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Orbit structures and incidence

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Abstract

Thom-polynomials can be approached variously. From a singularity theory viewpoint, they are characteristic classes representing preimages of certain singular subsets in vector bundles via generic sections. However, they are also equivariant Poincaré duals of invariant subsets of representations. A representation has the Incidence Property if it can be detected by Thom-polynomials, that whether an orbit is in the closure of another one or not. That is, $\mathrm{Tp}(\eta)|_\xi \neq 0$ if and only if $\xi \subset \overline{\eta}$ for every orbit η and ξ. Here we present a method for proving the Incidence Property and apply it to some geometrically interesting cases.

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Chapter 1

Introduction

Orbit structures of group actions, i. e. containment hierarchies of orbit closures, sometimes have important geometrical meanings. This usually happens, when geometrical objects are realized as orbit spaces of some actions. Widely investigated examples of this phenomenon are Schubert cells of Grassmanians or singularity classes of smooth maps. Here we present a method of determining orbit structures using Thom-polynomials.

First we state our main result anticipating a few definitions. We consider a representation $\rho : G \to \mathrm{GL}(V)$ of a complex Lie-group. We will introduce the following objects using ρ.

- a polynomial ring $\mathbb{Z}[x_1, \ldots, x_r]$, where r is the rank of G. This will be the equivariant cohomology ring of V.

- a polynomial ring $\mathbb{Z}[x_{\xi_1}, \ldots, x_{\xi_s}]$ for every orbit ξ, where s is the rank of the stabilizer subgroup of any point of ξ. This will be the equivariant cohomology ring of ξ.

- a restriction map $\mathbb{Z}[x_1, \ldots, x_r] \to \mathbb{Z}[x_{\xi_1}, \ldots, x_{\xi_s}]$, the image of which is denoted by $a|_\xi$ for every $a \in \mathbb{Z}[x_1, \ldots, x_r]$. This map will be the equivariant version of the usual restriction map between the cohomology rings of a space and a subspace.

- an element $\mathrm{Tp}(\eta)$ of $\mathbb{Z}[x_1, \ldots, x_r]$ for every orbit η. This element will be called the Thom-polynomial of η. Roughly speaking, it is the equivariant Poincaré dual of η.

So, $\mathrm{Tp}(\eta)|_\xi$ will be an element of $\mathbb{Z}[x_{\xi_1}, \ldots, x_{\xi_s}]$ for arbitrary orbits η and ξ. It is, by definition, the *incidence* class of η and ξ introduced in [Rim01]. It is zero by geometrical reasons, if $\overline{\eta} \not\supseteq \xi$. We prove, that with some additional assumptions this implication can be reversed.

Main Theorem *Consider a representation $\rho : G \to \mathrm{GL}(V)$, and two of its orbits η and ξ. If 0 is not in the convex hull of the weights of the induced representation $\rho_x : G_x \to \mathrm{GL}(N_x)$ of the stabilizer on the normal space of any point x of ξ, then*

$$\mathrm{Tp}(\eta)|_\xi \neq 0 \iff \overline{\eta} \supseteq \xi \tag{1.1}$$

Remark 1.1 ρ_x can be induced from $\rho|_{G_x}$, taking into account, that $\rho|_{G_x}$ maps $T_x\xi$ into itself and that $N_x \cong V/T_x\xi$ naturally.

Remark 1.2 The question, that whether (1.1) holds for any two orbits of a representation was originally asked in [Rim01]. More precisely, there it was asked for the special class of representations the orbits of which correspond to singularity classes of smooth maps. However, the question does make sense generally for any representation, the investigation of which is the main purpose of the article. If the answer is positive for a given representation, we say, that it has the Incidence Property.

Remark 1.3 The condition of Main Theorem turns out to be provable in some geometrically interesting cases. We present such results in Chapter 6. Particularly, we show in Section 6.3, that (1.1) holds for every two orbits of representations of Dynkin-quivers, examined in [FR02]. This result is a generalization of the formerly known result presented in Section 6.2. We apply also our results to special pairs of singularities in Section 6.4, thus giving a particular answer to the conjecture stated in [Rim01].

Remark 1.4 The proof of Main Theorem, apart from the definitions, consists of two main steps. First we prove in Chapter 3, that if $\overline{\eta} \supseteq \xi$, then $\mathrm{Tp}\eta|_\xi$ is equal to $\mathrm{Tp}(\eta \cap N_x) \in \mathbb{Z}[x_{\xi_1}, \ldots, x_{\xi_s}]$. That is, we reduce the computation of incidence to the computation of Thom-polynomials. Then we show, mainly in Chapter 5, that if the convex hull of the weights of the action of G_x on N_x does not contain 0, then there is no non-empty invariant subvariety ζ of N_x for which $\mathrm{Tp}(\zeta) = 0$. The combination of these two results yields the statement of our Main Theorem. The formal proof, which refers to previous chapters, can be found at the beginning of Chapter 6.

Thom-polynomials and their counterparts, which are usually named differently, are present in many areas of mathematics. They are also called equivariant Poincaré duals, equivariant cohomology classes, equivariant cycle classes or multidegree. In

Singularity Theory they are mainly viewed as characteristic classes representing preimages of certain singular subsets in vector bundles via generic sections. So as a consequence of the many existing definitions, a big part of the paper concerns with these different approaches and the connections between them.

Without trying to determine the paper where Thom-polynomials first appeared we mention some important articles. In Singularity Theory Thom initiated the program of examining singularities using Thom-polynomials [Tho56]. More recently their connection to group actions were clarified. This approach is present for example in [Kaz97], [FR04] or [Rim01]. In Algebraic Geometry they are introduced as equivariant cycle classes in equivariant Chow groups [EG98]. In Commutative Algebra their special cases are called multidegree (see [KM], [KMS] and [MS04])

The paper is organized as follows. In Chapter 2 we give a few equivalent definitions of Thom-polynomials. We do this, because the several areas where Thom-polynomials are used resulted many different definitions. We word the Incidence Property rigorously in Chapter 3, along with some technical statements concerning the notion of incidence. Then in Chapter 4 we clarify the connection between Thom-polynomials, and the algebraic invariant called multidegree, obtaining restrictions on how a Thom-polynomial can look like. Using these restrictions, in Chapter 5 we characterize the situation when Thom-polynomials are not zero, which by the results of Chapter 3 yields a sufficient condition on when the Incidence Property holds. Finally in Chapter 6 we fit together and apply our results.

Chapter 2

Thom-polynomials

In this chapter we define Thom-polynomials. We give a few definitions, and show the equivalence of the most of them. We do this, because although there are many definitions of Thom-polynomials presented in the literature, equivalences between them are not proven in many cases.

As a consequence, it is not necessary to read the whole chapter in order to understand the following ones. Every notion is defined in Section 2.2. Apart from that, the remarks of Section 2.5 are the most important. They are crucial for Chapter 3. We also use once in Chapter 4 the definition of Section 2.3, which is a variant of the one of Section 2.2. The other two sections will not be used later on. Section 2.1 is a short motivation of Thom-polynomials. There we present, why Singularity Theory is the main user of the notion. In Section 2.4 we show the definition of Kazarian. We do this only, because it is widely used in literature.

2.1 Singular point of view

In this section we present the Singularity Theory aspect of Thom-polynomials. We do not define and even less prove anything precisely here. However, one can prove the following statements as easy consequences of those of Section 2.2. We try to give here mostly a motivation of Thom-polynomials.

As we have already mentioned, from the singular point of view, Thom-polynomials are characteristic classes representing preimages of certain singular subsets in vector bundles, via generic sections. First we specify exactly what vector bundles and what singular subsets we examine. Consider, a representation $\rho : G \to \mathrm{GL}(V)$ of a complex Lie-group G on a complex vector space V. It determines a complex vector bundle $EG \times_\rho V$ over BG, where $EG \to BG$ is the universal G-bundle. Let η be a closed G-invariant subset of V, satisfying the technical assumptions of Definition 2.2.1. Roughly speaking these mean, that, modulo smaller dimensional parts, η is a pure dimensional, closed, G-invariant submanifold of V. For example, any algebraic subvariety of V and consequently the topological closure of any orbit satisfies these requirements. We define Thom-polynomials in this setting. That is, the Thom-polynomial $\mathrm{Tp}(\eta)$ of η will be the cohomology class of $H^*(BG)$ representing $\{x \in BG | s(x) \in EG \times_\rho \eta\}$ for a section s of $EG \times_\rho V$ transversal to $EG \times_\rho \eta$. Clearly, this will be also an obstruction for the existence of a section avoiding $EG \times_\rho \eta$.

If P is a smooth G-bundle over a smooth manifold M, then we can similarly construct $P \times_\rho V$ and $P \times_\rho \eta$, and also compute the cohomology class of $H^*(M)$ representing $\{x \in M | s(x) \in P \times_\rho \eta\}$ for a section s of $P \times_\rho V$ transversal to $P \times_\rho \eta$. Then this cohomology class will be the image of $\mathrm{Tp}(\eta)$ via $H^*(f)$, where $f : M \to BG$ is the classifying map of P.

For a more concrete and easy example, think of the case, when $G = \mathrm{GL}(n)$, ρ is the standard representation on $\mathbb{C}^n$ and $\eta = \{0\}$. Then $\{x \in M | s(x) \in P \times_\rho \eta\}$ is the 0-section of s, so the Thom-polynomial of η is the Euler-class of rank n.

2.2 Main definition

Here we give the precise definition of Thom-polynomials used later on in the article. As any characteristic classes, Thom-polynomials are elements of the cohomology ring $H^*(BG)$ of the classifying space of the underlying symmetry group. Instead of working with cohomology of BG, we can also work with equivariant cohomology of vector spaces, which is nothing but a technical issue. In this language, $\mathrm{Tp}(\eta)$ is an element of the G-equivariant cohomology ring $H^*_G(V) \cong H^*(BG)$ of V. This isomorphism holds, since for any G-space X the G- *equivariant cohomology* ring of X is $H^*_G(X) = H^*(EG \times_G X)$, by definition. We notice that, $H^*_G(.)$ is a contravariant functor of G-spaces and G-equivariant maps, thus G-equivariant maps induce ring homomorphisms between equivariant cohomology rings.

We define Thom-polynomials, through a technical argument. Our definition will coincide with the ones in [FR04], [FR03] and [FR02] and with the more algebro geometric one of [EG98] in case of complex algebraic varieties, but it will work on a larger class of subspaces. The reason for this wider definition is, that we will make at some point a localization, where we must step out of the class of complex varieties.

First, in our next definition we define the equivariantly stratifyable subsets of G-spaces, the largest class of subsets whose Thom-polynomials we define later.

From now on in this section M denotes a real smooth manifold and X denotes a subset of M. Sometimes, when indicated, it is also assumed, that there is a Lie-group G acting on M.

Definition 2.2.1 *A* stratification *of X is a set $\{X_i \subset M | i \in [k]\}$ of submanifolds of M for which*

- $\bigcup_{i=1}^{k} X_i = X$
- $\dim_{\mathbb{R}} X_i \leq \dim_{\mathbb{R}} X_k - 2$ *for every* $i \in [k-1]$
- $\bigcup_{i=1}^{j} X_i$ *is closed in* M*, for every* $j \in [k]$

X *is* stratifyable *if it admits a stratification. Then* $\dim X = \dim X_k$*, and* $\mathrm{codim} X = \mathrm{codim} X_k$.

If M is also a G-space, then a stratification $\{X_i \subset M | i \in [k]\}$ of X is G-invariant, if X_i is G-invariant, for every $i \in [k]$, and X is G-invariantly stratifyable*, if it has a G-invariant stratification.*

Remark 2.2.2

(i). Every closed submanifold is a stratifyable subset.

(ii). Every closed invariant submanifold is an invariantly stratifyable subset.

(iii). Every complex affine variety is a stratifyable subset of an adequate affine space. We can take X_k to be the smooth part, X_{k-1} the smooth part of the singular part, X_{k-2} the smooth part of the singular part of the singular part, etc.

(iv). Similarly, every invariant complex affine variety is an invariantly stratifyable subset.

(v). The closure of an orbit of a complex Lie group acting on a complex manifold is an invariantly stratifyable subset. It is the union of some orbits, which can be chosen to be the strata (i. e. the X_i's).

Our next step is to define the cohomology class represented by a stratifyable subset. For doing this, first we need the following lemma.

Lemma 2.2.3 *If M^m is orientable, N^n a closed orientable submanifold of M, ι the inclusion $M \setminus N \to M$ and d an integer, for which $0 \leq d \leq (m-n)-2$, then $H^d(\iota) : H^d(M) \to H^d(M \setminus N)$ is an isomorphism.*

Proof Let E be the total space of the normal bundle of N. Then using Thom-isomorphism and the assumption, that $d < m - n$, we obtain the following.

$$H^d(M|N) \cong H^d(E|N) \cong H^{d-(m-n)}(N) = 0 \tag{2.1}$$

Similarly, since $1 + d < m - n$, it also holds that

$$H^{d+1}(M|N) \cong H^{d+1}(E|N) \cong H^{d+1-(m-n)}(N) = 0 \tag{2.2}$$

So, according to (2.1) and (2.2) the long exact sequence of the pair $(M, M \setminus N)$ yields the statement of the lemma.

$$0 = H^d(M|N) \longrightarrow H^d(M) \xrightarrow{H^d(\iota)} H^d(M \setminus N) \longrightarrow H^{d+1}(M|N) = 0$$

■

From now on, we always suppose, that X is a stratifyable subset of M.

Definition 2.2.4 *The* cohomology class represented by X *is denoted by* $[X]$ *or* $[X \subset M]$*, and is defined via the following commutative diagram.*

$$\begin{tikzcd}
{[X]} \in \arrow[dddddd, mapsto] & H^d(M) \arrow[d, "\cong"] \\
 & H^d(M \setminus X_1) \arrow[d, "\cong"] \\
 & H^d(M \setminus (X_1 \cup X_2)) \arrow[d, "\cong"] \\
 & {} \\
 & \vdots \arrow[d, "\cong"] \\
 & {} \\
{[X_k]} \in & H^d(M \setminus (X_1 \cup X_2 \cup \cdots \cup X_{k-1}))
\end{tikzcd}$$

where $\{X_i \subset M | i \in [k]\}$ *is a stratification of* X*, the maps are induced by inclusions and are isomorphisms, because of Lemma 2.2.3, and* $[X_k]$ *is the cohomology class represented by the closed submanifold* X_k *of* $M \setminus (X_1 \cup X_2 \cup \cdots \cup X_{k-1})$.

Remark 2.2.5 Recall, that $[X_k] \in H^d(M \setminus (X_1 \cup X_2 \cup \cdots \cup X_{k-1}))$ is the image of the Thom-class of the normal bundle $N_M X_k$ of X_k via the map l of the following commutative diagram.

$$\begin{tikzcd}
H^d(N_M X_k|X_k) \arrow[dd, bend right=60, "l"'] \\
H^d(M \setminus (X_1 \cup X_2 \cup \cdots \cup X_{k-1})|X_k) \arrow[u, "\cong", "H^d(i)"'] \arrow[d, "H^d(j)"] \\
H^d(M \setminus (X_1 \cup X_2 \cup \cdots \cup X_{k-1}))
\end{tikzcd}$$

where i and j are the inclusions $(M \setminus (X_1 \cup X_2 \cup \cdots \cup X_{k-1}), \emptyset) \hookrightarrow (M \setminus (X_1 \cup X_2 \cup \cdots \cup X_{k-1}), M \setminus (X_1 \cup X_2 \cup \cdots \cup X_k))$ and $(N_M X_k, N_M X_k \setminus X_k) \hookrightarrow (M \setminus (X_1 \cup X_2 \cup \cdots \cup X_{k-1}), M \setminus (X_1 \cup X_2 \cup \cdots \cup X_k))$, respectively, and $H^d(j)$ is isomorphism, because of excision.

It might seem that Definition 2.2.4 is false, since it depends on the choice of stratification. The next lemma states that it is correct.

Lemma 2.2.6 $[X]$ *does not depend on the choice of stratification.*

Proof Choose two stratifications $\{X_i \subset M | i \in [k]\}$ and $\{Y_i \subset M | i \in [l]\}$. It can be supposed, that $k \geq l$. Define

$$\tilde{Y}_i = \begin{cases} \emptyset & \text{if } i \leq k-l \\ Y_{i+k-l} & \text{if } i > k-l \end{cases}$$

$\{Y_i \subset M | i \in [l]\}$ and $\{\tilde{Y}_i \subset M | i \in [k]\}$ define the same $[X]$, so it is enough to prove the statement for the later instead of the former.

Since $\dim X_k = \dim \tilde{Y}_k$, $[X]$ is an element of $H^d(M)$ independently of the choice of stratification. The fact, that it is always the same element of $H^d(M)$ follows from the diagram on Figure 2.1, using Lemma 2.2.3 many times. ■

Now we define transversality for stratifyable and invariantly stratifyable subspaces. Then we prove Proposition 2.2.8, which is the equivalent of the *Transversality Theorem* for stratifyable spaces. Later on we also prove a similar statement for equivariantly stratifyable subspaces.

Definition 2.2.7 *Suppose, that N is a smooth manifold, and $f : N \to M$ is a smooth map. f is* transversal *to X (i. e. $f \pitchfork X$), if X has a stratification $\{X_i \subset M | i \in [k]\}$, for which $f \pitchfork X_i$ for every $i \in [k]$. If N and M are G-spaces, f is G-equivariant and X is G-invariantly stratifyable, then this stratification of X must be also G-invariant.*

$$
\begin{tikzcd}
 & {[X]} \arrow[d, phantom, sloped, "\in"] \arrow[ddddddl, bend right, mapsto] \arrow[ddddddr, bend left, mapsto] & \\
 & H^d(M) \arrow[dl, "\cong"'] \arrow[d, "\cong"] \arrow[dr, "\cong"] & \\
H^d(M \setminus X_1) \arrow[r, "\cong"'] \arrow[d, "\cong"] & H^d(M \setminus (X_1 \cup \tilde{Y}_1)) \arrow[d, "\cong"] & H^d(M \setminus \tilde{Y}_1) \arrow[l, "\cong"] \arrow[d, "\cong"] \\
H^d(M \setminus (X_1 \cup X_2)) \arrow[r, "\cong"'] \arrow[d, "\cong"] & H^d(M \setminus (X_1 \cup X_2 \cup \tilde{Y}_1 \cup \tilde{Y}_2)) \arrow[d, "\cong"] & H^d(M \setminus (\tilde{Y}_1 \cup \tilde{Y}_2)) \arrow[l, "\cong"] \arrow[d, "\cong"] \\
\vdots \arrow[d, "\cong"] & \vdots \arrow[d, "\cong"] & \vdots \arrow[d, "\cong"] \\
\begin{array}{c} H^d(M \setminus (X_1 \cup X_2 \cup \\ \cup \cdots \cup X_{k-1})) \end{array} \arrow[r, "\cong"'] \arrow[d, phantom, sloped, "\ni"] & \begin{array}{c} H^d(M \setminus (X_1 \cup X_2 \cup \cdots \cup X_{k-1} \cup \\ \cup \tilde{Y}_1 \cup \tilde{Y}_2 \cup \cdots \cup \tilde{Y}_{k-1})) \end{array} \arrow[d, phantom, sloped, "\ni"] & \begin{array}{c} H^d(M \setminus (\tilde{Y}_1 \cup \tilde{Y}_2 \cup \\ \cup \cdots \cup \tilde{Y}_{k-1})) \end{array} \arrow[l, "\cong"] \arrow[d, phantom, sloped, "\ni"] \\
{[X_k]} \arrow[r, mapsto] & {[X_k \cap \tilde{Y}_k]} & {[\tilde{Y}_k]} \arrow[l, mapsto]
\end{tikzcd}
$$

Figure 2.1: *The diagram of the proof of Lemma 2.2.6.*

Proposition 2.2.8 *For every $f : N \to M$ smooth map, if $f \pitchfork X$, then $f^*[X] = [f^{-1}(X)]$.*

Proof Fix a stratification $\{X_i \subset M | i \in [k]\}$ of X, for which $f \pitchfork X_i$ for every $i \in [k]$. First, $f^{-1}(X)$ is stratifyable, since $\{f^{-1}(X_i) | i \in [k]\}$ is a stratification of $f^{-1}(X)$, because

- $f^{-1}(X_i)$ is a submanifold of N, since $f \pitchfork X_i$ for every $i \in [k]$.
- $\bigcup_{i=1}^{k} f^{-1}(X_i) = f^{-1}(X)$.
- $\bigcup_{i=1}^{l} f^{-1}(X_i) = f^{-1}\left(\bigcup_{i=1}^{l} X_i\right)$ is closed, since $\bigcup_{i=1}^{l} X_i$ is closed, and f is continuous.
- $\mathrm{codim} X_i = \mathrm{codim} f^{-1}(X_i)$, because $f \pitchfork X_i$ for every $i \in [k]$.

Second, the following commutative diagram, in which several times Lemma 2.2.3 and once the Transversality Theorem for submanifolds is used, proves the statement of the proposition.

$$\begin{tikzcd}
{[X]} \arrow[rrr, mapsto] \arrow[d, phantom, sloped, "\in"] \arrow[ddd, mapsto, bend right=60] & & & {[X_k]} \arrow[d, phantom, sloped, "\in"] \arrow[ddd, mapsto, bend left=60] \\
H^d(M) \arrow[r, "\cong"] \arrow[d] & H^d(M \setminus X_1) \arrow[r, "\cong"] \arrow[d] & \cdots \arrow[r, "\cong"] & H^d(M \setminus (X_1 \cup \cdots \cup X_{k-1})) \arrow[d] \\
H^d(N) \arrow[r, "\cong"] & H^d(N \setminus f^{-1}(X_1)) \arrow[r, "\cong"] & \cdots \arrow[r, "\cong"] & H^d(N \setminus f^{-1}(X_1 \cup \cdots \cup X_{k-1})) \\
{[f^{-1}(X)]} \arrow[rrr, mapsto] \arrow[u, phantom, sloped, "\in"] & & & {[f^{-1}(X_k)]} \arrow[u, phantom, sloped, "\in"]
\end{tikzcd}$$

■

These were the preliminary steps, and finally we arrived at the discussion of G-invariantly stratifyable subsets. So, from now on suppose, that a Lie-group G acts on M and η is a G-invariantly stratifyable subset of M. Denote $P \times_G \xi$ by $P\xi$, for any G-invariantly stratifyable subset ξ of M, and principle bundle P. We will use this notation later on, also in other sections and chapters, whenever it is evident which representation or action we are talking about.

In what follows, we investigate $[P\eta \subset PM]$. This is sensible, that is, $P\eta$ is stratifyable in PM, because if $\{\eta_i \subset M | i \in [k]\}$ is a G-invariant stratification of η, then $\{P\eta_i \subset PM | i \in [k]\}$ is a stratification of $P\eta$.

Before the definition of Thom-polynomials can be stated, we need one more proposition, Proposition 2.2.9, which is the basis of the definition.

Proposition 2.2.9 *If P and Q are smooth G-principal bundles, $f : P \to Q$ a G-equivariant smooth map (i. e. $P = f^*Q$) and $g = f \times_G M$, then $g^*([Q\eta \subset QM]) = [P\eta \subset PM]$.*

PROOF

$$[P\eta \subset PM] = [g^{-1}(Q\eta) \subset PM] = g^*([Q\eta \subset QM]$$

where the first equation is definition, while the second one holds because $g \pitchfork Q\eta$. This follows from the local triviality, since $T_{g(p)}(P\eta)$ contains the horizontal and $\mathrm{Im}(T_p g)$ contains the vertical tangent spaces of a local trivialization for every $p \in P$. ■

Definition 2.2.10 *From Proposition 2.2.9, it follows, that if M is contractible, then η determines a G-characteristic class. This characteristic class is the* Thom-polynomial $\mathrm{Tp}(\eta)$. *If η is not a closed set, then* $\mathrm{Tp}(\eta)$ *means* $\mathrm{Tp}(\overline{\eta})$. *($\overline{\eta}$ must be G-*

invariantly stratifyable.) If we want to emphasize the G-action or the ambient space, we write $\mathrm{Tp}_G(\eta)$ *or* $\mathrm{Tp}(\eta)^M$*, respectively.*

2.3 Variant of the main definition

Here we present an equivalent of Definition 2.2.10. Instead of realizing $\mathrm{Tp}(\eta)$ as a characteristic class, i. e. a system $\{[P\eta \subset PM] \mid P$ is a smooth G-principal bundle$\}$, we determine one bundle Q, which determines $\mathrm{Tp}(\eta)$. More precisely, for $k = \mathrm{codim}_{\mathbb{R}}\eta$, $H^k(Q/G) \cong H^k(BG)$ will hold, and thus $[Q\eta \subset QM]$ will define $\mathrm{Tp}(\eta) \in H^k(BG)$. From this it can be already seen, that we replace only Proposition 2.2.8, Proposition 2.2.9 and the surrounding definitions, by another argument, but the foregoing parts of Section 2.2 will still be used. Roughly speaking, we make a *finite dimensional approximation*, of the bundle $EG \to BG$, which is a standard trick and can be found for example in [Tot99] and [EG97].

We use the notations fixed in the previous sections, that is, G,M and η denotes the same thing as before.

Proposition 2.3.1 *There exists a representation $\rho_G : G \to \mathrm{GL}(U)$ of G on a vector space U, for which $NF_{\rho_G} := \{x \in U | x$ is in a non free orbit$\}$ is a positive codimensional subvariety of U.*

Example 2.3.2 If $G = \mathbb{C}^*$, then ρ_G of Proposition 2.3.1 can be taken to be the standard representation by left multiplication on $\mathbb{C}$. Similarly, if $G = (\mathbb{C}^*)^r$, then its standard representation on $\mathbb{C}^r$ is an eligible choice.

Proposition 2.3.3 *There exists a representation $\rho^i_G : G \to \mathrm{GL}(U^i)$, for which $\mathrm{codim}_{\mathbb{C}} NF_{\rho^i_G} \geq i$.*

Proof Take ρ_G of Proposition 2.3.1, and define ρ^i_G to be $\rho_G \oplus \rho_G \oplus \cdots \oplus \rho_G : G \to U^i$. ■

Example 2.3.4 In case of representations of Example 2.3.2 the standard representation of $\mathbb{C}^*$ on $\mathbb{C}^i$ and of $(\mathbb{C}^*)^r$ on $(\mathbb{C}^i)^r$ yields a representation like in Proposition 2.3.3.

Now fix an i, and $\rho^i_G : G \to \mathrm{GL}(U^i)$. Denote by NF and F, $NF_{\rho^i_G}$ and $U^i \setminus NF$, respectively. Suppose also, that the bundle $F \to F/G$ is locally trivial, i. e. it is a principal bundle. This is not a real restriction, because it holds automatically for compact G, and a non-compact G can always be replaced by its maximal compact subgroup, without changing the equivariant cohomology rings or the Thom-polynomials. (See Chapter 3 for the details.) Then the following lemma holds, which states roughly speaking, that F is a finite dimensional approximation of EG. That is, it admits a free G action and its lower homotopy groups are trivial.

Lemma 2.3.5 *$\pi_j(F) = 0$ for all $j \leq 2i - 2$.*

Proof Consider two maps $f, g : (I^j, \partial I^j) \to (F, pt)$. We prove, that f and g are homotopy equivalent, which will prove the statement. Since U^i is contractible, there is a homotopy $\phi : (I^j \times I, \partial I^j \times I) \to (U^i, pt)$ connecting f to g in U^i. ϕ can be approximated by a smooth map, and then homotoped to be transveral to every strata of NF. This means, that there exists a smooth map $\tilde{\phi} : (I^j \times I, \partial I^j \times I) \to (U^i, pt)$, for which $\tilde{\phi}_0$ and $\tilde{\phi}_1$ is homotopic to f and g, respectively in F, and $\tilde{\phi}$ is transversal to NF. But, since $j + 1 + \dim_{\mathbb{R}} NF < \dim_{\mathbb{R}} U^i$, transversality here means actually disjointness, that is $\mathrm{Im}\tilde{\phi} \subseteq F$ and thus f and g are homotopy equivalent in F. ∎

The property of a finite dimensional approximation of EG, what we really need is, that its classifying map induces isomorphisms on lower grades of cohomology. For F this is proved in the following lemma.

Lemma 2.3.6 *For the classifying map $f : F/G \to BG$ of F and $j \leq 2i - 2$, $H^j(f)$ is an isomorphism.*

Proof Fix an arbitrary $0 \leq j \leq 2i - 2$. Then from the naturality of the long exact sequence of the fibration we obtain the following.

$$\begin{tikzcd}
 & & 0 \arrow[d, equal] & & & 0 \arrow[d, equal] & \\
\cdots \arrow[r] & \pi_j(G) \arrow[r] \arrow[d, "\cong"] & \pi_j(F) \arrow[r] \arrow[d] & \pi_j(F/G) \arrow[r] \arrow[d, "\pi_j(f)"] & \pi_{j-1}(G) \arrow[r] \arrow[d, "\cong"] & \pi_{j-1}(F) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \pi_j(G) \arrow[r] & \pi_j(EG) \arrow[r] \arrow[d, equal] & \pi_j(BG) \arrow[r] & \pi_{j-1}(G) \arrow[r] & \pi_{j-1}(EG) \arrow[r] \arrow[d, equal] & \cdots \\
 & & 0 & & & 0 &
\end{tikzcd}$$

Using the 5-lemma it follows, that $\pi_j(f)$ is an isomorphism. That is, $\pi_j(\pi_f, F/G) = 0$, where π_f is the mapping cylinder of f. So, according to the relative Hurewitz-theorem $H_j(\pi_f, F/G) = 0$, from which follows the statement of the lemma, using the homology long exact sequence of the pair $(\pi_f, F/G)$ and the universal coefficient theorem. ∎

Finally, using Lemma 2.3.6, we can state the following proposition, which is nothing else, but the promised variant of Definition 2.2.10.

Proposition 2.3.7 $\mathrm{Tp}(\eta) = (f^*)^{-1}([F\eta \subset FM])$, *if* $\mathrm{codim}_{\mathbb{R}}\eta \leq 2i-2$ *and* M *is contractible.*

Example 2.3.8 In the special case of Example 2.3.2 and Example 2.3.4 Proposition 2.3.7 yields, that $\mathrm{Tp}(\eta)$ can be defined as $[\tau_k\eta \subset \tau_k M]$ for $G = \mathbb{C}^*$ and $[\tau_k^r\eta \subset \tau_k^r M]$ for $G = (\mathbb{C}^*)^r$, where τ_k and τ_k^r are the tautological $\mathbb{C}^*$ bundles over $\mathbb{CP}^k$ and $(\mathbb{CP}^k)^r$, respectively, and k is any integer not smaller than $(\mathrm{codim}_{\mathbb{R}}\eta)/2$.

2.4 Definition using Kazarian Spectral Sequence

In [Kaz97] Kazarian defines not only Thom-polynomials, but generally characteristic classes of singularities. Here we present this definition with small modifications, which must be made since our notion of stratification differs in some technical details from that of Kazarian.

Like in Section 2.3, we use some notations fixed in Section 2.2, that is, G and M means the same as there. The basic idea of the definition is, the following remark.

Remark 2.4.1 If $\{X_i | i \in [k-1]\}$ is a stratification of a stratifyable subset of M, then $\{M_i | i \in [k]\} = \{X_i | i \in [k-1]\} \cup \{M \setminus X_{k-1}\}$ is an almost stratification of M. That is, it fulfills every condition of Definition 2.2.1 except the dimensional gap of the second criteria.

We will call such almost stratification, like in Remark 2.4.1, a "stratification". Precisely, $\{M_i | i \in [k]\}$ is a *"stratification"* of M, if it satisfies every condition of Definition 2.2.1 with $X = M$, except the second one, and $\dim M_k = \dim M$, $\dim M_{k-1} < \dim M$ and $\dim M_i \leq M_{k-1} - 2$ for every $i \in [k-2]$.

So, according to Remark 2.4.1, we can actually restrict ourselves to "stratifications" of M, instead of examining stratifications of subsets of M. We define, in this manner, characteristic classes of "stratifications" of M, which are based on the spectral sequences of the next definition.

Definition 2.4.2 *The* spectral sequence of a "stratification" *of* M *is the cohomological spectral sequence associated to the filtration* $\{\mathcal{F}_i | i = 0, \ldots, \dim_{\mathbb{R}} M\}$*, where* $\mathcal{F}_i = \bigcup_{\mathrm{codim}_{\mathbb{R}} M_j \leq i} M_j$*. It is denoted by* $E_*^{*,*}(M)$*.*

Remark 2.4.3 $E_1^{p,q}(M) = H^{p+q}(\mathcal{F}_p, \mathcal{F}_{p-1})$, and $d_1 : E_1^{p,q}(M) \to E_1^{p+1,q}(M)$ is the boundary map $\delta : H^{p+q}(\mathcal{F}_p, \mathcal{F}_{p-1}) \to H^{p+1+q}(\mathcal{F}_{p+1}, \mathcal{F}_p)$ of the triple $(\mathcal{F}_{p+1}, \mathcal{F}_p, \mathcal{F}_{p-1})$.

Example 2.4.4

(i). If W is a $d > 1$ dimensional real vector space, $M_1 = \{0\} \subseteq W$ and $M_2 = W \setminus \{0\}$, then

$$\mathcal{F}_i = \begin{cases} W \setminus \{0\} & \text{if } 0 \leq i < d \\ W & \text{if } i = d \end{cases}$$

Consequently $E_1^{p,q} = 0$, unless $(p,q) = (0,0)$ or $(d,0)$, and $E_1^{0,0} \cong E_1^{d,0} \cong \mathbb{Z}$. So, $E_1^{*,*}$ is already the limit page of the spectral sequence (i.e. $E_1^{*,*} = E_\infty^{*,*}$).

(ii). If the "stratification" of M is also a CW-decomposition, then

$$E_1^{p,q} = \begin{cases} 0 & \text{if } q \neq 0 \\ \mathbb{Z}^{d_p} & \text{if } q = 0, \text{ where } d_p \text{ is the number of CW cells of dimension } p \end{cases}$$

Also, the differentials of the 0-th row of the first page are the boundary maps of the CW-cohomology. So, the 0-th row of the $E_2^{*,*}$ page contains the cohomology groups of M, and this page is already the limit page.

From now on in this section, we fix a "stratification" $\{M_i | i \in [k]\}$ of M. We use the following notations. $E_*^{*,*} := E_*^{*,*}(M)$, $X := \cup_{i=1}^{k-1} M_i$, $p := \operatorname{codim}_{\mathbb{R}} X$, and $X_i := M_i$ for every $i \in [k-1]$. We denote by $N_M X_k$ the normal bundle of X_k in M, which can be identified with a tubular neighborhood of X_k in M.

We use Definition 2.4.2 for two purposes. First, we define the cohomology class $[X]$ represented by X, and later on through a limit process we define characteristic classes of a "stratification". The next proposition does the preparation part of the work for the Kazarian variant of $[X]$.

Proposition 2.4.5

(i). $E_1^{p,0} \cong H^p(N_M X_{k-1} | X_{k-1})$ *naturally.*

(ii). $E_2^{p,0} \cong E_1^{p,0} / \operatorname{Im} d_1$*, and thus it is a homomorphic image of* $H^p(N_M X_{k-1} | X_{k-1})$*.*

Proof

(i). By definition of "stratification" hold the following equations.

$$E_1^{p,0} = H^p(\mathcal{F}_p, \mathcal{F}_{p-1}) = H^p(M_k \cup X_{k-1}, M_k) = H^p(M_k \cup X_{k-1} | X_{k-1})$$

Since X_{k-1} is a closed submanifold of $M_k \cup X_{k-1}$, excision can be used to the triple $(M_k \cup X_{k-1}, M_k, M_k \setminus N_{M_k \cup X_{k-1}} X_{k-1})$. That is, $H^p(M_k \cup X_{k-1} | X_{k-1}) \cong$

$H^p(N_{M_k \cup X_{k-1}} X_{k-1} | X_{k-1})$, from which the statement follows, since $N_M X_{k-1} = N_{M_k \cup X_{k-1}} X_{k-1}$.

(ii). Since $\{M_i | i \in [k]\}$ is a "stratification" of M, $\mathcal{F}_p = \mathcal{F}_{p+1}$. So, $E_1^{p+1,0} = H^{p+1}(\mathcal{F}_{p+1}, \mathcal{F}_p) = 0$, and thus $d_1 : E_1^{p,0} \to E_1^{p+1,0}$ is the zero map, which proves the statement. ■

Definition 2.4.6 $[X]_{\text{kz}} = l(\phi)$, *where* $\phi \in H^p(N_M X_{k-1} | X_{k-1})$ *is the Thom-class of* $N_M X_{k-1}$, *and* l *is defined via the following commutative diagram, in which* ι *is the natural embedding of* $E_2^{p,0}$ *into* $H^p(M)$, *and* j *is the map of Proposition 2.4.5 (ii).*

$$\begin{tikzcd} H^p(N_M X_{k-1} | X_{k-1}) \arrow[r, "j"] \arrow[rr, bend right, "l"'] & E_2^{p,0} \arrow[r, "\iota"] & H^p(M) \end{tikzcd}$$

Proposition 2.4.7 $[X]_{\text{kz}} = [X]$.

PROOF First let us recall and analyze how $[X]$ and $[X]_{\text{kz}}$ is defined.

$[X]$ is defined as the map of the Thom-class ϕ of $N_M X_{k-1}$ via $\tilde{l}$, where $\tilde{l}$ is defined in the above commutative diagram.

$$\begin{tikzcd} H^p(N_M X_{k-1} | X_{k-1}) \arrow[rrr, bend right, "\tilde{l}"'] & H^p(M_k \cup X_{k-1} | X_{k-1}) \arrow[l, "\cong"'] \arrow[r] & H^p(M_k \cup X_{k-1}) & H^p(M) \arrow[l, "\cong"'] \end{tikzcd}$$

$[X]_{\text{kz}}$ is defined as the map of ϕ via l, where l is the map of the following commutative diagram.

$$\begin{tikzcd} H^p(N_M X_{k-1} | X_{k-1}) \cong E_1^{p,0} = E_p^{p,0} \arrow[r, "p"] \arrow[rr, bend right, "l"'] & E_{p+1}^{p,0} = E_\infty^{p,0} \arrow[r, hook, "q"] & H^p(M) \end{tikzcd}$$

Where

- $E_1^{p,0} = E_p^{p,0}$, since $E_1^{i,j} = 0$, if $i = p+1, p-1, p-2, \ldots, 1$, and thus $E_s^{i,j} = 0$ for those indices of i and every value of j and s.
- $E_{p+1}^{p,0} = E_\infty^{p,0}$, since for $i \geq p+1$, d_i points from the $(p, 0)$ square outside of the upper right quadrant.
- p is the factorization by $\operatorname{Im}(d_p^{0,p-1})$
- q is the usual map of inclusion $E_\infty^{p,0} \hookrightarrow H^p(M)$ belonging to spectral sequences.

To understand the map q better, we dig a bit deeper into the definition of spectral sequences. For every i there is a long exact sequence, which has the following segment.

$$\begin{aligned}&\operatorname{Im}(H^{p-1}(\mathcal{F}_{p-i+1} \hookrightarrow \mathcal{F}_p)) \longrightarrow \operatorname{Im}(H^{p-1}(\mathcal{F}_{p-i} \hookrightarrow \mathcal{F}_{p-1})) \longrightarrow E_i^{p,0} \longrightarrow \\ &\longrightarrow \operatorname{Im}(H^p(\mathcal{F}_p \hookrightarrow \mathcal{F}_{p+i-1})) \longrightarrow \operatorname{Im}(H^p(\mathcal{F}_{p-1} \hookrightarrow \mathcal{F}_{p+i-2}))\end{aligned} \tag{2.3}$$

For $i = p + 1$ (2.3) looks like one of the rows in the commutative diagram of Figure 2.2. While the rest of the diagram explains the map q and how q is defined by (2.3).

If we constrict the equations in the diagram, the rows and the columns except the middle column become exact. In the middle column, the upper and the bottom half are exact separately. The diagram is mainly self explanatory, so we do not mention every detail. However me make some explanation.

- The lowest row is a segment of the long exact sequence of the pair $(\mathcal{F}_p, \mathcal{F}_{p-1})$.
- $\operatorname{Im}(H^p(\mathcal{F}_p \hookrightarrow M)) = \operatorname{Im}(H^p(\mathcal{F}_p \hookrightarrow \mathcal{F}_{2p}))$, and $\operatorname{Im}(H^p(\mathcal{F}_{p-1} \hookrightarrow M)) = \operatorname{Im}(H^p(\mathcal{F}_{p-1} \hookrightarrow \mathcal{F}_{2p-1}))$, since the map induced on p-th grade of cohomology by the inclusion $\mathcal{F}_i \hookrightarrow M$ is an isomorphism for $i \geq p$. $\operatorname{Ker}(H^p(\mathcal{F}_p \hookrightarrow M)) = 0$ and $H^p(\mathcal{F}_{p-1} \hookrightarrow M)) \cong H^p(\mathcal{F}_{p-1} \hookrightarrow \mathcal{F}_p))$ for the same reason.
- The line from $\operatorname{Ker}(H^p(\mathcal{F}_{p-1} \hookrightarrow M))$ to $\operatorname{Ker}(H^p(\mathcal{F}_{p-1} \hookrightarrow \mathcal{F}_p))$ is drawn brokenly only to enhance the visibility of the diagram.
- There are also exact sequences not ordered in one row or column. If we start from the top of the middle column, and turn to right at $\operatorname{Im}(H^p(\mathcal{F}_p \hookrightarrow M))$ or if we start from the bottom of that column and turn to the right at $H^p(\mathcal{F}_p)$, we obtain exact sequences.
- q is as drawn on the diagram, since this is its definition. More precisely, its definition is word by word the composition of maps through the route from $E_\infty^{p,0}$ to $H^p(M)$, which goes to $\operatorname{Im}(H^p(\mathcal{F}_p \hookrightarrow M))$ and then to $\operatorname{Ker}(H^p(\mathcal{F}_{p-1} \hookrightarrow M))$. See for example [Hat] page 6-7 for the definition.

According to Proposition 2.4.5 $H^p(\mathcal{F}_p, \mathcal{F}_{p-1}) = E_1^{p.0} \cong H^p(N_M X_k | X_k)$. So, $\phi \in H^p(\mathcal{F}_p, \mathcal{F}_{p-1})$ naturally. Both $[X]$ and $[X]_{\mathrm{kz}}$ is defined as the image of this element of $H^p(\mathcal{F}_p, \mathcal{F}_{p-1})$ in $H^p(M)$ in the diagram of Figure 2.2. The difference is only in the way of reaching $H^p(M)$. In the case of $[X]$, we go from $H^{p-1}(\mathcal{F}_p, \mathcal{F}_{p-1})$ to $H^p(\mathcal{F}_p)$, and then directly to $H^p(M)$ via the isomorphism. In the case of $[X]_{\mathrm{kz}}$ first we go up

$$\begin{tikzcd}
 & & 0 \arrow[d] & \\
 & 0 \arrow[r] & \mathrm{Ker}(H^p(\mathcal{F}_{p-1} \hookrightarrow M)) \arrow[r] \arrow[d, equal] \arrow[ddddd, dashed, bend right, "\cong"'] & H^p(M) \arrow[dddl, bend left=60, "\cong"] \\
 & 0 & \begin{array}{c}\mathrm{Ker}(H^p(\mathcal{F}_{p-1} \hookrightarrow M))/\\ /\mathrm{Ker}(H^p(\mathcal{F}_p \hookrightarrow M))\end{array} \arrow[d, "r"] & 0 \arrow[d] \\
0 \arrow[r] & E_\infty^{p,0} \arrow[r] \arrow[u] \arrow[d, equal] \arrow[uurr, bend left, "q"] & \mathrm{Im}(H^p(\mathcal{F}_p \hookrightarrow M)) \arrow[r] \arrow[d, equal] & \mathrm{Im}(H^p(\mathcal{F}_{p-1} \hookrightarrow M)) \arrow[d, equal] \\
0 \arrow[r] & E_{p+1}^{p,0} \arrow[r] & \mathrm{Im}(H^p(\mathcal{F}_p \hookrightarrow \mathcal{F}_{2p})) \arrow[r] \arrow[d, equal] & \mathrm{Im}(H^p(\mathcal{F}_{p-1} \hookrightarrow \mathcal{F}_{2p-1})) \arrow[d] \\
H^{p-1}(\mathcal{F}_{p-1}) \arrow[r, "\delta"] & H^p(\mathcal{F}_p, \mathcal{F}_{p-1}) \arrow[r] \arrow[u, "p"] & H^p(\mathcal{F}_p) \arrow[r] & H^p(\mathcal{F}_{p-1}) \\
 & H^{p-1}(\mathcal{F}_{p-1}) \arrow[u, "d_p=\delta"] & \mathrm{Ker}(H^p(\mathcal{F}_{p-1} \hookrightarrow \mathcal{F}_p)) \arrow[u] & \\
 & & 0 \arrow[u] &
\end{tikzcd}$$

Figure 2.2: *The diagram used in the proof of Proposition 2.4.7*

to $E_\infty^{p,0}$ and then through q to $H^p(M)$. That is, since the diagram is commutative, the statement is proved. ■

Suppose now, that M is contractible, admits a G action and all the M_i are G-invariant. Denote M_i by η_i for $i \in [k-1]$ and $\bigcup_{i=1}^{k-1} M_i$ by η. Then $\{PM_i | i \in [k]\}$ is a "stratification" of PM for every G-principal bundle P. This fact is used in the next definition, where a characteristic spectral sequence is made using spectral sequences "stratifications". Characteristic here means, that it contains only characteristic classes.

Definition 2.4.8 *The projective limit of the spectral sequences $E_*^{*,*}(PM)$ over all principal G-bundles P is the* characteristic spectral sequence $\tilde{E}_*^{*,*}$ *of the "stratification" $\{M_i | i \in [k]\}$ of M.*

Remark 2.4.9 The maps used to create the projective limit are the following. If $f : P \to Q$ is a G-equivariant map of G-bundles, then there is a map $f \times_G M$:

$PM \to QM$, for which $(f \times_G M)(PM_i) \subseteq QM_i$. Thus, because of the naturality of spectral sequences, there is a map from $E_*^{*,*}(QM)$ to $E_*^{*,*}(PM)$.

Remark 2.4.10 The projective limit in Definition 2.4.8 can be replaced by associating only to EG instead of every smooth G-bundle. Furthermore associating to EG means dealing with G-equivariant cohomology. So, if we replace cohomology with G-equivariant cohomology in Definition 2.4.2, we obtain the notion of characteristic spectral sequences, instead of spectral sequences of "stratifications".

Finally, we have arrived to this section's definition of Thom-polynomials. These are worded in the following propositions. The proofs of the propositions are easy consequences of the foregoing discussion, and thus we omit them from here.

Proposition 2.4.11 $\mathrm{Tp}(\eta) = [\eta]_{\mathrm{kz}}$ *in the characteristic spectral sequence* $\tilde{E}_*^{*,*}(M)$.

Remark 2.4.12 $\phi \in \tilde{E}_*^{*,*}$ is defined by the compatible system $\{\phi_P | P$ is a smooth G-principal bundle$\}$, where ϕ_P is the image of the Thom-class of $N_{PM}P\eta_{k-1}$ in $H^p(P(M_k \cup \eta_{k-1}), P\eta_{k-1}) = E_1^{p,0}(PM)$.

Proposition 2.4.13 $\mathrm{Tp}(\eta)$ *is the element of* $H_G^*(M)$ *determined by the compatible system* $\{[P\eta]_{\mathrm{kz}} | P$ *is a* G*-bundle*$\}$.

The difference between Proposition 2.4.11 and Proposition 2.4.13 is that in the first one we make the projective limit and then derive a $[.]_{\mathrm{kz}}$, while in the second one we do these steps in an opposite order. As we will shortly see, using Remark 2.4.10 the projective limit step can be omitted in the case when G is compact. Furthermore, the compactness of G here cause no difficulty, since G can always be replaced by one of its maximal compact subgroups. (See Chapter 3)

Remark 2.4.14 If G is compact, then a G-invariant Riemann metric can be introduced on M. In this case $N_M\eta_{k-1}$ can be realized as $Y = \{x \in M_k \cup \eta_{k-1} | \mathrm{d}(x, \eta_{k-1}) < \epsilon\}$ for an ϵ small enough. This is useful, since the invariance of η_{k-1} implies the invariance of Y. That is, the normal bundle of η_{k-1} is also invariant under G action, so $BY = BN_M\eta_{k-1}$ exists. ($BY = B \times_G Y$ by definition) So, ϕ of Proposition 2.4.11 can be defined as the image of the Thom-class of the bundle $BY \to B\eta_{k-1}$, in $H^p(B(M_k \cup \eta_{k-1}), B\eta_{k-1}) = H_G(M_k \cup \eta_{k-1}, \eta_{k-1}) = \tilde{E}_1^{p,0}$.

Summarizing, the projective limit step can be totally eliminated, by building $\tilde{E}_*^{*,*}$ in the manner of Remark 2.4.10 using equivariant cohomology, and taking ϕ to be the Thom-class of $BY \to B\eta_{k-1}$ (i. e. the "equivariant Thom-class" of $N_M\eta_{k-1} \to \eta_{k-1}$).

2.5 Further remarks

Definitions of Section 2.2 and 2.4 are not easily computable ones. The one in Section 2.3 is a bit better from that aspect, but the really computation oriented definition comes in Chapter 4 in the form of multidegree. However the one presented in [FR04] also turns out to be rather usable for computations. We do not state that definition here, but we indicate with the following proposition why it is the special case of ours. We use here also the notations of Section 2.2 (i. e. G, M and η), and we suppose, that M is contractible.

For an equivariant cohomology map $\iota^* : H_G^*(B) \to H_G^*(A)$ induced by a G-equivariant inclusion $\iota : A \hookrightarrow B$ we use the restriction sign, that is, $x|_A = i^*(x)$ for every $x \in H_G^*(B)$. Then the previously mentioned proposition is as follows.

Proposition 2.5.1 $\mathrm{Tp}(\eta)|_{M\setminus\eta} = 0$.

Proof The statement is the consequence of the fact that the same holds if we associate to G-principal bundles. Namely, if P is a G-principal bundle, then $([P\eta \subset PM])|_{P(M\setminus\eta)} = 0$.

This holds, since $[P\eta \subset PM]$ is the image of an element of $H^p(PM, P(M\setminus\eta))$ in the long exact sequence of the pair $(PM, P(M\setminus\eta))$. That is, $([P\eta \subset PM])|_{P(M\setminus\eta)}$ is the image of the same element through two maps in the same exact sequence, which implies that it is 0. ∎

Actually more is true, $\mathrm{Tp}(\eta)$ is the generator of $H_G^d(M) \cap \mathrm{Ker}(H_G^*(M\setminus\eta \hookrightarrow M))$ in many cases. This is used to define $\mathrm{Tp}(\eta)$ in [FR04].

We have one more easy corollary, which is rather important for the future.

Corollary 2.5.2 $\mathrm{Tp}(\eta)|_A = 0$ *for every G-invariant subset A of $M\setminus\eta$.*

Proof $.|_A$ can be factored through $.|_{M\setminus\eta}$ for any $A \subseteq M\setminus\eta$. ∎

Chapter 3

Incidence

In this chapter we formulate the Incidence Property precisely. Our main objective, throughout the whole article, is to give a usable sufficient condition on it. The result of this endeavor is stated in Main Theorem and proved at the beginning of Chapter 6. We also prove here a key lemma, in which the notion of incidence is reduced to the notion of Thom-polynomials. From now on until Chapter 6 we fix a representation ρ of a complex Lie-group G on a complex vector space V of complex dimension n.

The Incidence Property *ρ has the Incidence Property, if the following holds for every orbit η and ξ.*

$$\xi \subset \overline{\eta} \iff \mathrm{Tp}(\eta)|_\xi \neq 0 \tag{3.1}$$

Here, as at the end of the previous chapter, $\mathrm{Tp}(\eta)|_\xi = i_\xi^*(\mathrm{Tp}(\eta))$ is the image of $\mathrm{Tp}(\eta)$ by the ring homomorphism $i_\xi^* : H_G^*(V) \to H_G^*(\xi)$ induced by the G-equivariant inclusion of ξ into V. However it can be viewed also differently, since $H_G^*(\xi) = H_{G_x}^*(pt)$ for any $x \in \xi$. Through this correspondence i_ξ^* is equal to the map $H_G^*(V) = H^*(BG) \to H^*(BG_x) = H_G^*(\xi)$ induced by $Bi : BG_x \to BG$, where i is the inclusion $G_x \hookrightarrow G$.

We emphasize, that not every representation has the Incidence Property. So, the question is which representations have it. However, one direction of the statement of the Incidence Property always holds. If $\xi \not\subset \overline{\eta}$, then $\xi \subset V \setminus \overline{\eta}$, thus because of Corollary 2.5.2 $\mathrm{Tp}(\eta)|_\xi = 0$. So, during the verification of the Incidence Property we have to prove only that if $\xi \subset \overline{\eta}$ then $\mathrm{Tp}(\eta)|_\xi \neq 0$.

From now on in this chapter, we fix ξ and η, for which $\xi \subset \overline{\eta}$ and $\mathrm{codim}_{\mathbb{C}}(\xi) = k$. In the long distance we would like to prove that $\mathrm{Tp}(\eta)|_\xi \neq 0$ with some assumptions on ρ. But, first let us translate $\mathrm{Tp}(\eta)|_\xi$ to an ordinary Thom-polynomial in a normal space of a point of ξ. For doing this we replace G by one of its maximal compact subgroup G_c. Since $H^*_G(.)$ and $H^*_{G_c}(.)$ are the same functors and G_c Thom-polynomials are also the same as the G Thom-polynomials, we are allowed to do this. We emphasize that the orbits we are considering are still G orbits, which are G_c-invariantly stratifyable subsets, but generally not G_c orbits.

The next step is to choose a G_c invariant scalar product $\langle .,. \rangle$ on V. For such a scalar product the normal space $N = (T_x\xi)^\perp$ of ξ at a point x of ξ is a $G_{c,x}$ invariant subset of V. Here we note, that we can choose G_c, such that $G_{c,x}$ is also a maximal compact subgroup of G_x, i. e. $G_{x,c} = G_{c,x}$. Then $H^*_{G_x}(.)$ and $H^*_{G_{c,x}}(.)$ will be the same functors, as before in the case of $H^*_G(.)$ and $H^*_{G_c}(.)$. Such choice of G_c can be realized by first choosing a maximal compact subgroup of $G_{x,c}$ of G_x, and then extending this to a maximal compact subgroup G_c of G. This can be done by using standard Lie-theory.

We also note, that when we regard subspaces or vectors of T_xV as subsets or elements of V, we usually do it through the natural identification $\iota_x : T_xV \to V$, for which $\iota_x(0) = x$. Sometimes we also want to use the other natural identification $\iota_0 : T_xV \to V$, for which $\iota_0(0) = 0$, but then we denote this by underlining the subspace or vector.

The previously mentioned translation of incidence to Thom-polynomials is phrased in the following lemma.

Lemma 3.1 $\mathrm{Tp}_{G_{c,x}}(\eta \cap N) = \mathrm{Tp}_G(\eta)|_\xi$.

Now we can explain, why the whole machinery of maximal compact subgroups has been used. Without exchanging G to G_c, the invariance of $\eta \cap N$ would not be satisfied generally. In most cases even a normal space N can not be found, which is invariant under G_x action, and the case is even worse if we want $\eta \cap N$ to be invariant.

We also note, that there is no difference between taking η and $\overline{\eta}$ in Lemma 3.1. That is, $\overline{\overline{\eta} \cap N} = \overline{\eta \cap N}$ holds. This can be proven by using the following techniques of Lemma 3.3, i. e. localization to a neighborhood B of x, for which $\eta \cap B \pitchfork N \cap B$. For such a B it can be shown, that there is a continuous map $\pi : B \to B \cap N$, such that b and $\pi(b)$ are G-equivalent for every $b \in B$.

In order to prove Lemma 3.1, we have to deal with some other lemmas. First a lemma, which is the equivalent of the *Transversality Theorem* for invariantly stratifyable subsets.

Lemma 3.2 *If H is a Lie-group M and K are contractible smooth H spaces, $\zeta \subset M$ is H-invariantly stratifyable, $f : K \to M$ an H-equivariant smooth map and $f \pitchfork \zeta$, then* $\mathrm{Tp}(f^{-1}(\zeta)) = \mathrm{Tp}(\zeta)$

PROOF $f^{-1}(\zeta)$ is H-invariantly stratifyable, since if $\{\zeta_i \subset M | i \in [k]\}$ is an H-invariant stratification of ζ, for which $f \pitchfork \zeta_i$ for every $i \in [k]$, then $\{f^{-1}(\zeta_i) \subset K | i \in [k]\}$ is a H-invariant stratification of $f^{-1}(\zeta)$.

Consider a smooth H-principle bundle P. Then for $Pf = P \times_H f$, $(Pf)^* = \mathrm{id}$, because $(Pf)^*$ is homotopic equivalent to the identity map of P/H. Since $f \pitchfork \zeta$, $Pf \pitchfork P\zeta$, that is using Proposition 2.2.8 the following equation can be obtained.

$$[P(f^{-1})(\zeta) \subset PK] = [(Pf)^{-1}(\zeta) \subset PK] = (Pf)^*[P\zeta \subset PM] = [P\zeta \subset PM] \quad (3.2)$$

So, from (3.2) according to Definition 2.2.10 it follows, that $\mathrm{Tp}(f^{-1}(\zeta)) = \mathrm{Tp}(\zeta)$, which is the statement, we are supposed to prove. ∎

Second, we prove a lemma, which uses Lemma 3.2, and as it will be shown, nearly proves Lemma 3.1.

Lemma 3.3 $\mathrm{Tp}_{G_{c,x}}(\eta)|_N = \mathrm{Tp}_{G_{c,x}}(\eta \cap N)$

PROOF Let α be the map of the $G \times V \to V$ action, and $\alpha_z := \alpha|_{G\times\{z\}}$ for every $z \in V$. Then $\underline{\mathrm{Im} T_e\alpha_x} = \underline{T_x\xi} = \underline{N}^\perp$ holds. Let $y_1, \ldots, y_l \in T_eG$ be vectors, for which $\{\underline{T_e\alpha_x(y_i)}|i \in [l]\}$ is a base of $\underline{N}^\perp$, and let $x_1, \ldots, x_k$ be a base of $\underline{N}$. If we write the coordinates of $\underline{T_{\alpha_z}(y_1)}, \ldots, \underline{T_{\alpha_z}(y_l)}, x_1, \ldots, x_k$ in the columns of a matrix, we get a full rank matrix $\mathbf{A}(z)$ for every $z \in V$. $\mathbf{A} : N \to \mathbb{C}^{n\times n}$ is a continuous function, so there is a ball B of radius $\epsilon > 0$ around x, such that $\mathbf{A}(z)$ is full rank for every $z \in B \cap N$. That is, $N \pitchfork \mathrm{Im} T_e\alpha_z$ for every $z \in B \cap N$. Which means that, $N \cap B \pitchfork \overline{\eta} \cap B$, since $T_z\zeta = \mathrm{Im} T_e\alpha_z$ for every ζ orbit in $\overline{\eta}$ and $z \in \zeta$.

So, according to Lemma 3.2, if we regard the embedding $N \cap B \hookrightarrow B$ we obtain that

$$\mathrm{Tp}(\eta \cap B)^B = \mathrm{Tp}(\eta \cap N \cap B)^{N\cap B} \quad (3.3)$$

The inclusion $B \hookrightarrow V$ and $N \cap B \hookrightarrow N$ are also transversal, and $G_{c,x}$-equivariant, since $\langle .,. \rangle$ is $G_{c,x}$ invariant and x is a fix point of $G_{c,x}$. So, similarly we obtain that

$$\mathrm{Tp}(\eta)^V = \mathrm{Tp}(\eta \cap B)^B \quad (3.4)$$

and

$$\mathrm{Tp}(\eta \cap N \cap B)^{N\cap B} = \mathrm{Tp}(\eta \cap N)^N \quad (3.5)$$

So, from (3.3), (3.4) and (3.5) it follows, the statement of the lemma. ∎

PROOF OF LEMMA 3.1 It is enough to prove, that $\mathrm{Tp}_{G_{c,x}}(\eta \cap N) = \mathrm{Tp}_{G_c}(\eta)|_\xi$. Because of the remarks made after stating the Incidence Property, the statement of our lemma is meaningful. That is, $\mathrm{Tp}_{G_{c,x}}(\eta \cap N)$ and $\mathrm{Tp}_{G_c}(\eta)|_\xi$ live in the same space. We have the following commutative diagram

$$\begin{tikzcd}
H^*_{G_c}(V) \arrow[r, "(B\iota_{G_{c,x}})^*"'] \arrow[rrr, bend left, "\cdot|_\xi"] & H^*_{G_{c,x}}(V) \arrow[r, "\cong", "(\iota_N)^*"'] & H^*_{G_{c,x}}(N) \arrow[r, phantom, "\cong"] & H^*_{G_c}(\xi) \\
\Cup & \Cup & \Cup & \Cup \\
\mathrm{Tp}_{G_c}(\eta) \arrow[r, mapsto] \arrow[rrr, bend right, mapsto] & \mathrm{Tp}_{G_{c,x}}(\eta) \arrow[r, mapsto] & \mathrm{Tp}_{G_{c,x}}(\eta)|_N \arrow[r, phantom, "="] & \mathrm{Tp}_{G_c}(\eta)|_\xi
\end{tikzcd}$$

That is, it is enough to prove, that $\mathrm{Tp}_{G_{c,x}}(\eta)|_N = \mathrm{Tp}_{G_{c,x}}(\eta \cap N)$, which is the statement of Lemma 3.3 ∎

Chapter 4

Multidegree

Here we present the algebraic invariant called multidegree and its connections to Thom-polynomials. We do this, because it will yield important restrictions on the format of Thom-polynomials. First let us examine what we can say about the action of the subgroups of G.

If $F < G$, then the inclusion $\iota : F \hookrightarrow G$ induces a map $\iota^* : H^*_G(X) \to H^*_F(X)$, for any G-space X. Moreover ι^* is functorial and $\iota^*(\mathrm{Tp}_G(\eta)) = \mathrm{Tp}_F(\eta)$. If F is a maximal compact subgroup of G, then ι^* is just an isomorphism. If F is a maximal torus T^r of G, then ι^* is injective, therefore $\mathrm{Tp}_G(V)$ can be determined regarding just the T^r action. This fact is vital, since a T^r action is very special, which gives us a possibility to translate topology to algebra and combinatorics.

If T^r acts on V, then there is a decomposition of V into $V_1 \oplus \cdots \oplus V_n$, such that every V_i is a one dimensional invariant subspace of V. Then T^r acts on an arbitrary $x_i \in V_i$ with the formula

$$(\alpha_1, \ldots, \alpha_r) \cdot x_i = \alpha_1^{w_{i1}} \alpha_2^{w_{i2}} \ldots \alpha_r^{w_{ir}} x_i$$

We call the vector $(w_{i1}, \ldots, w_{ir})$ the i-th *weight vector* of our representation. Actually these weight vectors contain every information about Thom-polynomials. Every Thom-polynomial can be computed from them, which follows from the following discussion. We denote by $\underline{w}$ the vector $(w_1, \ldots, w_n)$ of weight vectors.

Consider the dual base $\epsilon_1, \ldots, \epsilon_n$ of V^*, that is, a base, such that $\epsilon_i(V_j) = 0$ for every $i \neq j$. Then T^r acts on the symmetric algebra $\mathrm{Sym}(V^*)$ by

$$(\alpha_1, \ldots, \alpha_r) \cdot \prod_{i=1}^{n} \epsilon_i^{m_i} = \prod_{j=1}^{r} \alpha_j^{-\sum_{i=1}^{n} m_i w_{ij}} \prod_{i=1}^{n} \epsilon_i^{m_i}$$

A polynomial ring, with such an action is called a $\mathbb{Z}^r$-graded ring. Precisely, a *$\mathbb{Z}^r$-graded polynomial ring* is a polynomial ring $S = \mathbb{C}[x_1, \dots, x_n]$ and a semigroup homomorphism

$$\deg : \{\prod_{i=1}^{n} x_i^{m_i} | m_i \in \mathbb{N}\} \to \mathbb{Z}^r \tag{4.1}$$

from the multiplicative semigroup of its monomials to the r times direct power of the additive group of integers. The similarity between the name of this homomorphism and the word degree is not an inadvertent coincidence. In the case of $r = 1$, (4.1) defines the notion of weighted degree. Define the homogeneous part S_a of degree a in S by $\deg^{-1}(a)$ for any $a \in \mathbb{Z}^r$. Then M is a *$\mathbb{Z}^r$-graded S-module* , if M is an S-module, $M = \bigoplus_{a \in \mathbb{Z}^r} M_a$ as a vector space, and $S_a M_b \subseteq M_{a+b}$ for every $a, b \in \mathbb{Z}^r$.

We define *multidegree* in this context. It is a function $\mathcal{C}$ from the set of $\mathbb{Z}^r$-graded modules to $\mathbb{Z}[e_1, \dots, e_r]$. Moreover this function is a unique function satisfying some criteria. We formulate these criteria here only in special cases. See [MS04] for the details. For formulating these criteria, first we need the notion of *monomial ordering.* It is an ordering on the monomials of S, such that $1 \leq p$ and if $p \leq r$, then $qp \leq qr$ for every monomial p, q and r of S. Clearly, a monomial ordering determines a *leading term* LT_p for every $p \in S$, the smallest term of p according to the monomial order.

Now let us see these criteria. We consider the special case when $M = S/I$ for some homogeneous ideal $I \lhd S$ (i. e. ideal generated by deg homogeneous polynomials). Then $V(I)$ is the variety defined by I, the components of which are $V_1, \dots, V_l$. Denote by I_i the vanishing ideal of V_i and by I_{in} the ideal $\{\mathrm{LT}_p | p \in I\}$. Every I_i is a prime ideal, so the localization S_{I_i} of S by I_i is meaningful. Since $I_i \supseteq I$, there is an ideal $\tilde{I}$ in S_{I_i} corresponding to I. There is also a maximal ideal m corresponding to I_i in S_{I_i}. We define mult_{I_i} in this context. It is the largest n for which there is a chain $\tilde{I} = I_1 \subsetneq I_2 \subsetneq \cdots \subsetneq I_{n-1} \subsetneq I_n = \mathrm{m}$ of ideals in S_{I_i}.

Then $\mathcal{C}(M)$ satisfies the following criteria.

(i). $\mathcal{C}(M) = \sum_{i=1}^{l} \mathrm{mult}_{I_i} \mathcal{C}(S/I_i)$

(ii). $\mathcal{C}(S/\langle x_{i_1}, \dots, x_{i_t} \rangle) = \prod_{j=1}^{t} \left(\sum_{k=1}^{r} \deg(x_{i_j})_k e_k \right)$

(iii). $\mathcal{C}(S/I) = \mathcal{C}(S/I_{in})$

If I is a monomial ideal, then $I_1, \dots, I_l$ are also monomial ideals. That is, in this case they are of form $\langle x_{i_1}, \dots, x_{i_t} \rangle$, since every monomial prime ideal has such form.

Thus, by applying to an arbitrary ideal $I \neq S$ first *(iii)*, then *(i)* and finally *(ii)*, we obtain a chain of decomposition of $\mathcal{C}(S/I)$, the result of which is that there is a non-zero $p \in \mathbb{N}[x_1, \ldots, x_n]$, for which the following holds.

$$\mathcal{C}(S/I) = p(\sum_{i=1}^{r} \deg(x_1)_i t_i, \sum_{i=1}^{r} \deg(x_2)_i t_i, \ldots, \sum_{i=1}^{r} \deg(x_n)_i t_i) \tag{4.2}$$

There is one step left. We have to find the connection between Thom-polynomials and multidegree. In our case $S = \mathrm{Sym}(V^*)$ and $\deg(x_i)_j = w_{ij}$. Denote by $W \cong \mathbb{R}^r$ the vector space where the weights live, the standard base of which is $e_1, \ldots, e_r$. Fix a homogeneous, or equivalently torus invariant, radical ideal $I \neq S$, which determines a non-empty invariant subvariety $\eta = V(I)$ of V. In [KMS] it is shown that $\mathcal{C}(S/I)$ determines the element of the *equivariant chow ring* $C^*_{T^r}$ of V which represents $V(I)$. Now we show that $C^i_{T^r}(V) \cong H^{2i}_{T^r}(V)$ naturally for $0 \leq i \leq n$, and every invariant subvariety represents the same element in both rings.

First of all, we notice, that although T^r can mean $(\mathbb{C}^*)^r$ and also $U(1)^r$, in our case the two choices are substantially the same. A $U(1)^r$ representation can be uniquely extended to a $(\mathbb{C}^*)^r$ representation, and a $(\mathbb{C}^*)^r$ representation contains a unique $U(1)^r$ subrepresentation. Even more, $\mathrm{Tp}_{(\mathbb{C}^*)^r}(\eta) \cong \mathrm{Tp}_{U(1)^r}(\eta)$ naturally, and Thom-polynomials are also the same in this situation. Since in [KMS] a $(\mathbb{C}^*)^r$ action is regarded when the foregoing statement is proven, from now on in this chapter we think of T^r as $(\mathbb{C}^*)^r$.

After these prerequisites we start the identification of $C^i_{T^r}(V)$ and $H^{2i}_{T^r}(V)$. Fix an integer $0 \leq i \leq n$. According to the definition of [EG98], $C^i_{T^r}(V) = C^i(\tau^r_n \times_\rho V)$, where τ^r_n is the tautological bundle over $(\mathbb{CP}^n)^r$. The definition of $H^{2i}_{T^r}(V)$ is quite the same, $H^{2i}_{T^r}(V) = H^{2i}(\tau^r_\infty \times_\rho V)$, where τ^r_∞ is the tautological bundle over $(\mathbb{CP}^\infty)^r$. Using ideas of Section 2.3 it can be seen, that these two definitions are even more similar. τ^r_n is the finite dimensional approximation of τ^r_∞, obtained from the standard representation of T^r on $(\mathbb{C}^{n+1})^r$. That is, the map ι^* induced on cohomology by the inclusion $\tau^r_n \times_\rho V \hookrightarrow \tau^r_\infty \times_\rho V$ is isomorphism. So, $H^{2i}_{T^r}(V)$ can be identified with $H^{2i}(\tau^r_n \times_\rho V)$ via ι^*.

Before proving anything let us also examine what element of these groups represents a T^r-invariant subvariety of V. Denote by $[Y]_C$ the element of the Chow-ring $C^*(X)$ of an algebraic variety X represented by a subvariety Y. Similarly, if an algebraic group G acts on X and Y is G-invariant, then denote by $[\eta]^G_C$ the element of $C^*_G(X)$ represented by Y (see [EG98] for the definition of the latter one).

Fix now an i complex codimensional T^r-invariant subvariety η of V. By definition

$[\eta]_C^{T^r} = [\tau_n^r \times_\rho \eta]_C$, and $\mathrm{Tp}_{T^r}(\eta) \in C^i(\tau_n^r \times_\rho V)$ is, according to Proposition 2.3.7, $[\tau_n^r \times_\rho \eta] \in H^{2i}(\tau_n^r \times_\rho V) \cong H^{2i}_{T^r}(V)$.

Summarizing, we obtained, that $C^i_{T^r}(V)$ and $H^{2i}_{T^r}(V)$ are the ordinary Chow groups and cohomology rings, respectively, of the same space (i. e. $\tau_n^r \times_\rho V$), and $[\eta]_C^{T^r}$ and $\mathrm{Tp}_{T^r}(\eta)$ are the elements of these two groups represented by the same subvariety (i. e. $\tau_n^r \times_\rho \eta$). Thus the following fact gives us the desired identification. (See [Ful98] for the details.)

Fact 4.1 *The* class homomorphism $cl : C^*(\tau_n^r \times_\rho V) \to H^{2*}(\tau_n^r \times_\rho V)$ *sending* $[X]_C$ *to* $[X]$ *for every irreducible subvariety is well-defined.*

Remark 4.2 Actually cl in [Ful98] is defined as a map to the Borel-Moore homology group $\overline{H}_{2n(r+1)-2i}(\tau_n^r \times_\rho V)$. However $\overline{H}_{2n(r+1)-2i}(\tau_n^r \times_\rho V)$ is naturally isomorphic to $H^{2i}(\tau_n^r \times_\rho V)$ via Thom-isomorphism, and moreover this identification takes the refined class η_X of X to $[X]$. This latter holds, because using the identification of $\overline{H}_{2n(r+1)-2i}(\tau_n^r \times_\rho V)$ and $H^{2i}(\tau_n^r \times_\rho V)$, η_X is the image of a generator of $H^{m-2n(r+1)+2i}(U, U \setminus X)$ via the map j of the following commutative diagram, for some m-dimensional contractible U containing $\tau_n^r \times_\rho V$.

$$\begin{tikzcd} H^{m-2n(r+1)+2i}(U, U\setminus X) \arrow[r] \arrow[rr, bend right, "j"'] & H^{m-2n(r+1)+2i}(U, U\setminus \tau_n^r \times_\rho V) \arrow[r, "\cong"] & H^{2i}(\tau_n^r \times_\rho V) \end{tikzcd}$$

This generator is chosen, so that if we leave out the singular part of X, it is mapped to the Thom-class of the normal bundle of the smooth part of X. It is not difficult to see, that this coincides with our definition of $[X]$.

Remark 4.3 The homomorphism cl gives us the needed identification, because it is actually an isomorphism. This holds, because both $C^*(\tau_n^r \times_\rho V)$ and $H^{2*}(\tau_n^r \times_\rho V)$ are isomorphic to the same factor of $\mathbb{Z}[e_1, \dots, e_r]$, and in both cases x_i is represented by the same subvariety $\pi^{-1}((\mathbb{CP}^n)^{i-1} \times \mathbb{CP}^{n-1} \times (\mathbb{CP}^n)^{r-i})$, where π is the bundle projection $\tau_n^r \times_\rho V \to (\mathbb{CP}^n)^r$.

So, we obtained, that multidegree and torus invariant Thom-polynomials actually coincide. By (4.2) we obtain the following proposition.

Proposition 4.4 *There is a non-zero* $p \in \mathbb{N}[x_1, \dots, x_n]$*, for which*

$$\mathrm{Tp}_{T^r}(\eta) = p\left(\sum_{i=1}^r w_{1i} e_i, \sum_{i=1}^r w_{2i} e_i, \dots, \sum_{i=1}^r w_{ni} e_i\right)$$

using the natural identification of $\mathrm{Sym}W$ *and* $H^*_{T^r}(V)$.

Chapter 5

Positive action

In this chapter we investigate, that at what criterion can the non-existence of zero Thom-polynomial be proven. We do this in order to use it later, together with Lemma 3.1, to prove the Main Theorem and as a consequence the Incidence Property for certain types of representations.

We fix also some notations here. From now on r denotes the rank of the maximal torus of G and W is the real r-dimensional vector space with base $e_1, \dots, e_r$ in which the weights live. B is the set of weights $\{w_1, \dots, w_n\}$.

The center point of the chapter is Theorem 5.2.1. There we prove the following characterization.

Main Result of the Chapter *There is no non-empty T^r invariant subvariety η of V for which $\mathrm{Tp}_{T^r}(\eta) = 0$ if and only if 0 is not contained in the convex hull of the weights.*

The condition, that 0 is not in the convex hull of the weights means, that every weight is contained in an open half space of W centered at 0. This can be imagined as somehow all the coordinates of T^r act in the same direction, that is, the action is in a sense *positive*. This motivates the name of the chapter.

The chapter is organized as follows. In Section 5.1 we make some preparations to prove in Section 5.2 a characterization of the non-existence of zero Thom-polynomials. Then in Section 5.3 we prove the statement which we will use directly in applications. It characterizes our condition when weights are of a special form.

5.1 Preparation

First we need some preparations about scalar products of tensor powers. Fix a scalar product, $\langle .,. \rangle$ on W. Then them map

$$\begin{array}{ccc} W^{2k} & \rightarrow & \mathbb{R} \\ (x_1,\ldots,x_k,y_1,\ldots,y_k) & \mapsto & \prod_{i=1}^{k} \langle x_i, y_i \rangle \end{array}$$

is a $2k$-linear map. Thus, by the category theoretic definition of tensor product, it induces a bilinear map

$$\begin{array}{cccc} \langle .,. \rangle_{\otimes}: & W^{\otimes k} \times W^{\otimes k} & \rightarrow & \mathbb{R} \\ & (\otimes_{i=1}^{k} x_i, \otimes_{i=1}^{k} y_i) & \mapsto & \prod_{i=1}^{k} \langle x_i, y_i \rangle \end{array}$$

Since $\langle .,. \rangle$ is a scalar product, there is a base U of W, such that $\langle x, y \rangle \geq 0$ and $\langle x, x \rangle > 0$ for any $x, y \in U$. Then $U^{\otimes k}$ is a base in $W^{\otimes k}$, and $\langle \tilde{x}, \tilde{y} \rangle_{\otimes} \geq 0$, $\langle \tilde{x}, \tilde{x} \rangle_{\otimes} > 0$ for any $\tilde{x}, \tilde{y} \in U^{\otimes k}$. Therefore, $\langle .,. \rangle_{\otimes}$ is a scalar product, which we denote also by $\langle .,. \rangle$ form now on.

We need a statement which states, that if some vectors are in an open half space in W, then their k-th symmetric powers are also in an open half space. Actually, this is fairly easy to prove in the k-th tensor product, so we use the following, well known inclusion.

$$\begin{array}{cccc} \iota: & \mathrm{Sym}^k V & \rightarrow & \left(V^{\otimes K}\right)^{S_k} \\ & \prod_{i=1}^{k} v_i & \mapsto & \sum_{\pi \in S_k} \otimes_{i=1}^{n} v_{\pi(i)} \end{array}$$

and we prove the desired statement in $W^{\otimes k}$.

Proposition 5.1.1 *If for an arbitrary $a \in W$, and the set of weights $B \subset W$, $\langle a, B \rangle > 0$ is fulfilled, then $\left\langle \iota\left(B^k\right), \iota\left(a^k\right)\right\rangle > 0$ also holds.*

Proof We are supposed to show, that if $x_1, \ldots, x_k \in B$, then

$$\left\langle \iota\left(\prod_{i=1}^{k} x_i\right), \iota\left(a^k\right) \right\rangle > 0$$

This holds, because

$$\left\langle \iota\left(\prod_{i=1}^{k} x_i\right), \iota\left(a^k\right)\right\rangle = \left\langle \sum_{\pi\in S_k} \otimes_{i=1}^{k} x_{\pi(i)}, \sum_{\pi\in S_k} a^{\otimes k}\right\rangle = k!\left\langle \sum_{\pi\in S_k} \otimes_{i=1}^{k} x_{\pi(i)}, a^{\otimes k}\right\rangle =$$
$$= k!\sum_{\pi\in S_k}\left\langle \otimes_{i=1}^{k} x_{\pi(i)}, a^{\otimes k}\right\rangle = k!\sum_{\pi\in S_k}\left(\prod_{i=1}^{k}\left\langle x_{\pi(i)}, a\right\rangle\right) > 0$$

since $\langle x_{\pi(i)}, a\rangle > 0$ for every i and π, by the assumption of the proposition. ■

5.2 Positive action and Thom-polynomials

Now, we can state, our characterization. It is important, that we work with the T^r action. Our proof does not work for arbitrary group actions. However, this is enough for our purpose, since $H^*_G(V)$ is embedded naturally into $H^*_{T^r}(V)$. (See Chapter 4.) It also has to be mentioned, that Theorem 5.2.1 yields three more equivalent assertions apart from the two given as main result in the introduction of the chapter. Although statement (ii) is mainly a technical statement, statements (iv) and (v) give interesting characterizations. Statement (v) will also be used later on in Section 6.4 and also strengthens the intuition that actions for which $0 \notin \langle B\rangle_{conv}$ should be called positive. It implies, that the action is positive if and only if it determines a $U(1)$ action with only positive weights on V.

Theorem 5.2.1 *If B is the set of weights $\{w_1, \ldots, w_n\}$, then the following are equivalent.*

(i). There is a non-empty T^r-invariant subvariety $\eta \subset V$, such that $\mathrm{Tp}(\eta) = 0$.

(ii). There is a $k \in \mathbb{Z}^+$, such that $0 \in \langle B^k\rangle_{conv}$.

(iii). $0 \in \langle B\rangle_{conv}$.

(iv). $\left(\mathrm{Sym}(V^*)^{T^r} \setminus \mathrm{Sym}^0(V^*)\right) \neq \emptyset$, *that is, there is a non-constant, invariant polynomial.*

(v). There is no homomorphism $U(1) \to T^r$, such that through this homomorphism $U(1)$ acts with only positive weights on V.

Proof *(i) ⇒ (ii)* We have a non- subvariety $\eta \subset V$, such that $\mathrm{Tp}(\eta) = 0$. By Proposition 4.4 there is a non-zero polynomial $p \in \mathbb{N}[x_1, \ldots, x_n]$, for which $p(\underline{w}) = Tp(\eta)$. Since $w_i \in \mathrm{Sym}^1 W$, $p_i(\underline{w})$ is homogeneous of degree i, for every i. Thus,

$p_i(\underline{w}) = 0$ for all i. As a special case $p_0 = p_0(\underline{w}) = 0$, and since $p \neq 0$, there is a $k \in \mathbb{Z}^+$, such that $p_k \neq 0$, but $p_k(\underline{w}) = 0$. Therefore, if we write p_k in the form

$$p_k(x_1, \ldots, x_n) = \sum_{I \in \mathrm{Part}(n,k)} a_I x^I$$

we obtain that $a_I \in \mathbb{N}$ for every $I \in \mathrm{Part}(n,k)$, not every $a_I = 0$, and

$$0 = \sum_{I \in \mathrm{Part}(n,k)} a_I w^I$$

Thus, $\sum\limits_{I \in \mathrm{Part}(n,k)} a_I \neq 0$, and consequently

$$0 = \sum_{I \in \mathrm{Part}(n,k)} \frac{a_I}{\sum\limits_{I \in \mathrm{Part}(n,k)} a_I} w^I$$

That is, $0 \in \langle w^I | I \in \mathrm{Part}(n,k) \rangle_{conv} = \langle B^k \rangle_{conv}$.

(ii) $\Rightarrow$ *(iii)* Suppose, that for a fixed $k \in \mathbb{Z}^+$, $0 \in \langle B^k \rangle_{conv}$, but $0 \notin \langle B \rangle_{conv}$. Then there is an $a \in W$, such that $\langle a, B \rangle > 0$. By Proposition 5.1.1 this means, that $\langle \iota\left(B^k\right), \iota\left(a^k\right) \rangle > 0$. That is, $0 \notin \langle \iota\left(B^k\right) \rangle_{conv}$, and as far as ι is an embedding, this implies that $0 \notin \langle B^k \rangle_{conv}$, which contradicts our assumption.

(iii) $\Rightarrow$ *(iv)* Since $0 \in \langle B \rangle_{conv}$, 0 can be written in the form $\sum\limits_{i=1}^{n} a_i w_i = 0$, where $a_i \geq 0$ and $\sum\limits_{i=1}^{n} a_i = 1$. The coordinates of w_i are integers, thus it can be supposed, that $a_i \in \mathbb{Q}$. Therefore, if we do not require $\sum\limits_{i=1}^{n} a_i$ to be 1 only, that there is an $a_i \neq 0$, then $a_i \in \mathbb{N}$ can be supposed. Then $\prod\limits_{i=1}^{n} x_i^{a_i}$ is constant under T^r action, since

$$(\alpha_1, \ldots, \alpha_r) \cdot \left(\prod_{i=1}^{n} x_i^{a_i}\right) = \prod_{i=1}^{n} \left(\left(\prod_{j=1}^{r} \alpha_j^{-w_{ij}}\right) x_i\right)^{a_i} =$$
$$= \prod_{j=1}^{r} \left(\alpha_j^{-\sum_{i=1}^{n} w_{ij} a_i}\right) \prod_{i=1}^{n} x_i^{a_i} = \prod_{i=1}^{n} x_i^{a_i}$$

This completes our proof, since we have found a non-constant, invariant polynomial.

(iv) $\Rightarrow$ *(i)* There is a $p \in \left((\mathrm{Sym}(V))^{T^r} \setminus \mathrm{Sym}^0 V\right)$. $p_0 \neq 0$ can be supposed, since if p is invariant under T^r-action, then $p + c$ is also invariant, for any $c \in \mathbb{Z}$. Let η be $\{v \in V | p(v) = 0\}$. Then $\eta \neq \emptyset$, since V is a complex vector space.

Now suppose, that $\mathrm{Tp}(\eta) \neq 0$. Then $\mathrm{Tp}(\eta)|_{\{0\}} = \mathrm{Tp}(\eta)$, since the embedding of $\{0\}$ into V is a homotopy equivalence. So, $\mathrm{Tp}(\eta)|_{\{0\}} \neq 0$. Thus, $0 \in \overline{\eta}$ according to Corollary 2.5.2, and as far as p is continuous $p(0) \in \overline{p(\eta)}$. But $p(\eta) = \{0\}$, thus $\overline{p(\eta)} = \{0\}$ also, however $p(0) = p_0 \neq 0$. This is a contradiction. Therefore, η is an invariant subset of V, for which $\mathrm{Tp}(\eta) = 0$.

(i) ⇒ (v) We prove that if there is a homomorphism $\phi : U(1) \to T^r$, such that through this homomorphism $U(1)$ acts with only positive weights on V, then $\mathrm{Tp}_{T^r}(\eta) \neq 0$ for every non-empty invariant subvariety η. Fix such a ϕ and η. In the following paragraph $U(1)$-equivariant cohomology and $U(1)$ Thom-polynomials will be meant using the action of $U(1)$ through ϕ.

ϕ induces a homomorphism $H_{T_r}(V) \to H_{U(1)}(V)$, which takes $\mathrm{Tp}_{T^r}(\eta)$ to $\mathrm{Tp}_{U(1)}(\eta)$. So, it is enough to prove that $\mathrm{Tp}_{U(1)}(V) \neq 0$. However the equivalence of the first four statements is already proven, which can be applied to the $U(1)$ action. So it is enough to prove, that the convex hull of the weights of $U(1)$ on V does not contain zero, which holds as a trivial consequence of that $U(1)$ acts with only positive weights on V.

(v) ⇒ (iii) We prove that if $0 \notin \langle B \rangle_{conv}$, then there is a homomorphism $\phi : U(1) \to T^r$, such that through this homomorphism a $U(1)$ acts with only positive weights on V. If $0 \notin \langle B \rangle_{conv}$, then the weights of T^r are all on one side of a hyperplane. The normal vectors of such hyperplanes form an open set in W, so there is a hyperplane H, the equation of which is rational:

$$\sum_{i=1}^{r} a_i e_i = 0 \qquad (a_i \in \mathbb{Q}) \tag{5.1}$$

By multiplying the equation with an integer, it can also be supposed, that all the a_i are in $\mathbb{Z}$, and the weights are on the positive side of H. Consider the $U(1)$ symmetry $\sum_{i=1}^{r} a_i e_i$, that is, the following homomorphism.

$$\begin{array}{rccc} \phi: & U(1) = \{\alpha \in \mathbb{C} \mid |\alpha| = 1\ \} & \to & T^r = \{(\alpha_1, \alpha_2, \ldots, \alpha_r) \in \mathbb{C}^r \mid |\alpha_i| = 1\ \} \\ & \alpha & \mapsto & (\alpha_1^{a_1}, \alpha_2^{a_2}, \ldots, \alpha_r^{a_r}) \end{array}$$

$U(1)$ acts through ϕ on x_j with weight $\sum_{i=1}^{r} a_i w_{ji}$. However this is positive, since it is the substitution of the j-th weight vector into (5.1), which is greater than zero, by the fact that all the weights are on the positive side of H. So, we have found a homomorphism ϕ with the desired property. ∎

5.3 Positive action for special weights

There are many situations where the weights are of form $e_i - e_j$. In fact, as it can be seen from Section 6.1, this always happens, when the product of two subgroups X and Y of $\mathrm{GL}(V)$ acts on V with the formula $(x, y) \cdot v = yvx^{-1}$. We will give more examples on situations where such weights occur, but first here we characterize in such situations the existence of zero Thom-polynomials.

Since now it is easier to determine a weight by giving the two coordinates where it is 1 and -1, we introduce $i_s, j_s \in [r]$, for which $w_s = e_{i_s} - e_{j_s}$. We will use this notation also in Chapter 6.

Proposition 5.3.1 *There is no non-empty T^r invariant subvariety η of V for which $\mathrm{Tp}(\eta) = 0$ if and only if there is an ordering " $>$ " on $[r]$, such that i_s" $>$ "j_s for every $s \in [n]$.*

Proof $\Leftarrow$ We are given a non-empty invariant subvariety η of V. We should prove that $\mathrm{Tp}(\eta) \neq 0$. It can be supposed, by permuting the coordinates of W, that the ordering " $>$ " on $[r]$ is the standard ordering. That is, $i_s > j_s$ for every $s \in [n]$.

Because of Proposition 4.4 $\mathrm{Tp}(\eta) = p(\underline{w})$ for a $0 \neq p \in \mathbb{N}[x_1, \dots, x_n]$. Introducing new variables $y_i = e_{i+1} - e_i$ for every $i \in [r-1]$, we obtain that

$$\mathrm{Tp}(\eta) = p\left(\sum_{s=j_1}^{i_1-1} y_s, \dots, \sum_{s=j_n}^{i_n-1} y_s\right) \tag{5.2}$$

However $\mathrm{Tp}(\eta)$ is an element of $\mathbb{Z}[e_1, \dots, e_r]$, according to (5.2) it can be also viewed as an element of $\mathbb{Z}[y_1, \dots, y_{r-1}]$. It is non-zero in $\mathbb{Z}[y_1, \dots, y_{r-1}]$, since we obtained it by substituting non-zero homogeneous polynomials with non-negative coefficients, sums of y_i's, into p. Thus, it is also non-zero in $\mathbb{Z}[e_1, \dots, e_r]$, since the substitution $y_i = e_{i+1} - e_i$ gives an inclusion of $\mathbb{Z}[y_1, \dots, y_{r-1}]$ into $\mathbb{Z}[e_1, \dots, e_r]$. So, $\mathrm{Tp}(\eta) \neq 0$, which is exactly what we were supposed to prove.

$\Rightarrow$ Let F be the oriented graph with vertices $V(F) = [r]$ and edges $E(F) = \{(i_s, j_s) | s \in [n]\}$. Suppose that there is no ordering " $>$ " on $[r]$. Then there is an oriented cycle in F, with vertices $v_1, \dots, v_l$ in the order of orientation. But then,

$$\sum_{i=1}^{l} \left(e_{v_i} - e_{v_{(i+1 \mod l)}}\right) = 0$$

That is, $0 \in \langle B \rangle_{conv}$, from which follows according to Theorem 5.2.1, that there is a non-empty T^r invariant subvariety η of V, for which $\mathrm{Tp}(\eta) = 0$. This is a contradiction which proves this direction of the proposition. ■

Chapter 6

Applications

Finally, we can glue together the results proven in earlier chapters. First we prove the Main Theorem, stated in Chapter 1. Until this we have not defined precisely what are the weights of a representation, however they were used in the statement of the theorem. They are the weights of the induced representation of an arbitrary maximal torus. This notion is well defined, since maximal tori are conjugate to each other.

Proof of Main Theorem Fix two orbits η and ξ of the representation ρ, for which $\overline{\eta} \supseteq \xi$. By Corollary 2.5.2 we are supposed to prove only, that $\mathrm{Tp}(\eta)|_\xi \neq 0$.

By Lemma 3.1 $\mathrm{Tp}_G(\eta)|_\xi = \mathrm{Tp}_{G_{c,x}}(\eta \cap N)$ for any $x \in \xi$ and an adequate normal space N of ξ centered at x. So, according to Theorem 5.2.1, if 0 is not in the convex hull of the weights of the induced representation $G_{c,x} \to \mathrm{GL}(N_x)$, then $\mathrm{Tp}_G(\eta)|_\xi \neq 0$. In order to finish the proof we have to prove only, that the weights of the representation of $G_{c,x}$ and G_x on N are the same. However, this holds by the fact that $G_{c,x}$ is a maximal compact subgroup of G_x and thus contains a maximal torus of G_x. ■

Specially, if every weight w of the representation of a stabilizer torus T^r on N is of form $e_{i_w} - e_{j_w}$, the criteria of the theorem can always be proven, by giving an ordering on $[r]$, such that for every weight w, i_w" > "j_w. This method will be used in Section 6.1, Section 6.2 and Section 6.3. For arbitrary weights the criteria can also be proven using the fifth statement of Theorem 5.2.1, by showing a homomorphism $U(1) \to G_x$ such that the induced representation of $U(1)$ has only positive weights. This will be the approach of Section 6.4.

There is only one thing left, the choice of N. We have to choose an N suitable for determining the weights of a maximal stabilizer torus. Actually, the weights are the same for every normal space of ξ at x, since all are equal to the weights of the action of T^r on $V/T_x\xi$. Thus, any normal space invariant under T^r action will be eligible. Luckily in our examples, every representation will have a standard scalar product invariant maximal torus. So, a linear subspace centered at x, perpendicular to ξ will be a satisfactory choice. Now, in the next sections, we use this technique, to prove the Incidence Property or its special case for some classes of representations.

6.1 The case of Giambelli-Thom-Porteous formula

First we examine the representation discussed in the Giambelli-Thom-Porteous formula. See [Tho56] and [FR04] for the details. In this case the statement of Incidence Property is not hard to prove. We present our proof here, just to show our technique working in an easy case.

We use the notations of [FR04]. We have an action of $\mathrm{GL}(n) \times \mathrm{GL}(n+k)$ on $\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^{n+k}) \cong \{(m_{ij})_{i=1,j=1}^{n+k,n}\}$, by $(R, L) \cdot X = LXR^{-1}$ for any $R \in \mathrm{GL}(n)$, $L \in \mathrm{GL}(n+k)$ and $X \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^{n+k})$. Σ_s is the orbit of

$$X_s = \begin{pmatrix} 0 & 0 \\ 0 & I_{n-s} \end{pmatrix}$$

and there are no other orbits. The maximal compact stabilizer subgroup of X_s is

$$G_s = \left\{ \left(\begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix}, \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} \right) \middle| (A, B, C) \in U(s) \times U(s+k) \times U(n-s) \right\}$$

$(N_{\Sigma_s})_{X_s}$ can be identified with the G_S invariant subspace

$$N_s = \left\{ \begin{pmatrix} M & 0 \\ 0 & I_{n-s} \end{pmatrix} \middle| M \in \mathrm{Hom}(\mathbb{C}^s, \mathbb{C}^{s+k}) \right\}$$

on which the natural action of G_s coincides with the action of the representation $\rho_{X_s}|_{G_s}$. The maximal torus of G_s is

$$\begin{aligned} T_s \quad = \quad \{ \quad & (\mathrm{diag}(a_1, \dots, a_s, c_1, \dots, c_{n-s}), \\ & \mathrm{diag}(b_1, \dots, b_{s+k}, c_1, \dots, c_{n-s})) \quad | \quad a_i, b_i, c_i \in U(1)\} \end{aligned}$$

Thus, if we put the coordinates of T_s in order $a_1, \dots, a_s, b_1, \dots, b_{s+k}, c_1, \dots, c_{n-s}$, then the weight of m_{ij} is $e_{s+i} - e_j$, that is the normal ordering of $[n+s+k]$ proves here the Incidence Property by Proposition 5.3.1, and the fact, that $j \leq s$.

6.2 Schubert-varieties of flag manifolds

Now we show the Incidence Property to the representation discussed in [FR03] and [KM] the full rank orbits of which correspond to the Schubert varieties of the n-dimensional Flag-manifold. Here we follow the notation of [FR03].

We are acting with $B^+ \times B^-$ on $\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n) \cong \{(m_{ij})_{i,j=1}^n\}$, where B^+ and B^- are the groups of $n \times n$ upper and lower triangular matrices, respectively, and $(R, L) \cdot M = LMR^{-1}$ for $R \in B^+$, $L \in B^-$ and $M \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n)$.

Then the orbits correspond to 0-1 matrices, every column and row of which contains at most one 1. There is a method of expanding such a matrix uniquely to the right and down to an $m \times m$ permutation matrix for any m large enough. This expansion is obtained by adding 1's to rows not containing 1's starting from the top, putting the 1 in the left-most position, where it is possible, outside of the upper-left $n \times n$ matrix. This way we can associate a $\pi \in S_m$ to an orbit, by setting $\pi(i) = j$, if there is a 1 in the (i, j) position of the permutation matrix obtained. If $\pi' \in S_{m'}$ is another permutation associated to the same orbit, and $m' \geq m$, then $\pi = \pi'|_{[m]}$.

For any m, $\pi \in S_m$, if possible, determines an orbit, uniquely, so from now on we denote orbits by permutations. The ambiguity of which permutation we choose will not effect the definitions and notions we use.

The maximal torus of the stabilizer group of an orbit π is

$$G_\pi = \{(\mathrm{diag}(\alpha_1, \ldots, \alpha_n), \mathrm{diag}(\alpha_{\pi(1)}, \ldots, \alpha_{\pi(n)}) | \alpha_i \in U(1) \ (i \in [n] \cup \pi([n]))\}$$

and the fiber of the normal bundle above M_π, the permutation matrix corresponding to π, can be identified with the subvariety

$$\{(m_{ij})_{i,j=1}^n | \ m_{ij} = 0 \text{ if } j \geq \pi(i), \text{ or } i \geq \pi^{-1}(j) \ \}$$

of $\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n)$ invariant under G_π action. Thus, the weight of m_{ij} is $e_{\pi(i)} - e_j$. That is, with the standard ordering of $[m]$ these weights fulfill the conditions of Proposition 5.3.1, since if m_{ij} is a coordinate of N_π, then $j < \pi(i)$.

6.3 Quiver representations

Here we show that the Incidence Property holds also for quiver representations. These are representations, the orbits of which correspond to isomorphia classes of algebra representations, and can be regarded also as some kind of generalizations of the representations discussed in Section 6.1. See [FR02] for more details. Here we introduce only the most necessary notions.

Consider an oriented graph Q of, so called, Dynkin type. Denote by Q_0 the set of its vertices and by Q_1 the set of its edges. If e is an edge of Q, then let e' and e'' denote the starting and the ending points of e, respectively.

If there is a $d : Q_0 \to \mathbb{N}$ function given we have a group $G = \bigoplus_{i \in Q_0} \mathrm{GL}(d(i))$ acting on $V = \bigoplus_{e \in Q_1} \mathrm{Hom}(\mathbb{C}^{d(e')}, \mathbb{C}^{d(e'')})$, by the formula

$$\left(\bigoplus_{i \in Q_0} A_i\right) \cdot \left(\bigoplus_{e \in Q_1} \phi_e\right) = \bigoplus_{e \in Q_1} \left(A_{e''} \phi_e A_{e'}^{-1}\right)$$

This is the representation we investigate here. The orbits correspond to the representations of the $\mathbb{C}Q$ algebra, with dimension vector d. Where the $\mathbb{C}Q$ algebra is the algebra, the (vector space) base of which consists of the oriented paths of Q (also the zero length ones), and if p and q are such paths then $pq = 0$ if the endpoint of p is not the starting point of q, otherwise pq is the concatenation of p and q. The homomorphisms ϕ_e $(e \in Q_1)$ give how the one long paths of $\mathbb{C}Q$ acts on V, and this determines the action of the whole $\mathbb{C}Q$.

We investigate only graphs of Dynkin type, since this is the case when there are only finitely many non-isomorphic indecomposable modules of $\mathbb{C}Q$. Denote them by l_r $(r \in R(Q))$ Do not mind what $R(Q)$ means, we handle it as a collection of indices of indecomposables. That is, every module M, or with other words representation, or orbit, has a unique form of $\sum_{r \in R(Q)} \nu_r l_r$ for some integer numbers ν_r by the Krull-Schmidt theorem. Then the maximal compact stabilizer subgroup of M is $G_M = \bigoplus_{r \in R(Q)} U(\nu_r)$. A normal slice is $N_M = \bigoplus_{r,s \in R(Q)} \mathrm{Hom}(\mathbb{C}^{\mu_r}, \mathbb{C}^{\mu_s})^{m_{rs}}$, where $m_{rs} = \dim \mathrm{Ext}_{\mathbb{C}Q}(l_r, l_s)$, and G_M acts on N_M with the rule

$$\left(\bigoplus_{r \in R(Q)} A_r\right) \cdot \left(\bigoplus_{r,s \in R(Q)} \bigoplus_{i=1}^{m_{rs}} \phi_{rs}^i\right) = \left(\bigoplus_{r,s \in R(Q)} \bigoplus_{i=1}^{m_{rs}} A_s \phi_{rs}^i A_r^{-1}\right)$$

(Here i is an index, not the sign of the i-th power.)

The maximal torus in $U(\mu_r)$ is $\mathrm{diag}(\alpha_{r,1}, \ldots, \alpha_{r,\mu_r})$. Therefore, the weights of the representation of G_M on $\mathrm{Hom}(\mathbb{C}^{\mu_r}, \mathbb{C}^{\mu_s})$ are $e_{s,i} - e_{r,j}$ for any $1 \le i \le \mu_s$ and $1 \le j \le \mu_r$. We prove that there is an ordering $\succ$ on $R(Q)$, such that if $m_{rs} \ne 0$, then $r \succ s$. Then define " $>$ " to be any ordering for which (s,i)" $>$ "(r,j), if $r \succ s$, $1 \le i \le \mu_s$ and $1 \le j \le \mu_r$. With such ordering the weights satisfy the conditions of

Proposition 5.3.1, since the presence of a weight $e_{s,i}-e_{r,j}$ imply, that $(s,i)" > "(r,j)$. So, we only have to prove what we have stated about $\succ$.

There is a partial self mapping map of $\{l_r | r \in R(Q)\}$ called *Auslander-Reiten translate*, denoted by τ, with the following properties.

(i). For every $r \in R(Q)$, there is a unique $n_r \in \mathbb{N}$, such that $\tau^{n_r} l_r$ is projective.

(ii). $\mathrm{Ext}_{\mathbb{C}Q}(l_r, l_s) = \mathrm{Ext}_{\mathbb{C}Q}(\tau l_r, \tau l_s)$ for every $r, s \in R(Q)$, where both τl_r and τl_s are defined.

Let $\succ$ be any ordering on $R(Q)$, for which if $n_r > n_s$, then $r \succ s$. Choose an r and an s, such that $m_{rs} = \dim \mathrm{Ext}_{\mathbb{C}Q}(l_r, l_s) \neq 0$. Suppose that $r \not\succ s$. Then $n_r \leq n_s$, and consequently $m_{rs} = \dim \mathrm{Ext}_{\mathbb{C}Q}(l_r, l_s) = \dim \mathrm{Ext}_{\mathbb{C}Q}(\tau^{n_r} l_r, \tau^{n_r} l_s) = 0$, since $\tau^{n_r} l_r$ is projective. This yields a contradiction which proves our statement about $\succ$.

6.4 Singularities

As it was mentioned in Chapter 1, the Incidence Property was asked originally for singularity classes of smooth maps. Here we deal with representations, the orbits of which correspond to such classes. These representations are parameterized by an integer number k. Actually we do not prove the Incidence Property for any k. Instead we give a criterion on ξ, which guaranties that for every η holds (3.1), that is

$$\mathrm{Tp}(\eta)|\xi \neq 0 \iff \overline{\eta} \supseteq \xi$$

We do not try to give an introduction to Singularity Theory. We only mention the most necessary facts. We refer to [Rim01] and [AGZV88] for more details.

First we summarize, what representations we are dealing with. A group G acts on M, where

- $M := \varepsilon(\infty, \infty + k) := \varinjlim \varepsilon(n, n+k)$, where
  - $\varepsilon(n, n+k) := \mathrm{Hol}((\mathbb{C}^n, 0), (\mathbb{C}^{n+k}, 0))$ is the smooth complex function germs from $\mathbb{C}^n$ to $\mathbb{C}^{n+k}$.
  - the direct limit is taken according to the unfolding maps

$$\begin{array}{rccc} u_a : & \varepsilon(n, n+k) & \to & \varepsilon(n+a, n+k+a) \\ & f & \mapsto & f \oplus \mathrm{id}_{\mathbb{C}^a} \end{array}$$

- $G := \mathcal{K}(\infty, \infty + k) := \lim\limits_{\to} \mathcal{K}(n, n + k)$, where
  - $\mathcal{K}(n, n + k) = \{(\phi, M) | \phi \in \mathrm{BiHol}(\mathbb{C}^n, 0), M$ is a germ$(\mathbb{C}^n, 0) \to \mathrm{BiHol}(\mathbb{C}^{n+k}, 0)\}$ is the contact group, where $\mathrm{BiHol}(\mathbb{C}^l, 0)$ denotes the group of biholomorphisms of $(\mathbb{C}^l, 0)$.
  - the limit is taken according to the maps:

$$\begin{array}{rccc} u_a: & \mathcal{K}(n, n+k) & \to & \mathcal{K}(n+a, n+k+a) \\ & (\phi, M) & \mapsto & (\phi \oplus \mathrm{id}_{\mathbb{C}^a}, (x, y) \mapsto M(x) \oplus \mathrm{id}_{\mathbb{C}^a}) \end{array}$$

- The action of G on M is defined as the limit of the action of $\mathcal{K}(n, n + k)$ on $\varepsilon(n, n + k)$ given by the formula

$$((\phi, M) \cdot f)(x) = (M(x) \circ f)(\phi^{-1}(x))$$

The orbits of this action are called singularities.

The space M of action in this case is in some aspects different from those of Section 6.1, Section 6.2 and Section 6.3. Although it is a vector space, that is contractible, it is not finite dimensional. So, there is a problem even with defining Thom-polynomials. However, this can be surmounted for finite codimensional orbits through a technical argument. For example, the approach of Section 2.4 can be used word by word. So, from now on we only consider finite codimensional singularities.

If ξ is a finite codimensional singularity, then every η for which $\overline{\eta} \supseteq \xi$ is also finite codimensional. Fix such orbits ξ and η. Using finite determinacy, $\mathrm{Tp}(\eta)|_\xi$ can be reduced to an incidence in the jet space, where the normal space and the stabilizer action coincides with the normal space and stabilizer action of the singularity. Furthermore the jet space is finite dimensional, so our previous results can be applied. That is, $\mathrm{Tp}(\eta)|_\xi \neq 0$ if the weights of a stabilizer torus on the normal space are contained in an open half space, or equivalently, if there is a stabilizer $U(1)$ action with positive weights.

Let us recall now, what are the normal space of ξ and its $U(1)$ symmetries. Consider a *genotype* $f : \mathbb{C}^n \to \mathbb{C}^{n+k}$, that is, a minimal dimensional element, of the orbit ξ. Denote by f_i the i-th coordinate function of f, and by e_j the function $(0, \ldots, 0, 1, 0, \ldots, 0)$ which has $n + k$ coordinates and is non-zero only in the j-th coordinate. Let I be the ideal

$$I = (\ f_i e_j\ ,\ \partial f/\partial x_l \mid i,j \in [n+k]\ ,\ l \in [n]\ )$$

and let $A = (\mathbb{C}[[x_1,\ldots,x_n]])^{n+k}/I$. If ξ is finite codimensional, then A is finite dimensional and has a basis which is the image of a set of the form $\{e_i | i \in [n+k]\} \cup B$, where $B = \{b_1,\ldots,b_s\}$ consists only of functions which are monomials in one coordinate and zero elsewhere.

The following definition defines a map called miniversal unfolding, the source space of which will be identified with the normal space of the singularity.

Definition 6.4.1 *The* miniversal unfolding *of f is the following map*

$$\begin{array}{cccc} \mu: & \mathbb{C}^n \times \mathbb{C}^s & \to & \mathbb{C}^{n+k} \times \mathbb{C}^s \\ & (x,y) & \mapsto & (f(x) + \sum_{i=1}^s y_i b_i, y) \end{array}$$

Remark 6.4.2 The miniversal unfolding could also be defined using a B which consists not only monomials. However for the sake of simplicity, we restrict here to the construction using only monomials.

Example 6.4.3

(i). For $n \geq 1$, A_n is the singularity with

$$\begin{array}{cccc} f: & \mathbb{C} & \to & \mathbb{C} \\ & x & \mapsto & x^{n+1} \end{array}$$

In that case $I = (\ (n+1)x^n\ ,\ x^{n+1}\ ) = (x^n)$ and thus $B = \{x, x^2, \ldots, x^{n-1}\}$. So μ in this case is the following function

$$\begin{array}{cccc} \mu: & \mathbb{C} \times \mathbb{C}^{n-1} & \to & \mathbb{C} \times \mathbb{C}^{n-1} \\ & (x,y) & \mapsto & (x^{n+1} + \sum_{i=1}^{n-1} y_i x^i, y) \end{array}$$

(ii). For $a \geq 2$, $I_{a,a}$ is the singularity with

$$\begin{array}{cccc} f: & \mathbb{C}^2 & \to & \mathbb{C}^2 \\ & (x,y) & \mapsto & (xy, x^a + y^a) \end{array}$$

In that case

$$\begin{aligned} I &= ((xy,0),(x^a+y^a,0),(0,xy),(0,x^a+y^a),(y,ax^{a-1}),(x,ay^{a-1})) = \\ &= ((0,x^a),(0,y^a),(0,xy),(y,ax^{a-1}),(x,ay^{a-1})) \end{aligned}$$

and thus

$$B = \{(0,x),(0,x^2),\dots,(0,x^{a-1}),(0,y),(0,y^2),\dots,(0,y^{a-1})\}$$

So, μ is the following function

$$\begin{array}{rccc} \mu: & \mathbb{C}\times\mathbb{C}\times\mathbb{C}^{a-1}\times\mathbb{C}^{a-1} & \to & \mathbb{C}\times\mathbb{C}\times\mathbb{C}^{a-1}\times\mathbb{C}^{a-1} \\ & (x,y,z,v) & \mapsto & (xy, x^a+y^a+\sum_{i=1}^{a-1} z_i x^i + \sum_{i=1}^{a-1} v_i y^i, z, v) \end{array}$$

We would like to use the fifth statement of Theorem 5.2.1 to prove the positivity of the stabilizer action on the normal space of ξ. We word the following proposition in this manner. We care about the action of only one $U(1)$.

Proposition 6.4.4 *The normal space of ξ can be identified with $\mathbb{C}^{n+s}$, thus that a stabilizer $U(1)$ is a $U(1)$ acting on $\mathbb{C}^{n+s}$, such that it has also an action on $\mathbb{C}^{n+k+s}$, for which*

$$\mu(y) = x\cdot(\mu(x^{-1}\cdot y)) \text{ for every } y\in\mathbb{C}^{n+s} \text{ and } x\in U(1) \tag{6.1}$$

So, $\mathrm{Tp}(\eta)|_\xi \neq 0$ if there is a $U(1)$ which acts both on $\mathbb{C}^{n+s}$ and $\mathbb{C}^{n+s+k}$, stabilizing μ, like in (6.1), and acting on $\mathbb{C}^{n+s}$ with only positive weights. The following discussion yields us a criteria on when such a $U(1)$ action can be chosen.

Consider a singularity, the genotype of which is a positively weighted homogeneous polynomial, that is a $p\in(\mathbb{C}[x_1,\dots,x_n])^{n+k}$ for which the following holds. There is a weight vector $(c_1,\dots,c_n)$, such that $c_i>0$ for $i\in[n]$ and for every $j\in[n+k]$ and every monomial $\prod_{i=1}^n x_i^{\alpha_i}$ of the j-th coordinate of p, $\sum_{i=1}\alpha_i c_i = d_j$, where d_j is a constant depending on j. We deal only with singularities which have positively weighted polynomial genotypes. So, after restricting to finite codimensional singularities this is a second tightening.

Then a $U(1)=\{\alpha\in\mathbb{C}||\alpha|=1\}$ acts on μ like in (6.1) with the following formulas.

$$\alpha\cdot(x,y) = (\alpha^{c_1}x_1,\dots,\alpha^{c_n}x_n,\alpha^{e_1}y_1,\dots,\alpha^{e_s}y_s) \text{ for every } (x,y)\in\mathbb{C}^n\times\mathbb{C}^s \tag{6.2}$$

$$\alpha\cdot(z,y) = (\alpha^{d_1}z_1,\dots,\alpha^{d_{n+k}}z_{n+k},\alpha^{e_1}y_1,\dots,\alpha^{e_s}y_s) \text{ for every } (z,y)\in\mathbb{C}^{n+k}\times\mathbb{C}^s$$

where e_l is $d_j - \sum_{i=1}^{n} \beta_i c_i$ in case b_l is a monomial $\prod_{i=1}^{n} x_i^{\beta_i}$ of the j-th coordinate of p.

Finally we can word the proposition which states, that for a class of singularities (3.1) holds. Although this class seems to be quite small, it contains nearly every famous singularity. More precisely, we have not found any named singularity, for which it is not satisfied. The proposition is the immediate consequence of Proposition 6.4.4 and the fifth statement of Theorem 5.2.1.

Proposition 6.4.5 *If the action of (6.2) is positive (i. e. all the $e_i > 0$), then for the singularity ξ determined by p and for every η :*

$$\mathrm{Tp}(\eta)|_{\xi} \neq 0 \iff \overline{\eta} \supseteq \xi$$

Example 6.4.6

(i). If $\xi = A_n$, then (6.2) has the form

$$\alpha \cdot (x, y) = (\alpha x, \alpha^n y_1, \alpha^{n-1} y_2, \ldots, \alpha^2 y_{n-1})$$

So, the criteria of Proposition 6.4.5 are satisfied.

(ii). If $\xi = I_{a,a}$, then (6.2) has the form

$$\alpha \cdot (x, y, z, v) = (\alpha x, \alpha y, \alpha^{a-1} z_1, \ldots, \alpha z_{a-1}, \alpha^{a-1} v_1, \ldots, \alpha v_{a-1})$$

So, the criteria of Proposition 6.4.5 in this case are also satisfied.

Appendix A

Notation

Here we present the notations used in the article. Everywhere in this appendix where they appear, n, r and k are positive integers, $\underline{y}$ is a vector $(y_1, \dots, y_n)$, for which every y_i is an element of an algebra, p is a polynomial, G and H are Lie-groups, X is a G-space, V and U are vector spaces, ρ is a $G \to \mathrm{GL}(n)$ representation, P is a G-principal bundle, A and B are smooth submanifolds of a smooth manifold C, g is a $D \to C$ smooth map, for an other smooth manifold D, f is a $G \to H$ homomorphism, Z is a set on which G acts, z is an element of Z and E and F are topological spaces.

Notation	Meaning
$\mathbb{N}$	natural numbers
$\mathbb{Z}^+$	positive integers
$[n]$	$\{1, 2, \dots, n\}$
$\mathrm{part}(r, k)$	r partitions of k, i. e. r long vectors $(i_1, \dots, i_r)$, for which $\sum_{j=1}^{r} i_j = k$ and $i_1 \leq i_2 \leq \dots \leq i_r$
p_k	k-th homogeneous part of p
$p(\underline{y})$	$p(y_1, y_2, \dots, y_n)$
y^I, where $I \in \mathrm{part}(n, k)$	$\prod_{j=1}^{n} y_j^{i_j}$
$g(A_1, \dots, A_n)$, where g is an operation of an algebraic structure A, and $A_i \subseteq A$,	$\{g(a_1, \dots, a_n) \mid a_i \in A_i\}$
pt	topological space consisting of one point
$P \times_G X = PX$	the X-bundle obtained by associating X to P with the G action

Notation	Meaning
$P \times_\rho V = PV$	the X-bundle obtained by associating X to P using the representation ρ
$A \pitchfork B$	A and B are transversal
$g \pitchfork B$	g is transversal to B
U^*	the dual vector space of U
$U^{\otimes k}$	the k-th tensor power of U
$U^{\otimes}$	the tensor algebra of U
Sym U	the symmetric algebra of U
$\mathrm{Sym}^k U$	the k-th symmetric power of U
$\langle Y \rangle_{conv}$, where $Y \subseteq U$	the convex hull of Y
$\mathrm{GL}(n)$	linear group of $\mathbb{C}^n$
$\mathrm{U}(n)$	unitary group of $\mathbb{C}^n$
S_n	the symmetric group on $[n]$
G_z	the stabilizer of z
Z^G	the fixed point of G on Z
$H^p(E \to F)$	the map induced on the p-th grade of cohomology by a certain $E \to F$ map
$H^p(E\|F)$, where $F \subseteq E$	$H^p(E, E \setminus F)$
ρ_x	the representation of G_x on the normal space of the orbit of x at x

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