Regular homotopy classes of singular maps

Thesis

Author: András Juhász

Supervisor: András Szűcs

Department of Analysis
Eötvös Loránd University
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Chapter 1

Introduction

This thesis is the collection of three separate papers centered around one main problem. In 1985 Pinkall [16] formed a semigroup from the immersions of all closed surfaces into $\mathbb{R}^3$ and computed this semigroup completely. My starting point was the following question posed by András Szűcs: How could one generalize Pinkall's work to (locally generic) singular maps of surfaces into $\mathbb{R}^3$?

To our surprise it turned out that when the maps do have singularities then the analogous semigroup structure becomes trivial. Moreover, this is also true for locally generic maps of n-manifolds into $\mathbb{R}^{2n-1}$ for all n different from 3. In order to show this and to understand what happens in dimension 3 I had to determine the set of path components of the space of such locally generic maps (i.e. maps of maximal rank except a finite set of Whitney umbrella singular points). This is done for $n = 2$ in the first paper, for all n different from 3 in the second one. In the third paper I determined the set of path components of the space of immersions of an oriented 3-manifold into $\mathbb{R}^3$.

Roughly speaking, an immersion is a smooth map that is locally an embedding, i.e., it might have self-intersection points. A regular homotopy of immersions is a smooth one-parameter family of such maps. The first step towards regular homotopy classification of immersions was the Whitney-Graustein theorem [26]. It states that the regular homotopy class of a regular plane curve is determined by its rotation. The breakthrough in this field was Hirsch-Smale theory [8] that reduced the problem of regular homotopy classification to pure homotopy theory, i.e., to an algebraic topological problem. This does not mean that the problem is settled because this homotopy theoretic problem is usually very hard to handle.

When we allow singularities then we face two problems. First, Hirsch-Smale theory does not apply to this situation, thus we cannot reduce the problem to homotopy theory. We have to examine the *geometry* of the singularities. Thus our methods need to be more geometric in nature and we also need a geometric classification of immersions. Here comes the second problem: such geometric classifications do not exist in many cases. Another approach would be to generalize Hirsch-Smale theory to singular maps. This has been done by

Gromov in many situations [7], but a suitable h-principle has not been published for locally generic maps of closed n-manifolds into $\mathbb{R}^{2n-1}$. Even if such an h-principle existed we would have to classify sections of the 2-jet bundle avoiding $\Sigma^{1,1}$ up to homotopy.

In Chapter 1 we carry out the classification of locally generic maps of n-manifolds into $\mathbb{R}^{2n-1}$ for $n = 2$ and in Chapter 2 for $n > 3$. The answer in these cases is surprisingly simple: the regular homotopy class of a map $f\colon M^n \to \mathbb{R}^{2n-1}$ is completely determined by the number of its singular points if f is not an immersion. The proof in the case $n > 3$ relies on a geometric classification of immersions of S^n into $\mathbb{R}^{2n-1}$ due to Ekholm [2]. However, this geometric classification becomes very different for immersions of S^3 and was done by Ekholm and Szűcs [4]. From this I conjectured that the regular homotopy class of a singular map of M^3 into $\mathbb{R}^5$ is *not* determined by the number of its singular points, another invariant comes into the picture. But first I needed a geometric definition of that invariant in the case of immersions. Such an invariant existed only for immersions with trivial normal bundle [19]. In Chapter 3 I extend that regular homotopy invariant to arbitrary immersions. Unfortunately the singular case is not yet settled for $n = 3$.

I would like to take this opportunity to thank Professor András Szűcs for our long and helpful discussions and for correcting this thesis.

Chapter 2

Regular homotopy classes of locally generic mappings

2.1 Introduction

Our work was motivated by the paper of U. Pinkall [16], which classifies immersions of compact surfaces into $\mathbb{R}^3$ up to regular homotopy, allowing diffeomorphisms of the source manifold M^2. So two immersions $f, g\colon M^2 \longrightarrow R^3$ are considered equivalent if there is a diffeomorphism φ of M^2 such that $f = g \circ \varphi$. This notion is different from regular homotopy, yielding an interesting classification of immersed surfaces using the Arf invariant. That paper also gives generators for the abelian semigroup of immersed surfaces with the connected sum operation. Professor András Szűcs asked me what happens with Pinkall's classification if we allow cross-cap (also called Whitney-umbrella) singularities. The notion of regular homotopy has to be revised and, unlike for immersions, for singular maps it turns out that all natural definitions are equivalent. In fact, we prove that for singular mappings (i.e. not immersions) of a closed connected surface the number of cross-caps totally determines the regular homotopy class, diffeomorphisms of the source manifold are not needed. Thus the approach of U. Pinkall and the classical regular homotopy classification give the same result for singular maps.

In the final section of our paper we present an application of the above result to the study of the path components of the space of those immersions of a closed connected surface into $\mathbb{R}^4$ whose projections into $\mathbb{R}^3$ are locally generic, i.e. may have cross-cap singularities.

It is a natural question if Theorem 2.2.2 generalizes to higher dimensions, for locally generic maps of a closed n-manifold M^n into $\mathbb{R}^{2n-1}$. Our methods of proof for Theorem 2.2.2 do not seem to work if $n > 2$ since they rely heavily on the results of surface topology. However, I could prove the general result in the case when $n > 3$ and M^n is 2-connected. I will publish this in a separate paper. I want to emphasize that the results of Section 3 easily generalize for any closed manifold M^n provided that the generalization of Theorem 2.2.2 holds true for M^n.

I would like to take this opportunity to express my gratitude to Professor András Szűcs for drawing my attention to this problem and for his constant support and encouragement. I would also like to thank Professor Balázs Csikós who read the first version of this paper and suggested several improvements.

2.2 The main result

Let us start by introducing the notion of a locally generic mapping of a closed n-manifold M^n into a $(2n-1)$-manifold N^{2n-1} for $n \geq 2$.

Definition 2.2.1. $f\colon M^n \longrightarrow N^{2n-1}$ is called *locally generic* if it is an immersion except for cross-cap singularities. The set of singular points of f in M^n is denoted by $S(f)$. ($|S(f)| < \infty$ because M^n is compact.) Let us denote by $\mathrm{Lgen}(M^n, N^{2n-1})$ the subspace of locally generic mappings in $C^\infty(M^n, N^{2n-1})$ endowed with the C^∞ topology.

Recall that a map $f\colon M^n \longrightarrow N^{2n-1}$ is called generic (or stable), if it is an immersion with normal crossings except in a finite set of points, moreover the singular points of f are non-multiple cross-cap points. This explains our terminology. Whitney [28] proved that the set of stable maps is dense open in $C^\infty(M^n, \mathbb{R}^{2n-1})$ with respect to the C^∞-topology. Studying the double point set of a generic mapping close to the locally generic mapping f in the C^∞-topology, one can easily verify that for M^n closed $|S(f)|$ is an even integer. (For the cross-caps are precisely endpoints of double-point curves.)

Definition 2.2.2. Two locally generic mappings $f, g\colon M^n \longrightarrow N^{2n-1}$ are called *regularly homotopic* (denoted by $f \sim g$) if there is a smooth mapping $H\colon M^n \times [0,1] \longrightarrow N^{2n-1}$ such that H_t is locally generic for each $t \in [0,1]$, moreover $H_0 = f$ and $H_1 = g$. Here $H_t(x) = H(x,t)$ for $x \in M^n$ and $t \in [0,1]$.

The following definition will be especially useful in the case $n = 2$.

Definition 2.2.3. Two locally generic mappings $f, g\colon M^n \longrightarrow N^{2n-1}$ are called *image-homotopic* if there is a diffeomorphism φ of M^n such that $f \circ \varphi$ is regularly homotopic to g. We denote this by $\mathrm{Im}(f) \sim \mathrm{Im}(g)$.

Proposition 2.2.1. *If $f \sim g$ or just $Im(f) \sim Im(g)$, then $|S(f)| = |S(g)|$. In fact, if $H\colon M^n \times [0,1] \longrightarrow N^{2n-1}$ is a regular homotopy, then $|S(H_t)| = |S(H_0)|$ for every $t \in [0,1]$.*

Proof. If φ is a diffeomorphism of M^n, then $S(f) = \varphi(S(f \circ \varphi))$, so $|S(f)| = |S(f \circ \varphi)|$. Thus we can suppose that $f \sim g$. From the definition of stability it is clear that every locally generic mapping h has a neighborhood U_h in the Whitney C^∞ topology such that for every $h' \in U_h$ we have $|S(h')| = |S(h)|$ (because every $p \in M^n$ has a neighborhood U such that $h|U$ is equivalent to $h'|U$ and since M^n is closed, see [5, p. 72]). Thus the function $|S(.)| : \mathrm{Lgen}(M^n, N^{2n-1}) \longrightarrow \mathbb{Z}$

is locally constant. So if H is a regular homotopy connecting f and g, this implies that H_t is a continuous path in $\mathrm{Lgen}(M^n, N^{2n-1})$, along which $|S(H_t)|$ is constant. $\square$

Remark 2.2.1. We have defined the notions of regular homotopy and image-homotopy between two locally generic maps f and g. Proposition 2.2.1 implies that if f and g are immersions then they are regularly homotopic, resp. image-homotopic as immersions iff they are those as locally generic maps.

If $H\colon M^2 \times [0,1] \longrightarrow \mathbb{R}^3$ is a regular homotopy between two locally generic mappings $f = H_0$ and $g = H_1$ and $k = |S(f)| = |S(g)|$, then there exists curves $\gamma_1, \dots \gamma_k\colon [0,1] \longrightarrow M^2$ such that $S(H_t) = \{\, \gamma_j(t)\colon 1 \le j \le k \,\}$ for every $t \in [0,1]$. We define the bijection $i_H\colon S(f) \longrightarrow S(g)$ the following way : for $1 \le j \le k$ let $i_H(\gamma_j(0)) = \gamma_j(1)$.

Now we can state the main result of this paper yielding a converse of Proposition 2.2.1 for singular mappings.

Theorem 2.2.2. *Let M^2 be a closed connected surface and suppose that*

$$f, g\colon M^2 \longrightarrow \mathbb{R}^3$$

are locally generic mappings with $|S(f)| = |S(g)| > 0$. Then $f \sim g$. Moreover for any bijection $i\colon S(f) \longrightarrow S(g)$ there exists a regular homotopy H connecting f and g such that $i_H = i$.

Let us now list a few interesting corollaries of this. A surprising consequence of Theorem 2.2.2 is the following: If U is the standard locally generic mapping of S^2 into $\mathbb{R}^3$ with two cross-cap points then for any two immersions $f, g\colon M^2 \longrightarrow \mathbb{R}^3$ the connected sums $f\#U$ and $g\#U$ become regularly homotopic as locally generic maps !

Another consequence of our theorem is that, given a closed connected surface M^2, we can easily produce a full list of representatives of all regular homotopy classes of locally generic maps $M^2 \longrightarrow \mathbb{R}^3$. (We only do this for singular maps, for immersions see [16].) First suppose that M^2 is orientable. If we denote by i_M the standard embedding of M^2 into $\mathbb{R}^3$ then

$$i_M \# \underbrace{U\# \dots \#U}_{n}$$

is a representative for the class of locally generic maps with $2n > 0$ singular points. Now suppose that M^2 is a non-orientable surface of genus g. We denote the Boy surface by B. Now

$$\underbrace{B\# \dots \#B}_{g} \# \underbrace{U\# \dots \#U}_{n}$$

is a locally generic mapping of M^2 into $\mathbb{R}^3$ with $2n$ cross-cap points.

Perhaps the following construction can be visualized more easily: Let us denote by V the well-known locally generic mapping of $\mathbb{R}P^2$ into $\mathbb{R}^3$ having

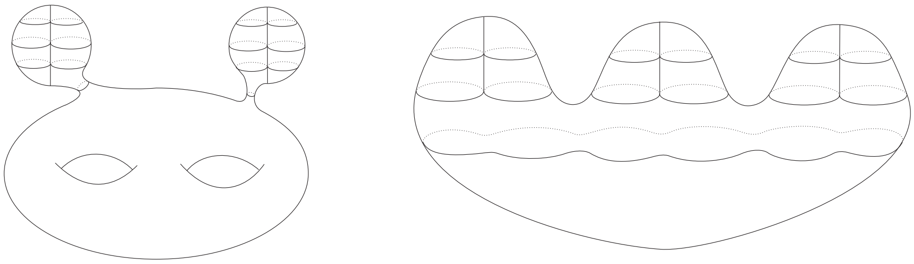

Figure 2.1:

two singular points ($V \sim B\#U$). Then any singular locally generic mapping is regularly homotopic to one of the form $i_N\#U\#\ldots\#U\#V\#\ldots\#V$, where N^2 is orientable. (The left side of Figure 2.1 depicts $A_2\#U\#U$, where A_2 is the orientable surface of genus 2. The right side of Figure 2.1 illustrates $B\#B\#B\#U\#U\#U \sim V\#V\#V$.)

Pinkall determined the abelian semigroup H of immersed surfaces in $\mathbb{R}^3$ with the connected sum operation (see [16]). If we consider the extended semigroup $\widetilde{H}$ of locally generic surfaces, then only U is needed as a new generator with the following new relations (using Pinkall's notation): $S\#U = T\#U$ and $B\#U = \overline{B}\#U$.

Note that $\widetilde{H} \setminus H$ is a sub-semigroup of $\widetilde{H}$ and it easily follows from Theorem 2.2.2 that it is isomorphic to $J \oplus \mathbb{Z}_+$, where J denotes the semigroup of (closed connected) surfaces. As a corollary we may conclude that the Grothendieck group of $\widetilde{H}$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.

2.3 Proof of the main result

The purpose of this section is to prove Theorem 2.2.2. First we recall the classification of immersions of an arbitrary two-dimensional manifold F^2 into $\mathbb{R}^3$ using Hirsch-Smale theory.

Theorem 2.3.1. *There is a 1-1 correspondence between the regular homotopy classes of immersions of a surface F^2 into $\mathbb{R}^3$ denoted by $Imm(F^2, \mathbb{R}^3)$ and $H^1(F^2; \mathbb{Z}_2)$.*

Proof. By Hirsch (see [8]) there is a weak homotopy equivalence between the space $\mathrm{Imm}(F^2, \mathbb{R}^3)$ and $\Gamma(\mu)$. Here $\Gamma(\mu)$ denotes the space of sections of the vector bundle $\mu = \mathrm{MONO}(TF^2, R^3)$ over F^2 whose fiber over $p \in F^2$ consists of all linear injections from T_pF to $\mathbb{R}^3$. Thus there is a bijection between the regular homotopy classes $\pi_0(\mathrm{Imm}(F^2, \mathbb{R}^3))$ and $\pi_0(\Gamma(\mu))$. Fix an arbitrary Riemannian metric on F^2 and let μ' be the bundle over F^2 whose general fiber over p is the space of orthogonal injections of T_pF into $\mathbb{R}^3$. Then the inclusion of μ' into μ is a fiber homotopy equivalence (see [16, p. 426]), thus $\pi_0(\Gamma(\mu)) = \pi_0(\Gamma(\mu'))$. Fixing a section $s \in \Gamma(\mu')$ every section $t \in \Gamma(\mu')$ can be obtained by the action of a unique element of $C(F^2, SO(3))$ on s. Thus $\Gamma(\mu')$ is homeomorphic to $C(F^2, SO(3))$, yielding $\pi_0(\Gamma(\mu)) = [F^2, SO(3)]$. Since $SO(3)$ is homeomorphic to RP^3, it follows from obstruction theory that

$$[F^2, SO(3)] = [F^2, RP^3] = [F^2, RP^\infty] = [F^2, K(\mathbb{Z}_2, 1)] = H^1(F^2; \mathbb{Z}_2),$$

where $K(\mathbb{Z}_2, 1) = RP^\infty$ is an Eilenberg-MacLane space. □

From now on M^2 denotes the closed connected surface mentioned in the statement of Theorem 2.2.2. If f and g are locally generic mappings of M^2 into $\mathbb{R}^3$ with $|S(f)| = |S(g)|$, then according to the lemma of homogeneity there exists a diffeotopy $\{\varphi_t \colon t \in [0,1]\}$ of M^2 such that $\varphi_0 = id_{M^2}$ and $\varphi_1(S(g)) = S(f)$. Since $f \circ \varphi_t$ provides a regular homotopy between f and $f \circ \varphi_1$, it is sufficient to prove Theorem 2.2.2 in the case $S(f) = S(g)$. Let $S(f) = S(g) = \{ p_1, \ldots, p_k \}$, where $k = |S(f)| > 0$ is an even integer. For each p_i choose a sufficiently small open neighborhood D_i diffeomorphic to an open 2-disc such that $f|D_i$ has the canonical form (x_1^2, x_2, x_1x_2) in an appropriate pair of local coordinate-systems centered at p_i and $f(p_i)$. Similarly $g|D_i$ should have the same canonical form in another pair of local coordinate-systems. Assume moreover that the discs $D_1, \ldots, D_k$ are pairwise disjoint. Denote by A the disjoint union $\coprod_{i=1}^k D_i$, then $F^2 = M^2 \setminus A$ is a two-manifold with boundary.

Lemma 2.3.2. *Suppose that f and g are locally generic mappings of the surface M^2 into $\mathbb{R}^3$ such that $S(f) = S(g)$. Choose open discs $D_1, \ldots, D_k \subset M^2$ centered at the points of $S(f)$ as above. Define $F^2 = M^2 \setminus \coprod_{i=1}^k D_i$. Then there exists a diffeomorphism d of the pair (M^2, F^2) such that the immersions $(f|F^2) \circ d$ and $g|F^2$ are regularly homotopic and d permutes $S(f)$. Moreover there is a diffeotopy d_t of M^2 with $d_0 = id_{M^2}$ and $d_1 = d$.*

Proof. According to Theorem 2.3.1 the regular homotopy classes of $f|F^2$ and $g|F^2$ correspond to cohomology classes $\alpha, \beta \in H^1(F^2; \mathbb{Z}_2)$. (These will be shown to be non-zero later.) We construct a diffeomorphism d of the pair (M^2, F^2) such that for the induced automorphism d^* of $H^1(F^2; \mathbb{Z}_2)$ it holds that $d^*(\alpha) = \beta$. Using Theorem 2.3.1 again this gives the required result $(f|F^2) \circ d \sim g|F^2$.

We first note that

$$H_1(F^2; \mathbb{Z}_2) = H_1(M^2; \mathbb{Z}_2) \oplus \overbrace{\mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2}^{k-1}, \tag{2.3.1}$$

as can be seen from the exact sequence of the pair (M^2, F^2). (Recall that $k = |S(f)|$.) For each i, $1 \le i \le k$ choose an embedded curve γ_i in F^2 around D_i. Denote its homology class by $[\gamma_i] = c_i$. The classes $c_1, \ldots, c_{k-1}$ can be chosen for the generators of the $(k-1)$ $\mathbb{Z}_2$ summands in 2.3.1. (Note that $c_1 + \cdots + c_k = 0$ since $\partial F^2 = \gamma_1 \cup \cdots \cup \gamma_k$.) According to Pinkall [16] we have that $\langle \alpha, c_i \rangle = 1$ and $\langle \beta, c_i \rangle = 1$ for every $i \in \{ 1, \ldots, k \}$, since γ_i has a neighborhood homeomorphic to $S^1 \times [0,1]$ which is mapped by f and also by g into a "figure eight$\times[0,1]$". Now we have two cases according to the orientability of M^2.

If M^2 is orientable of genus g we denote the standard generators of the homology group $H_1(M^2;\mathbb{Z}_2)$ by $a_1, b_1, \ldots, a_g, b_g$, and choose embedded curves $\varphi_1, \psi_1 \ldots \varphi_g, \psi_g$ in F^2 representing them. Define

$$H_a = \{\, 1 \le i \le g \colon \langle \alpha, a_i \rangle \ne \langle \beta, a_i \rangle \,\},$$

and

$$H_b = \{\, 1 \le i \le g \colon \langle \alpha, b_i \rangle \ne \langle \beta, b_i \rangle \,\}.$$

There is a simple (i.e. embedded) closed curve δ in F^2 that for each i, $1 \le i \le g$ intersects transversally in one point the curve φ_i if $i \in H_a$ and is disjoint from φ_i if $i \notin H_a$, moreover δ intersects transversally in one point ψ_i if $i \in H_b$ and is disjoint from ψ_i if $i \notin H_b$. Note that the homology class of δ will be

$$\left(\sum_{i \in H_a} b_i \right) + \left(\sum_{i \in H_b} a_i \right). \tag{2.3.2}$$

Such a δ exists because in $H_1(M^2;\mathbb{Z}_2)$ any class can be represented by a simple curve, and a simple curve representing the class 2.3.2 can be arranged to be transversal to all the curves φ_j and ψ_j for $j = 1, \ldots, k$ and to intersect each of them at most in one point. Now choose two points on δ very close to each other so that none of the curves φ_i and ψ_i for $1 \le i \le g$ intersects the shorter arc δ^* between them. Thus the following equalities hold :

$$|\delta \cap \varphi_i| = \begin{cases} 1 & \text{if } i \in H_a \\ 0 & \text{if } i \notin H_a \end{cases} \tag{2.3.3}$$

$$|\delta \cap \psi_i| = \begin{cases} 1 & \text{for } i \in H_b \\ 0 & \text{for } i \notin H_b \end{cases}$$

$$\delta^* \cap \varphi_i = \delta^* \cap \psi_i = \emptyset \quad \text{for } 1 \le i \le g.$$

Modify the arc δ^* (by a homology) so that it goes through the center of D_1 and D_2 avoiding the curves φ_i and ψ_i for $i = 1, \ldots, g$ as well as the discs D_i for $i = 3, \ldots, k$. This can be done since $M^2 \setminus \bigcup_{i=1}^{2g} (\varphi_i \cup \psi_i)$ is path-connected. From now on we will denote this new simple curve on M^2 by δ (see Figure 2.2). Note that the equalities 2.3.3 still hold. Now choose a tubular neighborhood T of δ such that $D_1, D_2 \subset T$ and for $i \in H_a$ the curve φ_i and for $j \in H_b$ the curve ψ_j intersects T in a line segment. (See the left side of Figure 2.3.) We also select a slightly wider tubular neighborhood $T' \supset T$. Define d on T to be a rotation of $T = S^1 \times [-1, 1]$ by 180° interchanging D_1 and D_2 and also interchanging p_1 and p_2. The diffeomorphism d acts identically on $M^2 \setminus T'$. On $T' \setminus T$, which is homeomorphic to $S^1 \times ([-2, 2] \setminus [-1, 1])$, define d as the rotation of $S^1 \times \{s\}$ by $(2 - |s|) \times 180^\circ$ for $s \in [-2, 2] \setminus [-1, 1]$ (see the right side of Figure 2.3). The diffeomorphism d is diffeotopic to id_{M^2} : construct $d_t \colon M^2 \longrightarrow M^2$ similarly to d, just take a rotation by $180^\circ \cdot t$ instead of rotating by 180° such that $d_t(p_1) \in \delta^*$ for every $t \in [0, 1]$.

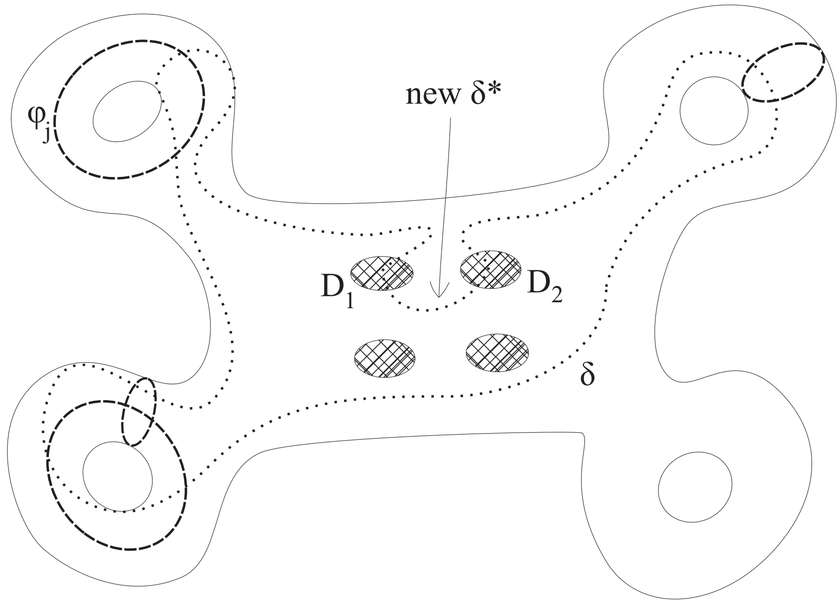

Figure 2.2:

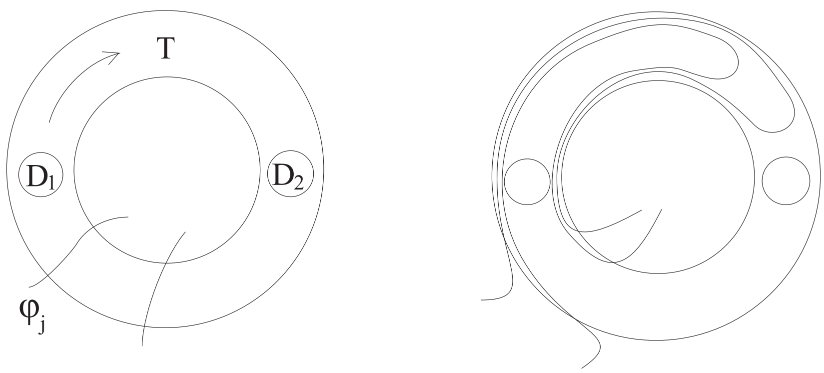

Figure 2.3:

For $i \notin H_a$ it is clear that d is the identity on the image of φ_i, thus $d_*(a_i) = a_i$. On the other hand if $i \in H_a$, then $d \circ \varphi_i$ is homologous to the connected sum of φ_i and γ_1 (surrounding D_1), thus $d_*(a_i) = a_i + c_1$. Similarly for $i \notin H_b$ we have $d_*(b_i) = b_i$, and if $i \in H_b$ then $d_*(b_i) = b_i + c_1$. Finally it holds that $d_*(c_2) = c_1$, $d_*(c_1) = c_2$ and if $i > 2$ then $d_*(c_i) = c_i$. (If $k = 2$ then $d_*(c_1) = c_2 = c_1$.) This can be verified by looking at the action of d on the curves $\gamma_1, \dots, \gamma_k$. Since d_* permutes the generators $c_1, \dots c_{k-1}$, for $1 \leq i \leq k-1$ we have $\langle \alpha, d_*(c_i) \rangle = 1$. (Recall that $\langle \alpha, c_i \rangle = 1$ and $\langle \beta, c_i \rangle = 1$ for every i.) Thus $\langle \alpha, d_*(c_i) \rangle = \langle \beta, c_i \rangle$. By the choice of H_a we see that for $i \in H_a$ it holds that $\langle \alpha, d_*(a_i) \rangle = \langle \alpha, a_i + c_1 \rangle = \langle \alpha, a_i \rangle + 1 = \langle \beta, a_i \rangle$ and for $i \notin H_a$ we have that $\langle \alpha, d_*(a_i) \rangle = \langle \alpha, a_i \rangle = \langle \beta, a_i \rangle$. A similar argument holds for $b_1, \dots, b_g$. Since $a_1, b_1, \dots, a_g, b_g$ and $c_1, \dots, c_{k-1}$ form a basis of $H_1(F^2; \mathbb{Z}_2)$, we have shown that $\langle d^*(\alpha), x \rangle = \langle \alpha, d_*(x) \rangle = \langle \beta, x \rangle$ for every $x \in H_1(F^2; \mathbb{Z}_2)$. Thus $d^*(\alpha) = \beta$. Hence $(f \circ d)|F^2$ is regularly homotopic to $g|F^2$ as required. Also d satisfies $d(S(f)) = S(f)$.

Now suppose that M^2 is a non-orientable surface of genus g, i.e. a sphere with g Moebius bands. Choose a curve φ_i in F^2 on the i-th Moebius band representing its homology generator for $1 \leq i \leq g$. Then $b_1 = [\varphi_1], \dots, b_g = [\varphi_g]$ together with $c_1, \dots, c_{k-1}$ is the standard basis of $H_1(F^2; \mathbb{Z}_2)$. Analogously to the orientable case let

$$H = \{\, 1 \leq i \leq g \colon \langle \alpha, b_i \rangle \neq \langle \beta, b_i \rangle \,\},$$

and similarly it is sufficient to construct a diffeomorphism d of the pair (M^2, F^2) with $d_*(c_i) = c_i$ for $1 \leq i \leq k$, $d_*(b_i) = b_i + c_1$ for $i \in H$ and $d_*(b_i) = b_i$ for $i \notin H$. It is enough to show that for any fix $1 \leq j \leq g$ we can find a diffeomorphism d_j of the pair (M^2, F^2) such that $d_{j*}(b_j) = b_j + c_1$ and that d_{j*} is identical on every other homology-generator. (Then $\prod_{j \in H} d_j$ is a good choice for d.)

For this end modify a small arc of φ_j using a homology such that it still lies in F^2 but gets close to D_1 and remains disjoint from all the other φ_i for $i \neq j$. (We shall call this modified curve φ_j also.) This is possible since $M^2 \setminus \bigcup_{i=1}^{g} \varphi_i$ is path-connected. Denote by T a tubular neighborhood of φ_j in M^2 containing D_1 and disjoint from D_i if $i > 1$ and from φ_i if $i \neq j$. (See Figure 2.4.) Then T is homeomorphic to the Moebius band. Also choose a slightly larger tubular neighborhood T' of T with similar properties. Now think of T as a rectangle with the vertical sides identified in the opposite direction and with D_1 in its center. Let d_j be the reflection of the rectangle T into its horizontal central axis going through the center of D_1 which is p_1. Then d_j induces an orientation-preserving diffeomorphism (a rotation) of $\partial T = S^1$, which can be extended to $T' \setminus T = S^1 \times [0, 1]$ being identical on $\partial T' = S^1 \times \{1\}$ as we have already seen. Finally d_j is identical on $M^2 \setminus T'$. This d_j maps the curve φ_j (which is the horizontal central line in the rectangle except that it avoids D_1 (see Figure 2.4)) to a curve homologous to the connected sum of φ_j and γ_1, thus $d_{j*}(b_j) = b_j + c_1$. Since φ_i for $i \neq j$ and γ_i for $i > 1$ are fixed by d_j, it satisfies the required conditions. Concerning c_1 we have $d_{j*}(c_1) = -c_1 = c_1$ since we are working with mod 2 coefficients.

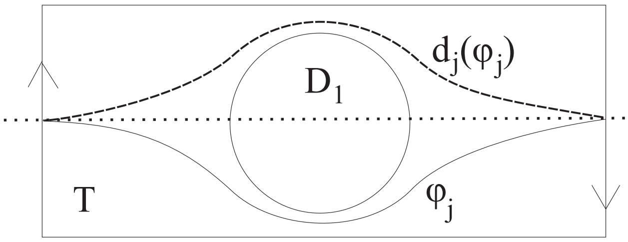

Figure 2.4:

The diffeomorphism d_j of M^2 is diffeotopic to id_{M^2}: Think of the Moebius band T as the factor space $S^1 \times [0,1]/_{p\times\{1\}=(-p)\times\{1\}}$ (thus we identify the opposite points of one boundary component of an annulus). In this model define $(d_j)_t$ on T as the diffeomorphism induced by the rotation of the annulus by $180^\circ \cdot t$ degrees. On $T \setminus T'$ define $(d_j)_t$ as before. Finally on $M^2 \setminus T'$ the diffeomorphism $(d_j)_t$ is the identity mapping. Then $d_j = (d_j)_1$ and $(d_j)_0 = id_{M^2}$ as required. □

As a consequence of the above proof we obtain the following proposition (for the definition of the mapping class group see [22]):

Corollary 2.3.3. *Suppose that F^2 is a surface of genus g with $k > 1$ boundary components, where k is even. We denote, like as before, the homology classes represented by the boundary components of F^2 in $H_1(F^2;\mathbb{Z}_2)$ by $c_1, \ldots, c_k$. The mapping class group $M(F^2)$ of F^2 acts on the set*

$$S = \{\, \alpha \in H^1(F^2, \mathbb{Z}_2) \colon \langle \alpha, c_i \rangle = 1,\ 1 \le i \le k-1 \,\}.$$

(If $\alpha \in S$ then $\langle \alpha, c_k \rangle = \langle \alpha, c_1 + \cdots + c_{k-1} \rangle = 1$ since k is even.) If M^2 is a closed surface of genus g then there is a homomorphism $m \colon M(F^2) \longrightarrow M(M^2)$ obtained by "filling in the holes". Then $\ker(m)$ acts transitively on S.

Lemma 2.3.4. *Let g and h be locally generic mappings of M^2 into $\mathbb{R}^3$ such that $S(g) = S(h)$ and $g|F^2 \sim h|F^2$, where F^2 is the complement of a small open neighborhood of the common singular set. Then $g \sim h$.*

Proof. Recall that $A = \coprod_{i=1}^k D_i$. Since for each i it holds that $h|D_i$ and $g|D_i$ have canonical forms in appropriate coordinate-systems, there is a regular homotopy H between h and a locally generic mapping $\widetilde{h}$ such that for each $t \in [0,1]$ we have $S(H_t) = S(h)$ and that $\widetilde{h}|A = g|A$. Thus $H|(F^2 \times [0,1])$ is a regular homotopy between the immersions $h|F^2$ and $\widetilde{h}|F^2$ showing that $\widetilde{h}|F^2 \sim g|F^2$. So we can suppose that $g|A = h|A$.

Let H be a regular homotopy between $g|F^2$ and $h|F^2$. For every $i = 1, \ldots, k$ fix a smaller concentric closed disc B_i in D_i (hence $p_i \in B_i \subset D_i$). Finally set $F_l = F^2 \cup \left(\bigcup_{i=1}^l D_i \right)$. We will define recursively a sequence of regular homotopies $H^l \colon F_l \times [0,1] \longrightarrow \mathbb{R}^3$ connecting $g_l = g|F_l$ and $h_l = h|F_l$ for

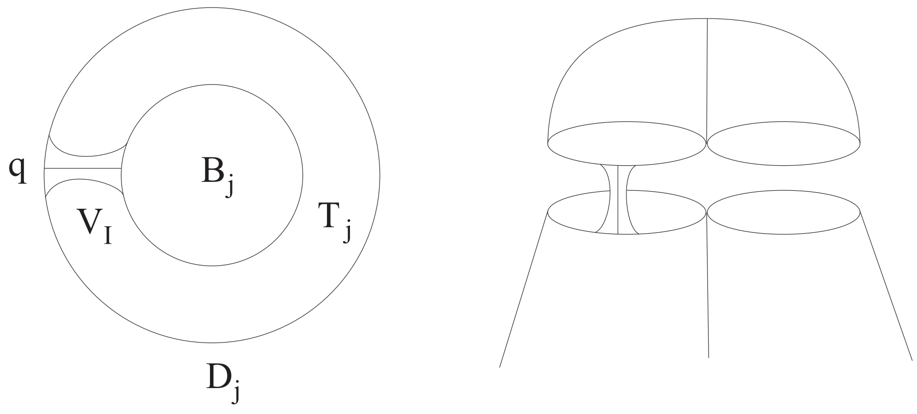

Figure 2.5:

$l = 0, \ldots, k$ with the property $H^0 = H$. Suppose that we have constructed H^l for $l < j$. Let q be a point in ∂D_j. For $t \in [0,1]$ there is a one-parameter family of elements $M_t \in GL(\mathbb{R}, 3)$ with $M_t \circ d_q H_t^{j-1} = d_q H_0^{j-1}$ and a vector $v_t \in \mathbb{R}^3$ with $M_t(H_t^{j-1}(q)) + v_t = H_0^{j-1}(q)$. Here $d_q H_t^{j-1}$ denotes the differential of the mapping H_t^{j-1} at the point q. Now define $H_t^j|F_{j-1}$ to be equal to $M_t \circ H_t^{j-1} + v_t$. Since $g(q) = h(q)$ and $d_q g = d_q h$, we have that $M_1 = id_{\mathbb{R}^3}$ and $v_1 = 0$, thus $H_1^j|F_{j-1} = h|F_{j-1}$. With this transformation of H^{j-1} we have achieved that $H_t^j(q) = H_0^j(q)$ and $d_q H_t^j = d_q H_0^j$ for every $t \in [0,1]$. Let U_q be a small closed neighborhood of q in F_{j-1} diffeomorphic to a closed 2-disc ($q \in \partial U_q$). Using a standard argument of S. Smale we can suppose that the homotopy H^j is kept fixed on a neighborhood of U_q (see Hirsch [8], Lemma 2.5 on page 249). On B_j define for every $t \in [0,1]$ the mapping $H_t^j|B_j = H_0^j|B_j = g|B_j$.

Denote the closure of the annulus $D_j \setminus B_j$ by T_j, and let I be a radial line in T_j containing q (see Figure 2.5). A tubular neighborhood $V_I \subset T_j$ of I is obtained by taking $V_I = (U_q \cap \partial D_j) \times I$. Let $H_t^j|V_I = g|V_I$ for every $t \in [0,1]$. This is possible since H^j is fix on U_q. We only have to define H^j on the closed two-cell $C_i = T_i \setminus \mathrm{Int}(V_I)$. For this purpose we will use Smale's lemma (see Theorem 1.1 on page 245 of [8] or Theorem 2.1 in [20]), which intuitively states the following: If we are given an immersed disk D^k in $\mathbb{R}^n$ such that $k < n$ and we deform the boundary of the disk and the normal derivatives along the boundary, then we can deform the whole disk at the same time so as to induce the given deformation on the boundary and normal derivatives. Since H^j is already defined on ∂C_i along with derivatives normal to ∂C_i we can use Smale's lemma to get a regular homotopy G on C_j with $G_0 = g|C_j$ and G prescribed along the boundary. Finally $G_1 \sim h|C_j$ since they coincide on ∂C_j and the obstruction is an element of $\pi_2(SO(3)) = 0$. Putting this homotopy after G we obtain the desired homotopy $H^j|C_j$. Thus we have constructed H^j on the whole manifold F_j. This shows that H^k is a regular homotopy connecting g and h. $\square$

Proof of Theorem 2.2.2. We have seen using the lemma of homogeneity that it is sufficient to prove Theorem 2.2.2 under the assumption $S(f) = S(g)$. By Lemma 2.3.2 there exists a diffeomorphism d such that $(f \circ d)|F^2 \sim g|F^2$ and $S(f \circ d) = S(f)$. Now applying Lemma 2.3.4 to $h = f \circ d$ and g we get $f \circ d \sim g$. But $f \circ d_t$ is a regular homotopy between f and $f \circ d$ proving that $f \sim g$.

It remains to show that the above regular homotopy H joining f and g can be chosen in such a way that it defines a prescribed bijection $i\colon S(f) \longrightarrow S(g)$, that is $i_H = i$. Recall that the bijection $i_H\colon S(f) \longrightarrow S(g)$ depends only on the choice of the diffeotopy φ_t mentioned before Lemma 2.3.2, that clearly might induce any prescribed bijection between $S(f)$ and $S(g)$. If M^2 is orientable, then the diffeomorphism $d\colon M^2 \longrightarrow M^2$ of Lemma 2.3.2 swaps the singular points p_1 and p_2 (i.e. $d(p_1) = p_2$, $d(p_2) = p_1$ and $d(p_i) = p_i$ for $i > 2$), and in the non-orientable case $d(p_i) = p_i$ for $1 \le i \le k$. Finally the homotopy constructed in Lemma 2.3.4 between $f \circ d$ and g is a singularity fixing homotopy in the sense of Definition 2.3.1. This completes the proof of Theorem 2.2.2. $\square$

The converse of Lemma 2.3.4 is true only in the following form:

Proposition 2.3.5. *Suppose that g and h are locally generic mappings of M^2 into $\mathbb{R}^3$ such that $S(g) = S(h)$. Denote by F^2 the complement of a small open neighborhood of the common singular set. Then $g \sim h$ implies that Im$(g|F^2) \sim$ Im$(h|F^2)$.*

Proof. Let H be a regular homotopy connecting g and h. Then

$$S(H_t) = \{\, p_1(t), \dots, p_k(t) \,\},$$

where $p_i(t)$ is a smooth curve in M^2. The lemma of homogeneity gives a diffeotopy φ_t of M^2 such that $\varphi_0 = id_{M^2}$ and $\varphi_t(S(H_0)) = S(H_t)$ for every $t \in [0,1]$. The homotopy $G_t = H_t \circ \varphi_t$ has the property that $S(G_t) = S(G_0)$ for $t \in [0,1]$ and connects g with $h \circ \varphi_1$. Since φ_1 permutes $S(g)$ (because $\varphi_1(S(g)) = \varphi_1(S(H_0)) = S(H_1) = S(g)$) we can choose φ_1 to map F^2 onto itself. $G_t|F^2$ is a regular homotopy between the immersions $g|F^2$ and $(h|F^2) \circ (\varphi_1|F^2)$ which means by definition that $\mathrm{Im}(g|F^2) \sim \mathrm{Im}(h|F^2)$. $\square$

Remark 2.3.1. Modify Definition 2.2.2 of regular homotopy the following way:

Definition 2.3.1. Locally generic mappings $f, g\colon M^2 \longrightarrow \mathbb{R}^3$ are *regularly homotopic through a singularity fixing homotopy* – notation $f \sim_s g$ – if $S(f) = S(g)$ and there exists a smooth mapping $H\colon M^2 \times [0,1] \to \mathbb{R}^3$ such that $H_0 = f$ and $H_1 = g$ and for every $t \in [0,1]$ the mapping H_t is locally generic with $S(H_t) = S(f)$. (That is the singular points are kept fixed.)

This gives a modification of the definition of image-homotopic maps:

Definition 2.3.2. Locally generic mappings $f, g\colon M^2 \longrightarrow \mathbb{R}^3$ are *image homotopic through a singularity fixing homotopy* – notation $\mathrm{Im}(f) \sim_s \mathrm{Im}(g)$ – if $S(f) = S(g)$ and there is a $d : M^2 \longrightarrow M^2$ diffeomorphism such that $f \circ d \sim_s g$. Note that $d(S(f)) = S(g) = S(f)$ and d can permute the points of $S(f)$.

Suppose that $S(f)$ or $S(g)$ is non-empty. The arguments above show that $\mathrm{Im}(f) \sim_s \mathrm{Im}(g)$ if and only if $|S(f)| = |S(g)|$. To prove this we only have to use diffeomorphisms instead of diffeotopies since Lemma 2.3.4 remains true using the new definition.

On the other hand $f \sim_s g$ implies that the immersions $f|(M^2 \setminus S(f))$ and $g|(M^2 \setminus S(f))$ are regularly homotopic. But there are locally generic mappings $f, g\colon M^2 \longrightarrow \mathbb{R}^3$ satisfying $S(f) = S(g)$ such that $f \sim g$ but $f|(M^2 \setminus S(f)) \nsim g|(M^2 \setminus S(f))$. Take for example $M^2 = RP^2$ and choose two arbitrary points $p, q \in RP^2$. Denote $RP^2 \setminus \{p, q\}$ by F^2. Using the notations of Lemma 2.3.2 we have that $H_1(F^2; \mathbb{Z}_2) = \langle f_1, c_1 \rangle$. Define the cohomology classes $\alpha, \beta \in H^1(F^2; \mathbb{Z}_2)$ by the equalities $\alpha(f_1) = 0$, $\beta(f_1) = 1$ and $\alpha(c_1) = \beta(c_1) = 1$. Then there exist locally generic mappings $f, g\colon RP^2 \longrightarrow \mathbb{R}^3$ satisfying $S(f) = S(g) = \{p, q\}$ such that $f|F^2$ and $g|F^2$ correspond to α and β using the bijection of Theorem 2.3.1 : Denote by U the locally generic mapping of S^2 to $\mathbb{R}^3$ with singular points p and q (this is unique up to singularity fixing homotopy). B is the famous Boy surface, $\overline{B}$ is the mirror image of B (see [16]). Then the connected sums $f = B \# U$ and $g = \overline{B} \# U$ satisfy the above conditions. Clearly $f|F^2 \nsim g|F^2$, thus $f \nsim_s g$. This provides examples of locally generic mappings f and g such that $\mathrm{Im}(f) \sim_s \mathrm{Im}(g)$, but $f \nsim_s g$.

2.4 Projections of regular homotopies

Suppose that M^2 is a closed connected surface. In the previous sections we examined the path-components of the space of locally generic mappings of M^2 into $\mathbb{R}^3$ endowed with the C^∞ topology. This space, denoted by $\mathrm{Lgen}(M^2, \mathbb{R}^3)$, is closely connected with the space $\mathrm{Imm}(M^2, \mathbb{R}^4)$ of immersions of M^2 into $\mathbb{R}^4$ (also considered with the C^∞ topology). This connection is realized by projections of $\mathbb{R}^4$ onto $\mathbb{R}^3$.

Definition 2.4.1. A mapping $\pi\colon \mathbb{R}^k \longrightarrow \mathbb{R}^l$ $(k > l)$ is a projection if it is linear and surjective. Denote the space of all projections from $\mathbb{R}^k$ to $\mathbb{R}^l$ by $\mathrm{Proj}(\mathbb{R}^k, \mathbb{R}^l)$.

Our starting point is the following result of Mather [12]:

Proposition 2.4.1. *Suppose that $F \in Imm(M^2, \mathbb{R}^4)$ is an immersion. Then for almost every $\pi \in Proj(\mathbb{R}^4, \mathbb{R}^3)$ (in the sense of Lebesgue measure) the mapping $\pi \circ F$ is locally generic.*

For every projection $\pi \in \mathrm{Proj}(\mathbb{R}^4, \mathbb{R}^3)$ let

$$\mathrm{Imm}_\pi(M^2, \mathbb{R}^4) = \left\{ F \in \mathrm{Imm}(M^2, \mathbb{R}^4) \colon \pi \circ F \in \mathrm{Lgen}(M^2, \mathbb{R}^3) \right\}$$

be the subspace of $\mathrm{Imm}(M^2, \mathbb{R}^4)$. There is a natural mapping

$$p_\pi\colon \mathrm{Imm}_\pi(M^2, \mathbb{R}^4) \longrightarrow \mathrm{Lgen}(M^2, \mathbb{R}^3)$$

defined by the formula $p_\pi(F) = \pi \circ F$ for every $F \in \mathrm{Imm}_\pi(M^2, \mathbb{R}^4)$.

In this section our aim is to examine the path-components of $\mathrm{Imm}_\pi(M^2, \mathbb{R}^4)$.

Definition 2.4.2. Two immersions $F, G \in \mathrm{Imm}_\pi(M^2, \mathbb{R}^4)$ are called *π-homotopic* (denoted by $F \sim_\pi G$) if they are in the same path-component of $\mathrm{Imm}_\pi(M^2, \mathbb{R}^4)$.

First let us recall that two immersions $F, G \in \mathrm{Imm}(M^2, \mathbb{R}^4)$ are regularly homotopic if and only if $e(F) = e(G)$, where $e(F)$ denotes the (twisted) Euler-number of the normal bundle of the immersion F. It is clear that if $F \sim_\pi G$ then $e(F) = e(G)$ and $|S(\pi \circ F)| = |S(\pi \circ G)|$. We are going to prove that if $S(\pi \circ F)$ (and $S(\pi \circ G)$) are non-empty then the converse also holds.

Suppose that $F, G \colon M^2 \longrightarrow \mathbb{R}^4$ are immersions. From Proposition 2.4.1 it is clear that for almost every projection $\pi \in \mathrm{Proj}(\mathbb{R}^4, \mathbb{R}^3)$ both $f = \pi \circ F$ and $g = \pi \circ G$ are locally generic. Fix such a projection π.

We are going to define the sign of every point in $S(f)$ (and in $S(g)$). (An equivalent definition can be found in [18]). Take a cross-cap point $p \in S(f)$. Choose orientations of $\mathbb{R}^4$ and of $\mathbb{R}^3$. We will define the sign of p as follows.

Fix local coordinates (x_1, x_2) on a neighborhood U_p of p and (y_1, y_2, y_3) centered at $f(p)$ such that f has the following normal form:

$$y_1 \circ f = x_1^2, \;\; y_2 \circ f = x_2, \;\; y_3 \circ f = x_1 x_2.$$

Suppose that U_p is so small that $F|U_p$ is an embedding. The sign of p will depend only on $F|U_p$ (thus the definition is local). Set $D_\varepsilon = \{y_1^2 + y_2^2 + y_3^2 \leq \varepsilon\} \subset \mathbb{R}^3$ and choose $\varepsilon > 0$ sufficiently small such that $\widetilde{D}_\varepsilon = f^{-1}(D_\varepsilon) \subset U_p$. The set $\widetilde{D}_\varepsilon$ is a closed disc neighborhood of p in M^2 and $\partial \widetilde{D}_\varepsilon = f^{-1}(\partial D_\varepsilon)$ (see Lemma 2.2 in [18]). Let L be the closure of the double point set of $f|\widetilde{D}_\varepsilon$, *i.e.* $L = \{\, x \in \widetilde{D}_\varepsilon \colon f^{-1}(f(x)) \cap \widetilde{D}_\varepsilon \neq \{x\} \,\} \cup \{p\}$. Then L is a one-dimensional smooth submanifold of $\widetilde{D}_\varepsilon$ and $L \cap \partial \widetilde{D}_\varepsilon$ consists of two points p_1 and p_2. We fix an orientation of $\partial \widetilde{D}_\varepsilon$ and take an oriented base (u) (resp. (v)) of the tangent space $T_{p_1}(\partial \widetilde{D}_\varepsilon)$ (resp. $T_{p_2}(\partial \widetilde{D}_\varepsilon)$). Then $(df_{p_1}(u), df_{p_2}(v))$ is a base of $T_q(\partial D_\varepsilon)$, where $q = f(p_1) = f(p_2)$. We may assume that $(df_{p_1}(u), df_{p_2}(v), \xi)$ is a positive basis of $T_q\mathbb{R}^3$, where ξ is the outward normal vector of ∂D_ε, exchanging p_1 and p_2 if necessary. Now orient L from p_2 to p_1.

Denote by ν a positive basis vector of T_pL. (If M^2 possess a Riemannian metric, then choose ν to be a unit-vector. This way ν is unique up to the orientation of $\mathbb{R}^3$.) Orient $\ker \pi$ in such a way that together with the orientation of $\mathrm{Im}\, \pi = \mathbb{R}^3$ we obtain the fixed orientation of $\mathbb{R}^4 = \mathrm{Im}\, \pi \oplus \ker \pi$. Using this direct sum decomposition of $\mathbb{R}^4$ the mapping $F \colon M^2 \longrightarrow \mathbb{R}^4$ can be written in the form $F = (\pi \circ F, F^*) = (f, F^*)$. Since F is an immersion at the point p and $p \in S(f)$, we have $dF^*(\nu) \neq 0$. After all this preparation we can now define the sign of p.

Definition 2.4.3. The cross-cap point p is *positive* if $dF^*(\nu) > 0$ and is *negative* if $dF^*(\nu) < 0$. We denote by $p(F)$ (resp. $n(F)$) the number of positive (resp. negative) cross-cap singularities of the locally generic mapping $f = \pi \circ F \colon M^2 \longrightarrow \mathbb{R}^3$.

The following proposition is a special case of Proposition 2.5 in [18].

Proposition 2.4.2. *Suppose that $\mathbb{R}^4$ and $\mathbb{R}^3$ are oriented. Then we always have*

$$e(F) = p(F) - n(F),$$

where $e(F) \in \mathbb{Z}$ is the (twisted) Euler-number of the normal bundle of the immersion F.

Thus the immersions F and G are regularly homotopic if and only if

$$p(F) - n(F) = e(F) = e(G) = p(G) - n(G).$$

On the other hand Theorem 2.2.2 states that if $S(f)$ and $S(g)$ are non-empty then $f \sim g$ if and only if

$$p(F) + n(F) = |S(f)| = |S(g)| = p(G) + n(G).$$

Comparing the preceding two chains of equations we have that if both f and g are singular then $[F \sim G$ and $f \sim g] \Leftrightarrow [p(F) = p(G)$ and $n(F) = n(G)]$. The following theorem implies that in this case we can even find a regular homotopy between F and G whose projection is a regular homotopy between f and g, i.e. $F \sim_\pi G$.

Theorem 2.4.3. *Suppose that M^2 is a closed connected surface, $F, G\colon M^2 \longrightarrow \mathbb{R}^4$ are immersions and $\pi\colon \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ is a projection such that $f = \pi \circ F$ and $g = \pi \circ G$ are both locally generic and singular. Then the following are equivalent:*

(1) *There exists a regular homotopy $H\colon M^2 \times [0,1] \longrightarrow \mathbb{R}^4$ between F and G such that $\widetilde{H} = \pi \circ H$ is a regular homotopy between f and g, i.e. $F \sim_\pi G$.*

(2) *The numbers of positive and negative cross-caps of f and g are the same, i.e. $p(F) = p(G)$ and $n(F) = n(G)$.*

Proof. First we prove the implication (1) $\Rightarrow$ (2). In this case $f \sim g$, thus using Proposition 2.2.1 we have that $|S(f)| = |S(g)|$. From Definition 2.4.3 it is clear that the signs of the singular points do not change during a regular homotopy. (Here we did not use the assumption that $|S(f)| > 0, |S(g)| > 0$.)

Now we are going to prove the implication (2) $\Rightarrow$ (1). Since $p(F) = p(G)$ and $n(F) = n(G)$ there exists a bijection $i\colon S(f) \longrightarrow S(g)$ that preserves the signs of the cross-cap points. By Theorem 2.2.2 there is a regular homotopy $\widetilde{H}$ between f and g such that $i_{\widetilde{H}} = i$. We shall construct a regular homotopy H between the immersions F and G such that $\pi \circ H = \widetilde{H}$ as follows: Choose an arbitrary Riemannian metric on M^2. In the paragraph preceding Definition 2.4.3 we saw that in this case for every $t \in [0,1]$ and $p \in S(\widetilde{H}_t)$ there is a unique positive unit-vector $\nu_t(p) \in T_pM^2$ tangent to the double-point curve L crossing p. The singular sets $S(\widetilde{H}_t)$ for $t \in [0,1]$ define curves $\gamma_1, \dots, \gamma_k$ on M^2 such that $S(H_t) = \{\, \gamma_j(t)\colon 1 \le j \le k \,\}$ for every $t \in [0,1]$. The points $\gamma_i(0)$ and $\gamma_i(1)$ have the same sign ($1 \le i \le k$). Suppose for example that for a fix i both $\gamma_i(0)$ and $\gamma_i(1)$ are positive cross-cap points. Introduce the notation $\nu_i(t) = \nu_t(\gamma_i(t))$, then $dF^*(\nu_i(0)) > 0$ and $dG^*(\nu_i(1)) > 0$. (Here F^* denotes the fourth coordinate function of F in $\mathbb{R}^4$.) Using the Levi-Civita connection of the Riemannian manifold M^2 we may consider the exponential mapping on M^2. Since $[0,1]$ is compact, there exists $\varepsilon > 0$ such that for every $t \in [0,1]$ the mapping $h\colon [-\varepsilon, \varepsilon] \times [0,1] \longrightarrow M^2$ satisfying

$$h(s,t) = \exp_{\gamma_i(t)}(s \cdot \nu_i(t))$$

is defined and for every $t \in [0,1]$ the mapping $h_t(s) = h(s,t)$ is an embedding of $[-\varepsilon, \varepsilon]$ into M^2 (h_t is a geodetic curve). Define the function H_t^* on $\operatorname{Im} h_t$ using the following formula:

$$H_t^*(h_t(s)) = (1-t) \cdot F^*(h_0(s)) + t \cdot G^*(h_1(s))$$

for $s \in [-\varepsilon, \varepsilon]$. Note that $H_0^* | \operatorname{Im} h_0 = F^* | \operatorname{Im} h_0$ and $H_1^* | \operatorname{Im} h_1 = G^* | \operatorname{Im} h_1$. Thus we can extend H^* to an open neighborhood of $\bigcup_{t \in [0,1]} (\operatorname{Im} h_t \times \{t\})$ in $M^2 \times [0,1]$ as a smooth function. From the construction of H_t^* it is clear that

$$dH_t^*(\nu_i(t)) = (1-t) \cdot dF^*(\nu_i(0)) + t \cdot dG^*(\nu_i(1)) > 0,$$

which implies that the mapping $H_t = (\widetilde{H}_t, H_t^*)$ is an immersion at the point $\gamma_i(t)$ for every $t \in [0,1]$. Repeat the preceding extension process for every $1 \leq i \leq k$ and afterwards extend the obtained H^* to the whole cylinder $M^2 \times [0,1]$ in such a way that $H_0^* = F^*$ and $H_1^* = G^*$. The mapping $H = (\widetilde{H}, H^*)$ is a regular homotopy connecting F and G whose projection is $\widetilde{H}$. □

Putting together our previous results we obtain the following theorem:

Theorem 2.4.4. *If $F, G \in Imm_\pi(M^2, \mathbb{R}^4)$ then*

$$F \sim_\pi G \Leftrightarrow [F \sim G \text{ and } \pi \circ F \sim \pi \circ G].$$

Proof. First we suppose that $S(\pi \circ F)$ or $S(\pi \circ G)$ is non-empty. Theorem 2.4.3 states that $F \sim_\pi G \Leftrightarrow [p(F) = p(G)$ and $n(F) = n(G)]$. We have seen in the paragraph preceding Theorem 2.4.3 that $[p(F) = p(G)$ and $n(F) = n(G)] \Leftrightarrow [F \sim G$ and $\pi \circ F \sim \pi \circ G]$.

Now we consider the case when both $S(\pi \circ F)$ and $S(\pi \circ G)$ are empty, i.e. $f = \pi \circ F$ and $g = \pi \circ G$ are immersions. If $f \sim g$ then any regular homotopy connecting f and g can be lifted to a regular homotopy between F and G, thus $F \sim_\pi G$. This proves the implication $F \sim_\pi G \Leftarrow [F \sim G$ and $\pi \circ F \sim \pi \circ G]$. The other implication is trivial. □

Chapter 3

Regular homotopy classes of singular maps

3.1 Introduction

This paper is a sequel to [11] in which locally generic maps of closed surfaces into $\mathbb{R}^3$ were classified up to regular homotopy. It turned out that for maps with at least one cross-cap point the number of singular points was the only regular homotopy invariant. The obstruction to constructing a regular homotopy was destroyed by pushing 1-cells of the surface through a singular point (see Figure 3.1).

In the present work we extend this result for locally generic maps of closed n-manifolds into $\mathbb{R}^{2n-1}$ in case $n \neq 3$ (Theorem 3.1.3). However, this is not simply an adaptation of the ideas of [11]. The idea of pushing $(n-1)$-simplices through a singular point originates from the previous work, but a lot of technical problems have to be dealt with. For example, if n is odd the Whitney-umbrella points have signs and the $(n-1)$-dimensional obstruction is $\mathbb{Z}$-valued. Moreover, a new type of obstruction appears if $n > 3$ that was not present in the case $n = 2$. This is an n-dimensional obstruction and is related to the double point set of the map. In eliminating this obstruction we make essential use of a result of T. Ekholm [3].

It seems that this second type of obstruction cannot be eliminated if $n = 3$. In fact, in addition to the number of singular points a new $\mathbb{Z}$-valued invariant comes into the picture. I intend to deal with this in a separate paper.

3.1.1 Preliminary definitions

Definition 3.1.1. Let M^n be a closed n-manifold and N^{2n-1} an arbitrary $(2n-1)$-manifold . A map $f\colon M^n \to N^{2n-1}$ is called *locally generic* if it is an immersion except for cross-cap (or Whitney-umbrella) singularities. Thus for any singular point p there exist coordinate systems $(x_1, \dots, x_n)$ about p and

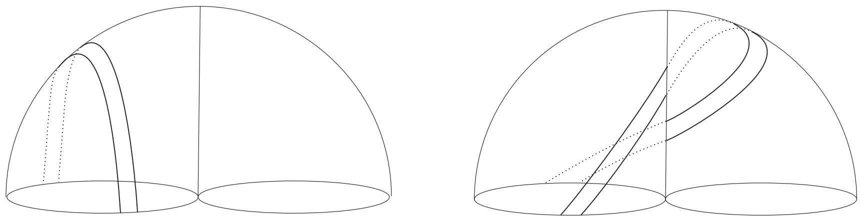

Figure 3.1: Pushing a 1-cell through a cross-cap point

$(y_1, \dots, y_{2n-1})$ about $f(p)$ such that f is given near p by

$$y_1 = x_1^2,\ y_i = x_i,\ y_{n+i-1} = x_1 x_i\ (i = 2, \dots, n). \tag{3.1.1}$$

Notation 3.1.1. *Let $L(M^n, N^{2n-1})$ denote the subspace of locally generic maps, $G(M^n, N^{2n-1})$ the subspace of generic maps and $Imm(M^n, N^{2n-1})$ the subspace of immersions in $C^\infty(M^n, N^{2n-1})$ endowed with the C^∞ topology. We introduce the notation $L'(M^n, N^{2n-1})$ for the space*

$$L(M^n, N^{2n-1}) \setminus Imm(M^n, N^{2n-1}).$$

For $f \in L(M^n, N^{2n-1})$ the set of singular points of the map f is denoted by $S(f)$.

Remark 3.1.1. Since M^n is compact and the singularities of f are isolated we have $|S(f)| < \infty$. It is well known that for M^n closed $|S(f)|$ is an even number.

For $n > 2$ a map $f\colon M^n \to N^{2n-1}$ is generic iff

- it is an immersion with normal crossings except in a finite set of points
- the singular points are non-multiple cross-cap points
- f has at most double crossings.

For $n = 2$ a generic f might also have triple points.

Definition 3.1.2. A *regular homotopy* is a path in the space $L(M^n, N^{2n-1})$. In other words, it is a smooth map $H\colon M^n \times [0,1] \to N^{2n-1}$ such that $H_t \in L(M^n, N^{2n-1})$ for every $t \in [0,1]$. Here $H_t(x) = H(x,t)$ for $x \in M^n$ and $t \in [0,1]$. The regular homotopy H is called *singularity fixing* if $S(H_t) = S(H_0)$ for every $t \in [0,1]$.

Definition 3.1.3. Two maps $f, g \in L(M^n, N^{2n-1})$ are called *regularly homotopic* if there exists a regular homotopy H such that $H_0 = f$ and $H_1 = g$. If H can be chosen to be singularity fixing then we say that f and g are *regularly homotopic through a singularity fixing homotopy.*

Notation 3.1.2. *The fact that f and g are regularly homotopic is denoted by $f \sim g$. Furthermore, $f \sim_s g$ denotes that f and g are regularly homotopic through a singularity fixing homotopy.*

3.1.2 The classification theorem

Theorem 3.1.3. *Suppose that $n > 3$ or $n = 2$. Let M^n be a closed n-manifold and $f, g \in L'(M^n, \mathbb{R}^{2n-1})$. Then*

$$f \sim g \Leftrightarrow |S(f)| = |S(g)| .$$

3.2 Known results used in the proof of Theorem 3.1.3

3.2.1 Smale's lemma and M^n-regular homotopies

We will use extensively the following result of Smale (this is Theorem 1.1 in [8], the original paper of Smale is [20]). First we need a few definitions.

Definition 3.2.1. Let $\mathcal{E} = \mathcal{E}_{k,m}$ be the space of all C^∞ immersions of D^k in $\mathbb{R}^m$ in the C^1 topology. Let $\mathcal{B} = \mathcal{B}_{k,m}$ be the set of pairs (g, g') where $g\colon S^{k-1} \to \mathbb{R}^m$ is a C^∞ immersion and $g'\colon S^{k-1} \to T\mathbb{R}^m$ is a C^∞ transversal field of g. $\mathcal{B}$ is topologized as a subspace of Cartesian product of the space of immersions $S^{k-1} \to \mathbb{R}^m$, in the C^1 topology, with the space of continuous maps $S^{k-1} \to T\mathbb{R}^m$, in the compact-open topology.

If $h \in \mathcal{E}$ then let $h'\colon S^{k-1} \to T\mathbb{R}^m$ be defined by $h'(x) =$ derivative of h along the radius at $x \in S^{k-1}$. I.e., if $r(x)$ is the unit tangent vector of D that is normal to S^{k-1} at x and which points away from the origin then $h'(x) = h_* r(x)$. We define the map $\pi\colon \mathcal{E} \to \mathcal{B}$ by the formula $\pi(h) = (h|S^{k-1}, h')$. It is clear that π is continuous. The following theorem is Theorem 1.1 in [8].

Theorem 3.2.1. *If $k < m$, then $\pi\colon \mathcal{E}_{k,m} \to \mathcal{B}_{k,m}$ has the covering homotopy property.*

The intuitive content of this theorem is as follows: If we are given an immersed disk D^k in $\mathbb{R}^m$ such that $k < m$ and we deform the boundary of the disk and the normal derivatives along the boundary, then we can deform the whole disk at the same time so as to induce the given deformation on the boundary and normal derivatives.

The following definitions were introduced by Hirsch in [8].

Definition 3.2.2. Let A be an arbitrary subset of the manifold M^n. Let $h\colon A \to Q^q$ and $h'\colon TM^n|A \to TQ^q$ (Q^q is a manifold) be continuous maps such that h' covers h. The pair (h, h') is called an *M^n-regular map*, or *M^n-immersion, of A in Q^q* if the following condition is satisfied: there is a neighborhood V of A in M^n and an immersion $l\colon V \to Q^q$ such that $dl|(TM|A) = h'$. It follows that $l|A = h$. We say that (h, h') is C^k if l can be chosen to be C^k.

Definition 3.2.3. Let $B \subset A \subset M^n$ be subsets. If $(r, r'), (s, s')\colon A \to Q^q$ are M^n-immersions such that $r|B = s|B$ and $r'|(TM^n|B) = s'|(TM^n|B)$, we say that (r, r') *and* (s, s') *are tangent on* B, and write this as $(r, r')|B = (s, s')|B$.

Definition 3.2.4. Let (r, r') and (s, s') be M^n-immersions of A in Q^q such that for a certain (possibly empty) subset B of A, (r, r') and (s, s') are tangent on B. We say that (r, r') and (s, s') are *M^n-regularly homotopic (rel B)* if there is a path (h_t, h'_t) in the space of all M^n-immersions of A in Q^q joining (r, r') to (s, s'), such that for each t, $(h_t, h'_t)|B = (r, r')|B$. Such a path is called an *M^n-regular homotopy (rel B)*, and it is C^k if every (h_t, h'_t) is C^k.

Notation 3.2.2. *The space of all C^∞ $\mathbb{R}^n$-immersions of D^k in $\mathbb{R}^q$ is denoted by $\mathcal{I}(k, q; n)$; the space of all C^∞ $\mathbb{R}^n$-immersions of S^{k-1} in $\mathbb{R}^q$ is denoted by $\mathcal{I}'(k, q; n)$. Now let $\pi_n : \mathcal{I}(k, q; n) \to \mathcal{I}'(k, q; n)$ be defined by*

$$\pi_n(f, f') = (f|S^{k-1}, f'|(T\mathbb{R}^n|S^{k-1})).$$

If $n = k$, this is the map $\pi : \mathcal{E}_{k,q} \to \mathcal{B}_{k,q}$ defined in Definition 3.2.1.
For $(f, f') \in \mathcal{I}(k, q; n)$ put $\Gamma_n(f, f') = \pi_n^{-1}(\pi_n(f, f'))$.

The following statement is Theorem 3.5 in [8]. It is a generalization of Smale's lemma to M^n-immersions. To avoid confusion we will refer to it also as Smale's lemma (despite the fact that it was proved by Hirsch).

Theorem 3.2.3. *π_n has the covering homotopy property if $k < q$.*

Theorem 3.2 in [8] gives an alternative description of M^n-immersions:

Notation 3.2.4. *$\{ e_1(x), \ldots, e_n(x) \}$ denotes the standard basis of $T_x\mathbb{R}^n$.*

Lemma 3.2.5. *There is a homeomorphism ζ between the space of $\mathbb{R}^n$-immersions $(h, h') : D^k \to \mathbb{R}^q$ and the space of pairs (l, Ψ), where $l \in Imm(D^k, \mathbb{R}^q)$ and Ψ is a transversal $(n-k)$-field along l. The homeomorphism is given by $\zeta(h, h') = (h, \Psi)$ where $\Psi(x) = h'\{e_{k+1}(x), \ldots, e_n(x)\}$. Moreover, (f, f') is C^k if and only if f and Ψ are C^k.*

An analogous result holds for S^{k-1} (see [8], Theorem 3.3):

Notation 3.2.6. *Denote the space of pairs (l, Ψ), where $l \in Imm(S^{k-1}, \mathbb{R}^q)$ and Ψ is a transversal $(n-k+1)$-field along l by*

$$Imm_{n-k+1}(S^{k-1}, \mathbb{R}^q).$$

Lemma 3.2.7. *There is a homeomorphism*

$$\chi : \mathcal{I}'(k, q; n) \to Imm_{n-k+1}(S^{k-1}, \mathbb{R}^q).$$

χ is given as follows: Let Φ be the normal $(n-k+1)$-field on S^{k-1} given by

$$\Phi(x) = \{r(x), e_{k+1}(x), \ldots, e_n(x)\},$$

where $r(x)$ is the outward unit normal to S^{k-1} in $\mathbb{R}^k$. Then $\chi(h, h') = (h, \Psi)$, where $\Psi(x) = f'(\Phi(x))$.

3.2.2 The obstructions τ and Ω

Hirsch defined in [8] an invariant $\tau(g')$ for each $(g, g') \in \mathcal{I}'(k, q; n)$. The vanishing of $\tau(g')$ implies that (g, g') comes from $\mathcal{I}(k, q; n)$.

Definition 3.2.5. $(f, f') \in \mathcal{I}'(k, q; n)$ is said to be *extendible* if there is a $(g, g') \in \mathcal{I}(k, q; n)$ such that $\pi_q(g, g') = (f, f')$.

Definition 3.2.6. Let $(f, f') \colon S^{k-1} \to \mathbb{R}^n$ be a C^∞ $\mathbb{R}^n$-immersion, i.e., $(f, f') \in \mathcal{I}'(k, q; n)$. The *obstruction to extending* (f, f'), denoted by $\tau(f') \in \pi_{k-1}(V_{q,n})$ is the homotopy class of the map $S^{k-1} \to V_{q,n}$ defined by

$$x \mapsto f'\{e_1(x), \ldots, e_n(x)\}.$$

The following lemma is Theorem 3.9 in [8]:

Lemma 3.2.8. *If $k < q$ and $\tau(f') = 0$ then (f, f') is extendible.*

Definition 3.2.7. Let $\Phi_n \colon \mathcal{I}(k, q; n) \to C^0(D^k, V_{q,n})$ be as follows:

$$\Phi_n(f, f')(x) = f'\{e_1(x), \ldots, e_n(x)\}.$$

Let $(f, f'), (g, g') \in \mathcal{I}(k, q; n)$, with $\pi_n(f, f') = \pi_n(g, g')$, so that $(f, f') \in \Gamma_n(g, g')$. Then $\Phi_n(f, f')$ and $\Phi_n(g, g')$ are maps $D^k \to V_{q,n}$ which are tangent on S^{k-1}.

Definition 3.2.8. Let A be a topological space, simple in dimension k. Let $f, g \colon D^k \to A$ and assume that $f(x) = g(x)$ if $x \in S^{k-1}$. Then $d(f, g) \in \pi_k(A)$ is represented by mapping the "top" hemisphere of S^k by f and the "bottom" one by g, assuming that the orientation of S^k is given by the coordinate frame $\{ e_1, \ldots, e_k \}$ at the "North" pole of S^k.

Definition 3.2.9.

$$\Omega(f', g') = d(\Phi_n(f, f'), \Phi_n(g, g')) \in \pi_k(V_{q,n})$$

is called the *obstruction to an $\mathbb{R}^n$-regular homotopy* (rel S^{k-1}) between (f, f') and (g, g'). (This is well defined since $V_{q,n}$ is simple in all dimensions.)

Remark 3.2.1. An explicit definition of $\Omega(f', g')$ is as follows: identify the upper and lower hemispheres of S^k with D^k. Let $\omega \colon S^k \to V_{q,n}$ be the map $\omega(x) = f'\{e_1(x), \ldots, e_n(x)\}$ if x is in the upper hemisphere, $\omega(x) = g'\{e_1(x), \ldots, e_n(x)\}$ if x is in the lower hemisphere. $\omega(x)$ is well defined on the equator because (f, f') and (g, g') agree on S^{k-1}. Then $\Omega(f', g')$ is the homotopy class of ω.

The following lemma is Theorem 4.3 in [8]:

Lemma 3.2.9. *Let $(f, f'), (g, g'), (h, h')$ be C^∞ $\mathbb{R}^n$-immersions of Δ^k in $\mathbb{R}^q$ which are all tangent on $\partial\Delta^k$. Then*
(a) $\Omega(f', g') + \Omega(g', h') = \Omega(f', h')$.
(b) $\Omega(f', f') = 0$.

(c) Given $\alpha \in \pi_k(V_{q,n})$ there exists (g, g') such that $\Omega(f', g') = \alpha$.

(d) Suppose that $\Omega(f', g') = 0$ and $k < q$. Let $F\colon \Delta^k \times I \to V_{q,n}$ be a homotopy (rel $\partial\Delta^k$) between the maps $F_0, F_1\colon \Delta^k \to V_{q,n}$ defined respectively by $x \mapsto f'\{e_i(x)\}$ and $x \mapsto g'\{e_i(x)\}$, $i = 1\ldots, n$. Then there is a C^∞ $\mathbb{R}^n$-regular homotopy (f_t, f_t') between (f, f') and (g, g') such that the map $\Delta^k \times I \to V_{q,n}$ defined by $(x, t) \mapsto f_t'(e_i(x))$ is homotopic (rel $(\partial\Delta^k \times I) \cup (\Delta^k \times \partial I)$) to F.

From the proof of Theorem 3.9 in [8] we can deduce the following

Theorem 3.2.10. *Suppose that the C^∞ $\mathbb{R}^n$-immersions $(f, f'), (g, g')\colon \partial\Delta^n \to \mathbb{R}^q$ are tangent on $(\partial\Delta^n) \setminus int(\Delta^{n-1})$, where Δ^{n-1} is an $(n-1)$-face of the n-simplex Δ^n. Then*

$$\Omega(f'|\Delta^{n-1}, g'|\Delta^{n-1}) = \tau(f') - \tau(g').$$

3.2.3 Hirsch's theorem and lemma

In [8] Hirsch proves the following theorem that reduces regular homotopy classification of immersions to homotopy theory.

Theorem 3.2.11. *Suppose that $n \leq q$ and let M^n and Q^q be manifolds such that M^n is open if $n = q$. Then the natural map*

$$Imm(M^n, Q^q) \to Mono(TM, TQ)$$

is a weak homotopy equivalence (so it induces a bijection in π_0).

The relative version of Hirsch's theorem also holds:

Theorem 3.2.12. *Let M^n and Q^q be as in Theorem 3.2.11. Suppose that $K \subset M$ is compact and V_K is a neighborhood of K. Given an immersion $h \in Imm(V_K, Q^q)$ with differential*

$$dh = \Phi\colon TM|V_K \to TQ|V_K,$$

there exists an immersion $l \in Imm(M^n, Q^q)$ such that $l|K = f$ and dl is homotopic to Φ.

As a consequence of Theorem 3.2.11 we obtain Hirsch's lemma:

Lemma 3.2.13. *Let $k \geq 1$ and $f \in Imm(M^n, \mathbb{R}^{n+k+1})$ be an immersion with normal bundle $\mu^k \oplus \varepsilon^1$, where ε^1 denotes the trivial line bundle over M^n. Let i denote the standard embedding $\mathbb{R}^{n+k} \to \mathbb{R}^{n+k+1}$. Then there exists an immersion $g \in Imm(M^n, \mathbb{R}^{n+k})$ with normal bundle μ^k such that $f \sim i \circ g$. During the regular homotopy a trivialization of the ε^1 component of $\nu(f \subset \mathbb{R}^{n+k+1})$ can be deformed simultaneously in the space of normal fields to be finally vertical along $i \circ g$.*

Remark 3.2.2. The relative version of Hirsch's lemma also holds, as can be seen by using the relative version of Hirsch's theorem.

Corollary 3.2.14. *If $k \geq 2$ then Lemma 3.2.13 gives a bijection*

$$b\colon \pi_0(Imm_1(M^n, \mathbb{R}^{n+k+1})) \to \pi_0(Imm(M^n, \mathbb{R}^{n+k})).$$

3.2.4 Whitney-umbrellas

In this chapter we sum up some ideas from [28]. Fix an orientation for $\mathbb{R}^{2n-1}$.

Definition 3.2.10. Suppose that N^n is an n-manifold with boundary and that $f\colon N^n \to \mathbb{R}^{2n-1}$ is a generic map (thus $f|\partial N^n$ is an embedding). Let $\nu\colon \partial N^n \to T\mathbb{R}^{2n-1}$ be a transversal vector field along $f|\partial N^n$ such that $-\nu$ points into $f(N^n)$. Then for ε sufficiently small define $\mathcal{L}_f(N)$ to be the linking number

$$\operatorname{lk}(f|(\partial N^n), f|(\partial N^n) + \epsilon\nu).$$

Remark 3.2.3. Notice that (f, ν) corresponds to the N^n-immersion $(f, df)|\partial N^n$.

As a consequence of Lemma 6 in [28] we obtain the following lemma.

Lemma 3.2.15. *Suppose that $f \in L(M^n, \mathbb{R}^{2n-1})$ and $p \in S(f)$. Choose a coordinate neighborhood U_p of p diffeomorphic to D^n such that $f|U$ is generic and has the form (3.1.1). Then $\mathcal{L}_f(U_p) = \pm 1$, moreover, the sign does not depend on the choice of U_p if n is odd. (Recall that for n even $\mathcal{L}_f$ is defined mod 2.)*

Notation 3.2.16. *Denote the map $(f, df)|\partial U_p \in \mathcal{I}'(n, 2n-1; n)$ by (w, w').*

Corollary 3.2.17. *For n even $\mathcal{L}_f(U_p) \equiv 1$ (mod 2) for every neighborhood U_p.*

In [28] signs are defined for Whitney-umbrellas if n is odd and $\mathbb{R}^{2n-1}$ is oriented. This can be done as follows (this is not the original definition of Whitney):

Definition 3.2.11. Suppose that $f \in L(M^n, \mathbb{R}^{2n-1})$ and $p \in S(f)$. Choose U_p as in Lemma 3.2.15. The sign of the Whitney-umbrella at p is then defined as

$$\operatorname{sgn}(p) = \mathcal{L}_f(U_p).$$

Notation 3.2.18. *If n is odd and $f \in L(M^n, \mathbb{R}^{2n-1})$ denote the set of positive (respectively negative) cross-cap points of f by $S_+(f)$ (respectively $S_-(f)$). Moreover let*

$$\#S(f) = \begin{cases} |S_+(f)| - |S_-(f)| & \text{if } n \text{ is odd,} \\ |S(f)| \bmod 2 & \text{if } n \text{ is even,} \end{cases}$$

where $|A|$ denotes the cardinality of the set A.

In [28] Theorem 2 states the following:

Lemma 3.2.19. *Let N^n be a compact n-manifold with boundary and*

$$f \in G(N^n, \mathbb{R}^{2n-1}).$$

Then $\mathcal{L}_f(N^n) = \#S(f)$.

Remark 3.2.4. If f is only locally generic, we may perturb it slightly to obtain a generic map. From this we can generalize Lemma 3.2.19 to locally generic maps.

As a trivial consequence we obtain

Corollary 3.2.20. *Let M^n be a closed n-manifold and $f \in L(M^n, \mathbb{R}^{2n-1})$. Then for n odd the algebraic number of singular points vanishes (i.e., $|S_+(f)| = |S_-(f)|$), while for n even, it vanishes mod 2 (i.e., $|S(f)|$ is even).*

We will also use Theorem 7 of [28], which states the following:

Theorem 3.2.21. *Let M^n be a compact and connected manifold with boundary and $f \colon M^n \to \mathbb{R}^{2n-1}$ be a smooth map which is an embedding in a neighborhood of ∂M^n. Suppose that $\mathcal{L}_f(M^n) = 0$ if n is odd and $\mathcal{L}_f(M^n) \equiv 0$ mod 2 if n is even. Then there exists an immersion $g \colon M^n \to \mathbb{R}^{2n-1}$ that is arbitrarily C^0-close to f and equals f in a neighborhood of ∂M^n.*

Finally, we present an equivalent definition of the sign of a Whitney-umbrella singularity (motivated by [18]). With an additional choice we can also define the sign of a singular point if n is even.

Definition 3.2.12. Let $f \in L(M^n, \mathbb{R}^{2n-1})$ and $p \in S(f)$. Fix an orientation of $\mathbb{R}^{2n-1}$. Then we define the sign of p as follows.

Choose local coordinates $(x_1, \ldots, x_n)$ about p and $(y_1, \ldots, y_{2n-1})$ about $f(p)$ such that f is given near p by equation (3.1.1). Let $D_\varepsilon = \{ y_1^2 + \cdots + y_{2n-1}^2 \leq \varepsilon \}$ for $\varepsilon > 0$ sufficiently small. Then Lemma 2.2 in [18] states that $B_\varepsilon = f^{-1}(D_\varepsilon)$ is a closed disc neighborhood of p in M^n. Choose an arbitrary orientation of $S_\varepsilon = \partial B_\varepsilon$. Denote by q the double point of $f|S_\varepsilon$ and let $f^{-1}(q) = \{ q_1, q_2 \}$. Moreover, let $\underline{v} = (v_1, \ldots, v_{n-1})$ be a positive basis of $T_{q_1} S_\varepsilon$ and $\underline{w} = (w_1, \ldots, w_{n-1})$ a positive basis of $T_{q_2} S_\varepsilon$. Orient ∂D_ε by its outward normal vector. Then p is called positive if $(df(\underline{v}), df(\underline{w}))$ is a positive basis of $T_q(\partial D_\varepsilon)$, and negative otherwise.

This definition does not depend on the choice of the orientation of S_ε. However, if n is even then the sign of p does depend on the ordering of the points q_1 and q_2, i.e., if we swap the two points then the sign of p also changes. For n odd the sign of p is independent of the ordering of q_1 and q_2. Thus fixing the orientation of $\mathbb{R}^{2n-1}$ defines the sign of a Whitney-umbrella point only if n is odd. If we also fix an ordering of the two branches of f meeting at the open double curve ending at $f(p)$ then this defines a sign of p if n is even.

It is easy to verify that Definition 3.2.11 and Definition 3.2.12 are equivalent.

3.3 Proof of Theorem 3.1.3

3.3.1 Outline

The proof of the case $n = 2$ can be found in [11]. The generalizations of those ideas are incorporated into the present proof for the case $n > 3$.

The implication

$$f \sim g \Rightarrow |S(f)| = |S(g)|$$

is Proposition 2.4 in [11]. The proof relies on the fact that the cross-cap singularity is stable and M^n is closed. Thus we will only prove the other implication.

For $n > 3$ the proof is divided into two main parts. Using Hirsch-Smale theory (for a reference see [8]) we will see that there are two obstructions to constructing a regular homotopy between two immersions from M^n to $\mathbb{R}^{2n-1}$. These obstructions can be eliminated in the presence of cross-cap points.

In the first part we will construct a regular homotopy that pushes the $(n-1)$-simplices of M^n through a singular point to destroy the first obstruction. In the second part we will first reduce the problem to the case $M^n = S^n$. To remove the second obstruction we will merge the closed double curves of f and g with the ones connecting the singular points. This can be done using a variant of the Whitney-trick. Finally, we replace f and g with immersions and we use an argument of [3] that shows that the Smale-invariant of an immersion of $S^n \to \mathbb{R}^{2n-1}$ is completely determined by the geometry of its self-intersection if $n \geq 4$.

3.3.2 Setup

From now on f and g denote the maps in the statement of Theorem 3.1.3. Thus $f, g \in L'(M^n, \mathbb{R}^{2n-1})$ and $|S(f)| = |S(g)|$. We also suppose that $n > 3$ and $\mathbb{R}^{2n-1}$ is oriented.

Using the lemma of homogeneity and Corollary 3.2.20 there exists a diffeotopy $\{d_t \colon t \in [0,1]\}$ of M^n such that $d_0 = id_{M^n}$, moreover

- $d_1(S(g)) = S(f)$ if n is even,
- $d_1(S_+(g)) = S_+(f)$ and $d_1(S_-(g)) = S_-(f)$ if n is odd.

This implies that $S(f \circ d_1) = S(g)$, moreover $S_+(f \circ d_1) = S_+(g)$ and $S_-(f \circ d_1) = S_-(g)$ if n is odd. Since $\{f \circ d_t \colon t \in [0,1]\}$ provides a regular homotopy connecting f and $f \circ d_1$ we might suppose that $S(f) = S(g)$, moreover $S_+(f) = S_+(g)$ and $S_-(f) = S_-(g)$ if n is odd.

If n is even then any two cross-caps are locally equivalent. For n odd any two cross-caps of the same sign are locally equivalent. Thus there is a neighborhood U of $S(f) = S(g)$ such that f is regularly homotopic to g on U. This regular homotopy can be extended to M^n using Smale's lemma. Thus we may suppose that $f|U = g|U$.

If U is chosen small enough then $f|U = g|U$ is generic. We perturb f and g slightly outside U, using a regular homotopy, to obtain generic maps. Thus we can also suppose that $f, g \in G(M^n, \mathbb{R}^{2n-1})$.

Choose a smooth simplicial decomposition of M^n so fine that for any n-simplex Δ^n containing a point p of $S(f) = S(g)$ we have $\Delta^n \subset U$ and $p \in \mathrm{int}(\Delta^n)$. (The proof of the existence of a smooth simplicial decomposition can be found in [14].) Thus $f|\Delta^n = g|\Delta^n$ for each Δ^n as above. For $p \in S(f)$ we also choose $\Delta^n \ni p$ so small that $f|\Delta^n = g|\Delta^n$ has the canonical form (3.1.1) in a coordinate neighborhood containing Δ^n and centered at p.

The fiber of the bundle $\mathrm{MONO}(TM, T\mathbb{R}^{2n-1})$ is homeomorphic to the Stiefel manifold $V_{2n-1,n}$ of n-frames in $\mathbb{R}^{2n-1}$. It is well known that $\pi_i(V_{2n-1,n}) = 0$ for $i < n-1$.

Thus there exists an M^n-regular homotopy connecting the M^n- immersion

$$(f, df)|\mathrm{sk}_{n-2}(M^n) \text{ with } (g, dg)|\mathrm{sk}_{n-2}(M^n).$$

Using Smale's lemma we can extend this M^n-regular homotopy to the whole manifold M^n keeping $f|U$ fixed. So we might suppose that

$$(f, df)|\mathrm{sk}_{n-2}(M^n) = (g, dg)|\mathrm{sk}_{n-2}(M^n).$$

3.3.3 The first obstruction

Our next task is to find an M^n-regular homotopy connecting (f, df) with (g, dg) on $sk_{n-1}(M^n)$. The obstruction $\Omega(df, dg)$ to finding a regular homotopy connecting (f, df) with (g, dg) on an $(n-1)$-simplex Δ^{n-1}, fixing the boundary $\partial\Delta^{n-1}$, lies in

$$\pi_{n-1}(V_{2n-1,n}) \approx \begin{cases} \mathbb{Z}_2 & \text{if } n \text{ is even,} \\ \mathbb{Z} & \text{if } n \text{ is odd.} \end{cases}$$

We now prove that the sum of the obstructions on the boundary of an n-simplex on which both f and g are immersions vanishes.

Lemma 3.3.1. *Suppose that the C^∞ $\mathbb{R}^n$-immersions (f, f') and (g, g') of $\partial\Delta^n$ into $\mathbb{R}^q$ $(q > n)$ are tangent on $sk_{n-2}(\Delta^n)$. Denote the faces of Δ^n by*

$$\Delta_0^{n-1}, \ldots, \Delta_n^{n-1},$$

oriented by the outward normal vectors of Δ^n and let

$$(f_i, f_i') = (f, f')|\Delta_i^{n-1}, \quad (g_i, g_i') = (g, g')|\Delta_i^{n-1} \ (i = 0, \ldots, n).$$

Then

$$\sum_{i=0}^{n} \Omega(f_i', g_i') = \tau(f') - \tau(g').$$

Proof. We define a sequence of C^∞ $\mathbb{R}^n$-immersions

$$(h_i, h_i') \colon \partial\Delta^n \to \mathbb{R}^{2n-1} \ (i = 0, \ldots, n+1)$$

such that $(h_0, h_0') = (f, f')$, $(h_{n+1}, h_{n+1}') = (g, g')$ and for $0 < i < n+1$ let

$$(h_i, h_i')|\Delta_j^{n-1} = \begin{cases} (g_j, g_j') & \text{if } j < i, \\ (f_j, f_j') & \text{if } i \le j \le n. \end{cases}$$

Then (h_i, h_i') is C^∞ since (f, f') and (g, g') are tangent on $sk_{n-2}(\Delta^n)$. Applying Theorem 3.2.10 to (h_i, h_i') and (h_{i+1}, h_{i+1}') we get that

$$\Omega(f_i', g_i') = \tau(h_i') - \tau(h_{i+1}').$$

Summing up these equations we obtain the required result. □

Corollary 3.3.2. *Let $q > n$ and $f, g \in Imm(\Delta^n, \mathbb{R}^q)$. Suppose that $(f, f') = (f, df)|\partial\Delta^n$ and $(g, g') = (g, dg)|\partial\Delta^n$ are tangent on $sk_{n-2}(\Delta^n)$. Using the notations of Lemma 3.3.1 it holds that*

$$\sum_{i=0}^{n} \Omega(f'_i, g'_i) = 0.$$

Proof. Since both (f, f') and (g, g') are extendible we have that $\tau(f') = 0$ and $\tau(g') = 0$. Now Lemma 3.3.1 gives the statement of our corollary. □

Lemma 3.2.7 implies that there is a homeomorphism

$$\chi\colon \mathcal{I}'(n, 2n-1; n) \to \mathrm{Imm}_1(S^{n-1}, \mathbb{R}^{2n-1}).$$

Notation 3.3.3. *Let $Emb_1(S^k, \mathbb{R}^l)$ denote the space of pairs (h, ν), where the map $h\colon S^k \to \mathbb{R}^l$ is an embedding and ν is a transversal 1-field along h.*

Definition 3.3.1. Two elements of $\mathrm{Imm}_1(S^{n-1}, \mathbb{R}^{2n-1})$ (or $\mathcal{I}'(n, 2n-1; n)$) are called *regularly homotopic* if they lie in the same path-component of the corresponding space. A *regular homotopy* is a path in $\mathrm{Imm}_1(S^{n-1}, \mathbb{R}^{2n-1})$ (or $\mathcal{I}'(n, 2n-1; n)$).

Proposition 3.3.4. *For every immersion $(f, \nu) \in Imm_1(S^{n-1}, \mathbb{R}^{2n-1})$ there exists an embedding $(h, \mu) \in Emb_1(S^{n-1}, \mathbb{R}^{2n-1})$ such that (f, ν) is regularly homotopic to (h, μ).*

Proof. The subspace of embeddings is dense open in $\mathrm{Imm}(S^{n-1}, \mathbb{R}^{2n-1})$. Thus we can perturb f by a regular homotopy to get an embedding h. If the perturbation is sufficiently small then ν can be deformed simultaneously as a transversal field along the regular homotopy. □

Definition 3.3.2. Suppose that $\mathbb{R}^{2n-1}$ is oriented. Let

$$\mathrm{lk}\colon \mathrm{Emb}_1(S^{n-1}, \mathbb{R}^{2n-1}) \to \begin{cases} \mathbb{Z} & \text{if } n \text{ is odd,} \\ \mathbb{Z}_2 & \text{if } n \text{ is even,} \end{cases}$$

be defined as follows: For $(f, \nu) \in \mathrm{Emb}_1(S^{n-1}, \mathbb{R}^{2n-1})$ choose $\varepsilon > 0$ sufficiently small so that $f + \varepsilon\nu$ is an embedding. Then let $\mathrm{lk}(f, \nu)$ be accordingly the $\mathbb{Z}$ or $\mathbb{Z}_2$ linking number $\mathrm{lk}(f, f + \varepsilon\nu)$.

Proposition 3.3.5. *lk is a regular homotopy invariant, i.e., if the maps $(f, \nu), (h, \mu) \in Emb_1(S^{n-1}, \mathbb{R}^{2n-1})$ are regularly homotopic then the equality $lk(f, \nu) = lk(h, \mu)$ holds.*

Proof. This is exactly Lemma 9 in [28]. □

Now we can extend the definition of lk from embeddings to immersions.

Definition 3.3.3. Suppose that $\mathbb{R}^{2n-1}$ is oriented. Let

$$\mathrm{lk}\colon \mathrm{Imm}_1(S^{n-1}, \mathbb{R}^{2n-1}) \to \begin{cases} \mathbb{Z} & \text{if } n \text{ is odd,} \\ \mathbb{Z}_2 & \text{if } n \text{ is even,} \end{cases}$$

be defined as follows: For $(f, \nu) \in \mathrm{Imm}_1(S^{n-1}, \mathbb{R}^{2n-1})$ choose an embedding (h, μ) in $\mathrm{Emb}_1(S^{n-1}, \mathbb{R}^{2n-1})$ regularly homotopic to (f, ν) (such an embedding exists because of Proposition 3.3.4). Then let $\mathrm{lk}(f, \nu) = \mathrm{lk}(h, \mu)$. This definition does not depend on the choice of (h, μ) because of Proposition 3.3.5.

Proposition 3.3.6. *The function lk defined above is a regular homotopy invariant.*

Proof. This is clear from Proposition 3.3.5 and Definition 3.3.3. $\square$

Notation 3.3.7. *Let* $I'_n = \pi_0(\mathcal{I}'(n, 2n-1; n))$ *and introduce the notation* $J'_n = \pi_0(\mathit{Imm}_1(S^{n-1}, \mathbb{R}^{2n-1}))$. *Proposition 3.3.6 implies that lk defines a function on* J'_n, *which we denote by* lk_*.

Lemma 3.3.8. *Let* $f \in L(D^n, \mathbb{R}^{2n-1})$ *and*

$$(f, f') = (f, df)|S^{n-1} \in \mathcal{I}'(n, 2n-1; n).$$

Then

$$lk(\chi(f, f')) = \#S(f).$$

(For the definition of χ *see Lemma 3.2.7.)*

Proof. Perturb f using a regular homotopy to get a generic map h (thus $h|S^{n-1}$ is an embedding). Let $(h, h') = (h, dh)|S^{n-1}$. Proposition 3.3.6 implies that $\mathrm{lk}(\chi(f, f')) = \mathrm{lk}(\chi(h, h'))$. Moreover, the number of singular points is also a regular homotopy invariant, thus $\#S(f) = \#S(h)$. By definition $\mathrm{lk}(\chi(h, h')) = \mathcal{L}_h(D^n)$. On the other hand Lemma 3.2.19 implies that $\mathcal{L}_h(D^n) = \#S(h)$. $\square$

Corollary 3.3.9. *Let* $(f, \nu) \in \mathit{Imm}_1(S^{n-1}, \mathbb{R}^{2n-1})$. *Then*

$$lk(f, -\nu) = -lk(f, \nu).$$

Proof. Choose two generic maps $f_0, f_1 \in G(D^n, \mathbb{R}^{2n-1})$ such that

$$\chi((f_i, df_i)|S^{n-1}) = (f, (-1)^i \nu)$$

for $i = 0, 1$. Then f_0 and f_1 fit together to define a map $h \in G(S^n, \mathbb{R}^{2n-1})$. Lemma 3.2.19 implies that $\#S(f_0) + \#S(f_1) = \#S(h) = 0$. Moreover, Lemma 3.3.8 implies that $\mathrm{lk}(f, \nu) = \#S(f_0)$ and $\mathrm{lk}(f, -\nu) = \#S(f_1)$. Putting together these results we get that $\mathrm{lk}(f, \nu) + \mathrm{lk}(f, -\nu) = 0$, as required. $\square$

Proposition 3.3.10. τ *(see Definition 3.2.6) is a regular homotopy invariant. Moreover, it induces a bijection* $\tau_*\colon I'_n \to \pi_{n-1}(V_{2n-1,n})$.

Proof. A regular homotopy of a map $(f, f') \in \mathcal{I}'(n, 2n-1; n)$ induces a homotopy of the map $S^{n-1} \to V_{q,n}$ defined by

$$x \mapsto f'\{e_1(x), \dots, e_n(x)\}.$$

Thus the homotopy class $\tau(f')$ is constant on the regular homotopy class of (f, f').

Hirsch's theorem (Theorem 3.2.11) implies that

$$\tau_*\colon \pi_0(\mathrm{Imm}_1(S^{n-1}, \mathbb{R}^{2n-1})) \to \pi_0(\mathrm{MONO}(TS^{n-1} \oplus \varepsilon^1, \mathbb{R}^{2n-1})) \approx \\ \approx \pi_{n-1}(V_{2n-1,n})$$

is a bijection (the bundle $TS^{n-1} \oplus \varepsilon^1$ is trivial). □

Definition 3.3.4. Define the *connected sum* operation # on elements x and y of $\mathrm{Imm}_1(S^{n-1}, \mathbb{R}^{2n-1})$ or $\mathcal{I}'(n, 2n-1; n)$ by joining them with a thin tube. This operation is also well defined on regular homotopy classes of maps: Suppose that $x \sim_H x_1$ and $y \sim_K y_1$. Smale's lemma implies that the regular homotopies H and K may be kept fixed on small disks. Now a regular homotopy L connecting $x\#x_1$ with $y\#y_1$ is defined by taking H and K, and joining them with a tube attached to the disks kept fixed.

Proposition 3.3.11. *The sets I'_n and J'_n (introduced in Notation 3.3.7) endowed with the connected sum operation form abelian groups. Moreover, the maps $\chi_*\colon I'_n \to J'_n$ and τ_* are group isomorphisms and lk_* is a group homomorphism.*

Proof. It is clear that $(I'_n, \#)$ and $(J'_n, \#)$ are abelian semigroups. Let $i\colon D^n \to \mathbb{R}^{2n-1}$ denote the standard embedding. Then $(i, i') = (i, di)|S^{n-1}$ (respectively $\chi((i, di)|S^{n-1}) = (i, r)$, where r is the outward normal field of S^{n-1} in $\mathbb{R}^n$) represents the identity of I'_n (respectively J'_n).

If $x, y \in I'_n$ then choose a representative (f, f') for x and (g, g') for y. Then $\tau(f'\#g')$ is the connected sum of the spheroids $\tau(f')$ and $\tau(g')$, which implies that $\tau_*(x\#y) = \tau_*(x) + \tau_*(y)$. Since (i, i') is extendible, $\tau_*(i') = 0$. Proposition 3.3.10 states that τ_* is a bijection. Thus τ_* is a semigroup isomorphism that takes the identity of I'_n to the identity of $\pi_{n-1}(V_{2n-1,n}) \approx \mathbb{Z}$ or $\mathbb{Z}_2$. This implies that I'_n is a group isomorphic to $\mathbb{Z}$ or $\mathbb{Z}_2$ (depending on the parity of n) and τ_* is a group isomorphism.

Since χ is a homeomorphism, $\chi_*\colon I'_n \to J'_n$ is a bijection. Moreover, from the definition of χ it is easy to see that χ_* is a semigroup homomorphism. We saw above that χ_* takes the identity (i, i') of I'_n to the identity (i, r) of J'_n. Thus J'_n is also a group isomorphic to $\mathbb{Z}$ or $\mathbb{Z}_2$ and χ_* is a group isomorphism.

We finally show that lk_* is a group homomorphism from J'_n to $\mathbb{Z}$ or $\mathbb{Z}_2$. Suppose that $x, y \in J'_n$ and choose representatives (f_1, ν_1) and (f_2, ν_2) for x, respectively y, such that f_1 and f_2 can be separated by a hyperplane in $\mathbb{R}^{2n-1}$. Also choose $h_1, h_2 \in G(D^n, \mathbb{R}^{2n-1})$ so that $\chi((h_i, dh_i)|S^{n-1}) = (f_i, \nu_i)$ for $i = 1, 2$. Denote the boundary sum $h_1 \natural h_2$ with h. Then $\chi((h, dh)|S^{n-1}) = (f_1, \nu_1)\#(f_2, \nu_2)$ and $\#S(h) = \#S(h_1) + \#S(h_2)$. Using Lemma 3.3.8 we get that $\mathrm{lk}((f_1,\nu_1)\#(f_2,\nu_2)) = \mathrm{lk}(f_1,\nu_1)+\mathrm{lk}(f_2,\nu_2)$, i.e., $\mathrm{lk}(x\#y) = \mathrm{lk}(x) + \mathrm{lk}(y)$. □

The following lemma is the key to connecting the works of Hirsch and Whitney.

Lemma 3.3.12. *There exists an isomorphism*

$$\alpha\colon \pi_{n-1}(V_{2n-1,n}) \to \begin{cases} \mathbb{Z} & \text{if } n \text{ is odd,} \\ \mathbb{Z}_2 & \text{if } n \text{ is even,} \end{cases}$$

such that

$$lk \circ \chi = \alpha \circ \tau.$$

I.e., the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{I}'(n,2n-1;n) & \xrightarrow{\chi} & Imm_1(S^{n-1},\mathbb{R}^{2n-1}) \\ \tau\big\downarrow & & lk\big\downarrow \\ \pi_{n-1}(V_{2n-1,n}) & \xrightarrow{\alpha} & \mathbb{Z} \text{ or } \mathbb{Z}_2. \end{array}$$

Remark 3.3.1. If n is even then α might be omitted from the formula since $\mathbb{Z}_2$ has only one automorphism.

Proof. We saw that τ and lk are regular homotopy invariants, i.e., they are constant on the path-components of their domains. They define maps

$$\tau_*\colon I'_n = \pi_0(\mathcal{I}'(n,2n-1;n)) \to \pi_{n-1}(V_{2n-1,n})$$

and

$$\mathrm{lk}_*\colon J'_n = \pi_0(\mathrm{Imm}_1(S^{n-1},\mathbb{R}^{2n-1})) \to \begin{cases} \mathbb{Z} & \text{if } n \text{ is odd,} \\ \mathbb{Z}_2 & \text{if } n \text{ is even.} \end{cases}$$

Proposition 3.3.11 implies that I'_n and J'_n are abelian groups with the connected sum operation, moreover, χ_*, τ_* are isomorphisms and lk_* is a homomorphism. We also know that I'_n and J'_n are isomorphic to $\mathbb{Z}$ or $\mathbb{Z}_2$, according to the parity of n.

lk_* is an epimorphism because of Lemma 3.2.15:

$$\mathrm{lk}(\chi(w,w')) = \pm 1.$$

So lk_* is a $\mathbb{Z}\to\mathbb{Z}$ or $\mathbb{Z}_2\to\mathbb{Z}_2$ epimorphism, and thus it is also an isomorphism. Now define α to be $\mathrm{lk}_*\circ\chi_*\circ(\tau_*)^{-1}$. □

From Theorem 3.2.10 and Lemma 3.3.12 we get the following

Proposition 3.3.13. *Suppose that $(f,\nu),(g,\mu) \in Imm_1(\partial\Delta^n,\mathbb{R}^q)$ agree on $(\partial\Delta^n)\setminus int(\Delta^{n-1})$, where Δ^{n-1} is an $(n-1)$-face of the n-simplex Δ^n. Then for $(f,f') = (\chi^{-1}(f,\nu))|\Delta^{n-1}$ and $(g,g') = (\chi^{-1}(g,\mu))|\Delta^{n-1}$*

$$\alpha(\Omega(f',g')) = lk(f,\nu) - lk(g,\mu).$$

Remark 3.3.2. Δ^{n-1} is co-oriented by the outward normal vector of Δ^n. Then

$$(\chi^{-1}(f,\nu))|\Delta^{n-1} = \zeta^{-1}((f,\nu)|\Delta^{n-1}),$$

where ζ is the homeomorphism of Lemma 3.2.5. Thus $\mathrm{lk}(f,\nu) - \mathrm{lk}(g,\mu)$ depends only on $(f,\nu)|\Delta^{n-1}$ and $(g,\mu)|\Delta^{n-1}$.

Definition 3.3.5. For $(f,f'),(g,g') \in \mathcal{I}(n-1,2n-1;n)$ that are tangent on $\partial\Delta^n$ let

$$o(f',g') = \alpha(\Omega(f',g')).$$

Corollary 3.3.14. *Let Δ^{n-1} be a face of the n-simplex Δ^n and let (f,f') and (g,g') be C^∞ $\mathbb{R}^n$-immersions of Δ^{n-1} into $\mathbb{R}^{2n-1}$ tangent on $\partial\Delta^{n-1}$. Choose extensions of (f,f') and (g,g') to $\partial\Delta^n$ that are tangent on $\partial\Delta^n \setminus \Delta^{n-1}$. Then*

$$o(f',g') = lk \circ \chi(f,f') - lk \circ \chi(g,g').$$

Lemma 3.3.15. *Let (f,f') and (g,g') be as in Corollary 3.3.14 and (w,w') be the boundary of a Whitney-umbrella of sign ε (as in Notation 3.2.16). Then*

$$o(f'\#w',g') = o(f',g') + \varepsilon.$$

Proof. Using extensions as in Corollary 3.3.14 we get that

$$o(f'\#w',g') = \mathrm{lk} \circ \chi((f,f')\#(w,w')) - \mathrm{lk} \circ \chi(g,g').$$

Using the fact that lk and χ are additive we get that this equals

$$\mathrm{lk} \circ \chi(f,f') + \mathrm{lk} \circ \chi(w,w') - \mathrm{lk} \circ \chi(g,g').$$

The definition of the sign of an umbrella implies that $\mathrm{lk} \circ \chi(w,w') = \varepsilon$. □

Proposition 3.3.16. *Let (f,f') and (g,g') be as in Corollary 3.3.14. Then (f,f') and (g,g') are regularly homotopic (rel $\partial\Delta^{n-1}$) iff $o(f',g') = 0$.*

Proof. This is a trivial consequence of Lemma 3.2.9. □

Remark 3.3.3. Lemma 3.3.15 and Proposition 3.3.16 show us how Whitney-umbrellas destroy obstructions to moving f to g on an $(n-1)$-simplex (rel boundary). We have to take the connected sum of f with several copies of the boundary of an umbrella. This will be done by a diffeotopy of M^n that pushes the $(n-1)$-simplex through a singular point.

Lemma 3.3.17. *Using the notations of Lemma 3.3.1*

$$\sum_{i=0}^{n} o(f_i',g_i') = lk \circ \chi(f,f') - lk \circ \chi(g,g').$$

In particular, if (f,f') and (g,g') are both extendible then

$$\sum_{i=0}^{n} o(f_i',g_i') = 0.$$

Proof. This is a trivial consequence of Lemma 3.3.1. □

Remark 3.3.4. Lemma 3.3.17 shows us that the obstruction o has the cocycle property, i.e., the sum of the obstructions on the boundary of an n-simplex, on which both f and g are immersions, vanishes.

3.3.4 Pushing (n-1)-simplices through Whitney-umbrellas

Now we are going to use the apparatus developed above for the proof of Theorem 3.1.3. Since until now we only worked in a standard simplex in Euclidean space, we have to globalize our results to the triangulated manifold M^n. We do *not* require M^n to be oriented.

Definition 3.3.6. Suppose that Δ^n is an n-simplex of M^n. Then we can define the map $\chi = \chi_{\Delta^n}$ taking M^n-immersions of $\partial\Delta^n$ to $\mathrm{Imm}_1(\partial\Delta^n, \mathbb{R}^{2n-1})$ using the outward normal vectors $r(x)$ of $\partial\Delta^n$ (corners rounded off) as follows: Let (f, f') be an M^n-immersion of $\partial\Delta^n$ into $\mathbb{R}^{2n-1}$. Then let $\chi(f, f') = (f, \nu)$, where $\nu(x) = f'(r(x))$.

Definition 3.3.7. Suppose that Δ^{n-1} is an $(n-1)$-simplex of M^n that is co-oriented. Then we can define the map $\zeta = \zeta_{\Delta^{n-1}}$ taking M^n-immersions of Δ^{n-1} to $\mathrm{Imm}_1(\Delta^{n-1}, \mathbb{R}^{2n-1})$ as follows: Let $r(x)$ be a positive normal field along Δ^{n-1}. If (f, f') is an M^n-immersion of Δ^{n-1} into $\mathbb{R}^{2n-1}$ then let $\zeta(f, f') = (f, \nu)$, where $\nu(x) = f'(r(x))$.

Corollary 3.3.14 implies that the following definition makes sense:

Definition 3.3.8. Suppose that Δ^{n-1} is an $(n-1)$-simplex of M^n that is co-oriented and let (f, f') and (g, g') be M^n-immersions of Δ^{n-1} into $\mathbb{R}^{2n-1}$ tangent on $\partial\Delta^{n-1}$. Then the obstruction $o(f', g')$ to finding a regular homotopy of (f, f') to (g, g') (rel $\partial\Delta^{n-1}$) is given as follows: Choose an n-simplex Δ^n of M^n such that $\Delta^{n-1} \subset \partial\Delta^n$. Extend (f, f') and (g, g') to $\partial\Delta^n$ to be tangent on $\partial\Delta^n \setminus \Delta^{n-1}$. Also choose a normal field $s(x)$ along $\partial\Delta^n$ that agrees with the co-orientation of Δ^{n-1}. Now define $o(f', g')$ to be $\mathrm{lk}(f, f' \circ s) - \mathrm{lk}(g, g' \circ s)$.

Remark 3.3.5. If we change the co-orientation of Δ^{n-1} then $o(f', g')$ changes sign: To see this we choose the same extensions of (f, f') and (g, g') to the boundary $\partial\Delta^n$ of the same n-simplex Δ^n. Then we must take $-s$ instead of s because of the changed co-orientation of Δ^n. Now using Corollary 3.3.9 we get that

$$\mathrm{lk}(f, f' \circ (-s)) - \mathrm{lk}(g, g' \circ (-s)) = -\mathrm{lk}(f, f' \circ s)) + \mathrm{lk}(g, g' \circ s) = -o(f', g').$$

Let f and g be the two locally generic maps of Theorem 3.1.3. We have supposed that (f, df) and (g, dg) are tangent on $\mathrm{sk}_{n-2}M^n$. If n is odd choose a co-orientation $O_{\Delta^{n-1}}$ for every $(n-1)$-simplex Δ^{n-1} of M^n so that $o(df, dg) \leq 0$. This can be done because of Remark 3.3.5. Let $k = |S(f)| = |S(g)|$ and denote by $\Delta_1^n, \ldots, \Delta_k^n$ the n-simplices of M^n that contain any point of $S(f) = S(g)$.

We are now going to construct an oriented curve γ on M^n that intersects each simplex Δ^{n-1} of M^n in the positive direction (according to $O_{\Delta^{n-1}}$) in exactly $|o(df, dg)|$ points. Notice that if $\Delta^{n-1} \subset \partial\Delta_i^n$ for some $1 \leq i \leq k$ then on Δ^{n-1} we have $o(df, dg) = 0$ since $f|\Delta_i^n = g|\Delta_i^n$. Thus γ avoids the simplices Δ_i^n. This γ may be thought of as the dual of the obstruction to deform f to g on the $(n-1)$-skeleton of M^n.

To obtain γ choose a set of $|o(df, dg)|$ points $P_{\Delta^{n-1}}$ on each $(n-1)$-simplex of M^n. Now take an n-simplex Δ^n and denote its faces by $\Delta_0^{n-1}, \ldots, \Delta_n^{n-1}$, moreover let $(f_i, f_i') = (f, df)|\Delta_i^{n-1}$, $(g_i, g_i') = (g, dg)|\Delta_i^{n-1}$, $O_i = O_{\Delta_i^{n-1}}$ and $P_i = P_{\Delta_i^{n-1}}$ for $0 \leq i \leq n$. Lemma 3.3.17 implies that if each Δ_i^{n-1} is co-oriented by the outward normal vector of Δ^n (denote this co-orientation by U_i) then

$$\sum_{i=0}^{n} o(f_i', g_i') = 0. \tag{3.3.1}$$

If we consider Δ_i^{n-1} with the co-orientation U_i then $o(f_i', g_i') < 0$ implies that $U_i = O_i$ and $o(f_i', g_i') > 0$ implies that $U_i = -O_i$. Give a minus sign to each point of P_i if $U_i = O_i$ and a plus sign otherwise. Then equation (3.3.1) is equivalent to the statement that the sum of the signs in $\bigcup_{i=0}^n P_i$ equals 0. Now let $\gamma \cap \Delta^n$ be given as follows: Make a bijection between the $+$ and $-$ points of $\bigcup_{i=0}^n P_i$ and connect each pair of points with an embedded curve segment oriented from $+$ to $-$. We do this so that these curve segments are pairwise disjoint. This is possible since $n \geq 3$.

Doing this for each n-simplex we obtain an oriented embedded curve γ with the required intersection property. Now we make γ connected by taking the connected sum of its components: Let for example γ_1 and γ_2 be two components of γ. Then choose two points $p_1 \in \gamma_1$ and $p_2 \in \gamma_2$. Join p_1 and p_2 with an embedded curve η that avoids Δ_i^n for $1 \leq i \leq k$ and also each $(n-2)$-simplex and intersects each $(n-1)$-simplex transversally. Then take two parallel curves η_1 and η_2 close to η and orient them according to the orientations of γ_1 and γ_2. If η intersects an $(n-1)$-simplex Δ^{n-1} at a point x then η_1 and η_2 will intersect Δ^{n-1} in different directions near x.

Similarly, we might modify γ so that it goes through exactly one point p of $S(f)$, and let the Whitney-umbrella of f at p be positive if n is odd. To do this, choose a small embedded curve containing p and join it to γ as above. We will also call this modified curve γ. Let us suppose that the n-simplex containing p is Δ_1^n.

Thus it will hold true that the connected embedded curve γ intersects *algebraically* each $(n-1)$-simplex Δ^{n-1} in $|o(df, dg)|$ points if we consider Δ^{n-1} with the co-orientation $O_{\Delta^{n-1}}$. Moreover, γ contains the cross-cap point $p \in \Delta_1^n$ and is disjoint from Δ_i^n for $1 < i \leq k$.

Let ν_γ be a thin tubular neighborhood of γ in M^n. We are going to construct a diffeotopy $\{\Psi_t \colon 0 \leq t \leq 1\}$ of ν_γ such that $\Psi_0 = id_{\nu_\gamma}$ and for every $t \in [0, 1]$ the diffeomorphism Ψ_t is the identity in a neighborhood of $\partial\nu_\gamma$. Thus we can extend Ψ_t to the whole manifold M^n to be the identity outside ν_γ.

The diffeomorphism Ψ_t is constructed as follows: Denote by T the identity of $\mathbb{R}^{n-1}$ if γ is orientation preserving, and let T be a reflection in a hyperplane of $\mathbb{R}^{n-1}$ if γ is orientation reversing. Then ν_γ is diffeomorphic to the factor space

$$\Gamma = \mathbb{R} \times D_2^{n-1} \big/ _{(x,y)\sim(x+1,T(y))} ,$$

where $D_2^{n-1} \subset \mathbb{R}^{n-1}$ is the disc of radius 2. Denote by $p\colon \mathbb{R} \times D_2^{n-1} \to \Gamma$ the projection. Let $\lambda\colon \mathbb{R} \to \mathbb{R}$ be a C^∞ function such that $\lambda(x) = 0$ for $x < \varepsilon$ and $\lambda(x) = 1$ for $x > 1 - \varepsilon$ (where $\varepsilon < 1$ is a small positive constant). First we define a diffeotopy $\{\phi_t\colon 0 \le t \le 1\}$ of $\mathbb{R} \times D_2^{n-1}$ with the formula

$$\phi_t(x,y) = \begin{cases} (x+t, y) & \text{if } \|y\| \le 1, \\ (x + \lambda(s)t, y) & \text{if } \|y\| = 2 - s. \end{cases}$$

$(x,y) \sim (x+1, T(y))$ implies that $(x+c, y) \sim (x+c+1, T(y))$ for any number c. Thus the diffeomorphism ϕ_t factors through the projection p to a diffeomorphism Ψ_t of $\Gamma \approx \nu_\gamma$. I.e., the following diagram is commutative:

$$\begin{array}{ccc} \mathbb{R} \times D_2^{n-1} & \xrightarrow{\phi_t} & \mathbb{R} \times D_2^{n-1} \\ {\scriptstyle p}\big\downarrow & & {\scriptstyle p}\big\downarrow \\ \nu_\gamma & \xrightarrow{\Psi_t} & \nu_\gamma . \end{array}$$

Since Ψ_1 is the identity on $p(\mathbb{R} \times D_1^{n-1})$, the diffeomorphism Ψ_1 is the identity on a thinner tubular neighborhood of γ. Thus $\Psi_1(p) = p$. Moreover, let F denote a fiber of ν_γ (diffeomorphic to D^{n-1}) with a normal framing v (in M^n) in the direction determined by γ; and S a small sphere around p contained in ν_γ with the outward normal framing r. Then the framed submanifold $(\Psi_1(F), d\Psi_1(v))$ is equal to the connected sum $(F, v)\#(S, r)$. On the other hand, the framed submanifold $(\Psi_1(F), d\Psi_1(-v)) = (F, -v)\#(S, -r)$ (see Figure 3.2).

Now look at the regular homotopy $f \circ \Psi_t$ connecting f with $h = f \circ \Psi_1$. Then $S(h) = S(f)$, moreover $S_\pm(h) = S_\pm(f)$ if n is odd. We also get that

$$(h, dh)|F = ((f, df)|F)\#((f, df)|S).$$

Notice that $(f, df)|S$ is the M^n-immersion (w, w') of Notation 3.2.16 and if we co-orient S by r then $\mathrm{lk} \circ \chi(w, w') = 1$ since the sign of p is 1. Thus if we co-orient F by v, as above, then Lemma 3.3.15 gives that $o(dh|F, df|F) = 1$. On the other hand, if we co-orient F by $-v$ then $o(dh|F, df|F) = -1$ (see Figure 3.2). So if we take an $(n-1)$-simplex Δ^{n-1} of M^n then $o(dh|\Delta^{n-1}, df|\Delta^{n-1})$ is the algebraic intersection of Δ^{n-1} and γ. From the construction of γ we know that

$$\Delta^{n-1} \cap \gamma = \left| o(df|\Delta^{n-1}, dg|\Delta^{n-1}) \right| .$$

Thus on Δ^{n-1}

$$o(dh, dg) = o(dh, df) + o(df, dg) = |o(df, dg)| + o(df, dg) = 0,$$

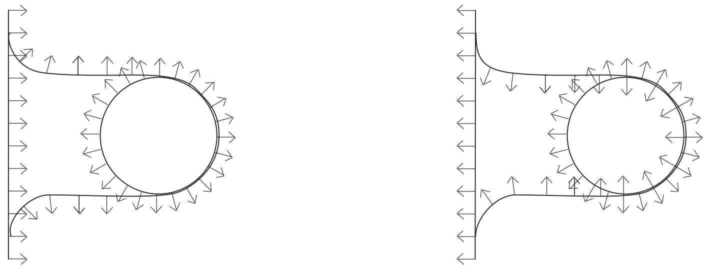

Figure 3.2: Co-orientations and the connected sum

since the co-orientation $O_{\Delta^{n-1}}$ was chosen so that $o(df, dg) \leq 0$.

Consequently, there is no obstruction to finding an M^n-regular homotopy between h and g on the $(n-1)$-skeleton of M^n. Since f is regularly homotopic to h, we suppose from now on that $f = h$. Therefore we can suppose that the maps f and g coincide on (a neighborhood of) $\mathrm{sk}_{n-1}(M^n)$.

3.3.5 The second obstruction

Let us recall that $\Delta_1^n, \ldots, \Delta_k^n$ denote those n-simplices of M^n which contain a singular point of f, where $k = 2l$ is even. (Each $\mathrm{int}(\Delta_i^n)$ contains a single Whitney umbrella point. If n is odd then $\Delta_1^n, \ldots, \Delta_l^n$ contain a positive Whitney umbrella, $\Delta_{l+1}^n, \ldots, \Delta_{2l}^n$ a negative one for both f and g. Moreover, $f|\Delta_i^n = g|\Delta_i^n$ for $1 \leq i \leq k$.) Let

$$U = \bigcup_{i=1}^{k} \mathrm{int}(\Delta_i^n).$$

Take the manifold with boundary $N^n = M^n \setminus U$ and let $\hat{f} = f|N^n$ and $\hat{g} = g|N^n$. We know from the previous section that $\hat{f}$ and $\hat{g}$ are regularly homotopic on $\mathrm{sk}_{n-1}(N^n)$. The secondary obstruction to finding a regular homotopy between $\hat{f}$ and $\hat{g}$, not fixing the boundary, lies in the cohomology group $H^n(N^n; \pi_n(V_{2n-1,n}))$ with twisted coefficients. However, this group is 0 since an n-manifold without closed components is homotopy equivalent to an $(n-1)$-dimensional simplicial complex.

So there exists a regular homotopy connecting $\hat{f}$ with $\hat{g}$, but not necessarily fixing ∂N^n. Using Smale's lemma we can extend this regular homotopy to the whole manifold M^n, fixing a small neighborhood V $(\subset U)$ of $S(f) = S(g)$. (During this regular homotopy on the "collar" $\overline{U} \setminus V$ the map f might become twisted, i.e., it may differ from g.)

Hence we might suppose that $f|N^n = g|N^n$ and $f|V = g|V$, but it might happen that $f|(U \setminus V) \neq g|(U \setminus V)$. If n is odd then $|S_+(f)| = |S_+(g)| = l$, let us suppose that

$$S_+(f) = S_+(g) \subset \bigcup_{1 \leq i \leq l} \Delta_i^n,$$

while

$$S_-(f) = S_-(g) \subset \bigcup_{l+1 \le i \le k} \Delta_i^n.$$

Since up to this point we have not used the assumption $n > 3$, we can formulate and prove the following result that is weaker than Theorem 3.1.3 and will not be used in its proof, but will be needed in dealing with the exceptional case $n = 3$.

Theorem 3.3.18. *Suppose that $n > 2$ and let M^n be a closed n-manifold. Moreover, let $f, g \in L'(M^n, \mathbb{R}^{2n-1})$ be two locally generic maps that satisfy $|S(f)| = |S(g)|$. Then there exists a map $f' \in L'(M^n, \mathbb{R}^{2n-1})$ and a subset $D \subset M^n$ diffeomorphic to the disk D^n such that $f \sim f'$, the maps $f'|D$ and $g|D$ are immersions, and $f'|(M^n \setminus D) = g|(M^n \setminus D)$.*

Proof. Let U and V be as above (where $S(g) \subset V \subset U$). I.e., U is the union of the interiors of the simplices Δ_i^n for $1 \le i \le k$ (where $k = |S(f)| = |S(g)|$) and f is regularly homotopic to a map f' such that $f'|(M^n \setminus U) = g|(M^n \setminus U)$ and $f'|V = g|V$. We can also suppose that $V_i = \Delta_i^n \cap V$ is diffeomorphic to an n-disc for $1 \le i \le k$. Let us take a small neighborhood U_i of Δ_i^n that is diffeomorphic to D^n. From now on we will denote by U the union $\cup_{i=1}^k U_i$. For each i choose a 1-simplex $e_i \subset U_i \setminus V_i$ connecting a point of ∂U_i with a point of ∂V_i and let E_i be a thin tubular neighborhood of e_i. Then $D_i = U_i \setminus (V_i \cup E_i)$ is diffeomorphic to D^n. Since $n > 2$ there is no obstruction to constructing a regular homotopy between $f'|E_i$ and $g|E_i$ fixing the ends of the tube E_i. Thus (using Smale's lemma) we can suppose that $f'|E_i = g|E_i$ for $1 \le i \le k$. Now we connect D_i and D_{i+1} for $1 \le i \le k-1$ with a tube $T_i \subset M^n \setminus U$ diffeomorphic to $I \times D^{n-1}$. (Thus each T_i is an n-dimensional 1-handle attached to D_i and D_{i+1}.) Let

$$D = \left(\bigcup_{i=1}^k D_i \right) \cup \left(\bigcup_{j=1}^{k-1} T_j \right).$$

Then D is diffeomorphic to D^n and clearly $f'|D$ and $g|D$ are immersions, finally $f'|(M^n \setminus D) = g|(M^n \setminus D)$. □

Let $F_1, \dots, F_l$ be pairwise disjoint subsets of M^n such that F_i is diffeomorphic to the closed n-disc D^n, moreover F_i contains Δ_i^n and Δ_{l+i}^n in its interior for $1 \le i \le l$. We will show that for $1 \le i \le l$ the locally generic maps $f_i = f|F_i$ and $g_i = g|F_i$ are regularly homotopic keeping a neighborhood of ∂F_i fixed. This would finish the proof of Theorem 3.1.3. From now on let $1 \le i \le l$ be fixed.

It is apparent from the choice of F_i that $S(f_i) = S(g_i)$ and $|S(f_i)| = |S(g_i)| = 2$. Moreover, if n is odd we have that $S_+(f_i) = S_+(g_i)$ and $S_-(f_i) = S_-(g_i)$ are both 1-element sets. Thus $\#S(f_i) = 0$ and $\#S(g_i) = 0$. Lemma 3.2.19 implies that $\mathcal{L}_{f_i}(F_i) = 0$ and $\mathcal{L}_{g_i}(F_i) = 0$. Using Lemma 3.3.12 we get that

$$0 = \mathcal{L}_{f_i}(F_i) = \operatorname{lk} \circ \chi((f_i, df_i)|\partial F_i) = \alpha \circ \tau((f_i, df_i)|\partial F_i).$$

Since α is an isomorphism, $\tau((f_i, df_i)|\partial F_i) = 0$ and from Lemma 3.2.8 we can see that the map $(f_i, df_i)|\partial F_i$ is extendible. Similarly, $(g_i, dg_i)|\partial F_i$ is also extendible. (This also follows from Theorem 3.2.21.)

Let S^n_+ (respectively S^n_-) denote the northern (respectively southern) hemisphere of S^n and we identify S^n_+ with F_i. Then we get from the previous paragraph that there exist two locally generic maps $f'_i, g'_i \in L(S^n, \mathbb{R}^{2n-1})$ such that $f'_i|S^n_+ = f_i$, $g'_i|S^n_+ = g_i$, moreover $f'_i|S^n_- = g'_i|S^n_-$ is an immersion. Thus it is sufficient to prove the following theorem:

Theorem 3.3.19. *Let $f, g \in L(S^n, \mathbb{R}^{2n-1})$ be locally generic maps and suppose that $f|S^n_- = g|S^n_-$ are immersions, $|S(f)| = |S(g)| = 2$ and $S(f) = S(g)$, moreover $S_+(f) = S_+(g)$ if n is odd. Then there exists a singularity fixing regular homotopy connecting f and g that is fixed on S^n_-, i.e., $f \sim_s g$ (rel S^n_-).*

We will use the following lemma in the proof of Theorem 3.3.19:

Lemma 3.3.20. *Let $f, g\colon S^n \to \mathbb{R}^{2n-1}$ be locally generic maps such that $S(f) = S(g) \subset int(S^n_+)$ and $f|S^n_- = g|S^n_-$ are immersions. If $f \sim_s g$ then $f \sim_s g$ (rel S^n_-).*

Proof. Choose a point $x \in \partial S^n_-$ and denote the differential $(df)_x = (dg)_x$ by j. Let $S = S(f) = S(g)$ and

$$\mathcal{B} = \{\, h \in \mathrm{Imm}(S^n_-, \mathbb{R}^{2n-1})\colon (dh)_x = j \,\},$$

moreover

$$\mathcal{E} = \{\, h \in L(S^n, \mathbb{R}^{2n-1})\colon S(h) = S \text{ and } (dh)_x = j \,\}.$$

We define a map $p\colon \mathcal{E} \to \mathcal{B}$ by the formula $p(h) = h|S^n_-$ for $h \in \mathcal{E}$.

We will now show that p has the covering homotopy property. For this end choose a subset $B \subset S^n_+$ such that B is diffeomorphic to an open n-disc centered at the North pole N of S^n, moreover $S \subset B$. Denote by R the intersection of $S^n_+ \setminus B$ with the geodesic arc connecting N and x. Let $K = S^n_+ \setminus (B \cup R)$, then $\mathrm{int}(K)$ is diffeomorphic to an open n-disc. Let P be a polyhedron, $\psi\colon P \times I \to \mathcal{B}$ and $\phi_0\colon P \times \{0\} \to \mathcal{E}$, such that $\psi|(P \times \{0\}) = p \circ \phi_0$. We must extend ϕ_0 to $\phi\colon P \times I \to \mathcal{E}$ so that $\psi = p \circ \phi$, i.e., the following diagram is commutative:

$$\begin{array}{ccc} P \times I & \xrightarrow{\phi} & \mathcal{E} \\ {\scriptstyle \mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ P \times I & \xrightarrow{\psi} & \mathcal{B}. \end{array}$$

For $r \in P$ and $t \in I$ we define $\phi(r,t)|(\overline{B} \cup R)$ to be equal to $\phi_0(r)|(\overline{B} \cup R)$. Moreover, let $(d\phi(r,t))|R = (d\phi_0(r))|R$. This is possible since $(\overline{B} \cup R) \cap S^n_- = \{x\}$ and $(d\psi(r,t))_x = j$ for every $r \in P$ and $t \in I$. The map $\phi_0(r)|K$ is an immersion for every $r \in P$ and $\phi(r,t)$ is already defined (together with normal derivatives) on ∂K for every $r \in P$ and $t \in I$. Thus using Smale's lemma (Theorem 3.2.1) $\phi(r,t)$ can be extended to K as an immersion for every $r \in P$ and $t \in I$.

It is well known that the space $\mathcal{B}$ is contractible, thus its homotopy groups are all trivial. Take the fiber of p defined by $\mathcal{F} = p^{-1}(f|S^n_-)$, then by assumption $f, g \in \mathcal{F}$. Denote by i the inclusion of $\mathcal{F}$ into $\mathcal{E}$. Let us look at the following part of the homotopy exact sequence of the fibration $p\colon \mathcal{E} \xrightarrow{\mathcal{F}} \mathcal{B}$:

$$\pi_1(\mathcal{B}) \longrightarrow \pi_0(\mathcal{F}) \xrightarrow{i_*} \pi_0(\mathcal{E}) \xrightarrow{p_*} \pi_0(\mathcal{B}).$$

Since $\pi_1(\mathcal{B}) = 0$ and $\pi_0(\mathcal{B}) = 0$, the morphism i_* is an isomorphism. By assumption $f \sim_s g$, moreover, a singularity fixing regular homotopy H connecting f and g can be chosen so that $(dH_t)_x = j$ for every $t \in [0,1]$. Thus $i(f)$ and $i(g)$ lie in the same path-component of $\mathcal{E}$, i.e., they represent the same element of $\pi_0(\mathcal{E})$. Using the fact that i_* is a monomorphism we get that f and g lie in the same path-component of $\mathcal{F}$. This means that there exists a singularity fixing regular homotopy connecting f and g that is fixed on S^n_-. $\square$

3.3.6 Merging double curves

The idea of the procedure of eliminating the second obstruction (and achieving the coincidence of f and g – after a regular homotopy – also on the last part of M^n, i.e., on U) is as follows: In [3] Ekholm showed that the regular homotopy class of an immersion $S^n \to \mathbb{R}^{2n-1}$ is completely determined by the behavior of the map in a small neighborhood of the double curves if $n \geq 4$. We are going to reduce the problem to this result of Ekholm. First we make f and g coincide in a neighborhood of their singular sets. Then using Theorem 3.2.21 we will get to the case of the immersions of Ekholm's theorem. To manipulate the double curves of f and g we will us the following construction.

In the presence of Whitney umbrella singular points we can merge the closed double curves with the double curves connecting the singular points. The merging is done by using the following construction motivated by the Whitney-trick (for a reference see [13] or [27]). (Let us recall that in the Whitney trick one defines a standard model and then one shows that it can be embedded into the manifold under consideration.)

The construction. In our case the standard model S is chosen to be the standard model of the Whitney-trick (two arcs in D^2 that intersect in two points) times the interval $[-1,1]$ (see the left side of Figure 3.3). Thus we have two surfaces P_1 and P_2 in $D^2 \times [-1,1]$, both diffeomorphic to the square, that intersect in two line segments l_1 and l_2. The regular homotopy in this model is the identity near $\partial(D^2 \times [-1,1])$ and is the isotopy of the standard model of the Whitney trick (l_2 is pushed above l_1) near $D^2 \times \{0\}$. As a result of this deformation the double points are removed near $D^2 \times \{0\}$, and thus the double curves of the final arrangement coincide with the connected sum $l_1 \# l_2$ (see the right side of Figure 3.3).

Let p and q be two double points of the generic map $f\colon N^n \to \mathbb{R}^{2n-1}$, where N^n is any compact connected manifold with boundary. All we have to do is to embed $D^2 \times [-1,1] \times \mathbb{R}^{n-2} \times \mathbb{R}^{n-2}$ into $\mathbb{R}^{2n-1}$ so that $P_1 \times \mathbb{R}^{n-2} \times \{0\}$ and

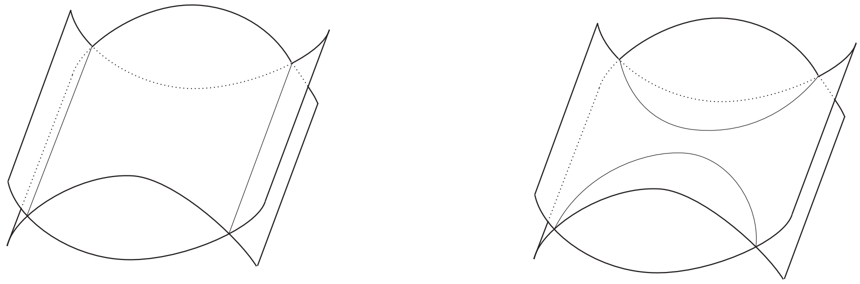

Figure 3.3: The standard model

$P_2 \times \{0\} \times \mathbb{R}^{n-2}$ map to $\mathrm{Im}(f)$, moreover $l_1 \times \{0\}$ maps on the double curve through p and $l_2 \times \{0\}$ maps on the double curve through q.

Let $f^{-1}(p) = \{p_1, p_2\}$ and $f^{-1}(q) = \{q_1, q_2\}$. Choose an embedded curve c_i in N^n connecting p_i and q_i and let $C_i = f(c_i)$ for $i = 1, 2$. We might suppose that $C_1 \cap C_2 = \{p, q\}$ and no other double or singular point of f lies on c_1 and c_2. Thus $f|c_1$ and $f|c_2$ are embeddings. Let D_i be an embedded (tubular) neighborhood of C_i in $f(N^n)$ diffeomorphic to D^n for $i = 1, 2$. The intersection $D_1 \cap D_2 = L_1 \sqcup L_2$, where $p \in L_1$ and $q \in L_2$.

Next we choose a Whitney disk W containing C_1 and C_2 as in [13]. Since $\dim(\mathrm{Im}(f)) + \dim(W) = n + 2 < 2n - 1$ (because $n \geq 4$), if W is in general position then $W \cap \mathrm{Im}(f) = C_1 \cup C_2$. We might also suppose that W is transversal to L_1 and L_2. Since from now on we will work only in a neighborhood of W, we might restrict our attention to $D_1 \cup D_2$ and forget about the rest of $\mathrm{Im}(f)$.

Construct a Riemannian metric on $\mathbb{R}^{2n-1}$ for which $D_1 \perp D_2$. Choose an orientation of D_1 and a co-orientation of D_2 in $\mathbb{R}^{2n-1}$. Let

$$\alpha_1, \ldots, \alpha_{n-1} \in \nu_p(D_2 \subset \mathbb{R}^{2n-1}) \subset T_p D_1$$

be a positive normal frame of D_2. Let $\tilde{\nu}(p) \in T_p L_1$ be taken so that $\tilde{\nu}(p), \alpha_1, \ldots, \alpha_{n-1}$ is a positive basis of $T_p D_1$. Similarly, let

$$\beta_1, \ldots, \beta_{n-1} \in \nu_q(D_2 \subset \mathbb{R}^{2n-1}) \subset T_q D_1$$

be a positive normal frame of D_2 and $\tilde{\nu}(q) \in T_q L_2$ be chosen so that $\tilde{\nu}(q), \beta_1, \ldots, \beta_{n-1}$ is a *negative* basis of $T_q D_1$.

We can extend $\tilde{\nu}$ to $C_1 \cup C_2$ so that $\tilde{\nu}|C_i$ is tangent to D_i since $n \geq 3$. Next we extend $\tilde{\nu}$ from $C_1 \cup C_2$ to a normal field of W: Since $\nu(W \subset \mathbb{R}^{2n-1})$ is the trivial bundle and $C_1 \cup C_2$ is homotopy equivalent to S^1, the obstruction to extend $\tilde{\nu}$ is an element of $\pi_1(V_{2n-3,1}) = \pi_1(S^{2n-4})$. But this group is 0 since $2n - 4 > 1$ if $n \geq 3$. Taking the exponential map along $\tilde{\nu}$ on W we can embed the standard model $S = D^2 \times [-1, 1]$ into $\mathbb{R}^{2n-1}$. Denote the image of S by U.

Orient C_1 from p to q and denote by $\tau(x)$ the positive unit tangent vector of C_1 at x. Moreover, let $\nu(p) = \tau(p)$ and $\nu(q) = -\tau(q)$. Since $n \geq 4$ we can extend ν along C_2 as a normal field of D_2.

Choose vectors $\xi_1(p), \dots, \xi_{n-2}(p)$ in $T_p D_1$ so that

$$\tilde{\nu}(p), \nu(p), \xi_1(p), \dots, \xi_{n-2}(p)$$

is a positive basis of $T_p D_1$. Using parallel translation extend $\xi_1, \dots, \xi_{n-2}$ to C_1. Then

$$\tilde{\nu}(q), \nu(q), \xi_1(q), \dots, \xi_{n-2}(q) \in T_q D_1$$

is a negative basis since $\nu(q) = -\tau(q)$. Thus from the definition of $\tilde{\nu}(q)$ we get that $\nu(q), \xi_1(q), \dots, \xi_{n-2}(q)$ is a positive basis of $\nu_q(D_2 \subset \mathbb{R}^{2n-1})$. Moreover, $\nu(p), \xi_1(p), \dots, \xi_{n-2}(p)$ is a positive basis of $\nu_p(D_2 \subset \mathbb{R}^{2n-1})$ and $\nu|C_2$ is a continuous vector field. All this together implies that the $(n-2)$-frames $\underline{\xi}(p)$ and $\underline{\xi}(q)$ can be extended onto C_2 to be transversal to D_2 and ν.

We can extend the frame $\underline{\xi}$ to U: The pair $(U, C_1 \cup C_2)$ is homotopy equivalent (deformation retracts onto) (D^2, S^1). Thus the obstruction lies in $\pi_1(V_{2n-4,n-2}) = 0$ since $n \geq 4$ implies that $1 < (2n-4) - (n-2)$.

Finally we construct an $(n-2)$-frame $\underline{\eta}$ on U that is tangent to D_2 on C_2. Look at the bundle of orthogonal $(n-2)$-frames on U that are perpendicular to U and $\underline{\xi}$. Since U is contractible, this bundle is trivial; let $\underline{\eta}$ be any section. Notice that $\nu(U \cap D_2 \subset U)$ is spanned by the vector field $\nu|C_2$ (the subset $U \cap D_2$ corresponds to P_2 in the standard model S). Because $\underline{\eta}|C_2 \perp \underline{\xi}, U$, we get that

$$\underline{\eta}|C_2 \perp \langle \underline{\xi}, \nu|C_2 \rangle = \nu(D_2 \subset \mathbb{R}^{2n-1})|C_2.$$

Thus $\underline{\eta}|C_2$ is tangent to D_2, as required.

Using the exponential map on the fields $\underline{\xi}$ and $\underline{\eta}$ along U we can embed $S \times \mathbb{R}^{n-2} \times \mathbb{R}^{n-2}$ into $\mathbb{R}^{2n-1}$ as described above. $\square$

The regular homotopy constructed above is clearly singularity fixing.

3.3.7 The proof of Theorem 3.3.19

Using Lemma 3.3.20 we only have to prove that $f \sim_s g$. With a singularity fixing regular homotopy we can perturb f and g to obtain generic maps. Let us recall (see the beginning of the previous section) that the main idea of the proof is to reduce the problem to the case of immersions and then apply Ekholm's theorem.

Let l_f (respectively l_g) denote the closure of the double curve of f (respectively g) that connects the two singular points of f (respectively g). With the aid of the above construction we merge using a singularity fixing regular homotopy all closed double curves of f (respectively g) with l_f (respectively l_g). From now on we will suppose that l_f is the only double curve of f and l_g is the only double curve of g. Let $m_f = f^{-1}(l_f)$ and $m_g = g^{-1}(l_g)$. Since $n \geq 4$ there exists a diffeotopy $\{\, d_t \colon t \in [0,1] \,\}$ of S^n that fixes $S(f) = S(g)$ and takes m_g to m_f. Composing f with $\{d_t\}$ we can suppose that $m_f = m_g$. Let $m = m_f = m_g$ and $S(f) = S(g) = \{p, q\}$. If n is even we have to be careful when choosing the diffeotopy $\{d_t\}$: If we order the two components of $m \setminus \{p, q\}$ then this defines signs of the singular points of f and g (see Definition 3.2.12), and it might happen that $S_+(f) \neq S_+(g)$ (the equality $\#S(f) = \#S(g) = 0$ still holds). In this case we compose f with a diffeotopy of S^n that fixes p and q and swaps the two components of $m \setminus \{p, q\}$ in order to swap $S_+(f)$ and $S_-(f)$. This is possible if $n \geq 4$. Thus if n is even we also fix an ordering of the components of $m \setminus \{p, q\}$ and suppose that for the induced signs $S_\pm(f) = S_\pm(g)$.

Next we compose f with a diffeotopy of $\mathbb{R}^{2n-1}$ that moves l_f to l_g. Thus we can suppose that $f|m = g|m$ and let $l = l_f = l_g$. Using Lemma 2.14 of [2] we can straighten f and g close to their self intersection $\operatorname{int}(l)$, i.e., we can achieve that they agree with their normal derivatives on a neighborhood of m.

We now sketch the idea of the construction that follows: The singular points of f and g coincide and have the same signs if n is odd. The sign is the only local isotopy invariant a Whitney-umbrella singularity can have. Moreover, any two bundles over an interval are equivalent. Since m is an orientation preserving curve in S^n the two cross-caps at the end of l_f and l_g are in the same "relative position". So f and g are isotopic if restricted to a small neighborhood of m, fixing $S(f) = S(g)$. This isotopy can be extended to an ambient isotopy of $\mathbb{R}^{2n-1}$. So we might suppose that f and g agree in a small open tubular neighborhood V of their common double locus m. The exact details are as follows:

Since S^n is orientable, the normal bundle $\nu(m \subset S^n)$ is trivial. Thus we can choose a normal framing $\underline{\nu}$ of m in S^n. Notice that $\ker(df_p) = \ker(dg_p) = T_pm$ and $\ker(df_q) = \ker(dg_q) = T_qm$, so $df(\underline{\nu}_p)$ and $dg(\underline{\nu}_p)$ are $(n-1)$-frames in $T_{f(p)}\mathbb{R}^{2n-1} = T_{g(p)}\mathbb{R}^{2n-1}$. Similarly, $df(\underline{\nu}_q)$ and $dg(\underline{\nu}_q)$ are $(n-1)$-frames in $T_{f(q)}\mathbb{R}^{2n-1} = T_{g(q)}\mathbb{R}^{2n-1}$. Moreover, $\underline{\nu}_f = df(\underline{\nu}|(m \setminus \{p, q\}))$ and $\underline{\nu}_g = dg(\underline{\nu}|(m \setminus \{p, q\}))$ provide $(2n-2)$-frames along $\operatorname{int}(l)$.

Suppose that $S_+(f) = S_+(g) = \{p\}$ and $S_-(f) = S_-(g) = \{q\}$. Let $a = f(p) = g(p)$ and $b = f(q) = g(q)$, then a and b are the endpoints of the curve l. Orient l from a to b, this induces an orientation of $\nu(l \subset \mathbb{R}^{2n-1})$. (Recall that we fixed an orientation of $\mathbb{R}^{2n-1}$ to define the signs of the Whitney-umbrellas.) Then from Definition 3.2.12 of the signs of the singular points we can see that $\left\{ \underline{\nu}_f, \underline{\nu}_g \right\}$ provides a negative basis of $\mu = \nu\left(\operatorname{int}(l) \subset \mathbb{R}^{2n-1}\right)$. Thus there exists a homotopy $T\colon \operatorname{int}(l) \times I \to GL_+(2n-2, \mathbb{R})$ such that for every $x \in \operatorname{int}(l)$ the transformation $T(x, 0) = \mathrm{id}_{\mathbb{R}^{2n-2}}$ and $(\underline{\nu}_f)_x T(x, 1) = (\underline{\nu}_g)_x$. The homotopy T can be extended to an ambient isotopy of f that takes f to g on a small open tubular neighborhood V of m, and this isotopy is identical outside a neighborhood of l. Thus we can suppose that $f|V = g|V$.

Let $D \subset V$ be diffeomorphic to the closed n-disc D^n, so that $p, q \in \operatorname{int}(D)$ and $f|\partial D$ $(= g|\partial D)$ is an embedding. Since the only double locus of f and g is $m \subset V$, the distances $d_1 = d(f(D), f(S^n \setminus V))$ and $d_2 = d(g(D), g(S^n \setminus V))$ are positive. Let $0 < \varepsilon < \min(d_1, d_2)$. Using Theorem 3.2.21 choose a generic immersion $h\colon D \to \mathbb{R}^{2n-1}$ that agrees with $f|\partial D$ in a neighborhood of ∂D, moreover the C^0-distance of h and $f|D$ is $< \varepsilon$. (This is possible since $\#S(f|D) = 0$ and $f|\partial D$ is an embedding.) Denote by f_1 (respectively g_1) the immersion of S^n in $\mathbb{R}^{2n-1}$ that agrees with f (respectively g) on $S^n \setminus D$ and with h on D. From the choice of h we can see that $f(S^n \setminus V) \cap h(D) = \emptyset$ and $g(S^n \setminus V) \cap h(D) = \emptyset$. Moreover, since $f|(S^n \setminus V)$ and $g|(S^n \setminus V)$ are embeddings that do not intersect $f(V \setminus D) = g(V \setminus D)$, the double loci of f_1 and g_1 are contained in V. But $f_1|V = g_1|V$, thus f_1 and g_1 agree in a neighborhood of their common double locus. Theorem 1 in [3] implies that the regular homotopy class of a generic immersion $S^n \to \mathbb{R}^{2n-1}$ can be expressed in terms of the geometry of the self intersection if $n \geq 4$. Thus $f_1 \sim g_1$.

Since $f_1|D = g_1|D$, the regular homotopy connecting f_1 and g_1 can be chosen to be constant on D, i.e., $f_1 \sim g_1$ (rel D). This gives us a regular homotopy $f \sim g$ (rel D), thus $f \sim_s g$.

3.3.8 An application

In part 4 of [11] the $n = 2$ case of Theorem 3.1.3 was applied to projections of immersions of surfaces into $\mathbb{R}^4$. Knowing Theorem 3.1.3 the same results can be generalized without modification of the proof.

Let us first recall some definitions.

Definition 3.3.9. Let

$$\mathrm{Imm}_\pi(M^n, \mathbb{R}^{2n}) = \left\{ F \in \mathrm{Imm}(M^n, \mathbb{R}^{2n}) \colon \pi \circ F \in L(M^n, \mathbb{R}^{2n-1}) \right\}$$

be the subspace of $\mathrm{Imm}(M^n, \mathbb{R}^{2n})$ formed by those immersions whose projection in $\mathbb{R}^{2n-1}$ is locally generic. Two immersions $F, G \in \mathrm{Imm}_\pi(M^n, \mathbb{R}^{2n})$ are called *π-homotopic* (denoted by $F \sim_\pi G$) if they are in the same path-component of $\mathrm{Imm}_\pi(M^n, \mathbb{R}^{2n})$.

The generalization of Theorem 4.7 of [11] is then the following.

Theorem 3.3.21. *If $n = 2$ or $n > 3$ and $F, G \in Imm_\pi(M^n, \mathbb{R}^{2n})$ then*

$$F \sim_\pi G \Leftrightarrow [F \sim G \text{ and } \pi \circ F \sim \pi \circ G].$$

Chapter 4

A geometric classification of immersions of 3-manifolds into 5-space

4.1 Introduction

Hirsch-Smale theory [8] reduces the problem of regular homotopy classification of immersions to homotopy theory. However, this homotopy theoretic problem is usually hard to deal with. In the case of immersions of oriented 3-manifolds into $\mathbb{R}^5$ this homotopy theoretic problem was solved by Wu [25] using algebraic topological methods (also see [1]).

However, it remains a problem to determine the regular homotopy class of a given immersion from its geometry. By geometry we mean the structure of the "singularities" of the map. For example, double points are such "singularities", and indeed, Smale [21] showed that for $n > 1$ the regular homotopy class of an immersion $S^n \looparrowright \mathbb{R}^{2n}$ is completely determined by the number of its double points (modulo 2 if n is odd). A similar classification was carried out by Ekholm [3] for immersions of S^k into $\mathbb{R}^{2k-1}$ for $k \geq 4$.

The whole picture changes when we consider immersions of S^3 into $\mathbb{R}^5$. Hughes and Melvin [9] showed that there are infinitely many embeddings $S^3 \hookrightarrow \mathbb{R}^5$ that are pairwise not regularly homotopic to each other. Therefore one can not determine the regular homotopy class from the "singularities" since an embedding has no such. Ekholm and Szűcs [4] came over this problem using "singular Seifert surfaces" bounded by the immersions. For an immersion $f\colon M^3 \looparrowright \mathbb{R}^5$ a singular Seifert surface is a generic map $F\colon W^4 \to \mathbb{R}^5$ of a compact orientable manifold W^4 with boundary M^3 such that $\partial F = f$. In [4] it is shown that for $M^3 = S^3$ the Smale invariant of f can be computed from the singularities of F. Later Saeki, Szűcs and Takase [19] generalized these results for immersions $f\colon M^3 \looparrowright \mathbb{R}^5$ with trivial normal bundle (for oriented M^3). The invariant introduced in [19] corresponds to the 3-dimensional obstruction to a regular homotopy between two such immersions. Our present paper generalizes the results of [19] to arbitrary immersions $f\colon M^3 \looparrowright \mathbb{R}^5$.

We will consider the set $\mathrm{Imm}(M^3,\mathbb{R}^5)_\chi$ of immersions with fixed normal Euler class $e(\nu_f) = \chi \in H^2(M^3;\mathbb{Z})$ and construct a $\mathbb{Z}_{2d(\chi)}$-valued regular homotopy invariant i for this set of immersions, where $d(\chi)$ denotes the divisibility of χ. The construction of the invariant i will also make use of a singular Seifert surface F. In [19] F had to be an immersion near the boundary, but we (have to and) will allow arbitrary generic maps. If $\chi = 0$ and F is an immersion near the boundary then the construction of the invariant i agrees with the one introduced in [19]. We will also show that whenever $f, g\colon M^3 \looparrowright \mathbb{R}^5$ are regularly homotopic on a neighborhood of the 2-skeleton of M^3 then $i(f) = i(g)$ iff f and g are regularly homotopic. This shows that i corresponds to the 3-dimensional obstruction to a regular homotopy between f and g. (Note that there is an invariant which determines the regular homotopy class of the restriction of an immersion to a neighborhood of the 2-skeleton of M^3. This invariant was called the Wu invariant in [19], see below.)

Regular homotopy classes of immersions of oriented 3-manifolds into $\mathbb{R}^5$ endowed with the connected sum operation form a semigroup whose structure we will also determine. Finally, an exact sequence will be defined that relates $\mathrm{Imm}[M^3,\mathbb{R}^5]$ to $\mathrm{Imm}[M^3,\mathbb{R}^6]$ and $[M^3,S^2]$.

4.2 Preliminaries

First let us recall the result of Wu [25] that classifies immersions of an oriented 3-manifold M^3 into $\mathbb{R}^5$ up to regular homotopy.

Theorem 4.2.1. *The normal Euler class χ of an immersion $f\colon M^3 \looparrowright \mathbb{R}^5$ is of the form $2c$ for some $c \in H^2(M^3;\mathbb{Z})$ and for any $c \in H^2(M^3;\mathbb{Z})$ there is an immersion f such that $\chi = 2c$. Furthermore,*

$$\mathit{Imm}[M^3,\mathbb{R}^5]_\chi \approx \coprod_{c\in H^2(M^3;\mathbb{Z})\,,\,2c=\chi} H^3(M^3;\mathbb{Z})/(2\chi \cup H^1(M^3;\mathbb{Z})),$$

where $\mathit{Imm}[M^3,\mathbb{R}^5]_\chi$ is the set of regular homotopy classes of immersions with normal Euler class $\chi \in H^2(M^3;\mathbb{Z})$ and $\cup$ represents the cup product, moreover the symbol $\approx$ denotes a bijection.

Remark 4.2.1. For $\chi \in H^2(M^3;\mathbb{Z})$ let $d(\chi) \in \mathbb{Z}$ denote the divisibility of χ, so that χ equals $d(\chi)$ times a primitive class in $H^2(M^3;\mathbb{Z})$ modulo torsion, and $d(\chi) = 0$ if χ is of finite order. Then Poincaré duality implies that

$$H^3(M^3;\mathbb{Z})/(2\chi \cup H^1(M^3;\mathbb{Z})) \approx \mathbb{Z}_{2d(\chi)}.$$

If f is an immersion of M^3 into $\mathbb{R}^5$ with normal Euler class χ then let us introduce the notation $d(f)$ for $d(\chi)$.

Notation 4.2.2. *For $\chi \in H^2(M^3;\mathbb{Z})$ let $\Gamma_2(\chi)$ denote the set $\{ c \in H^2(M^3;\mathbb{Z})\colon 2c = \chi \}$. Throughout this paper we will use the notation $M^3_\circ$ for the punctured 3-manifold $M^3 \setminus D^3$, where $D^3 \subset M^3$ is a closed 3-disc. Then the 2-skeleton $sk_2(M^3)$ is a deformation retract of $M^3_\circ$.*

Theorem 4.2.1 can also be applied to the open manifold $M^3_\circ$. Since the cohomology group $H^3(M^3_\circ;\mathbb{Z}) = 0$ we obtain a bijection

$$\bar{c}\colon \operatorname{Imm}[M^3_\circ, \mathbb{R}^5]_\chi \to \Gamma_2(\chi).$$

Thus for an immersion $f\colon M^3 \looparrowright \mathbb{R}^5$ the invariant $c(f) = \bar{c}(f|M^3_\circ) \in \Gamma_2(\chi)$ describes the regular homotopy class of $f|M^3_\circ$. Following [19] we will call $c(f)$ the Wu invariant of the immersion f.

To get a complete description of $\operatorname{Imm}[M^3, \mathbb{R}^5]_\chi$ we will construct a $\mathbb{Z}_{2d(\chi)}$-valued invariant i such that the map

$$(c, i)\colon \operatorname{Imm}[M^3, \mathbb{R}^5]_\chi \to \Gamma_2(\chi) \times \mathbb{Z}_{2d(\chi)}$$

will be a bijection. The invariant i is constructed in a geometric manner and is an extension of the invariant defined in [19] for $\chi = 0$.

Next let us recall Theorem 1.1(a) in [4]. Let $f\colon S^3 \looparrowright \mathbb{R}^5$ be an immersion and V^4 an arbitrary compact oriented 4-manifold with $\partial V^4 = S^3$. The map f extends to a generic map $F\colon V^4 \to \mathbb{R}^5$ which has no singular points near the boundary ∂V^4 since the normal bundle ν_f of f is trivial. This map F has isolated cusps, each one having a sign. Let us denote by $\#\Sigma^{1,1}(F)$ their algebraic number and let $\Omega(f)$ be the Smale invariant of f. The following formula was proved in [4].

Theorem 4.2.3.

$$\Omega(f) = \frac{1}{2}(3\sigma(V^4) + \#\Sigma^{1,1}(F)).$$

The proof of this theorem relies on the following proposition ([24], Lemma 3).

Lemma 4.2.4. *Let X^4 be a closed oriented 4-manifold and $g\colon X^4 \to \mathbb{R}^5$ a generic map. Then $3\sigma(X^4) + \#\Sigma^{1,1}(g) = 0$.*

For the sake of completeness we will also recall from [19] the definition of the invariant i for immersions with trivial normal bundle. First we need a preliminary definition.

Definition 4.2.1. Let M^3 be a closed oriented 3-manifold. We denote by $\alpha(M^3)$ the dimension of the $\mathbb{Z}_2$ vector space $\tau H_1(M^3;\mathbb{Z}) \otimes \mathbb{Z}_2$, where $\tau H_1(M^3;\mathbb{Z})$ is the torsion subgroup of $H_1(M^3;\mathbb{Z})$.

Definition 4.2.2. Let $f\colon M^3 \looparrowright \mathbb{R}^5$ be an immersion with trivial normal bundle. Let W^4 be any compact oriented 4-manifold with $\partial W^4 = M^3$ and $F\colon W^4 \to \mathbb{R}^5$ a generic map nonsingular near the boundary such that $F|\partial W^4 = f$. (We can choose such a generic map F since f is an immersion with trivial normal bundle.) Denote the algebraic number of cusps of F by $\#\Sigma^{1,1}(F)$. Then let

$$i(f) = \frac{3}{2}(\sigma(W^4) - \alpha(M^3)) + \frac{1}{2}\#\Sigma^{1,1}(F).$$

It is proved in [19] that $i(f)$ is always an integer and a regular homotopy invariant.

In the following sections we will extend the above regular homotopy invariant i to arbitrary immersions. If $f\colon M^3 \looparrowright \mathbb{R}^5$ has non-trivial normal bundle then we have to give up the assumption that the singular Seifert-surface F is an immersion near the boundary. Thus we will use an arbitrary generic map $F\colon W^4 \to \mathbb{R}^5$ such that $\partial F = f$. The singular set $\Sigma^1(F)$ of such an F is a 2-dimensional submanifold of W^4 with boundary $C(F) = \partial\Sigma^1(F) \subset M^3$. If we orient $\ker(dF)|C(F)$ so that it points into W^4 and project it into TM^3 then we obtain a normal field $\nu(F)$ along $C(F)$. We will define the rotation of $\nu(F)$ around $C(F)$ modulo $4d(f)$ and denote this by $R(F)$. The double of the extended invariant will be defined to be

$$I(f) = 3(\sigma(W^4) - \alpha(M^3)) + \#\Sigma^{1,1}(F) + R(F) \in \mathbb{Z}_{4d(f)}.$$

We will show that $I(f)$ is always even, thus it defines an element $i(f) \in \mathbb{Z}_{2d(f)}$ using the natural embedding $\mathbb{Z}_{2d(f)} \hookrightarrow \mathbb{Z}_{4d(f)}$.

4.3 Rotation

Throughout this paper M^3 will denote a fixed closed connected and oriented 3-manifold.

Notation 4.3.1. *A pair (C,ν) will always stand for an oriented 1-dimensional submanifold C of M^3 and a nowhere vanishing normal field ν along C.*

Definition 4.3.1. Let $\chi \in H^2(M^3;\mathbb{Z})$ and let C_0 and C_1 be 1-dimensional oriented submanifolds of M^3 with normal fields ν_0 and ν_1 such that $PD[C_0] = PD[C_1] = \chi$. (Here PD denotes Poincaré duality.) Then we can define the *rotation difference* $\mathrm{rd}((C_0,\nu_0),(C_1,\nu_1)) \in \mathbb{Z}_{2d(\chi)}$ of (C_0,ν_0) and (C_1,ν_1) as follows. Since $[C_0] = [C_1]$ and

$$H_1(M^3;\mathbb{Z}) \approx H^2(M^3;\mathbb{Z}) \approx [M^3, \mathbb{C}P^\infty],$$

there exists an oriented cobordism $K^2 \subset M^3 \times I$ between $C_0 \subset M^3 \times \{0\}$ and $C_1 \subset M^3 \times \{1\}$. Let ν be a generic normal field along K^2 that extends ν_0 and ν_1. Then a sign can be given to each zero of ν since M^3 is oriented. Now we define $\mathrm{rd}((C_0,\nu_0),(C_1,\nu_1))$ to be the algebraic number of zeroes of ν modulo $2d(\chi)$. Equivalently, $\mathrm{rd}((C_0,\nu_0),(C_1,\nu_1))$ is the self intersection of K in $M^3 \times I$ modulo $2d(\chi)$ if perturbed in the direction of ν.

Remark 4.3.1. The rotation difference is the obstruction to the existence of a framed cobordism between the framed submanifolds (C_0, ν_0) and (C_1, ν_1) of M^3. Using the Pontrjagin construction this corresponds to the obstruction to a homotopy between two maps of M^3 to S^2. This situation was first examined in [17]. It is easy to see that $\mathrm{rd}((C_0,\nu_0),(C_1,\nu_1)) = 0$ iff (C_0,ν_0) and (C_1,ν_1) are framed cobordant. Thus we obtain a bijection

$$[M^3, S^2] \approx \coprod_{\chi \in H^2(M^3;\mathbb{Z})} H^3(M^3;\mathbb{Z})/2\chi \cup H^1(M^3;\mathbb{Z}).$$

Proposition 4.3.2. *In Definition 4.3.1 above the rotation difference is well defined, i.e., it does not depend on the choice of K and ν.*

Proof. Let K, ν and K', ν' be as in Definition 4.3.1. We glue together $M^3 \times I$ and $-M^3 \times I$ along their boundaries so that we obtain the double $D(M^3 \times I) = M^3 \times S^1$. Place K into the half of $M^3 \times S^1$ corresponding to $M^3 \times I$ and K' into the other half. Then we obtain a closed oriented surface $F = K \cup -K'$ in $M^3 \times S^1$ and a normal field $\mu = \nu \cup \nu'$ along F. Since $H^*(S^1;\mathbb{Z})$ is a torsion free $\mathbb{Z}$-module we can apply Künneth's theorem and we get that

$$H^2(M^3 \times S^1;\mathbb{Z}) \approx H^1(M^3;\mathbb{Z}) \otimes H^1(S^1;\mathbb{Z}) \oplus H^2(M^3;\mathbb{Z}) \otimes H^0(S^1;\mathbb{Z}).$$

Thus the Poincaré dual of F can be written in the form

$$PD[F] = x \times \alpha + y \times 1 \in H^2(M^3 \times S^1;\mathbb{Z}),$$

where $x \in H^1(M^3;\mathbb{Z})$ and $y \in H^2(M^3;\mathbb{Z})$, moreover α denotes the generator of $H^1(S^1;\mathbb{Z})$ and 1 the generator of $H^0(S^1;\mathbb{Z})$ given by the orientation of S^1. Note that for $1 \in S^1$ the dual class of $M^3 \times \{1\} \subset M^3 \times S^1$ is

$$PD[M^3 \times \{1\}] = PD[M^3] \times PD[\{1\}] = 1 \times \alpha \in H^1(M^3 \times S^1;\mathbb{Z}).$$

Moreover,

$$PD[F \cap (M^3 \times \{1\})] = PD[C_1 \times \{1\}] = PD[C_1] \times PD[\{1\}] = \chi \times \alpha.$$

On the other hand

$$\begin{aligned} PD[F \cap (M^3 \times \{1\})] &= PD[F] \cup PD[M^3 \times \{1\}] = \\ (x \times \alpha + y \times 1) \cup (1 \times \alpha) &= x \times \alpha^2 + y \times \alpha. \end{aligned}$$

Since $\alpha^2 = 0$ we get that $y \times \alpha = \chi \times \alpha$. Using Künneth's theorem again we obtain the equality $y = \chi$. Thus we get that

$$PD[F] \cup PD[F] = (x \times \alpha + \chi \times 1)^2 = (2x \cup \chi) \times \alpha$$

since $\alpha^2 = \chi^2 = 0$ and $x \cup \chi = \chi \cup x$ because the degree of χ is 2. So the self intersection of F in $M^3 \times S^1$ equals $\langle (2\chi \cup x) \times \alpha, [M^3 \times S^1] \rangle = \langle 2\chi \cup x, [M^3] \rangle \in 2d(\chi)\mathbb{Z}$. If we perturb F in the direction of μ we get that the self intersection of K with respect to ν equals the self intersection of K' with respect to ν' modulo $2d(\chi)$. $\square$

Proposition 4.3.3. *If* $[C_0] = [C_1] = [C_2] \in H_1(M^3, \mathbb{Z})$ *then*

$$rd((C_0, \nu_0), (C_1, \nu_1)) + rd((C_1, \nu_1), (C_2, \nu_2)) = rd((C_0, \nu_0), (C_2, \nu_2)).$$

Definition 4.3.2. For each $a \in H_1(M^3; \mathbb{Z})$ fix a pair (C_a, ν_a) such that $[C_a] = a$. Then for $[C] = a$ let $r(C, \nu) = \mathrm{rd}((C, \nu), (C_a, \nu_a))$.

Corollary 4.3.4. *If* $[C_0] = [C_1]$ *then*

$$r(C_0, \nu_0) - r(C_1, \nu_1) = rd((C_0, \nu_0), (C_1, \nu_1)).$$

Definition 4.3.3. We can define the mod 2 rotation difference denoted by $\mathrm{rd}_2((C_0, \nu_0), (C_1, \nu_1))$ for unoriented C_0 and C_1 just as in Definition 4.3.1 but allowing the cobordism K to be non-orientable and counting the self intersection of K in $M \times I$ only modulo 2. The proof that this is well defined is analogous to the oriented case. It is clear that the epimorphism $\mathbb{Z}_{2d(\chi)} \to \mathbb{Z}_2$ takes rd to rd_2. The mod 2 rotation r_2 is defined just like r.

Unfortunately we will have to lift the invariants rd and r to $\mathbb{Z}_{4d(\chi)}$. To be able to do this we need more structure on M^3 then just a framed submanifold. We will use this additional structure to restrict the homology class of the cobordism K so that the surface F in the proof of Proposition 4.3.2 will represent an even homology class and thus x will always be even (since χ is even). So the self intersection of F will be divisible by $4d(\chi)$ instead of just $2d(\chi)$.

Notation 4.3.5. *Fix a cohomology class* $\chi \in H^2(M^3; \mathbb{Z})$*. Let* ε^3_M *denote the 3-dimensional trivial bundle over* M^3 *and let* $t, v \in \Gamma(\varepsilon^3_M)$ *be two generic non-zero sections of* ε^3_M*. Furthermore, suppose that the 2-dimensional oriented subbundle* $t^\perp < \varepsilon^3_M$ *has Euler class* χ*. If we project* v *into* $t^\perp$ *we obtain a section* $w \in \Gamma(t^\perp)$ *that vanishes along a curve* $C \subset M^3$ *and we orient* C *so that* $PD[C] = e(t^\perp)$*. In particular,* t *and* v *are linearly dependent exactly at the points of* C*. Finally let* ν *be a non-zero normal field along* C*. In the future we will denote such a structure on* M^3 *by a quadruple* (C, ν, t, v) *and the set of these structures by* $N(M^3, \chi)$*.*

Remark 4.3.2. Since $PD[C]|_2 = w_2(\varepsilon^3_M) = 0 \in H^2(M^3; \mathbb{Z}_2)$, the cohomology class $\chi = PD[C]$ is of the form $2c$ for some $c \in H^2(M^3; \mathbb{Z})$. This can be seen from the long exact sequence associated to the coefficient sequence $\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_2$. Thus $N(M^3, \chi) = \emptyset$ if χ is not of the form $2c$.

Definition 4.3.4. Suppose that $a_0 = (C_0, \nu_0, t_0, v_0)$ and $a_1 = (C_1, \nu_1, t_1, v_1)$ are elements of $N(M^3, \chi)$, where $\chi = 2c$. Then we will define their rotation difference $\mathrm{Rd}(a_0, a_1) \in \mathbb{Z}_{4d(\chi)}$ as follows. We will consider a_i to be in $N(M^3 \times \{i\}, \chi)$ for $i = 0, 1$. Let $t, v \in \Gamma(\varepsilon^3_{M \times I})$ be generic non-zero sections extending t_i and v_i for $i = 0, 1$. Denote by K the 2-dimensional submanifold of $M^3 \times I$ where t and v are linearly dependent. Let w denote the projection of v into the 2-dimensional oriented subbundle $t^\perp < \varepsilon^3_{M \times I}$. Then w is zero exactly at the points of K, thus it defines an orientation of K. With this orientation K is an oriented cobordism between C_0 and C_1. Let ν denote a normal field of K that extends both ν_0 and ν_1. Now we define $\mathrm{Rd}(a_0, a_1)$ to be the algebraic number of zeroes of ν modulo $4d(\chi)$. Equivalently, $\mathrm{Rd}(a_0, a_1)$ is the self intersection of K in $M^3 \times I$ modulo $4d(\chi)$ if perturbed in the direction of ν.

Proposition 4.3.6. *In Definition 4.3.4 the rotation difference is well defined. I.e., it does not depend on the extensions t, v and ν.*

Proof. Let t, v, ν and t', v', ν' be as in Definition 4.3.4. The sections t, v are linearly dependent over K and t', v' are dependent over K'. Just as in the proof of Proposition 4.3.2 we will place K, ν and K', ν' in the two halves of the double $D(M^3 \times I) = M^3 \times S^1$ and place t, v and t', v' in the two halves of the trivial bundle $\varepsilon^3_{M\times S^1}$. Let F denote the oriented surface $K \cup -K'$ and by μ the normal field along F obtained from ν and ν'. Moreover, let $T = t \cup t'$ and $V = v \cup v'$. Then $T, V \in \Gamma(\varepsilon^3_{M\times S^1})$ are linearly dependent exactly over F, thus $PD[F]|_2 = w_2(\varepsilon^3_{M\times S^1}) = 0 \in H^2(M^3 \times S^1; \mathbb{Z}_2)$. Using the coefficient sequence $\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_2$ we get that $PD[F]$ is of the for $2b$ for some $b \in H^2(M^3 \times S^1; \mathbb{Z})$. Since $PD[F]$ is of the form $x \times \alpha + \chi \times 1$ where $\chi = 2c$ we get that there exists an element $z \in H^1(M^3; \mathbb{Z})$ such that $x = 2z$. Thus $PD[F]^2 = (4z \cup \chi) \times \alpha$ which implies that the self intersection of F is divisible by $4d(\chi)$. If we perturb F in the direction of μ we get that the self intersection of K with respect to ν equals the self intersection of K' with respect to ν' modulo $4d(\chi)$. □

Remark 4.3.3. The surface K represents the dual of the Stiefel-Whitney class of the bundle $\varepsilon^3_{M\times I}$ relative to the sections t_i, v_i given over $M^3 \times \{0, 1\}$. I.e.,

$$PD[K]|_2 = w_2(\varepsilon^3_{M\times I}; t_i, v_i),$$

which is an element of $H^2(M^3 \times I, ((M^3 \setminus C_0) \times \{0\}) \cup ((M^3 \setminus C_1) \times \{1\}); \mathbb{Z}_2)$ since v and t are linearly independent over $((M^3 \setminus C_0) \times \{0\}) \cup ((M^3 \setminus C_1) \times \{1\})$. Using Lefschetz duality we get that the relative homology class $[K]|_2 \in H^2(M^3 \times I, C_0 \times \{0\} \cup C_1 \times \{1\}; \mathbb{Z}_2)$ is independent of the choice of t and v. If we choose a simplicial subdivision of M^3 so that $\mathrm{sk}_1(M^3) \cap C_i = \emptyset$ for $i = 0, 1$ then w_2 is the obstruction to extending the map $(t_i, v_i)\colon \mathrm{sk}_1(M^3 \times \{0, 1\}) \to V_2(\mathbb{R}^3)$ to $\mathrm{sk}_2(M^3 \times I)$. So the homology class $[K]|_2$ and thus $\mathrm{Rd}(a_0, a_1)$ depends only on the homotopy class of the map

$$(t_i, v_i)|\mathrm{sk}_1(M^3)\colon \mathrm{sk}_1(M^3) \to V_2(\mathbb{R}^3)$$

for $i = 0, 1$. For the sake of completeness we note that if the extension t is given then

$$PD[K] = e(t^\perp; w_i) \in H^2(M^3 \times I, (M^3 \setminus C_0) \times \{0\} \cup (M^3 \setminus C_1) \times \{1\}; \mathbb{Z}).$$

So we have obtained the following proposition.

Proposition 4.3.7. *Suppose that (C_0, ν_0) and (C_1, ν_1) are framed submanifolds of M^3 and let $a_0, b_0, a_1, b_1 \in N(M^3, \chi)$ be of the form $a_i = (C_i, \nu_i, t^a_i, v^a_i)$ and*

$b_i = (C_i, \nu_i, t_i^b, v_i^b)$ for $i = 0, 1$. Moreover, suppose that $sk_1(M^3) \cap C_i = \emptyset$ and $(t_i^a, v_i^a)|sk_1(M^3)$ is homotopic to $(t_i^b, v_i^b)|sk_1(M^3)$ as maps into $V_2(\mathbb{R}^3)$ for $i = 0, 1$. Then the following equality holds:

$$Rd(a_0, a_1) = Rd(b_0, b_1).$$

Proof. Let t^a and v^a be generic extensions of t_i^a, respectively v_i^a over $M^3 \times I$ and denote by K^a the submanifold of $M^3 \times I$ where t^a and v^a are linearly dependent. We obtain the sections t^b and v^b of $\varepsilon^3_{M\times I}$ and the submanifold $K^b \subset M^3 \times I$ in a similar way. Then, according to Remark 4.3.3, we get that

$$PD[K^a]|_2 = w_2(\varepsilon^3_{M\times I}; t_i^a, v_i^a) = w_2(\varepsilon^3_{M\times I}; t_i^b, v_i^b) = PD[K^b]|_2,$$

since w_2 is the obstruction to extending a map into $V_2(\mathbb{R}^3)$ from $\mathrm{sk}_1(M^3 \times I)$ to $\mathrm{sk}_2(M^3 \times I)$ and $(t_i^a, v_i^a)|\mathrm{sk}_1(M^3 \times \{i\})$ is homotopic to $(t_i^b, v_i^b)|\mathrm{sk}_1(M^3 \times \{i\})$. Thus if F denotes the submanifold of $M^3 \times S^1 = D(M^3 \times I)$ obtained by piecing together K^a and K^b we get that $PD[F]|_2 = w_2(\varepsilon^3_{M\times S^1}) = 0$, so we can proceed as in the proof of Proposition 4.3.6. □

Proposition 4.3.8. *If $a_i = (C_i, \nu_i, t_i, v_i) \in N(M^3, \chi)$ for $i = 0, 1$ then*

$$rd((C_0, \nu_0), (C_1, \nu_1)) \equiv Rd(a_0, a_1) \mod 2d(\chi).$$

Proposition 4.3.9. *If $a_0, a_1, a_2 \in N(M^3, \chi)$ then*

$$Rd(a_0, a_1) + Rd(a_1, a_2) = Rd(a_0, a_2).$$

Definition 4.3.5. For each $\chi \in H^2(M^3; \mathbb{Z})$ of the form $\chi = 2c$ fix an element $a_\chi \in N(M^3, \chi)$. Then for each $a \in N(M^3, \chi)$ define the dotation $R(a) \in \mathbb{Z}_{4d(\chi)}$ to be $\mathrm{Rd}(a, a_\chi)$.

Corollary 4.3.10. *If $a_0, a_1 \in N(M^3, \chi)$ then $Rd(a_0, a_1) = R(a_0) - R(a_1)$.*

4.4 The orientation of Σ^1

Now let us recall a special case of Lemma 6.1 of [4]. Let $F\colon W^4 \to \mathbb{R}^5$ be a generic map of a compact orientable manifold. Then the singularity set $\Sigma(F)$ of F is a 2-dimensional submanifold of W^4 which is not necessarily orientable.

Lemma 4.4.1. *The line bundles $\det(T\Sigma(F))$ and $\ker(dF)$ over $\Sigma(F)$ are isomorphic.*

Definition 4.4.1. Let π denote the projection of $\mathbb{R}^{m+1}$ onto $\mathbb{R}^m$. A map $f\colon N^n \to \mathbb{R}^m$ is called *prim* if there exists an immersion $f'\colon N^n \looparrowright \mathbb{R}^{m+1}$ such that $\pi \circ f' = f$.

Corollary 4.4.2. *If $F\colon W^4 \to \mathbb{R}^5$ is a generic prim map then $\Sigma(F) \subset W^4$ is an orientable surface.*

Proof. Let s denote the sixth coordinate function of F', i.e., $F' = (F, s)$. Since F' is non-singular, the function s is non-degenerate along $\ker(dF)$. Thus we can orient $\ker(dF)$ so that the derivative of s in the positive direction of $\ker(dF)$ is positive. But the orientability of $\ker(dF)$ implies the orientability of $\Sigma(F)$ by Lemma 4.4.1. $\square$

The following definition, motivated by Corollary 4.4.2, gives an explicit isomorphism Ψ between $\ker(dF)$ and $\det(T\Sigma(F))$.

Definition 4.4.2. Let W^4 be a compact oriented manifold with possibly non-empty boundary and let $F\colon W^4 \to \mathbb{R}^5$ be a generic map. For $p \in \Sigma(F)$ choose a small neighborhood $U_p \subset W^4$ of p in which $\ker(dF)$ is orientable. Put $F_p = F|U_p$ and choose an orientation o_p of $\ker(dF_p)$. Then there exists a smooth function $s\colon U_p \to \mathbb{R}$ such that the derivative of s in the direction of o_p is positive. (First construct s along $\Sigma(F_p)$ near $\Sigma^{1,1}(F_p)$ then extend it to a tubular neighborhood of $\Sigma(F_p)$.) The map $F_p' = (F_p, s)\colon U_p \looparrowright \mathbb{R}^6$ is an immersion. If U_p is chosen sufficiently small then we can even suppose that F_p' is an embedding. Denote by e_6 the sixth coordinate direction in $\mathbb{R}^6$ and let $\nu_6\colon U_p \to T\mathbb{R}^6$ denote the vector field along F_p' defined by the formula $\nu_6(x) = e_6 \in T_{F_p'(x)}\mathbb{R}^6$ for $x \in U_p$. Projecting ν_6 into the normal bundle of F_p' we obtain a normal field μ_6 along F_p' that vanishes exactly at the points of $\Sigma(F_p)$. Perturb F_p' in the direction of μ_6 to obtain an embedding F_p''. Then orient $\Sigma(F_p)$ as the intersection of F_p' and F_p'' in $\mathbb{R}^6$. Here $\mathbb{R}^6$ is considered with its standard orientation. This orientation of $\Sigma(F_p)$ does not depend on the choice of the function s, since if s_1 and s_2 are two such functions then for $0 \le t \le 1$ the convex combination $(1-t)s_1 + ts_2$ also satisfies the conditions for s.

If we reverse the orientation of $\ker(dF_p)$, i.e. if we orient it by $-o_p$, then we can choose $-s$ instead of s. Thus we obtain the embedding $(F_p, -s)$, which is the reflection of F_p' in the hyperplane $\mathbb{R}^5$. Denote this reflection by $R\colon \mathbb{R}^6 \to \mathbb{R}^6$ (i.e., $R(x_1, \dots, x_5, x_6) = (x_1, \dots, x_5, -x_6)$). Then $(F_p, -s) = R \circ F_p'$. The vector field $dR \circ \nu_6$ along $R \circ F_p'$ points in the direction $-e_6$ and $R \circ F_p''$ is the perturbation of $R \circ F_p'$ in the direction of $dR \circ \mu_6$. But in this case we should perturb $R \circ F_p'$ in the direction of $-(dR \circ \mu_6)$. We obtain the same orientation if we look at the intersection $(R \circ F_p'') \cap (R \circ F_p')$ instead. Since the intersection is 2-dimensional and U_p is 4-dimensional we get that $(R \circ F_p'') \cap (R \circ F_p') = (R \circ F_p') \cap (R \circ F_p'')$ in the oriented sense. The orientations of $\Sigma(F_p)$ defined by the intersections $F_p' \cap F_p''$ and $(R \circ F_p') \cap (R \circ F_p'')$ are opposite. This can be seen from the following argument: For $0 \le t \le 1$ denote by R_t the rotation of the hyperplane $\mathbb{R}^6$ in $\mathbb{R}^7$ around $\mathbb{R}^5$ by the angle πt. The orientation of $(R_t \circ F_p') \cap (R_t \circ F_p'')$ in $R_t(\mathbb{R}^6)$ changes continuously as t goes from 0 to 1. The orientations of the hyperplanes $R_1(\mathbb{R}^6)$ and $\mathbb{R}^6$ are opposite, thus the reflection R changes the orientation of the intersection $F_p' \cap F_p''$.

So we have defined an isomorphism Ψ_p between $\ker(dF_p)$ and $\det(T\Sigma(F_p))$ for every $p \in W^4$ in a compatible way (i.e., $\Psi_p|(U_p \cap U_q) = \Psi_q|(U_p \cap U_q)$ for $p, q \in W^4$). These local isomorphisms define a global isomorphism Ψ between $\ker(dF)$ and $\det(T\Sigma(F))$.

4.5 The invariant

In this section we will give a geometric formula for the 3-dimensional obstruction to the existence of a regular homotopy between two immersions of M^3 into $\mathbb{R}^5$. This generalizes the results of [19] to immersions with non-trivial normal bundle.

Definition 4.5.1. Let W^4 be a compact oriented manifold with boundary M^3 and $F\colon W^4 \to \mathbb{R}^5$ a generic map such that $f = F|M^3$ is an immersion. Recall that $\Sigma(F)$ denotes the set of singular points of F. Let us denote by $C(F) \subset M^3$ the 1-dimensional submanifold $\partial\Sigma(F)$. Choose a trivialization τ of $\ker(dF)|C(F)$ so that it points into the interior of W^4. This is possible since f is non-singular (and so $\ker(dF)$ never lies in TM^3). Then τ is normal to $\Sigma(F)$ because F is generic and thus $\Sigma^{1,1}(F) \cap C(F) = \emptyset$. So if we project τ into TM along $\Sigma(F)$ we obtain a nowhere vanishing normal field $\nu(F)$ in $\nu(C(F) \subset M^3)$.

Let U denote a small collar neighborhood of $C(F)$ in $\Sigma(F)$. Then clearly U is orientable. Using the isomorphism Ψ of Definition 4.4.2 the trivialization τ of $\ker(dF)$ induces an orientation of U. Thus $C(F) \subset \partial U$ is also oriented. So we have assigned a pair $(C(F), \nu(F))$ to F as in Notation 4.3.1. Let $r(F) = r(C(F), \nu(F))$.

Notation 4.5.1. *For $\chi \in H^2(M^3;\mathbb{Z})$ let us denote by $Imm(M^3,\mathbb{R}^5)_\chi$ the space of immersions with normal Euler class χ.*

Fix a cohomology class $\chi \in H^2(M^3;\mathbb{Z})$. Our aim is to define an invariant $i\colon \pi_0\left(\mathrm{Imm}(M^3,\mathbb{R}^5)_\chi\right) \to \mathbb{Z}_{2d(\chi)}$.

Proposition 4.5.2. *Let $f \in Imm(M^3,\mathbb{R}^5)_\chi$ and let $F\colon W^4 \to \mathbb{R}^5$ be a generic map such that $\partial F = f$. Then $[C(F)] = D\chi$.*

Proof. Let κ denote an inner normal field of M^3 in W^4 that extends τ (see Definition 4.5.1). Then $dF \circ \kappa$ is a vector field along f that is tangent to f exactly at the points of $C(F) = \partial\Sigma(F)$. (If $p \in C(F)$ then the rank of $(dF)_p$ is 3, moreover $dF|(T_pM^3) = df$ is non-degenerate. Thus $dF(\kappa_p) \in df(T_pM^3)$.) So if we project $dF \circ \kappa$ into the normal bundle of f we obtain a normal field of f that vanishes along $C(F)$. To see that $C(F)$ represents the normal Euler class of f, we have to know that it is oriented suitably.

Using the notations of Definition 4.5.1 we choose a function $s\colon W^4 \to \mathbb{R}$ such that the derivative of s in the direction of κ (and thus τ) is positive and $s|M^3 \equiv 0$. Then there exists a collar neighborhood V of M^3 in W^4 such that $F' = (F,s)|V$ is an immersion. Denote by $\nu_{F'}$ the normal bundle of F' in $\mathbb{R}^6$ and by ν_f the normal bundle of f in $\mathbb{R}^5$. Then $\nu_{F'}|M^3 = \nu_f$ as oriented bundles, since s is increasing along κ (here $\mathbb{R}^5$ and $\mathbb{R}^6$ are considered with their standard orientations). By Definition 4.4.2 the surface of singular points $U = \Sigma(F|V)$ is oriented as the self-intersection of F' in $\mathbb{R}^6$, or more precisely, as the intersection of the zero section and a generic section of $\nu_{F'}$. Moreover, $C(F)$ is oriented as the boundary of U. Thus $C(F)$ is the self-intersection of the zero section of $\nu_{F'}|M^3 = \nu_f$, so it is dual to the Euler class $e(\nu_f) = \chi$. (Here we used the naturality of the Euler class.) ∎

Definition 4.5.2. Let $f \in \mathrm{Imm}(M^3, \mathbb{R}^5)_\chi$ and let $F\colon W^4 \to \mathbb{R}^5$ be generic such that $\partial F = f$. Denote the algebraic number of cusps of F by $\#\Sigma^{1,1}(F)$ (for the definition see [4]). Then let $j(f) = 3\sigma(W^4) - 3\alpha(M^3) + \#\Sigma^{1,1}(F) + r(F) \in \mathbb{Z}_{2d(\chi)}$.

Note that if $\chi = 0$ and F is an immersion near ∂W^4 then $r(F) = 0$. Thus in this case $j(f)$ agrees with the double of the invariant introduced in [19] (see Definition 4.2.2).

Theorem 4.5.3. *j(f) is well defined, i.e., it does not depend on the choice of the generic map F. Moreover, if f_0 and f_1 are regularly homotopic then $j(f_0) = j(f_1)$.*

Proof. For $i \in \{0,1\}$ let $F_i\colon W_i^4 \to \mathbb{R}^5$ be a generic map such that $\partial F_i = f_i$. Choose a regular homotopy $\{h_t : 0 \le t \le 1\}$ connecting f_0 and f_1. This defines an immersion $H\colon M^3 \times I \to \mathbb{R}^5 \times I$ by the formula $H(x,t) = (h_t(x), t)$. Also choose a closed collar neighborhood U_i of M^3 in W_i^4 and a diffeomorphism $d_i\colon U_i \to M^3 \times [0,\varepsilon]$ for $i = 0,1$. Let $p\colon M^3 \times [0,\varepsilon] \to [0,\varepsilon]$ denote the projection onto the second factor. If ε (i.e., U_i) is sufficiently small then $p \circ d_i$ is non-degenerate along $\ker(dF_i)$ for $i = 0,1$ since $\ker(dF_i)$ never lies in TM^3. Let s_i be an arbitrary smooth extension of $p \circ d_i$ over W_i^4. Now let $F_0' = (F_0, -s_0)\colon -W_0^4 \to \mathbb{R}^6$ and $F_1' = (F_1, s_1 + 1)\colon W_1^4 \to \mathbb{R}^6$. Then F_i' is an immersion on U_i. Notice that $H|(M^3 \times \{0\}) = F_0'|(\partial W_0^4)$ and $H|(M^3 \times \{1\}) = F_1'|(\partial W_1^4)$.

Denote by κ_i the inner normal field of W_i^4 along $M^3 = \partial W_i^4$ and by v_6 the sixth coordinate direction in $\mathbb{R}^6$. Then the inner product $\langle dF_0'(\kappa_0), v_6\rangle < 0$ and $\langle dF_1'(\kappa_1), v_6\rangle > 0$. Furthermore, if λ_i denotes the inner normal field of $M^3 \times I$ along $M^3 \times \{i\}$ for $i = 0,1$ then $\langle dH(\lambda_0), v_6\rangle > 0$ and $\langle dH(\lambda_1), v_6\rangle < 0$. So $dH(\lambda_i)$ is homotopic to $-dF_i(\kappa_i)$ in the space of vector fields normal to $H|(M^3 \times \{i\})$. Using Smale's lemma there exists a regular homotopy of H fixed on the boundary $M^3 \times \{0,1\}$ that induces the above homotopy of normal fields. Denote by H' the result of this regular homotopy of H. Then F_0', H' and F_1' fit together to a smooth map F' of $W^4 = -W_0^4 \cup (M^3 \times I) \cup W_1^4$ into $\mathbb{R}^6$ that is an immersion on $M^3 \times I$. Let $\pi\colon \mathbb{R}^6 \to \mathbb{R}^5$ denote the projection map. Then by a small perturbation of H' we can achieve that $F = \pi \circ F'$ is generic.

Since $G = F|(M^3 \times I) = \pi \circ H'$ is prim, the singular surface $\Sigma(G)$ is oriented and a trivialization τ of $\ker(dG)$ is given. If we project τ into $\nu(\Sigma(G) \subset M^3 \times I)$ we obtain a normal field ν along $\Sigma(G)$ that vanishes exactly at the cusps of G, i.e., where τ is tangent to $\Sigma(G)$. So $\#\Sigma^{1,1}(G)$ is equal to the algebraic number of zeroes of ν, which in turn is congruent to $\mathrm{rd}((C(F_0), \nu(F_0)), (C(F_1), \nu(F_1))) = r(F_0) - r(F_1)$ modulo $2d(\chi)$ by Definition 4.3.1.

Now using the result of Szűcs [24] that $3\sigma(W^4) + \#\Sigma^{1,1}(F) = 0$ we get that

$$- \big(3\sigma(W_0^4) + \#\Sigma^{1,1}(F_0) + r(F_0)\big) + \big(3\sigma(W_1^4) + \#\Sigma^{1,1}(F_1) + r(F_1)\big) = 3\sigma(W^4) + \#\Sigma^{1,1}(-F_0 \cup G \cup F_1) = 0. \quad (4.5.1)$$

In the special case $f_0 = f_1 = f$ this implies that $j(f)$ is well defined, and for f_0 and f_1 arbitrary (but regularly homotopic) we get that j is a regular homotopy invariant. $\square$

Proposition 4.5.4. *For any immersion $f\colon M^3 \looparrowright \mathbb{R}^5$ the invariant $j(f)$ is always an even element of $\mathbb{Z}_{2d(\chi)}$, i.e., it is mapped to 0 by the epimorphism $\mathbb{Z}_{2d(\chi)} \to \mathbb{Z}_2$.*

Proof. Choose an immersion $f_1 \in \mathrm{Imm}(M^3, \mathbb{R}^5)_0$ and denote f by f_0. Since in [19] it is proved that $j(f_1)$ is always even ($j(f_1) = 2i(f_1)$ for i as in Definition 4.2.2) it is sufficient to prove that $j(f_0) \equiv j(f_1) \mod 2$. Choose a singular Seifert surface $F_i\colon W_i \to \mathbb{R}^5$ for f_i ($i = 0, 1$) and let $G\colon M^3 \times I \to \mathbb{R}^5$ be a generic map such that $-F_0 \cup G \cup F_1$ is a smooth map on $-W_1 \cup (M^3 \times I) \cup W_2$. Then by equation 4.5.1 above it is sufficient to prove that $r(F_0) - r(F_1) \equiv \#\Sigma^{1,1}(G) \mod 2$. Since f_1 has trivial normal bundle we may choose F_1 to be an immersion in a neighborhood of ∂W_1^4. So G is an immersion in a neighborhood of $M^3 \times \{1\}$, moreover $r(F_1) = 0$. The difference between the present situation and the proof of Theorem 4.5.3 is that now $\ker(dG)$ might be non-orientable. Using Definition 4.3.3 we get that $r_2(F_0) - r_2(F_1) \equiv r(F_0) - r(F_1) \mod 2$. Let ν denote a generic normal field along $\Sigma^1(G)$ that extends both $\nu(F_0)$ and $\nu(F_1)$. By definition $r_2(F_0) - r_2(F_1)$ equals the mod 2 number of zeroes of ν. Thus we only have to prove that $\left|\nu^{-1}(0)\right| \equiv \#\Sigma^{1,1}(G) \mod 2$.

From now on we will denote $\Sigma^1(G)$ by K and the line bundle $\ker(dG) < T(M^3 \times I)|K$ by l. Then l is tangent to K exactly at the points of $\Sigma^{1,1}(G)$. For $\varepsilon > 0$ sufficiently small let $\widetilde{K}$ denote the sphere bundle $S_\varepsilon l$. If ε is sufficiently small then the exponential map of $M^3 \times I$ defines an immersion $s\colon \widetilde{K} \to M^3 \times I$ so that the double points of s correspond exactly to the points of $\Sigma^{1,1}(G)$. So we have to prove that $|D_2(s)| \equiv \left|\nu^{-1}(0)\right| \mod 2$. By Lemma 4.4.1 the surface $\widetilde{K}$ is the orientation double cover of K, in particular $\widetilde{K}$ is oriented and a sign can be given to each double point of s (here we also use that $\dim(\widetilde{K})$ is even). The sign of a double point of s is the opposite of the sign of the corresponding cusp of G (since the sign of a cusp is defined as the self intersection of K). Thus $\#D_2(s) = -\#\Sigma^{1,1}(G)$. Let $p\colon \widetilde{K} \to K$ denote the covering map. Then $p^*\nu_K \approx \nu_s$, thus $p^*\nu$ defines a section $\widetilde{\nu}$ of ν_s. From the construction of $\widetilde{\nu}$ it is clear that $\#\widetilde{\nu}^{-1}(0) = 2\#\nu^{-1}(0)$.

If we perturb s in the direction of $\widetilde{\nu}$ we get a self intersection point of s for each element of $\widetilde{\nu}^{-1}(0)$ and two self intersection points for each double point of s. Thus

$$s \cap (s + \varepsilon\widetilde{\nu}) = \#\widetilde{\nu}^{-1}(0) + 2D_2(s) = 2(\#\nu^{-1}(0) - \#\Sigma^{1,1}(G)).$$

So we only have to show that the left hand side is divisible by 4.

From now on we will work in a fixed tubular neighborhood T of $C(F_0) \subset M^3$. Note that $\partial(s, \widetilde{\nu}) = (C(F_0) + \varepsilon\nu(F_0), \nu(F_0)) \cup (-C(F_0) - \varepsilon\nu(F_0), \nu(F_0)) \subset T \times \{0\}$. Let us denote by C the one-dimensional submanifold $(C(F_0) + \varepsilon\nu(F_0)) \cup (-C(F_0) - \varepsilon\nu(F_0))$ of M^3. Then $s \cap (s + \varepsilon\widetilde{\nu}) = r(C, \nu(F_0)) \in \mathbb{Z}$ since $C \sim C(F_0) - C(F_0)$ is null homologous in M^3. We define an embedding $e\colon C(F_0) \times [-\varepsilon, \varepsilon] \to T$ by the formula $e(x, t) = x + t\nu(F_0)$. Then $E = \mathrm{Im}(e)$ is a 2-dimensional oriented submanifold of T with boundary C. Thus $r(C, \nu(F_0)) = E \cap (C + \nu(F_0))$ where the right hand side is considered to be a generic intersection (each fiber of E is parallel to $\nu(F_0)$). Let n be a small non-zero vector field along $C(F_0)$ orthogonal to $\nu(F_0)$. Then $E \cap (C + \nu(F_0) + n) = \emptyset$ (this can be verified by inspecting each fiber of T), thus $r(C, \nu(F_0)) = 0$. So we get that $\#\nu^{-1}(0) = \#\Sigma^{1,1}(G)$, not just a mod 2 congruence. $\square$

Remark 4.5.1. A small improvement on the proof of Proposition 4.5.4 yields an interesting result: Let $G\colon M^3 \times [0,1] \to \mathbb{R}^5$ be a generic map connecting the immersions f_0 and f_1. Let K denote the singular surface of G and let ν_i be a trivialization of $\ker(dG)|(M^3 \times \{i\})$ for $i = 0, 1$. Then $\#\Sigma^{1,1}(G)$ is equal to the relative twisted normal Euler class $e(\nu_K; \nu_0, \nu_1)$.

Note that if $\partial K = C_0 \cup C_1$ then by definition

$$\mathrm{rd}_2((C_0, \nu_0), (C_1, \nu_1)) \equiv e(\nu_K; \nu_0, \nu_1) \mod 2.$$

If in particular K is an oriented cobordism between C_0 and C_1 then

$$\mathrm{rd}((C_0, \nu_0), (C_1, \nu_1)) = e(\nu_K; \nu_0, \nu_1) = \#\Sigma^{1,1}(G);$$

here C_i is oriented by ν_i using the isomorphism Ψ (see Definition 4.4.2).

Thus j may take only $d(\chi)$ different values if $d(\chi) > 0$. (Since j is additive if a connected sum is taken with an immersion of a sphere (Lemma 4.6.1) and $j(g)$ can be any even number for $g\colon S^3 \looparrowright \mathbb{R}^5$ it follows that j is an epimorphism onto $2\mathbb{Z}_{2d(\chi)}$.) But Theorem 4.2.1 implies that there are exactly $2d(\chi)$ regular homotopy classes with normal Euler class χ. So j describes the regular homotopy class of f only up to a $2:1$ ambiguity. To resolve this problem we will lift the invariant $j \in \mathbb{Z}_{2d(\chi)}$ to an invariant $I \in \mathbb{Z}_{4d(\chi)}$. It follows from Proposition 4.5.4 that I is always an even element, thus it defines an invariant $i \in \mathbb{Z}_{2d(\chi)}$ by the embedding $\mathbb{Z}_{2d(\chi)} \hookrightarrow \mathbb{Z}_{4d(\chi)}$.

Notation 4.5.5. *Let $f \in Imm(M^3, \mathbb{R}^5)_\chi$ and let $F\colon W^4 \to \mathbb{R}^5$ be a singular Seifert surface for f. If κ denotes the inner normal field along ∂W^4 then let $\bar{w}(F) \in \Gamma(\nu_f)$ be the projection of $dF(\kappa)$ into ν_f. If $\bar{t} \in \Gamma(\varepsilon^1_M)$ denotes a trivialization of ε^1_M then we can consider $\bar{t}$ and $\bar{w}(F)$ to be sections of $\nu_f \oplus \varepsilon^1_M$. Let $\bar{v}(F) = \bar{w}(F) + \bar{t} \in \Gamma(\nu_f \oplus \varepsilon^1_M)$.*

From now on we will fix a spin structure $s_M \in \mathrm{Spin}(M^3)$. If we consider $\mathbb{R}^5$ with its unique spin structure then for every immersion $f\colon M^3 \looparrowright \mathbb{R}^5$ a spin structure $s(f)$ is induced on ν_f by s_M. Then $s(f)$ is equivalent to a trivialization $\tau(f)\colon \varepsilon^3_M|\mathrm{sk}_2(M) \to \nu_f \oplus \varepsilon^1_M|\mathrm{sk}_2(M^3)$ up to homotopy. Since $\pi_2(SO(3)) = 0$ the trivialization $\tau(f)$ extends to an isomorphism $\tau(f)\colon \varepsilon^3_M \to \nu_f \oplus \varepsilon^1_M$, but this extension is not unique because $\pi_3(SO(3)) \neq 0$.

Definition 4.5.3. Using the above notations let $t(f), v(F) \in \Gamma(\varepsilon^3_M)$ be defined by the formulas $t(f) = \tau(f)^{-1} \circ \bar{t}$ and $v(F) = \tau(f)^{-1} \circ \bar{v}(F)$. Denote by $a(F)$ the quadruple $(C(F), \nu(F), t(f), v(F)) \in N(M^3, \chi)$. Then define $R(F) \in \mathbb{Z}_{4d(\chi)}$ to be $R(a(F))$. Since the homotopy class of the map $(t(f), v(F))|\mathrm{sk}_2(M^3)\colon \mathrm{sk}_2(M^3) \to V_2(\mathbb{R}^3)$ is independent of the choice of the extension of $\tau(f)$ to ε^3_M Proposition 4.3.7 implies that $R(F)$ is also independent of $\tau(f)$ and depends only on s_M.

Remark 4.5.2. Proposition 4.3.8 implies that $r(F) \equiv R(F) \mod 2d(\chi)$.

Now we can finally define a complete regular homotopy invariant.

Definition 4.5.4. For $f \in \mathrm{Imm}(M^3, \mathbb{R}^5)_\chi$ and a singular Seifert surface F let $I(f) \in \mathbb{Z}_{4d(\chi)}$ be defined as $3\sigma(W^4) - 3\alpha(M^3) + \#\Sigma^{1,1}(F) + R(F)$. (Recall that we have fixed a spin structure s_M on M^3 for the definition of $R(F)$.) Remark 4.5.2 above implies that $j(f) \equiv I(F) \mod 2d(\chi)$. Thus by Proposition 4.5.4 we get that $I(F)$ is always an even element of $\mathbb{Z}_{4d(\chi)}$. Let us denote by $\frac{1}{2}$ the isomorphism from $2\mathbb{Z}_{4d(\chi)}$ to $\mathbb{Z}_{2d(\chi)}$. Then let $i(F) = \frac{1}{2}I(F)$.

Clearly $j(f) = 2i(f)$ for every $f \in \mathrm{Imm}(M^3, \mathbb{R}^5)_\chi$.

Theorem 4.5.6. *I(f) is well defined, i.e., it does not depend on the choice of the generic map F. Moreover, if f_0 and f_1 are regularly homotopic then $I(f_0) = I(f_1)$.*

Proof. Using the notations of the proof of Theorem 4.5.3 we only have to show that the surface $K = \Sigma(G) \subset M^3 \times I$ satisfies Definition 4.3.4. I.e., there exist generic sections t and v of $\varepsilon^3_{M\times I}$ that extend $t(f_i)$ and $v(F_i)$ for $i = 0, 1$ and are linearly dependent exactly over K. The regular homotopy between f_0 and f_1 defines the immersion $H\colon M^3 \times I \looparrowright \mathbb{R}^5 \times I$. For $i \in \{0, 1\}$ there is a canonic isomorphism $\varphi_i\colon \nu_H|(M^3 \times \{i\}) \to \nu_{f_i}$. Let $\bar{w} \in \Gamma(\nu_H)$ denote the projection of the sixth coordinate vector $v_6 \in \mathbb{R}^6$ into ν_H. Then $\varphi_i \circ (\bar{w}|M^3 \times \{i\}) = \bar{w}(F_i)$. Moreover, $K = \bar{w}^{-1}(0)$ and the orientation of K is defined as the self intersection of H if perturbed in the direction of $\bar{w}$. Define $\bar{t}$ to be a trivialization of the $\varepsilon^1_{M\times I}$ component of the bundle $\nu_H \oplus \varepsilon^1_{M\times I}$ and let $\bar{v} = \bar{w} + \bar{t}$. Then $\bar{t}$ and $\bar{v}$ are linearly dependent exactly at the points of K. Note that $t(f_i) = \tau(f_i)^{-1} \circ (\varphi_i \oplus \mathrm{id}_{\varepsilon^1})(\bar{t}|M^3 \times \{i\})$ and $v(F_i) = \tau(f_i)^{-1} \circ (\varphi_i \oplus \mathrm{id}_{\varepsilon^1})(\bar{v}|M^3 \times \{i\})$. Thus we only have to define a trivialization $\tau\colon \varepsilon^3_{M\times I} \to \nu_H \oplus \varepsilon^1_{M\times I}$ such that

$$\tau|(M^3 \times \{i\}) = \varphi_i^{-1} \circ \tau(f_i) \quad \text{for} \quad i = 0, 1. \tag{4.5.2}$$

The spin structure $s_M \in \mathrm{Spin}(M^3)$ and the unique spin structure on I define a spin structure on $M^3 \times I$. Together with the unique spin structure of $\mathbb{R}^6$ we get a spin structure s_H on ν_H. When s_H is restricted to $M^3 \times \{i\}$ we get back the spin structure s_M. Thus s_H defines a trivialization

$$\tau_H\colon \varepsilon^3_{M\times I}|\mathrm{sk}_2(M^3 \times I) \to (\nu_H \oplus \varepsilon^1_{M\times I})|\mathrm{sk}_2(M^3 \times I)$$

satisfying equation 4.5.2 over the 2-skeleton of $M^3 \times \{0, 1\}$. Note that the trivialization $\tau(f_i)$ is only well defined over $\mathrm{sk}_2(M^3)$ and that we can choose an arbitrary extension over M^3 in order to define the rotation difference. Thus we only have to extend τ_H to a trivialization τ of $\nu_H \oplus \varepsilon^1_{M\times I}$ and then define $\tau(f_i)$ by formula 4.5.2.

First we extend τ_H to $\mathrm{sk}_3(M^3 \times I) \setminus \mathrm{sk}_3(M^3 \times \{1\})$. This is possible since the obstruction to extending the trivialization over a 3-simplex from its boundary lies in $\pi_2(SO(3)) = 0$. If σ^3 is a 3-simplex of M^3 then we can extend τ_H to $\sigma^3 \times I$ since it is given only on $\partial(\sigma^3 \times I) \setminus (\sigma^3 \times \{1\})$. Thus we have obtained the required extension τ of τ_H. $\square$

4.6 Connected sums and completeness of the invariant i

Lemma 4.6.1. *If $f \in Imm(M^3, \mathbb{R}^5)_\chi$ and $g \in Imm(S^3, \mathbb{R}^5)$ then $c(f\#g) = c(f)$, in particular $e(\nu_{f\#g}) = e(\nu_f)$. Moreover*

$$i(f\#g) = i(f) + (i(g) \mod 2d(\chi)) \in \mathbb{Z}_{2d(\chi)}.$$

Proof. Since $c(f)$ describes the regular homotopy class of $f|(M^3_\circ)$ it is trivial that $c(f\#g) = c(f)$. Let F be a singular Seifert surface of f and G of g such that G is an immersion near the boundary. Then the result follows by inspecting the boundary connected sum $F \natural G$ and the fact that $C(G) = \emptyset$. □

Theorem 4.6.2. *Suppose that the immersions $f_0, f_1 \in Imm(M^3, \mathbb{R}^5)$ are regularly homotopic on $M^3 \setminus D$, where $D \subset M^3$ is diffeomorphic to the closed disc D^3 (i.e., $c(f_0) = c(f_1)$). Then $i(f_0) = i(f_1)$ implies that f_0 is regularly homotopic to f_1.*

Proof. The proof consists of two cases according to the value of $d(\chi)$.

If $d(\chi) > 0$ then i takes values in $\mathbb{Z}_{2d(\chi)}$ which is a finite group. Theorem 4.2.1 implies that there are exactly $2d(\chi)$ regular homotopy classes with a fixed Wu invariant c. Thus we only have to show that the invariant i restricted to immersions with Wu invariant c is an epimorphism onto $\mathbb{Z}_{2d(\chi)}$. For this end choose an immersion $f \in \mathrm{Imm}(M^3, \mathbb{R}^5)$ such that $c(f) = c$. In [4] it is shown that $i\colon \mathrm{Imm}[S^3, \mathbb{R}^5] \to \mathbb{Z}$ is a bijection. Thus Lemma 4.6.1 implies that $c(f\#g) = c(f) = c$ for every $g \in \mathrm{Imm}(S^3, \mathbb{R}^5)$, moreover $i\colon \{\, f\#g\colon g \in \mathrm{Imm}(S^3, \mathbb{R}^5) \,\} \to \mathbb{Z}_{2d(\chi)}$ is surjective.

If $d(\chi) = 0$ then i maps into $\mathbb{Z}$. Using Smale's lemma we can suppose that $f_0|(M^3 \setminus D) = f_1|(M^3 \setminus D)$. The normal bundles of $f_0|D$ and $f_1|D$ in $\mathbb{R}^5$ are trivial, choose a trivialization for both of them. Let τ_0 be a non-zero normal field along $f_0|D$. Then $\tau_0|\partial D$ considered in the trivialization of the normal bundle of $f_1|D$ is a map $(\tau_0|\partial D)\colon \partial D \to S^1$. Since ∂D is homeomorphic to S^2 and $\pi_2(S^1) = 0$ the normal field $\tau_0|\partial D$ can be extended to a normal field τ_1 of $f_1|D$. Thus τ_i is a normal field of $f_i|D$ for $i = 0, 1$ and $\tau_0|\partial D = \tau_1|\partial D$.

Next choose an oriented compact manifold W_0^4 with boundary M^3. We push D into the interior of W_0^4 fixing the boundary ∂D to obtain a 3-disc $D_1 \subset W_0^4$ so that $\partial D = \partial D_1$ and $M_1^3 = (M^3 \setminus D) \cup D_1$ is a smooth submanifold of W_0^4. If we throw out the domain bounded by D and D_1 in W_0^4 we obtain a 4-dimensional submanifold W_1^4 of W_0^4 with boundary M_1^3. Clearly W_0^4 is diffeomorphic to W_1^4.

We can choose a generic map $F_0\colon W_0^4 \to \mathbb{R}^5$ with the following three properties:

1. $F_0|M^3 = f_0$ and $F_0|M_1^3 = f_1$ (where M_1^3 is identified with M^3 by a diffeomorphism keeping $M^3 \setminus D$ fixed).

2. F_0 is an immersion in a neighborhood of D and D_1.

3. If κ_0 denotes the inner normal field of D in W_0^4 and κ_1 denotes the inner normal field of D_1 in W_1^4 then $dF_0 \circ \kappa_0 = \tau_0$ and $dF_1 \circ \kappa_1 = \tau_1$.

Let $F_1 = F_0|W_1^4$. Then (2) implies that $C(F_0) = C(F_1) \subset M^3 \setminus D$, moreover $\nu(F_0) = \nu(F_1)$. In particular, the normal Euler class of f_0 and f_1 coincide. Thus $R(F_0) = R(F_1)$. Since $\sigma(W_0^4) = \sigma(W_1^4)$, we get that

$$0 = i(f_0) - i(f_1) = \#\Sigma^{1,1}(F_0|(W_0^4 \setminus W_1^4)).$$

Choose diffeomorphisms $d_0\colon S_+^3 \to D$ and $d_1\colon S_+^3 \to D_1$, where S_+^3 denotes the northern hemisphere of S^3. Then the immersion $F_0 \circ d_i$ can be extended to an immersion $f_i'\colon S^3 \looparrowright \mathbb{R}^5$ for $i = 0, 1$ so that $f_0'|S_-^3 = f_1'|S_-^3$. (This is possible since $j^1(f_0)|\partial D = j^1(f_1)|\partial D$.) Now repeating the same argument as above for f_0' and f_1', we obtain that

$$i(f_0') - i(f_1') = \#\Sigma^{1,1}(F_0|(W_0^4 \setminus W_1^4)).$$

(Note that τ_0 and τ_1 have a common extension over S_-^3.) Thus $i(f_0') - i(f_1') = 0$, so using [4] we get that f_0' and f_1' are regularly homotopic. But this implies that there exists a regular homotopy between f_0' and f_1' that is fixed on S_-^3 (see [10], Lemma 3.33). So $f_0|D$ and $f_1|D$ are regularly homotopic keeping the 1-jets on the boundary fixed, which completes the proof that f_0 and f_1 are regularly homotopic. □

Corollary 4.6.3. *The map*

$$(c, i)\colon \mathit{Imm}[M^3, \mathbb{R}^5] \to \coprod_{c \in H^2(M^3;\mathbb{Z})} \mathbb{Z}_{4d(c)}$$

is a bijection.

We get more structure on the set of regular homotopy classes of immersions of oriented 3-manifolds into $\mathbb{R}^5$ if we endow it with the connected sum operation. Let us introduce the notation

$$I(3,5) = \left\{\, [f]\colon [f] \in \mathrm{Imm}[M^3, \mathbb{R}^5] \text{ for } M^3 \text{ oriented} \,\right\}.$$

Then $(I(3,5), \#)$ is a semigroup whose structure is described in the following theorem.

Theorem 4.6.4. *Let M_1^3 and M_2^3 be oriented 3-manifolds. Then*

$$H^2(M_1^3 \# M_2^3; \mathbb{Z}) \approx H^2(M_1^3; \mathbb{Z}) \oplus H^2(M_2^3; \mathbb{Z}). \tag{4.6.1}$$

If $f_i \in \mathit{Imm}(M_i^3, \mathbb{R}^5)$ for $i = 1, 2$ then

$$c(f_1 \# f_2) = c(f_1) \oplus c(f_2) \in H^2(M_1^3 \# M_2^3; \mathbb{Z}). \tag{4.6.2}$$

Moreover, if χ_i denotes the normal euler class of f_i and χ the normal euler class of $f_1 \# f_2$ then $d(\chi) = \gcd(d(\chi_1), d(\chi_2))$. Finally,

$$i(f_1 \# f_2) = (i(f_1) \mod 2d(\chi)) + (i(f_2) \mod 2d(\chi)) \in \mathbb{Z}_{2d(\chi)}, \tag{4.6.3}$$

where $i(f_i) \in \mathbb{Z}_{2d(\chi_i)}$ for $i = 1, 2$.

Proof. Equation 4.6.1 follows from the fact that $H^2(M_i^3;\mathbb{Z}) \approx H^2(M_i^3 \setminus D^3;\mathbb{Z})$ (see the long exact sequence of the pair $(M_i^3, M_i^3 \setminus D^3)$) and the Mayer-Vietoris exact sequence for $M_1^3 \# M_2^3 = (M_1^3 \setminus D^3) \cup (M_2^3 \setminus D^3)$.

Equation 4.6.2 can be seen from the description of $c(f_i)$ as the regular homotopy class of $f_i|\mathrm{sk}_2(M^3)$. Since $\chi = \chi_1 \oplus \chi_2$ the statement about $d(\chi)$ is trivial.

Finally, equation 4.6.3 is obtained by taking the boundary connected sum $F_1 \natural F_2$ of singular Seifert surfaces F_1 and F_2 for f_1, respectively f_2. $\square$

4.7 Immersions of M^3 into $\mathbb{R}^6$ with a normal field

Let $\mathrm{Imm}_1(M^3, \mathbb{R}^6)$ denote the space of immersions of M^3 into $\mathbb{R}^6$ with a normal field ν. Moreover, let $\mathrm{Imm}_1[M^3, \mathbb{R}^6] = \pi_0(\mathrm{Imm}_1(M^3, \mathbb{R}^6))$ be the set of regular homotopy classes of such immersions with normal fields. If we fix a trivialization of TM^3 then Hirsch's theorem [8] implies that the natural map $\mathrm{Imm}_1(M^3, \mathbb{R}^6) \to C(M^3, V_4(\mathbb{R}^6))$ is a weak homotopy equivalence.

For $f \in \mathrm{Imm}(M^3, \mathbb{R}^5)$ let $\iota(f) \in \mathrm{Imm}_1(M^3, \mathbb{R}^6)$ be the immersion f with the constant normal field defined by the sixth coordinate vector in $\mathbb{R}^6$. Thus ι is an embedding of $\mathrm{Imm}(M^3, \mathbb{R}^5)$ into $\mathrm{Imm}_1(M^3, \mathbb{R}^6)$. As a special case of Hirsch's compression theorem we have the following proposition.

Proposition 4.7.1. $\iota_*\colon$ *Imm*$[M^3, \mathbb{R}^5] \to$ *Imm*$_1[M^3, \mathbb{R}^6]$ *is a bijection.*

Proof. The embedding $\mathbb{R}^5 \hookrightarrow \mathbb{R}^6$ induces an embedding $V_3(\mathbb{R}^5) \hookrightarrow V_4(\mathbb{R}^6)$ and thus a map $\psi\colon [M^3, V_3(\mathbb{R}^5)] \to [M^3, V_4(\mathbb{R}^6)]$ that makes the following diagram commutative.

$$\begin{array}{ccc} \mathrm{Imm}[M^3, \mathbb{R}^5] & \xrightarrow{\iota_*} & \mathrm{Imm}_1[M^3, \mathbb{R}^6] \\ \downarrow & & \downarrow \\ [M^3, V_3(\mathbb{R}^5)] & \xrightarrow{\psi} & [M^3, V_4(\mathbb{R}^6)]. \end{array}$$

By Hirsch's theorem the vertical arrows are bijections, thus it is sufficient to prove that ψ is also a bijection. To see this consider the fibration $V_3(\mathbb{R}^5) \to V_4(\mathbb{R}^6) \to S^5$. Then from the homotopy exact sequence of this fibration we get that the homomorphism $\pi_i(V_3(\mathbb{R}^5)) \to \pi_i(V_4(\mathbb{R}^6))$ is an isomorphism for $i \leq 3$ and this implies that ψ is a bijection. $\square$

The natural forgetful map $\varphi\colon \mathrm{Imm}_1(M^3, \mathbb{R}^6) \to \mathrm{Imm}(M^3, \mathbb{R}^6)$ is a Serre fibration.

Proposition 4.7.2. *For any immersion* $f\colon M^3 \looparrowright \mathbb{R}^6$ *the normal bundle* ν_f *is trivial.*

Proof. Since M^3 is spin the normal bundle ν_f is also spin, thus it is trivial over the 2-skeleton of M^3. Such a trivialization can be extended to the 3-simplices of M^3 because $\pi_2(SO(3)) = 0$. $\square$

So φ is surjective and the fiber of φ is homotopy equivalent to $\Gamma(\nu_f) = C(M^3, S^2)$. Thus the end of the homotopy exact sequence of φ looks like as follows:

$$\pi_1(\mathrm{Imm}(M^3, \mathbb{R}^6)) \longrightarrow [M^3, S^2] \longrightarrow \mathrm{Imm}_1[M^3, \mathbb{R}^6] \xrightarrow{\varphi_*} \mathrm{Imm}[M^3, \mathbb{R}^6] \longrightarrow 0.$$

By Hirsch's theorem there is a bijection $\mathrm{Imm}[M^3, \mathbb{R}^6] \approx [M^3, V_3(\mathbb{R}^6)]$. Since $V_3(\mathbb{R}^6)$ is 2-connected and $\pi_3(V_3(\mathbb{R}^6)) \approx \mathbb{Z}_2$ we get from obstruction theory that $[M^3, V_3(\mathbb{R}^6)] \approx H^3(M^3; \mathbb{Z}_2) \approx \mathbb{Z}_2$. It is well known that for $f \in \mathrm{Imm}(M^3, \mathbb{R}^6)$ the regular homotopy class of f is determined by the number of its double points $D(f)$ modulo 2. This gives a geometric interpretation of the map φ_*: for $(f, \nu) \in \mathrm{Imm}_1(M^3, \mathbb{R}^6)$ the regular homotopy invariant $\varphi_*(f, \nu)$ is equal to $D(f)$ modulo 2.

How can we determine the value of $\varphi_* \circ \iota_*(g)$ for a generic $g \in \mathrm{Imm}(M^3, \mathbb{R}^5)$? This question was answered in [23], let us recall that result now. The self-intersection set $A(g)$ of g is a closed 1-dimensional submanifold of M^3 and $g(A(g))$ is also a closed 1-dimensional submanifold of $\mathbb{R}^5$. We say that a component C of $g(A(g))$ is non-trivial if the double cover $g|g^{-1}(C) \colon g^{-1}(C) \to C$ is non-trivial, i.e., if $g^{-1}(C)$ is connected. Let the number of non-trivial components be denoted by $\delta(g)$. In [23] Szűcs proved the following.

Theorem 4.7.3. *Suppose that $f \colon M^3 \to \mathbb{R}^6$ is a generic immersion and that $\pi \colon \mathbb{R}^6 \to \mathbb{R}^5$ is a projection such that $g = \pi \circ f$ is also a generic immersion. Then*

$$D(f) \equiv \delta(g) \mod 2.$$

Note that the immersions f and g above are regularly homotopic in $\mathbb{R}^6$. Thus for any generic g we have that $\varphi_* \circ \iota_* = \delta$.

Now we are going to determine the group $\pi_1(\mathrm{Imm}(M^3, \mathbb{R}^6))$. Using Hirsch's theorem we get that it is isomorphic to $\pi_1(C(M^3, V_3(\mathbb{R}^6))) = [SM^3, V_3(\mathbb{R}^6)]$. Here SM^3 denotes the suspension of M^3 and is a 4-dimensional CW complex. The space $V_3(\mathbb{R}^6)$ is 2-connected and $\pi_3(V_3(\mathbb{R}^6)) \approx \mathbb{Z}_2$. Moreover, $\pi_4(V_3(\mathbb{R}^6)) \approx 0$. This can be seen as follows: From the homotopy exact sequence of the fibration $V_3(\mathbb{R}^6) \to V_4(\mathbb{R}^7) \to S^6$ we get that $\pi_4(V_3(\mathbb{R}^6)) \approx \pi_4(V_4(\mathbb{R}^7))$. It was shown by Paechter [15] that for $k \geq 4$ the isomorphism $\pi_k(V_k(\mathbb{R}^{2k-1})) \approx 0$ holds if $k \equiv 0 \mod 4$. Thus obstruction theory yields that $[SM^3, V_3(\mathbb{R}^6)] \approx H^3(SM^3; \mathbb{Z}_2) \approx H^2(M^3; \mathbb{Z}_2)$.

Putting together the above results we obtain the following theorem.

Theorem 4.7.4. *The following sequence is exact:*

$$H^2(M^3; \mathbb{Z}_2) \longrightarrow [M^3, S^2] \longrightarrow \mathit{Imm}[M^3, \mathbb{R}^5] \xrightarrow{\delta} \mathbb{Z}_2 \longrightarrow 0.$$

Remark 4.7.1. If we fix a trivialization of TM^3 then non-zero vector fields (or equivalently, oriented 2-plane fields) on M^3 correspond to maps $M^3 \to S^2$. Thus the set of homotopy classes of oriented 2-plane fields on M^3 is equal to $[M^3, S^2]$ which was determined in Remark 4.3.1. A geometric classification of such oriented 2-plane fields, avoiding the use of a trivialization of TM^3, was carried out by Gompf (see [6], section 4). A complete set of homotopy invariants, similar to those introduced in our present paper, were obtained in [6]. Gompf's result and the regular homotopy classification of immersions of M^3 into $\mathbb{R}^5$ are related by Theorem 4.7.4.

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