems are studied in a large number of papers by E. Szemerédi, J. Komlós, M. Simonovits, R. Yuster.

The thesis concentrates on the third topic, studying on the number of edge disjoint cliques. For a graph G of order n and size m (which are given), let $edk_k(G)$ denote the maximum number of edge disjoint K_k's in G. Let $edk_k(n,m)$ denote the minimum of $edk_k(G)$ for all graphs G of order n and size m.

In the first chapter, generally speaking, we will prove $edk_3(n, \frac{n^2}{4} + m) \geq \frac{5}{9}m - O(n)$, $edk_4(G) \geq \frac{5}{18}m - o(n^2)$ and $edk_k(n, t_{k-1,n} + m) \geq \frac{1}{\binom{k}{2}-(k-2)}m$ for $k \geq 5$. The latter could be refined by method applied in the first two, however the result would not be much better and still far from the possibly best $edk_k(G) \geq \frac{2}{k}m - o(n^2)$.

The second chapter discusses the case the graph contains $t_{r-1}(n) + m$ edges, where $m = o(n^2)$. There are almost m cliques can be found. Generally,

$$edk_k(n, t_{k-1}(n) + m) = m - O(\frac{m^2}{n^2})$$

and

$$edk_k(n, t_{k-1}(n) + m) = m$$

if m is "linear".

The last chapter is miscellaneous and contains three small topics. The first two deal a bit with fractional method. On the third we are likely to have some notes upon the relation between $edk_k(G)$ and other structures of the graph.

Thanks
I am very grateful to my advisor Győri Ervin for giving me this pretty topic. Without his help and encouragement I could not have finished the work.

Chapter 1

For $n = t(k-1)+r, 0 \le r \le k-2$, let $T_{k-1,n}$ denote the complete (k-1)-partite graph of n vertices such that r color classes contain $t+1$ vertices and $k-r-1$ color classes contain t vertices. Then $T_{k-1,n}$, which has $t_{k-1,n} := |E(T_{k-1,n})| = [n(n-t) - r(t+1)]/2$ edges, does not contain any complete subgraph K_k of k vertices. Turán proved that any other graph of n vertices with at least $t_{k-1,n}$ edges contains K_k as a subgraph. $T_{k-1,n}$ is called the $(k-1)-$partite Turán graph.
Results of Erdős, Goodman and Pósa [16] ($k = 3$) and Bollobás [2] ($k \ge 4$) imply that

T 1. *The edge set of every graph of* n *vertices can be decomposed into at most* $t_{k-1,n}$ *edge disjoint* K_k*'s and edges. If* $k \ge 3$ *then* $T_{k-1,n}$ *is the only extremal graph. For* $k = 3$*, the complete graph* K_4, K_5 *and the graphs* $T_{2,n}(n = 1, 2, ..)$ *are the extremal graphs.*

The results can be formulated in the following form

$$edk_k(n, t_{k-1,n} + m) \ge \frac{1}{\binom{k}{2}-1} m \text{ if } k \ge 3.$$

In a paper by Győri and Tuza [6], the result was improved ($k \ge 4$).
Let $p_k(G)$ =min$\{\sum_1^m |V(G_i)| : G_i$'s are K_k and edges,$\cup_i^m E(G_i) = E(G)\}$. If G is K_k-free then $p_k(G) = 2|E(G)|$ and, in particular, $p_k(T_{k-1,n}) = 2t_{k-1,n}$. The authors stated that

T 2. *If* $k \ge 4$ *and* G *is an arbitrary graph of* n *vertices then* $p_k(G) \le 2t_{k-1,n}$ *and equality holds only for* $T_{k-1,n}$.

Which means

$$edk_k(n, t_{k-1,n} + m) \ge \frac{1}{\binom{k}{2}-\frac{k}{2}} m \quad \text{if } k \ge 4$$

equality holds only for Turán graphs.
($k \ge 4$ is supposed because $p_3(G)$ is not always smaller than $2t_2(n)$. If we cover the complete graph K_{6m-2} by edge disjoint triangles and edges, the number of edges used in the cover is at least $3m$, so, $p_3(K_{6m-2}) \ge |E(K_{6m-2})| + 3m = 2t_{2,6m-2} + 1$)
By using the same method, the result can be improved a bit.

Theorem 1.1. *($n \geq k \geq 3$)*

$$edk_k(n, t_{k-1,n} + m) \geq \frac{1}{\binom{k}{2} - (k-2)} m$$

and only $T_{k-1,n}$, for $k = 3$, K_4,K_5, $T_{2,n}$ are the extremal examples.

To prove this, we need the a result of Hajnal and Szemerédi[1] which has been noticed in the preface.

T 3. *Let G be a graph of n vertices such that the degree of every vertex in G is at least d. Then there exist vertex disjoint complete subgraphs G_1,...,G_{n-d} of G such that $\cup_1^{n-d} V(G_i) = V(G)$ and the numbers of the vertices of the subgraphs G_i differ from each other at most one, i.e,$|V(G_i)|$ is $\lfloor \frac{n}{n-d} \rfloor$ or $\lceil \frac{n}{n-d} \rceil$ for $1 \leq i \leq n-d$.*

Proof of Theorem 1.1. The case $k = 3$ was settled. We prove for $k \geq 4$ by induction on n.
If $n = k$ the statement holds obviously. Suppose that it holds for graphs of at most $n - 1$ vertices.
Let

$$n - 1 = t(k-1) + r, 0 \leq r \leq k-2, t \geq 1.$$

Then

$$t_{k-1,n} - t_{k-1,n-1} = n - t - 1 = t(k-2) + r.$$

Consider an arbitrary graph G of n vertices and $t_{k-1}(n) + m$ edges. Let x be a vertex of G with minimum degree, $d := d_G(x)$.
If $d \leq n - 1 - t$, then

$$edk_k(G) \geq edk_k(G - x) \geq \frac{1}{\binom{k}{2} - (k-2)} (t_{k-1,n} + m - d - t_{k-1,n-1})$$

$$\geq \frac{1}{\binom{k}{2} - (k-2)} m.$$

Equality holds iff $d = n - t - 1$, $G - x = T_{k-1,n-1}$, and $m = 0$, so $G = T_{k-1,n}$ (otherwise Turán's theorem implies $edk_k(G) \geq 1$)
From now on, suppose

$$d \geq n - t = t(k-2) + r + 1.$$

In this case we will prove the strict inequality.
Because of the minimality of $d(x)$, the degree of every vertex in $G(N(x))$(the induced graph by neighbors of x) is at least $2d - n$.
Applying **T 3** to $G(N(x))$, we obtain that there exist vertex disjoint complete subgraphs

G_1,..,G_{n-d} of order $\lfloor \frac{d}{n-d} \rfloor$ or $\lceil \frac{d}{n-d} \rceil$ and $\cup_1^{n-d} V(G_i) = V(N(x))$.
Since

$$\frac{d}{n-d} \geq \frac{t(k-2)+r+1}{t} > k-2$$

There are two cases to be considered:

Case 1. There exists G_i with $|V(G_i)| = k-2$.
Suppose $|V(G_i)| = k-1$ for $i = 1, ..., p$ and $|V(G_i)| = k-2$ for $i = p+1, ..., n-d$ where $p \leq n-d-1$.

$$d = p(k-1) + (n-p-d)(k-2)$$

implies that

$$p = d(k-1) - n(k-2). \tag{1.1}$$

Note that $p \geq 1$ (otherwise, $d = \frac{n(k-2)}{k-1}$ and thus $\frac{d}{n-d} = k-2$, contradiction.)

The graph $H := (V(G) - x, E(G) \setminus \cup_1^p E(G_i))$ has $t_{k-1,n} + m - d - p\binom{k-1}{2}$ edges. By the induction hypothesis,

$$edk_k(G) \geq p + edk_k(H) \geq p + \frac{1}{\binom{k}{2} - (k-2)}(t_{k-1,n} + m - d - p\binom{k-1}{2} - t_{k-1,n-1})$$

$$= \frac{1}{\binom{k}{2} - (k-2)}m + \frac{1}{\binom{k}{2} - (k-2)}(p + t(k-2) + r - d) > \frac{1}{\binom{k}{2} - (k-2)}m$$

Because

$$p + t(k-2) + r - d = p - 1 + t(k-2) + r + 1 - d$$

$$= p - 1 + \frac{n(k-2)}{k-1} + \frac{r+1}{k-1} - d = p - 1 - \frac{p}{k-1} + \frac{r+1}{k-1}$$

- If $p \geq 2$, then $p - 1 - \frac{p}{k-1} \geq p - 1 - \frac{p}{2} = \frac{p-2}{2} \geq 0$, the sum is positive.

- If $p = 1$, then the sum above may be 0 when $r = 0$ and $H = T_{k-1,n-1}$, leading to $n = t(k-1) + 1$ and $d = \frac{n(k-2)+1}{k-1} = t(k-2) + 1$. Since $n > k$ we see that $t \geq 2$ and so $d < n - 1$.

In H every color is of t vertices and every vertex is of $t(k-2)$ degree. After adding x and restoring the deleted edges to make G, there is still at least one vertex of that degree, which is smaller than that of x, contradicts the supposition about x.
Case 2. $|V(G_i)| \geq k-1$ for $i = 1, ..., n-d$. So, $d \geq (k-1)(n-d)$
Let

$$d = s(k-1) + q, \quad 0 \leq q \leq k-2. \tag{1.2}$$

Then $1 \leq n - d \leq s \leq t$ by **1.2**. The minimum degree in $G(N(x))$ is at least

$$d - (n - d) \geq d - s$$

and so, according to **T 3**, there exist vertex disjoint complete subgraphs $G_1, G_2, ..., G_s$ of $k-1$ vertices in $G(N(x))$ so that $\cup_1^s V(G_i) \subseteq V(N(x))$.
Again, using the induction hypothesis for the graph

$$H := (V(G) - x, E(G) \setminus \cup_1^s E(G_i) \cup E(x))$$

of $t_{k-1,n} + m - d - s\binom{k-1}{2}$ edges, we have

$$edk_k(G) \geq s + edk_k(H)$$

$$\geq s + \frac{1}{\binom{k}{2} - (k-2)}(t_{k-1,n} + m - d - s\binom{k-1}{2}) = \frac{1}{\binom{k}{2} - (k-2)}m$$

$$+\frac{1}{\binom{k}{2} - (k-2)}(s + t(k-2) + r - d) > \frac{1}{\binom{k}{2} - (k-2)}m.$$

Here

$$s + t(k-2) + r - d = s + t(k-2) + r - s(k-1) - q$$

$$= t(k-2) - s(k-2) + r - q = (t-s)(k-2) - q + r > 0$$

Let's explain why this happens.

• if $t = s$, then the sum is not negative, and could be zero if $q = r$, that is $d = n - 1$, and $H = T_{k-1,n-1}$. However, in H, each color does not contain edges because we have deleted from them, but we only deleted edges that form edge disjoint K_{k-1} in $G - x$. So each color is of $k-1$ size, it means $n = (k-1)^2 + 1$. It is easy to verify that in this graph, a copy of $K_{(k-1)^2+1})$ where $k \geq 4$, there are at least k, and then more than $k - 1 = \frac{1}{\binom{k}{2} - (k-2)}m$ edge disjoint $K_k's$.

• if $t > s$ then $q \leq k-2$, the sum is not negative. It would be 0 if $t = s+1, q = k-2, r = 0$, and $H = T_{k-1,n-1}$, so $d = (t-1)(k-1) + k - 2 = t(k-1) - 1 = n - 2$. But the in H, again, every vertex is of $t(k-2)$ degree. After adding x and the deleted edges to make G, since there is one vertex is not $x's$ neighbor, its degree remains unchanged, in addition, $t(k-1) - 1 > t(k-2)$, so x could not be a vertex of minimum degree in G □

The result of Győri and Tuza is proved for $k \geq 4$ already. As we have noticed, the K_{6m-2} example disproves the similar assertion for $k = 3$: $p_3(G) \leq 2t_{2,n}$. However, it might be possible that:

Conjecture. *For every graph of n vertices, $p_3(n) \le \frac{1}{2}n^2 + o(n^2)$.*
($\Leftrightarrow edk_3(t_2(n)+m) \ge \frac{2}{3}m - o(n^2)$).

Győri and Tuza [6] proved

T 4. $p_3(n) \le \frac{9}{16}n^2$.

We prove the following improvement

Theorem 1.2. *$edk_3(n, t_2(n)+m) \ge \frac{5}{9}m - O(n)$.*

Before proving the theorem, we need

T 5. *Let $k_3(G)$ denote the number of triangles in an arbitrary graph G of n vertices. Then G contains at least $\frac{k_3(G)}{n-2} - O(n)$ edge disjoint triangles.*

Proof of T 5. It has been shown proven by Spencer [10] that in the complete graph K_n of n vertices there can be found a (partial) Steiner system $\mathbf{S_n}$ of $s_n = \frac{1}{3}\binom{n}{2} - O(n)$ edge disjoint triangles.
Let G be a graph of n vertices and denote by $\mathbf{T}$ the set of triangles of $G(\text{then}|\mathbf{T}| = k_3(G))$.
For an arbitrary permutation π of $V(G)$, put $t_\pi = |\mathbf{T} \cap \pi(\mathbf{S_n})|$ where $\pi(\mathbf{S_n})$ is the image of $\mathbf{S_n}$ after applying π.
For each pair T, T' of triangles satisfying $T \in \mathbf{T}$ and $T' \subset K_n$ there are exactly $6(n-3)!$ permutations such that $T = \pi(T')$. On the other hand, $\mathbf{S_n}$ contains s_n of the triangles of K_n, therefore the average value of t_π is $k_3(G)s_n/\binom{n}{3}$. Consequently, there exists a π for which $\pi(\mathbf{S_n})$ contains at least $k_3(G)s_n/\binom{n}{3}$ triangles of G. □

T 6. *Every graph of n vertices and e edges contains at least $\frac{e}{3n}(4e - n^2)(\frac{1}{n-2}) - O(n)$ edge disjoint triangles.*

Proof of T 6. Without loss of generality, we replace $t_2(n)$ by $\frac{n^2}{4}$.
If $e \le \frac{n^2}{4}$, there is nothing to prove.
Otherwise, if $\frac{n^2}{4} < e \le \binom{n}{2}$, the result of Moon and Moser [11] states that G contains at least

$$k_3 \ge \frac{e}{3n}(4e - n^2)$$

triangles, therefore **T 5** implies the validity of our theorem. □

Proof of Theorem 1.2. Induction on n.
The statement is obvious for $n \le 5$.
Assume that Theorem 1.2 holds for graphs of at most $n-1$ vertices.
Let G be an arbitrary graph of order n and size $e = \frac{n^2}{4} + m$
Case 1. $e \ge \frac{5n^2}{12}$ then **T 6** implies

$$edk_3(G) \ge \frac{e}{3n^2}(4e - n^2) - O(n)$$

$$\geq \frac{\frac{5}{12}n^2}{3n^2}4m - O(n) = \frac{5}{9}m - O(n)$$

Case 2. $e < \frac{5n^2}{12}$.
Denote by d the minimum degree, then $d < \frac{5}{6}n$. Let x be a vertex of G with this minimum degree, $N(x)$ be the graph induced by $x's$ neighbors. We distinguish two subcases.

Subcase 2.1 If $d \leq \frac{n}{2}$ then

$$edk_3(G) \geq edk_3(G-x) \geq \frac{5}{9}(e - d - \frac{(n-1)^2}{4}) - O(n-1)$$

$$= \frac{5}{9}(e - \frac{n^2}{4}) + \frac{5}{18}(n-2d) - O(n) \geq \frac{5}{9}m - O(n).$$

Subcase 2.2 If $d > \frac{n}{2}$, in $N(x)$ there exist vertex disjoint complete subgraphs G_1,...,G_{n-d} of order $\lfloor \frac{d}{n-d} \rfloor$ or $\lceil \frac{d}{n-d} \rceil$ satysfying $\cup_1^{n-d} V(G_i) = V(N(x))$.
Suppose that $|V(G_i)| = \lceil \frac{d}{n-d} \rceil$ for $i = 1,..,p$ and $|V(G_i)| = \lfloor \frac{d}{n-d} \rfloor$ for $i = p+1, ..., n-d$ where $p \leq n-d-1$
If $\lfloor \frac{d}{n-d} \rfloor = 1$, then $\frac{n}{2} < d < \frac{2n}{3}$ and $p = 2d - n$.
We see that

$$edk_3(G) \geq p + edk_3(G-x, e-d-p) \geq p + \frac{5}{9}(e-d-p-\frac{(n-1)^2}{4}) - O(n-1)$$

$$= \frac{5}{9}(e - \frac{n^2}{4}) + p + \frac{5}{18}n - \frac{5}{9}(d+p) - O(n)$$

$$= \frac{5}{9}m - O(n) + \frac{6d-3n}{18} \geq \frac{5}{9}m - O(n).$$

If $\lfloor \frac{d}{n-d} \rfloor \geq 2$ then let

$$d := 2s + q (q = 0, 1), \tag{1.3}$$

In $G(N(x))$ every degree is at least $d-s (2d-n \geq d-s)$, so there exist s endpoint disjoint edges. Therefore

$$edk_3(G) \geq s + edk_3(n-1, e-d-s)$$

$$\geq s + \frac{5}{9}(e - 3s - q - \frac{(n-1)^2}{4}) - O(n-1)$$

$$= \frac{5}{9}(e - \frac{n^2}{4}) - O(n) + \frac{5}{18}n - \frac{2}{3}s$$

$$\geq \frac{5}{9}m - O(n)$$

since $2s \leq d < \frac{5}{6}n$. We are done. □

Corollary. $p_3(n) \leq \frac{5}{9}n^2 + O(n)$.

Proof of corollary. Obviously, $p_3(G) = 2e - 3edk_3(G)$. Using both

$$edk_3(G) \geq \frac{4m(m - \frac{n^2}{4})}{3n^2}$$

and

$$edk_3(G) \geq \frac{5}{9}m - O(n)$$

for every graph G of size e more than $\frac{n^2}{4}$ we obtain

$$p_3(G) \leq n^2 \frac{m}{n^2}(3 - 4\frac{m}{n^2})$$

and also

$$\leq \frac{m}{3} + \frac{5}{12}n^2 + O(n)$$

either. The maximum point of the smaller value between them does not exceed $\frac{5}{9}n^2+O(n)$, being when $\frac{m}{n^2}$ is about $\frac{3}{8}$ □

I believe that by induction, as well as by combination of estimates of $edk_3(G)$ depending on the magnitude of e and $\delta(G)$, we might work out better results.

For $k = 4$, Theorem 1.1 gives us $edk_4(n, t_3(n)+m) \geq \frac{1}{4}m$. Using the analogous technique, this result can be refined.

Theorem 1.3.

$$edk_4(n, m + t_3(n)) \geq \frac{5}{18}m - o(n^2).$$

To settle this theorem, a general result of Moon and Moser dealing the number of cliques is needful. Let $t(K_r, G) = \frac{hom(K_r,G)}{n^r}$ where $hom(K_r, G)$ denote the number of homomorphism of K_r to G, that is $t(K_r, G) = \frac{r!k_r(G)}{n^r}$.

T 7. $r\frac{t(K_r,G)}{t(K_{r-1},G)} \leq (r-1)\frac{t(K_{r+1},G)}{t(K_r,G)} + 1$.

Proof of T 7. H^i_{r+1} denotes the graph remaining after deleting i edges sharing a common vertex in K_{r+1}. It can be shown that $t(K_{r-1}, G)t(H^1_{r+1}, G) \geq t(K_r, G)^2$ (from the positive semidefinite property of a matrix, [13])
Consequently,

$$r\frac{t(K_r, G)}{t(K_{r-1}, G)} \leq r\frac{t(H_{r+1}, G)}{t(K_r, H)}.$$

Applying the super-modular property(i.e $t(F+e, G)+t(F+f, G) \leq t(F, G)+t(F+e+f, G)$) $r-1$-times, we obtain that $rt(H^1_{r+1}) \leq (r-1)t(K_{r+1}) + t(K_r)$. □

Corollary. *for $k \geq 2$*

$$k_k(G) \geq \frac{1}{k!}\prod_{1}^{k-1}(1-i(1-2q))n^k \ \textit{where}\ q = \frac{e}{n^2}. \tag{1.4}$$

Proof of corollary. There is nothing to prove if $e < \frac{1}{2}(1-\frac{1}{k-1})n^2$. Suppose that $e \geq \frac{1}{2}(1-\frac{1}{k-1})n^2$. Let $b_{i+1} := \frac{t_{i+1}(G)}{t_i(G)}$, $i = 1,..,k-1$, $b_1 := 1$. Note that $b_i \leq 1 (i = 1,..,k)$ and $t_k(G) = \prod_1^k b_i$.
The inequality 1.4 implies

$$rb_r \leq (r-1)b_{r+1} + 1, \text{ so } r(1-b_r) \geq (r-1)(1-b_{r+1}).$$

therefore

$$1 - b_i \leq \prod_1^{i-2} \tfrac{r+1}{r}(1-b_2) = (i-1)(1-\tfrac{2e}{n^2}) = (i-1)(1-2q),\ b_i \geq 1-(i-1)(1-2q).$$

□

Corollary.

$$k_4(G) \geq \frac{1}{6}q(3q-1)(4q-1)n^4 \ \textit{where}\ q = \frac{e}{n^2} \geq \frac{1}{3}. \tag{1.5}$$

We still need the general form of the **T 5**

T 8. *G contains at least $\frac{(r-2)!k_r(G)}{n^{r-2}} - o(n^2)$ edge disjoint K_r.*

Proof of this theorem. According to a result of V. Rödl [18], There is a system of edge disjoint K_r's which covers almost all the edges of the complete graph K_n but $o(n^2)$. Denote this system by $\mathbf{S_n}$, the left steps are quite similar to what we have done in **T 5** □

Proof of Theorem 2.2. We will use the same technique as before. For convenience, let $t_3(n)$ be $\frac{n^2}{3}$. To explain why the constant is $\frac{5}{18}$, let us replace it by α and try to find the possibly largest value of α.
For small n, clearly $edk_4(G) \geq \alpha m - o(n^2)$. Suppose that it holds for every graph of at most $n-1$ vertices. Consider an arbitrary graph G of n vertices and let x be a vertex of G with minimum degree, $d(x) = d$.
If d is relatively small then the induction step can be done easily as follows. In case of $d < \frac{2n}{3}$, we have

$$edk_4(G) \geq edk_4(G-x) \geq \alpha(e - d - \frac{(n-1)^2}{3}) - o((n-1)^2)$$
$$\geq \alpha m - o(n^2).$$

It is still easy when $\frac{2n}{3} \leq d < \frac{3n}{4}$ because applying the **T 3** to $G(N(x))$, there exist vertex disjoint subgraphs G_1,...,G_{n-d} of order 2 or 3 which cover the vertices of $G(N(x))$. The number of $K_3's$ in $\{G_1,...,G_{n-d}\}$ is $3d-2n$, therefore

$$edk_4(G) \geq p + edk_4(G-x) \geq p + \alpha(e-d-3p-\frac{(n-1)^2}{3}) - o(n^2)$$

$$= \alpha(e-\frac{n^2}{3}) + (3-10\alpha)(d-\frac{2n}{3}) - o(n^2) \geq \alpha m$$

Providing that

$$\alpha \leq \frac{3}{10}.$$

So by this way we can't hope to find α larger than $\frac{3}{10}$
From now on let $d \geq \frac{3n}{4}$.
Set $d = 3s + q$, where $0 \leq q \leq 2$. Again, since every vertex in $G(N(x))$ has a degree at least $d - s$, there exist s vertex disjoint K_3 in $G(N(x))$, as a consequence

$$edk_4(G) \geq s + edk_4(n-1, e-d-3s))$$

$$\geq s + \alpha(e-d-3s-\frac{(n-1)^2}{3})) - o((n-1)^2)$$

$$\geq \alpha m - o(n^2) + \frac{2\alpha}{3}(n - \frac{d(6\alpha-1)}{2\alpha}).$$

Thus if $e \leq \frac{\alpha}{6\alpha-1}n^2$ then we accept this induction step. Otherwise the formula stated in **T 8** for $r = 4$ gives,

$$edk_4(G) \geq \frac{1}{3}q(3q-1)(4q-1)n^2 - o(n^2),$$

here $q = \frac{e}{n^2} \geq \frac{\alpha}{6\alpha-1}$ and $q \geq \frac{1}{3}$ also.
Under these conditions, the inequality

$$\tfrac{1}{3}q(3q-1)(4q-1) \geq \alpha(q-\tfrac{1}{3}) \text{ holds when } \alpha \leq \tfrac{5}{18}.$$

□

A possible generalization of these theorems is

Conjecture. $edk_k(n, t_{k-1}(n) + m) \geq \frac{2}{k}m - o(n^2)$.

Remarks.
Let $p(G)$ denote the minimum of $\sum |V(G_i)|$ over all decompositions of G into edge disjoint cliques $G_1, G_2, ..$ (i.e. the subgraphs G_i's are pairwise edge disjoint cliques such that $E(G) = \cup E(G_i)$ and they are not of given sizes). It was conjectured by G.O.H. Katona and T. Tarján and was proven by F.R.K. Chung, E. Győri and A.V. Kostochka that

T 9.

$$p(G) \leq 2t_2(n)$$

equality holds if and only if $G \simeq T_2(n)$.

The proof is now simple. So if G is K_4 free then $edk_3(G) \geq \frac{2}{3}m$, It is still easy to show that in this case $edk_3(G) \geq \frac{3}{4}m$
P. Erdős suggested to study a similar weigh-function. For any graph G, let $p^*(G)$ denote the minimum of $\sum(|V(G_i)|-1)$ over all decompositions of G into pairwise edge disjoint cliques $G_1, G_2, \ldots$. A maybe too optimistic conjecture is:

Conjecture. *For every graph* G *of* n *vertices,*

$$p^*(G) \leq t_2(n).$$

This conjecture seems to be just a bit stronger than the T 9, but it is not the case. A special case of it is still open, as well.

Conjecture. *Every* K_4*-free graph of* n *vertices and* $t_2(n)+m$ *edges contains* m *edge disjoint triangles.*

Only the following even weaker special case is settled by Győri.

T 10. *Every 3-colorable graph of* n *vertices and* $t_2(n)+m$ *edges contains* m *edge disjoint triangles.*

Proof of **T 10.** Suppose A, B, C with cardinals $a \geq b \geq c$ be the three color parts of $V(G)$. For M is an arbitrary subset of A with cardinals b let $edk_3(M,B,C)$ denote the maximum edge disjoint triangles of G restricted on M, B, C. Then $edk_3(M,B,C) \geq bc - n_{BC} - n_{BM} - n_{CM}$ where $n_{XY} := |X||Y| - e_G(X,Y)$(the number of not-$G$ graph edges between X and Y)
So the average

$$\frac{\sum_{M\subset A} edk_3(M,B,C)}{\binom{a}{b}} \geq bc - n_{BC} - \frac{n_{AC}\binom{a-1}{b-1} + n_{AB}\binom{a-1}{b-1}}{\binom{a}{b}}$$

$$= bc - n_{BC} - \frac{b}{a}(n_{AC}+n_{AB}) \geq bc - (n_{BC}+n_{AC}+n_{AB})$$

$$= bc - (ab+bc+ca-m-t_2(n)) = m + t_2(n) - a(b+c) \geq m.$$

□

To end up the chapter, we examine the graph G of order n, formed by adding m edges to the $T_{2,n}$.
Let m_1, m_2 be the number of added edges in the red, as well as in the blue color of G, let $c_i = \frac{m_i}{n^2}$, $c := c_1 + c_2 = \frac{m}{n^2}$. We estimate the number of edge disjoint triangles as follows. Suppose there are m_1 not monochromatic edge disjoint triangles that cover the m_1 red edges and m_2 not monochromatic edge disjoint triangles that cover the m_2 blue edges. By permuting the vertices in each color we receive many such covering by triangles. Let us choose one of those examples randomly and consider the triangles as vertices, adjancency means that the triangles share some edge. We are finding maximal independent set of vertices. Note that a graph of n edges and e edges have at least $n - e$ independent set of vertices. By counting the expectation of the size of the independent set, since each pair of red-blue edges add 4 to the amount, it is at least

$$m_1 + m_2 - \frac{4m_1m_2}{\frac{n}{2}^2} \geq m - 4\frac{m^2}{n^2}$$

As we can see, when $m = o(n^2)$, the estimate works and almost the best, results in about m. However, when $m = O(n^2)$ it is unusable because we just used the not monochromatic triangles, for example if $m \geq \frac{1}{8}n^2$ then we win not more than $\frac{m}{2}$ triangles.
It is likely that there are at least $\frac{2}{3}m$ edge disjoint triangles in G(would be a corollary of the first conjecture.) However I have no idea to verify this fact. A special case, when the m edges form a triangle-free subgraph in the two colors (therefore only not-monochromatic triangles exist) might be examined by using the fractional results discussed in Chapter 3.

Similarly, the result can be verified for $k \geq 3$, that is there exists a constant c_k such that after adding m edges to $T_{k-1,n}$ we must have $m - c_k\frac{m^2}{n^2}$ edge disjoint $K_k's$.

Chapter 2

If we add m edges to the Turán graph $T_{k-1}(n)$ arbitrarily then the resulting graph contains at most m edge disjoint cliques of k vertices because each clique must contain at least one new edge. that is,

$$edk_k(n, t_{k-1}(n) + m) \leq m$$

If m is large enough, it may occur that we can't find m or asymptotically m edge disjoint cliques of k vertices.
To examine the case when m is small, first we need the following([12])

Let k be an integer ≥ 3, and G be a graph of order n so that $t_{k-1}(n) \leq |E(G)|$. We write $|E(G)|$ in the form

$$|E(G)| = (1 - \frac{1}{t})\frac{n^2}{2}$$

Let d be $\lfloor t \rfloor$, set $m = |E(G)| - t_d(n)$ then

T 11. *("stability theorem")*
For C be an arbitrary constant there exist positive constants $\Delta = \Delta(C)$ and $D = D(C)$ such that if $0 < m < \Delta n^2$ and

$$k_k(G) \leq \binom{t}{k}(\frac{n}{t})^k + Cmn^{k-2}$$

then there exists a $K_d(n_1, .., n_d)$ such that $\sum n_i = n, |n_i - \frac{n}{d}| < D\sqrt{m}$, and G can be obtained by adding to and deleting from this $K_d(n_1, n_2, .., n_d)$ less than Dm edges.

Especially, for $k = d + 1$, suppose that $m = E(G) - t_{k-1}(n)$ is much smaller than n^2, the theorem becomes

Corollary If $k_k(G) \leq Cmn^{k-2}$ then there exists a $K_{k-1}(n_1, .., n_{k-1})$ such that $\sum n_i = n, |n_i - \frac{n}{k-1}| < D\sqrt{m}$ and G can be obtained by adding to and deleting from it less than Dm edges.

Now we are ready to prove

Theorem 2.1. *$edk_k(n, t_{k-1}(n) + m) = m - O(\frac{m^{\frac{3}{2}}}{n})$ if $m = o(n^2)$.*

Proof of theorem 2.1. Set C be any constant larger than 1.
If

$$k_k(G) \geq Cmn^{k-2}$$

then by the proof used in **T.8**, G contains

$$\frac{k_k(G)k!(n-k)!(\frac{\binom{n}{2}}{\binom{k}{2}} - o(n^2))}{n!} \geq \frac{k_k(G)n^2(k-2)!(1-o(1))}{n^k}$$
$$= Cm(k-2)!(1-o(1))$$

edge disjoint copies of K_k, which is more than m provided that n is sufficiently large, and we are done.
Otherwise, by the corollary,

$$k_k(G) \leq Cmn^{k-2}$$

implies there exists $K_{k-1}(n_1, .., n_{k-1})$ such that $\sum n_i = n, |n_i - \frac{n}{k-1}| < D\sqrt{m}$ and G can be obtained by adding to and deleting from it less than Dm edges. Suppose that the number of deleted edges is m_{del}. Let m_{nom} be the number of monochromatic G-edges, then

$$O(m) = m_{nom} \geq m_{del} + m. \tag{2.1}$$

Set m_i be the number of monochromatic edges in i-th color. In each color we choose $\frac{n}{k-1} - D\sqrt{m}$ vertices such that the number of monochromatic edges in it maximum, that is it contains at least

$$(\frac{\frac{n}{k-1} - D\sqrt{m}}{\frac{n}{k-1} + D\sqrt{m}})^2 m_i$$

chromatic edges. Therefore the chosen vertices form a Turán graph(with colors' of size $\frac{n}{k-1} - D\sqrt{m})$ but at most m_{del} edges have been deleted and the number of the remaining monochromatic edges, m'_{nom}, is obviously

$$\geq (\frac{\frac{n}{k-1} - D\sqrt{m}}{\frac{n}{k-1} + D\sqrt{m}})^2 m_{nom} = m_{nom} - O(\frac{m^{\frac{3}{2}}}{n}). \tag{2.2}$$

According to the result noted at the end of the previous chapter, there are at least

$$m'_{nom} - O(\frac{m'_{nom}}{\frac{n}{k-1} - D\sqrt{m}})^2 - m_{del}$$

edge disjoint copies of K_k, Taking account into 2.1, 2.2 we see that

$$edk_k(G) \geq m - O(\frac{m^{\frac{3}{2}}}{n})$$

□

For the triangle-case(and similarly for $k \geq 4$), let us produce a direct proof of

$$edk_3(n, t_2(n) + m) = m - O(\frac{m^{\frac{3}{2}}}{n})$$

if $m = o(n^2)$.

Proof of . Roughly speaking, we will prove that either we can find m edge disjoint triangles or the graph is similar to a graph obtained from $T_2(n)$ by adding m edges to it. Delete the edges of edge disjoint triangles of maximum number and let G^* denote the resulting triangle-free graph. If we found m edge disjoint triangles then we are done. So we assume that

$$|E(G^*)| > t_2(n) - 2m$$

Choose a maximum size spanning bipartite graph G_0 of G^*(with two coloring A, B), let $G_1 = (V, E(G^*) - E(G_0))$, i.e the spanning subgraph of the monochromatic edges. Note that $d_{G_0}(v) \geq d_{G_1}(v)$ (otherwise by changing the color of v we would obtain a greater size spanning bipartite subgraph)

Proposition

$$E(G_1) = O(\frac{m^{\frac{3}{2}}}{n})$$

Proof

Since

$$\frac{1}{|E(G^*)|} \sum_{uv \in E(G^*)} (d_G^*(u) + d_G^*(v)) = \frac{1}{|E(G^*)|} \sum_{v \in V} (d_G^*(v))^2$$

$$\geq \frac{n}{|E(G^*)|} (\frac{\sum_{v \in V} d_G^*(v)}{n})^2$$

$$= \frac{4|E(G^*)|}{n} \geq n - \frac{8m}{n}.$$

There is an edge $uv \in E(G^*)$ such that $d_G^*(u) + d_G^*(v) \geq n - \frac{8m}{n}$.

Let H_0 be a spanning bipartite graph of G^* which

i) contains the edges incident to u or v.

ii) $d_{H_0}(w) \geq d_{H_1}(w)$ if w is not u nor v neighbor, $H_1 = (V, E(G^*) - E(H_0))$.

There exists such H_0, for example a maximum size spanning bipartite subgraph satisfying i). We have

$$E(H_1) \leq d\frac{8m}{n}$$

Where d is the maximum degree in H_1 because the vertices that are not adjacent to u or v represent all the edges of H_1.

If x is a vertex that $d_{H_1}(x) = d$ then the graph G^* does not contain any edge joining $N_{H_0}(x)$ and $N_{H_1}(x)$, so

$$t_2(n) - 2m \leq |E(G^*)| = |E(H_0)| + |E(H_1)|$$

$$\leq t_2(n) - dH_1(x)dH_0(x) + d\frac{8m}{n}$$

$$\leq t_2(n) - d^2 + d\frac{8m}{n}$$

Thus,

$$d \leq \frac{4m}{n} + \sqrt{\frac{16m^2}{n^2} + 2m} \leq (\sqrt{2} + o(1))\sqrt(m)$$

and

$$|E(H_1) = O(\frac{m^{\frac{3}{2}}}{n}).$$

which implies that $E(G_1) = O(\frac{m^{\frac{3}{2}}}{n})$ by the choice of G_0.
The proposition implies that

$$|E(G_0)| \geq t_2(n) - 2m - O(\frac{m^{\frac{3}{2}}}{n}) = t_2(n) - (2 + o(1))m$$

and

$$|A|, |B| \geq \frac{n}{2} - O(\sqrt{m}).$$

From here we may proceed as above. □

However, by more careful techniques, we can also prove that

Theorem 2.2. *If* $m = |E(G)| - t_{k-1}(n) = o(n^2)$ *then either* G *contains* m *edge disjoint* $K_k^\prime s$ *or there is an* $(k-1)$*-coloring* $\{V_1, ..V_{k-1}\}$ *of* $V(G)$ *such that the number of monochromatic edges is* $t_{k-1}(n) - O(1)\frac{m^2}{n^2}$ *and so* $|V_i| = \frac{n}{r-1} + O(1)\frac{m}{n}$ *for* $i = 1, .., k-1$.

Using this theorem, it is easy to prove that

Theorem 2.3. *(Győri [8])*

$$edk_k(n, t_{k-1}(n) + m) = m - O(\frac{m^2}{n^2}) \ \ if \ \ m = o(n^2)$$

Corollary. There is a function $f(c)(= c_0c^2)$ such that every graph of n vertices and $t_{k-1}(n) + cn$ edges contains $cn - f(c)$ edge disjoint $K_k's$.

It can be proved that for a given order n and size $t_{k-1}(n) + m$ $(m = o(n^2))$, $edk_k(G)$ is minimum if G is obtained from $T_{k-1}(n)$ by adding edges so that $\lfloor \frac{m}{n} \rfloor$ vertices of each color join to all vertices.
It implies
T 12. There is a constant c that $edk_3(t_2(n) + m) \leq m - c\frac{m^2}{n^2}$ if $2n \leq m = o(n^2)$.(and we can do also for $k \geq 4$, so the $O(\frac{m^2}{n^2})$ factor is tight in the 2.3.)
Proof. It is enough to estimate the $edk_3(G)$ where G is obtained from $T_2(2k)$ by adding $m = an - a(a+1), 2 \leq a = o(n)$ edges so that a vertices of each color should be joined to all other vertices. Let the two colors be U, V and A_U, A_V be the two set of a vertices. For a set S of edge disjoint triangles let G_0 be the graph containing monochromatic edges that belong to the not chromatic triangles of the set.
Let m_0, m_1 be the number of not chromatic and chromatic triangles of the set.
The chromatic triangles contain 3, the not chromatic ones contain 1 new edges, so

$$m_0 + 3m_1 \leq m$$

It implies that

$$|S| = m_0 + m_1 \leq \frac{m}{3} + \frac{2m_0}{3}$$

Obviously

$m_0 = \sum_{u_i \in A_U} d_{G_0}(u_i) + \sum_{v_i \in A_V} d_{G_0}(v_i) - |G_0\text{-edges having both endpoints in } A_U \text{ or } A_V|$

For $u \in A_U$ we have taken at least $2(d_{G_0}(u) + a - k)$ not monochromatic edges that have an endpoint in A_V to cover $d_{G_0}(u)$ not monochromatic triangles at u, the sum using to cover all can be at least(each edge can be counted twice)

$$\sum_{u_i \in A_U} (d_{G_0}(u_i) + a - k) = \sum_{u_i \in A_U} d_{G_0}(u_i) + a^2 - ak$$

On the other hand we have to use at least $\sum_{v_i \in A_V} d_{G_0}(v_i)$ not monochromatic edges to cover A_V-vertices, hence

$$\sum_{v_i \in A_V} d_{G_0}(v_i) \leq ak - (\sum_{u_i \in A_U} d_{G_0}(u_i) + a^2 - ak)$$

that is

$$\sum_{u_i \in A_U} d_{G_0}(u_i) + \sum_{v_i \in A_V} d_{G_0}(v_i) \leq 2ak - a^2 = m + a$$

Therefore if the number of $|G_0$-edges having both endpoints in A_U or A_V is at least $\frac{a^2}{2}$ then $m_0 \leq m + a - \frac{a^2}{2}$.
Otherwise, of less than $\frac{a^2}{2}$ edges in A_U and A_V are used in the not monochromatic triangles. Thus $m_0 \leq m - \frac{a^2}{2}$.
So

$$edk_3(G) \leq m - \frac{a^2}{3} + O(a) = m - O(\frac{m^2}{n^2}).\Box$$

Another relating problem, questioned by P.Erdős, is

Problem. Determine the maximum m such that every graph of n vertices and $t_2(n) + m$ edges has m edge disjoint triagles. The answer follows

Theorem 2.4. *(Győri [7])*

$$edk_3(n, t_2(n) + m) = m \begin{array}{l} if \;\; m \leq 2n - 10 \;\; for \;\; odd \;\; n \\ if \;\; m \leq \frac{3}{2}n - 5 \;\; for \;\; even \;\; n \end{array}$$

providing that n is sufficiently large.

Examples show that the upper bounds for m in Theorem 2.4 are sharp.

Example 1. $n = 2k$
Add $3k - 4$ edges to $T_2(n)$ so that two vertices of color class A and one vertex of color class B should be joined to all vertices. If the $k - 1$ edges in B are covered by $k - 1$ edge disjoint triangles then deleting these $3k - 3$ edges, every vertex in A has $k - 2$ neighbors in B except one which has k ones. Since two vertices in A have $k - 1$ neighbors in A, we can't find $2k - 3$ edges disjoint triangles covering the $2k - 3$ edges incident to these two vertices.
This is the unique extremal example.

Example 2. $n = 2k + 1$
Take the Turán graph $T_2(n)$ with color classes V_1 and V_2 of k and $k + 1$ vertices, respectively. Take one vertex in V_1 and join it to all vertices in V_1. Then, take three vertices, say x_1,x_2,x_3 in V_2, and join them to each other and to $k - 3$(except one) other vertices in V_2.(The set of $k - 3$ vertices do not have to be the same for the vertices x_1, x_2, x_3) It is easy to see that the resulting graph G of n vertices and $t_2(n) + 2n - 9$ edges does not contain $2n - 9$ edge disjoint triangles.
We have only three extremal examples.

The continuing of **2.4** follows

Theorem 2.5. *(Győri [8])*

$$edk_k(n, t_{k-1}(n) + m) = m \ \ if \ \ m \leq \lfloor 3\frac{n+1}{r-1} \rfloor - 5 \ \ for \ \ r \geq 4$$

provided that n is sufficently large.

The following example shows that the upper bound for m in Theorem 2.5 is sharp.

Example 3.
Consider two color classes V_i and V_j of $T_(k-1)(n)$ of $\lfloor \frac{n+1}{r-1} \rfloor$ vertices. Add $3\lfloor \frac{n+1}{r-1} \rfloor$ - 4 edges to $T_(k-1)(n)$ so that two elements of V_i and one element of V_j should be joined to all vertices. If the $\lfloor \frac{n+1}{r-1} \rfloor - 1$ edges in V_j are covered by $\lfloor \frac{n+1}{r-1} \rfloor - 1$ edge disjoint $K_k's$ then deleting these $(\lfloor \frac{n+1}{r-1} \rfloor - 1)\binom{n}{2}$ edges, every vertex in V_i has $\lfloor \frac{n+1}{r-1} \rfloor - 2$ neighbors in V_j except one which has $\lfloor \frac{n+1}{r-1} \rfloor$ neighbors. Since two elements of V_i have $\lfloor \frac{n+1}{r-1} \rfloor - 1$ neighbors in V_i, we cannot find $2\lfloor \frac{n+1}{r-1} \rfloor - 3$ edge disjoint $K_k's$ covering the $2\lfloor \frac{n+1}{r-1} \rfloor - 3$ edges incident to these two vertices.
This is the only extremal graph.

Remarks
Another important characterization of the Turán graphs is
Let $G_{k-1,m}(n)$ denote a graph obtained from $T_{k-1}(n)$ by adding m edges so that the new edges belong to the same class having maximum number of vertices(i.e. $\lceil \frac{n}{k-1} \rceil$) and the new edges do not form triangles, if this is possible. Then there exist a constant $c_k > 0$ such that for $m < c_k n$, if G is a graph of order n and size $t_{k-1}(n) + m$ then

$$k_k(G) \geq k_k(G_{k-1,m}(n)) = m \prod_{0 \leq i \leq k-3} \lfloor \frac{n+i}{k-1} \rfloor \tag{2.3}$$

Problem Determine the constant c_k in the theorem above.

If we add $p+1$ or more edges to the first class of $K_{k-1}(p+1, p, .., p, p-1)$, then each new edge will be contained only in $(p-1)p^{k-3}$ $K_k's$, and we can verify that 2.3 does not hold. So $c_k \leq \frac{1}{k-1}$

Theorem 2.6. *[12] If $e(G) = t_{k-1}(n) + m$, where $m < \lfloor \frac{n}{k-1} \rfloor$, then*

$$k_k(G) \geq m \prod_{0 \leq i \leq k-3} \lfloor \frac{n+i}{k-1} \rfloor$$

For example, $k = 3$ gives

If $e(G) = t_2(n) + l, 0 < l \leq \frac{n}{2}$, then G contains at least $l\lfloor \frac{n}{2} \rfloor$ triangles.

Here we reproduce a simple fact proving for the case $l \leq \frac{n}{4}$, as follows[3]

We assume that n is even(similar of the case n odd)
Let t^k (k=1,2,3) be the number of triangles in K_n that contain exactly k edges of $\overline{G}$. Thus $t^0 = k_3(G)$.
Let λ_v be the number of edges between $N(v)$ and $V(G - N(v)$. For uv edge, denote by t^k_{uv} the number of triangles in K_n that contain uv and k edges of $\overline{G}$. By definitions, it is obviously that

$$\sum_v \lambda_v = 2t^2, \tag{2.4}$$

$$\sum_{uv \in E(G)} t^0_{uv} = 3t^0, \tag{2.5}$$

$$\sum_{uv \in E(G)} t^2_{uv} = t^2, \tag{2.6}$$

$$n = d(u) + d(v) - t^0_{uv} + t^2_{uv}, \tag{2.7}$$

provided that $uv \in E(G)$.
Summing the last equality over all edges uv, and by 2.5, 2.6, we have

$$nm = \sum_u d(u)^2 - 3t^0 + t^2, \tag{2.8}$$

Since

$$\sum_{uv \in E(G)} (d(u) + d(v)) = \sum_u d(u)^2.$$

By putting $p(u) = d(u) - \frac{n}{2}$ we obtain that

$$\sum_u d(u)^2 = \sum_u p(u)^2 + n \sum_u p(u) + \frac{n^3}{4} = \sum_u p(u)^2 + sn(m - \frac{n^2}{4}) + \frac{n^3}{4} = \sum_u p(u)^2 + 2ln + \frac{n^3}{4}.$$

Together with 2.4 and 2.8, it gives

$$3t^0 = \sum_u (p(u)^2 + \frac{\lambda(u)}{2}) + ln.$$

This relation gives the assertion of the theorem unless

$$\sum_u (p(u)^2) + \frac{\lambda(u)}{2} \leq \frac{ln}{2}. \tag{2.9}$$

Thus, we may assume 2.9 holds. Then for some vertex u

$$p(u)^2 + \frac{\lambda(u)}{2} \leq \frac{l}{2}. \tag{2.10}$$

The number of edges in G with both end vertices in $N(u)$ or $V(G) - N(u)$ is

$$\frac{n^2}{4} + l - (\frac{n^2}{4} - p(u)^2 - \lambda(u)) = l + p(u)^2 + \lambda(u).$$

Each such edge belongs to at least

$$\frac{n}{2} - |p(u)| - \lambda(u) \geq \frac{n}{2} - p(u)^2 - \lambda(u)$$

G-triangles (which have the third vertex in the other color). Therefore

$$k_3(G) > ((l + p(u)^2) + \lambda(u))(\frac{n}{2} - p(u)^2 - \lambda(u)) =$$

$$l\frac{n}{2} + (p(u)^2 + \lambda(u))(\frac{n}{2} - l - p(u)^2 - \lambda(u))$$

$$\geq l\frac{n}{2} + (p(u)^2 + \lambda(u))(\frac{n}{2} - 2l) > \frac{ln}{2}\square$$

Chapter 3

Part 1.

In the beginning of **Chapter 1** we have defined:

$$p_k(G) = \min\{\sum_1^m |V(G_i)| : G_i\text{'s are } K_k \text{ and edges}, \cup_i^m E(G_i) = E(G)\}.$$

The weight of each $K_k's$ copies is k and 2 is that of edges. We also proved that

$$p_k(G) \leq 2t_{k-1,n}$$

for $k \geq 4$.
In the first part of this chapter, we use the fractional method from [5]. We will study the problem when the weight of $K_k's$ copies is a given integer s less than $k(k-1)$.
Let's start with some definitions.
For G is a graph of n vertices, a ψ^* function on subgraphs H of G where $H \simeq K_k$ or an edge to $[0,1]$ is called fractional full-packing if

$$\sum_{e \in H} \psi^*(H) = 1$$

for every edge e.

Definition. $|\psi_s^*| := \sum_{H \simeq K_k} s\psi^*(H) + \sum_{H \simeq K_2} 2\psi^*(H)$.

Definition. $\psi_s^*(G) := min\{|\psi_s^*|, \psi^* \text{ is fractional full-packing of } G\}$.

A ψ fractional full-packing to $\{0,1\}$ is called integer full-packing.
$\psi_s(G) := min\{|\psi_s|, \psi \text{ is an integer full-packing of } G\}$.
It is obvious that $\psi_s(G) \geq \psi_s^*(G)$, the following shows that the difference is small

Theorem 3.1.

$$\psi_s(G) - \psi_s^*(G) = o(n^2)$$

for all graphs G.

The theorem in fact is a direct corollary of a deep application of the Szemerédi regularity lemma [19], as follows
Let H_0 be any fixed graph, let $\nu_{H_0}(G)$ denote the maximum size of a set of pairwise edge disjoint copies of H_0 in G and $\nu^-_{H_0}(G)$ denote the maximum value of $\sum_{H \in G, H \simeq H_0} \psi(H)$, where ψ runs over all functions on copies of H satisfying $\sum_{e \in H} \psi(H) \leq 1$ for every edge e of G(i.e fractional packing). Then

$$\nu^-_{H_0}(G) - \nu_{H_0}(G) = o(|V(G)|^2)$$

for all graphs G.
This theorem makes it easier to study the $edk_{H_0}(G)$ when the structure of the graph is regular.

Proof of Theorem 3.1. We see that
$\psi^*_s(G) = min\{\sum_{H \simeq K_k} s\psi^*(H) + 2\sum_{H \simeq K_2} \psi^*(H) : \psi^*$ is a fractional full-packing of $G\}$= $\min\{2\sum_e \sum_{H \simeq K_k \ or \ edge, e \in H} \psi^*(H) - (k(k-1)-s)\sum_{H \simeq K_k} \psi^*(H) :$ ψ^* is a fractional full-packing of $G\} = 2e - (k(k-1)-s)\nu^-_{K_k}(G)$, since $max \sum_{H \simeq K_k} \psi^*(H) = \nu^-_{K_k}(G)$.
Similarly, $\psi_s(G) = min\{\sum_{H \simeq K_k} s\psi(H) + 2\sum_{H \simeq K_2} \psi(H) : \psi$ is an integer full-packing of $G\}$= $\min\{2\sum_e \sum_{H \simeq K_k \ or \ edge, e \in H} \psi(H) - (k(k-1)-s)\sum_{H \simeq K_k} \psi(H) :$ ψ is an integer full-packing of $G\} = 2e - (k(k-1)-s)\nu_{K_k}(G)$
due to $max \sum_{H \simeq K_k} \psi(H) = \nu_{K_k}(G)$. □

Definition. $p^*_s(n) := max\{|\psi^*_s(G)|, G$ is of order $n\}$, $p_s(n) := max\{|\psi_s(G)|, G$ is of order $n\}$.
Corollary. $p_s(n) - p^*_s(n) = o(n^2)$
The $p_k(n)$ has been investigated. By the corollary we can study $p^*_s(n)$ instead of $p_s(n)$. The $p^*_s(n)$ can be studied easily.

Theorem 3.2. *The $\frac{p^*_s(n)}{n(n-1)}$ sequence is monotone decreasing.*

Proof of Theorem 3.2. Let G be a graph of order $n+1$ where $|\psi^*_s(G)| = p^*_s(n+1)$.
For every vertex v of G let G_v is the $G - v$ graph, and $\psi^*_{G_v}$ is its optimal fractional full-packing.
Then, the $\xi^* := \frac{1}{n-1}\sum_v \psi^*_{G_v}$ is also a fractional full-packing in$G$.
Obviously,

$$|\xi^*_s| = \frac{1}{n-1}\sum_v |\psi^*_{G_v}| \leq \frac{n+1}{n-1} p^*_s(n)$$

and $|\xi^*_s| \geq min\{|\psi^*|, \psi^*$ is a fractional full-packing of $G\} = \psi^*_s(G) = p^*_s(n+1)$.

□

Corollary. *The limit of $\frac{p^*_s(n)}{n^2}$, and also, of $\frac{p_s(n)}{n^2}$ exists when $n \longrightarrow \infty$, they are equal.*

We have another relation

Theorem 3.3. $p_s^*(kn) \leq k^2 p_s^*(n) + sn + \lfloor \frac{R(k,k)-1}{k} \rfloor (k^2 - k - 2 - s)^+$
where $R(k,k)$ is the appropriate Ramsey number.

Proof of Theorem 3.3. Let G_{kn} be a graph of order kn where $|\psi_s^*(G_{kn})| = p_s^*(kn)$.
We can divide the graph into n groups $G_1, G_2, .., G_n$ of k vertices so that each of the first $n - \lfloor \frac{R(k,k)-1}{k} \rfloor$ groups is K_k or its complement.
For each $v_1, v_2, .., v_n$ vertices from each group, define $\psi^*_{v_1,v_2,..v_n}$ be an optimal fractional full-packing of the graph formed by $v_1, v_2, ..v_n$. Let ψ_i^* be an optimal fractional full-packing of the $i - th$ group(graph formed by the vertices in group).
Clearly,

$$\xi^* := \frac{1}{k^{n-2}} \sum_{v_i \in G_i} \psi^*_{v_1,v_2,..v_n} + \sum_i \psi_i^*$$

is a fractional full-packing in G_{kn}, therefore $|\xi_s^*| \geq p_s^*(kn)$
On the other hand,

$$|\xi_s^*| \leq \frac{1}{k^{n-2}} k^n p_s^*(n) + (n - \lfloor \frac{R(k,k)-1}{k} \rfloor) s + \lfloor \frac{R(k,k)-1}{k} \rfloor max\{s, k(k-1)-2\}$$

$$= k^2 p_s^*(n) + sn + \lfloor \frac{R(k,k)-1}{k} \rfloor (k^2 - k - 2 - s)^*,$$

and we are done. □

Let c_s^k be the limit in Theorem 3.2.
In Chapter one we have shown some lower bounds of c_k^k, not knowing there exist or not.
As $k = 3$ and $s = 3$, since $R(3,3) = 6$, the Theorem 3.4 shows:

$$p_3^*(3n) \leq 9 p_3^*(n) + 3n + 1. \tag{3.1}$$

The $c_3^3 \leq \frac{5}{9}$ is a result in chapter one.

Theorem 3.4. $c_3^3 \leq \frac{p_3^*(n)}{n^2} + \frac{1}{2n} + \frac{1}{8n^2}$, *and hence* $c_3^3 \leq \frac{p_3(n)}{n^2} + \frac{1}{2n} + \frac{1}{8n^2}$ *too*

Proof of Theorem 3.4. From 3.1,

$$p_3^*(3^{k+1}n) \leq 9^{k+1} p_3^*(n) + \sum_{i=0}^{k} 9^{k-i}(3^{i+1}n + 1).$$

This implies

$$c_3^3 \leq \frac{p_3^*(3^{k+1}n)}{3^{k+1}n(3^{k+1}n - 1)} = \frac{3^{k+1}n}{3^{k+1}n - 1} \frac{p_3^*(3^{k+1}n)}{9^{k+1}n^2}$$

$$\leq \frac{3^{k+1}n}{3^{k+1}n-1}(\frac{p_3^*(n)}{n^2}+\frac{1}{3n}\sum_{i=0}^{k}3^{-i}+\frac{1}{n^2}\sum_{i=0}^{k}9^{-i-1}).$$

Also the limit of the latter term is $\frac{p_3^*(n)}{n^2}+\frac{1}{2n}+\frac{1}{8n^2}$ when k tends to infinity □

Similarly, the $k=4, s=4$ case($R(4,4)=18$):

$$p_4^*(4n) \leq 16p_4^*(n)+4n+24. \tag{3.2}$$

and we have

Theorem 3.5. $c_4^4 \leq \frac{p_4^*(n)}{n^2}+\frac{1}{3n}+\frac{1}{10n^2}$. *Consequently,* $c_4^4 \leq \frac{p_4(n)}{n^2}+\frac{1}{3n}+\frac{1}{10n^2}$ *too.*

Proof of theorem 3.5. From 3.2

$$p_4^*(4^{k+1}n) \leq 16^{k+1}p_4^*(n)+\sum_{i=0}^{k}16^{k-i}(4^{i+1}n+24)$$

Which leads to

$$c_4^4 \leq \frac{p_4^*(4^{k+1}n)}{4^{k+1}n(4^{k+1}n-1)} = \frac{4^{k+1}n}{4^{k+1}n-1}\frac{p_4^*(4^{k+1}n)}{16^{k+1}n^2}$$

$$\leq \frac{4^{k+1}n}{4^{k+1}n-1}(\frac{p_4^*(n)}{n^2}+\frac{1}{16n}\sum_{i=0}^{k}4^{1-i}+\frac{24}{16n^2}\sum_{i=0}^{k}16^{-i-1}).$$

The limit of the latter term is $\frac{p_4^*(n)}{n^2}+\frac{1}{3n}+\frac{1}{10n^2}$ when k tends to infinity □

By help of such computation we can determine the value of $\frac{p_3(n)}{n^2}$, $\frac{p_4(n)}{n^2}$...when n is large enough, I think the constants $\frac{5}{9}$, $\frac{5}{18}$... can be improved this way.

Remarks.
Let k be unchanged. For $x \in [0, k(k-1)]$, and n is given, let G_x be an optimal graph(or one of),i.e $|\psi_{G_x}| = p_x(n)$.
We set $m_x := E(G_x)$, T_x:= maximal number of edge disjoint $K_k's$ in G_x (in fact m_x, T_x depend on G_x), then for any $r \leq s$ in the $[0, k(k-1)]$ interval the followings hold
(a) $T_s \geq T_r$,$m_s \geq m_r$ and

$$(k(k-1)-r)(T_s-T_r) \geq 2(m_s-m_r) \geq (k(k-1)-s)(T_s-T_r)$$

(b) c_x is a continuously monotone, convex function.
(c)

$$(s-r)T_s \geq p_s(n)-p_r(n) \geq (s-r)T_r.$$

Proof of **(a).** By the maximality of $p_s(n)$
$p_s(n) \geq |\psi_s(G_r)| = 2m_r - (k(k-1)-s)T_r$, but $p_s(n) = 2m_s - (k(k-1)-s)T_s$, So

$$2(m_s - m_r) \geq (k(k-1)-s)(T_s - T_r)$$

By the maximality of $p_r(n)$
$p_r(n) \geq |\psi_r(G_s)| = 2m_s - (k(k-1)-r)T_s$, but $p_r(n) = 2m_r - (k(k-1)-r)T_r$, So

$$2(m_s - m_r) \leq (k(k-1)-r)(T_s - T_r)$$

(b) $p_r(n) \geq |\psi_r(G_s)| \geq \frac{r}{s}|\psi_s(G_s)| = \frac{r}{s}p_s(n)$, thus $c_s \geq c_r \geq \frac{r}{s}c_s$, and therefore it is continuous.
If $t = \alpha r + (1-\alpha)s$, we see that

$$p_t(n) = |\psi_t(G_t)| = 2m_t - (k(k-1)-t)T_t =$$

$$\alpha(2m_t - (k(k-1)-r)T_t) + (1-\alpha)2m_t - (k(k-1)-s)T_t \leq \alpha p_r(n) + (1-\alpha)p_s(n).$$

(c) $p_s(n) - p_r(n) = 2m_s - (k(k-1)-s)T_s - 2m_r + (k(k-1)-r)T_r$, and we are done by **(b)**. □

Clearly $c_0 = 1 - \frac{1}{k-1}, c_{k(k-1)} = 1$

Corollary.

$$\frac{c_s - c_r}{s-r} \leq \frac{1}{k(k-1)}.$$

If the conjecture $edk_k(G) \geq \frac{2}{k}m - o(n^2)$ is true, then $c_{k(k-2)} = 1 - \frac{1}{k-1}$(equivalent), which means the function c_x is constant in $[0, k(k-2)]$ and linear in $[k(k-2), k(k-1)]$.

Part 2.

In any 2-coloring of K_n, where $n > R(k,k)$, there are monochromatic $K_k's$. We can ask about the minimum number of such monochromatic $K_k's$.
Here we will consider the number of edge disjoint monochromatic $K_k's$.
Definition. Let $N(n,k)$ be the minimum number of pairwise edge disjoint monochromatic complete subgraphs K_k in any 2-coloring of the edges of K_n.

Theorem 3.6. $\lim_{n\to\infty} \frac{N(n,k)}{n(n-1)}$ *exists* .

Proof of theorem 3.6.
1. If $c_k = sup_n \frac{N(n,k)}{n(n-1)}$, then for any given small $\epsilon > 0$, there is an $m = m(\epsilon)$ such that

$$\frac{N(m,k)}{m(m-1)} \geq (1-\epsilon)c_k m(m-1).$$

Then for n sufficiently large, the edges of K_n can be packed with $(1-\epsilon)\frac{\binom{n}{2}}{\binom{m}{2}}$ edge disjoint $K_m's$ by [18]. As a consequence, for n large

$$N(n,k) \geq (1-\epsilon)\frac{\binom{n}{2}}{\binom{m}{2}}(1-\epsilon)c_k m(m-1) \geq (1-\epsilon)^2 c_k n(n-1).$$

Therefore we are done, and the limit equals $sup_n \frac{N(n,k)}{n(n-1)}$

2. Let $N^-(n,k)$ be the minimum possible value of $\nu^-_{K_k}(G) + \nu^-_{K_k}(\overline{G})$ over all 2-colorings $G \cup \overline{G}$ of the edges of K_n.

According to a result in Part 1.,

$$N^-(n,k) - N(n,k) = o(n^2).$$

The $N^-(n,k)'s$ have the following property

The sequence $\frac{N^-(n,k)}{n(n-1)}$ is increasing in n.

Because considering 2-coloring the edges of complete graph K_{n+1}, for every $1 \leq i \leq n+1$ we can find a fractional packing w_i of monochromatic K_k of the complete graph received from K_{n+1} after deleting it's i-th vertex; the total value of w_i is at least $N^-(n,k)$ by definition.

Each edge of K_{n+1} belongs to $n-1$ such complete subgraphs, so the $w := \frac{1}{n-1}\sum_i w_i$ is also a fractional packing for K_{n+1}. But its total value is at least $\frac{n+1}{n-1}N^-(n,k)$. Therefore

$$N^-(n+1,k) \geq \frac{n+1}{n-1}N^-(n,k).$$

The $\frac{N^-(n,k)}{n(n-1)}$ is bounded by $\frac{2}{k(k-1)}$ for every n. Therefore by the above, this sequence converges.

□

Let c_k be this constant; The main problems are to estimate c_k for every k.

For the case $k=3$, if we consider the 2-coloring of K_n determined by $T_2(n)$ and its complement(two monochromatic cliques) then there are approximately

$$2\frac{\binom{n/2}{2}}{3} = \frac{n^2}{12} + o(n^2)$$

edge disjoint monochromatic triangles, thus $c_3 \leq \frac{1}{12}$. This example led to a conjecture of P. Erdős.

Conjecture:

$$c_3 = \frac{1}{12}.$$

The value of $N(11,3)$ is 6. Therefore, by the first method above,

$$c_3 = sup_n \frac{N(n,3)}{n(n-1)} \geq \frac{N(11,3)}{11.10} = \frac{3}{55}.$$

Another way to find estimate for c_3 is by second method and techniques used in the first part.

Theorem 3.7. *([5])* $N^-(3n,3) \geq 9N^-(n,3) + n - 1$.

Proof of theorem 3.7. Consider such a 2-coloring of K_{3n} that the total value of its maximal packing is $N^-(3n,3)$. Since 6 vertices contain a monochromatic triangle we can find $n-1$ vertex disjoint monochromatic triangles. Let $T_1, T_2, .., T_{n-1}$ denote these triples and T_n be the remaining three vertices. For any 3^n distinct copies of K_n(from each set each vertex) there is a fractional packing w_i, and because each edge connecting different $T_j's$ is covered exactly 3^{n-2} times by such $w_i's$, the $w := 3^{-(n-2)}\sigma_i w_i$ is a fractional packing of the connecting edges. By adding $T_i's$ to this packing, we receive a packing of K_{3n}, whose value is at least

$$3^{-(n-2)}.3^n N^-(n,3) + (n-1) = 9N^-(n,3) + n - 1$$

Iterating this result will lead to
Corollary

$$c_3 \geq \frac{N^-(n,3)}{n^2} + \frac{1}{6n} - \frac{1}{8n^2}$$

for every $n \geq 1$
The corollary provides that, with the easy fact $N^-(7,3) \geq N(7,3) = 2$,

$$c_3 \geq \frac{N^-(7,3)}{7^2} + \frac{1}{6.7} - \frac{1}{8.7^2} = \frac{73}{1176} > \frac{1}{16.11} > \frac{3}{55}$$

The up-to-now best result of c_3 is due to Sudakov and Keevash: $c_3 > \frac{1}{12.88}$
Similarly, we might study the case $k \geq 4$. It seems that these bounds will depend on the Ramsey numbers $R(k,k)'s$.

Remark
The conjecture may be weakened by asking whether there are approximately $\frac{1}{12}n^2$ triangles in the G graph whose complement is triangle-free.
We may ask the related question, let $N'(n,k)$ be the minimum of the maximum of the two monochromatic edge disjoint $K_k's$ number in any 2-coloring of the edges of K_n. The construction based on the blowup of C_5 shows that the number of monochromatic edge disjoint triangles in either of the colors is about $\frac{n^2}{20}$, and this led to the conjecture of

Jacobson
Conjecture If n is sufficiently large, then

$$N'(n,3) = \frac{n^2}{20} + o(n^2).$$

Obviously, $N'(n,k) \geq \frac{1}{2}N(n,k)$.

Part 3.

In this part, we are discussing on the relations between the graph's structures.
For F is a fixed graph, let $\Delta_F(G)$ be the minimum number of copies of F and edges which fully pack $E(G)$, $\Delta_F^+(G)$ be the minimum number of copies of F and edges which cover $E(G)$ and finally, $\mu_F(G)$ be the maximum number of edges in the F-free subgraphs of G. Clearly,

$$\Delta_F(G) \geq \Delta_F^+(G)$$

According to [16], $\Delta_{K_k}(G) \leq t_{k-1,n}$ for any graph G of order n

for $F = K_3$, we have

Theorem 3.8. *([9])* $\Delta_{K_3}^+(G) \leq \mu_{K_3}(G)$

Shortly, for every e edge $\in E - M$(M is a maximal triangle free subgraph) we find an f edge $\in M$ so that e and f are edges of a triangle, and no $e's$ have the same f. The Hall condition can be verified by using the maximality of M and the fact that every graph contains a triangle free subgraph of size at least it's one half.
More generally, the theorem is true for all F has the chromatic number not less than 3(one can check that for some bipartite $F's$ the theorem does not stand, on the other hand, for these F the size of a F-free subgraph always be $o(n^2)$), and the proof of this is quite similar to the triangle case,

Theorem 3.9. $\Delta_F^+(G) \leq \mu_F(G)$ *with* $\chi(F) \geq 3$ *and equality holds iff* G *is* $F-$*free.*

For the case $F = K_k (k \geq 3)$ and G contains a $k-$clique, the equality does not hold, indeed on these conditions

Theorem 3.10. $\Delta_F^+(G) \leq \mu_F(G) - \binom{k}{2} + 2$

The following conjectures [9] fit to our topic

Conjecture

$$\Delta_{K_3}(G) \leq \mu_{K_3}(G)$$

This result would be a generalization of [16] stating that the edges of a graph on n vertices can be covered by at most $\lfloor \frac{n^2}{4} \rfloor$ pairwise disjoint edges and triangles.
Conjecture

$$\Delta_F(G) \leq \mu_F(G)$$

if $\chi(F) \geq 3$?

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