Pontrjagin-Thom construction for singular maps with negative codimension

Boldizsár Kalmár

degree-thesis

Consultant

András Szűcs

August 26, 2005

Contents

Introduction

This paper is about cobordisms of fold maps with negative codimension. The study of maps with negative codimension includes the study of the possible singular fibers as well. In the case of codimension -1 the singular fibers are one-dimensional complexes, i.e. graphs. Kushner, Levine and Porto [**6, 7**] started to study these graphs and the behaviour of singularities using these graphs. Their work has been continued by Saeki and Yamamoto [**11, 12, 19**].

To study -1-codimensional singularities another tool is the Stein factorization of a map which is a simpler form of the map, which allows us to forget the singular fibers and emphasizes other essential properties like the alteration of the number of connected components of the fibers. Besides singular fibers, Stein factorization was studied by Kushner, Levine and Porto [**6, 7**] in order to describe the singularities of fold maps from 3-dimensional manifolds into the plane.

The goal of this paper is to join the methods of singular fibers, Stein factorization and Pontrjagin-Thom construction (see [**10, 13, 18**]) in order to obtain cobordism invariants of fold maps, compute the cobordism groups and give geometrical interpretations of the results.

Singular fibers, Stein factorization and Pontrjagin-Thom construction are three extensive subjects, and all of them are based on a classification of the singularities. The singularities occurred in this paper was classified by Saeki and Yamamoto [**11, 12**] by using singular fibers.

Pontrjagin-Thom construction for fold maps with negative codimension is a main result of this paper, while the extended version of Pontrjagin-Thom construction for singular maps with positive codimension is a result of Rimányi and Szűcs [**10, 13**]. Concerning Pontrjagin-Thom construction with negative codimension, many ideas come from the construction for positive codimension.

In this paper there are three chapters. They sequence shows a progression in the abstraction and a formation of a method, the work of the author in his last years.

In the first chapter we compute the cobordism group of Morse functions on unoriented surfaces. Ikegami and Saeki [**4**] have proved that the cobordism group of Morse functions on oriented surfaces is an infinite cyclic group. Their method is applicable with a little modification to the computation of the cobordism group of Morse functions on unoriented surfaces. We prove that this group is isomorphic to the direct sum of the infinite cyclic group and the finite group of order two.

In the second chapter we compute the cobordism group of Stein factorizations of fold maps from oriented 3-manifolds into the plane using a Pontrjagin-Thom-type construction. Using this result we compute the oriented cobordism group of fold maps from 3-manifolds into the plane. It turns out that these two groups are isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. We have the analogous results for bordisms as well.

In the third chapter, using a Pontrjagin-Thom-type construction we compute some other cobordism groups of fold maps with prescribed singularities with codimension -1. The results show strong relations with cobordism groups of immersions and stable homotopy groups.

The results of this paper are strongly based on the results of O. Saeki and A. Szűcs. Besides them the author would like to thank the contribution to all the people who had strong influence through useful questions and conversations on his last years.

Special thanks to A. Stipsicz for the help and encouragement, to O. Saeki for the thorough correcting of the first chapter and for A. Szűcs for the persistent and particularly interesting teachings in the topology, for the active help during the consultations and for correcting the English drafting.

CHAPTER 1

Cobordism group of Morse functions on unoriented surfaces

1. Introduction

Ikegami and Saeki [**4**] initiated the study of the oriented cobordism group of Morse functions on oriented surfaces. Summarizing in an informal language, two Morse functions $f_0 \colon M_0 \to \mathbb{R}$ and $f_1 \colon M_1 \to \mathbb{R}$ on closed surfaces M_0 and M_1 are *cobordant*, if there exists a smooth map $F \colon X \to \mathbb{R} \times [0,1]$ with only fold singularities (i.e. a fold map; fold singularities are the higher dimensional analogues of the non-degenerate critical points of Morse functions; for a precise definition of a fold map, see §2) from a compact 3-manifold X with $\partial X = M_0 \amalg M_1$, and F restricted to ∂X is identified with $f_0 \amalg f_1$, where $\amalg$ denotes the disjoint union (while the normal direction is mapped onto the normal direction isomorphically). Such an F is a *cobordism* between f_0 and f_1. One can define the *cobordism group* of Morse functions on surfaces, which describes the structure of Morse functions on surfaces from the viewpoint of the cobordism relation. Ikegami and Saeki [**4**] have transformed the problem of Morse functions and their cobordisms into a problem of maps from graphs to $\mathbb{R}$ and maps from 2-dimensional polyhedra to $\mathbb{R} \times [0,1]$. This transformation has been realized by using the Stein factorization: the Stein factorization of a cobordism between two Morse functions provides a cobordism in an abstract sense between the Stein factorizations of the Morse functions (called also the Reeb functions).

The computation of the oriented cobordism group of Morse functions on oriented surfaces [**4**] involves two steps:

(1) computing the abstract cobordism group of abstract Reeb functions, which are maps of graphs to $\mathbb{R}$ like Reeb functions, where a cobordism between two such maps is a map from a compact 2-dimensional polyhedron to $\mathbb{R} \times [0,1]$ like the Stein factorization of a cobordism between two Morse functions,
(2) showing that the abstract cobordism group is isomorphic to the cobordism group of Morse functions.

This method enables us to solve the problem in a purely combinatorial context.

More precisely step (1) consists of the following. A cobordism between two Morse functions is a generic map of a compact 3-dimensional manifold into $\mathbb{R} \times [0,1]$. The local structure of the Stein factorizations of such maps is completely described in [**6, 7**]. Using this description, Ikegami and Saeki [**4**] gave an alternative description of an abstract cobordism of Reeb functions, which is more algorithmic: if we change a function on a graph by a finite iteration of simple local moves together with a homotopy, we get a function cobordant to the original function in the abstract sense. Using this observation, Ikegami and Saeki [**4**] gave representatives of the elements of the abstract cobordism group, and proved that it is isomorphic to $\mathbb{Z}$. In step (2) [**4**] realizes an abstract cobordism between Reeb functions of two Morse functions as the Stein factorization of a fold map of an oriented 3-manifold.

In this chapter, we follow the same line of arguments in order to compute the cobordism group of Morse functions[1] on unoriented surfaces. However, we have to modify the method of [**4**]: in step (1), when we want to compute the abstract cobordism group, we have to consider functions and maps coming from the Stein factorizations of Morse functions on *nonorientable* surfaces and those of fold maps of *nonorientable* 3-manifolds, respectively. Levine [**7**] describes these maps completely. With these new cases, we get an equivalent algorithmic description of an abstract cobordism, which differs from [**4**] only in some new moves on a few new types of Reeb graphs. It turns out that the method of the computation of the abstract cobordism group given in [**4**] is applicable in our case as well: we give a set of representatives of the elements of this group by using the algorithm of [**4**], and we show that the group is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_2$. Step (2) also needs some modifications. On the one hand this step becomes simpler since we are allowed to use nonorientable 3-manifolds for the realization of abstract cobordisms between Reeb functions. On the other hand this step becomes more complicated, since the method used in [**4**] cannot be applied here. An abstract cobordism between two given Reeb functions is a map from a compact 2-dimensional polyhedron to $\mathbb{R} \times [0,1]$. In order to realize the neighbourhood of an appropriate 1-skeleton of this polyhedron as a Stein factorization of a fold map of a 3-manifold X with boundary, one can apply the method described in [**4**], but this realization is not a cobordism between the two given Reeb functions. To obtain a cobordism we attach 2-disk bundles and their fold maps to the appropriate part of the boundary of X, as will be described in §4.

2. Preliminaries

DEFINITION 2.1. Let Q^q and N^n be smooth manifolds of dimensions q and n respectively $(q \geq n)$. Let $p \in Q^q$ be a singular point of f. A smooth map $f\colon Q^q \to N^n$ has a *fold singularity* at the singular point p, if we can write f in some local coordinates

[1]The author was informed, when this chapter had already been written, that Ikegami [**3**] had computed the analogous group in each dimension. His method is quite different and more sophisticated than our approach.

at p and $f(p)$ in the form

$$f(x_1,\dots,x_q)=(x_1,\dots,x_{n-1},\pm x_n^2\pm\cdots\pm x_q^2).$$

A smooth map $f\colon Q^q\to N^n$ is called a *fold map*, if f has only fold singularities. A smooth function on a manifold which has only fold singularities is called a *Morse function.*

Let $\amalg$ denote the disjoint union.

DEFINITION 2.2. Two Morse functions $f_0\colon M_0\to\mathbb{R}$ and $f_1\colon M_1\to\mathbb{R}$ on closed surfaces M_0 and M_1 are *cobordant*, if there exists a smooth map $F\colon X\to\mathbb{R}\times[0,1]$ with only fold singularities from a compact 3-manifold X with $\partial X=M_0\amalg M_1$ such that $F\mid_{M_0\times[0,\varepsilon)}=f_0\times\mathrm{id}_{[0,\varepsilon)}$ and $F\mid_{M_1\times(1-\varepsilon,1]}=f_1\times\mathrm{id}_{(1-\varepsilon,1]}$, where $M_0\times[0,\varepsilon)$ and $M_1\times(1-\varepsilon,1]$ are small collar neighbourhoods of ∂X with the identifications $M_0=M_0\times\{0\}$, $M_1=M_1\times\{1\}$, and $\mathrm{id}_A(a)=a$ $(a\in A)$ for an arbitrary set A.

We call the map F a *cobordism* between f_0 and f_1.

This clearly defines an equivalence relation.

DEFINITION 2.3. We can define a group operation on the set of the cobordism classes of Morse functions as follows. If $[f_0\colon M_0\to\mathbb{R}]$ and $[f_1\colon M_1\to\mathbb{R}]$ are cobordism classes of Morse functions, then the *sum* of $[f_0]$ and $[f_1]$ is the cobordism class represented by the Morse function $f_0\amalg f_1\colon M_0\amalg M_1\to\mathbb{R}$.

It is easy to show that the cobordism class of the Morse function $f_0\amalg f_1$ does not depend on the choice of the representatives f_0 and f_1.

This group operation is clearly commutative, so we have an abelian group, which we denote by $\mathcal{C}ob_f(2,-1)$, where the lower index "f" comes from "fold map".

REMARK 2.4. One can define the analogous cobordism groups $\mathcal{C}ob_f(q,k)$ $(q\in\mathbb{N}$, $k\in\mathbb{Z}$, $-q\le k\le 0)$ of fold maps $f\colon Q^q\to\mathbb{R}^{q+k}$ from closed q-dimensional manifolds Q^q to $\mathbb{R}^{q+k}$ as well. The integer k is called the *codimension* of the map f.

For $k\ge 0$ there are many results concerning the cobordism groups $\mathcal{C}ob_\tau(q,k)$, where τ is a set of singularity types, the elements of the group $\mathcal{C}ob_\tau(q,k)$ are cobordism classes of smooth maps with only singularities in τ, and a cobordism between two such maps has only singularities in τ. See for example [**5, 10, 13, 16**].

The aim of this chapter is to prove

THEOREM 2.5. *The cobordism group $\mathcal{C}ob_f(2,-1)$ is isomorphic to $\mathbb{Z}\oplus\mathbb{Z}_2$.*

We use the notion of the Stein factorization of a smooth map $f\colon Q^q\to N^n$, where Q^q and N^n are smooth manifolds of dimensions q and n respectively $(q\ge n)$. Two points $p_1, p_2 \in Q^q$ are *equivalent* if and only if p_1 and p_2 lie on the same component of an f-fiber. Let W_f denote the quotient space of Q^q with respect to this equivalence relation and $q_f \colon W_f \to N^n$ the quotient map. Then there exists a unique continuous map $\bar{f} \colon W_f \to N^n$ such that $f = \bar{f} \circ q_f$. The space W_f or the factorization of the map f into the composition of q_f and $\bar{f}$ is called the *Stein factorization* of the map f. We call the map $\bar{f}$ the *Stein factorization* of the map f as well.

In the case of a Morse function $f \colon Q^q \to \mathbb{R}$ on a closed manifold Q^q, W_f is a graph (since every regular fiber is a 1-codimensional submanifold in Q^q and the set of critical points of f is finite and discrete), which is called the *Reeb graph*, and the function $\bar{f} \colon W_f \to \mathbb{R}$ is called the *Reeb function* of the Morse function f.

We may assume that the Morse function f is in general position, that is, every fiber contains at most one critical point, since a small perturbation of the function f is a Morse function cobordant to f.

3. Abstract cobordism

Let $f \colon M \to \mathbb{R}$ be a Morse function on a closed surface M. One can see (for example, see [**7, 11**]) that the Reeb function of f in a neighbourhood of the q_f-image of a critical point is equivalent to one of the functions as depicted in Fig. 1. In case (a) the corresponding critical point of f is a local minimum or a local maximum, in case (b) the corresponding critical point of f has index 1 with sign $+1$ or -1, and a sign is associated to each vertex of degree three as in Fig. 1. Case (c) can occur only on a nonorientable surface. The corresponding critical point of f is also of index 1. Note that the Reeb function restricted to an edge of the Reeb graph is always an embedding.

DEFINITION 3.1. We say that a smooth map $F \colon X \to \mathbb{R}^2$ from a closed 3-manifold X has a *definite fold singularity* at a singular point $p \in X$ if we can write F in some local coordinates at p and $F(p)$ in the form

$$F(x_1, x_2, x_3) = (x_1, x_2^2 + x_3^2),$$

and we say that F has an *indefinite fold singularity* at a singular point $p \in X$ if we can write F in some local coordinates at p and $F(p)$ in the form

$$F(x_1, x_2, x_3) = (x_1, x_2^2 - x_3^2).$$

We say that F has a *simple indefinite fold singularity* at a singular point $p \in X$ if the map F has an indefinite fold singularity at p and each component of $F^{-1}(F(p))$ contains at most one singular point. Furthermore, F has a *nonsimple indefinite fold singularity*

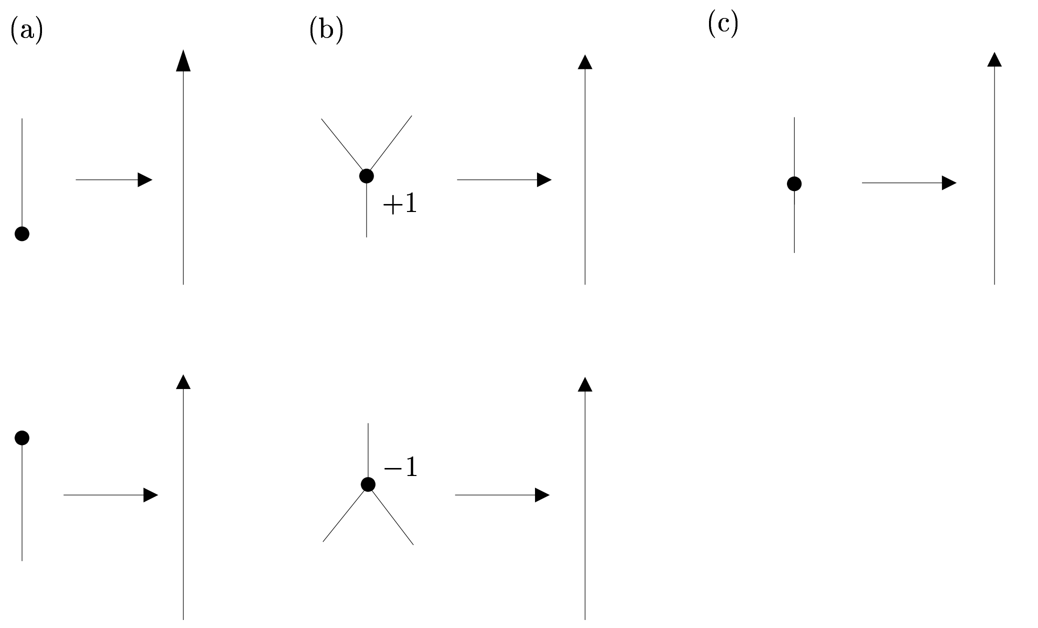

FIGURE 1. The Reeb function of a Morse function f in a neighbourhood of the q_f-image of a critical point

at p if F has an indefinite fold singularity at p and there is a component of $F^{-1}(F(p))$ which contains at least two singular points.

Let $F\colon X \to \mathbb{R}^2$ be a generic smooth map of a closed 3-manifold X with only fold singularities. By [**7**] the Stein factorization $\bar{F}\colon W_F \to \mathbb{R}^2$ of F in a neighbourhood of the q_F-image of a singular point is equivalent to one of the maps as depicted in Fig. 2, where the maps in question are projections to the plane of the paper. The pictures (f)–(i) can occur only if X is nonorientable. The lines in bold indicate the q_F-image of the set of singular points of F, which is a 1-dimensional submanifold of X and is immersed into $\mathbb{R}^2$ by F. The dot in bold indicates the q_F-image of the singular point under study. Case (a) corresponds to a regular value, case (b) corresponds to the image of a definite fold singularity, case (c) corresponds to the image of a simple indefinite fold singularity, cases (d) and (e) correspond to a nonsimple indefinite fold singularity, and these exhaust all the possibilities when X is orientable. Case (f) corresponds to the image of a simple indefinite fold singularity, and cases (g), (h), and (i) correspond to a nonsimple indefinite fold singularity. Details can be found in [**4**] , [**6**] and [**7**].

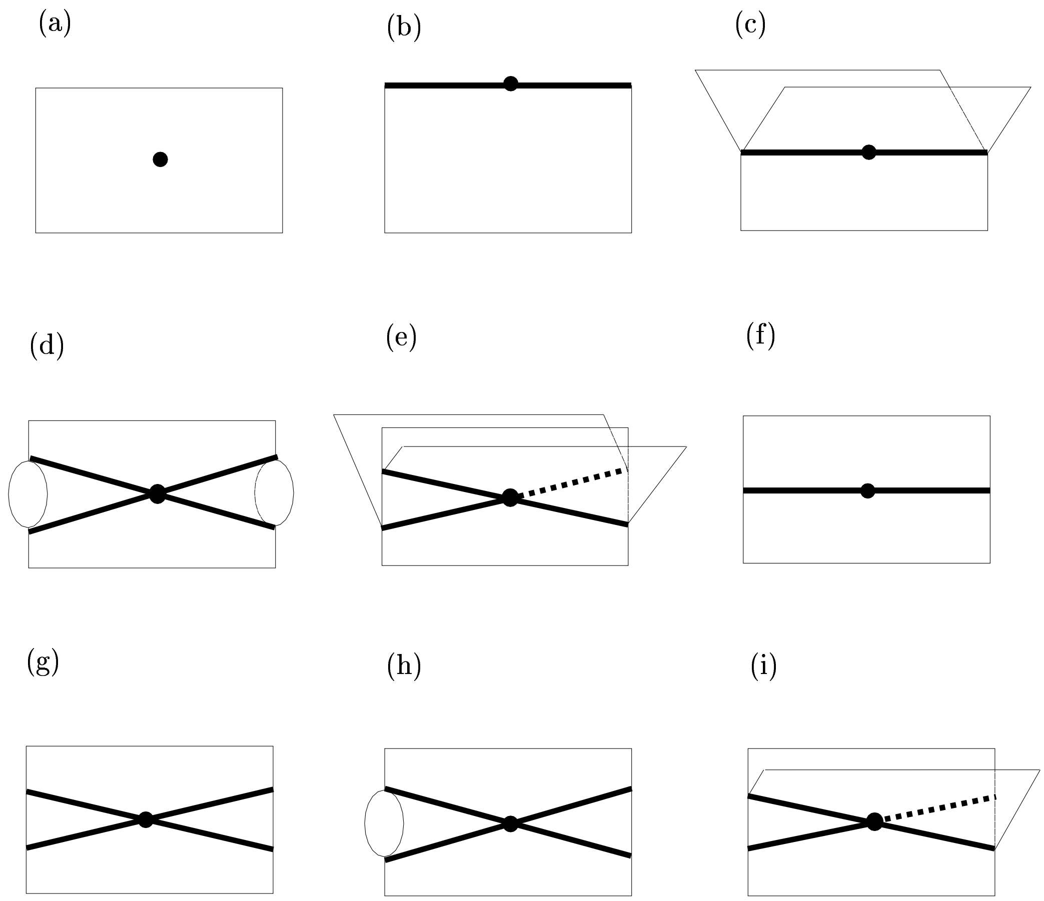

FIGURE 2. The Stein factorization of a fold map F in a neighbourhood of the q_F-image of a singular point

Based on these observations, we give a definition of abstract Reeb functions and their cobordism group analogous to [**4**] as follows.

DEFINITION 3.2. Let G be a finite graph such that each of its vertices is of degree 1, 2 or 3. A continuous function $r\colon\ G \to \mathbb{R}$ is said to be an *abstract Reeb function* if r is an embedding on each edge, and r is equivalent in a neighbourhood of every vertex to one of the functions as depicted in Fig. 1.

DEFINITION 3.3. Let $r_i\colon\ G_i \to \mathbb{R},\ i = 0, 1$, be abstract Reeb functions in the sense of Definition 3.2. We say that r_0 and r_1 are *cobordant* if there exists a continuous map $R\colon\ P \to \mathbb{R} \times [0, 1]$ of a compact 2-dimensional polyhedron P such that

(i) $G_i = R^{-1}(\mathbb{R} \times \{i\})$, $i = 0, 1$, are subcomplexes of P with regular neighbourhoods of the forms $G_0 \times [0, \varepsilon)$ and $G_1 \times (1-\varepsilon, 1]$, where G_i corresponds to $G_i \times \{i\}$, $i = 0, 1$,

(ii) $R|_{G_0 \times [0,\varepsilon)} = r_0 \times \mathrm{id}_{[0,\varepsilon)}$ and $R|_{G_1 \times (1-\varepsilon,1]} = r_1 \times \mathrm{id}_{(1-\varepsilon,1]}$,

(iii) in a neighbourhood of each point of $P \setminus (G_0 \cup G_1)$, the polyhedron P and the map R are equivalent to one of the pictures as depicted in Fig. 2.

Furthermore, we call the map $R\colon P \to \mathbb{R} \times [0,1]$ a *cobordism* between r_0 and r_1.

This clearly defines an equivalence relation.

Definition 3.4. We can define a group operation on the set of the cobordism classes of abstract Reeb functions as follows. If $[r_0\colon G_0 \to \mathbb{R}]$ and $[r_1\colon G_1 \to \mathbb{R}]$ are cobordism classes of abstract Reeb functions, then the *sum* of $[r_0]$ and $[r_1]$ is the cobordism class represented by the abstract Reeb function $r_0 \amalg r_1\colon G_0 \amalg G_1 \to \mathbb{R}$.

It is easy to show that the cobordism class of the abstract Reeb function $r_0 \amalg r_1$ does not depend on the choice of the representatives r_0 and r_1.

Let us denote the cobordism group of abstract Reeb functions by $\mathcal{A}$.

Proposition 3.5. *The cobordism group $\mathcal{A}$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_2$.*

Proof. By the facts explained in [**4**], we see that two abstract Reeb functions $r_0\colon G_0 \to \mathbb{R}$ and $r_1\colon G_1 \to \mathbb{R}$ are cobordant if and only if we can deform r_0 to r_1 by a finite iteration of the eleven local moves as depicted in Fig. 3 up to a homotopy in the space of abstract Reeb functions.

By [**4**] we can transform every abstract Reeb function whose domain (source) graph has no vertices of degree 2, using the first seven local moves of Fig. 3 together with a homotopy in the space of abstract Reeb functions, into one of the abstract Reeb functions as depicted in Fig. 4, where n stands for the sum of the signs over all vertices of degree 3.

Lemma 3.6. *Every abstract Reeb function is cobordant to one of the functions as depicted in Figs.* 4 *and* 5.

Proof. With the help of the local moves (j) and (k) of Fig. 3, we can apply the algorithm of [**4**] even if a graph has vertices of degree 2, since we can lower any vertex of degree 2 through a vertex of degree 3. Furthermore by the local move (i) of Fig. 3, we can arrange so that the graph has at most one vertex of degree 2. □

Now let us define a map $\sigma\colon \mathcal{A} \to \mathbb{Z} \oplus \mathbb{Z}_2$ from the cobordism group of abstract Reeb functions as follows. For a given abstract Reeb function r put

$$\sigma(r) = (t(r), d(r)) \in \mathbb{Z} \oplus \mathbb{Z}_2,$$

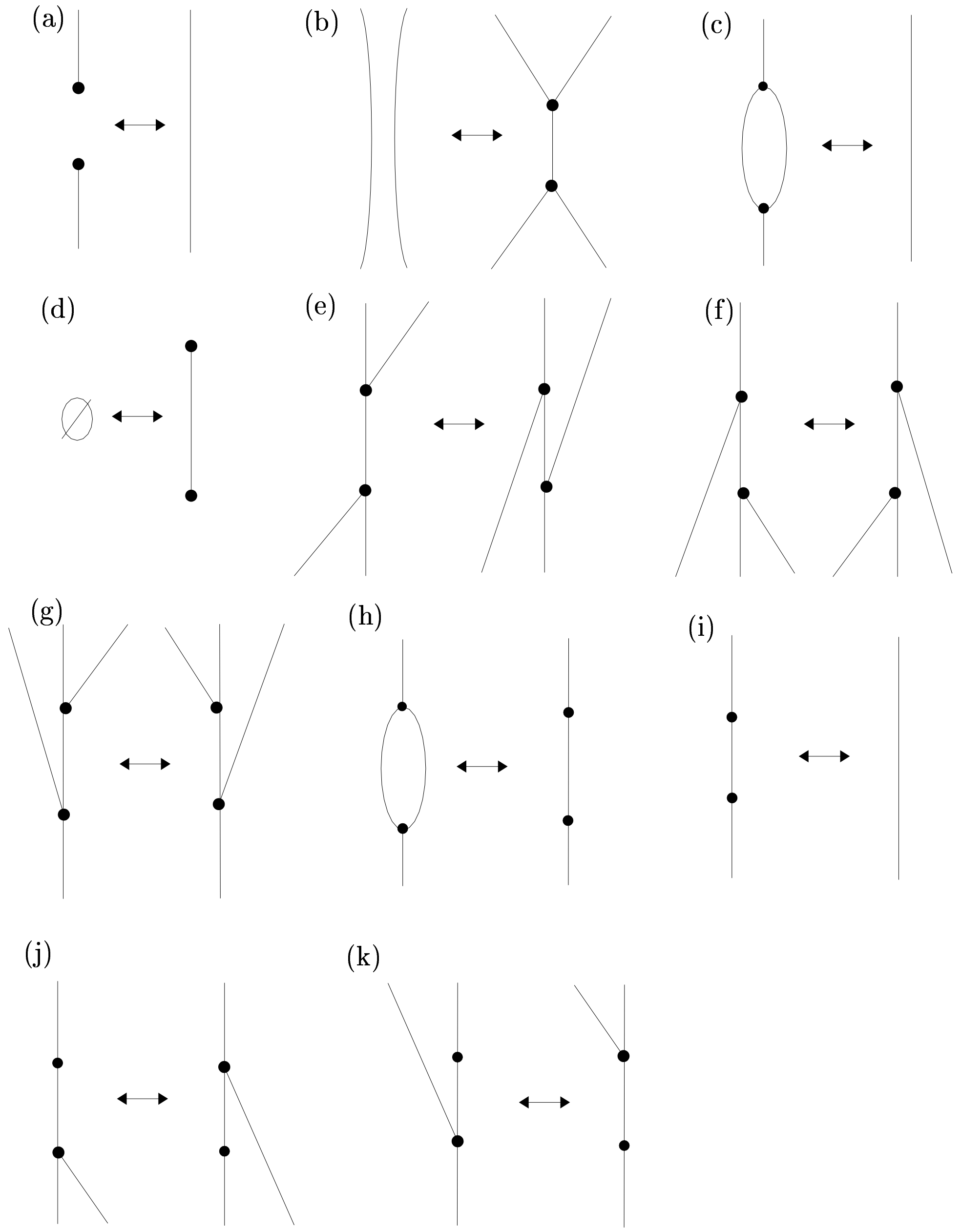

FIGURE 3. Eleven local moves for abstract Reeb functions

where $t(r) \in \mathbb{Z}$ is the sum of the signs over all vertices of degree 3, and $d(r) \in \mathbb{Z}_2$ is the number of vertices of degree 2 modulo two. Then $\sigma(r)$ depends only on the cobordism class of an abstract Reeb function r, since the local moves of Fig. 3 or a homotopy in the space of abstract Reeb functions do not change the values of t nor d. So σ defines a

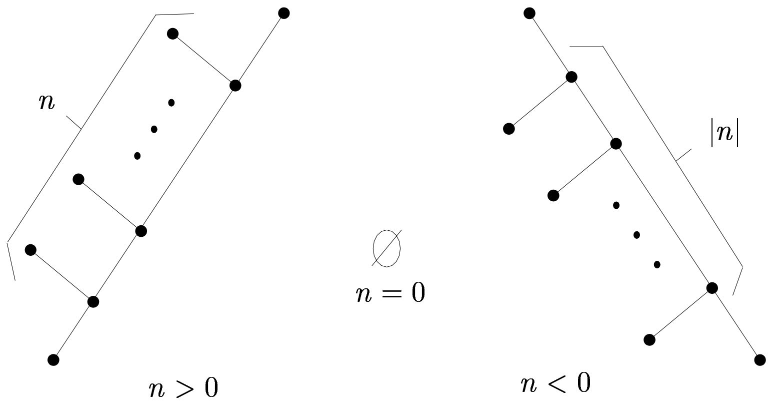

FIGURE 4. First family of representatives of the elements of the cobordism group $\mathcal{A}$

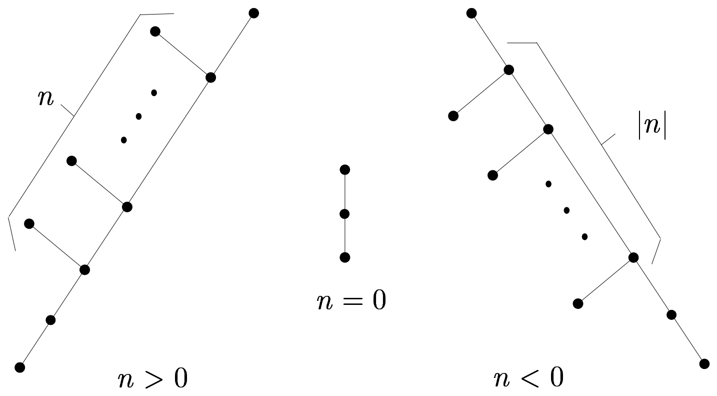

FIGURE 5. Second family of representatives of the elements of the cobordism group $\mathcal{A}$

well-defined map of the set of cobordism classes, which is clearly a homomorphism, and by Lemma 3.6 it is an isomorphism. This completes the proof of Proposition 3.5. $\square$

4. Relation between cobordism and abstract cobordism

If we have a cobordism $F\colon X \to \mathbb{R}\times[0,1]$ between two Morse functions $f_0\colon M_0 \to \mathbb{R}$ and $f_1\colon M_1 \to \mathbb{R}$ on closed surfaces, then the Stein factorization $\bar{F}\colon W_F \to \mathbb{R}\times[0,1]$

of the possibly perturbed F provides a cobordism between the abstract Reeb functions $r_0\colon G_0 \to \mathbb{R}$ and $r_1\colon G_1 \to \mathbb{R}$ associated with the Morse functions f_0 and f_1 respectively, where $r_j = \bar{f}_j$ and $G_j = W_{f_j}$, $j = 0, 1$ (for details, see [**4**]). So the following definition makes sense.

DEFINITION 4.1. Let $\varphi\colon \mathcal{C}ob_f(2,-1) \to \mathcal{A}$ be the map associating to the cobordism class of a Morse function f the cobordism class of its Reeb function $\bar{f}\colon W_f \to \mathbb{R}$. This is clearly a well-defined homomorphism between the two cobordism groups $\mathcal{C}ob_f(2,-1)$ and $\mathcal{A}$.

PROPOSITION 4.2. *The homomorphism φ is an isomorphism.*

PROOF. The map φ is clearly surjective. For the injectivity let $R\colon P \to \mathbb{R} \times [0,1]$ be a cobordism between two abstract Reeb functions $r_0\colon G_0 \to \mathbb{R}$ and $r_1\colon G_1 \to \mathbb{R}$, and suppose that r_0 and r_1 correspond to Morse functions f_0 and f_1 on closed surfaces M_0 and M_1 respectively.

It is sufficient to show that there exists a cobordism F between the given Morse functions f_0 and f_1. For this, it suffices to show that

(1) there exists a cobordism $\tilde{R}\colon \tilde{P} \to \mathbb{R} \times [0,1]$ between the two Reeb functions $r_0\colon G_0 \to \mathbb{R}$ and $r_1\colon G_1 \to \mathbb{R}$, and
(2) there exists a fold map $\tilde{F}\colon \tilde{X} \to \mathbb{R} \times [0,1]$ from a compact 3-manifold $\tilde{X}$ with boundary such that $\tilde{F}$ is a composition of a map $\tilde{X} \to \tilde{P}$ and the map $\tilde{R}$, and $\tilde{R}$ is identified with the Stein factorization of $\tilde{F}$. In this case we say that the map $\tilde{F}$ is *over* $\tilde{R}$.

Starting in the same way as in [**4**], we consider the decomposition

$$P = N(Q) \cup N(V) \cup N(\Sigma) \cup T$$

as follows. We put $Q = G_0 \cup G_1$, and let $N(Q)$ $(\cong (G_0 \times [0,\varepsilon]) \amalg (G_1 \times [1-\varepsilon, 1]))$ be the regular neighbourhood of Q in P. Let V be the set of the points of P which have regular neighbourhoods as in (d), (e), (g), (h), and (i) of Fig. 2, and let $N(V)$ be its small regular neighbourhood in P. Let Σ be the set of the points of P which have regular neighbourhoods as in (b)–(i) of Fig. 2, and let $\tilde{N}(\Sigma)$ be its regular neighbourhood in P. Then $N(\Sigma)$ is the closure of $\tilde{N}(\Sigma) \setminus (N(Q) \cup N(V))$. Let T be the closure of $P \setminus (N(Q) \cup N(V) \cup N(\Sigma))$. Note that T is a compact surface with boundary.

By the construction of Mata-Lorenzo [**8**] we can construct a 3-dimensional compact manifold X with boundary and a map $F\colon X \to \mathbb{R} \times [0,1]$ over $R|_{N(Q)\cup N(V)\cup N(\Sigma)}$ such that $\partial X = M_0 \amalg M_1 \amalg S$, $F|_{M_0 \times [0,\varepsilon)} = f_0 \times \mathrm{id}_{[0,\varepsilon)}$, $F|_{M_1 \times (1-\varepsilon,1]} = f_1 \times \mathrm{id}_{(1-\varepsilon,1]}$, and S is a 2-dimensional manifold fibered over ∂T: in fact, we consider $(M_0 \times [0,\varepsilon]) \amalg (M_1 \times [1-\varepsilon, 1])$

over $N(Q)$, and over $N(V) \cup N(\Sigma)$ we consider a manifold similar to that constructed in [**4**].

It is not clear whether we can extend X and F like in [**4**] over T so that the extension is a cobordism between the given Morse functions f_0 and f_1, since now S can be nonorientable. From [**7**] T is an orientable surface with boundary and $R|_T \colon T \to \mathbb{R} \times [0,1]$ is an immersion. Over ∂T we have an S^1-bundle ($= S$). Summarizing in an informal language, we have constructed a 3-manifold X with boundary $M_0 \amalg M_1 \amalg S$ and its fold map into $\mathbb{R} \times [0,1]$ whose Stein factorization gives a neighbourhood of the 1-skeleton of the 2-dimensional polyhedron P, where 1-skeleton means $Q \cup \Sigma$. It remained to extend this construction over the 2-cells of P, that is over T. This was easy in the oriented case considered in [**4**], since S was a trivial circle bundle over the boundary of T in P. In the present case S may contain a nonorientable circle budle. Such a nonorientable circle bundle is a fibration of the Klein bottle over S^1. This fibration can be extended to a fold map of the (nonorientable) solid Klein bottle $S^1 \times_{\mathbb{Z}_2} D^2$ to the annulus $S^1 \times [0,1]$ with only definite fold singularities, where the boundary Klein bottle $S^1 \times_{\mathbb{Z}_2} S^1$ is mapped onto $S^1 \times \{1\}$ by the above fibration, and the zero section $S^1 \times_{\mathbb{Z}_2} \{0\}$ coincides with the set of the singular points and is mapped onto $S^1 \times \{0\}$ diffeomorphically. (For this, it suffices to consider the function $(x,y) \mapsto x^2 + y^2$ on each 2-disk fiber.) Attaching such a solid Klein bottle or its orientable version to X along each component of S, we obtain a compact 3-dimensional manifold and its fold map into $\mathbb{R} \times [0,1]$, which gives a cobordism between the More functions f_0 and f_1. This completes the proof of Proposition 4.2 and Theorem 2.9. □

REMARK 4.3. We see that the two functions as depicted in Fig. 6 represent generators of the cobordism group of abstract Reeb functions. The function (a) has infinite order, while the function (b) has order 2. We can realize the function (a) as the Reeb function associated with a height function of a 2-sphere appropriately embedded in $\mathbb{R}^3$ (see [**4**]), and the function (b) can be realized as the Reeb function associated with a Morse function on $\mathbb{R}P^2$ with three critical points whose indices are 0, 1 and 2.

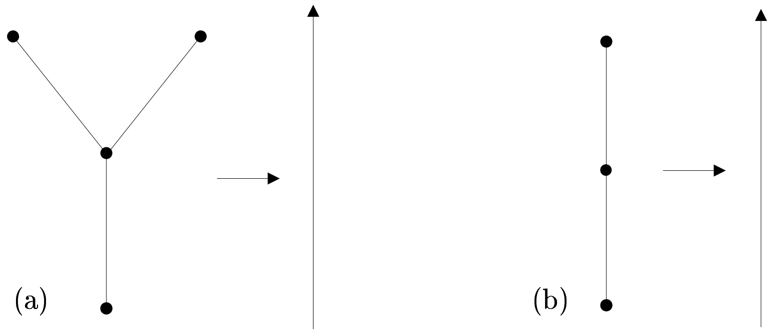

FIGURE 6. Two generators of the abstract cobordism group

CHAPTER 2

Cobordism group of fold maps from oriented 3-manifolds into the plane

1. Introduction

There are many results concerning the cobordism groups of embeddings, immersions, and smooth maps with given type of singularities. One method to study these groups is the Pontrjagin-Thom construction [**18**] and its extended version for cobordisms of smooth maps with prescribed singularities. In the case of positive codimension this method was developed by Rimányi and Szűcs [**10, 13**].

In the case of negative codimension an approach is to study the cobordism group of abstract analogues of Stein factorizations, and the relationship between the Stein factorization and the cobordism group of smooth maps. By this method Ikegami and Saeki have computed the cobordism group of Morse functions on oriented surfaces [**4**].

An other approach is to reduce a general map to a map without prescribed singularities by methods of eliminating these singularities. Using this method Ikegami [**3**] has computed the cobordism group of Morse functions on manifolds.

There was an idea of Szűcs that we could make a Pontrjagin-Thom-type construction for Stein factorizations, this is the goal of the present chapter.

The chapter is organized as follows. In §2 we give some fundamental definitions concerning cobordism, and we state our main theorems. In §3 we give some other technical definitions and prove lemmas. In §4 we construct the classifying space for Stein factorizations using a Pontrjagin-Thom-type construction. In §5 we compute some homotopical properties of this classifying space. We prove our main theorems and we give some geometrical invariants of the cobordism classes. In §6 we prove the analogous results for bordism groups.

2. Preliminaries

Definition 2.1. Let Q^q and N^n be smooth manifolds of dimensions q and n respectively $(q \geq n)$. Let $p \in Q^q$ be a singular point of a smooth map $f\colon Q^q \to N^n$. The smooth map f has a *fold singularity* at the singular point p, if we can write f in some local coordinates at p and $f(p)$ in the form

$$f(x_1, \dots, x_q) = (x_1, \dots, x_{n-1}, \pm x_n^2 \pm \dots \pm x_q^2).$$

A smooth map $f\colon Q^q \to N^n$ is called a *fold map*, if f has only fold singularities.

DEFINITION 2.2. Let Q^q and N^n be smooth manifolds of dimensions q and n respectively $(q \geq n)$. A smooth map $f\colon Q^q \to N^n$ at the fold singularity $p \in Q^q$ has a *definite fold singularity* if we can write f in some local coordinates at p and $f(p)$ in the form

$$f(x_1, \dots, x_q) = (x_1, \dots, x_{n-1}, x_n^2 + \dots + x_q^2),$$

otherwise f has an *indefinite fold singularity* at the fold singularity $p \in Q^q$. Let $S_1(f)$ denote the set of indefinite fold singularities of f in Q^q.

Let $FOLD$ denote the set of the fold maps $f\colon M^3 \to \mathbb{R}^2$, where M^3 runs over the set of all closed oriented 3-dimensional manifolds. Let $\amalg$ denote the disjoint union.

DEFINITION 2.3. Two fold maps $f_0\colon M_0 \to \mathbb{R}^2$ and $f_1\colon M_1 \to \mathbb{R}^2$ on closed oriented 3-dimensional manifolds M_0 and M_1 are *cobordant*, if there exists a smooth map $F\colon X \to \mathbb{R}^2 \times [0,1]$ with only fold singularities from a compact oriented 4-dimensional manifold X with $\partial X = M_0 \amalg -M_1$ such that $F\mid_{M_0 \times [0,\varepsilon)} = f_0 \times id_{[0,\varepsilon)}$ and $F\mid_{M_1 \times (1-\varepsilon,1]} = f_1 \times id_{(1-\varepsilon,1]}$, where $M_0 \times [0,\varepsilon)$ and $M_1 \times (1-\varepsilon, 1]$ are small collar neighbourhoods of ∂X with the identifications $M_0 = M_0 \times \{0\}$, $M_1 = M_1 \times \{1\}$, and $id_A(a) = a$ $(a \in A)$ for an arbitrary set A.

We call the map F a *cobordism* between f_0 and f_1.

This clearly defines an equivalence relation on the set $FOLD$.

DEFINITION 2.4. We can define a group operation on the set of the cobordism classes as follows. If $[f_0\colon M_0 \to \mathbb{R}^2]$ and $[f_1\colon M_1 \to \mathbb{R}^2]$ are cobordism classes of fold maps, then the *sum* of $[f_0]$ and $[f_1]$ is the cobordism class represented by the fold map $f_0 \amalg f_1\colon M_0 \amalg M_1 \to \mathbb{R}^2$. It is easy to show that the cobordism class of the fold map $f_0 \amalg f_1$ does not depend on the choice of the representatives f_0 and f_1.

This group operation is clearly commutative, so we have an abelian group, which we denote by $\mathcal{C}ob_f(3,-1)$ or simply by $\mathcal{C}ob$, where the lower index "f" comes from "fold map".

REMARK 2.5. One can define the analogous cobordism groups $\mathcal{C}ob_f(q,k)$ $(q \in \mathbb{N}$, $k \in \mathbb{Z}$, $-q \leq k \leq 0$ $)$ of fold maps $f\colon Q^q \to \mathbb{R}^{q+k}$ from closed oriented q-dimensional manifolds Q^q to $\mathbb{R}^{q+k}$ as well. The integer k is called the *codimension* of the map f. There was a result of Ikegami and Saeki [**4**] that $\mathcal{C}ob_f(2,-1) = \mathbb{Z}$.

For $k \geq 0$ there are many results concerning the cobordism groups $\mathcal{C}ob_\tau(q,k)$, where τ is a set of singularity types, the elements of the group $\mathcal{C}ob_\tau(q,k)$ are cobordism classes of smooth maps with only singularities in τ, and a cobordism between two such maps has only singularities in τ. See for example [**5, 10, 13, 14, 16, 17**].

We use the notion of the Stein factorization of a smooth map $f \colon Q^q \to N^n$, where Q^q and N^n are smooth manifolds of dimensions q and n respectively $(q \geq n)$. Two points $p_1, p_2 \in Q^q$ are *equivalent* if and only if p_1 and p_2 lie on the same componenet of an f-fiber. Let W_f denote the quotient space of Q^q with respect to this equivalence relation and $q_f \colon W_f \to N^n$ the quotient map. Then there exists a unique continuous map $\bar{f} \colon W_f \to N^n$ such that $f = \bar{f} \circ q_f$. The space W_f or the factorization of the map f into the composition of q_f and $\bar{f}$ is called the *Stein factorization* of the map f. We call the map $\bar{f}$ also the *Stein factorization* of the map f as well. Note that if f is a smooth map from a closed q-dimensional manifold into $\mathbb{R}^{q-1}$, then its Stein factorization W_f is a compact $q-1$-dimensional CW complex.

The local forms of Stein factorizations of fold maps from 3-manifolds into $\mathbb{R}^2$ and of fold maps from oriented 4-manifolds into $\mathbb{R}^3$ are completely described in [**6, 7**] and [**11**, Chapter 4], for a short description of these local forms in this chapter, see the end of §2. The Stein factorizations of fold maps from 3-manifolds into $\mathbb{R}^2$ and of fold maps from oriented 4-manifolds into $\mathbb{R}^3$ are locally equivalent to one of these local forms.

The following notions are the analogues of Reeb functions and cobordisms between Reeb functions introduced in [**4**].

Definition 2.6. A continuous map $g \colon W \to \mathbb{R}^2$ is called an *abstract Stein factorization*, if W is a compact 2-dimensional CW complex such that W and the map g are locally equivalent to a Stein-factorization of a fold map $f \colon M^3 \to \mathbb{R}^2 \in FOLD$.

Let us denote the set of abstract Stein factorizations by $ASTEIN$.

Definition 2.7. Let $g_i \colon W_i \to \mathbb{R}^2$, $i = 0,1$, be abstract Stein factorizations in the sense of Definition 2.3. We say that g_0 and g_1 are *cobordant* if there exists a continuous map $G \colon P \to \mathbb{R}^2 \times [0,1]$ of a compact 3-dimensional polyhedron P such that

(i) $W_i = G^{-1}(\mathbb{R}^2 \times \{i\})$, $i = 0,1$, are subcomplexes of P with regular neighbourhoods of the forms $W_0 \times [0,\varepsilon)$ and $W_1 \times (1-\varepsilon,1]$, where W_i corresponds to $W_i \times \{i\}$, $i = 0,1$,

(ii) $G\,|_{W_0 \times [0,\varepsilon)} = g_0 \times id_{[0,\varepsilon)}$ and $G\,|_{W_1 \times (1-\varepsilon,1]} = g_1 \times id_{(1-\varepsilon,1]}$,

(iii) in a neighbourhood of each point of $P \setminus (W_0 \cup W_1)$, the polyhedron P and the map G are equivalent to one of the local forms of Stein factorizations of fold maps from oriented 4-manifolds into $\mathbb{R}^3$.

Furthermore, we call the map $G\colon P \to \mathbb{R}^2 \times [0,1]$ a *cobordism* between g_0 and g_1.

This clearly defines an equivalence relation on the set of abstract Stein factorizations $ASTEIN$.

DEFINITION 2.8. We can define a group operation on the set of the cobordism classes of abstract Stein factorizations as follows. If $[g_0\colon W_0 \to \mathbb{R}^2]$ and $[g_1\colon W_1 \to \mathbb{R}^2]$ are cobordism classes of abstract Stein factorizations, then the *sum* of $[g_0]$ and $[g_1]$ is the cobordism class represented by the abstract Stein factorization $g_0 \amalg g_1\colon W_0 \amalg W_1 \to \mathbb{R}^2$. It is easy to show that the cobordism class of the abstract Stein factorization $g_0 \amalg g_1$ does not depend on the choice of the representatives g_0 and g_1.

Let us denote the cobordism group of abstract Stein factorizations by $\mathcal{C}ob_S$.

The map $\Upsilon\colon FOLD \to ASTEIN$ which associates to a fold map $f \in FOLD$ its Stein factorization, induces a homomorphism $\upsilon\colon \mathcal{C}ob \to \mathcal{C}ob_S$.

We can define the *bordism* and the *bordism group* by replacing the spaces $\mathbb{R}^2$ and $\mathbb{R}^2 \times [0,1]$ with an arbitrary closed oriented surface and an oriented 3-manifold with boundary respectively. Let $\mathcal{B}or$ denote the bordism group of fold maps from oriented 3-manifolds into oriented surfaces.

The main results of this chapter are the followings.

THEOREM 2.9. *The cobordism group of abstract Stein factorizations $\mathcal{C}ob_S$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.*

THEOREM 2.10. *The homomorphism $\upsilon\colon \mathcal{C}ob \to \mathcal{C}ob_S$ is an isomorphism,*

whose proof is based on the method described in §3 and some results of O. Saeki [**11**, Chapter 6].

The important corollary of Theorems 2.9 and 2.10 is that the cobordism group $\mathcal{C}ob$ of fold maps from closed oriented 3-dimensional manifolds into $\mathbb{R}^2$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

We have the analogous result for bordisms as well.

THEOREM 2.11. *The bordism group $\mathcal{B}or$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.*

Now let us describe the local forms of Stein factorizations of maps which will be important for us. The details can be found in [**6, 7**] and [**11**, Chapter 4].

Let us first define the notations of the following singularities. In Fig. 1 one can see the canonical singularity $\sigma_{1,1}\colon s_{1,1} \to I$. In the Fig. 2 we can see the canonical singularities $\sigma_{2,1}\colon s_{2,1} \to I^2$, $\sigma_{2,2}\colon s_{2,2} \to I^2$, $\sigma_{2,3}\colon s_{2,3} \to I^2$, respectively.

In the Fig. 3 we can see the singular fibers of the possible singularities $\tilde{\sigma}_{3,j}\colon \tilde{s}_{3,j} \to I^3$ of fold maps $M^4 \to \mathbb{R}^3$ which will be relevant for us. Let $\sigma_{3,j}\colon s_{3,j} \to I^3$ be the Stein factorizations of these singularities.

By [**6, 7**] and [**11**, Chapter 4] we can state the following theorem.

THEOREM 2.12. *Let $v\colon A \to \mathbb{R}$ be a stable fold map (i.e., a Morse function) from an oriented surface A. Then its Stein factorization $\bar{v}\colon W_v \to \mathbb{R}$ in a neighbourhood of the q_v-image of an indefinite fold singularity of v is locally equivalent to the canonical singularity as depicted in Fig. 1.*

Let $f\colon M \to \mathbb{R}^2$ be a stable fold map from a closed oriented 3-dimensional manifold M. Let $y \in \mathbb{R}^2$ be the f-image of an indefinite fold singularity of f. Then the Stein factorization $\bar{f}\colon W_f \to \mathbb{R}^2$ in a neighbourhood of $\bar{f}^{-1}(y) \cap q_f(S_1(f))$ is locally equivalent to the suspension of the canonical singularity as depicted in Fig. 1, or it is locally equivalent to one of the canonical singularities as depicted in Fig. 2.

Let $F\colon X \to \mathbb{R}^3$ be a stable fold map from a closed oriented 4-dimensional manifold X. Let $z \in \mathbb{R}^3$ be the F-image of an indefinite fold singularity of F. Then the Stein factorization $\bar{F}\colon W_F \to \mathbb{R}^3$ in a neighbourhood of $\bar{F}^{-1}(z) \cap q_F(S_1(F))$ is locally equivalent to the second suspension of the canonical singularity as depicted in Fig. 1, or it is locally equivalent to one of the first suspensions of the canonical singularities as depicted in Fig. 2, or it is locally equivalent to one of the canonical singularities $\sigma_{3,j}\colon s_{3,j} \to I^3$ $(j = 1 \ldots 8)$.

In §4 we construct a topological space Γ whose homotopic groups give cobordism groups. The role of these canonical singularities in this chapter is that the space Γ consists of blocks of bundles whose fibers are these canonical singularities.

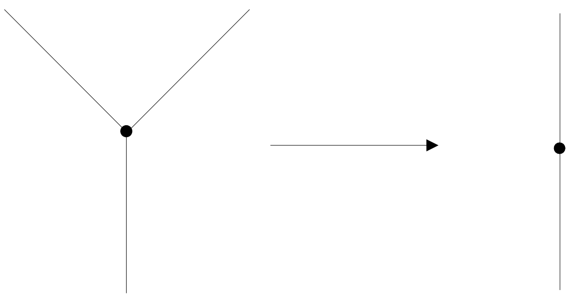

FIGURE 1. The canonical singularity $\sigma_{1,1}\colon s_{1,1} \to I$

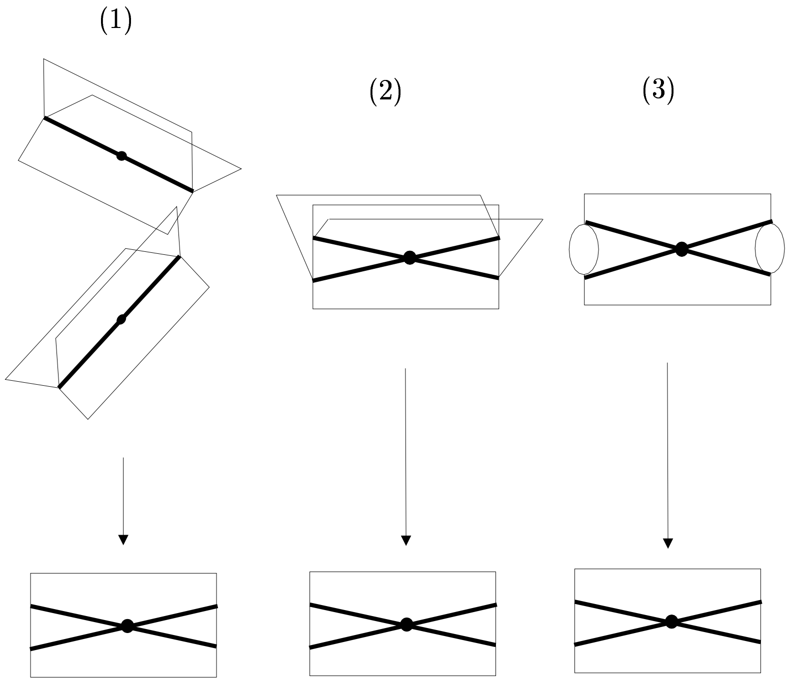

FIGURE 2. The canonical singularities $\sigma_{2,j}\colon\ s_{2,j} \to I^2\ \ (j = 1, 2, 3)$

3. Making holes

Our first goal is the computation of the group $\mathcal{C}ob_S$. We shall call an abstract Stein factorization *simple*, or *punctured* if its image is contained in a small tubular neighbourhood of the image of its (indefinit) singular set. It is defined analogously a *simple cobordism* (or *punctured cobordism*) of two abstract Stein factorizations. In this section we show that the cobordism group $\mathcal{C}ob_T$ of simple abstract Stein factorizations under simple cobordisms is isomorphic to $\mathcal{C}ob_S$. Moreover the natural map $\mathcal{C}ob_T \to \mathcal{C}ob_S$ turns out to be an isomorphism. (Recall that by Stein factorization in this chapter we always mean a Stein factorization of a fold map of a closed 3-manifold in $\mathbb{R}^2$. By abstract Stein factorizations we mean the abstract analogue of such a Stein factorization. By cobordism of fold maps we mean a fold map $(X^4, \partial X^4) \to (\mathbb{R}^2 \times [0,1], \partial(\mathbb{R}^2 \times [0,1]))$.

LEMMA 3.1. *The natural map $\mathcal{C}ob_T \to \mathcal{C}ob_S$ of the cobordism group of simple abstract Stein factorizations into the group of abstract Stein factorizations is an isomorphism.*

PROOF. Let $g\colon W \to \mathbb{R}^2 \in ASTEIN$ be an abstract Stein factorization. Let $S_g \subseteq W$ be the set of the "indefinit" singular points of g: the singular points in W where g has the form of a Stein factorization of a fold map $f\colon M^3 \to \mathbb{R}^2\ \in FOLD$ around its *indefinit*

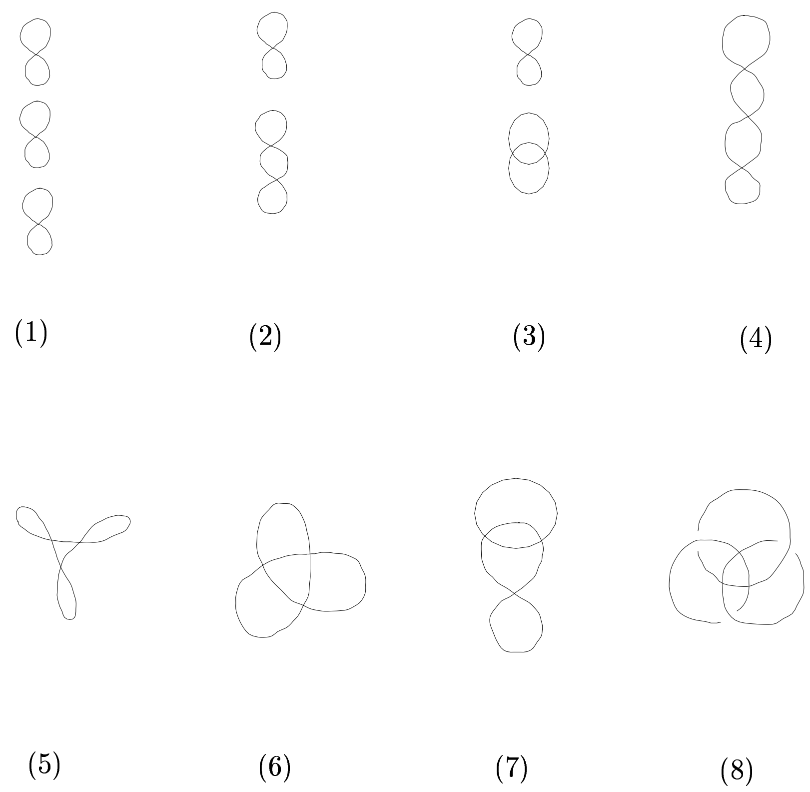

FIGURE 3. The singular fibers of the singularities $\tilde{\sigma}_{3,j}\colon\ \tilde{s}_{3,j} \to I^3$ $(j = 1 \dots 8)$

singularity. $g(S_g)$ is an immersed closed 1-dimensional manifold in $\mathbb{R}^2$. Let $U(S_g)$ be a small regular neighborhood of S_g in W. Let g' be the restriction of g to $W \setminus U(S_g)$. The map $g'\colon W \setminus U(S_g) \to \mathbb{R}^2$ is an immersion of a 2-dimensional manifold with boundary into $\mathbb{R}^2$. We want to cut out this map g' from the map g, more precisely we define an abstract Stein factorization $h \in ASTEIN$ cobordant to g (denoted by $h \sim g$) with domain $D_h = \overline{U(S_g)}$ such that $h\colon U(S_g) \to \mathbb{R}^2$ is equal to the restriction of g to $U(S_g)$.

In order to see that h is cobordant to g let us consider a function $\Phi\colon W \to \mathbb{R}$ such that $\Phi \geq 0$, $\Phi < 1/2$, $\Phi|_{U(S_g)} = 0$, $\Phi|_{\partial(W \setminus U(S_g))} = 0$, and $\Phi|_{int(W \setminus U(S_g))} > 0$. Since $g(\partial U(S_g))$ is an immersed 1-dimensional manifold which has a normal-field pointing to $g(int(W \setminus U(S_g)))$, such a Φ exists.

Let us consider the set $W \times [0,1]$. Then $\{\Phi \leq t\} \times \{t\} \subseteq W \times \{t\} \subseteq W \times [0,1]$, where $0 \leq t \leq 1$.

Let V denote the set $\bigcup_{0 \leq t \leq 1} \{\Phi \leq t\} \times \{t\}$. Clearly $V \subseteq W \times [0,1]$. If we choose a nice Φ, then $G := (g \times id_{[0,1]}) \mid_V \colon V \to \mathbb{R}^2 \times [0,1]$ will be an abstract cobordism between g and the map $h := g \mid_{U(S_g)}$, because $\{\Phi \leq 1\} = W$ and $\{\Phi = 0\} = \overline{U(S_g)}$.

The described method can be applied to a cobordism $G \colon X \to \mathbb{R}^2 \times [0,1]$ between two abstract Stein factorizations in $ASTEIN$, because if S_G denotes the corresponding set for G, then $G(\partial U(S_G))$ is an immersed 2-dimensional manifold in $\mathbb{R}^2 \times [0,1]$, which has a normal-field pointing to $G(int(X \setminus U(S_G)))$ (see for example in [**11**], we can see it from the list of the possible singularities), so the function Φ exists.

Hence if g_1 and g_2 are two abstract Stein factorizations which are cobordant by a cobordism G, then we can puncture the cobordism G using the described method, and so we obtain a punctured cobordism holed between h_1 $(\sim g_1)$ and h_2 $(\sim g_2)$.

Let us define the map $\Theta \colon ASTEIN \to ASTEIN$ which associates to a map $g \in ASTEIN$ the map h $(\sim g)$ obtained by the method described above. Let $\mathcal{C}ob_T$ denote the cobordism group of the punctured abstract Stein factorizations. Then Θ induces a homomorphism $\Psi \colon \mathcal{C}ob_S \to \mathcal{C}ob_T$, because we can puncture a cobordism as well. Ψ is clearly surjective, and Ψ is injective since we have made the holes using a cobordism. $\square$

So it is enough to compute the group $\mathcal{C}ob_T$.

REMARK 3.2. The number of singular fibers of a representative of an element of a cobordism class in $\mathcal{C}ob_T$ is at most two, and the number of singular fibers of a cobordism between two elements of $\mathcal{C}ob_T$ is at most three.

REMARK 3.3. It is easy to see that for a given element $[f \colon M \to \mathbb{R}^2] \in \mathcal{C}ob$ we can find an element $[\tilde{f} \colon M \to \mathbb{R}^2] \in \mathcal{C}ob$ such that f is cobordant by a cobordism F to $\tilde{f}$ and the Stein factorization of $\tilde{f}$ is simple. To obtain $\tilde{f}$ and the cobordism F we have to apply the method of making holes for the Stein factorization $\bar{f} \colon W_f \to \mathbb{R}^2$ of the map f. Using the notations of this section, $\tilde{f}$ is equal to

$$f \mid_{q_f^{-1}(\{\Phi = 0\})}$$

if we glue to $\partial q_f^{-1}(\{\Phi = 0\})$ and to f an appropriate 2-disk bundle and its map to $\mathbb{R}^2$ (see Chapter 1), and the cobordism F is equal to

$$(f \times id_{[0,1]}) \mid_{(q_f \times id_{[0,1]})^{-1}(V)}$$

if we glue to the appropriate part of $\partial (q_f \times id_{[0,1]})^{-1}(V)$ and to $f \times id_{[0,1]}$ an appropriate 2-disk bundle and its map to $\mathbb{R}^2 \times [0,1]$ similarly to the previous.

In the following, $g \in \mathcal{C}ob_T$ can denote a class of cobordism and a map too.

4. The classifying space

We will prove the following:

THEOREM 4.1. *There exists a space Γ such that $\pi_2(\Gamma) = \mathcal{C}ob_T$.*

The construction of the space Γ is very similar to the constructions of the classifying spaces of Szűcs, see for exemple [**14, 17**]. The transversality and the uniqueness up to homotopy are included in the same way at the relevant points in our case as well.

PROOF. Let $g \in \mathcal{C}ob_T$ be a simple abstract Stein factorization. Let I^k denote the cube $[-1,1]^k$. Let Σ_k denote the k-codimensional submanifold of $g(S_g)$. If we choose $U(S_g)$ small enough in Section 3, then we can suppose that a small regular neighborhood of the set $g(S_g)$ coincides with $g(U(S_g))$ in $\mathbb{R}^2$. Such a sufficiently small neighborhood of $g(S_g)$ can be induced from the classifying space of codimension 1 immersions. So we get a block-partition $\{T_k\}_k$ of $g(U(S_g))$, where we can identify the block T_k with the total space of the k-dimensional normal-bundle of $\Sigma_k \setminus T_{k+1}$, and we can identify the fiber of T_k with I^k, and I^k is adjacent to I^{k+1}, if the block T_k is adjacent to T_{k+1}. Since $U(S_g)$ is small, I^i and I^j are disjoint if they are not adjacent because of the block-structure. We note that we can say the same in the case of a simple cobordism G instead of a simple abstract Stein factorization $g \in \mathcal{C}ob_T$.

Let $\sigma_{k,j}\colon s_{k,j} \to I^k$ denote the k-codimensional canonical singularity with type j of a map $g \in \mathcal{C}ob_T$ or of a cobordism in $\mathcal{C}ob_T$, where j is an element of the set of the distinct k-codimensional singularities, $k = 1,2,3$. Let us partition the block T_k into the blocks $T_{k,j}$, where the zero-section of $T_{k,j}$ (which we denote by $\Sigma_{k,j}$) is the image of singularities of type j.

Let $AUT(\sigma_{k,j})$ denote the group of automorphisms of the singularity $\sigma_{k,j}$. Let $Aut(\sigma_{k,j})$ be the subgroup of $AUT(\sigma_{k,j})$, where the elements of $Aut(\sigma_{k,j})$ act linearly on $[-1,1]^k$. Let $Perm(\sigma_{k,j})$ be the subgroup of $Aut(\sigma_{k,j})$, where the elements of $Perm(\sigma_{k,j})$ act on $[-1,1]^k$ by permuting the factors and reflecting in the coordinate hyperplanes.

LEMMA 4.2. *Let $\xi \to B^b$ be a $\sigma_{k,j}$-bundle over a b-dimensional manifold B, $k = 1, j = 1, b \in \mathbb{N}$, or $k = 2, j = 1,2,3, b = 1$. Then its structure group can be reduced to $Aut(\sigma_{k,j})$. Moreover its structure group can be reduced to $Perm(\sigma_{k,j})$.*

PROOF. Let $g \in AUT(\sigma_{k,j})$. Then g consists of two homeomorphisms g_1 and g_2 such that the diagram
$$\begin{tikzcd} s_{k,j} \arrow[r, "g_2"'] \arrow[d] & s_{k,j} \arrow[d] \\ I^k \arrow[r, "g_1"] & I^k \end{tikzcd}$$
commutes, where the two maps $s_{k,j} \to I^k$ are equal to the canonical singularitiy $\sigma_{k,j}$. We define a stratification on I^k as follows. Let H_n^{k-1},

$n = 1 \dots k$, denote the coordinate hyperplanes in I^k, and let P_l^i, $i = 0 \dots k, l = 1 \dots r_i$ denote the r_i i-dimensional parts of the natural stratification of I^k induced by $\cup_n H_n^{k-1}$. There exists over a point of every $int(P_l^{k-1})$ a point in $s_{k,j}$ which has a neighbourhood of the form $\curlyvee$, so g_1 maps P_l^{k-1} to some $P_{l'}^{k-1}$, $l = 1 \dots r_{k-1}$, so the homeomorphism g_1 preserves the stratification.

It is clear that for a given deformation of g_1 through homeomorphisms which preserve the stratification, there exists a unique deformation parametrized by t through homeomorphisms of g_2 such that the pair $(g_1, g_2)_t$ is an element of $AUT(\sigma_{k,j})$ at every $t \in [0,1]$, since $s_{k,j}$ is a covering over every stratum. So it is enough to give a deformation only of the homeomorphism g_1 through homeomorphisms which preserve the stratification.

When $\sigma_{k,j} = \sigma_{1,1}$, we can reduce the structure group of the bundle $\xi \to B$ as follows. The total space ξ is a bundle map $\pi\colon \eta \to \lambda$ where η is a total space of an $s_{1,1}$-bundle, λ is a total space of an I^1-bundle, and their structure groups are the actions of the automorphisms in $AUT(\sigma_{1,1})$ on $s_{1,1}$ and on I^1 respectively. We can reduce the structure group of λ to the trivial group, since the direction of the π-image of the upper arms of the fibers $\curlyvee$ is welldefined in λ, so we can define the parallel translations of the zero section of λ, and this gives a trivialization of λ.

When $\sigma_{k,j} = \sigma_{2,1}$, $\sigma_{k,j} = \sigma_{2,2}$, or $\sigma_{k,j} = \sigma_{2,3}$, it is enough to give a deformation of the stratification preserving homeomorphism $g_1\colon I^2 \to I^2$, since the dimension of the manifold B is equal to 1, so a glueing map of a local trivialization of the bundle $\xi \to B^1$ is an automorphism of $\sigma_{2,j}$, $j = 1, 2, 3,$.

We deform g_1 by the standard way as follows. At first the result of the deformation $G\colon (x,t) \mapsto g_1(x)(|x|t + |g_1(x)|(1-t))/|g_1(x)|$, $x \in I^2, x \neq 0, t \in [0,1]$, is a homeomorphism $g_1' := G(\cdot, 1)\colon I^2 \to I^2$ which preserves the norm. Then there exists a deformation whose result is a homeomorphism g_1'' which preserves the norm and maps a line $[0, x]$ into a line $[0, g_1''(x)]$. Then g_1'' induces homeomorphisms on $\partial P_l^2 \cap \partial I^2$, $l = 1 \dots 4$, which we can clearly deform into norm preserving homeomorphisms, which induce a g_1''' on I^2. This completes the proof. $\square$

Let $\xi_{k,j} \to B_{k,j}$ be the universal bundle with structure group $Aut(\sigma_{k,j})$. From this we can induce the $\sigma_{k,j}$-bundle with total space $g^{-1}(T_{k,j}) \to T_{k,j}$:

$$\begin{tikzcd}
(g^{-1}(T_{k,j}) \to T_{k,j}) \arrow[r] \arrow[d, "\sigma_{k,j}"] & \xi_{k,j} \arrow[d, "\sigma_{k,j}"] \\
\Sigma_{k,j} \arrow[r] & B_{k,j}
\end{tikzcd}$$

From the total spaces $\xi_{k,j}$ we can construct a universal punctured abstract Stein factorization for which we shall apply a version of the Pontrjagin-Thom construction.

More precisely we construct $\xi_{k,j}$ as follows. Given a group G let $EG \to BG$ denote the universal G-budle. The bundle $\xi_{k,j} \to B_{k,j}$ will be the bundle

$$EAut(\sigma_{k,j}) \times_{Aut(\sigma_{k,j})} (\sigma_{k,j}\colon\ s_{k,j} \to I^k) \to BAut(\sigma_{k,j}),$$

(for the definition of $\sigma_{k,j}$ and $s_{k,j}$ see Figs. 1, 2, 3) where we can write the total space $\xi_{k,j}$ in an other form

$$EAut(\sigma_{k,j}) \times_{Aut(\sigma_{k,j})} s_{k,j} \to EAut(\sigma_{k,j}) \times_{Aut(\sigma_{k,j})} I^k. \tag{4.1}$$

What are the glueing-maps? Let us define the boundary of $\xi_{k,j}$ as the subset

$$EAut(\sigma_{k,j}) \times_{Aut(\sigma_{k,j})} (\partial\sigma_{k,j}\colon\ \partial s_{k,j} \to \partial I^k)$$

of $\xi_{k,j}$. This is an abstract Stein factorization, the singularities of which have codimension less than k. To induce this boundary from the space constructed from the blocks $\xi_{i',j'}$ $(i' < k)$, we have to puncture it. Then we can induce the glueing-map.

So we have a universal punctured abstract Stein factorization $\xi := (\{\xi_{k,j}\}_{k,j}, \{\varphi_{k,j}\}_{k,j})$ constructed from the blocks $\xi_{k,j}$ and the glueing-maps $\varphi_{k,j}$.

If we construct in the described way this universal map ξ, we have the target space of ξ constructed from the target spaces

$$\Gamma_{k,j} := EAut(\sigma_{k,j}) \times_{Aut(\sigma_{k,j})} I^k$$

of $\xi_{k,j}$, see (5.1).

The most important part of this universal map (universal punctured abstract Stein factorization) ξ is its target space, since the homotopy groups of this target space provide us the corresponding cobordism groups according to the (generalized) Pontrjagin-Thom construction. The target space of ξ will be the space Γ in Theorem 4.1: if we have a punctured abstract Stein factorization $g\colon W^2 \to \mathbb{R}^2 \in \mathcal{C}ob_T$ then we obtain a map $\tilde{s}\colon \mathbb{R}^2 \to \Gamma$ by inducing the $\sigma_{k,j}$-bundles of g from Γ as described above, so we have a map $s\colon S^2 \to \Gamma$ which is an element of $\pi_2(\Gamma)$. And if we have a map $s\colon S^2 \to \Gamma$, then after the appropriate identifications we can glue a punctured abstract Stein factorization from the $\sigma_{k,j}$-bundles $s^*(\xi_{k,j}) \to s^{-1}(\ \textit{zero section of}\ \Gamma_{k,j}\ )$ induced by the map s from Γ. This can be applied for homotopies and for cobordisms as well. This completes the proof of Theorem 4.1. □

REMARK 4.3. Clearly $\pi_1(\Gamma)$ is equal to the cobordism group of punctured abstract Reeb-functions which is clearly isomorphic to the cobordism group of abstract Reeb-functions by the method of making holes. This cobordism group was computed in [**4**].

REMARK 4.4. In the proof of Theorem 4.1 we have constructed the universal space of *punctured* abstract Stein-factorizations, but in the same way we could construct the universal space of *any* abstract Stein-factorization: the universal space for $\mathcal{C}ob_S$. In this latter case we have to glue the blocks of the total spaces of universal bundles with structure groups of the automorphisms of the canonical (multi)singularities. There are infinitely many types of canonical (multi)singularity, since an abstract Stein factorization in $\mathcal{C}ob_S$ can have any number of fibers. Hence we have to glue infinitely many blocks by this construction. This is the advantage of the method of making holes.

5. Computing $\pi_2(\Gamma)$

Now we turn to the computation of $\pi_2(\Gamma)$. We will see, that we don't have to know precisely the automorphisms of $\sigma_{k,j}$, it will be enough to know that some special automorphisms exist in these groups. At first we describe the space Γ and we introduce some notation.

Description of the space Γ. We have only one singularity of codimension 1, the canonical map $\sigma_{1,1}\colon \curlyvee \to I$. $Aut(\sigma_{1,1})$ is $\mathbb{Z}_2$, the generator of which interchanges the two upper arms of $\curlyvee$. So the universal $\sigma_{1,1}$-bundle $\xi_{1,1} \to B_{1,1}$ is equal to the bundle $E\mathbb{Z}_2 \times_{\mathbb{Z}_2} (\sigma_{1,1}\colon \curlyvee \to I) \to \mathbb{RP}^\infty$. So $\Gamma_{1,1} = \mathbb{RP}^\infty \times I$, see at the end of the proof of Theorem 4.1. We have to contract the boundary of $\Gamma_{1,1}$ to one point according to the Pontrjagin-Thom construction, so in this first step of the construction of Γ we have $(\mathbb{RP}^\infty \times I)\big/_{\mathbb{RP}^\infty\times\{-1\}\ \amalg\ \mathbb{RP}^\infty\times\{1\}} = S^1 \vee S\mathbb{RP}^\infty$. Put $\Gamma_1 = S^1 \vee S\mathbb{RP}^\infty$.

Now we have to glue the three spaces of 2-codimensional singularities to Γ_1. These are $\Gamma_{2,1}$, $\Gamma_{2,2}$, and $\Gamma_{2,3}$ corresponding to the canonical singularities $\sigma_{2,1}$, $\sigma_{2,2}$ and $\sigma_{2,3}$.

So we get a space

$$\Gamma_2 := \Gamma_1 \cup_{\varphi_{2,1}} \Gamma_{2,1} \cup_{\varphi_{2,2}} \Gamma_{2,2} \cup_{\varphi_{2,3}} \Gamma_{2,3}$$

where $\varphi_{2,j}$ $(j = 1, 2, 3)$ are the glueing-maps, and from this space we can induce every $g \in \mathcal{C}ob_T$. But we want to induce the cobordisms too, so we have to glue the 8 spaces corresponding to the 3-codimensional singularities, see [**11**].

As far as we are considering only cobordisms $W^3 \to \mathbb{R}^2 \times [0, 1]$ (what is the case in the present chapter), we can replace the universal blocks corresponding to the codimension 3 singularities $EAut(\sigma_{k,j}) \times_{Aut(\sigma_{k,j})} (\sigma_{k,j}\colon\ s_{k,j} \to I^k)$ just by one fiber $\sigma_{k,j}\colon\ s_{k,j} \to I^k$ for each $j = 1, 2, \ldots 8$. Hence in order to obtain the 2-homotopy type of the universal target space it is enough to attach to the space Γ_2 eight cubes $I^3_1, \ldots I^3_8$ by the glueing maps corresponding to the boundaries of the maps $\sigma_{3,j}$ for $j = 1, 2, \ldots 8$. So

$$\Gamma = \Gamma_2 \cup_{\varphi_{3,1}} \Gamma_{3,1} \cup \ldots \cup_{\varphi_{3,8}} \Gamma_{3,8}.$$

We will see that the glueing-maps have a very simple structure.

Let $\tilde{\Gamma}_1$ denote the space $\Gamma_1 \cup_{\varphi_{2,1}} \Gamma_{2,1} \cup_{\varphi_{3,1}} \Gamma_{3,1}$. Let $\tilde{\Gamma}$ denote the space $\tilde{\Gamma}_1 \cup_{\varphi_{2,2}} \Gamma_{2,2} \cup_{\varphi_{2,3}} \Gamma_{2,3} \cup_{\varphi_{3,2}} \Gamma_{3,2} \cup_{\varphi_{3,3}} \Gamma_{3,3}$.

REMARK 5.1. Let us compute $\pi_1(\Gamma)$. One can see that the boundaries of the canonical singularities $\sigma_{2,j}$, $j = 1, 2, 3$, are nullcobordant in the cobordism group $\pi_1(\Gamma_1)$. Hence the glueing maps $\varphi_{2,j}$, $j = 1, 2, 3$, restricted to a fiber are nullhomotopic. So the 1-homotopy type of the universal space is $\Gamma_1 \vee S^2 \vee S^2 \vee S^2$ where the three S^2 come from the fibers of $\Gamma_{2,j}$, $j = 1, 2, 3$. So $\pi_1(\Gamma) = \pi_1(S^1 \vee S\mathbb{RP}^\infty \vee S^2 \vee S^2 \vee S^2) = \mathbb{Z}$, and by Remark 4.3 we have an other proof for [**4**].

Computation of $\pi_2(\Gamma)$. The steps of the computation of $\pi_2(\Gamma)$ are the following:

(i) computation of $\pi_2(\tilde{\Gamma}_1)$
(ii) computation of $\pi_2(\tilde{\Gamma}_1 \cup_{\varphi_{2,2}} \Gamma_{2,2} \cup_{\varphi_{3,2}} \Gamma_{3,2})$
(iii) computation of $\pi_2(\tilde{\Gamma})$
(iv) computation of $\pi_2(\Gamma)$.

PROPOSITION 5.2. $\pi_2(\tilde{\Gamma}_1) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

PROOF. $\pi_2(\tilde{\Gamma}_1)$ is the cobordism group of oriented 1-manifolds immersed into the plane with a line-bundle over them, since $Aut(\sigma_{1,1})$ is equal to Z_2. The cobordisms are oriented 2-manifolds with boundary immersed into $(\mathbb{R}^2 \times [0,1], \partial(\mathbb{R}^2 \times [0,1]))$ with a line-bundle over them. Let $\Omega_1(\mathbb{RP}^\infty)$ denote the 1-dimensional oriented bordism group of $\mathbb{RP}^\infty$ and let $\mathcal{I}mm^{SO}(1,1)$ denote the oriented cobordism group of 1-manifolds immersed into $\mathbb{R}^2$. Recall that $\Omega_1(\mathbb{RP}^\infty)$ is equal to Z_2 and $\mathcal{I}mm^{SO}(1,1)$ is equal to Z_2.

There is a homomorphism $\varrho\colon\ \pi_2(\tilde{\Gamma}_1) \to \Omega_1(\mathbb{RP}^\infty)$ which associates to an element of $\pi_2(\tilde{\Gamma})$ the bordism class of its line-bundle, and there is a homomorphism $\psi\colon\ \pi_2(\tilde{\Gamma}_1) \to \mathcal{I}mm^{SO}(1,1)$ which associates to an element of $\pi_2(\tilde{\Gamma}_1)$ the immersion of its source-manifold.

The homomorphism $\varrho \oplus \psi\colon\ \pi_2(\tilde{\Gamma}_1) \to \Omega_1(\mathbb{RP}^\infty) \oplus \mathcal{I}mm^{SO}(1,1)$ is clearly surjective. It is injective, since every element of $\pi_2(\tilde{\Gamma}_1)$ can be represented by line-bundles over embedded circles and trivial line-bundles over a finite set of figure "8" immersions. $\square$

LEMMA 5.3. *We can change the singularity $\sigma_{2,2}$ in each fiber of $\xi_{2,2}$ as it is shown in Fig. 4, so that a neighbourhood of the zero section (the hatched disk in Fig. 4) does not intersect the image of the changed punctured abstract Stein factorization. The glueing map $\varphi_{2,2}\colon \partial\Gamma_{2,2} \to \Gamma_1$ is homotopic to a constant map.*

PROOF. The modification depicted in Fig. 4 is invariant under the structure group $Aut(\sigma_{2,2})$ of $\xi_{2,2}$, because $Aut(\sigma_{2,2})$ is isomorphic to $\mathbb{Z}_2$, the generator of which is a reflection on I^2. The glueing map $\varphi_{2,2}\colon \partial\Gamma_{2,2} \to \Gamma_1$ is homotopic to a constant map, since the modified $\xi_{2,2}$ shows that $\partial\xi_{2,2}$ is nullcobordant. ☐

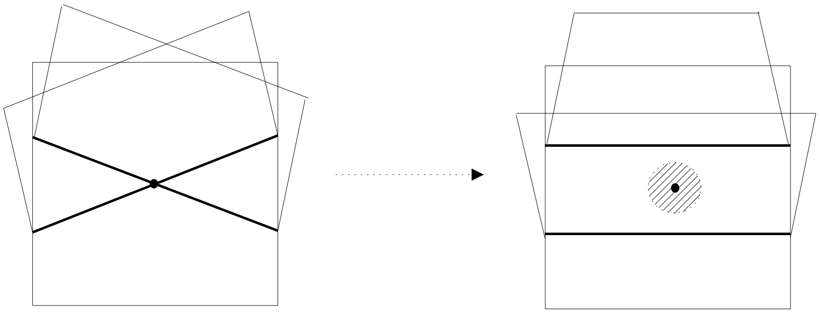

FIGURE 4. The change of $\sigma_{2,2}$

LEMMA 5.4. *We can change the singularity $\sigma_{2,3}$ in each fiber of $\xi_{2,3}$ as it is shown in Fig. 5, so that a neighbourhood of the zero section (the hatched disk in Fig. 5) does not intersect the image of the changed punctured abstract Stein factorization. The glueing map $\varphi_{2,3}\colon \partial\Gamma_{2,3} \to \Gamma_1$ is homotopic to a constant map.*

PROOF. It is easy to see that the modification depicted in Fig. 5 is invariant under the structure group $Aut(\sigma_{2,3})$. ☐

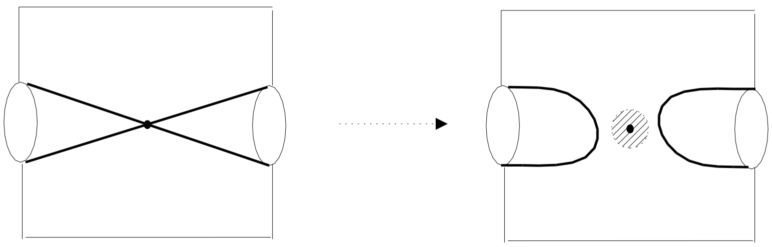

FIGURE 5. The change of $\sigma_{2,3}$

Given any vectorbundle $\xi \to B$ let $T\xi$ denote the Thom-space of the total space.

LEMMA 5.5. *Let ξ^n be an n-dimensional vector bundle of the form $\xi^n = PG \times_\varrho R^n$, where ϱ is a linear representation of the Lie-group G. Suppose that ϱ contains an orientation reversing involution element of $GL(\mathbb{R}^n)$. Then $\pi_n(T\xi)$ is isomorphic to Z_2.*

PROOF. The elements of the cobordism group $\pi_n(T\xi)$ are represented by a finite set of points in $\mathbb{R}^n$. If ϱ contains an orientation reversing involution element g of $GL(\mathbb{R}^n)$, then we can join any two points by an arc in $\mathbb{R}^n \times [0,1]$ with normal bundle admitting the group $\mathbb{Z}_2 = \{\mathrm{id}, g\} \subset GL(\mathbb{R}^n)$ as structure group. □

LEMMA 5.6. $\pi_2(T\Gamma_{2,2}) = \pi_2(T\Gamma_{2,3}) = \mathbb{Z}_2$.

PROOF. We apply Lemma 5.5 for $\xi = \Gamma_{2,2}$ and $\xi = \Gamma_{2,3}$. □

PROPOSITION 5.7. $\pi_2(\tilde{\Gamma}_1 \cup_{\varphi_{2,2}} \Gamma_{2,2} \cup_{\varphi_{3,2}} \Gamma_{3,2}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

PROOF. The corollary of the modification of the singularity $\sigma_{2,2}$ (see Fig. 4) is that we can induce by a new glueing map $\tilde{\varphi}_{2,2}$ from the classification space $\tilde{\Gamma}_1$ the neighbourhood U_∂ of the boundary of $\xi_{2,2}$ (the complement of the hatched disk in Fig. 4) which contains the changed punctured abstract Stein factorization. One can see that the glueing map $\tilde{\varphi}_{2,2}$ restricted to the boundary of the tubular neighbourhood of the zero section (the hatched disk in Fig. 4) is a constant map. So after the modification of the singularity we can interpret the glued block $\xi_{2,2}$ as a classification space for embeddings of codimension 2 manifolds with normal bundle equipped with the structure group $Aut(\sigma_{2,2})$. When we change the singularity, we change the corresponding singularities on the boundary of $\xi_{3,2}$ in the same way, see Fig. 6, where p_1 and p_2 in the figure denote the two singularities of type $\sigma_{2,2}$, and the dotted circles bound the neighbourhoods U_{p_1} and U_{p_2} which are subjected to the modification of the singularities. The punctured abstract Stein factorizations c_1 and c_2 in Fig. 6 are nullcobordant in the group $\pi_2(\tilde{\Gamma}_1)$, hence we can interpret the glued block $\xi_{3,2}$ as a block which guarantees that an embedding of a codimension 2 manifold with normal bundle structure $Aut(\sigma_{2,2})$ can traverse the image of a punctured abstract Stein factorization consisting of singular fibers $\sigma_{1,1}$. The cobordism group of embeddings of codimension 2 manifolds with normal bundle structure $Aut(\sigma_{2,2})$ is equal to $\pi_2(T\Gamma_{2,2})$. So Lemma 5.6 and the argument above show that $\pi_2(\tilde{\Gamma}_1 \cup_{\varphi_{2,2}} \Gamma_{2,2} \cup_{\varphi_{3,2}} \Gamma_{3,2})$ is the cobordism group $\pi_2(\tilde{\Gamma}_1) \oplus \pi_2(T\Gamma_{2,2}) = \mathbb{Z}_2^3$. □

PROPOSITION 5.8. $\pi_2(\tilde{\Gamma}) = \mathbb{Z}_2^4$.

PROOF. Lemma 5.6 and an argument similar to the proof of Proposition 5.7 show that $\pi_2(\tilde{\Gamma}_1 \cup_{\varphi_{2,2}} \Gamma_{2,2} \cup_{\varphi_{2,3}} \Gamma_{2,3} \cup_{\varphi_{3,2}} \Gamma_{3,2} \cup_{\varphi_{3,3}} \Gamma_{3,3})$ is the cobordism group $\pi_2(\tilde{\Gamma}_1) \oplus \pi_2(T\Gamma_{2,2}) \oplus \pi_2(T\Gamma_{2,3}) = \mathbb{Z}_2^4$. □

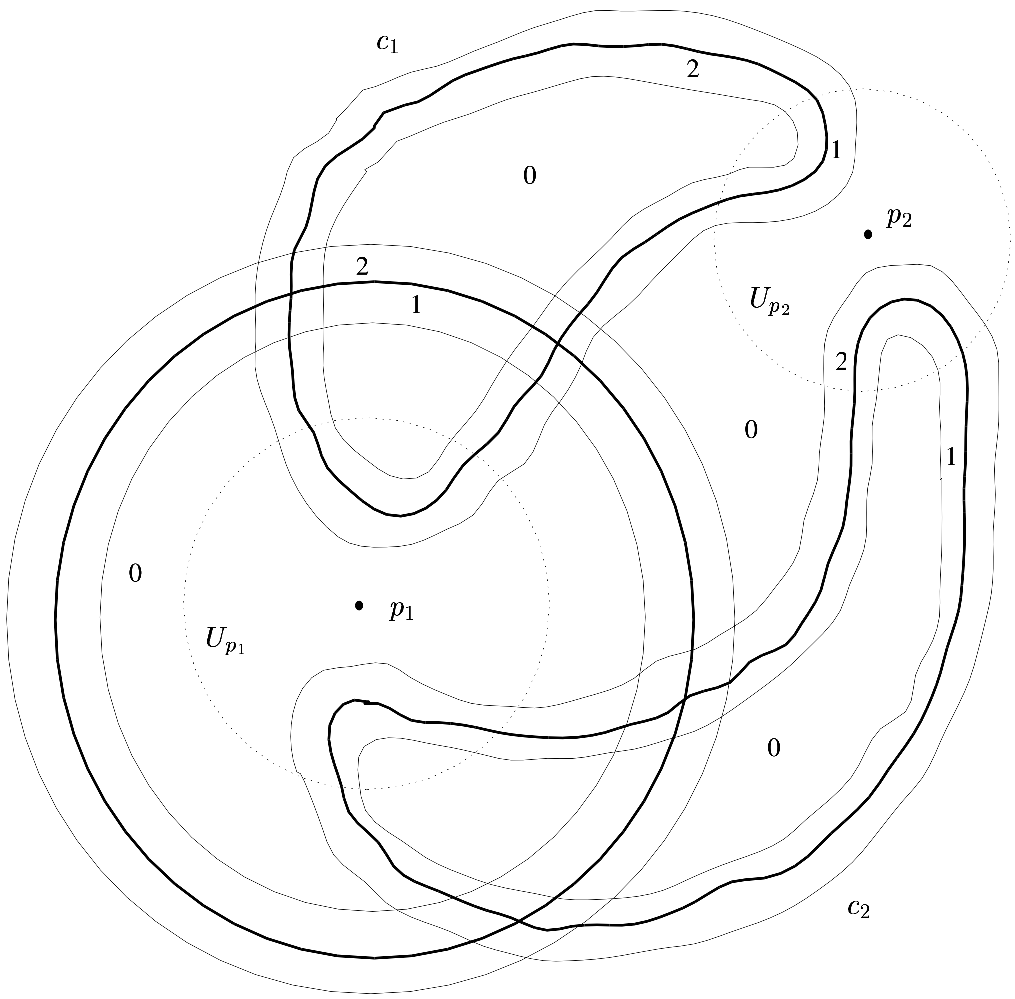

FIGURE 6. The boundary of $\xi_{3,2}$ after the modification of the singularities of type $\sigma_{2,2}$ p_1 and p_2

PROPOSITION 5.9. $\pi_2(\Gamma) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

PROOF. Let $4 \leq j \leq 8$. If we glue the space $\Gamma_{3,j}$ to the space $\tilde{\Gamma}$, $\pi_2(\tilde{\Gamma})$ will be factorised by the element $[\varphi_{3,j}] \in \pi_2(\tilde{\Gamma})$. Let (y,ξ), η, θ denote the images of the generators of $\pi_2(\tilde{\Gamma}_1)$, $\pi_2(T\Gamma_{2,2})$, $\pi_2(T\Gamma_{2,3})$ respectively in $\pi_2(\tilde{\Gamma})$. Then after a little computation we get the following list, which shows for each j what represents the glueing map $\varphi_{3,j}\colon \partial\Gamma_{3,j} \to \tilde{\Gamma}$ in $\pi_2(\tilde{\Gamma})$:

4: $[\varphi_{3,4}] = \xi + 5\eta$
5: $[\varphi_{3,5}] = 6\eta$
6: $[\varphi_{3,6}] = 3\eta + 3\theta + y$
7: $[\varphi_{3,7}] = 4\eta + 2\theta$
8: $[\varphi_{3,8}] = 6\theta$

The only interesting cases are $j = 4$ and $j = 6$, because for the other values of j one has $[\varphi_{3,j}] = 0 \in \pi_2(\tilde{\Gamma})$. For $j = 4$ we get the equation $\xi + 5\eta = \xi + \eta = 0$, so $\xi = \eta$. For $j = 6$ we get $3\eta + 3\theta + y = \eta + \theta + y = 0$. The first equation identifies ξ and η, the second identifies η and $\theta + y$. Hence $\pi_2(\Gamma) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. □

COROLLARY 5.10. *The cobordism group $\mathcal{C}ob_T$ of simple abstract Stein factorizations is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.*

PROOF. By Theorem 4.1 the cobordism group $\mathcal{C}ob_T$ is isomorphic to $\pi_2(\Gamma)$. □

COROLLARY 5.11. *The cobordism group $\mathcal{C}ob_S$ of abstract Stein factorizations is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.*

PROOF. By Lemma 3.1 $\mathcal{C}ob_S$ is isomorphic to $\mathcal{C}ob_T$. □

COROLLARY 5.12. *The cobordism group $\mathcal{C}ob$ of fold maps from closed oriented 3-dimensional manifolds into $\mathbb{R}^2$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.*

PROOF. From [**4**] we know that the homomorphism $\upsilon\colon \mathcal{C}ob \to \mathcal{C}ob_S$ is surjective. So we have only to show the injectivity. Let $[f_1\colon M_1 \to \mathbb{R}^2]$ and $[f_2\colon M_2 \to \mathbb{R}^2]$ be two elements of $\mathcal{C}ob$. Let $[g_1\colon W_1 \to \mathbb{R}^2]$ and $[g_2\colon W_2 \to \mathbb{R}^2]$ denote the abstract cobordism classes of their Stein factorizations, respectively. It is sufficient to show that if the abstract Stein factorizations g_1 and g_2 are cobordant by a cobordism G then the fold maps f_1 and f_2 are cobordant.

By Remark 3.3 there exists $[\tilde{f}_1]$ and $[\tilde{f}_2]$ in $\mathcal{C}ob$ such that $[\tilde{f}_1] = [f_1]$, $[\tilde{f}_2] = [f_2]$, and the Stein factorizations $\tilde{g}_1$ and $\tilde{g}_2$ of the fold maps $\tilde{f}_1$ and $\tilde{f}_2$ respectively are elements of $\mathcal{C}ob_T$.

We can suppose that the cobordism G between $\tilde{g}_1$ and $\tilde{g}_2$ is a simple cobordism. Now we can construct a map $F\colon X \to \mathbb{R}^2 \times [0,1]$ from an oriented 4-manifold X such that F is a cobordism between $\tilde{f}_1$ and $\tilde{f}_2$ and the Stein factorization of F is the cobordism G.

The steps of the construction of the map F are the analogous steps of the construction desrcibed in [**11**, Chapter 6]. So at first we construct oriented 4-manifolds P_k and their fold maps $f_k\colon P_k \to \tilde{N}(\Sigma_k)$ onto the tubular neighbourhoods $\tilde{N}(\Sigma_k)$ of the k-codimensional submanifolds Σ_k, $k = 1, 2, 3$, of the set $G(S_G)$ (see Section 3), and then we glue these fold maps together.

We have to note that the canonical singularities $\tilde{\sigma}_{2,2}$ and $\tilde{\sigma}_{2,3}$ whose Stein factorizations are $\sigma_{2,2}$ and $\sigma_{2,3}$ respectively (see [**7**]) have an automorphism α_j which acts as a reflection on I^2 and as an orientation reversing diffeomorphism on the source manifold of $\tilde{\sigma}_{2,j}$, $j = 2, 3$. If we choose the group $AUT(\sigma_{2,3})$ such that it is generated by this element α_3 and by the elements which come from the automorphism group of the singularity $\tilde{\sigma}_{2,3}$ and which act as an orientation preserving diffeomorphism on the source manifold of $\tilde{\sigma}_{2,3}$, then we can construct the oriented 4-manifolds and the fold maps mentioned above.

After we have constructed and glued these maps, we have an oriented 4-manifold $\tilde{X}$ glued from the parts P_k, $k = 1, 2, 3$, such that $\partial\tilde{X} = M_1 \amalg -M_2 \amalg S$ where S is a total space of an oriented S^1-bundle over an oriented surface T immersed into $\mathbb{R}^2 \times [0, 1]$.

Now we can glue to S the total space D of a 2-disc bundle over T whose boundary is S, and we can construct a fold map from D into $\mathbb{R}^2 \times [0, 1]$, by a similar method as described in Chapter 1. This completes the proof. $\square$

REMARK 5.13. By the proof of Theorem 2.9 we can represent the three non zero elements of $\mathcal{C}ob_S$ by the maps (a), (b), and (c) shown on Fig. 7. In (a) the $\curlyvee$-bundle is non-trivial, in (b), (b') and (c) it is trivial. The case (b) is cobordant to the case (b').

The suitable generators of the group $\mathcal{C}ob$ for the generators in Fig. 7 are fold maps from 3-manifolds whose Stein factorizations are the maps as depicted in Fig. 7.

We can construct these 3-manifolds by the method of Mata-Lorenzo described in [**8**]. So these 3-manifolds are $S^1 \times S^2$ in the case (a), (b), and (c) as well, as one can see easily.

REMARK 5.14. We can define two homomorphisms $\mathcal{C}ob \to \mathbb{Z}_2$ which are complete invariants of $\mathcal{C}ob$. For a given $[f\colon M^3 \to \mathbb{R}^2] \in \mathcal{C}ob$ let $\psi([f])$ be the number of the self-intersections modulo two of the f-image of the indefinit singular points of f. We can define a second homomorphism $\varrho\colon \mathcal{C}ob \to \mathbb{Z}_2$ as follows. At first we define a line-bundle γ over S_1, where S_1 denotes the set of all indefinit fold points of f in M^3. S_1 is an embedded 1-manifold in M^3. Let $p \in S_1$, and suppose that $f^{-1}(f(p)) \cap S_1$ consists of one point, that is, p is a simple indefinit fold point. Then the singular fiber $f^{-1}(f(p))$ has the form of an "8", whose position defines a fiber of a line-bundle over the simple indefinit singular points in M^3. In S_1 in a neighbourhood of a singular fiber of a singular point of type $\sigma_{2,2}$ the position of the "8" is welldefined as one can see in [**7**], and in a neighbourhood of a singular fiber of a singular point of type $\sigma_{2,3}$ the position of the "8" is welldefined modulo two. So we have a line-bundle γ over S_1 which can be trivial or not, and which gives the "position of the fiber 8" in the above sense. The first Stiefel-Whitney number $w_1(\gamma)$ of this line-bundle γ is an element in Z_2. Now let $\varrho([f]) \in Z_2$ be the sum

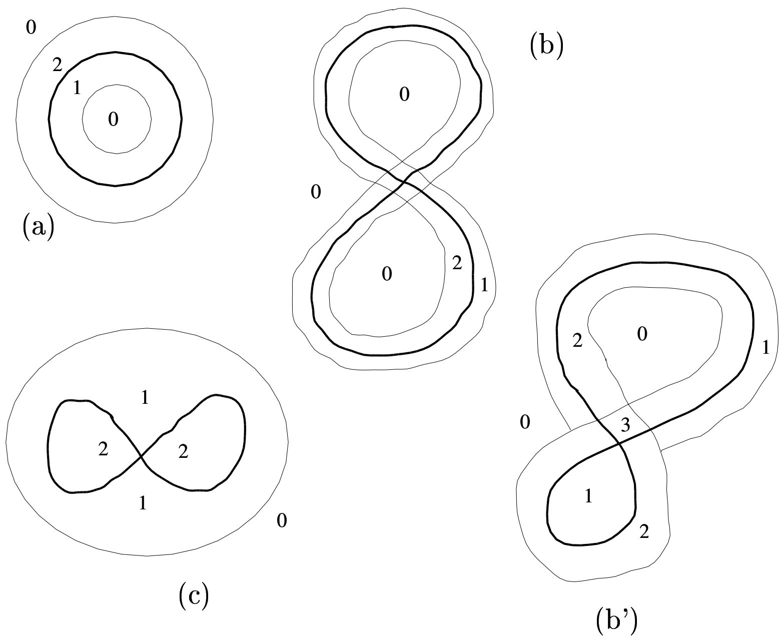

FIGURE 7. The non zero elements of $\mathcal{C}ob_S$

of $w_1(\gamma)$ and the number of singular points of type $\sigma_{2,3}$ modulo two. By Proposition 5.9, Corollary 5.12 and Remark 5.13 $\psi \oplus \varrho\colon \mathcal{C}ob \to \mathbb{Z}_2 \oplus \mathbb{Z}_2$ is a welldefined homomorphism on $\mathcal{C}ob$, and it is an isomorphism.

6. Computing the oriented bordism group $\Omega_2(\Gamma)$

It is clear that the methods described in the previous sections can be applied for oriented 2-manifolds as target spaces in place of $\mathbb{R}^2$ and oriented 3-manifolds with boundary as cobordisms in place of $\mathbb{R}^2 \times [0,1]$ as well. For a space X let $\Omega_n(X)$ denote the n-dimensional oriented bordism group of X. Note that the group $\Omega_2(\Gamma)$ is equal to the oriented bordism group of punctured abstract Stein factorizations whose target spaces are closed oriented surfaces, and the group $\Omega_1(\Gamma)$ is equal to the oriented bordism group of punctured Reeb functions whose target spaces are closed oriented 1-manifolds.

We note that the oriented bordism group $\Omega_1(\Gamma)$ is equal to $\mathbb{Z}$ since $\Omega_1(\Gamma) = H_1(\Gamma) = \mathbb{Z}$, and its generator can be represented by the map as depicted in Fig. 8. By the method described in [**4**] $\Omega_1(\Gamma)$ is equal to the bordism group of Morse functions on oriented surfaces.

LEMMA 6.1. $\Omega_2(\tilde{\Gamma}_1) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

PROOF. The proof is like that of Proposition 5.2. We can define the similar homomorphisms $\varrho\colon \Omega_2(\tilde{\Gamma}_1) \to \Omega_1(\mathbb{RP}^\infty)$ and $\psi\colon \Omega_2(\tilde{\Gamma}_1) \to \Omega_2(QMSO(1))$ such that the

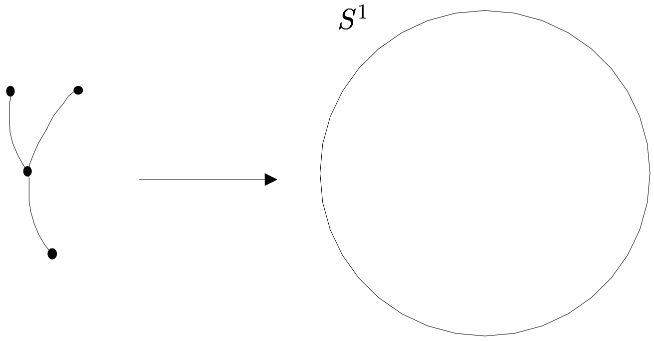

FIGURE 8. The generator of $\Omega_1(\Gamma)$

homomorphism $\varrho \oplus \psi\colon \Omega_2(\tilde{\Gamma}_1) \to \Omega_1(\mathbb{RP}^\infty) \oplus \Omega_2(QMSO(1))$ is an isomorphism, where $\Omega_2(QMSO(1))$ denotes the 2-dimensional oriented bordism group of 1-codimensional immersions of oriented manifolds which is isomorphic to $\mathbb{Z}_2$. $\square$

Recall that the equality $\Omega_2(X \vee Y) = \Omega_2(X) \oplus \Omega_2(Y)$ holds, where X and Y are CW complexes, and recall that for a glueing map $\varphi\colon \partial D^3 \to X$ the group $\Omega_2(X \cup_\varphi D^3)$ is equal to $\Omega_2(X)$ factorized by the element $[\varphi\colon S^2 \to X] \in \Omega_2(X)$. One can see this directly from the Mayer-Vietoris sequence for bordism groups.

Similarly to Lemma 5.6 we have

LEMMA 6.2. $\Omega_2(T\Gamma_{2,2}) = \Omega_2(T\Gamma_{2,3}) = \mathbb{Z}_2$.

PROOF. The proof is analogous to that of Lemma 5.6. $\square$

THEOREM 6.3. $\Omega_2(\Gamma) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

PROOF. Let $\tilde{\Gamma}_2$ denote the space $\tilde{\Gamma}_1 \cup_{\varphi_{2,2}} \Gamma_{2,2} \cup_{\varphi_{2,3}} \Gamma_{2,3}$. By Lemma 5.3, Lemma 5.4, Lemma 6.2, and Lemma 6.1 the bordism group $\Omega_2(\tilde{\Gamma}_2) = \mathbb{Z}_2^4$. By computations similar to the proof of Proposition 5.9 the glueing maps $\varphi_{3,2}\colon \Gamma_{3,2} \to \tilde{\Gamma}_2$ and $\varphi_{3,3}\colon \Gamma_{3,3} \to \tilde{\Gamma}_2$ represents zero in $\Omega_2(\tilde{\Gamma}_2)$. So $\Omega_2(\tilde{\Gamma}) = \mathbb{Z}_2^4$. By an analogue of Proposition 5.9 we have $\Omega_2(\Gamma) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. $\square$

By Theorem 6.3 we have the following.

COROLLARY 6.4. *The bordism group $\mathcal{B}or$ of fold maps from oriented 3-manifolds into oriented surfaces is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. The natural map $\mathcal{C}ob \to \mathcal{B}or$ is an isomorphism.*

CHAPTER 3

Cobordism group of fold maps with codimension -1

1. Introduction

Pontrjagin-Thom construction is an elementary method to study cobordisms. It has many versions, Pontrjagin's construction for cobordisms of framed embeddings, Thom's construction for cobordisms of manifolds, Wells method [**18**] for cobordisms of immersions, etc.

In the 70s Szűcs started to extend Pontrjagin-Thom construction for maps with singularities [**13**] in a way suggested by Gromov. After many steps Szűcs arrived with his student Rimányi at the construction for maps with singularities in positive codimension [**10**]. The fruit of this work is many results concerning singular maps [**9, 14, 15, 17**].

In spite of its benefits in positive codimension it has not been such a construction yet in the negative codimensional case. For example by works of Ando [**1**] and Saeki [**11**] there exist universal constructions for studying cobordisms of maps with singularities in negative codimension as well, but these are different methods from Pontrjagin-Thom-type classifying spaces.

Szűcs asked about the way of making a Pontrjagin-Thom-type construction for singular maps in negative codimension. For Stein factorizations of such maps we showed a construction in Chapter 2 when the codimension was equal to -1. For other negative codimensions the appropriate classifying spaces are much more complicated than the classifying spaces for positive codimensions.

In this chapter we show a construction for fold maps of codimension -1 as well because of the simplicity of the finite character of this case. In other negative codimensions applications of such classifying spaces are more complicated.

As application we compute some cobordism groups of fold maps with prescribed singularities. It turns out that these groups are strongly related to cobordism groups of immersions and to stable homotopic groups.

The chapter is organized as follows. In §2 we give some basic definitions and we summarize results concerning fold maps in codimension -1. In §3 we state and prove some lemmas related to symmetries of fold singularities. In §4 we give definitions of cobordisms and we introduce further notions and notations. In §5 we describe the construction of the Pontrjagin-Thom-type classifying space and we show some relation between fold maps and their Stein factorizations. In §6 we compute some cobordism groups as an application of the classifying space.

2. Preliminaries

Definition 2.1. Let Q^q and N^n be smooth manifolds of dimensions q and n respectively $(q \geq n)$. Let $p \in Q^q$ be a singular point of a smooth map $f\colon Q^q \to N^n$. The smooth map f has a *fold singularity* at the singular point p, if we can write f in some local coordinates at p and $f(p)$ in the form

$$f(x_1, \dots, x_q) = (x_1, \dots, x_{n-1}, \pm x_n^2 \pm \dots \pm x_q^2).$$

A smooth map $f\colon Q^q \to N^n$ is called a *fold map*, if f has only fold singularities.

Definition 2.2. Let Q^q and N^n be smooth manifolds of dimensions q and n respectively $(q \geq n)$. A smooth map $f\colon Q^q \to N^n$ at the fold singularity $p \in Q^q$ has a *definite fold singularity* if we can write f in some local coordinates at p and $f(p)$ in the form

$$f(x_1, \dots, x_q) = (x_1, \dots, x_{n-1}, x_n^2 + \dots + x_q^2),$$

otherwise f has an *indefinite fold singularity* at the fold singularity $p \in Q^q$. Let $S_1(f)$ denote the set of indefinite fold singularities of f in Q^q.

For fixed integers q and n, $q \geq n \geq 0$, let $FOLD(q,n)$ ($FOLD^O(q,n)$) denote the set of the fold maps $f\colon Q^q \to \mathbb{R}^n$, where Q^q runs over the set of all closed q-dimensional manifolds (closed *oriented* q-dimensional manifolds, the upper index "O" comes from "oriented").

Stein factorization. We use the notion of the Stein factorization of a smooth map $f\colon Q^q \to N^n$, where Q^q and N^n are smooth manifolds of dimensions q and n respectively $(q \geq n)$. Two points $p_1, p_2 \in Q^q$ are *equivalent* if and only if p_1 and p_2 lie on the same componenet of an f-fiber. Let W_f denote the quotient space of Q^q with respect to this equivalence relation and $q_f\colon W_f \to N^n$ the quotient map. Then there exists a unique continuous map $\bar{f}\colon W_f \to N^n$ such that the diagram

$$\begin{tikzcd} Q^q \arrow[rr, "f"] \arrow[dr, "q_f"'] & & N^n \\ & W_f \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

commutes or in other words $f = \bar{f} \circ q_f$. The space W_f or the factorization of the map f into the composition of q_f and $\bar{f}$ is called the *Stein factorization* of the map f. We call the map $\bar{f}$ also the *Stein factorization* of the map f as well. Note that if f is a smooth map from a closed q-dimensional manifold into N^n, then its Stein factorization W_f is a compact n-dimensional CW complex.

The following notion is the analogue of *abstract Reeb function* introduced in [**4**].

DEFINITION 2.3. Let $ASTEIN(q,n)$ denote the set of the continuous maps $g\colon W^n \to \mathbb{R}^n$ where W^n is a compact n-dimensional CW complex such that W^n and the map g are locally equivalent to a Stein-factorization of a fold map $f\colon Q^q \to \mathbb{R}^n \in FOLD(q,n)$. An element of $ASTEIN(q,n)$ is called an *abstract Stein factorization.*

In this chapter we are interested in the case when the codimension $n-q$ is equal to -1. So let $FOLD_q$, $FOLD_q^O$, $ASTEIN_q$ denote the set $FOLD(q,q-1)$, $FOLD^O(q,q-1)$, $ASTEIN(q,q-1)$ respectively.

Local forms of fold maps. The local forms of singularities of stable fold maps in $FOLD_q$, $q = 2,3,4$, $FOLD_5^O$, their singular fibers and the local forms of their Stein factorizations are completely described in [**6, 7**], [**11**, Chapter 4], [**12**] and [**19**].

In this section we summarize the classification of the singularities which will be important for us. Without mentioning, throughout the chapter we consider *stable* fold maps. For singular fibers we use the notations of [**11**], [**12**] and [**19**].

DEFINITION 2.4. Let J denote the interval $[-1,1]$.

Let $\sigma_{\mathfrak{F}}\colon s_{\mathfrak{F}} \to J^k$ denote the *canonical singularity of singular fiber denoted by* $\mathfrak{F}$. For example, for $\mathfrak{F} = \mathrm{II}^2$ the singularity σ_{II^2} has a singular fiber of type II^2 (for the notation II^2 see [**11**, Chapter 4], [**12**] , [**19**]).

Let $\bar{\sigma}_{\mathfrak{F}}\colon w_{\mathfrak{F}} \to J^k$ denote the Stein factorization of the singularity $\sigma_{\mathfrak{F}}$. We call the map $\bar{\sigma}_{\mathfrak{F}}$ the *canonical Stein factorization singularity* $\bar{\sigma}_{\mathfrak{F}}$ or simply the *canonical singularity* $\bar{\sigma}_{\mathfrak{F}}$.

Let $\kappa(\mathfrak{F})(=k)$ denote the codimension of the $\sigma_{\mathfrak{F}}$-image of the singular fiber $\mathfrak{F}$.

We note that by this definition the singularity $\sigma_{\mathfrak{F}}$ has no circle components in its singular fiber.

For example, if we have a Morse function f on an orientable surface, we have an indefinite critical point of f (which is a saddle-point) such that the corresponding singular fiber is equivalent to an "8", then the function f restricted to a neighbourhood of the singular fiber corresponds to the canonical singularity $\sigma_{\mathrm{I}^1}\colon s_{\mathrm{I}^1} \to J$, its Stein factorization (which is a function "$\curlyvee \to J$") is the singularity $\bar{\sigma}_{\mathrm{I}^1}$ and $\kappa(\mathrm{I}^1) = 1$, see Fig. 2.

In the Fig. 2 we can see the canonical singularities $\bar{\sigma}_{\mathrm{II}^{1,1}}\colon w_{\mathrm{II}^{1,1}} \to J^2$, $\bar{\sigma}_{\mathrm{II}^2}\colon w_{\mathrm{II}^2} \to J^2$, $\bar{\sigma}_{\mathrm{II}^3}\colon w_{\mathrm{II}^3} \to J^2$, respectively.

In the Fig. 3 we can see the singular fibers of the possible singularities $\sigma_{\mathfrak{F}}\colon s_{\mathfrak{F}} \to J^{\kappa(\mathfrak{F})}$ of fold maps $Q^q \to \mathbb{R}^{q-1}$, $q = 2 \dots 5, \kappa(\mathfrak{F}) = 1 \dots 4$, which will be relevant for us.

By [**6, 7**], [**11**, Chapter 4] and [**12**] we can state the following theorem.

THEOREM 2.5. *Let $f\colon Q^q \to \mathbb{R}^{q-1}$ be an element of $FOLD_q^O$, $q = 2 \dots 5$. Let $p \in Q^q$ be an indefinite singular point of f, i.e., $p \in S_1(f)$. Then there exists a neighbourhood U of $f^{-1}(f(p))$ in Q^q such that U contains only the singular fibers of the singular points in $S_1(f)$ and $f|_U$ is equivalent to $\sigma_{\mathfrak{F}} \times \mathrm{id}_{\mathbb{R}}^{q-1-\kappa(\mathfrak{F})}$ where $\sigma_{\mathfrak{F}}$ is a canonical singularity of singular fiber $\mathfrak{F}$ of the following types.*

(a) I^1
(b) $\mathrm{II}^{1,1}, \mathrm{II}^2, \mathrm{II}^3$
(c) $\mathrm{III}^{1,1,1}, \mathrm{III}^{1,2}, \mathrm{III}^{1,3}, \mathrm{III}^4, \dots \mathrm{III}^8$
(d) $\mathrm{IV}^{1,1,1,1}, \mathrm{IV}^{1,1,2}, \mathrm{IV}^{1,1,3}, \mathrm{IV}^{2,2}, \mathrm{IV}^{2,3}, \mathrm{IV}^{3,3}, \mathrm{IV}^{1,4}, \dots \mathrm{IV}^{1,8}, \mathrm{IV}^9, \dots \mathrm{IV}^{22}$

REMARK 2.6. Let f be a map in $FOLD_q$. About the map f we suppose that it can be stratified so that each stratum is a $\sigma_{\mathfrak{F}}$-bundle for some canonical singularity $\sigma_{\mathfrak{F}}$. For example, by stability conditions [**2**, Chapter I, §3 and Chapter II, §5] we can have such a stratification. Note that for low dimension q this holds.

DEFINITION 2.7. We can define the canonical singularities whose singular fibers are the disjoint unions of the singular fibers in Theorem 2.5. So we have the canonical singularities of singular fibers of types $\mathrm{V}^{1,1,1,1,1}, \mathrm{VI}^{1,1,1,1,1,1}, \dots, \mathrm{V}^{1,2,3}, \dots \mathrm{VI}^{3,3,3}, \dots$, etc. If a singularity $\sigma_{\mathfrak{F}}$ has a disconnected singular fiber, then we call $\sigma_{\mathfrak{F}}$ a *multisingularity.*

DEFINITION 2.8. Let $\sigma_{\mathfrak{F}}\colon s_{\mathfrak{F}} \to J^k$ be a canonical singularity where the manifold $s_{\mathfrak{F}}$ is orientable. If we choose an orientation on $s_{\mathfrak{F}}$ and on the manifold J^k in a standard way as well. then we say that the canonical singularity $\sigma_{\mathfrak{F}}\colon s_{\mathfrak{F}} \to J^k$ is *oriented.* From now on we fix an orientation for these canonical singularities.

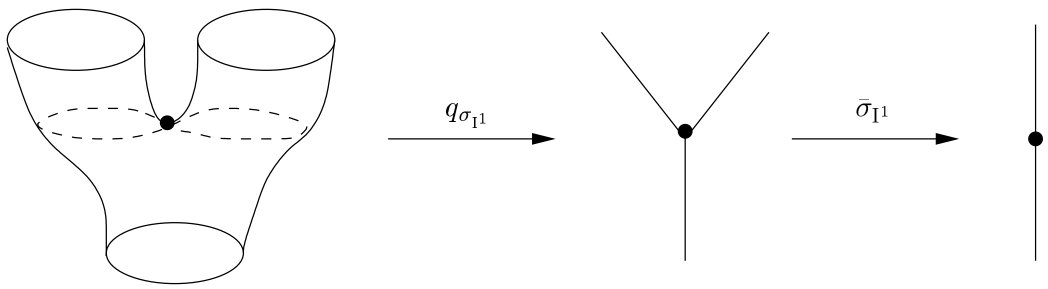

FIGURE 1. Stein factorization of the canonical singularity $\sigma_{\mathrm{I}^1}\colon s_{\mathrm{I}^1} \to J$

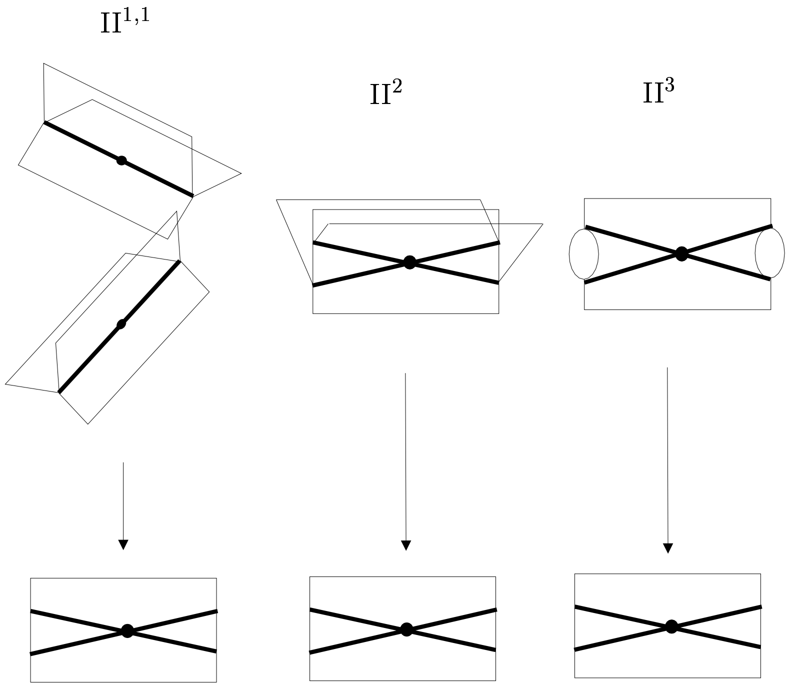

FIGURE 2. The canonical singularities $\bar{\sigma}_{\mathfrak{F}}\colon\, w_{\mathfrak{F}} \to J^2$ $(\mathfrak{F} = \mathrm{II}^{1,1}, \mathrm{II}^2, \mathrm{II}^3)$

3. Symmetries of singularities

Let $AUT(\sigma_{\mathfrak{F}})$ denote the group of automorphisms of the singularity $\sigma_{\mathfrak{F}}$. Let $g \in AUT(\sigma_{\mathfrak{F}})$. Then g consists of two diffeomorphisms g_1 and g_2 such that the diagram

$$\begin{array}{ccc} s_{\mathfrak{F}} & \xrightarrow[g_2]{} & s_{\mathfrak{F}} \\ \downarrow & & \downarrow \\ J^k & \xrightarrow{g_1} & J^k \end{array}$$

commutes, where the two vertical maps $s_{\mathfrak{F}} \to J^k$ are equal to the canonical singularitiy $\sigma_{\mathfrak{F}}$.

Let $AUT^O(\sigma_{\mathfrak{F}})$ denote the subgroup of automorphisms of g where the diffeomorphisms g_1 and g_2 both preserve or both reverse the orientations of the manifolds $s_{\mathfrak{F}}$ and J^k.

Let $Aut^{(O)}(\sigma_{\mathfrak{F}})$ be the subgroup of $AUT^{(O)}(\sigma_{\mathfrak{F}})$, where the elements of $Aut^{(O)}(\sigma_{\mathfrak{F}})$ act linearly on $[-1,1]^k$ and act effectively on the Stein factorization $\bar{\sigma}_{\mathfrak{F}}$. Let $Perm^{(O)}(\sigma_{\mathfrak{F}})$ be the subgroup of $Aut^{(O)}(\sigma_{\mathfrak{F}})$, where the elements of $Perm^{(O)}(\sigma_{\mathfrak{F}})$ act on $[-1,1]^k$ by permuting the factors and reflecting in the coordinate hyperplanes.

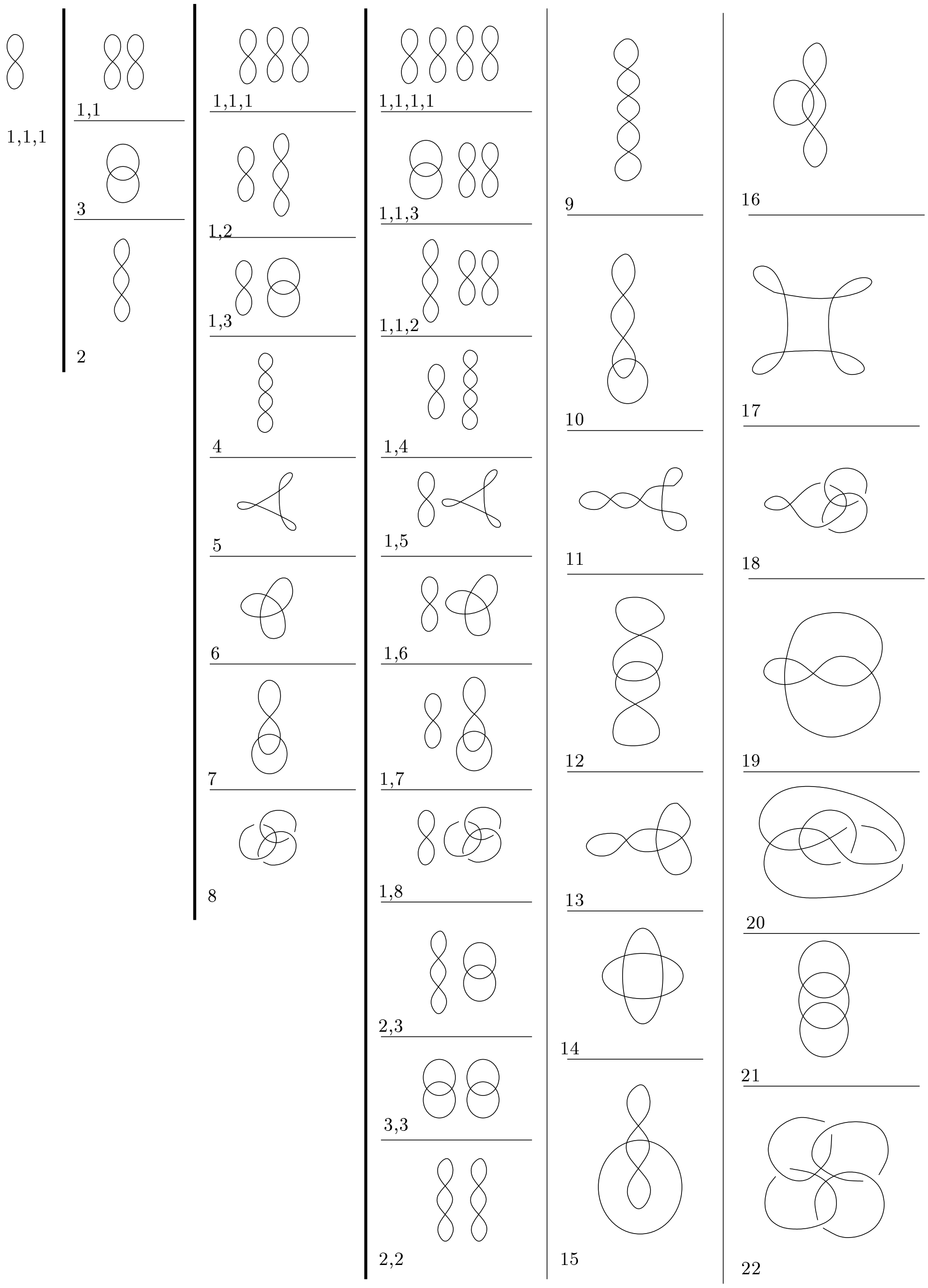

FIGURE 3. The singular fibers of the singularities $\sigma_{\mathfrak{F}}$, $\mathfrak{F} = \mathrm{I}^1 \dots \mathrm{IV}^{22}$

PROPOSITION 3.1. *Let $\xi \to B$ be a $\sigma_{\mathfrak{F}}$: $s_{\mathfrak{F}} \to J^{\kappa(\mathfrak{F})}$-bundle over a manifold B with structure group $AUT(\sigma_{\mathfrak{F}})$, i.e. the total space ξ is a fiber preserving map from the total space F of an $F \to B$ $s_{\mathfrak{F}}$-bundle to the total space Γ of a $\Gamma \to B$ $J^{\kappa(\mathfrak{F})}$-bundle such that the map ξ is equal to the canonical singularity $\sigma_{\mathfrak{F}}$ on each fiber of F.*

Then the structure group of the $\sigma_{\mathfrak{F}}$-bundle $\xi \to B$ can be reduced to $Aut(\sigma_{\mathfrak{F}})$. Moreover its structure group can be reduced to $Perm(\sigma_{\mathfrak{F}})$.

PROOF. We define a stratification on J^k as follows. Let H_n^{k-1}, $n = 1 \dots k$, denote the coordinate hyperplanes in J^k, and let P_l^i, $i = 0 \dots k, l = 1 \dots r_i$ denote the r_i i-dimensional parts of the natural stratification of J^k induced by $\cup_n H_n^{k-1}$. There exists over a point of every $int(P_l^{k-1})$ a point in $s_{k,j}$ which has a neighbourhood of the form $\curlyvee$, so g_1 maps P_l^{k-1} to some $P_{l'}^{k-1}$, $l = 1 \dots r_{k-1}$, so the diffeomorphism g_1 preserves the stratification.

Since the Stein factorization $\bar{s}_{\mathfrak{F}}$ is a covering over every stratum, it is clear that for a given deformation (parametrized by t) of g_1 through diffeomorphisms which preserve the stratification there exists a deformation parametrized by t through diffeomorphisms of g_2 such that the pair $(g_1, g_2)_t$ is an element of $AUT(\sigma_{\mathfrak{F}})$ at every $t \in [0,1]$, and it is clear that the action of $(g_1, g_2)_t$ on the Stein factorization $\bar{\sigma}_{\mathfrak{F}}$ is determined by the deformation of g_1.

We need to reduce the structure group $AUT(\sigma_{\mathfrak{F}})$ to the group $Aut(\sigma_{\mathfrak{F}})$. First we linearize the structure group of the $\Gamma \to B$ $J^{\kappa(\mathfrak{F})}$-bundle.

LEMMA 3.2. *If we restrict the map ξ to a neighbourhood of the singular points in $\xi^{-1}($ zero section of Γ), we get an immersion into the $J^{\kappa(\mathfrak{F})}$-bundle Γ. So the structure group of the $J^{\kappa(\mathfrak{F})}$-bundle $\Gamma \to B$ can be reduced to the ortogonal group.* □

By this lemma the proof of the proposition is almost complete, still we have to note that the subgroup of the group $AUT(\sigma_{\mathfrak{F}})$ where the elements act identically on the Stein factorization $\bar{\sigma}_{\mathfrak{F}}$ can be retracted into the identity automorphism of $AUT(\sigma_{\mathfrak{F}})$, since the singular fiber $\mathfrak{F}$ is a 1-dimensional complex which does not contain circle components, and an automorphism which act identically on the Stein factorization $\bar{\sigma}_{\mathfrak{F}}$ keeps the vertices of the singular fiber $\mathfrak{F}$ unchanged and maps the edges into themselves. □

PROPOSITION 3.3. *For a given singularity $\sigma_{\mathfrak{F}}$ the group $Aut^O(\sigma_{\mathfrak{F}})$ can be reduced to the groups in the next table.*

I^1	II^2	II^3	III^4	III^5	III^6	III^7	III^8
$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	S_3	0	$\mathbb{Z}_3$

PROOF. We have to examine the automorphisms of the singular fibers. Details are left to the reader. □

4. Cobordism of fold maps

Let us first give the classical definitions of cobordism. Later we give a definition more convenient and sophisticated. We define an equivalence relation on the set $FOLD(q,n)$ $(FOLD^O(q,n))$ as follows. Let $\amalg$ denote the disjoint union.

DEFINITION 4.1. Two fold maps $f_0\colon Q_0^q \to \mathbb{R}^n$ and $f_1\colon Q_1^q \to \mathbb{R}^n$ on closed q-dimensional manifolds (closed oriented q-dimensional manifolds) Q_0^q and Q_1^q are *cobordant* (*oriented cobordant*), if there exists a smooth map $F\colon X^{q+1} \to \mathbb{R}^n \times [0,1]$ with only fold singularities from a compact $q+1$-dimensional manifold (compact oriented $q+1$-dimensional manifold) X^{q+1} with $\partial X^{q+1} = Q_0^q \amalg Q_1^q$ ($\partial X^{q+1} = Q_0^q \amalg -Q_1^q$) such that $F\mid_{Q_0^q \times [0,\varepsilon)} = f_0 \times id_{[0,\varepsilon)}$ and $F\mid_{Q_1^q \times (1-\varepsilon,1]} = f_1 \times id_{(1-\varepsilon,1]}$, where $Q_0^q \times [0,\varepsilon)$ and $Q_1^q \times (1-\varepsilon,1]$ are small collar neighbourhoods of ∂X^{q+1} with the identifications $Q_0^q = Q_0^q \times \{0\}$, $Q_1^q = Q_1^q \times \{1\}$, and $id_A(a) = a$ $(a \in A)$ for an arbitrary set A.

We call the map F a *cobordism* between f_0 and f_1.

This clearly defines an equivalence relation on the set $FOLD(q,n)$ $(FOLD^O(q,n))$.

DEFINITION 4.2. We can define a group operation on the set of the cobordism classes as follows. If $[f_0\colon Q_0^q \to \mathbb{R}^n]$ and $[f_1\colon Q_1^q \to \mathbb{R}^n]$ are cobordism classes of fold maps, then the *sum* of $[f_0]$ and $[f_1]$ is the cobordism class represented by the fold map $f_0 \amalg f_1\colon Q_0^q \amalg Q_1^q \to \mathbb{R}^n$. It is easy to show that the cobordism class of the fold map $f_0 \amalg f_1$ does not depend on the choice of the representatives f_0 and f_1.

This group operation is clearly commutative, so we have an abelian group, which we denote by $\mathcal{C}ob_f(q, n-q)$ $(\mathcal{C}ob_f^O(q, n-q))$ where the lower index "f" comes from "fold map" and the upper index "O" comes from "oriented".

REMARK 4.3. The integer $n-q$ is called the *codimension* of the map f. There was a result of Ikegami and Saeki [**4**] that $\mathcal{C}ob_f^O(2,-1) = \mathbb{Z}$. In Chapter 1 we proved that $\mathcal{C}ob_f(2,-1) = \mathbb{Z} \oplus \mathbb{Z}_2$ and in Chapter 2 poved that $\mathcal{C}ob_f^O(3,-1) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Ikegami computed the groups $\mathcal{C}ob_f(q, 1-q)$ and $\mathcal{C}ob_f^O(q, 1-q)$ (which are the cobordism groups of Morse functions) for arbitrary integers $q \geq 0$ [**3**].

For $k \geq 0$ there are many results concerning the cobordism groups $\mathcal{C}ob_\tau(q,k)$, where τ is a set of singularity types, the elements of the group $\mathcal{C}ob_\tau(q,k)$ are cobordism classes of smooth maps with only singularities in τ, and a cobordism between two such maps has only singularities in τ. See for example [**5, 10, 13, 14, 16, 17**].

The following notion is the analogue of *cobordism between abstract Reeb functions* introduced in [**4**].

DEFINITION 4.4. Let q and n be integers, $q \geq n \geq 0$. Let $g_i \colon W_i^n \to \mathbb{R}^n$, $i = 0, 1$, be abstract Stein factorizations in $ASTEIN(q,n)$ in the sense of Definition 2.3. We say that g_0 and g_1 are *cobordant* if there exists a continuous map $G \colon P^{n+1} \to \mathbb{R}^n \times [0,1]$ of a compact $n+1$-dimensional polyhedron P^{n+1} such that

(i) $W_i^n = G^{-1}(\mathbb{R}^n \times \{i\})$, $i = 0, 1$, are subcomplexes of P^{n+1} with regular neighbourhoods of the forms $W_0^n \times [0, \varepsilon)$ and $W_1^n \times (1-\varepsilon, 1]$, where W_i^n corresponds to $W_i^n \times \{i\}$, $i = 0, 1$,
(ii) $G\mid_{W_0^n \times [0,\varepsilon)} = g_0 \times id_{[0,\varepsilon)}$ and $G\mid_{W_1^n \times (1-\varepsilon,1]} = g_1 \times id_{(1-\varepsilon,1]}$,
(iii) in a neighbourhood of each point of $P^{n+1} \setminus (W_0^n \cup W_1^n)$, the polyhedron P^{n+1} and the map G are equivalent to one of the local forms of Stein factorizations of fold maps from $q+1$-manifolds into $\mathbb{R}^{n+1}$.

Furthermore, we call the map $G \colon P^{n+1} \to \mathbb{R}^n \times [0,1]$ a *cobordism* between g_0 and g_1.

This clearly defines an equivalence relation on the set of abstract Stein factorizations $ASTEIN(q,n)$.

DEFINITION 4.5. We can define a group operation on the set of the cobordism classes of abstract Stein factorizations $ASTEIN(q,n)$ as follows. If $[g_0 \colon W_0^n \to \mathbb{R}^n]$ and $[g_1 \colon W_1^n \to \mathbb{R}^n]$ are cobordism classes of abstract Stein factorizations in $ASTEIN(q,n)$, then the *sum* of $[g_0]$ and $[g_1]$ is the cobordism class represented by the abstract Stein factorization $g_0 \amalg g_1 \colon W_0^n \amalg W_1^n \to \mathbb{R}^n$. It is easy to show that the cobordism class of the abstract Stein factorization $g_0 \amalg g_1$ does not depend on the choice of the representatives g_0 and g_1.

Let us denote the cobordism group of abstract Stein factorizations $ASTEIN(q,n)$ by $\mathcal{C}ob^S(q, n-q)$ where the upper index "S" comes from "Stein factorization".

In this chapter we are interested in the case when the codimension $n-q$ is equal to -1. So let $\mathcal{C}ob_f(q)$, $\mathcal{C}ob_f^O(q)$, $\mathcal{C}ob^S(q)$ denote the -1-codimensional case $\mathcal{C}ob_f(q,-1)$, $\mathcal{C}ob_f^O(q,-1)$, $\mathcal{C}ob^S(q,-1)$ respectively.

Next we prove a lemma which says that in the group $\mathcal{C}ob_f^{(O)}(q)$ we have to care only about the maps which have singular fibers without circle components. We shall call a map in $FOLD_q$ *punctured* if its image is contained in a small tubular neighbourhood of the image of its indefinite singular set. It is defined analogously a *punctured cobordism* of two punctured fold maps.

LEMMA 4.6. *Let f be a map in $FOLD_q$. Then there exists a map f' in $FOLD_q$ such that*

(i) the fold map f' is cobordant to the fold map f,
(ii) the map f' is punctured,

(iii) the restriction of f to a neighbourhood of its indefinite singular set $S_1(f)$ is equal to the the restriction of f' to a neighbourhood of its indefinite singular set $S_1(f')$.

PROOF. As desrcibed in Chapter 2 by the method of making holes we have such a map f'. □

By Lemma 4.6 we have that the cobordism group of punctured maps in $FOLD_q^{(O)}$ under punctured cobordisms is isomorphic to the cobordism group $\mathcal{C}ob_f^{(O)}(q)$. Hence $\mathcal{C}ob_f^{(O)}(q)$ can be identified with the cobordism classes of punctured fold maps (under punctured cobordisms).

REMARK 4.7. Let f be a punctured map in $FOLD_q$. If we cut the neighbourhood of its definite singular points, we get a fold map from a manifold with boundary, and the boundary is an S^1-bundle over the image of the boundary. Conversely, if we have a fold map with codimension -1 from a manifold with boundary (without singularities on the boundary), then the boundary is an S^1-bundle over its image. Then we can glue to the boundary a fold map from a D^2-bundle, thus we get a fold map without boundary. So we can consider the group $\mathcal{C}ob_f^{(O)}(q)$ as a cobordism group of punctured fold maps *without* definite singular points, i.e., fold maps without definite singular points from manifolds with boundary whose images are contained in small tubular neighbourhoods of the images of their indefinite singular sets.

Cobordism with prescribed singularities. In this section we introduce some notion which are analogues of "τ-maps" of the positive codimensional case [**10**] and we give a more sophisticated definition of cobordism.

DEFINITION 4.8. Let $\sigma_{\mathfrak{F}}$ be a canonical singularity and let $\mathcal{G}$ be a subgroup of $Aut(\sigma_{\mathfrak{F}})$. We call the pair $(\sigma_{\mathfrak{F}}, \mathcal{G})$ a *global singularity.*

DEFINITION 4.9. Let f be an element of $FOLD_q$. By the stratification of the neighbourhood of the image of f (see Reamark 2.6) the neighbourhoods of the strata and their pre-images together can be considered as bundles of canonical singularities. Let τ_f be a set of the global singularities $(\sigma_{\mathfrak{F}}, \mathcal{G}_{\mathfrak{F}})$ where each $\sigma_{\mathfrak{F}}$ is a singularity of f and the corresponding $\mathcal{G}_{\mathfrak{F}}$ is the structure group of the bundle of this singularity. We call the set τ_f the *global singularity set of f.*

DEFINITION 4.10. Let τ_0 and τ_1 be two sets of global singularities. $\tau_0 \prec \tau_1$ if and only if for every $(\sigma_{\mathfrak{F}}, \mathcal{G}_{\mathfrak{F}}) \in \tau_0$ there exists an element $(\sigma'_{\mathfrak{F}}, \mathcal{G}'_{\mathfrak{F}}) \in \tau_1$ such that $\sigma_{\mathfrak{F}} = \sigma'_{\mathfrak{F}}$ and $\mathcal{G}_{\mathfrak{F}}$ is a subgroup of $\mathcal{G}'_{\mathfrak{F}}$.

DEFINITION 4.11. Let τ be a set of global singularities. Let f be an element of $FOLD_q$. We say that f is a *τ-map* if the global singularity set $\tau_f \prec \tau$.

DEFINITION 4.12. Let τ and τ' be two sets of global singularities. Two τ-maps f_0 and f_1 in $FOLD_q^{(O)}$ on closed (oriented) q-dimensional manifolds Q_0^q and Q_1^q are *(oriented) τ'-cobordant*, if there exists a τ'-map $F\colon X^{q+1} \to \mathbb{R}^n \times [0,1]$ from a compact (oriented) $q+1$-dimensional manifold X^{q+1} with $\partial X^{q+1} = Q_0^q \amalg (-)Q_1^q$ such that $F\mid_{Q_0^q \times [0,\varepsilon)} = f_0 \times id_{[0,\varepsilon)}$ and $F\mid_{Q_1^q \times (1-\varepsilon,1]} = f_1 \times id_{(1-\varepsilon,1]}$, where $Q_0^q \times [0,\varepsilon)$ and $Q_1^q \times (1-\varepsilon,1]$ are small collar neighbourhoods of ∂X^{q+1} with the identifications $Q_0^q = Q_0^q \times \{0\}$, $Q_1^q = Q_1^q \times \{1\}$, and $id_A(a) = a$ $(a \in A)$ for an arbitrary set A.

We call the map F an *(oriented) τ'-cobordism* between f_0 and f_1.

For two global singularity sets τ and τ' let us denote the cobordism group of τ-maps under τ'-cobordisms by $\mathcal{C}ob_{(\tau,\tau')}^{(O)}(q)$. For $\tau = \tau'$ let us $\mathcal{C}ob_{(\tau,\tau)}^{(O)}(q)$ denote by $\mathcal{C}ob_{\tau}^{(O)}(q)$.

REMARK 4.13. Note that the group $\mathcal{C}ob_{(\tau,\tau')}(q)$ is equal to the group $\mathcal{C}ob_{(\tau,\tau')}^{O}(q)$, if the automorphisms of the global singularities $(\sigma_{\mathfrak{F}}, \mathcal{G})$ are elements in $Aut^O(\sigma_{\mathfrak{F}})$.

We have the analogous definitions for *global abstract Stein factorization singularities*, *τ-abstract Stein factorizations* and their *τ'-cobordisms* using the canonical Stein factorization singularities (see Definition 2.4) and their automorphisms. Let us denote the cobordism group of τ-abstract Stein factorizations by $\mathcal{C}ob_{\tau}^{S}(q)$.

REMARK 4.14. As a consequence of Lemma 4.6 and Remark 4.7 we can suppose that the cobordism group $\mathcal{C}ob_{(\tau,\tau')}^{(O)}(q)$ contains only punctured maps with boundary without definite singular points as representatives of cobordism classes and the cobordisms are similar punctured cobordisms as well. In other words as far as we are considering cobordisms of fold maps with codimension -1 we have to care only about the indefinite singularities with singular fibers without circle components.

DEFINITION 4.15. Let τ_1 denote the set of the global singularity $(\sigma_{\mathrm{I}^1}, Aut(\sigma_{\mathrm{I}^1}))$ and its multisingularities. Let τ_2 denote the set of global singularities in τ_1, $(\sigma_{\mathrm{II}^l}, Aut(\sigma_{\mathrm{II}^l}))$ and their multisingularities. Let τ_3 denote the set of global singularities in τ_2, $(\sigma_{\mathrm{III}^l}, Aut(\sigma_{\mathrm{III}^l}))$ and their multisingularities, and so on, we define the set of global singularities τ_n for all integer n. Let $\mathcal{C}ob_n^{(O)}(q)$ denote the group $\mathcal{C}ob_{\tau_n}^{(O)}(q)$.

Note that $\mathcal{C}ob_n^{(O)}(n) = \mathcal{C}ob_f^{(O)}(n)$ and $\mathcal{C}ob_n^{(O)}(q)$ is a cobordism group where the maps and cobordisms can have types of singularities only which are types of singularities of maps in $FOLD^O(n+1)$.

If τ is the set whose elements are the global singularity $(\sigma_{\mathrm{I}^1}, \{\mathrm{id}\})$ (i.e., the automorphism group of the global singularity is equal to the group consisting only of the identical automorphism) and its multisingularities, then $\mathcal{C}ob_\tau(n)$ is equal to the group $\mathcal{I}mm^{fr}(n-2,1)$ which is the group $\pi^S(n-2)$.

5. The classifying space for τ-maps

Given a group G let $EG \to BG$ denote the universal G-budle. For a global singularity $(\sigma_{\mathfrak{F}}, \mathcal{G})$ we construct a bundle $\xi_{\mathfrak{F}}^{\mathcal{G}} \to B_{\mathfrak{F}}^{\mathcal{G}}$ as follows. Let $\xi_{\mathfrak{F}}^{\mathcal{G}} \to B_{\mathfrak{F}}^{\mathcal{G}}$ be the bundle

$$E\mathcal{G} \times_{\mathcal{G}} (\sigma_{\mathfrak{F}}\colon\ s_{\mathfrak{F}} \to J^k) \to B\mathcal{G},$$

where we can write the total space $\xi_{\mathfrak{F}}^{\mathcal{G}}$ in an other form

$$E\mathcal{G} \times_{\mathcal{G}} s_{\mathfrak{F}} \to E\mathcal{G} \times_{\mathcal{G}} J^k. \tag{5.1}$$

Let us define the boundary $\partial \xi_{\mathfrak{F}}^{\mathcal{G}}$ of the total space $\xi_{\mathfrak{F}}^{\mathcal{G}}$ as the subset

$$E\mathcal{G} \times_{\mathcal{G}} (\partial\sigma_{\mathfrak{F}}\colon\ \partial s_{\mathfrak{F}} \to \partial J^k)$$

of $\xi_{\mathfrak{F}}^{\mathcal{G}}$. This is a fold map, the singularities of which have codimension less than $\kappa(\mathfrak{F})$.

We can define a partial ordering on the set of global singularities.

Definition 5.1. We say that the global singularity $\mathfrak{a} = (\sigma_{\mathfrak{F}_0}, \mathcal{G}_0)$ is *smaller* than the global singularity $\mathfrak{b} = (\sigma_{\mathfrak{F}_1}, \mathcal{G}_1)$ $(\mathfrak{a} \prec \mathfrak{b})$ if $\kappa(\mathfrak{F}_0) < \kappa(\mathfrak{F}_1)$, $\partial\xi_{\mathfrak{F}_1}^{\mathcal{G}_1}$ has a singularity of type $\sigma_{\mathfrak{F}_0}$ and the structure group of the singularity $\sigma_{\mathfrak{F}_0}$ in $\partial\xi_{\mathfrak{F}_1}^{\mathcal{G}_1}$ is a subgroup of the structure group $\mathcal{G}_0$.

Note that for any group $\mathcal{G}_1$ the relation $(\sigma_{\mathfrak{F}_0}, Aut(\sigma_{\mathfrak{F}_0})) \prec (\sigma_{\mathfrak{F}_1}, \mathcal{G}_1)$ holds if and only if the boundary $\partial\sigma_{\mathfrak{F}_1}$ of the canonical singularity $\sigma_{\mathfrak{F}_1}$ has a singularity of type $\sigma_{\mathfrak{F}_0}$.

Definition 5.2. We say that a global singularity set τ is *closed* if for any global singularity $(\sigma_{\mathfrak{F}}, \mathcal{G})$ in τ for every singularity of type $\sigma'_{\mathfrak{F}}$ of the map $\partial\xi_{\mathfrak{F}}^{\mathcal{G}}$ there exists a global singularity $(\sigma'_{\mathfrak{F}}, \mathcal{G}')$ in τ such that $(\sigma'_{\mathfrak{F}}, \mathcal{G}') \prec (\sigma_{\mathfrak{F}}, \mathcal{G})$.

Note that for a map $f \in FOLD_q$ the global singularity set τ_f is closed.

Conversely, for a global singularity set τ which is closed we can construct by induction a map ξ_τ which is a τ-map in the sense that τ_ξ is equal to τ as follows.

The following construction of the map ξ_τ is very similar to the constructions of the classifying spaces of Szűcs [**14, 17**] and the universal map of Rimányi and Szűcs [**10**]. We suggest to the reader to study and understand the construction in [**10**] before reading the following definition. In our case the transversality and the uniqueness up to homotopy are included without mentioning.

Definition 5.3. Let τ be a closed global singularity set. For a global singularity $(\sigma_{\mathfrak{F}}, \mathcal{G})$ in τ with $\kappa(\mathfrak{F}) = 1$, the map $\partial\xi_{\mathfrak{F}}^{\mathcal{G}}$ has no singularities. Let ξ_τ^1 denote the map $D(\xi_{\mathfrak{F}}^{\mathcal{G}})/S(\xi_{\mathfrak{F}}^{\mathcal{G}})$.

Suppose that we have constructed the map ξ_τ^k. Then for every global singularity $(\sigma_{\mathfrak{F}}, \mathcal{G})$ in τ with $\kappa(\mathfrak{F}) = k+1$, the map $\partial\xi_{\mathfrak{F}}^{\mathcal{G}}$, after we have punctured it, has only global singularities in $\tau_{\xi_\tau^k}$ because τ is closed. From the appropriate stratification of $\partial\xi_{\mathfrak{F}}^{\mathcal{G}}$ we have a glueing map $\varphi_{\mathfrak{F}}^{\mathcal{G}}\colon \partial\xi_{\mathfrak{F}}^{\mathcal{G}} \to \xi_\tau^k$ which restricted to the parts of the stratification is a bundle-map into the total spaces $\xi_{\mathfrak{F}'}^{\mathcal{G}'} \subseteq \xi_\tau^k$ for every global singularities $(\sigma_{\mathfrak{F}'}, \mathcal{G}') \prec (\sigma_{\mathfrak{F}}, \mathcal{G})$. Now we have $\xi_\tau^{k+1} = \xi_\tau^k \cup_{\varphi_{\mathfrak{F}}^{\mathcal{G}}} \xi_{\mathfrak{F}}^{\mathcal{G}}$ where $(\sigma_{\mathfrak{F}}, \mathcal{G})$ runs over the set of global singularities in τ with $\kappa(\mathfrak{F}) = k+1$.

Let ξ_τ denote the limit $\xi_\tau^1 \subseteq \xi_\tau^2 \subseteq \ldots \subseteq \xi_\tau^k \subseteq \ldots$.

The map ξ_τ defined in Definition 5.3 has a target space Γ_τ glued from the target spaces

$$\Gamma_{\mathfrak{F}}^{\mathcal{G}} := E\mathcal{G} \times_{\mathcal{G}} J^k$$

of $\xi_{\mathfrak{F}}^{\mathcal{G}}$, see (5.1).

THEOREM 5.4. *The space Γ_τ is the classifying space of τ-maps. The homotopy group $\pi_q(\Gamma_\tau)$ is equal to the cobordism group $\mathcal{C}ob_\tau(q+1)$.*

PROOF. The homopoty groups of Γ_τ provide us the corresponding cobordism groups according to the (generalized) Pontrjagin-Thom construction as follows.

Let f be a τ-map in $FOLD_{q+1}$. By Remark 4.14 we can suppose that f is a punctured map without definite singularities (and with boundary). Then we obtain a map $\mathfrak{S}\colon S^q \to \Gamma_\tau$ as follows. We note that the total space of a $\sigma_{\mathfrak{F}}\colon s_{\mathfrak{F}} \to J^{\kappa(\mathfrak{F})}$-bundle is a $s_{\mathfrak{F}}$-bundle. Let $U_{\mathfrak{F}(f)}$ denote the tubular neighbourhood of the f-image of the singularities of type $\sigma_{\mathfrak{F}}$ in $\mathbb{R}^q$. By a map $\mathfrak{S}_{\mathfrak{F}}\colon U_{\mathfrak{F}(f)} \to \Gamma_{\mathfrak{F}}^{\mathcal{G}}$ we induce the total space of the $\sigma_{\mathfrak{F}}$-bundle (which is an $s_{\mathfrak{F}}$-bundle) of f from the block $\Gamma_{\mathfrak{F}}^{\mathcal{G}}$ of Γ_τ for every global singularity $(\sigma_{\mathfrak{F}}, \mathcal{G}')$ in τ_f. Then we glue these maps $\mathfrak{S}_{\mathfrak{F}}\colon U_{\mathfrak{F}(f)} \to \Gamma_\tau$ to a map $\mathfrak{S}\colon S^q \to \Gamma_\tau$ by the generalized Pontrjagin-Thom construction.

Conversely, let $\mathfrak{S}\colon S^q \to \Gamma_\tau$ be an element of $\pi_q(\Gamma_\tau)$. Then we can glue a punctured fold map of a $q+1$-dimensional manifold with boundary without definite singularities from the total spaces of the $\sigma_{\mathfrak{F}}$-bundles $\mathfrak{S}^*(\xi_{\mathfrak{F}}^{\mathcal{G}}) \to \mathfrak{S}^{-1}(\textit{zero section of } \Gamma_{\mathfrak{F}}^{\mathcal{G}})$.

This can be applied for homotopies and cobordisms as well. □

REMARK 5.5. Clearly we can repeat the above methods and constructions for Stein factorization singularities. For a global abstract Stein factorization singularity set τ let Γ_τ^S denote the classifying space of τ-abstract Stein factorizations.

From a more abstract viewpoint we can repeat the above methods and constructions for the canonical Stein factorization diagrams

$$\begin{tikzcd} s_{\mathfrak{F}} \arrow[rr, "\sigma_{\mathfrak{F}}"] \arrow[dr] & & J^{\kappa(\mathfrak{F})} \\ & w_{\mathfrak{F}} \arrow[ur, "\bar{\sigma}_{\mathfrak{F}}"'] & \end{tikzcd}$$

as fibers instead of canonical singularities.

Let τ be a set of global singularities and let $\bar{\tau}$ be the corresponding set of global abstract Stein factorization singularities where the global abstract Stein factorization singularity $(w_{\mathfrak{F}}, \mathcal{G}) \in \bar{\tau}$ is the Stein factorization of the global singularity $(\sigma_{\mathfrak{F}}, \mathcal{G}) \in \tau$.

Let $\Upsilon\colon FOLD_q^{(O)} \to ASTEIN_q$ be the natural map that maps a fold map $f \in FOLD_q^{(O)}$ to its Stein factorization. Then Υ induces a homomorphism $\upsilon\colon \mathcal{C}ob_\tau^{(O)}(q) \to \mathcal{C}ob_{\bar{\tau}}^S(q)$.

Proposition 5.6. *The homomorphism υ is an isomorphism.*

Proof. The classifying space Γ_τ and the classifying space Γ_τ^S are the same because the corresponding structure groups of the global singularities in τ and $\bar{\tau}$ are the same. $\square$

Corollary 5.7. *When we want to compute a cobordism group of fold maps, it is sufficient to work with abstract Stein factorizations where the structure group of an abstract Stein factorization singularity $\bar{\sigma}_{\mathfrak{F}}$ is the same as that of the singularity $\sigma_{\mathfrak{F}}$.*

In other words when we want to compute the homotopy groups of Γ_τ, it is sufficient to work with the space Γ_τ^S and abstract Stein factorizations equipped with the appropriate structure groups.

6. Computation of cobordism groups

For a global singularity $\mathfrak{a} = (\sigma_{\mathfrak{F}}, \mathcal{G})$ let $\mathcal{I}mm^{\mathcal{G}}(q - \kappa(\mathfrak{F}), \kappa(\mathfrak{F}))$ or $\mathcal{C}ob_{\mathfrak{a}}(q+1)$ denote the cobordism group of immersions of $q - \kappa(\mathfrak{F})$-dimensional manifolds in $\mathbb{R}^q$ with normal structure induced from the bundle $\Gamma_{\mathfrak{F}}^{\mathcal{G}}$. Note that the action of the automorphism group $\mathcal{G}$ may be not effective on $J^{\kappa(\mathfrak{F})}$. This group is equal to the cobordism group of -1-codimensional fold maps from $q+1$-dimensional manifolds with boundary where the image is contained in a small tubular neighbourhood of the singularities of type $\sigma_{\mathfrak{F}}$ and its multisingularities. This gives reason for the notation $\mathcal{C}ob_{\mathfrak{a}}(q+1)$.

According to the Pontrjagin-Thom type construction for maps (see [**10**], Definition 5.3 and Theorem 5.4) a map into a closed manifold P^q is nullcobordant if and only if the corresponding map from P^q (in Theorem 5.4 the manifold P^q was the sphere S^q) is homotopic to a constant map. This can be applied in many ways related to the fold map $\partial \xi_{\mathfrak{F}}^{\mathcal{G}}$ and the corresponding (glueing) map $\varphi_{\mathfrak{F}}^{\mathcal{G}}|_{\partial \Gamma_{\mathfrak{F}}^{\mathcal{G}}}$. An option is to apply it for some suspensions of the map $\partial \xi_{\mathfrak{F}}^{\mathcal{G}}$ where $\mathfrak{F}$ is a multisingularity. This is a lemma of Szűcs. The following lemma is another option with a viewpoint analogous to Szűcs's.

LEMMA 6.1. *Let τ be a closed set of global singularities which contains all the multisingularities of its global singularities. Let $\mathfrak{a} = (\sigma_{\mathfrak{F}}\colon\ s_{\mathfrak{F}} \to J^k, \mathcal{G})$ be a global singularity which is not an element of τ and which is not a multisingularity. Let τ' be the set of global singularities consisting of*

(1) $\mathfrak{a}$
(2) all the global singularities in τ
(3) all the multisingularities formed from the global singularities in (1) and (2).

Note that the group $\pi_q(\Gamma_{\tau'}/\Gamma_\tau)$ is equal to the group $\mathcal{C}ob_{\mathfrak{a}}(q+1)$.

Suppose that we can modify the map $\sigma_{\mathfrak{F}}$ to a new map $\sigma'_{\mathfrak{F}}$ such that

(i) this modification is invariant under the structure group $\mathcal{G}$,
(ii) the map $\partial\sigma_{\mathfrak{F}}$ is equal to the map $\partial\sigma'_{\mathfrak{F}}$,
(iii) the singularities of $\sigma'_{\mathfrak{F}}$ are the same as the singularities of $\partial\sigma_{\mathfrak{F}}$,
(iv) the image of the new map $\sigma'_{\mathfrak{F}}$ is contained in the complement of a neighbourhood of the origin in J^k.

Then the cobordism group $\mathcal{C}ob_{\tau'}(q+1)$ is isomorphic to the group $\mathcal{C}ob_{\mathfrak{a}}(q+1) \oplus \mathcal{C}ob_\tau(q+1)$, or in other words the cobordism group $\mathcal{C}ob_{\tau'}(q+1)$ is isomorphic to the group $\mathcal{I}mm^{\mathcal{G}}(q-k,k) \oplus \mathcal{C}ob_\tau(q+1)$.

PROOF. The modification of $\sigma_{\mathfrak{F}}$ shows that the glueing map $\varphi^{\mathcal{G}}_{\mathfrak{F}}\colon\ \partial\xi^{\mathcal{G}}_{\mathfrak{F}} \to \xi_\tau$ restricted to the space $\partial\Gamma^{\mathcal{G}}_{\mathfrak{F}}$ is homotopic to a constant map in a natural way such that this homotopy extends to the blocks of multisingularities. □

Let $\widetilde{\mathcal{I}mm^{fr}}(q-1,1)$ denote the cobordism group of immersions of pairs $V^{q-2} \subset U^{q-1}$ where U^{q-1} is a closed framed $q-1$-dimensional manifold immersed into $\mathbb{R}^q$ and V^{q-2} is a (probably non-oriented) $q-2$-dimensional manifold embedded into U^{q-1}.

PROPOSITION 6.2. *The cobordism group $\widetilde{\mathcal{I}mm^{fr}}(q-1,1)$ is isomorphic to the cobordism group $\mathcal{C}ob^O_1(q+1)$.*

PROOF. For a fold map $f\colon\ Q^{q+1} \to \mathbb{R}^q$ in $FOLD^O_{q+1}$ with only singularities of type σ_{I^1} and its multisingularities the automorphism group $Aut(\sigma_{\mathrm{I}^1}) = \mathbb{Z}_2$ can be considered as a structure group of a line-bundle over the indefinite singular set $\Sigma(f)$ of the map f. This line-bundle can be considered as a 1-codimensional submanifold of $\Sigma(f)$. So the cobordism group $\mathcal{C}ob^O_1(q+1)$ can be considered as the cobordism group $\widetilde{\mathcal{I}mm^{fr}}(q-1,1)$. □

Let $\pi^s(q-1)$ denote the $(q-1)$th stable homotopic group of spheres. Note that this group is equal to the cobordism group $\mathcal{I}mm^{fr}(q-1,1)$ which is the cobordism group of 1-codimensional immersions into $\mathbb{R}^q$ with one normal frame. Recall that the cobordism group $\mathcal{I}mm(q-2,1)$ of 1-codimensional immersions into $\mathbb{R}^{q-1}$ is equal to the cobordism group $\mathcal{I}mm^{\varepsilon^1\oplus\gamma^1}(q-2,2)$ of 2-codimensional immersions into $\mathbb{R}^q$ with a direct sum of one frame and an arbitrary line bundle as normal structure.

THEOREM 6.3. *$\mathcal{C}ob_1^O(q+1)$ (for the notation see Definition 4.15) is isomorphic to $\pi^s(q-1)\oplus\mathcal{I}mm(q-2,1)$.*

PROOF. By Proposition 6.2 we have to compute the group $\widetilde{\mathcal{I}mm^{fr}}(q-1,1)$. 1-codimensional submanifolds in immersed $q-1$-dimensional manifolds can be considered as a set of type of 2-codimensional "singularity" which has normal structure $\varepsilon^1\oplus\gamma^1$ and which is glued to the framed immersed $q-1$-dimensional manifold by the line-bundle corresponding to γ^1 as its normal bundle in the immersed $q-1$-dimensional manifold. We can apply Lemma 6.1 with τ equal to framed 1-codimensional immersions as singular sets and $(\sigma_{\mathfrak{F}},\mathcal{G})$ equal to the 2-codimensional singularity with $\varepsilon^1\oplus\gamma^1$ as normal structure. The group $\mathcal{I}mm^{\mathcal{G}}(q-\kappa(\mathfrak{F}),\kappa(\mathfrak{F}))$ is equal to the group $\mathcal{I}mm^{\varepsilon^1\oplus\gamma^1}(q-2,2)$. So by Lemma 6.1 the group $\widetilde{\mathcal{I}mm^{fr}}(q-1,1)$ is equal to $\mathcal{I}mm^{fr}(q-1,1)\oplus\mathrm{Imm}^{\varepsilon^1\oplus\gamma^1}(q-2,2)$ which is equal to the group $\pi^s(q-1)\oplus\mathcal{I}mm(q-2,1)$. $\square$

LEMMA 6.4. *Let $\mathfrak{F}$ be a fiber of type II^2 or II^3. Then the singularity $\sigma_{\mathfrak{F}}$ satisfies the conditions (i) – (iv) in Lemma 6.1.*

PROOF. By Fig. 4 and Fig. 5 we can modify the singularities σ_{II^2} and σ_{II^3}. $\square$

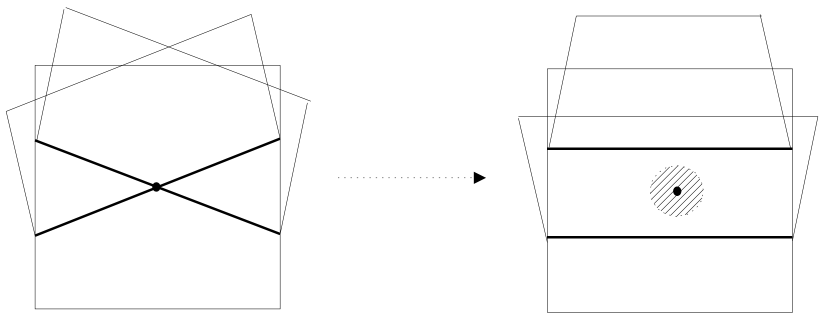

FIGURE 4. The modification of σ_{II^2}

THEOREM 6.5. *$\mathcal{C}ob_2^O(q+1)$ (for the notation see Definition 4.15) is isomorphic to $\pi^s(q-1)\oplus\mathcal{I}mm(q-2,1)^3\oplus\mathcal{I}mm^{\gamma^1\times\gamma^1}(q-3,2)$.*

PROOF. By Lemma 6.1 and Lemma 6.4 the cobordism group $\mathcal{C}ob_2^O(q+1)$ is equal to the group $\mathcal{C}ob_1^O(q+1)\oplus\mathcal{C}ob_{\mathfrak{a}}(q+1)\oplus\mathcal{C}ob_{\mathfrak{b}}(q+1)$ where the global singularity $\mathfrak{a}=(\sigma_{\mathrm{II}^2},Aut^O(\sigma_{\mathrm{II}^2}))$ and the global singularity $\mathfrak{b}=(\sigma_{\mathrm{II}^3},Aut^O(\sigma_{\mathrm{II}^3}))$.

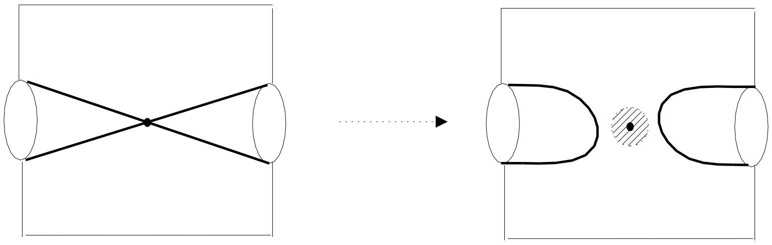

FIGURE 5. The modification of σ_{II^3}

The group $\mathcal{C}ob_{\mathfrak{a}}(q+1)$ is equal to $\mathcal{I}mm^{\varepsilon^1\oplus\gamma^1}(q-2,2)$ since $\kappa(\mathrm{II}^2)=2$ and $Aut^O(\sigma_{\mathrm{II}^2})=\mathbb{Z}_2$ acts on J^2 as the structure group $\varepsilon^1\oplus\gamma^1$.

A computation similar to the one in the proof of Theorem 6.3 shows that the group $\mathcal{C}ob_{\mathfrak{b}}(q+1)$ is equal to $\mathcal{I}mm^{\varepsilon^1\oplus\gamma^1}(q-2,2)\oplus\mathcal{I}mm^{\varepsilon^1\oplus\gamma^1\times\gamma^1}(q-3,3)$ since $Aut^O(\sigma_{\mathrm{II}^3})=\mathbb{Z}_2\oplus\mathbb{Z}_2$ acts on J^2 as the structure group $\varepsilon^1\oplus\gamma^1$ and on s_{II^3} as an involution which is the identity on J^2. □

THEOREM 6.6. *$\mathcal{C}ob_3^O(4)$ (for the notation see Definition 4.15) is isomorphic to $\mathbb{Z}_2^l\oplus\mathbb{Z}^3$ where $1\le l\le 3$.*

PROOF. By Theorem 6.5 the group $\mathcal{C}ob_2^O(4)$ is equal to the group $\pi^s(2)\oplus\mathcal{I}mm(1,1)^3\oplus\mathcal{I}mm^{\gamma^1\oplus\gamma^1}(0,2)=\mathbb{Z}_2^5$.

Recall that by the Pontrjagin-Thom construction the group $\mathcal{C}ob_2^O(4)$ is isomorphic to the homotopy group $\pi_3(\Gamma_{\tau_2})$ and the group $\mathcal{C}ob_3^O(4)$ is isomorphic to the group $\pi_3(\Gamma_{\tau_3})$. We obtain the space Γ_{τ_3} by glueing the appropriate blocks to the space Γ_{τ_2}. Let us denote by τ_2' the set of global singularities of τ_2, $(\sigma_{\mathrm{III}^4},Aut^O(\sigma_{\mathrm{III}^4}))$, $(\sigma_{\mathrm{III}^6},Aut^O(\sigma_{\mathrm{III}^6}))$ and their multisingularities.

Now let us examine the homotopical sequence

$$\pi_3(\Gamma_{\tau_2})\xrightarrow{\delta}\pi_3(\Gamma_{\tau_2'})\xrightarrow{\beta}\pi_3(\Gamma_{\tau_2'},\Gamma_{\tau_2})\xrightarrow{\alpha}\pi_2(\Gamma_{\tau_2})\xrightarrow{\varphi}\pi_2(\Gamma_{\tau_2'})$$

We know the followings. By Chapter 2 $\pi_2(\Gamma_{\tau_2})=\mathbb{Z}_2^4$, $\pi_2(\Gamma_{\tau_2'})=\mathbb{Z}_2^2$ and φ is an epimorphism, so $\operatorname{im}\alpha=\mathbb{Z}_2^2$. The group $\pi_3(\Gamma_{\tau_2'},\Gamma_{\tau_2})=\pi_3(\Gamma_{\tau_2'}/\Gamma_{\tau_2})=\pi_3(T\Gamma_{\mathrm{III}^4}^{Aut^O(\sigma_{\mathrm{III}^4})})\oplus\pi_3(T\Gamma_{\mathrm{III}^6}^{Aut^O(\sigma_{\mathrm{III}^6})})=\mathbb{Z}_2^2$ because the singularities σ_{III^4} and σ_{III^6} both have automorphisms which are orientation reversing on J^3. Hence α is a monomorphism, so $\beta=0$ and δ is an epimorphism. Therefore the group $\pi_3(\Gamma_{\tau_2'})$ is a factor of the group $\pi_3(\Gamma_{\tau_2}) = \mathbb{Z}_2^5$. In Chapter 2 we saw that factorizing out the group $\pi_2(\Gamma_{\tau_2}) = \mathbb{Z}_2^4$ by the homotopy classes of the attaching maps of the two singularities σ_{III^4} and σ_{III^6} we obtain the group $\pi_2(\Gamma_{\tau_2'}) = \mathbb{Z}_2^2$ because two independent relations come from the boundaries of these two singularities for the elements in $\pi_2(\Gamma_{\tau_2})$. We can check that these two independent relations are satisfied in the case of the group $\pi_3(\Gamma_{\tau_2'})$ as well, i. e. there is a homomorphism $h_\circ \colon \pi_2(\Gamma_{\tau_2'}) \to \pi_3(\Gamma_{\tau_2'})$ where $h_\circ$ is the composition by the Hopf-map $h \colon S^3 \to S^2$. Hence $\pi_3(\Gamma_{\tau_2'}) = \mathbb{Z}_2^l$ where $1 \leq l \leq 3$. The inequality $1 \leq l$ comes from the following.

First note that there exists a natural map $r \colon \mathcal{C}ob_4^O(4) \to \mathcal{I}mm(2,1)$ which maps the image of the singular set of a representative of an element in $\mathcal{C}ob_4^O(4)$ to its cobordism class in the group $\mathcal{I}mm(2,1) = \mathbb{Z}_8$. The image of the map r restricted to the group $\mathcal{C}ob_1^O(4)$ is clearly equal to $\mathbb{Z}_2$ since the image of the singular set of a representative of an element in $\mathcal{C}ob_1^O(4)$ is an oriented immersed surface. But the r-image (without restricted the map r) of the singular set of a representative of an element in $\mathcal{C}ob_1^O(4)$ cannot be equal to 0 since the group $\mathcal{I}mm^{SO}(2,1) = \mathbb{Z}_2$ is a subgroup of the group $\mathcal{I}mm(2,1) = \mathbb{Z}_8$.

Now we can obtain the space Γ_{τ_3} by glueing the appropriate blocks to the space $\Gamma_{\tau_2'}$. Namely these are the blocks of the global singularities $(\sigma_{\mathrm{III}^5}, Aut^O(\sigma_{\mathrm{III}^5}))$, $(\sigma_{\mathrm{III}^7}, Aut^O(\sigma_{\mathrm{III}^7}))$, $(\sigma_{\mathrm{III}^8}, Aut^O(\sigma_{\mathrm{III}^8}))$ and the corresponding multisingularities in order to get the space Γ_{τ_3}.

Similarly to the singularities σ_{II^2} and σ_{II^3} we can easily modify the singularities σ_{III^5} and σ_{III^8} since their automorphism groups are equal to $\mathbb{Z}_3$ which is represented by the permutations of the axes in J^3, see Figure 6 (the automorphism group of the singularity σ_{III^7} is equal to 0, so the corresponding glueing map is homotopic to a constant map as we can see in Chapter 2).

Since the group $\pi_3(\Gamma_{\mathfrak{F}}^{Aut^O(\sigma_{\mathfrak{F}})})$ is equal to $\mathbb{Z}$ ($\mathfrak{F} = \mathrm{III}^5, \mathrm{III}^7, \mathrm{III}^8$), by Lemma 6.1 we have that $\pi_3(\Gamma_{\tau_3}) = \pi_3(\Gamma_{\tau_2'}) \oplus \mathbb{Z}^3$. This completes the proof. □

REMARK 6.7. By checking the boundaries of the singularities $\sigma_{\mathfrak{F}}$ with $\kappa(\mathfrak{F}) = 4$ we see similarly to Chapter 2 that the group $\mathcal{C}ob_4^O(4)$ is a factor of the group $\mathcal{C}ob_3^O(4)$, and this group $\mathcal{C}ob_3^O(4)$ must be equal to $\mathbb{Z}_2^m \oplus \mathbb{Z}$ for some $1 \leq m \leq 3$. By [**12**] the component $\mathbb{Z}$ must be equal to the 4-dimensional oriented cobordism group of the source manifolds. For a fold map $f \colon M^4 \to \mathbb{R}^3 \in FOLD_4$ this shows a relation modulo 8 between the signature of the source manifold, the cobordism class of the f-image of the singular set (which is an element in $\mathbb{Z}_8$) and the $\mathbb{Z}_2$-components of the cobordism group $\mathcal{C}ob_f(4)$.

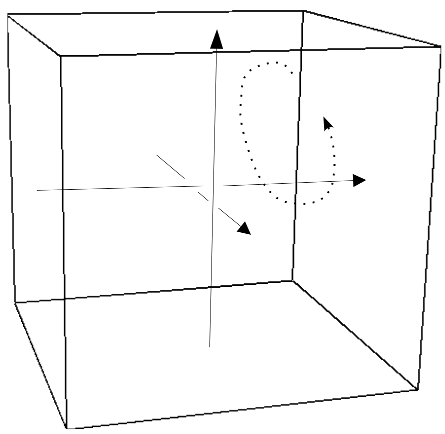

FIGURE 6. The action of $\mathbb{Z}_3$

Bibliography

[1] Y. Ando, Cobordism of maps without prescribed singularities, eprint arXiv:math/0412234.

[2] C. G. Gibson, K. Wirthmüller, A. A. du Plessis and E. J. N. Looijenga, Topological stability of smooth mappings, Lect. Notes in Math. **552**, Springer-Verlag, 1976.

[3] K. Ikegami, Cobordism group of Morse functions on manifolds, Hiroshima Math. J. **34** (2004), 211–230.

[4] K. Ikegami and O. Saeki, Cobordism group of Morse functions on surfaces, J. Math. Soc. Japan **55** (2003), 1081–1094.

[5] U. Koschorke, Vector fields and other vector bundle morphisms - a singularity approach, Lect. Notes in Math. **847**, Springer-Verlag, 1981.

[6] L. Kushner, H. Levine and P. Porto, Mapping three-manifolds into the plane I, Bol. Soc. Mat. Mexicana **29** (1984), 11–33.

[7] H. Levine, Classifying immersions into $\mathbb{R}^4$ over stable maps of 3-manifolds into $\mathbb{R}^2$, Lect. Notes in Math. **1157**, Springer-Verlag, 1985.

[8] W. Motta, P. Porto and O. Saeki, Stable maps of 3-manifolds into the plane and their quotient spaces, Proc. London Math. Soc. **71** (1995), 158–174.

[9] R. Rimányi, Thom polynomials, symmetries and incidences of singularities, Inv. Math. **143** (2001), 499–521.

[10] R. Rimányi and A. Szűcs, Generalized Pontrjagin-Thom construction for maps with singularities, Topology **37** (1998), 1177–1191.

[11] O. Saeki, Topology of singular fibers of differentiable maps, Lect. Notes in Math. **1854**, Springer-Verlag, 2004.

[12] O. Saeki and T. Yamamoto, Singular fibers of stable maps and signatures of 4-manifolds, preprint.

[13] A. Szűcs, Analogue of the Thom space for mappings with singularity of type Σ^1 (in Russian), Math. Sb. (N.S.) **108 (150)** (1979), 433–456, 478; English translation: Math. USSR-Sb. **36** (1980), 405–426.

[14] A. Szűcs, Cobordism groups of immersions with restricted self-intersection, Osaka J. Math. **21** (1984), 71–80.

[15] A. Szűcs, Elimination of singularities by cobordism, Contemporary Mathematics **354** (2004), 301–324.

[16] A. Szűcs, On the cobordism group of Morin maps, Acta Math. Hungar. **80** (1998), 191–209.

[17] A. Szűcs, Topology of $\Sigma^{1,1}$-singular maps, Math. Proc. Camb. Phil. Soc. **121** (1997), 465–477.

[18] R. Wells, Cobordism of immersions, Topology **5** (1966), 281–294.

[19] T. Yamamoto, Classification of singular fibres of stable maps from 4-manifolds to 3-manifolds and its applications, preprint.