Recognition algorithms for median graphs

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Introduction

Median graphs comprise an interesting class of graphs which has been extensively investigated in the last 20 years. They can be characterized as retracts of hypercubes or as isometric subgraphs of hypercubes satisfying certain convexity properties. We shall denote the class of median graphs by M and the class of isometric subgraphs of hypercubes by PC. For the class of hypercubes we shall use the notation H. Furthermore, it will be convenient to denote almost median graphs and premedian graphs, first introduced in [IK2] by PM and AM.

The following hierarchy holds for these classes:

$$H \subset M \subset AM \subset PM \subset PC.$$

Hypercubes can be recognized in $O(m)$ time, which is optimal, and partial cubes in $O(mn)$ time. We shall show that median graphs can be recognized in $O(m\log n^{1.41})$ time and premedian graphs in $O(m\log n)$ time. This improves the best known previously established complexities of $O(m\sqrt{n})$ for median graphs and $O(mn)$ for premedian graphs.

Interestingly median graphs can be embedded into hypercubes in $O(m\{log)n)$ time but currently we can only recognize them in $O(m\log n^{1.41})$ time.

Motivations for isometric embeddings into hamming graphs has come from communication theory (Graham and Pollak [GP])and in linguistics (Firsov [FV]).

The theory of isometric embeddings into Cartesian product graphs nowadays is also an extensively investigated part of graph theory. Below we give 3 interesting open problems connected to the characterization of partial binary Hamming graphs.

Problem 1 Is every regular partial cube a Cartesian product of cycles and K_2's?

Problem 2 Is every premedian regular graph a hypercube?

(It was proved by Mulder that it is true for the class of median graphs.)

Conjecture A graph is a partial binary Hamming graph if and only if it is bipartite and every inervallum of it is convex.

Embedding algorithms are also useful in the solving large linear systems, which describe flow problems. The reason is, if we can embed the graph into a Hamming graph we define in this way coordinatization on its vertices and this data structure is more handy.

For the graphs, which can not be isometrically embedded into a Hamming graph it may be useful to cat the graph onto parts each of which can be isometrically embedded into a Hamming graph and consider on every part the above described data structure. Giving an efficient algorithm for such a cutting of non-embeddable graphs is also an interesting open problem.

An another area where partial Hamming graphs appear naturally is the dynamic location problem. A motivation for the problem comes from optimization theory, in particular from on-line sequential search problems.

Suppose a system is represented by a connected graph G. Let us assume the existence of a common resource which is located at some vertex s of G and can be moved to any other vertex. The vertex s is called the *state* of the system. Each request consists of a vertex r in which some processing must be done. Let the cost of such an operation be $d(r, s)$, where s is the current state of the system. After processing a request we may move the resource to another vertex of G. Assume that the cost of moving the resource from a state s to a state s' to be equival to $d(s, s')$.

Let $s_0 = r_0$ be the *initial state* of the system and let $r_1, r_2, ..., r_k$ be a *request sequence* to G. Our goal is to find a *state sequence* $s_1, s_2, ..., s_k$ such that the *cost* of the state sequence to the request sequence

$$\sum_{i=1}^{k}(d(s_{i-1}, s_i) + d(s_i, r_i))$$

is as small as possible. The *dynamic location problem* is to find such a state sequence. If the request is known in advance, the problem can be solved by standard dynamic programming approach. However, we want to solve the problem on-line, since in general we have only incomplete knowledge of future requests.

An algorithm for the dynamic location problem *works within window* k, if for each i the choice of s_i depends only on r_j for $j < i + k$. The window index, or windex of graph G , $WX(G)$, is the smallest k such that there exists an optimal algorithm for the dynamic location problem that works within window k. If no such k exists then $WX(G) = \infty$.

Consider for example the graph $K_4 - e$. Let $V(K_4 - e) = \{u, v, w, z\}$ and let u and z be the nonadjacent vertices. Assume that $s_{i-1} = v$ for some $i \geq 1$ and consider the request sequence $\sigma = wuzuz...uz$, where $w = r_i$, $u = r_{i+1}$,... It is easy to see that if the first request after σ will be w then s_i must be w and if the first request after σ will be v we must set $s_i = v$. It follows that $WX(K_4 - e) = \infty$.

Proposition. For every graph with at least one edge, $WX(G) \geq 2$.

Proposition. For any $n \geq 2$, $WX(K_n) = n$.

Proposition. If G is a retract of a Hamming graph then $WX(G) < \infty$.

Proposition. $WX(G) = 2$ if and only if G is median.

The recognition algorithms for partial binary Hamming graphs are closely related to the embedding algorithms. In particular with the algorithm for recognition of premedian graphs we will obtain by the way an isometric embedding of a premedian graph into a hypercube.

Recently closed relationship was discovered between median graphs and triangle-free graphs. In particular it can be shown that there are as many median graphs as triangle-free graphs. If $M(m,n)$ denotes the complexity of checking whether a graph G with m edges and n vertices is median then the complexity of checking whether G is triangle-free is at most $M(m,m)$ and conversely, the complexity of checking whether a given graph is a median graph is at most $O(m\log n) + T(m\log n)$, where $T(m,n)$ is the complexity of finding all triangles of the graph. [IKM]

An another nice application of the partial binary Hamming graphs and in particular of median graphs is in organic chemistry.

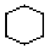
A topological index is a numerical quantity derived in an unambiguous manner from the structural graph of a moleculae. Chemists employ structural formulae in communicating information on molecules and their structure. The first topological index in chemistry was introduced by Harold Wiener [WH]in 1947 to study the boiling point of paraffins.

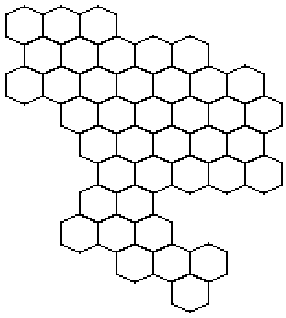
The Wiener index $W(G)$, of a graph G is just the sum of the distances between all pairs of vertices of G. That is

$$W(G) = \frac{1}{2} \sum_{u \in V(G)} \sum_{v \in V(G)} d_G(u,v).$$

Since then the Wiener index W has been used to explain various chemical and physical properties of molecules and to correlate the structure of molecules with their biological activity. It found numerous applications in the modeling of a variety of physico-chemical, pharmacological and biological properties of organic molecules, and its theory is equally well developed; for recent reviews see [GYLL],[KG],[NTM].

It is clear that pharmacological activity of a substance is often related to some of its physico-chemical properties. Thus, it is not surprising that attempts have been made to use the Wiener index in designing new drugs. Lucovits [L] established correlations between W and cytostatic and antihistaminic activities of certain pharmacologically interesting compounds. In





Benzoid system

a recent study he employed W for estimating the partition coefficient of octan-1-ol and water, a parameter of profound importance for forecasting pharmacological activities of many compounds.

The Wiener index found several noteworthy applications in polymer chemistry. Polymers (natural or synthetic) are essential for life (for example, proteins, nucleic acids) and civilized life (for example, plastics, composites, elastomers). The melting points and other physical properties of polymers (e.g. of polyethylene) were predicted on the basis of their W-values [BL],[GYLL].

A hexagon is a plane figure of six sides and six angles Two regular hexagons are said to be congruent if their edges are equal in size. A benzenoid system is a connected geometric figure obtained by arranging congruent regular hexagons in a plane, so that two hexagons are either disjoint or have a common edge.

We shall call a connected plane graph G a benzenoid system if G has no cut vertices, and in which every interior region is bounded by a hexagon. Hexagonal systems provide the graph representation of the so-called benzenoid hydrocarbons, a class of substances of outstanding importance in chemistry.

Theorem A benzenoid system is a partial binary Hamming graph.

Let G be a benzenoid system with n vertices and let $\alpha : V(G) \to \}0,1\{^k$ be a Hamming labeling of G. For $i = 1, 2, ..., k$ let n_i be the number of vertices u of G such that the i-th component of $\alpha(u)$ is equal to 1. The following and the related results can be found in [KGM]:

Proposition A Wiener index of G is equal to

$$W(G) = \sum_{i=1}^{k} n_i(n - n_i)$$

There have been number of methods developed for the computation of the Wiener index of a given graph G. Among them original method of Wiener, method based on the distance matrix and method based on the adjacency matrix. In addition there are computer methods for obtaining the Wiener index of a (molecular) graph.

An important class of graphs in organic chemistry form the so called grid graphs, which are also planar median graphs. A grid graph is a spanned subgraph of a complete grid.

Premedian and median graphs

All graphs considered in this work are finite undirected graphs without loops or multiple edges.

A subgraph H of a graph G is called *isometric* if $d_H(u,v) = d_G(u,v)$ for all $u, v \in V(H)$.

A graph is *Hamming graph*, if is a Cartesian product of complete graphs. In the case when a graph is a Cartesian product of several copies of K_2, we call the graph a *binary Hamming graph* or *hypercube*. The isometric subgraphs of Hamming graphs are called *partial Hamming graphs* and the isometric subgraphs of hypercubes are called *partial binary Hamming graphs*.

Let $\sum$ be a finite alphabet and let w_1 and w_2 be words of equal length over $\sum$. Then the *Hamming distance* between w_1 and w_2 is the number of positions k in w_1 and w_2 such that the k-th symbol in w_1 differs from the k-th symbol in w_2.

The *Cartesian product* $G \times H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ and $(a,x)(b,y) \in E(G \times H)$ whenever $ab \in E(G)$ and $x = y$ or $a = b$) and $xy \in E(H)$. The Cartesian product is commutative, associative and K_1 is a unit. Also $G \times H$ is connected if and only if both G and H are connected. For $G_1 \times G_2 \times ... \times G_k$ we shall also wright $\prod_{i=1}^{k} G_i$.

For $G = \prod_{i=1}^{k} G_i$ let $p_i : V(G) \to V(H)$ $i \in \{1, 2, ..., k\}$ be the natural projection of G onto the i-th factor G_i, i.e. for $v = (v_1, v_2, ..., v_k) \in V(G)$ we set $p_i(v) = v_i \in V(G_i)$.

The following observation is starting point in studying isometric subgraphs of Cartesian products of graphs.

Lemma1.1*For* $G = \prod_{i=1}^{k} G_i$, $k \geq 1$, *let* $u = (u_1, ..., u_k)$ *and* $v = (v_1, v_2, ..., v_k)$ *be arbitrary vertices of* G. *Then* $d_G(u,v) = \sum_{i=1}^{k} d_{G_i}(u_i, v_i)$.

Proof. Follows from the definitions.

With lemma it is an easy exercise to characterize the partial Hamming graphs:

Lemma1.2 *The vertices of G can be labeled such that the distance between two vertices in G is the Hamming distance if and only if G is an isometric subgraph of a Cartesian product of complete graphs i.e. if and only if G is a partial Hamming graph.*

Hence if G is a k-dimensional hypercube, that is G is a Cartesian product of k copies of K_2 we can mark its points with $0-1$ vectors of length k, such that for two different points u and v of the graph $d_G(u,v) = \langle m(u), 1-m(v)\rangle$ where $m(u)$ and $m(v)$ are the marking vectors of u and v respectively.

For every two points x, y of graph G let I_{xy} denote the union of shortest paths connecting x and y and call it the *connecting interval of x and y.*

A subgraph H of graph G is called *convex* if for all pairs of points $a, b \in H$ the connecting interval I_{ab} is contained in H.

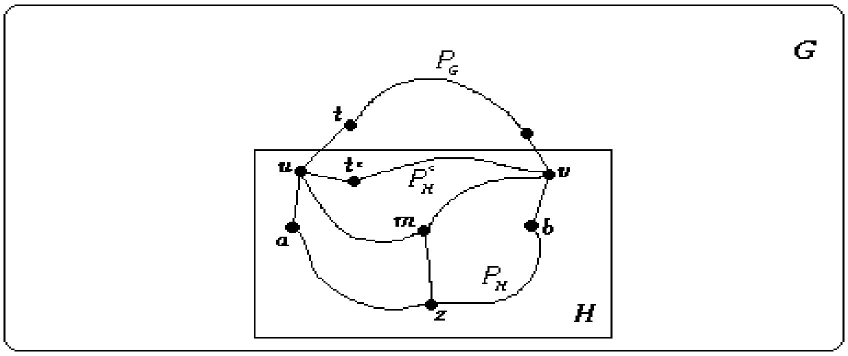
The point m is called the *median* of points a, b and c if
$d(a,m) + d(m,b) = d(a,b)$
$d(a,m) + d(m,c) = d(a,c)$ and $d(b,m) + d(m,c) = d(b,c)$

It is easy to see that a graph is bipartite if every triple of its points has a median. Indeed: Suppose on the contrary, that the graph has a circuit of odd length. Consider a shortest odd circuit, and consider points a, b and c of it such, that $d(a,b) = 1$ and $d(a,c) = d(b,c)$. Then the triple a, b, c doesn't have median. Contradiction.

Let H be a subgraph of a graph G. The boundary σH of H in G is the set of all edges xy of G with $x \in H$ and $y \notin H$.

A subgraph H of graph G is called *2-convex* if for all pairs of points $a, b \in H$ with $d_G(a,b) = 2$ the connecting interval I_{ab} is contained in H.

Lemma 1.3 *Let G be a bipartite graph in which every triple of vertices has a median. Then a subgraph H of G is convex if and only if it is a 2-convex connected.*
Proof. Suppose that H is a 2-convex connected subgraph of G. By induction on d_H we prove that for every pair of points $x, y \in V(H)$ the interval $I_{xy} \subset$



H. The initial step of induction, the case $d_H(x, y) = 2$ follows from the prerequisites of the lemma. Say we proved the statement for $d_H(x, y) < d$. Then consider points a, b such that $d_H(a, b) = d$ and $I_{ab} \not\subset H$. Take a shortest path in G which doesn't belong to H and connects a and b. Denote this path by P_G. Besides take a shortest path in H that connects a and b and mark it by P_H. Observe points u, v such that $u, v \in P_G \cap H$ and choose them to be endpoints of some edges of σU. Now take a point $z \in P_H$ so that $d_H(u, z) < d$ and $d_H(v, z) < d$ holds. Since u, v and z have a median, and because of the induction hypothesis, the median and the corresponding shortest paths lie in H. For u and v (the case $u = a$ and $v = b$ also may occur) we have $d_H(u, v) = d_G(u, v) \leq d$. If $d_H(u, v) = d_G(u, v) < d$ holds, we immediately get a contradiction with the inductive supposition. Thus, for the points u and v denote by P'_G the part of the path P_G, that connects u and v, and note by P'_H a fixed shortest connecting path in H. Take the two neighbors of u in $P'_G \cup P'_H$ and mark them by t and t' with $t' \in V(H)$. Observe the median of the triple t, t' and v. Obviously it can not be t'. Because of 2-convexity of H the median must belong to $G \setminus H$. On the other hand by the inductive supposition $I_{t'v} \subset H$. Contradiction. •

The relation Θ :

Let $e = xy$ and $f = uv$ be edges of a graph G. Then $e\Theta f$ if

$$d_G(x, u) + d_G(y, v) \neq d_G(x, v) + d_G(y, u)$$

The relation Θ is a well defined, reflexive and symmetric relation. We denote its transitive closure by Θ^*.

Lemma 1.4 *Let P be a shortest path in a graph G. Then no two different edges of P are in relation Θ.*

Proof. Let $p_0, p_1, ..., p_n$ be a shortest path, and let $e = p_i p_{i+1}$ and $f = p_j p_{j+1}$ with $i < j$

Then $d(p_i, p_j) + d(p_{i+1}, p_{j+1}) = (d(p_{i+1}, p_j) + 1) + (d(p_i, p_{j+1}) - 1)$,

hence e is not in relation Θ with f. •

Lemma 1.5 *Suppose that P is a path connecting the endpoints of an edge e. Then P contains an edge f with $e\Theta f$.*

Proof. Let $p_0, p_1, ..., p_n$ be the connecting path for $e = p_n p_0$.
Denote $e_i = p_{i-1} p_i$ for $i = 1, 2, ..., n$.

Let $\mu(e, e_i) = d(p_n, p_{i-1}) + d(p_0, p_i) - d(p_n, p_i) - d(p_0, p_{i-1})$.

Consider $s = \sum_{i=1}^{m} \mu(e, e_i)$.

Clearly $s = 2$, which means that at least one of the summands $\mu(e, e_i)$ is $\neq 0$. Thus $e\Theta e_i$. •

Lemma 1.4$'$ *Let P be a path in a graph G. If no two different edges of P are in relation Θ then P is a shortest connecting path for its endpoints.*

Proof. Suppose indirectly that P is a path on which no two different edges of P are in relation Θ but it is not a shortest path. Consider a shortest connecting path P' with the same endpoints. Then $P \cup P'$ contains a circuit C for which $|P \cap C| > |P' \cap C|$ by Lemma 1.5 for every edge $e \in P \cap C$ exists an $f \in E(C)$ with $e\Theta f$. But since $|P \cap C| > |P' \cap C|$ there must be two edges $e', e'' \in P \cap C$ for which $e'\Theta e''$. •

Corollary 1.1 *Neighboring edges of a triangle-free graph can not be in relation Θ.*

In particular, *Neighboring edges of a bipartite graph can't be in relation Θ.*

Lemma 1.6 *Let G be bipartite and $e\Theta f$ for two edges $e = uv$ and $f = xy$ of G. Then the notation can be chosen such that $d(u,x) = d(v,y) = d(u,y) - 1 = d(v,x) - 1$.*

Proof. Clearly $d(u,x) \neq d(u,y)$, because otherwise the bipartite graph G would contain a circuit of odd length (which goes through u, x and y). As x and y are connected by an edge $d(u,x)$ and $d(u,y)$ this can differ by at most one. Let the notation be chosen such that $d(u,y) = d(v,y) + 1$. By the same argument as before we have $d(v,x) \neq d(v,y)$.
Then $d(v,y) \leq d(v,u) + d(u,x) = 1 + d(u,x) = 1 + d(u,y) - 1$
and thus $d(u,y) = d(v,y) + 1$ •

Convexity lemma. *An induced connected subgraph H of a bipartite graph G is convex if and only if no edge of σH is in relation Θ to an edge in H.*

Proof. Indirectly suppose that H is convex and there are edges $uv = e \in H$ and $xy = f \in \sigma H$ with $e\Theta f$, where $x \in H$ and $y \notin H$. By Lemma 1.6 we can suppose, that notation is chosen so that f is on a shortest path from v to x, in contradiction to the convexity of H.

For the converse suppose that no edge of σH is in relation Θ to an edge in H. Let $a, b \in H$ be two arbitrary vertices. Suppose indirectly, that there is a shortest connecting path P in G , which is not contained in H. As a and b are in H the path P contains an edge $e \in \sigma H$. By Lemma 1.4 no edge of P is in relation Θ with e. Denote by Q a path connecting a and b which is contained in H. Then by Lemma 1.5 e is in relation Θ to an edge in $P \cup Q \setminus e$ and thus to an edge in $Q \subset H$, in contradiction to our assumption. •

For an edge uv of a graph G let

$$W_{uv} = \{w | w \in V(G), d_G(w, u) < d_G(w, v)\}.$$

Lemma 1.7 *Let $e = uv$ be an edge of a bipartite graph G and let*

$$F_{uv} = \{f | f \in E(G), e\Theta f\}$$

Then $G \setminus F_{uv}$ has exactly two connected components and they are induced by the vertex sets W_{uv} and W_{vu}. Furthermore, every shortest path P from u to a vertex x in W_{uv} is contained in W_{uv}.

Proof. $G \setminus F_{uv}$ is disconnected because there can be no path in $G \setminus F_{uv}$ from u to v.

Let $x \in W_{uv}$ and let P be a shortest path connecting u and x. Then vP is a shortest path that connects v and x. Hence, from the Lemma 1.5 no edge of P is in relation Θ with e, that is P connects u and x in $G \setminus F_{uv}$. So W_{uv} and W_{vu} are connected components in $G \setminus F_{uv}$.

Since G is bipartite, for an arbitrary edge uv and a point w holds $d(w, u) \neq d(w, v)$. Therefore W_{uv} and W_{vu} form a partition of $V(G)$. •

Corollary 1.2*Let G be a bipartite graph with transitive Θ then for any edge ab of the graph the subgraphs $\langle W_{ab} \rangle$ and $\langle W_{ba} \rangle$ are convex, i.e in a partial binary Hamming graph for any edge ab the subgraphs $\langle W_{uv} \rangle$ and $\langle W_{vu} \rangle$ are convex.*

Proof. Because of transitivity of Θ no edge of $\langle W_{uv} \rangle$ is in relation Θ to an edge of F_{ab}. Since G is bipartite means that $\langle W_{uv} \rangle$ is convex. •

Let G be a Cartesian product graph and $(v_1, v_2, ..., v_k)(u_1, u_2, ..., u_k) \in V(G)$ connected by an edge e in the graph. Let denote by $c(e)$ the coordinate i for which $v_i \neq u_i$.

Lemma 1.8 *Let $G = \prod_{i=1}^{k} G_i$ and let $e, e' \in E(G)$*

(i) *If $c(e) = c(e') = i$ and $p_i(e) = p_i(e')$ then $e\Theta e'$*

(ii) *If $c(e) \neq c(e') = i$ then $e\Theta e'$ does not hold.*

Consider the product $GtimesH$ of two graphs. Then Lemma 1.8 (i) claims that if $ab \in E(G)$, then $(a, x)(b, x)$ is in relation Θ with $(a, y)(b, y)$, for any $x, y \in V(H)$. Lemma 1.8 (ii) on the other hand says that if $ab \in E(G)$ and $xy \in E(H)$, then $(a, z)(b, z)$ is not in relation Θ with $(c, x)(c, y)$ for any $c \in V(G)$ and $z \in V(H)$, i.e. edges of different Θ-classes with respect to Θ in the product graph are not in relation Θ. In other words, every theta class (with respect to the product graph) is the union of one or more Θ-classes with respect to Θ^*.

Theorem 1.1[Winkler] *A graph G is a partial binary Hamming graph if and only if G is bipartite and $\Theta = \Theta^*$.*

Proof. Suppose that G is bipartite and $\Theta = \Theta^*$. Let $e_1 = x_1y_1$, $e_2 = x_2y_2$,..., $e_k = x_ky_k$, $k \geq 1$ be representatives of each equivalence class of Θ^*. Let $\alpha : V(G) \to Q_k = \{0, 1\}^k$ be the following embedding: for $v \in V(G)$ let the i-th coordinate of $\alpha(v)$ be 0 if $v \in W_{x_iy_i}$ and 1 if $v \in W_{y_ix_i}$. We'll show that α is an isometric embedding.

α is injective, because taking two points a and b and a shortest path connecting them, for initial edge of the path a and b are in different W-sets. Let $uv \in E(G)$ and suppose $uv\Theta e_i$. Then $\alpha(u)$ and $\alpha(v)$ differ in the i-th coordinate. Furthermore, if $j \neq i$ then $d(u, x_j) + d(v, y_j) = d(u, y_j) + d(v, x_j)$. Thus $\alpha(u)$ and $\alpha(v)$ have the same j-th coordinate, and α maps edges to edges.

Let P be a shortest path between u and v, then as we have seen in Lemma 1.4, no two different edges of P are in relation Θ and thus $\alpha(u)$ and $\alpha(v)$ differ in just as many coordinates as the number of edges of P. Hence $d_G(u, v) = d_{Q_k}(u, v)$, namely the embedding is isometric.

Now assume that G is an isometric subgraph of a hypercube. Then G is clearly bipartite. As we know from foregoing discussion we can mark the points of G with $0-1$ vectors of length k, such that for two different points u and v of the graph $d_G(u, v) = \langle m(u), 1 - m(v)\rangle$ where $m(u)$ and $m(v)$ is the marking vector of u and v respectively. For two adjacent points u and v let $c(u, v) = i$ if $m(u)$ and $m(v)$ differ in the i-th coordinate.

Define a relation R on $E(G)$ as follows. For $e = uv$ and $e' = ab$ let eRe' if and only if $c(u, v) = c(a, b)$. It's easy to see that if eRe' then $e\Theta e'$ and if $c(u, v) \neq c(a, b)$ then e and e' are not related by Θ. Thus $R = \Theta$ and as R is transitive $\Theta = \Theta^*$ follows. •

For an edge ab of a graph G let

$U_{ab} = \{u \in W_{ab} | u \text{ is adjacent to a vertex in } W_{ab}\}$.

Lemma 1.9 *If G is triangle-free and has transitive Θ, then F_{uv} is a matching, and induces isomorphism between $\langle U_{uv}\rangle$ and$\langle U_{vu}\rangle$.*

Proof. If G is a triangle-free graph with transitive Θ then it is bipartite. Indeed: suppose indirectly, that the graph contains odd-length circuits. Observe one such circuit C of smallest length. Say $|C| = 2d+1$ with $d > 1$. As follows from Lemma 1.5 for every edge e of C exists an edge $f \in E(C)$ such that $e\Theta f$. Since $|C|$ is odd for an edge $e' \in E(C)$ there are two such edges: f and f'. Two of the edges e', f and f' must be in distance smaller than d, that is they must be on a shortest length connecting path. But since Θ is transitive, we have $f\Theta f'$ $e'\Theta f$ and $e\Theta f'$. Contradiction with Lemma 1.4.

Now, from Lemma 1.7 we know, that F_{uv} is a matching. It remains to show, that F_{uv} induces an isomorphism between $\langle U_{uv}\rangle$ and$\langle U_{vu}\rangle$. Let a and b be endpoints of an edge in U_{uv} and let a' and b' be two points in U_{vu} such that aa' and bb' are edges of F_{uv} . Then because Θ is transitive we have $aa'\Theta bb'$, thus path $a'a, ab, bb'$ is not a shortest connecting path in G. Hence a' and b' are adjacent. •

Let us call a graph G *median graph* if in it every triple of vertices a, b and c has an unique median, that is the intersection of the intervals $I_{ab} \cap I_{ac} \cap I_{bc}$ consists of one point.

For example grid graphs or hypercubes are median graphs.

Let us call a bipartite graph G *premedian* if it has transitive Θ and for every edge ab of G the sets U_{ab} and U_{ba} are connected.

Later we shall see that median graphs are also premedian.

The relation δ:

For edges $e, f \in E(G)$ let $e\delta f$ if e and f are opposite edges of a square in G or $e = f$.

Clearly δ is reflexive and symmetric and is contained in Θ, and hence the transitive closure δ^* is contained in Θ^*.

Theorem 1.2 *A bipartite graph is premedian if and only if* $\Theta = \delta^*$

Proof. Assume that G is a premedian graph. From Lemma 1.9 follows that only an edge uv with $u \in U_{ab}$ and $v \in U_{ba}$ can be in relation Θ to ab. Since U_{ab} is connected every edge uv in relation Θ to ab is also in relation δ^* to ab because the distances between endpoints of ab and the respective endpoints of uv must be equivalent. Thus, $\Theta \subseteq \delta^*$. Since $\delta^* \subseteq \Theta^* = \Theta$ we conclude that $\Theta = \delta^*$.

On the other hand, if $\Theta = \delta^*$ i.e. Θ is transitive, hence we infer that $\Theta = \Theta^*$. It remains to show that $\langle U_{ab} \rangle$ and $\langle U_{ba} \rangle$ are connected. Let an edge uv with $u \in U_{ab}$ and $v \in U_{ba}$ be in relation Θ to ab. Since $\Theta = \delta^*$ there are paths $u = u_0u_1...u_k = a$ and $v = v_0v_1...v_k = b$ such that all $u_iv_i, 0 \leq i \leq k$ are in $E(G)$. But then $u = u_0u_1...u_k = a$ is in $\langle U_{ab} \rangle$ and $v = v_0v_1...v_k = b$ is in $\langle U_{ba} \rangle$. As uv was arbitrary chosen $\langle U_{ab} \rangle$ and $\langle U_{ba} \rangle$ are connected. •

Corollary 1.3 *Let G be a premedian graph. For every cycle C of G and any edge e of C there is another edge f such that $e\delta^*f$.*

Proof. Apply Lemma 1.5 and Theorem 1.2. •

The isometric subgraphs of premedian graphs are not necessary premedian. For example for the graph $K_2 \times K_2 \times K_2$ such a subgraph is C_6. But if G is premedian and H is a convex subgraph of G then H is premedian. this assertion is also true for subgraphs of median graphs, i.e. a convex subgraph of a median graph is also median.

Theorem 1.3 *Let G be a premedian graph and let $ab \in E(G)$,then*

(1) G is bipartite and contains no $K_{2,3}$ as an induced subgraph.

(2) G can be isometrically embedded in a hypercube.

(3) F is a matching on U_{ab} and U_{ba} that defines an isomorphism between $\langle U_{ab} \rangle$and $\langle U_{ba} \rangle$.

(4) $\langle W_{ab} \rangle$ and W_{ba} are convex subgraphs of G and are also premedian.

Proof. Statement (1) one can easily check.

(2) follows from the transitivity of Θ, using Theorem 1.1.

(3) follows from Lemma 1.9 and the fact that $\Theta = \delta^*$.

(4) The convexity follows from Corollary 1.2 thus $\langle W_{ab} \rangle$ and $\langle W_{ba} \rangle$ are premedian. •

Theorem 1.3′ *Let G be a median graph and let $ab \in E(G)$,then*

(1) G is bipartite and contains no $K_{2,3}$ as an induced subgraph.

(2) G contains no convex Q_3^-.

(3) Each shortest path from a to $u \in U_{ab}$ is completely in $\langle U_{ab} \rangle$.

(4) $\langle U_{ab} \rangle, \langle U_{ba} \rangle, \langle W_{ab} \rangle$ and $\langle W_{ba} \rangle$ are convex subgraphs of G and $\langle W_{ab} \rangle$ and $\langle W_{ba} \rangle$ are median graphs.

(5) F_{ab} is a matching on U_{ab} and U_{ba} that defines an isomorphism between $\langle U_{ab} \rangle$ and $\langle U_{ba} \rangle$.

(6) G can be isometrically embedded in a hypercube.

Proof.

The statements (1) and (2) one can easily check.

(3) Let P be a shortest path from a to u which is not completely in $\langle U_{ab} \rangle$. By Lemma 1.7 P lies completely in $\langle W_{ab} \rangle$. Let w be the vertex closest to u which is not in U_{ab}. Without loss of generality we can assume u to be adjacent to w. Let v be the neighbor of u in U_{ba}. Clearly $d_G(b, w) = d_G(a, u) = d_G(b, v)$ and $d_G(w, v) = 2$. Hence the median m of b, v and w must be on shortest path from b to v and adjacent to w (and v). Then by Corollary 1.2 m must be in $\langle W_{ba} \rangle$, but w is not adjacent to any vertex in $\langle W_{ba} \rangle$.

(4) First we show that $\langle W_{ab} \rangle$ and $\langle W_{ba} \rangle$ are median graphs. This is true, because any shortest path connecting points u and v in $\langle W_{ab} \rangle$ remains in $\langle W_{ab} \rangle$. Indeed, let P be a shortest path, which connects points u and v and is not contained in $\langle W_{ab} \rangle$. This path would have to contain two edges of F_{ab}, which is not possible by Lemma 1.4. By Corollary 1.2 the subgraphs $\langle W_{ab} \rangle$ and $\langle W_{ba} \rangle$ are convex.

Now we show that $\langle U_{ab} \rangle$ and $\langle U_{ba} \rangle$ are isometric subgraphs of G.

Let $u, w \in U_{ab}$. Consider the median m of a, u, w. It must be on the shortest path P from a to u and Q from a to w. By (3) on the other hand, the parts of P from u to m and Q from m to w form a shortest path from u to w which is in U_{ab}.

Suppose $\langle U_{ab} \rangle$ is not convex . By the Convexity lemma there is an edge $f = xy$ in $\langle W_{ab} \rangle$ with $x \in U_{ab}$ and $y \notin U_{ab}$ which is in relation Θ to an edge in U_{ab}, say $g = uv$. We can assume $d_G(x, u) = d_G(y, v)$. Let v' and x' be the neighbors of v and x in $\langle U_{ba} \rangle$. Clearly, $d(v, x) = d(v', x')$ and $d(x', y) = 2$. As in the proof of (3) we now see that v', x', and y have no median.

(5)Since median graph is bipartite by Lemma 1.7 F_{ab} is a matching. To show that F_{ab} induces an isomorphism, let $uv, xy \in F_{ab}$, where $u, x \in U_{ab}$

$v, y \in U_{ba}$ and u, x are connected by an edge, but not v, y. Let m be the median of b, v and y. It can be in U_{ab} and it is also a median of a, v and y. It is easy to see that the median of m' of a, u and x must be u or x, say u. Moreover, since vy is not an edge and since $vuxy$ is a path of length 3, it must be a shortest path, and hence m' is also a median of a, v, y, contradiction.

(6)From (4) and (5) follows that Θ is transitive, thus by Theorem 1.1 the graph G is partial hypercube. $\bullet$

Theorem 1.4 [Bandelt,1982] *A graph G is a median graph if and only if G is bipartite and for every edge ab of G, the sets U_{ab} and U_{ba} are convex.*

We will prove a little bit stronger result:

Theorem 1.4$'$ *A graph G is a median graph if and only if G is bipartite and for every edge ab of G, the sets U_{ab} and U_{ba} are 2-convex.*

Proof of Theorem 1.4$'$. First we prove that for arbitrary edge ab the generated subgraphs $\langle U_{ab} \rangle$ and $\langle U_{ba} \rangle$ are isometric and Θ defines isomorphism between $\langle U_{ab} \rangle$ and $\langle U_{ba} \rangle$.

By induction on $d_{U_{ab}}(a, x)$ we will show that for every vertex $x \in U_{ab}$ taking a shortest connecting path $a = x_1 x_2 x_3 ... x_k = x$ in U_{ab} and considering edges $x_1 x'_1, x_2 x'_2, ..., x_k x'_k \in F_{ab}$ $x'_1 x'_2, x'_2 x'_3, ..., x'_{k-1} x'_k \in E(G)$ holds. And that $d_{U_{ab}}(a, x) = d_G(a, x)$.

In the following discussion we will use the notation $d(y, z)$ for $d_{U_{ab}}(y, z)$.

Suppose that $x \in U_{ab}$ and $d(a, x) = 1$. Then as ax'_1 and xx'_2 are in relation Θ vertices x'_1 and x'_2 must be connected in the graph.

If $x \in U_{ab}$ is a vertex on distance 2 from a, that is $d(a, x) = 2$. Then by 2-convexity of $\langle U_{ab} \rangle$ the inner vertex x_2 of any shortest connecting $a-x$ path must be in U_{ab}. Considering the edge $x_2 x'_2 \in F_{ab}$ vertices x'_1 and x'_2 must be connected, as we have seen. As $d(x'_1, x'_3) = 2$ by Lemma 1.4$'$ we conclude that two edges of the path $x'_1 x'_2 x_2 x_3 x'_3$ are in relation Θ. Clearly $x'_1 x'_2 \not\Theta x_2 x'_2$, $x_2 x_3 \not\Theta x_2 x'_2$ and $x'_1 x'_2 \not\Theta x_2 x_3$ because they are edges of a shortest path. Then $x_3 x'_3$ must be in relation Θ with $x'_1 x'_2$ or $x'_2 x_2$ or $x_2 x_3$. Obviously $x_3 x'_3 \not\Theta x_2 x_3$. $x_3 x'_3 \not\Theta x'_1 x'_2$ because by Lemma 1.7 $F_{x_3 x'_3}$ is a matching. Thus $x_3 x'_3 \not\Theta x_2 x'_2$ and then $x'_2 x'_3$ must be in $E(G)$.

In the above discussed cases $d(a, x) = d_G(a, x)$ certainly holds.

Suppose now that we have proved the statement for all edges ab and vertices y with $d(a, y) < d$. Let x be a vertex with $d(a, x) = d$. Taking a shortest connecting path $a = x_1 x_2 x_3 ... x_d = x$ in U_{ab} we have by induction $x'_1 x'_2, x'_2 x'_3, ..., x'_{d-1} x'_d \in E(G)$ with $x_1 x'_1, x_2 x'_2, ..., x_d x'_d \in F_{ab}$. If $x_d x'_d \Theta x_i x'_i$ for some $(i = 2, 3, ..., d-1)$ we are ready by induction. So assume that $x_d x'_d \not\Theta x_i x'_i$ for $(i = 2, 3, ..., d-1)$.

case 1 : If $d_G(a, x) = d$ then $x'_1 x_1 x_2 x_3 ... x_{d-1} x_d$ is a shortest path in G thus no two edges of it are in relation Θ. The path $x'_1 x_1 x_2 x_3 ... x_{d-1} x_d x'_d$ is not shortest thus $x_d x'_d$ must be in relation Θ with one of $x'_1 x_1, x_1 x_2, x_2 x_3, ..., x_{d-1} x_d x'_d$ and clearly $x_d x'_d$ can be in relation at most with one of them. Observe that $x_d x'_d \not\Theta x_{d-1} x_d$ and $x_d x'_d \not\Theta x'_{i-1} x'_i (i = 2, 3, ..., d-1)$ because else $F_{x_d x'_d}$ must disconnect the circuit $x_{i-1} x'_{i-1} x'_i x_i$ so we get $x_d x'_d \Theta x_{i-1} x_i (i = 2, 3, ..., d-1)$ which is a contradiction. Observe path $x'_1 x'_2 x'_3, ..., x'_{d-1} x_{d-1}, x_d x'_d$. This path is not shortest and by the above discussion $x_d x'_d$ must be in relation Θ with one of $x'_1 x'_2, x'_2 x'_3, ..., x'_{d-1} x_{d-1}, x_{d-1} x_d$ and we've seen that it can be only edge $x'_{d-1} x_{d-1}$. Thus x'_{d-1} and x'_d must be connected in G. Note that $d_{U_{ab}} = d_G$ immediately follows from the fact that no two edges of $x'_1 x_1 x_2 x_3 ... x_{d-1} x_d$ are in relation Θ.

case 2: If $d_G(a, x) < d$ using the inductive supposition one can easily see that $d_G(a, x) = d-2$. Since $x'_1 x_1 x_2 x_3 ... x_{d-1}$ is still a shortest path the edge $x_{d-1} x_d$ must be in relation with some edge on it, and because of $d_G(a, x) = d - 2$ with exactly one edge. Suppose $x_{d-1} x_d \Theta x_{i-1} x_i$. Then $x_i x_{i+1} x_{i+2} ... x_{d-1} x_d$ is a shortest path and thus $x'_i x_i x_{i+1} x_{i+2} ... x_{d-1} x_d$ is also a shortest path. The path $x'_i x_i x_{i+1} x_{i+2} ... x_{d-1} x_d x'_d$ is not shortest, thus $x_d x'_d$ is in relation Θ with an another edge of it. It can't be $x_i x_{i+1}, x_{i+1} x_{i+2}, ..., x_{d-1} x_d$ (because then path $x'_1 x_1 x_2 x_3 ... x_{d-1} x_d x'_d$ would have two edges and three other edges in relation with one another). Thus $x'_i x_i \Theta x_d x'_d$ and we are ready by induction.

Now we know that F_{ab} defines isomorphism between $\langle U_{ab} \rangle$ and $\langle U_{ba} \rangle$ and that the above subgraphs are isometric. Note that $\langle W_{ab} \rangle$ must be convex then. This also means that Θ_G restricted to $\langle W_{ab} \rangle$ or $\langle W_{ba} \rangle$ is just the same as $\Theta_{\langle W_{ab} \rangle}$ or $\Theta_{\langle W_{ba} \rangle}$ respectively. Thus observing spanned subgraphs $\langle W_{ab} \rangle$ and $\langle W_{ba} \rangle$ the conditions of Theorem must hold in them.

We'll continue the proof by induction on $|V(G)|$. Suppose we've proved the Theorem for case when $|V(G)| < n$ and assume that $|V(G)| = n$.Then by inductive supposition $\langle W_{ab} \rangle$ and $\langle W_{ba} \rangle$ are median graphs. Take three points $a, x, c \in V(G)$. Consider the edge ab with point b from a shortest connecting $a - x$ path. Then $x \in W_{ba}$.
case 1: $c \in W_{ba}$. Then clearly the median m of points b, x, c in $\langle W_{ba} \rangle$ is also a median for a, x, c.
case 2: $c \in W_{ab}$. Taking a point $p \neq\in U_{ba}$ of a shortest connecting $a - x$ path. (it may be x if $x \in U_{ba}$) and the corresponding point $p' \in U_{ab}$, that is $pp' \in F_{ab}$, the median m of points a, c, p' is also a median for a, x, c.

Now remains to show that the median is unique.

As every three points has a median, by Lemma 1.3 we have seen that convexity in this case is equivalent to 2-convexity, thus $\langle U_{ab} \rangle$ and $\langle U_{ba} \rangle$ are convex. It's easy to see that in this case Θ is transitive. Suppose indirectly that points a, x, c has two medians m and m'. Because of convexity of $\langle W_{ab}\rangle$ and $\langle W_{ba}\rangle$ in the *case 1* $m' \in W_{ba}$ in *case 2* $m' \in W_{ab}$ and m is on the same distance from a, x and c as m'. Consider a shortest connecting path between m and m and take an edge e' of it. By Lemma 1.5 there are edges e, f, g on paths $m - a - m'$, $m - x - m'$ and $m - x - m'$ respectively, which are in relation Θ with e'. Thus one of paths $a - m - x$, $a - m' - x$, $c - m - x$, $c - m' - x$, $c - m - a$, $c - m' - a$ contains two edges which are in relation Θ with e'. Say f and g. By the transitivity we get $f\Theta g$ which is a contradiction, because the above mentioned paths are shortest paths and thus by Lemma 1.4 it can't contain two edges which are in relation Θ. •

Theorem 1.5 *A graph is a median graph if and only if it is premedian and contains no convex* Q_3^-.

Proof. As it comes from preliminaries if a graph is median graph, it is premedian and doesn't contain convex Q_3^- as it's a forbidden subgraph for a median graph.

For the converse suppose that G is a premedian graph, that contains no convex Q_3^-. Using Theorem 4', it's enough to show, that for arbitrary edge ab of the graph the sets U_{ab} are 2-convex.

On the contrary, suppose that some set U_{ab} is not 2-convex. From the foregoing discussion we know that F_{ab} is a matching, which joint the graph in two parts: W_{ab} and W_{ba} and induces isomorphism between $\langle U_{ab}\rangle$ and$\langle U_{ba}\rangle$.Let u and v be two vertices of U_{ab} such that $d(u, v) = 2$ and there is a shortest connecting path P which doesn't lay in U_{ab}. Note by x the vertex of P_1, which is in $W_{ab} \setminus U_{a,b}$. Let Q_1 be a shortest $u - v$ path that lies in U_{ab}. If Q_1 is longer than 2, than observe the edge $e = ux$. As comes from Corollary 1 there is an edge $wy = f \in Q_1 \cup P_1 \setminus e$ such that $e\delta^* f$. For this two edges there are paths $w = u_0 u_1 ... u_k = u$ and $y = x_0 x_1 ... x_k = x$ such that all $u_i x_i, 0 \leq i \leq k$ are in $E(G)$. Observe the smallest index i for which $u_i x_i \in E(\delta U_{ab})$. Thus as $ux \in E(U_{ab})$ the edge $u_{i-1} x_{i-1} \in E(U_{ab})$. Thus choosing i as above for points u_i and x_{i-1} there exist a connecting path Q of length 2 in U_{ab} and a connecting path P of length 2 which is not in U_{ab}. Modifying the notations: there exists points $u, x \in U_{ab}$ and points $w \in U_{ab}$ and $v \notin U_{ab}$ such that $uw, xw, uv, xv \in E(G)$. Let u', x' and w' be the vertices of U_{ba} such that uu', xx' and $ww' \in F_{ab}$. We will show that the graph $H = \langle\{v, u, x, w, u', x', w'\}\rangle$ is a convex Q_3^-.

We first consider the pairs of vertices in H in distance 2. Observe that w and v are the only common neighbors for u and x because G is a $K_{2,3}$ free graph. For the same reason u and x have no other common neighbors than w and v. As F_{ab} is matching it is easy to see that the only paths of length

2 between the pairs of vertices $\{v, u'\}, \{v, x'\}, \{u, w'\}, \{w, u'\}, \{x, w'\}, \{w, x'\}$ are those induced by H. Assume that there is a path $v' - w'' - x'$, where $w' \neq w''$. w'' can't be contained in U_{ba} because in this case the corresponding point in U_{ba} and vertices u, w, v, x would be vertices of $K_{2,3}$. If $w'' \in W_{ba} \setminus U_{ba}$ then by the transitivity of Θ $vx\Theta w''x'$ which is impossible as they lay on a shortest length path.

It remains to consider the pairs of vertices in H on distance 3. By the symmetry it is enough to consider the pairs $\{v, w'\}$ and $\{u, x'\}$. As F_{ab} is a matching, it follows that $d_G(v, w') = d_G(u, x') = 3$. Suppose that there is a path $v - v' - v'' - w'$ where $v' \in U_{ab}, v' \neq x$ and $v'' \in U_{ba}, v'' \neq x', u'$. Then the edge $v'v'' \in F_{ab}$ and as Θ is transitive v' is adjacent to w. But then points v, w, u, x, v' are vertices of $K_{2,3}$. Analogously a path $u - v' - v'' - x'$ where $v' \in U_{ab}, v' \neq x, w$ and $v'' \in U_{ba}, v'' \neq w', u'$ would give us another $K_{2,3}$. So we have found a convex Q_3^- which contradicts the supposition.•

Finding triangles and quadrangles in a graph

This section is inspired to prepare the ground for the recognition algorithms for premedian and median graphs and contains that determining and listing aid-algorithms for triangles and quadrangles that we shall use later. For a given graph G let note by n the number of vertices in G and by m the number of edges.

Let denote by ω the smallest real number such that $\forall\, k \in \mathbf{N}$ two $k \times k$-matrices with real entries can be multiplied in $O(k^{\omega})$ steps. This value is called the *exponent of matrix multiplication.*

It is known that $w < 2,376$.

Lemma 2.1[N.Alon, R.Yuster and U.Zwick] *Deciding whether a graph G=(V,E) contains a triangle, and finding one if it does, can be done in*

$$O(m^{2\omega/(\omega+1)}) = O(m^{1.41}) \text{ time.}$$

Proof. Let $\Delta = m^{(\omega-1)/(\omega+1)}$. If a vertex has more than Δ neighbors let us call it a vertex of *high degree* and a vertex of *low degree* otherwise. Consider all paths of length 2 in G whose interior vertex is of low degree. There are $m \cdot \Delta$ such paths, and they can be found in $O(m \cdot \Delta)$ time. For each such path, check whether it's endpoints are connected. If no triangle is found this way, then any triangle in G must be composed of three high-degree vertices. As there are at most $2m/\Delta$ high-degree vertices, we can check whether there exists such a triangle using matrix multiplication in $O((m/\Delta)^{\omega})$ time. The total complexity of the algorithm is therefore

$$O(m \cdot \Delta + (m/\Delta)^{\omega}) = O(m^{2\omega/(\omega+1)})$$

This completes the proof •

Corollary 2.1 *Let G be a graph with m edges. Then the complexity of deciding in how many triangles every edge of G is contained is $O(m^{2\omega/(\omega+1)})$.*

Proof. For the vertices of low degree all triangles containing them are constructed by the above described procedure. We can thus determine within the given complexity in how many such triangles every edge is contained.

Using matrix multiplication we can determine the number of length 2 connecting paths via vertex of high degree for every pair of vertices of high degree. If such a pair of vertices is connected by an edge this number is the number of triangles consisting only of vertices of high degree containing this edge.

Summing up the numbers of this different types of triangles in which every edge is contained we obtain the total number of triangles in which every edge is contained. •

For a graph G the *arboricity* $a(G)$ is the minimum number of edge-disjoint forests into which G can be decomposed.

Nash-Williams [NW] showed that:

Lemma 2.2

$$a(G) = max(\lceil |E(H)|/|(V(H)| - 1)\rceil : H \subset G)$$

Lemma 2.3 *Let a graph G have n vertices and m edges. Then*

(1) $a(G) \leq \lceil (2m+n)^{\frac{1}{2}}/2 \rceil$

(2) $a(G) \leq \lceil n/2 \rceil$

(3) $a(G) \leq 3 if\ G is\ planar$

Proof. Suppose that $H \subseteq G$ is the subgraph for which

$$a(G) = |E(H)|/|(V(H)| - 1).$$

Denote $|E(H)| = q$ and $|V(H)| = p$. Let k be the number of edges of a complete graph with p vertices, that is $k = p(p-1)/2$.

If $k \leq m$, we have

$$a(G) = \lceil q/(p-1) \rceil \leq \lceil k/(p-1) \rceil = \lceil p/2 \rceil = \lceil (2k+p)^{\frac{1}{2}}/2 \rceil \leq \lceil (2m+n)^{\frac{1}{2}}/2 \rceil.$$

If $k \geq m$

$$a(G) = \lceil q/(p-1) \rceil \leq \lceil m/(p-1) \rceil \leq \lceil \{mk/(p-1)^2\}^{\frac{1}{2}} \rceil = \lceil \{(m(p-1)+m)/2(p-1)\}^{\frac{1}{2}} \rceil \leq$$

$$\leq \lceil \{m/2 + k/2(p-1)\}^{\frac{1}{2}} \rceil \leq \lceil (2m+n)^{\frac{1}{2}}/2 \rceil.$$

(2) comes from the Lemma 7. If G is planar $|E(G)| \leq 3|V(G)| - 3$ and then by Lemma 7 we have

$$a(G) \leq max(\lceil (3p-3)/(p-1) \rceil : H \subseteq G) = 3. \bullet$$

Lemma 2.4 *If a graph G=(V,E) has n vertices and m edges, then*

$$\sum (min\{d(u), d(v)\} : (u,v) \in E) \leq 2a(G)m$$

Proof. Let F_i $(1 \leq i \leq a(G))$ be the edge-disjoint spanning forests of G such that $E(G) = \cup(E(F_i) : 1 \leq i \leq a(G))$. Associate each edge of F_i with a vertex of G as follows: choose an arbitrary vertex u of each tree T in forest F_i as the root of T and regard T as a rooted tree with root u in which all the edges are directed from the root to the descendants and associate each edge e of tree T with the head vertex h(e) of e. Thus, every vertex of F_i except the roots is associated with exactly one edge of F_i.

Then we have

$$\sum (min\{d(u), d(v)\} : (u,v) \in E) \leq$$

$$\sum (\sum d(h(e)) : e \in E(F_i)) : 1 \leq i \leq a(G)) \leq$$

$$\sum (\sum (d(v) : v \in V) : 1 \leq i \leq a(G)) = 2a(G)m \bullet$$

Now let turn to finding triangles in a graph. Taking advantage of the fact that each triangle containing vertex v corresponds to an edge joining two neighbors of v we get a very simple algorithm, which lists triangles in a graph, and has complexity $O(a(G)m)$.

Algorithm List $\triangle$

begin

```
renumber the vertices v_1, v_2, ..., v_n of G in such a way that
d(v_1) ≥ d(v_2) ≥ ... ≥ d(v_n)
   for i := 1 to n − 2  do
     begin
        {find all the triangles containing vertex v_i, each of these
        triangles corresponds to an edge joining two neighbors of v_i}
(1):    mark all the vertices adjacent to v_i
            for each marked vertex u  do
              begin
(2):            for each marked vertex w adjacent to u  do
                  if w is marked
                    then print out triangle v_i, u, w
(3):          erase the mark from u
              end
              {delete v_i from G so that no duplication occurs.}
(4):          delete v_i from G and let G be the resulting graph
     end
end
```

Theorem 2.1 *The algorithm List $\triangle$ runs in $O(a(G)m)$ time.*

Proof. Clearly the degrees of vertices can be computed in $O(m)$ time. Since the degree of any vertex is at most $n-1$, one can arrange the vertices in $O(n)$ time.

We well use doubly linked adjacency lists as data structure to represent the graph G. The two copies of each edge uv, one in the list of u another in the list of v are also doubly linked. Using such a data structure, we can delete a vertex v from G in $O(d(v))$ time, and scan all vertices adjacent to v in $O(d(v))$ time.

Now consider the time required by the i-th iteration of the outmost **for** statement. Statements (1),(3) and (4) require $O(d(v_i))$ time. Statement (2) requires at most $O(\sum d(u) : u \in N(v_i)))$ time, where $d(u)$ denotes the degree of vertex u in the original graph and $N(v_i)$ denotes the set of neighbors of v_i in the current graph. Therefore the total running time T of the algorithm is bounded as follows:

$$T \leq O(m) + O(n) + \sum(O(d(v_i)) : v_i \in V) + \sum(d(u) : u \in N(v_i)).$$

Since v_i has the largest $d(v_i)$ among the vertices in the current graph, we have $d(u) \leq d(v_i)$ for each $u \in N(v_i)$. Since v_i is deleted at (4), each edge of G is involved exactly once in the double summations above. Thus we have

$$T \leq O(m) + O(n) + O(\sum(min\{d(u), d(v)\} : uv \in E(G))).$$

Using Lemma 2.4 we have $\leq O(a(G)m)$. $\bullet$

Corollary 2.2 *If G is a planar graph, the complexity of algorithm List$\triangle$ is $O(m)$.*

Proof. Since the algorithm List$\triangle$ runs in $O(a(G)m)$ time and by Lemma 2.3 we have $a(G) \leq 3$, for the complexity of the algorithm holds $O(a(G)m) \leq O(m)$. $\bullet$

The benefit of the above algorithm may be intuitively explained as follows: since we delete the vertices one by one in the largest degree order, the graph tends to become sparse soon. This also prevents the edges incident to a vertex of large degree of being scanned very often.

Using the above searching strategy we give an efficient algorithm for finding all the quadrangles in a graph.

If vertices $u_1, u_2, ..., u_l (l \geq 2)$ are all adjacent to two common vertices v and w, that is these $l+2$ vertices induce a complete bipartite graph $K_{2,l}$ then any quadruple $(v, u_i, w, u_j), 1 \leq i < j \leq l$, forms a quadrangle. Thus even in a planar graph, there may exist $O(n^2)$ quadrangles. Instead of listing these quadrangles individually, we list the triples $(v, w, \{u_1, u_2, ..., u_k\})$ representing them altogether.

In our algorithm for each vertex v of the graph, to find all the quadrangles containing v : for each vertex w within distance two from v, the algorithm finds all such $u_1, u_2, ..., u_k$ which are adjacent to both v and w, and stores them in a set $U[w]$. When the quadrangles containing v have been found, v is deleted to avoid the duplications.

Algorithm List $\diamondsuit$

begin
  count the vertices $v_1, v_2, ..., v_n$ of G in a way that
  $d(v_1) \geq d(v_2) \geq ... \geq d(v_n)$
    for each vertex $v \in V$ **do** $U(v) := \emptyset$
    for $i := 1$ **to** n **do**
     begin
       for each vertex u adjacent to v_i **do**
       begin
          for each vertex $w \neq v_i$ adjacent to u **do**
           begin
              $U[w] := U[w] \cup \{u\}$
           end
         for each vertex w with $|U[w]| \geq 2$ **do**

print out the triple $(v_i, w, U[w])$

for each vertex w with $U[w] \neq \emptyset$ **do** $U[w] := \emptyset$

delete the vertex v_i from G and let G be the new graph

end

end

end

Theorem 2.2 *Algorithm List $\diamondsuit$ represents all the quadrangles in a connected graph G in $O(a(G)m)$ time.*

Proof. The tenet of algorithm List $\diamondsuit$ is practically the same as we have seen in algorithm List $\triangle$. It's easy to see that it runs within above limited time complexity. $\bullet$

Expansion procedures

In this part we will give a useful characterization for partial cubes, premedian and median graphs. In particular we will prove the results of W.Imrich and S.Klavžar, which are the basic theorems for many recognition algorithms: they showed that premedian graphs are precisely those graphs that can be obtained from a one-vertex graph by connected expansion procedures and the median graphs are the graphs that can be obtained by convex expansion procedures from one-vertex graph.

Let G be a graph and let V_1 and V_2 be vertex subsets of G such that $V_1 \cap V_2 \neq \emptyset$ and $V_1 \cup V_2 = V(G)$. Assume in addition that $\langle V_1 \rangle$ and $\langle V_2 \rangle$ are isometric subgraphs of G and that there is no edge between a vertex of $V_1 \setminus V_2$ and $V_2 \setminus V_1$.

An *expansion* of a graph G (with respect to V_1 and V_2) is a graph H obtained from G in the following way:

(1) Replace each vertex $v \in V_1 \cap V_2$ by vertices v_1 and v_2 and join them by an edge.

(2) Join v_1 and v_2 to all neighbors of v in $V_1 \setminus V_2$ and $V_2 \setminus V_1$ respectively.

(3) If $v, u \in V_1 \cap V_2$ are adjacent in G, then join v_1 to u_1 and v_2 to u_2.

A *contraction* is just the inverse of expansion, that is G is a *contraction* of the graph H.

An *expansion procedure* is a sequence of expansions.

Observe that in the expansion procedure no condition is on the intersection $S = V_1 \cap V_2$ (except that it is not empty). We call an expansion *connected expansion* if S is connected, a *convex expansion* if S is convex.

Lemma 3.1 *Every contraction of a partial binary Hamming graph is a collapsion of one Θ-class.*

Proof. Consider a contraction of a partial binary Hamming graph G by a matching F and denote the parts of $G \setminus F$ by V_1 and V_2.

Consider two edges ab and $cd \in F$ with $a, c \in V(V_1)$ and $b, d \in V(V_2)$. Because V_1 and V_2 must be isometric after contraction, $d(a, c) = d(b, d)$, since ab and cd were arbitrary chosen, this equality holds for every pair of edges in F.

Let denote $d = d_(a, c) = d_(b, d)$. Then $d_G(b, c)$ can not be smaller than $d + 1$, because in that case we have $d_G(b, c) \leq d - 1$ and the shortest path connecting b with c must contain a matching-edge. Let denote it xy with $x \in V_2$ and $y \in V_1$. Then $d(b, x) = d(a, y)$, thus $d(a, c) = d(a, y) + d(y, c) < d + 1$, contradiction. Thus $ab\Theta cd$, i.e. all the edges in the matching are in relation Θ.

Clearly no edge of that Θ-class can be missing from F, because then F could not cut the graph.

A Θ-class in a partial binary Hamming graph clearly fulfills the prerequisites for contraction. •

Theorem 3.1[Chepoi,1988] *A graph G is a partial binary Hamming graph if and only if G can be obtained from a one-vertex graph by a sequence of expansions.*

Proof. First we will show, that after a contraction step in a partial binary Hamming graph we get a partial binary Hamming graph.

Consider an isometric embedding of our partial binary Hamming graph G into a hypercube H. The contraction step on G induces a contraction step on H (by contracting the corresponding Θ-class). After the contraction step we get an isometric embedding of the contracted graph G into a smaller dimension hypercube, thus the graph obtained after contraction is also partial binary Hamming graph.

Suppose that G is a partial binary Hamming graph. We will show that a one-vertex graph can be obtained from G by a sequence of contractions (this is equivalent to the right-hand side of the theorem).

Consider a partial binary Hamming graph. Use contraction steps on it while it has edges! Clearly at the end we shall get a one-vertex graph.

Suppose that G can be obtained from a one-vertex graph by a sequence of expansions. Taking in each of expansion steps $V_1 = V_2 = V(G')$ in the current graph, we will obtain a hypercube and an isometric embedding of G into it. •

Theorem 3.2 [Imrich and Klavžar,1997] *A graph G is a premedian graph if and only if G can be obtained from a one-vertex graph by a sequence of connected expansions.*

Proof. By the result of Winkler (Theorem 1.1) premedian graphs are partial binary Hamming graphs and by Theorem 3.1 the graph G can be obtained from a one-vertex graph by the expansion procedure. By Lemma 3.1 every contraction of partial binary Hamming graph is equivalent to collapsing one Θ class. As Θ is transitive, by Lemma 1.9 we know that F_{ab} is a matching and induces isomorphism between $\langle U_{ab}\rangle$ and$\langle U_{ba}\rangle$. Hence an expansion is connected if an only if $\langle U_{ab}\rangle$ is connected. •

Theorem 3.3 [Mulder,1978] *A graph G is a median graph if and only if G can be obtained from a one-vertex graph by a sequence of convex expansions.*

Proof. By Theorem 1.4 a bipartite graph G is median if and only if for every edge ab the sets $\langle U_{ab}\rangle$ and $\langle U_{ba}\rangle$ are convex. By Theorem 3.1 the graph G is a partial binary Hamming graph if and only if it can be obtained from a one-vertex graph by a sequence of convex expansions. Since by Lemma 3.1 every contraction of a partial binary Hamming graph is a collapsion of one Θ-class, we conclude that an expansion is convex if and only if the sets $\langle U_{ab}\rangle$ and $\langle U_{ba}\rangle$ are convex, which complete the proof. •

For $u \in V(G)$ let $N(u)$ be the set of all vertices adjacent to u.

Theorem 3.4*Let G be a connected bipartite graph, and let $ab \in E(G)$. Suppose that following properties hold:*

(1) F_{ab} is a matching that defines an isomorphism between $\langle U_{ab}\rangle$ and $\langle U_{ba}\rangle$

(2) $\langle U_{ab}\rangle$ and $\langle U_{ba}\rangle$ are connected.

(3) For any $u \in W_{ab} \setminus U_{ab}$ and $v \in W_{ba} \setminus U_{ba}$,
$|N(u) \cup U_{ab}| \leq 1$ and $|N(v) \cup U_{ba}| \leq 1$

Then G is median if and only if $\langle W_{ab}\rangle$ and $\langle W_{ba}\rangle$ are median graphs.

Before giving the proof let us show two lemmas.

Lemma 3.2 *Let G be a median graph and H be a convex subgraph of G. Then for any $v \in G$ there exists a unique $u \in H$ such that $d(v, H) = d(v, u)$.*

Proof. Clearly the statement holds for $v \in H$. Assume that $v \in G \setminus H$ and that $d(v, H) = d(v, u) = d(v, w)$ for two different vertices $v, w \in H$. Let m be the median of triple u, v, w. (The median is unique!) As H is convex, $m \in H$ and one of the following inequalities holds: $d(v, H) \leq d(v, m) < d(v, u)$ or $d(v, H) \leq d(v, m) < d(v, w)$. Contradiction. •

Lemma 3.3 *Let G be a median graph and H be a convex subgraph of G. For any $v \in G$ and $w \in H, d(v, w) = d(v, u) + d(u, w)$, where $u \in H$ is the unique vertex satisfying $d(v, H) = d(v, u)$.*

Proof. If $u = v$ or $u = w$ the lemma holds. For $u \neq v$ and $u \neq w$, note that $v \neq w$ and let m be the median of triple u, v, w. Hence $d(v, u) = d(v, m) + d(m, u)$. By convexity, $m \in H$ and therefore either $m = u$ or $d(v, m) < d(v, u)$, a contradiction. Thus we have $m = u$ and $d(v, w) = d(v, u) + d(u, w)$. •

Proof of Theorem 3.3. Suppose that G is a median graph, by the Theorem 1.3′ W_{ab} and W_{ba} are convex subgraphs of G and are median.

For the converse note first that Lemma 1.3 and (2),(3) implies that $\langle U_{ab} \rangle$ and $\langle U_{ba} \rangle$ are convex in W_{ab} and W_{ba} respectively. We will show that for arbitrary triple v_1, v_2, v in $V(G)$ there is an unique median in G. If v_1, v_2 and v belong to W_{ab} or W_{ba} we only must to show that the median is unique, that is no vertex of W_{ba} can be a median for a triple from W_{ab} and conversely. Suppose, that $v_1, v_2, v \in W_{ab}$ has a median $m' \in W_{ba}$. Let P_1 be a shortest connecting path for v_1 and m'. Then P_1 must contain an edge of F_{ab}, note it by u_1u_1'. Let P_2 be a shortest connecting path for v_2 and m' and P for v and m'. Note the corresponding matching edges by u_2u_2' and uu' respectively. Assume that $u_1, u_2, u \in W_{ab}$ and $u_1', u_2', u \in W_{ba}$. Then m' is also a median for u_1', u_2', u and since $\langle U_{ba} \rangle$ is convex $m' \in U_{ba}$. Because of (1) the vertex m for which $mm' \in F_{ab}$ is a median for u_1, u_2, u But then we have $d(v_1, m) + d(v_2, m) + d(v, m) < d(v_1, m') + d(v_2, m') + d(v, m')$, thus m' can't be a median, contradiction.

Now we can assume without loss of generality that $v \in W_{ab}$ and $v_1, v_2 \in W_{ba}$. Let u be the vertex in U_{ab} such that $d(v, U_{ab}) = d(v, u)$. Let uu' be the matching edge, where $u \in U_{ab}$. Let m be the unique median of triple v_1, v_2, u'. By the fact that any shortest path from v_1 or v_2 to v must contain a matching edge and by using Lemma 3.2 we get:

$$d(v_1, v) = d(v_1, m) + d(m, m') + d(m'u') + d(u', u) + d(u, v)$$

$$d(v_2, v) = d(v_2, m) + d(m, m') + d(m'u') + d(u', u) + d(u, v)$$

where $m' \in U_{ba}$ satisfies $d(m, U_{ba}) = d(m, m')$. Thus m is a median for v_1, v_2, v. Suppose that for v_1, v_2, v exists an another median m^*. Then m^* must belong to a shortest path between v_1 and v_2. Note that m^* would be a median for v_1, v_2, u' also. Thus as W_{ba} is a median graph, $m = m^*$. •

Theorem 3.5 *Let G be a connected bipartite graph, and let $ab \in E(G)$. Suppose that the following properties hold:*

(1) F_{ab} is a matching that defines an isomorphism between $\langle U_{ab}\rangle$ and $\langle U_{ba}\rangle$

(2) $\langle U_{ab}\rangle$ and $\langle U_{ba}\rangle$ are connected.

(3) For any two matching edges aa' and $bb' \in F_{ab}$ with $a, a' \in U_{ab}$ and $b, b' \in U_{ba}$ the distances $d_{W_{ab}}(a, b) = d_{W_{ba}}(a', b')$

Then G is premedian if and only if $\langle W_{ab}\rangle$ and $\langle W_{ba}\rangle$ are premedian graphs.

Proof. If G is a premedian graph, it clearly fulfills the above listed properties.

Suppose that (1),(2),(3) holds and $\langle W_{ab}\rangle$ and $\langle W_{ba}\rangle$ are premedian graphs. Then F_{ab} clearly satisfies the conditions for a connected expansion step and in the contracted graph every Θ-class will consist of union of a theta class in $\langle W_{ab}\rangle$ and a Θ-class in $\langle W_{ba}\rangle$. Hence the contracted graph is premedian and thus the initial graph G. •

Recognizing premedian graphs

In this section we will give an algorithm which is due to W.Imrich and S.Klavžar. They showed, that premedian graphs can be recognized in $O(m\log n)$ time. This algorithm is improvement of the former which has complexity $O(mn)$.

By Theorem 1.2 we know, that a bipartite graph is premedian if and only if $\Theta = \delta^*$.

For a given graph G we can decide whether it is bipartite in $O(m)$ time (for example by determining a BFS-tree and checking that no edge lays in a breath-level).

After checking the bipartiteness the algorithm begins to determine δ^* (rejecting by-the-way some incongruous cases), then checks whether $\Theta = \delta^*$ holds using the characterization of premedian graphs (Theorem 3.5).

For determining δ^* we shall use the algorithm List$\diamondsuit$, which lists all squares of the graph, and thus determines δ and hence δ^*.

If the graph G is premedian, $a(G) \leq \log n$. For to prove it we recall the result of Nash-Williams (Lemma 2.2), but first we need the following observation:

Lemma 5.1 *Let H be a nontrivial subgraph of hypercube. Then*

$$|E(H)| \leq \frac{|\mathrm{V(H)}|}{2} \log_2 |(V(H)|$$

Proof. Denote hypercubes of dimension D by K^D.

Suppose that H is a subgraph of a d-dimensional hypercube.

Then we can score up $K^d = K^{d-1} \times K_2$. So K_d is a graph obtained from two copies of K^{d-1} (note them by C_1 and C_2) , vertices of which are in bijection and the corresponding vertices are connected.

We shall continue the proof by induction on the dimension of hypercube. Suppose we have proved the statement of Lemma 5.1 in the case when the dimension of hypercube is smaller than d. Then we have:

$$|E(H)| \leq |E(H \cap C_1)| + |E(H \cap C_2| + |E(C_1)|.$$

Denoting $|E(C_1)| = \frac{n}{2}$

$$|E(H \cap C_1)| \leq \frac{1}{2}(\frac{n}{2} \log\frac{n}{2}) \quad \text{and} \quad |E(H \cap C_2)| \leq \frac{1}{2}(\frac{n}{2} \log\frac{n}{2})$$

So we have:

$$|E(H)| \leq 2 \cdot \frac{1}{2}(\frac{n}{2} \log_2\frac{n}{2}) + \frac{n}{2} = \frac{n}{2} \log_2 n. \bullet$$

Now from $a(G) = max(\lceil |E(H)|/|(V(H)| - 1) \rceil : H \subset G)$ and $|E(H)| \leq \frac{|\mathrm{V(H)}|}{2} \log_2 |(V(H)|$ the fact that $a(G) \leq \log n$ follows.

As the algorithm List$\diamondsuit$ runs in $O(ma(G))$ time, its complexity is $O(m \log n)$ if G is premedian.

With the following observation we can reject all that graphs for which $a(G) > \log n$, under running of the algorithm List$\diamondsuit$.

The number of squares in a premedian graph is not bigger than $m \log n$. Indeed, the number of squares in which a single edge uv is contained is bounded by $min\{d_G(u), d_G(v)\}$, because a premedian graph doesn't contain $K_{2,3}$. By Lemma 2.4 $\sum(min\{d(u), d(v)\} : (u, v) \in E) \leq 2a(G)m$, hence the number of quadrangles in the graph is bounded by $m \log n$.

Thus, if the algorithm finds too many quadrangles G cannot be a premedian graph and is discarded.

Keeping in mind that we wish $\Theta = \delta^*$, let us recall Theorem 4.2, that declares that a graph is premedian if and only if it can be obtained from one-point graph by a sequence of connected expansions.

Now for any given edge $f = uv$ we define sets $F_{uv}^{\delta^*}, W_{uv}^{\delta^*}$ and $U_{uv}^{\delta^*}$ as follows. We first set

$$F_{uv}^{\delta^*} = \{f | f \in E(G), e\delta^* f\}$$

Then we consider the subgraph induced by the set of endpoints of the edges in $F_{uv}^{\delta^*}$. Removal of $F_{uv}^{\delta^*}$ from it should disconnect it into two isomorphic components. The component containing u will be denoted by $U_{uv}^{\delta^*}$ and the other one by $U_{vu}^{\delta^*}$. Furthermore, the edges in $F_{uv}^{\delta^*}$ should induce isomorphism between these components. All these properties can be checked by standard MAKE-SET, UNION and FIND-SET operations in $O(m\log n)$ time. All graphs not satisfying these conditions will be discarded.

Finally, we have to define the graphs $W_{uv}^{\delta^*}$ and $W_{vu}^{\delta^*}$. They are the components of $G \setminus F_{uv}^{\delta^*}$ containing u and v respectively. If they are not distinct, the graph will be rejected.

Suppose we can show that for an edge uv that $W_{uv}^{\delta^*}$ and $W_{vu}^{\delta^*}$ are distinct premedian graphs with $\delta^*_{W_{uv}} = \delta^*|W_{uv}$ and $\delta^*_{W_{vu}} = \delta^*|W_{vu}$. Then we contract every edge of $F_{uv}^{\delta^*}$ to a single vertex and set $V_1 = V(W_{uv}^{\delta^*})$ and $V_2 = V(W_{vu}^{\delta^*})$. Because the corresponding edges of $U_{uv}^{\delta^*}$ and $U_{vu}^{\delta^*}$ is identical by our construction the distances between the neighbors a', b' of a, b in $U_{vu}^{\delta^*}$ measured in $W_{vu}^{\delta^*}$. Thus V_1 and V_2 satisfy the conditions for connected expansion and we can conclude that G is a premedian graph. We call such an operation a basic connected expansion step.

This is exactly the procedure we will follow. Let $E_1, E_2, ..., E_k$ be the equivalence classes of $E(G)$ with respect to δ^*. We note on the way that one can discard all graphs for which $k \geq n$ because Θ^* has at most $n-1$ equivalence classes.

Moreover, we observe that

$$\sum_{uv \in E(G)} |U_{uv}^{\delta^*}| = 2\sum_{j=1}^{k} |E_j| = 2m$$

.

We shall now define sets $F_i = \cup_{j=1}^{i} E_j$ for $i = 1, 2, ..., k$ and let G_i be the graph on $V(G)$ with $E(G_i) = F_i$. Clearly $G = G_k$. Setting $G_0 = (V(G), \emptyset)$ we then have

$$G_i = G_{i-1} \cup E_i \text{ for } i = 1, 2, ..., k.$$

We now observe that all components of G_0 are premedian graphs. Suppose all components of G_{i-1} are premedian graphs. We now add the edges of E_i, that is, the edges of some $F_{uv}^{\delta^*}$ to G_{i-1}.

If there exists a single edge e in $U_{uv}\delta^*$ both endpoints of which are in one and the same component of G_{i-1}, then this edge e must be in relation Θ to an edge of another δ^*-class by Lemma 1.3. In this case we discard G because it cannot be a premedian graph.

Let H be a component of G_i and suppose removal of the edges from E_i disconnects H into more than two components. These components must be components of G_{i-1}. It is not hard to see that in this case there must be two adjacent vertices a, b in a component, say C_1 of G_{i-1}, such that a is connected via an edge aa' in E_i to a component C_2 of G_{i-1} and b via an edge bb' to a component C_3 of G_{i-1}, where $C_2 \neq C_3$. But then aa' and bb' are not adjacent, which is not possible by previously checked isomorphism of graphs $\langle U_{uv}^{\delta^*}\rangle$ with the corresponding $\langle U_{vu}^{\delta^*}\rangle$. Thus, there can be just two components, they are isomorphic and the isomorphism preserves the Θ-classes.

We can therefore perform basic connected expansion steps unless edges are detected both endpoints of which are in the same component of G_{i-1}, but in this case the graph is discarded.

Thus, all graphs accepted by our algorithm are premedian and only premedian graphs are accepted.

Again we observe that the operations we have to perform are:

(i) MAKE-SET to form the initial sets consisting of n disjoint vertices

(ii) FIND-SET to find out in which set a given vertex is contained

(iii) UNION to join disjoint sets.

For every one of the m edges in the graph we have to perform two FIND-SET operations and possibly one UNION. It is well known that this operations can be executed within time complexity $O(m \log n$.

Naturally, when we determine the Θ-classes of the graph, i.e. δ^* we also obtain an isometric embedding of the graph into a hypercube. thus we get:
Corollary 3.1 *A premedian graph can be embedded into a hypercube in $O(m log n)$ time.* •

Recognizing median graphs

In this section we will give two – so far the fastest – algorithms for deciding, whether a graph G is median. Both of them are due to W.Imrich and S.Klavžar.

Algorithm IK1:

To check whether a graph G is a median graph, we first use the algorithm of the last section to decide whether it is premedian. Then, by Theorem 1.4 it is sufficient to check whether for every edge ab set $\langle U_{ab}\rangle$ is 2-convex.

For a given set U_{ab} one can check the 2-convexity on the following way:

Scan all vertices $x \in U_{ab}$ and mark all neighbors of x not in U_{ab}. Subgraph $\langle U_{ab} \rangle$ is 2-convex if and only if no vertex is marked twice. Since G has n vertices the total time for one such procedure is at most $O(n)$.

But when the set U_{ab} is not so large, namely $|U_{ab}| < \sqrt{n}$ we can decide faster whether it is 2-convex by using the Convexity lemma (and checking the convexity instead of weaker 2-convexity):

Scan the vertices $x \in U_{ab}$ one after the other and check for every Θ-class occuring in whether an edge e in the boundary σU_{ab} is in that Θ-class. With a proper data structure each of these tests can be performed in constant time. In $\langle U_{ab} \rangle$ cannot occur more than $|U_{ab}|$ pieces of Θ-classes, because a BFS-forest must contain a representative from each Θ-class. Hence total time used for this procedure is bounded by $O(|U_{ab}|)^2$.

Since $\sum_{U_{uv}} |U_{uv}| = m$ the number of sets U_{ab} with $|U_{ab}| \geq \sqrt{n}$ is smaller than $\frac{m}{\sqrt{n}}$. Thus the time used for processing these sets is at most $O(n) \cdot \frac{m}{\sqrt{n}} = O(m\sqrt{n})$.

Let S denote the set of all U_{ab}-s with $|U_{ab}| < \sqrt{n}$. Then the total time needed to check these graphs for convexity with the Convexity lemma is

$$\sum(|U_{uv}|^2 : U_{uv} \in S) \leq \sqrt{n} \sum(|U_{uv}| : U_{uv} \in S) \leq n^{0.5} m$$

Thus, the total time used by algorithm is $O(m\sqrt{n})$. •

The algorithm IK1 is a nice short algorithm which is interesting, because the former one of the same complexity (J.Hagauer, W.Imrich, S.Klavžar [HIK]) is rather complicated. Let refer on it as algorithm IKH. It is based on the Theorem which is a little bit weaker than Theorem 3.4. It claims that if for a connected bipartite graph G the following 3 properties hold:

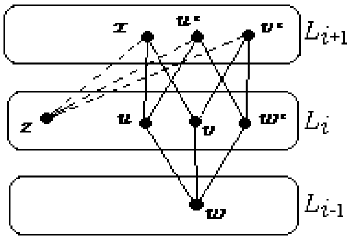
(1) F_{ab} is a matching that defines an isomorphism between $\langle U_{ab} \rangle$ and $\langle U_{ba} \rangle$

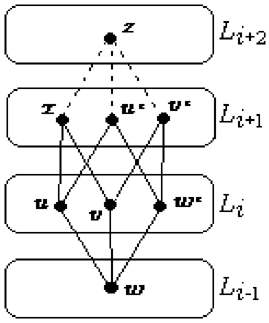
(2) $\langle U_{ab} \rangle$ and $\langle U_{ba} \rangle$ are isometric.

(3) For any $u \in W_{ab} \setminus U_{ab}$ and $v \in W_{ba} \setminus U_{ba}$,
$|N(u) \cup U_{ab}| \leq 1$ and $|N(v) \cup U_{ba}| \leq 1$

Then G is median if and only if $\langle W_{ab} \rangle$ and $\langle W_{ba} \rangle$ are median graphs.

The main idea of the algorithm IKH is to decompose a graph into two parts, check if the conditions of Theorem are fulfilled and recursively repeat the procedure until we end up with a single vertex graph. P.K Jha and G.





Slutzki [JS] used a similar idea, but instead of decomposing a graph they contract it along F_{ab}.

Algorithm IKH at first checks the conditions (1) and (2) for every Θ-class, then condition (3). We will now describe an approach for testing condition (3) which uses algorithm for determining all triangles in a graph.

Suppose that we have tested conditions (1) and (2) by algorithm IKH. Consider a breadth first search tree T with root a. Suppose that a vertex x from $W_{ab} \setminus U_{ab}$ has two neighbors in U_{ab} say u and v. Because the condition (1) was also tested before there are vertices u', v' and w' in U_{ba} which are adjacent to u, v and w respectively, such that these 6 vertices together with x induce a Q_3^-.

Let $L_0, L_1, ...$ be the distance levels of the tree T and assume that $x \in L_{i+1}$. Then we know that u and v both belong to L_i as condition (2) of Theorem has already been tested at this stage. Also $w \in L_{i-1}$. We thus have the situation depicted in figure.

Then there must not exist a vertex $z \in L_{i+2}$ adjacent to x, u' and v' because then x must be in U_{ab}. (Clearly the case $z \in L_i$ can not appear because then the points u, v, a would have two medians.) So we only have to check if the graph contains a so-positioned Q_3^- .

We can do this test using an algorithm for listing all triangles of a given graph. Let H_i be the graph on the vertex set L_i where two vertices of H_i are adjacent, if they have a common neighbor in L_{i+1}. In the Lemma 5.1 below we will show that H_i has at most $|L_{i+1}|\log^2 n$ edges. Thus all the graphs H_i together have at most $n\log^2 n$ edges. We now use the algorithm for listing all triangles of the graph.

For each triangle of H_i we only have to check whether the corresponding three vertices in L_{i+1} have a common neighbor z in L_{i+2}. But as we have already determine the Θ-classes and thus an embedding of G into a hypercube by the previous steps of the algorithm IKH.

Suppose that we have an algorithm of complexity $T(m,n)$ which finds all triangles in a given graph with n vertices and m edges. By Lemma 4.1 we know that for nontrivial subgraphs of hypercubes, and thus also for median graphs $m = O(n\log n)$. As already mentioned , all steps of algorithm IKH except testing of (3) require $O(m\log n)$ time. The above approach for testing (3) takes $T(n\log^2 n, n)$ time. thus we have proved the following theorem.

Theorem 5.1 *Let T(m,n) be the complexity of finding all triangles of a given graph with m edges and n vertices. Then the complexity of checking whether a graph G on n vertices and m edges is a median graph is at most* $O(mlogn) + T(mlogn, n)$.

For a given breadth first search tree T with root a let direct the edges of T from a and call them *up-edges*, the other edges let direct to a and let call them *down-edges*.

Lemma 5.1 *Let G be a median graph and let $v \in V(G)$ have k down-edges with respect to the vertex a. Then $|G| \geq 2^k$.*

Proof. By Theorem 1.3′ G can be isometrically embedded in a s-dimensional hypercube for some $s \geq 1$. Let l be a corresponding labeling, i.e., for any $u, v \in V(G)$, $d_G(u,v) = H(l(u), l(v))$, where $H(x,y)$ denotes the Hamming distance of the 0-1 strings x and y. We may assume that $l(a) = 00...0$.

As v has k down-edges with respect to a, $l(v)$ differs from $l(a)$ in at least k coordinates ($k \leq s$). Therefore $l(v)$ contains at least k non-zero components and we may without loss of generality assume that the first k coordinates of $l(v)$ equal 1.

Let vu_i, $1 \leq i \leq k$, be down-edges of v, where the i/th coordinate of $l(u_i)$ is 0. To complete the proof we claim that for each binary string x that agrees with $l(v)$ on the components $k+1, k+2, ..., s$ there is a vertex $u \in V(G)$ with $l(u) = x$. Let t be the number of zeros in the first k coordinates of x. The claim will be proved by induction on t.

For $t = 0$ and $t = 1$ the claim clearly holds. Let x be a string with zeros in the components $i_1, i_2, ..., i_t$, $2 \leq t \leq k$, $i_t \leq k$. By the induction hypothesis, there exists a vertex $w \in V(G)$ such that $l(w)$ has zeros in the components $i_1, i_2, ..., i_{t-1}$. Consider the vertices a, w and u_{i_t}. Their median has the label in which a coordinate is obtained as a majority over the corresponding components in $l(a), l(v)$ and $l(u_{i_t})$. Thus x the median of $l(a), l(v)$ and $l(u_{i_t})$. •

Corollary 5.1 *Planar median graphs can be recognized in linear time.*

Proof. Let G be a graph on n vertices with m edges. We wish to show that the complexity of checking whether G is a planar median graph is $O(m+n)$. As it is well-known that connectedness, bipartiteness and planarity can be checked in linear time we can assume that G is a connected, planar bipartite graph given by its adjacency list and that we wish to check whether it is a median graph. We further observe that a distance-tree can be found in linear time and that down-degree can be checked and found in linear time too.

We now consider an embedding of G in the plane and the subgraph X_i spanned by L_{i+1} and L_i. In L_{i+1} there may be vertices if degree one two or three in X_i. Let w be a vertex in L_{i+1} of degree three and a ,b, c be its neighbors in L_i. We split w into three vertices x, y, z and replace the edges aw, bw, cw by ax, ay, by, bz, cz, cx. We do this for every vertex of degree 3. Clearly the new graph $X_i^{\prime}$ obtained in this way is still planar. Moreover, every vertex of X_i not in L_i has degree one or two. We now delete the vertices of degree one and replace every path $x_1x_2x_3$, where $x_1, x_2 \in L_i$ and $x_2 \in L_{i+1}$ by a single edge x_1x_3. This way we obtain the graph H_i from the construction in the proof of Theorem 5.1.

Proceeding as in the proof of Theorem 5.1 we have to find the triangles in the H_i and perform certain checks, the complexity of these operations being determined by the complexity finding all triangles. Now, the number of triangles in a planar graphs can be found in linear time and the proof is completed by the observation that the total number of edges in the H_i is at most $3n$. •

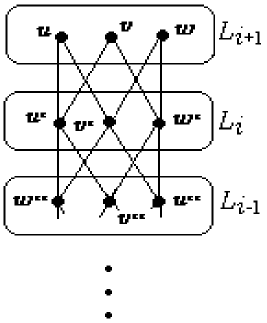
Theorem 5.2 [W.Imrich and S.Klavžar]*Deciding if a given graph G is median can be done in $O(mlogn)^{2\omega/(\omega+1)}$ time, where by ω we denote the exponent of matrix multiplication.*

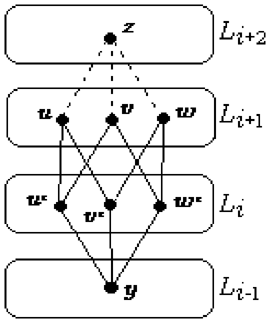
Proof.

Algorithm IK2:

Consider a given graph G. We first check if G is premedian with the algorithm of the former part. This algorithm determines also δ^* (and checks that $\delta^* = \Theta$).

Let us consider again a breadth first search tree T with distance levels $L_0, L_1, \ldots$. Observe, that if G is a median graph where b and c are two vertices in L_i which are adjacent to a vertex x in L_{i+1} then it must be a vertex y in L_{i-1} which is adjacent to both of b and c (it is definitely the median of triple a, b, c). Let call this property of median graph by down-closure.





It is easy to see that if G satisfies the down-closure property then every subgraph Q_3^- in it must contain a hexagon vertices of which are in two neighboring levels of T.

Suppose we can find a hexagon $uv'wu'vw'u$, where $u, v, w \in L_{i+1}$ and $u', v', w' \in L_i$.

We first note that it is not possible that there exists a vertex $y \in L_{i+1}$which is adjacent to u', v' and w'. For, suppose such a vertex y exists.
Let $y \neq v$ and let $v'' \in L_{i-1}$ be a vertex adjacent to both u' and v'; we know by down-closure that such a vertex exists. But then u', v', together with y, v, v'' span a $K_{2,3}$ which is not possible because G is premedian.

Furthermore, let u'' be the unique vertex in L_{i-1} adjacent to v', w' and w'' the unique vertex in L_{i-1} adjacent to u' and v'. Suppose $u'' = v''$. Then u', v' together with w, w'', u'' span a $K_{2,3}$. Thus either $u'' = v'' = w''$ or u'', v'' and w'' are three distinct vertices. Continuing this procedure we construct vertices u^{i-k+1}, v^{i-k+1},w^{i-k+1} in level L_{i-k} such that u^{i-k}, v^{i-k+1},w^{i-k}, $v^{i-k}, w^{i-k+1}, u^{i-k}$ is hexagon and such that there is no vertex z which is adjacent to u^{i-k}, v^{i-k}, w^{i-k}.

However, as G is finite, there must be a level, say L_{i-k+1} such that $u^{i-k+2} = v^{i-k+2} = w^{i-k+2}$. But then we have found an induced Q_3^- in G which is not contained in an induced Q_3 and G is not a median graph unless $k = 3, u'' = v'' = w''$ and there is a vertex $z \in L_{i+2}$ adjacent to u, v and w.

Since $u'' = v'' = w''$ if such a vertex $z \in L_{i+2}$ adjacent to u, v and w exists, we conclude that G is a median graph if and only if to every hexagon $uv'wu'vw'u$, where $u, v, w \in L_{i+1}$ and $u', v', w' \in L_i$ there exist a vertex $z \in L_{i+2}$ adjacent to u, v and w.

Let H_i be the graph on the vertex set L_i where two vertices of H_i are adjacent, if they have a common neighbor in L_{i+1}. Then every hexagon

$uv'wu'vw'u$ where $u, v, w \in L_{i+1}$ and $u', v', w' \in L_i$ corresponds to a triangle in H_i and G is a median graph if and only if there is a vertex z in L_{i+2} adjacent to u, v and w for every such triangle.

From Lemma 5.1 the down-degree of every vertex in G is at most $\log n$. Therefore H_i has at most $|L_{i+1}|\log^2 n$ edges, from which we conclude that the union of all graphs H_i has at most $n\log^2 n$ edges.

On the other hand, given an edge in H_i we can very easily find out in how many triangles it is contained that correspond to hexagon $uv'wu'vw'u$, where $u, v, w \in L_{i+1}$ and $u', v', w' \in L_i$ such that there is a vertex$z$ in L_{i+1} adjacent to u, v and w.

The complexity of finding the numbers of these special triangles is $O(m(\log)^2 n)$. We thus only have to find out in how many triangles every edge of H_i is contained. If the numbers of special triangles and all triangles coincide for every edge, then G is a median graph.

Now the proof is completed by invoking Corollary 2.1 the number of triangles in which every edge of the H_i is contained can be find within the specified time complexity. •

Although, the algorithm IK2 is assimptotically better than IK1, it improves the complexity when $n > 10^{40}$, so it has only theoretical magnitude.

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