On the fundamental group of posets

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ABSTRACT. We give a combinatorial definition for fundamental group on posets. We prove that the fundamental group of every idempotent subalgebra of a finite poset is trivial iff no crown is retract of an idempotent subalgebra. This implies that if a poset admits Taylor terms then the fundamental group of every idempotent subalgebra is trivial. This combinatorial homotopy group has a topological application, too: $\mathbb{Z}$ is the homomorphic image of the fundamental group of a finite simplicial complex X if and only if X has S^1 as retract.

1. INTRODUCTION

Partially ordered sets and order-primal algebras are strongly related to one another, but they were almost always investigated independently. Given a poset P, the order-primal algebra on P is an algebra with base set P and whose operations are the monotone operations on P. Order-primal algebras are good test-algebras for conjectures in universal algebra. Fix a finite structure of finite type T and consider the following so-called restricted constraint satisfaction problem (shortly CSP): Given a finite structure of the same type S is there a homomorphism from S to T? The structure theory of posets came recently into the centre of interest: It was shown in [8] and [12] that every CSP problem is polynomial-time equivalent to a poset retraction problem. Bulatov, Krokhin and Jeavons [4] conjectured that the CSP problem for a finite structure (poset) P is in polynomial-time decidable iff the algebra whose base operations are the relation preserving (monotone), idempotent functions admits Taylor-terms. In the case when this algebra does not admit any Taylor term the CSP problem is NP-complete [17]. The typical posets admitting no Taylor terms are the crowns, every monotone, idempotent operation on a crown is a projection [6].

Here, we present a few results on the algebraic properties of P: For varieties of algebras congruence distributivity is equivalent to the existence of Jónsson operations [11], congruence modularity is equivalent to the existence of Gumm operations [10]. Davey has showed in [5] that a directed poset admits Jónsson operations if and only if it admits Gumm operations. McKenzie has extended Davey's result to all finite posets [20]. Larose and Zádori [15] gave seven equivalent conditions for the existence of Jónsson and Gumm operations, for example the existence of a near unanimity function (shortly NUF) in the case of finite connected posets. They presented the first theorem of combinatorial flavour of the topic, the above conditions are equivalent to the following one: Every so-called idempotent P-subalgebra is dismantlable. The first attempt to "algebraisize" posets was in [7]: Duffus and Rival defined order-varieties as classes of posets closed under retraction and product. Later, in [13] the authors extended the definition of Duffus and Rival. They defined order varieties as a class of posets closed under retractions, product and idempotent subalgebras. They extend the results of Zádori and Larose mentioned above. Namely, it is proved that a finite connected poset P admits a NUF if and only if the two element antichain is not in the order-variety generated by P, that is every poset in this order-variety is connected.

Topological properties of posets were always examined [3]. In general every poset can be viewed as a topological space, the points are the elements of the poset, and the open sets are the ideals. The continuous maps are exactly the monotone maps of the poset. This ideal topology helps in the algebraic examination of the poset [16] [22] [23]. The other kind of topological spaces strongly related to a poset are simplicial complexes [3]. To every poset we can correspond a simplicial complex called its order complex whose simplices are the chains of the poset. And to every simplicial complex we can correspond the poset of simplices ordered by containment. Order complexes helped to prove results in graph theory [18] [1]. The most known problem of topological nature about posets is to determine posets of fixed point property [2].

We introduce a combinatorial definition of fundamental group on posets. The loops of the fundamental group are the cycles of the poset. Basically we would like for two cycles to be equivalent if they are compared. The concatenation is a well defined product on the equivalence classes of the cycles, and they form a group with this product. This group turns out to be isomorphic to the fundamental group of the order complex or the ideal topology. And for every simplicial complex X the poset of simplices has an order complex homeomorphic to X. Hence every group can be represented as the fundamental group of a poset. A great practical advantage of the combinatorial definition is that an element of the fundamental group can be represented by a cycle in the poset. Finite cycles are much easier to deal with than loops in a topological space, even in finite ones. For example the ideal topology corresponding to the two element chain is a two element topological space having an open and a closed point. And there is a loop that maps the Cantor set to the closed point and its complement to the open point, we can say this loop goes through infinitely many times both points. We build up the theory of fundamental groups using the terminology of posets, paths, cycles, product etc. We get the theorems for fundamental group – like the Van Kampen theorem – from the ideal topology.

Finally, we will have a combinatorial theory of fundamental groups involving the usual covering theorems, for example the connection between subgroups of the fundamental group and covering posets. And there is a strong, "one to one" correspondence between simplicial complexes, simplicial maps, coverings etc. and posets, monotone maps, poset coverings etc. Surprisingly this makes possible to prove a new topological theorem about fundamental group. As a consequence of our similar result for posets we get that $\mathbb{Z}$ is the homomorphic image of the fundamental group of the finite simplicial complex X iff X has S^1 as retract.

In Section 2 we write the usual definitions about posets, retractions and idempotent operations. Readers familiar with posets can skip this section. Similarly, Section 3 consists of well known theorems about fundamental group and coverings. In Section 4 we introduce the combinatorial definition of fundamental group. We prove the usual theorems about fundamental group and covering projections using the ideal topology. The fundamental group of a poset is isomorphic to the fundamental group of its order complex. The last section contains the two new results.

If $\mathbb{Z}$ is the homomorphic image of the fundamental group of a finite poset P then a crown is a retract of a finite covering poset of P. The group $\mathbb{Z}$ is the homomorphic image of the fundamental group of a finite simplicial complex X iff S^1 is a retract of X. There is no nicer theorem in the case of posets: There is a poset P of sixteen elements such that its fundamental group is $\mathbb{Z}$ but no crown is a retract of P.

Every connected idempotent subalgebra of the finite poset P is simply connected iff no crown is a retract of an idempotent subalgebra of P. This gives a proof for the fact that if a poset admits Taylor-terms then the fundamental group of every idempotent subalgebra is trivial. (For another proof see [16].)

2. Definitions, notations

Let P and Q be posets. A map $f : P \to Q$ is monotone if it is order-preserving. The poset of all monotone maps $f : P \to Q$ ordered pointwise is denoted by Q^P. The identity map of P is denoted by id_P. We say that Q is a subposet of P, or P contains Q, if Q is a subset of P and the ordering of Q is the ordering of P restricted to Q. Two elements of P, x and y are comparable if $x \leq y$ or $x \geq y$. A path from x to y is a sequence $\{x = x_1, \ldots, x_n = y\}$ such that $x_i \in P$ for every i, and the elements x_i and x_{i+1} are comparable for $i = 1, \ldots, n-1$. A poset P is connected if for every $x, y \in P$ there is a path from x to y. A map $r \in P^P$ is a retraction if $r^2 = r$. A subset R of P is a retract of P if there exists a retraction $r \in P^P$ such that $r(P) = R$. A retract R of P is a proper retract if $R \neq P$. A retract R of P is a dismantle retract if there exist retractions $r_0, r_1, \ldots, r_k$ such that $P = r_0(P) \supset r_1(P) \supset \cdots \supset r_k(P) = R$ and the retractions r_{i-1} and r_i are comparable in P^P for $i = 1, \ldots, k$.

An operation $f \in P^{P^n}$ is an idempotent operation if $f(x, x, \ldots, x) = x$. The projection on the kth variable is denoted by π_k. We denote by $I^n(P)$ the set of all n-ary idempotent operations on P as a subposet of P^{P^n}. A subposet $Q \subseteq P$ is an idempotent subalgebra of P if Q is preserved by every idempotent operation on P. We denote by P_∞ the infinite path, its base set is $\{p_i : i \in \mathbb{Z}\}$ and $p_{2i-1} \leq p_{2i} \geq p_{2i+1}$. The crown C_{2n} is a poset of base set $\{c_1, \ldots, c_{2n}\}$, and $c_{2i} \leq c_{2i+1} \geq c_{2i+2} (1 \leq i \leq n), c_1 \geq c_{2n}$. A subposet $C \subseteq P$ is a chain if any two elements of C are compared. The neighboorhood of the element $x \in P$ is a subposet $N(x) = \{y \in P : y \text{ and } x \text{ are compared }\}$. A subposet I is called an ideal if $x \in I$ and $y \leq x$ imply $x \in I$. An element $x \in P$ is an atom if x is not minimal and $y < x$ implies that y is minimal.

3. Topological spaces and the fundamental group

This section consists of general theorems about topological spaces. We need these theorems only for path-connected and locally path-connnected spaces, mainly for ideal topologies. Good references are [21], [9] and [19].

The topological space of all continuous mappings of the space X into the space Y is $H(X, Y)$. The closed interval $[0; 1]$ is denoted by I. We say that the maps $f, g \in H(X, Y)$ are homotopic and write $f \sim g$ whenever there exists a continuous map $\phi : X \times I \to Y$ such that $\phi(x, 0) \equiv f$ and $\phi(x, 1) \equiv g$, the mapping ϕ is called a homotopy between f and g. An abstract simplicial complex K consists of a set V of vertices and a set S of finite subsets of V called simplices such that
(a) Any set containing exactly one vertice is a simplex.
(b) Any nonempty subset of a simplex is a simplex.
A simplex containing exactly $q + 1$ vertices is a q-simplex. The q-skeleton of the complex is the union of the q-simplices. Let K_1 and K_2 denote simplicial complexes. A map $f : K_1 \to K_2$ is simplicial if the image of every simplex is a simplex. By a simplicial complex we

always mean the geometric realization of an abstract simplicial complex. Every connected simplicial complex is path-connected and locally path-connected.

A path in X is a map $\alpha \in H(I, X)$. The point $\alpha(0)$ is the origin of α, and $\alpha(1)$ is the end, respectively. If there are paths α and β such that the end of α coincide with the origin of β then define the product path $\alpha * \beta$ in the following way:

$$\alpha * \beta(t) = \begin{cases} \alpha(2t) & \text{if } 0 \le t \le \frac{1}{2} \\ \beta(2t-1) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

Two paths γ_1 and γ_2 are said to be homotopic if there is a homotopy map ϕ between them such that the mappings $\phi(0,t)$ and $\phi(1,t)$ are constant and we write $\gamma_1 \sim \gamma_2$. For any two points x_0, x_1 the relation $\sim$ is an equivalence relation on the set of paths from x_0 to x_1. The resulting equivalence classes are called path classes, and the class containing the path α is denoted by $[\alpha]$. The product $*$ is associative on the path classes.
A loop is a path of the same origin and end. Fix a point $x_0 \in X$. Let us consider all loops of origin x_0. The loop class of $f * g$ depends only on the loop class of f and the equivalence class of g. The loop classes form a group with the product $*$, the constant x_0 loop is the unit and the inverse of the loop class $[\alpha]$ is $[\beta]$ for $\beta(t) = \alpha(1-t)$. The group does not depend on the choice of the basepoint if X is path-connected. For $x_0, x_1 \in X$ and a path γ of origin x_0 and endpoint x_1 an isomorphism $f : \pi_1(X, x_0) \to \pi_1(X, x_1)$ can be defined in the following way: $f(\alpha) = \gamma * \alpha * \gamma^{-1}$ where $\gamma^{-1}(t) = \gamma(1-t)$. This group is called the fundamental group of X, and it is denoted by $\pi_1(X)$. A space X is simply connected (1-connected) if X is connected and $\pi_1(X)$ is trivial. Two loops are free homotopic if there is a homotopy ϕ between them such that $\phi(0,t) = \phi(1,t)$ for every t. One of the most important theorems about fundamental group is the Van Kampen theorem.

Theorem 1. *Let X denote a path-connected space with path-connected open subspaces X_1 and X_2 such that $X_1 \cap X_2$ is path-connected and $X = X_1 \cup X_2$. Let $i_j : X_j \subset X$ $(j = 1, 2)$ denote the injective homomorphism, $x_0 \in X_1 \cap X_2$. Then $\pi_1(X, x_0)$ is isomorphic to the quotient group of the free product of $\pi_1(X_1, x_0)$ and $\pi_1(X_2, x_0)$ by the normal subgroup generated by $\{i_{1*}(\gamma) \circ i_{2*}(\gamma)^{-1} : \gamma \in \pi_1(X_1 \cap X_2, x_0)\}$.*

Lemma 2. *Let X and Y denote path-connected topological spaces. Then $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$.*

Lemma 3. *Let X denote a path-connected topological space. Two loops in $\pi_1(X)$ are free homotopic iff the elements of the group $\pi_1(X)$ belonging to them are conjugated.*

A path-connected space T is called a covering space over the path-connected space X if there is given a mapping $\pi : T \to X$ such that for every $x \in X$ there exists an open neighboorhood of x, $U \subseteq X$ for which $\pi^{-1}(U)$ is homeomorphic to $U \times D$ by the homeomorphism ϕ where D is a discrete set and $\pi = \pi_U \circ \phi$ holds (π_U denotes the projection $\pi_U : U \times D \to U$). The mapping π is called a covering (projection).

Theorem 4. *Let $\pi : T \to X$ be a covering. Let us be given an arbitrary space Z, a mapping $f : Z \to T$ and a homotopy $\varphi : Z \times I \to X$ such that $\pi \circ f = \varphi|_{Z\times\{0\}}$. Then there exists a unique homotopy $F : Z \times I \to T$ such that $F|_{Z\times\{0\}} = f$.*

Lemma 5. *Let $\pi : T \to X$ be a covering, $x \in X$, $t \in \pi^{-1}(X)$ and let $f : I \to X$ be a path for which $f(0) = x$. Then there exists a unique path $g : I \to T$ such that $g(0) = t$ and $\pi \circ g = f$.*

The covering projection π induces a monomorphism $\pi_* : \pi_1(T) \to \pi_1(X)$. The covering is regular if $\pi_*(\pi_1(T))$ is a normal subgroup of $\pi_1(X)$. A covering projection $f : X \to Y$ is an universal covering if for every covering $g : Z \to Y$ there is a covering $h : X \to Z$ such that $f = g \circ h$ holds, and X is called the universal covering space of Y.

Theorem 6. *A simply connected covering space of a path-connected and locally path-connected space is an universal covering space.*

Two universal covering spaces of a path-connected and locally path-connected space are homeomorphic.

Theorem 7. *Consider the covering $\pi : T \to X$ be a covering, and a point $x \in X$. The pre-image $\pi^{-1}(x)$ is in one-to-one correspondence with the cosets of $\pi_1(X)$ by the subgroup $\pi_*(\pi_1(T))$.*

The correspondence is the following: Fix an $x_0 \in \pi^{-1}(x)$, and correspond to every loop α the endpoint of the lift of α of origin x_0, this is well defined on the cosets.

Theorem 8. *For any path-connected topological space and subgroup $G \leq \pi_1(X, x_0)$ there exists a covering $\pi : T \to X$ such that $\pi_*(\pi_1(T, t_0)) = G$ for a suitable point $t_0 \in \pi^{-1}(x_0)$.*

T can be the quotient space of the path classes of origin x_0 factorized by the equivalence $\alpha \equiv \beta$ iff $\alpha^{-1} \circ \beta \in G$.

Let $\pi : X \to Y$ denote a covering projection. A self-equivalence of π is a homeomorphism $h : X \to Y$ such that $\pi = \pi \circ h$, we call this group the group of covering transformations of π.

A group of homeomorphisms of X, G is discontinuous if the orbits of G are discrete subsets of X. G is properly discontinuous if every $x \in X$ has an open neighboorhood U such that the points of U are in different orbits of G. A properly discontinuous group is discontinuous and acts without fixpoints. The group of covering transformations is properly discontinuous. A covering projection is regular iff the group of covering transformations is transitive on $f^{-1}(y)$ for every $y \in Y$. We need the next statements to prove the similar claims for posets.

Theorem 9. *Let G denote a properly discontinuous subgroup on the simply connected topological space U. Then there exist a topological space X and an universal covering $f : U \to X$ such that its covering transformation group is exactly G.*

Theorem 10. *Let X and Y denote path-connected topological spaces, $\phi : Y \to X$ is a covering. The map ϕ induces a monomorphism $\pi_* : \pi_1(Y) \to \pi_1(X)$. If the subgroup $\pi_*(\pi_1(Y))$ is normal then the factor group $\pi_1(X)/\pi_1(Y)$ is isomorphic to the group of covering transformations of ϕ.*

Fix the basepoint $x_0 \in X$. Correspond to a loop α the unique covering transformation what maps to every point $y \in \pi^{-1}(x_0)$ the end of the lift of α of origin y. This gives a well defined isomorphism on the factor group. We denote this covering transformation by ϕ_α.

4. The fundamental group of posets

A useful topology is on the base set of a poset P the so called ideal topology $T(P)$, its open sets are the order-ideals (subsets $I \subseteq P$ satisfying $x \leq y \in I$ implies $x \in I$). Let Q be a poset. The map $f : P \to Q$ is monotone iff it is continuous in the ideal topology. The topological space $T(P)$ is (path-)connected iff P is connected. Two monotone maps $f, g \in P^Q$ are homotopic if there is a path in P^Q connecting them, this path is called a homotopy path. A monotone map $\gamma : P_\infty \to P$ is called a path if there are integers a and b such that $\gamma(p_i) = \gamma(p_j)$ for every $i, j \leq a$ and $\gamma(p_i) = \gamma(p_j)$ for every $i, j \leq b$. The maximal such a and minimal b is denoted by $a(\gamma)$ and $b(\gamma)$. (For the constant path $a(\gamma) = b(\gamma) = 0$.) The origin and end(point) of the path γ is $\gamma(p_{a(\gamma)})$ and $\gamma(p_{b(\gamma)})$, respectively. The length of the path γ is $length(\gamma) = b(\gamma) - a(\gamma)$. A loop is a path whose origin and endpoint is the same. Two paths α and β are homotopic if there is a homotopy path of paths connecting them such that every path in the homotopy path has the same origin and endpoint. Two loops are (fixed) homotopic if they are homotopic as paths. Two loops are free homotopic if there is a homotopy path in P^{P_∞} connecting them such that every map is a loop in the homotopy path. The path α is denoted by the infinite vector $\bar{x}$ where $x_i = \alpha(p_i)$. (Hence α and β have the same origin and endpoint.) Assume for the paths α and β that the endpoint of α is the origin of β. Their product is the path $\alpha * \beta$:

$$\alpha{*}\beta(p_i) = \begin{cases} \alpha(p_i) \text{ if } i \leq b(\alpha) \\ \beta(i - a(\beta) + b(\alpha)) \text{ if } i > b(\alpha) \text{ and } 2|a(\alpha) - b(\beta) \\ \beta(i - a(\beta) + b(\alpha + 1) \text{ if } i > b(\alpha) \text{ and } 2|a(\alpha) - b(\beta) + 1. \end{cases}$$

Lemma 11. *For the path α let β denote the path $\beta(p_i) = \alpha(p_{i-2})$ if $i > j$, $\beta(p_j) = \alpha(p_{j-1})$ and $\beta(p_i) = \alpha(p_i)$ if $i < j$. Then α and β are fixed homotopic.*

Proof. Consider the loop $\alpha = (\ \dots\ , x_{a(\alpha)}, \dots\ , x_{b(\alpha)}, \dots\ )$. Then $\alpha =$
$(\ \dots\ , x_{a(\alpha)}, \dots\ x_{b(\alpha)-3}, x_{b(\alpha)-2}, x_{b(\alpha)-1}, x_{b(\alpha)}, x_{b(\alpha)}, x_{b(\alpha)}, \dots\ ) \sim$
$(\ \dots\ , x_{a(\alpha)}, \dots\ x_{b(\alpha)-3}, x_{b(\alpha)-2}, x_{b(\alpha)-1}, x_{b(\alpha)-1}, x_{b(\alpha)-1}, x_{b(\alpha)}, \dots\ ) \sim$
$(\ \dots\ , x_{a(\alpha)}, \dots\ x_{b(\alpha)-3}, x_{b(\alpha)-2}, x_{b(\alpha)-1}, x_{b(\alpha)-1}, x_{b(\alpha)-1}, x_{b(\alpha)}, \dots\ ) \sim$
$(\ \dots\ , x_{a(\alpha)}, \dots\ x_{b(\alpha)-3}, x_{b(\alpha)-2}, x_{b(\alpha)-2}, x_{b(\alpha)-2}, x_{b(\alpha)-1}, x_{b(\alpha)}, \dots\ ) \sim$
$\dots \sim \beta$. □

Corollary 12. *The product $*$ is associative on the paths.*

Lemma 13. *Suppose $\alpha_1 \sim \alpha_2$ and $\beta_1 \sim \beta_2$ for four paths, the origin of α_i coincides with the end of β_j $(1 \leq i, j \leq 2)$. Then $\alpha_1 * \beta_1 \sim \alpha_2 * \beta_2$.*

Proof. The paths α_1 and α_2 are compared via the fixed homotopy path $\alpha_1 = f_1, f_2, \dots\ , f_m = \alpha_2$ and $\beta_1 = g_1, g_2, \dots\ , g_n = \beta_2$. We may suppose by Lemma 11 that there is a K such that $b(f_i) < K < a(g_j)$ for every $1 \leq i \leq m$, $1 \leq j \leq n$. Consider the paths $F_i(p_k) = f_i(p_k)$ if $k \leq K$, and $F_i(p_k) = \beta_1(p_k)$ if $k > K$; $G_i(p_k) = g_i(p_k)$ if $k > K$, and $G_i(p_k) = \alpha_2(p_k)$ if $k \leq K$. The maps $F_0, F_1, \dots\ , F_n, G_1, \dots\ G_{m-1}, G_m$ form a fix homotopy path connecting $\alpha_1 * \beta_1$ and $\alpha_2 * \beta_2$. □

Hence the product $*$ is well defined on the fixed homotopy classes.

Lemma 14. *Consider the points $s_i, v_i \in I$ such that $s_{2i-1} < s_{2i} \leq s_{2i+1}, v_{2i-1} < v_{2i} \leq v_{2i+1}, lim_{i\to-\infty} t_i = lim_{i\to-\infty} v_i = 0, lim_{i\to\infty} t_i = lim_{i\to\infty} v_i = 1 (i \in \mathbb{Z})$. Let γ denote a path in the poset P, α and β the following paths in $T(P)$:*

$$\alpha(x) = \begin{cases} \gamma(p_{2k}) \textit{ if } s_{2k} \leq x \leq s_{2k+1} \\ \gamma(p_{2k+1}) \textit{ if } s_{2k+1} < x < s_{2k+2}, \end{cases}$$

$$\beta(x) = \begin{cases} \gamma(p_{2k}) \ \textit{if } v_{2k} \leq x \leq v_{2k+1} \\ \gamma(p_{2k+1}) \ \textit{if } v_{2k+1} < x < v_{2k+2}. \end{cases}$$

Then the paths α and β are homotopic.

Proof. A homotopy between α and β is ϕ:

$$\phi(x,t) = \begin{cases} \gamma(p_{2k}) \text{ if } ts_{2k} + (1-t)v_{2k} \leq x \leq ts_{2k+1} + (1-t)v_{2k+1} \\ \gamma(p_{2k+1}) \text{ if } ts_{2k+1} + (1-t)v_{2k+1} < x < ts_{2k+2} + (1-t)v_{2k+2}. \end{cases}$$

□

Lemma 15. *Consider the map φ that corresponds to a path γ in P a path in $T(P)$ in the following way:*

$$\varphi(\gamma)(t) = \begin{cases} \varphi(p_{4-2k}) \ \textit{if } t = \frac{1}{k}, k > 1 \\ \varphi(p_{2k-4}) \ \textit{if } t = 1 - \frac{1}{k}, k > 1 \\ \varphi(p_{3-2k}) \ \textit{if } \frac{1}{k+1} < t < \frac{1}{k}, k > 1 \\ \varphi(p_{2k-3}) \ \textit{if } 1 - \frac{1}{k} < t < 1 - \frac{1}{k+1}, k > 1. \end{cases}$$

Then φ gives a one-to-one correspondence between the path classes of P and the path classes of $T(P)$. And φ is compatible with the path product.

Proof. Suppose that the paths α and β are homotopic, the homotopy path is $\alpha = \gamma_0 \geq \gamma_1 \leq \ldots \leq \gamma_{2k} = \beta$. Then

$$\phi(x,t) = \begin{cases} T(\gamma_{2j})(x) \text{ if } t = \frac{j}{k} \\ T(\gamma_{2j-1})(x) \text{ if } \frac{j-1}{k} < t < \frac{j}{k} \end{cases}$$

is a homotopy path between $T(\alpha)$ and $T(\beta)$. The map φ is surjective: Choose an arbitrary path γ in $T(P)$. For every $0 \leq x \leq 1$ there is an open interval I_x such that $x \in I_x$ and $\gamma(y) \leq \gamma(x)$ for every $y \in I_x$. There are finitely many intervals $I_{x_1}, \ldots, I_{x_k}$ covering the interval I, suppose that $I_{x_i} \cap I_{x_j} = \emptyset$ iff $|i-j| \geq 2$, $0 \in I_{x_1}$ and $1 \in I_{x_k}$. Choose an $\varepsilon > 0$ and elements y_i such that $(y_i - \varepsilon, y_i + \varepsilon) \subset I_{x_i} \cap I_{x_j}$, and $\gamma(y_i) \geq \gamma(y)$ for every $y \in (y_i - \varepsilon, y_i + \varepsilon)$ $(0 \leq i \leq k)$. Moreover, we may assume $y_0 = 0$ and $y_k = 1$. Now the path

$$\beta(x) = \begin{cases} \gamma(x_i) \text{ if } y_{i-1} + \varepsilon \leq x \leq y_i - \varepsilon \\ \gamma(y_i) \text{ if } y_i - \varepsilon < x < y_i + \varepsilon \end{cases}$$

is homotopic to γ by the following homotopy ϕ:

$$\phi(x,t) = \begin{cases} \beta(x) \text{ if } t = 1 \\ \gamma(x) \text{ if } t \neq 1. \end{cases}$$

Define the path α in P:

$$\alpha(p_i) = \begin{cases} \gamma(x_j) \text{ if } i = 2j, 0 \leq j \leq k \\ \gamma(y_j) \text{ if } i = 2j+1, 0 \leq j \leq k-1 \\ \gamma(0) \text{ if } i \leq 0 \\ \gamma(1) \text{ if } i \geq k+1 \end{cases}$$

The paths $T(\alpha)$ and β are homotopic.

Now consider the paths α and β in P, and assume that $T(\alpha)$ and $T(\beta)$ are homotopic, denote the homotopy by ϕ. We will prove that α and β are homotopic, too. Using double indices for every $0 \leq t \leq 1$ there are open intervals $I_{x_1^t}, \dots, I_{x_{k(t)}^t}$ and J_t such that $x_i^t \in I_{x_i^t}$ for every i, $t \in J_t$ and $\phi(y,z) \leq \gamma(x_i^t, t)$ for every $y \in I_{x_i^t}$ and $z \in J_t$. The interval I is covered by finitely many open intervals $J_{t_1}, \dots J_{t_k}$ such that $t_i \in J_{t_i}$, we may suppose $t_1 = 0, t_k = 1$ and $J_{t_i} \cap J_{t_j} = \emptyset$ iff $|i-j| \geq 2$. For every $i \leq k$ choose a number $u_i \in J_{i-1} \cap J_i$. And choose for every $t = t_i$ or $t = u_i$ open intervals $I_{x_1^t}, \dots, I_{x_{k(t)}^t}$ such that the above conditions will hold and $I_{x_l^t} \cap I_{x_j^t} = \emptyset$ iff $|l-j| \geq 2$. Moreover let $y_{2i-1}^t, y_{2i}^t \in I_{x_i^t} \cap I_{x_{i+1}^t}$ s.t. for every $y_{2i-1}^t < x < y_{2i}^t$ the inequality $\phi(x,t) \leq \phi(\frac{y_{2i-1}^t + y_{2i}^t}{2}, t)$ holds, and assume that the intervals (y_{2i-1}^t, y_{2i}^t) are disjoint. Let $Z = \{y_i^t : t = t_j \text{ or } t = u_j \text{ for some } j, \text{ and } 1 \leq i \leq k(t)\} = \{z_1, z_2, \dots, z_{2m}\}$, where $z_i < z_{i+1}$ holds. Since the intervals are disjoint for every $i \leq m$ we have $(z_{2i-1}, z_{2i}) = (y_{2j-1}^t, y_{2j}^t)$ for some t and j. Define the paths γ_i in P in the following way:

$$\gamma_i(p_j) = \begin{cases} \phi(x_l^{t_i}, t_i) \text{ if } y_{2l-2}^{t_i} \leq x_l^{t_i} \leq y_{2l-1}^{t_i} \text{ for some } y_{2l-2}^{t_i} = z_{2p-2} \\ \quad \text{and } y_{2l-1}^{t_i} = z_{2q-1} \text{ where } 2p-2 \leq j < 2q-1 \\ \phi(\frac{y_{2l-1}^{t_i} + y_{2l}^{t_i}}{2}, t_i) \text{ if } y_{2l-1}^{t_i} = z_j \\ \phi(0, t_i) \text{ if } j < 0 \\ \phi(1, t_i) \text{ if } j > 2m \end{cases}$$

and similarly the paths δ_i for the points $y_l^{u_i}$. Clearly γ_1 is homotopic to α and γ_k is homotopic to β. We succeed to prove that γ_i is homotopic to δ_i and δ_i is homotopic to γ_{i+1} for every i. Fix an i and a number $y_{2j-1}^{t_i} = z_{2l+1}$. Consider the paths in P:

$$\underline{\gamma}_i(p_h) = \begin{cases} \gamma_i(p_{2l+1}) \text{ if } h = 2l+1 \\ \gamma_i(p_{2l}) \text{ if } h < 2l+1 \\ \gamma_i(p_{2l+1}) \text{ if } h > 2l+1. \end{cases}$$

$$\underline{\delta}_i(p_h) = \begin{cases} \delta_i(p_{2l+1}) \text{ if } h = 2l+1 \\ \gamma_i(p_{2l}) \text{ if } h < 2l+1 \\ \gamma_i(p_{2l+1}) \text{ if } h > 2l+1. \end{cases}$$

We shall prove that these paths are homotopic. The path in $T(P)$: $\varepsilon(t) = \phi(\frac{y_{2j-1}^{t_i}+y_{2j}^{t_i}}{2}, tt_i + (1-t)u_i)$ connects $\gamma_i(p_{2l+1})$ and $\delta_i(p_{2l+1})$ and $\varepsilon(t) \leq \gamma_i(p_{2l}), \gamma_i(p_{2l+2})$ holds for every t. Hence there is a path in P, $\underline{\epsilon}$ for which $T(\underline{\varepsilon}) = \varepsilon$, moreover every element of ε is less or equal than $\gamma_i(p_{2l})$ and $\gamma_i(p_{2l+2})$, what is easy to see from the proof of the surjectivity. Suppose $\underline{\varepsilon}(p_0) = \gamma_i(p_{2l+1})$ and $\underline{\varepsilon}(p_h) = \delta_i(p_{2l+1})$. Now the paths $\varepsilon_h(p_v) = \gamma_i(p_v)$ if $v \neq 2l+1$ and $\varepsilon_h(p_{2l+1}) = \underline{\varepsilon}(p_h)$ connect γ_i and δ_i. The product of the paths $\underline{\gamma}_i$ is γ_i. And the product of the paths $\underline{\delta}_i$ is greater or equal than δ_i, hence it is homotopic δ_i. We get similarly that δ_i and γ_{i+1} are homotopic. □

Remark 16. The fact that α and β are free homotopic iff $T(\alpha)$ and $T(\beta)$ are free homotopic can be proved by a little change of the proof.

Consider the loops in P of the same end x_0. These loop-classes build a group which is isomorphic to $\pi_1(T(P), x_0)$, an isomorphism is induced by φ. The constant loop is the unit and the inverse of the loop α is β: $\beta(p_i) = \alpha(p_{-i})$. If P is connected we get isomorphic groups for different basepoints x_0 and x_1. And an isomorphism can be defined in the same way as for topological spaces: For a path γ of origin x_0 and end x_1 the "conjugation" with γ gives an isomorphism.

Remark 17. We can define this group using monotone maps from crowns $C_{2n} (n \in \mathbb{N})$ instead of P_∞.

The next propositions are the consequences of Lemma 15 and the similar claims for topological spaces: Lemma 3, Lemma 2 and Theorem 1.

Proposition 18. *Two loops in $\pi_1(P)$ are free homotopic iff the elements of the group $\pi_1(P)$ belonging to them are conjugated.*

Proposition 19. *Let P and Q denote connected posets. Then the fundamental group $\pi_1(P \times Q)$ is isomorphic to $\pi_1(P) \times \pi_1(Q)$.*

Proof. The product of the topological spaces $T(P)$ and $T(Q)$ is exactly $T(P \times Q)$. □

Theorem 20. *The posets $P_1, P_2, P = P_1 \cup P_2$, and $P_1 \cap P_2$ are connected, P_1 and P_2 are ideals in P. The injective homomorphism $P_j \subset P$ $(j = 1, 2)$ is denoted by i_j, the element x_0 is in P. Now $\pi_1(P, x_0)$ is* *isomorphic to the quotient group of the free product of* $\pi_1(P_1, x_0)$ *and* $\pi_1(P_2, x_0)$ *by the normal subgroup generated by* $\{i_{1*}(\gamma) = i_{2*}(\gamma) : \gamma \in \pi_1(P_1 \cap P_2)\}$.

We say that the monotone map $f : P \to Q$ is a covering if for every $q \in Q$ its neighboorhood $N(q)$ is isomorphic with every connected component of $f^{-1}(N(q))$ with the isomorphism f. The monotone map f is a covering iff it is a covering in topological sense (using the ideal topologies). A path α in P is the lift of the path β in Q if $f \circ \alpha = \beta$.

Lemma 21. *Let* $f : P \to Q$ *denote a covering, and* α *is a path in* Q, $\alpha(a(\alpha)) = f(p)$ *for* $p \in P$. *Then there exists uniquely a path* β *in* P *such that it is the lift of the path* α, *and* $\beta(p_{a(\beta)}) = p$.

Proof. We define β in the following way. $\beta(p_i) = \beta(a(\beta))$ if $i \le a(\beta)$. Suppose that $\beta(p_{i-1})$ is defined. Then there is a unique $x \in N(\beta(p_i))$ such that $f(x) = \alpha(p_i)$, let us define $\beta(p_i) = x$. For $i, j \ge b(\beta)$ we clearly have $\beta(p_i) = \beta(p_j)$. Hence β is a path for we have $a(\alpha) = a(\beta)$, $b(\alpha) = b(\beta)$, and there is no other lift of α. □

Lemma 22. *Let* $f : P \to Q$ *denote a covering,* α_1 *and* α_2 *are paths of the same origin in* Q, $\alpha_1 \le \alpha_2$, β_i *is a lift of* α_i $(1 \le i \le 2)$, β_1 *and* β_2 *have the same origin. Then* $\beta_1 \le \beta_2$.

Proof. We succeed to prove the lemma for the case $\alpha_1(p_i) = \alpha_2(p_i)$ if $i \ne 0$. We have seen in the proof of the previous lemma that $\beta_1(p_{-1}) = \beta_2(p_{-1})$, hence $\beta_1(p_0) \le \beta_2(p_0)$. Since $\beta_1(p_0) \le \beta_2(p_0) \le \beta_2(p_1)$, and $f(\beta_2(p_1)) = \alpha_1(p_1) = \alpha_2(p_1)$ we have $\beta_1(p_1) = \beta_2(p_1)$, and $\beta_1(p_i) = \beta_2(p_i)$ for every $i \ge 1$. □

This gives the next propositon.

Proposition 23. *Let* $f : P \to Q$ *denote a covering,* α_1 *and* α_2 *are fixed homotopic paths in* Q, $\alpha = \gamma_1, \gamma_2, \dots \gamma_n = \alpha_2$ *is a fixed homotopy path connecting them. Then their lifts,* β_1 *and* β_2 *are fixed homotopic, and the lift of the homotopy path* $\gamma_1, \dots, \gamma_n$ *is a fixed homotopy path connecting them.*

A simply connected poset is a connected poset whose fundamental group is trivial. A (monotone) covering $f : P \to Q$ is an universal covering if for every covering $g : S \to Q$ there is a covering $h : P \to S$ such that $f = g \circ h$. And P is called the universal covering poset of P.

Theorem 24. *For any subgroup* $G \le \pi_1(P, x_0)$ *there exists a covering* $\pi : T \to P$ *such that* $\pi_*(\pi_1(T, t_0)) = G$ *for a suitable point* $t_0 \in \pi^{-1}(x_0)$.

Proof. T can be the factor poset of the path classes of origin x_0 factorized by the equivalence $C_1 \equiv C_2$ iff there is an $\alpha \in C_1$ and $\beta \in C_2$ such that $\alpha^{-1} \circ \beta \in G$. The ordering of the factor poset is the following: $C_1 \leq C_2$ iff there is an $\alpha \in C_1$ and $\beta \in C_2$ such that $\alpha \leq \beta$. □

Let $f : Q \to P$ denote a covering map. A covering transformation is an automorphism ϕ of the poset Q such that $f = f \circ \phi$. An automorphism subgroup G of a poset Q is discontinuous if for every $q \in Q$ we have $N(q) \cap N(\phi(q)) = \emptyset$, or $\phi(q) = q$. The group G is properly discontinuous if it is discontinuous and acts without fixpoints. The group of covering transformations is properly discontinuous. A regular covering is a covering for which the group of covering transformations is transitive on the pre-image of every element. The group G is (properly) discontinuous iff it is (properly) discontinuous in topological sense.

Theorem 25. *Every connected poset P has a unique simply connected covering poset U. And this is an universal covering poset.*

Proof. Fix a $x_0 \in P$, and consider the paths of origin x_0, denote the set of their fix homotopy classes by C_{x_0}. Take the next partial ordering on C_{x_0}: $C_1 \leq C_2$ if there are paths $\alpha \in C_1$ and $\beta \in C_2$ such that $\alpha \leq \beta$ (as monotone maps). The covering corresponds to a class C the endpoint of the paths in C. The proof is the same as for topological spaces. The covering is universal iff $T(U)$ is an universal covering space of $T(P)$, what comes from Theorem 6. And it is unique since $T(P)$ is path-connnected and locally path-connected. □

The next two theorems comes from Theorem 9, Theorem 10, the similar statements for topological spaces. But we write the constructions since we shall use them.

Theorem 26. *Let G denote a properly discontinuous subgroup of the simply connected poset U. Then there exist a poset P and an universal covering $f : U \to P$ such that its covering transformation group is exactly G.*

Proof. Consider the equivalence relation $\equiv$ on U: $q_1 \equiv q_2$ iff there is a $\phi \in G$ such that $\phi(q_1) = q_2$. There is a partial ordering on the equivalence classes: $C_1 \leq C_2$ iff there is a $q_1 \in C_1$, and $q_2 \in C_2$ such that $q_1 \leq q_2$. This partial ordering is antisymmetric since G consists of properly discontinuous transformations. Let P be the poset of the equivalence classes, the covering corresponds to an element of P its equivalence class. □

Theorem 27. *Let S and P denote connected posets, $\phi : S \to P$ is a covering. The map ϕ induces an injective homomorphism $\pi_* : \pi_1(S) \to$*

$\pi_1(P)$. If the subgroup $\pi_*(\pi_1(S))$ is normal then $\pi_1(P)/\pi_1(S)$ is isomorphic to the group of covering transformations of φ.

Proof. Fix a loop γ in P. Then there is a unique automorphism of S that maps the origin of every lift of γ to its endpoint, we denote this automorphism by ϕ_γ. The correspondence $\alpha \to \phi_\alpha$ gives an isomorphism of the factor group. □

The order complex of the poset P is a simplicial complex $\triangle(P)$ whose vertices are the elements of P and whose simplices are corresponding to the chains of P. Consider the following map $f : \triangle(P) \to T(P)$: For $x \in X$ choose the minimal simplex containing x and the minimal element y of the poset P which is a vertice of this cell, and define $f(x) = y$. Now the map f is a weak homotoopy equivalence: For every homotopy group f induces an isomorphism between $\pi_n(\triangle(P))$ and $\pi_n(T(P))$ (what we only use in the case $n = 1$). Moreover f gives a one to one correspondence between path classes in $\triangle(P)$ of which origin and endpoint is a vertice and path classes in $T(P)$. For a simplicial complex the poset of simplices is a poset of which base set is the set of simplices ordered by containment.

Proposition 28. *For every (finite) simplicial complex, X there is a (finite) poset P, namely the poset of simplices such that X and $\triangle(P)$ are homeomorphic.*

Proof. The elements of the poset are corresponding to the cells of X. And for two elements $a, b \in P$ we have $a \leq b$ iff the cell corresponding to b contains the cell corresponding to a. We can get the simplicial complex $\triangle(P)$ from the simplicial complex X taking a plus vertice in the interior of every cell, and plus cells as simplexes with given vertices. □

Theorem 29. *The fundamental group of a finite, connected poset is finitely presented, and for every (finitely presented) group G there is a (finite) P such that $\pi_1(P) = G$.*

Proof. The fundamental group of P is isomorphic to the fundamental group of $\triangle(P)$ and this is finitely presented. For every (finitely presented) group there is a (finite) simplicial complex of isomorphic fundamental group, and the poset of simplices has the same fundamental group. □

We say that the map $f : P \to Q$ is comparable if for every $x, y \in P$ to be compared the elements $f(x), f(y) \in Q$ are compared, too. Especially, every monotone map is comparable. The next Lemma shows

the connection between comparable maps on P and (simplicial) maps on $\triangle(P)$.
Lemma 30. *Let P and Q denote posets, $\triangle(P)$ and $\triangle(Q)$ are their order complexes. Every comparable map $\phi : P \to Q$ induces a simplicial map $\phi_* : \triangle(P) \to \triangle(Q)$. And every (simplicial) map $\xi : \triangle(P) \to \triangle(Q)$ induces a comparable map $\xi_* : P \to Q$.*

Remark 31. If ϕ is retraction, covering, dismantle retraction then it induces retraction, covering, deformation. Hence a dismantle retract of the poset P has the same fundamental group as P.

5. Main results

Lemma 32. *Suppose that f_0 is a minimal element in the poset P. The subposet $F = \{f_0, f_1, \dots\}$ is a finite or infinite fence of endpoint f_0, $d_F(f_0, f_i) = i$ for every i. For every $f \in F$ the distance of f_0 and f is the same in the fence and the poset P : $d_F(f_0, f) = d_P(f_0, f)$. Then F is a retract of P by the retraction r:*

$$r(x) = \begin{cases} f_{d(f_0,x)} & \textit{if } d(f_0, x) \leq |F| - 1 \\ f_{|F|-1} & \textit{if } d(f_0, x) \geq |F| \end{cases}$$

Consider an abelian group A, a connected poset P and a homomomorphism $\varphi : \pi_1(P, x_0) \to A$. For a loop α choose an arbitrary loop β of end x_0 free homotopic with α, and define $\varphi(\alpha) = \varphi(\beta)$. This extension is well defined by Proposition 18. And for loops of the same end, β_1 and β_2 the equality $\varphi(\beta_1)\varphi(\beta_2) = \varphi(\beta_1 * \beta_2)$ holds.

Lemma 33. *Suppose for the finite, connected poset P, that there is an integer-valued map, φ on the loops of P, such that if the loops γ_1 and γ_2 are free homotopic then $\varphi(\gamma_1) = \varphi(\gamma_2)$, and if γ_1 and γ_2 have the same end then $\varphi(\gamma_1 * \gamma_2) = \varphi(\gamma_1) + \varphi(\gamma_2)$. Then there is a loop α such that the quotient $\frac{|\varphi(\beta)|}{length(\beta)}$ ($length(\beta) \neq 0$) has its maximum in α.*

Proof. We claim that this quotient has a maximum taken by a circle. If we have a loop that goes through an element of P more than once, than we can split the loop in two shorter loops, and the quotient is more or equal for one of the loops than for the original loop. Since P is finite this quotient has a maximum for circles, and this is the maximum for every loops. $\square$

Lemma 34. *Let P denote a connected poset, and $x_0 \in P$ a minimal element. Suppose that there is an integer-valued map, φ on the loops of P, such that if the loops γ_1 and γ_2 are free homotopic then $\varphi(\gamma_1) = \varphi(\gamma_2)$, and if the end of γ_1 and γ_2 coincides then $\varphi(\gamma_1 * \gamma_2) = \varphi(\gamma_1) + \varphi(\gamma_2)$. Moreover there is a loop of endpoint x_0, α such that $\varphi(\alpha) = 1$, and for every nonconstant loop β the inequality $\frac{|\varphi(\beta)|}{length(\beta)} \leq \frac{1}{length(\alpha)}$ holds. Then α is a crown as a subposet of P, and it is a retract of P.*

Proof. **Claim:** The cycle α is a crown.

We should prove for any two elements of α compared that they are neighbours on the cycle, too. Let us take two elements of α compared, and connecting them split the loop α in two smaller loops β_1 and β_2. We have $length(\beta_1)+length(\beta_2) = length(\alpha)+2$ and $\varphi(\beta_1)+\varphi(\beta_2) = \varphi(\alpha)$. For an index $1 \leq i \leq 2$ we get $\varphi(\beta_i) \neq 0$, now $\frac{|\varphi(\beta_i)|}{length(\beta_i)} \leq \frac{1}{length(\alpha)}$ implies $length(\beta_i) \geq length(\alpha)$. Hence $length(\beta_{3-i}) \leq 2$, the chosen compared elements are neighbours on the cycle.

Denote by S the covering poset of P corresponding to the subgroup $ker\varphi$. The covering $\tau : S \to P$ is regular, the corresponding properly discontinuous group is isomorphic to $\mathbb{Z}$ and it can be generated by ϕ_α. Fix an element $s_0 \in \tau^{-1}(q_0)$ and introduce the notation $s_i = \phi_\alpha^i(s_0)$. Every s_i is minimal since x_0 is minimal. Now fix an $i \in \mathbb{Z}$, and consider for every $j \geq 0$ the lift of α^j of origin s_i. For $j > k > 0$ the lift of α^k is a subpath of the lift of α^j. Let R_i denote the union of these paths.

Claim: The subposet R_i is an infinite fence of end s_i. It is a retract of P and the retraction r_i can be defined in the same way as in Lemma 32.

R_i is really the image of the infinite fence by a monotone map it is the lift of the infinite path $\cup_{j\geq 0}\alpha^j$ of origin s_i. For $j > i$ and a path γ connecting s_i and s_j, we have $\varphi(\tau(\gamma)) = j - i$ and $length(\tau(\gamma)) = length(\gamma)$. Hence by the choice of α we have the inequality $length(\gamma) \geq length(\alpha)(j - i) = length(\alpha^{j-i})$. Now $d_{R_i}(s_i, s) = d_S(s_i, s)$ for every $s = s_j$, and hence for every $s \in R_i$. The subposet R_i is really a fence of endpoint s_i, and we can use Lemma 32.

Claim: The subposet $R = \cup_i R_i$ is a retract of S. And there is a retraction r such that $r \circ \phi_\alpha = \phi_\alpha \circ r$.

We claim that the sequence $r_i(s)$ stabilizes for every $s \in S$ if $i \to -\infty$. For $s \in S$ let us choose an integer $M > \frac{d(s_0,s)}{length(\alpha)}$. If $i < j$ than $d(s_i, s) \leq d(s_j, s) + (j - i)length(\alpha)$. Now $d(r_i(s), s_M)$ is monotone decreasing as a function of i ($i \leq 0$), and $d(s_0, s_M) \geq d(s_0, s)$. These imply that $d(r_i(s), s_M)$, and hence $r_i(s)$ stabilizes ($i \to -\infty$). Let us define $r(s)$ to be this element of S. The map r is clearly retraction, and the retract is R. The equality $r_{i+1} \circ \phi_\alpha = \phi_\alpha \circ r_i$ implies $r \circ \phi_\alpha = \phi_\alpha \circ r$. Now r is a retraction on the poset of the orbits of S by the automorphisms subgroup generated by ϕ_α, this means on P. The image of R by the covering projection τ (what maps a point to its orbit) is exactly the loop α, hence α is a retract of P. The retraction ρ can be defined in the following way: For $x \in P$ choose an arbitrary element $s \in \tau^{-1}(x)$ and $\rho(x) = \tau(r(s))$. This is a well defined monotone map, what fixes the elements of α since r fixes R. Now, the loop α is the retract of P by the retraction ρ. ☐

Theorem 35. *Suppose for the finite, connected poset P, that $\mathbb{Z}$ is a homomorphic image of $\pi_1(P)$. Then there is a finite covering poset of P that has a crown as retract.*

Proof. Denote the extension of the homomorphism by φ, and consider the quotient $\frac{|\varphi(\gamma)|}{length(\gamma)}$ for every nonconstant loop γ. This quotient takes its maximum by Lemma 33 in a loop α, we may suppose that the origin of α is minimal. Clearly $\varphi(\alpha) \neq 0$. Let Q denote the covering poset of P corresponding to the subgroup $\varphi^{-1}(\varphi(\alpha)\mathbb{Z})$, and τ the covering. Now $\psi = \frac{\varphi \circ \tau}{\varphi(\alpha)}$ is the extension of the homomorphism $\frac{\varphi}{\varphi(\alpha)}|_{\pi_1(Q)} : \pi_1(Q) \to \mathbb{Z}$. The lift of α is a loop β. The quotient $\frac{\psi(\gamma)}{length(\gamma)}$ has its maximum in β, and $|\psi(\beta)| = 1$. Lemma 34 implies that the subposet of Q, β is a crown and it is a retract of Q. ☐

We can say more about simplicial complexes, for which we need the following theorem (for a proof see [14]).

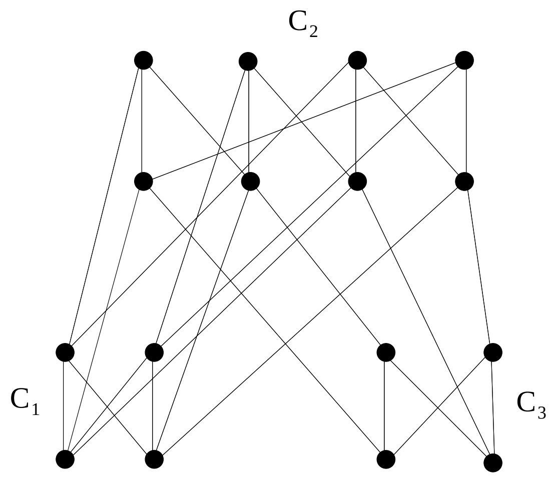
Theorem 36. *Let X denote a finite, connected simplicial complex, and suppose that $\mathbb{Z}$ is the homomorphic image of $\pi_1(X)$. Then there is a homomorphism $\varphi : \pi_1(X) \to \mathbb{Z}$ and a not self-crossing loop α such that $\varphi(\alpha) = 1$, moreover α is the union of paths such that every subpath is contained in a simplex of X and every subpath is linear.*

Theorem 37. *Let X denote a finite, connected simplicial complex, and suppose that $\mathbb{Z}$ is the homomorphic image of $\pi_1(X)$. Then S^1 is a retract of X.*

Proof. We may suppose by Theorem 36 that there is a not self-crossing loop α which is the union of 1-simplices, and a homomorphism $\varphi : \pi_1(P) \to \mathbb{Z}$ for which $\varphi(\alpha) = 1$ holds. In the poset of faces P every loop is free homotopic to a loop of minimal elements and atoms of the same length: A loop is clearly free homotopic to a loop of which every second element is minimal. And if two minimal elements in P has a common upper bound then the corresponding vertices both are contained in a simplex of X, hence they are contained in a 1-simplex. The element of P corresponding to this simplex is an atom. Consequently for every loop there is a not longer loop free homotopic with it of which every second element is atom or minimal alternately.

For such a loop β denote the corresponding loop in X what is the union of 1-simplices by $\underline{\beta}$. Now $\frac{length(\beta)}{2}$ equals to the length of $\underline{\beta}$ as the length of a path in the 1-skeleton as in a graph. Choose a 1-simplex of X which is not in α. Take a plus vertice in this 1-simplex and consider the barycentric subdivision of every simplex containing this 1-simplex. Denote this simplicial complex by X'. Every loop in X' is free homotopic to a loop in the 1-skeleton of X which is not longer than the original (as a path in the 1-skeleton of X). Repeat this procedure for every 1-simplex of X not in α, this means do it for the 1-simplices it is divided in N-times for a N large enough, denote by X' the simplicial complex we get. In the 1-skeleton of X' has every circle not contained by α of length at least N. Hence the quotient $\frac{|\varphi(\gamma)|}{length(\gamma)}$ defined for every nonconstant cycle γ in the 1-skeleton of X is maximal for the cycle α if N is large enough, since it takes its maximum in a circle. The similar quotient defined in the poset of simplices P' is maximal for the loop α' corresponding to α. Now α' is a retract of P'. Hence $\triangle(\alpha')$ is a retract of $\triangle(P')$, α' is a retract of X. $\square$

Remark 38. The similar claim is not true for posets. There is a poset P of 16 elements such that $\pi_1(P) = \mathbb{Z}$, and no crown is a retract of P. (See the figure)



Proof. Denote the loops corresponding to the crowns C_1, C_2 and C_3 by α, β and γ, respectively. The crown C_1 is a dismantle retract of the subposet $C_1 \cup C_2$, and the crown C_3 is a dismantle retract of the subposet $C_2 \cup C_3$: The first element of α is less than the first and fifth element of β, the second element of α is less than the second and fifth element of β ... The loop $\gamma = (a_1, a_2, a_3, a_4)$ where a_1 and a_3 are minimal elements is homotopic to $(a_1, a_2, a_2, a_2, a_3, a_4, a_4, a_4)$ and this loop is clearly homotopic (less) than β. Hence $\pi_1(C_1 \cup C_2) = \pi_1(C_2 \cup C_3) = \mathbb{Z}$, the loop β is free homotopic to α^2 and to γ. By the dual of Theorem 20 we get that $\pi_1(P) = \mathbb{Z}$ and α is free homotopic to a generator loop. Suppose for a contradiction that a crown is a retract of P. Every retraction r induces an idempotent epimorphism of $\pi_1(P)$ to the subgroup $\pi_1(r(P))$. Hence the homomorphism induced by a retraction to a crown (its fundamental group is $\mathbb{Z}$) should be identical, and the image of the loop of length γ should be free homotopic to the square of the generator of the retract crown by a homotopy path in the crown. But the length of $r(\gamma)$ is 4, and a loop in a crown homotopic to the square of the crown should have length at most twotimes the length of the crown, a contradiction. $\square$

Theorem 39. *For a finite connected poset P the following conditions are equivalent:*

1. *Every connected idempotent subalgebra of P is simply connected.*
2. *There is no crown as retract of an idempotent subalgebra in P.*

Proof. (1) implies (2) is trivial.
Now, let us assume that (1) does not hold. We would like to construct a crown as a retract of an idempotent subalgebra of P. So, let us assume that there is a crown C that the corresponding loop is not null homotopic in $\langle C \rangle$, where $\langle C \rangle$ denotes the idempotent subalgebra generated by C. Let C be the shortest of these crowns, its length is even, say $2n$. Thus every crown and so every loop shorter than $2n$ is null homotopic. Now, there is an idempotent operation, let us call it f, of arity k such that $f(C^k) = \langle C \rangle$. We would like to show a monotone idempotent map, a retraction from $\langle C \rangle$ to C. For every $1 \leq i \leq k$ we want to define the map r_i in the following way:
For an $a \in \langle C \rangle$ let

$$r_i(a) = \pi_i f^{-1}(a).$$

We claim that at least for one index, i, the map r_i is well defined and monotone, hence C is a retract of $\langle C \rangle$. Let us assume to the contrary that for every i either

(a) r_i is not well defined or
(b) r_i is well defined, but not monotone

In case (a) it means that there are $\underline{v}_1$ and $\underline{v}_2$ in C^n such that $f(\underline{v}_1) = f(\underline{v}_2)$ and the ith coordinates of the two vectors differ that is $(\underline{v}_1)_i \neq (\underline{v}_2)_i$. Let $(\underline{v}_1)_i = a \neq b = (\underline{v}_2)_i$. In C there are two "shortest" paths between a and b, both of them shorter than $2n$ such that their disjoint union is C. This gives a cycle C_i in C^k through $(\underline{v}_1)$ and $(\underline{v}_2)$ such that

$$\pi_j(C_i) \sim \begin{cases} \text{C} & \text{if } i = j \\ \text{null homotopic} & \text{if } i \neq j \end{cases}$$

Now, C_i is (freely) homotopic to the cycle consisting of the points $\{x_0, x_0, \ldots x_0, c_i, x_0 \ldots, x_0 \mid c_i \in C\}$. Moreover $f(C_i)$ intersects itself at the point $f(\underline{v}_1) = f(\underline{v}_2)$, hence $f(C_i)$ is a concatenation of two loops both shorter then $2n$. Suppose that the basepoint of the fundamental group is $f(\underline{v}_1) = f(\underline{v}_2)$. Now C is the product of two null homotopic loops. We obtained the following:

Claim: $f(C_i)$ is the product of two loops both shorter than $2n$, hence $f(C_i)$ is null homotopic.

The situation is slightly different in case (b). If r_i is well defined but not monotone, then there are $\underline{v}_1$ and $\underline{v}_2$ in C^n such that $a = f(\underline{v}_1) \leq f(\underline{v}_2) = b$ and $(\underline{v}_1)_i \not\leq= (\underline{v}_2)_i$. Again, we have a cycle C_i of length $2n$ through $\underline{v}_1$ and $\underline{v}_2$ homotopic to $\{x_0, x_0, \ldots x_0, c_i, x_0 \ldots, x_0 \mid c_i \in C\}$. Now, in $f(C_i)$, there are two comparable points, namely $a = f(\underline{v}_1) \leq f(\underline{v}_2) = b$. The pair (a, b) cuts $f(C_i)$ into two loops. Again, we would like to prove that

Claim: $f(C_i)$ is the product of two null homotopic loops, hence $f(C_i)$ is null homotopic.

This is fairly obvious if the distance of $(\underline{v}_1)_i$ and $(\underline{v}_2)_i$ is $d \geq 2$ in the crown. Indeed, in this case the new edge cuts the cycle into two shorter loops of lengths $d + 1$ and $2n - d + 1$.

If $d = 1$, $(\underline{v}_1)_i = c_j > c_{j+1} = (\underline{v}_2)_i$. The loop C_i splits into two paths of length 2 and $2n$, their images are loops. We succeed to prove that the loop of length $2n$ is null homotopic. We claim that the elements of this loop do not form a crown. Else we can see from the construction of C_i that $\pi_i \circ f^{-1}$ is an isomorphism between this loop and C, where f^{-1} corresponds to an element of the crown a unique element of the path of length $2n$. But for $a \leq b$ we have $\pi_i \circ f^{-1}(a) = c_j > c_{j+1} = \pi_i \circ f^{-1}(b)$, a contradiction. The loop is not a crown, hence it can be splitted into shorter loops, it is null homotopic.

Hence we got k loops, $C_1, \ldots C_k$ such that C_i is homotopic to the loop $(x_0, x_0, \ldots x_0, C, x_0 \ldots, x_0)$, suppose that $(x_0, \ldots , x_0)$ is the basepoint. The product of these loops is homotopic to the diagonal loop D of C^k. Then the product of $f(C_i)$ is homotopic to $f(D)$. By the idempotency of f the image $f(D)$ is homotopic to C (in fact, $f(D) = C$). But by the claims each $f(C_i)$ is null homotopic, hence their product is null homotopic as well, contradicting the assumption that C is not null homotopic. Thus f is well defined for at least one coordinate and C is a retract of $\langle C \rangle$. $\square$

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