Multiple point manifolds of immersions

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Contents

1 Introduction

All manifolds and maps in this thesis will be smooth, unless otherwise mentioned. Self intersection sets of immersed manifolds are a popular research subject since the sixties. They can be investigated from many different viewpoints. One could try to approach the problem geometrically. Using surgery methods Banchoff proved that for a generic immersion $f : M^2 \to \mathbb{R}^3$ the Euler characteristic of M is congruent modulo 2 to the number of triple points of f (see [1]). Using surgery methods we shall prove a new result about the self intersection sets of immersed spheres.

Let us consider an immersion $f : M^m \to N^n$. The self intersection set of f as a subset of M represents a homology class in $H_*(M)$. One can hope to express this class in terms of f and the manifolds. On the other hand this set can be given a manifold structure itself, so its topological invariants should be expressable by f.

The first results about the homology class of the self intersection set was published by Lashof and Smale in 1959 (see [9]). They presented a formula expressing the homology class of the r-tuple self intersection set for any natural number r. Their arguments were incorrect when $r \geq 3$. Herbert corrected their formula in 1975 (see [7]). In 1980 Ronga gave a new and shorter proof of Herbert's formula (see [13]).

In this thesis we shall present two different approaches, viewpoints, to multiple point sets. Using the homology viewpoint we present proofs and applications of Herbert's formula. In the first section we give the basic definitions and notations used throughout the thesis and show Ronga's original proof. In the second section we give a new proof of the formula based on ideas of András Szűcs. The third sections will show an application of the formula. The idea behind this application is to use the formula in a generalized homology and cohomology theory, and then translate the result back to ordinary cohomology, thereby obtaining new formulas. In the last section we use the geometric viewpoint and characterize the self intersection sets of a generic $S^2 \to \mathbb{R}^3$ immersion without triple points.

I would like to thank András Szűcs for the interesting topic, for supplying me with papers and for his many suggestions and corrections concerning the thesis.

2 Ronga's proof

2.1 Definitions and notation

Let V^n be an n-dimensional manifold. We shall denote its k-fold product by $V^{(k)}$.

Definition 2.1. The narrow diagonal of $V^{(k)}$ is $\delta_V(k) = \{(y, \ldots, y) : y \in V\}$ and the fat diagonal of $V^{(k)}$ is $\Delta_V(k) = \{(x_1, \ldots, x_k) \in V^{(k)} : \exists i, j \text{ such that } x_i \neq x_j\}$.

Definition 2.2. An immersion $f : V^n \to W^m$ is said to be proper if the preimage of any compact set is compact.

Remark. An immersion of a compact manifold is automatically proper.

Definition 2.3. A immersion $f : V^n \to W^m$ is said to be self-transverse if $f^k : V^{(k)} \to W^{(k)}$ is transversal to $\delta_W(k)$ outside $\Delta_V(k)$. This means that whenever $f(x_1) = \cdots = f(x_k) = y, x_i \neq x_j$, then the vector spaces $\mathrm{Im}(df_{x_1}), \ldots, \mathrm{Im}(df_{x_k})$ are in general position in TW_y.

Now we construct the multiple point manifolds of a proper self-transverse immersion $f : V^n \to W^m$. For any integer k let

$$N_k(f) = \{y \in W : |f^{-1}(y)| = k\}, \qquad M_k(f) = f^{-1}(N_k)$$

denote the k-tuple points of f in V and W respectively. Further let

$$\hat{M}_k(f) = \{(x_1, \ldots, x_k) \in V^{(k)} \setminus \Delta_V(k) : f(x_i) = f(x_j)$$

denote the nontrivial representatives of k-tuple points. The self-transverseness of f implies that $\hat{M}_k(f)$ is a manifold. Clearly, the symmetric group S_k acts freely on $\hat{M}_k(f)$, similarly the group S_{k-1} acts freely on the last $k-1$ coordinates of $\hat{M}_k(f)$, so we can define the following factor manifolds:

$$\tilde{N}_k(f) = \hat{M}_k(f)/S_k, \qquad \tilde{M}_k(f) = \hat{M}_k(f)/S_{k-1}$$

These are called the multiple point manifolds of f. The reason for this name will be explained below. The class of $(x_1, \ldots, x_k) \in \hat{M}_k(f)$ will be denoted by $[x_1, \ldots, x_k]$ and $(x_1, [x_2, \ldots, x_k])$ in $\tilde{N}_k(f)$ and $\tilde{M}_k(f)$ respectively. We can define natural maps from the multiple point manifolds onto the closure of the self intersection sets with the following formulas:

$$f_k : \tilde{M}_k(f) \to V, \qquad f_k(x_1, [x_2, \ldots, x_k]) = x_1$$

$$g_k : \tilde{N}_k(f) \to W, \qquad g_k([x_1, \ldots, x_k]) = f(x_1)$$

These maps are immersions, thus they realize the set of k-tuple points as an immersed manifold. Self intersections of these manifolds occur only in the at least $k+1$-tuple points of f. The next lemma summarizes the properties of f_k and g_k.

Lemma 1. *f_k and g_k are proper immersions with normal bundles $\nu_{g_k} = (\nu_f^{(k)}|_{\hat{M}_k(f)})/S_k$ and $\nu_{f_k} = (0 \times \nu_f^{(k-1)}|_{\hat{M}_k(f)})/S_{k-1}$.*

Proof. Clearly $\hat{M}_k(f) = (f^k)^{-1}(\delta_W(k)) \setminus \Delta_V(k)$. Further f^k is transversal to $\delta_W(k)$ outside $\Delta_V(k)$. Thus $T(\hat{M}_k(f))_{(x_1,\dots,x_k)} = \{(v_1,\dots,v_k) \in T(V)^{(k)}_{(x_1,\dots,x_k)} : df_{x_i}(v_i) = df_{x_j}(v_j)\}$. The differential of f_k takes $(v_1,\dots,v_k)$ into v_1, so if it is zero on a vector, then $df_{x_1}(v_1) = 0$ so every $df_{x_i}(v_i) = 0$, but f is an immersion so then every $v_i = 0$. That is, the differential of f_k can only be zero on the zero vector, thus f_k is an immersion. A similar argument shows that g_k is an immersion as well. The normal bundles are clearly as stated.

To see the properness of f_k and g_k we show that $\hat{M}_k(f)$ is closed in $V^{(k)}$. It is obviously closed in $V^{(k)} \setminus \Delta_V(k)$, so it is sufficient to show that it is separated from the fat diagonal. Suppose on the contrary that there exist sequences $\{x^i\}, \{y^i\} \subset V$ such that $f(x^i) = f(y^i), x^i \neq y^i$, and $\lim_{i\to\infty} x^i = \lim_{i\to\infty} y^i = x$. Writing f locally as a map $f : \mathbb{R}^n \to \mathbb{R}^m$ we may assume by choosing a subsequence that $\lim_{i\to\infty}(x^i - y^i)/\|x^i - y^i\| = v \in \mathbb{R}^n, \|v\| = 1$. This however would mean that $df_x(v) = 0$ contradicting the fact the f is an immersion. □

Finally we introduce some notation about homology. In this section H_* and H^* will denote any generalized homology and corresponding cohomology theories that have Poincaré duality. When talking about homologies and cohomologies of a space X, will denote the duality in both directions by D_X. If every object can be oriented in the given theory, we will take integer coefficients, otherwise we will take $\mathbb{Z}_2$ coefficients.

Consider a map $f : A \to B$ between manifolds. Besides $f_* : H_*(A) \to H_*(B)$ and $f^* : H^*(B) \to H^*(A)$, there is also an associated map $f_! : H^*(A) \to H^*(B)$ which is called the Gysin homomorphism, and it is defined by $f_! = D_B \circ f_* \circ D_A$. The following is a simple and well-known fact about the Gysin homomorphism:

Lemma 2. *$f_!(f^*(x)) = f_!(1) \cdot x$ holds for any $x \in H^*(B)$.*

Remark. For any vector bundle $E \xrightarrow{\mathbb{R}^n} X$ there is $e(E) \in H^n(X)$, the Euler class of the bundle. If $v : X \to E$ is a section transversal to the zero section of E, then the Euler class of E can be computed as the Poincaré dual of the set of zeroes of v. Let $Y = \{x \in X : v(x) = x\}$ and $j : Y \hookrightarrow X$ be the inclusion. Now $e(E) = D_X([Y]) = D_X(j_*([Y])) = D_X(j_*(D_Y(1))) = j_!(1)$ where the first $[Y]$ denotes the homology class in X that is represented by Y, while the second $[Y]$ denotes the fundamental class of Y.

2.2 Sub-cartesian diagrams

In this section we introduce sub-cartesian diagrams. They are the key ingredient in Ronga's proof. These diagrams generalize the notion of clean intersection of Quillen (see [10])

Definition 2.4. A commutative diagram of proper immersions:

$$\begin{tikzcd} Z \arrow[r, "f_B"] \arrow[d, "f_A"] & B \arrow[d, "\beta"] \\ A \arrow[r, "\alpha"] & X \end{tikzcd}$$

is said to be sub-cartesian if

(i) $f_A \times f_B : Z \to A \times B$ is an embedding onto $\{(a,b) \in A \times B : \alpha(a) = \beta(b)\}$.

(ii) the following sequence is exact:

$$0 \to TZ \xrightarrow{d(f_A \times f_B)} f_A^* TA \times f_B^* TB \xrightarrow{(d\alpha - d\beta)} f_A^* \alpha^* TX$$

The first condition of the definition says that Z is the intersection of A and B where multiple intersections (that is: points that are multiple for α or β) are counted with appropriate multiplicity. The second condition says that the intersection is clean in the terminology of Quillen, that is, the tangent space of the intersection manifold is locally the same as the intersection of the two tangent spaces.

Definition 2.5. In a sub-cartesian diagram $E = \mathrm{coker}(d\alpha - d\beta)$ is called the excess vector bundle.

Locally the rank of E indicates how far is the intersection of A and B being transversal. Thus generally neither Z nor E will have constant dimension. The diagram is cartesian if and only if E is zero, that is when α and β are transversal to each other.

The main step of the proof is the following lemma, which describes the cohomological properties of a sub-cartesian diagram.

Lemma 3. *$\alpha^*\beta_!(c) = f_A!(e(E)\cdot f_B^*(c))$ holds for any $c \in H^*(B)$.*

Proof. We will think of Z as a subset of $A \times B$. We can equip X with a Riemannian metric, and then we can identify E with the orthogonal complement of $\mathrm{Im}(d\alpha - d\beta)$ in $f_A^*\alpha^*TX$. We denote by $e : TX \to X$ the exponential map associated to the metric. For every $x \in X$ there is an open neighbourhood U_x of $0 \in TX_x$ such that $e_x = e|_{U_x}$ is a diffeomorphism onto an open neighbourhood of x.

Let Ω be a closed tubular neighbourhood of Z in $A \times B$. Let $\partial\Omega$ denote its boundary. In order to represent the Euler class of E, we take a section $v : Z \to E$ that is transversal to the zero section. We denote by $\bar{E}$ and $\bar{v}$ the extensions of E and v to Ω, with $\bar{E}$ being a sub-bundle of $TX' = p_A^*\alpha^*TX|_\Omega$. Here p_A means the projection of Ω to its first coordinate.

If Ω is small enough, then for any $(a,b) \in \Omega$ we have $\beta(b) \in e_{\alpha(a)}(U_{\alpha(a)})$. So setting $w(a,b) = e_{\alpha(a)}^{-1}(\beta(b))$ defines a section of TX' over Ω. In other words $w(a,b)$ is the tangent vector of X in $\alpha(a)$ that "points towards" $\beta(b)$. So if Ω is small enough, then $w(a,b)$ is always in the region spanned by TA and TB, so $w(a,b) \notin \bar{E}_{(a,b)}$ when $(a,b) \notin Z$. Thus we can add w and $\bar{v}$ and this will not change the zero-set of the section: Let $\bar{w} = w + \bar{v}$, and let $Z'' = \{(a,b) \in \Omega : \bar{w}(a,b) = 0\} = \{(a,b) \in Z : v(a,b) = 0\}$.

Since the intersection of A and B is clean, $\bar{w}$ is transversal to the zero section of TX'. Now setting $F : \Omega \to X \times X, F(a,b) = (\alpha(a), e_{\alpha(a)}(\bar{w}(a,b))$ we obtain a slight perturbation of $\alpha \times \beta$ that is transversal to the diagonal of $X \times X$. Further obviously $F^{-1}(\delta_X(2)) = Z''$ holds. So now we realized the zero-set of v as a transversal preimage of the diagonal in $X \times X$ by a map that is almost $\alpha \times \beta$.

On the other hand we can make α transversal to β by perturbing it slightly. The perturbed map will be denoted $\alpha' : A \to X$. So now $F' = \alpha' \times \beta : A \times B \to X \times X$ is also transversal to the diagonal. Set $Z' = (\alpha' \times \beta)^{-1}(\delta_X(2))$. The following diagram is cartesian:

$$\begin{array}{ccc} Z' & \xrightarrow{f'_B} & B \\ \Big\downarrow{\scriptstyle f'_A} & & \Big\downarrow{\scriptstyle \beta} \\ A & \xrightarrow{\alpha'} & X \end{array}$$

where f'_A and f'_B are the projections, so by the well known properties of duality we have $\alpha'^*\beta_!(c) = f'_{A!}f'^*_B(c)$.

Consider the inclusions $i : Z \hookrightarrow \Omega, i' : Z' \hookrightarrow \Omega, i'' : Z'' \hookrightarrow \Omega, j : Z'' \hookrightarrow Z$, the projection $p_A : \Omega \to A$ and the associated Gysin homomorphisms $i_! : H^*(Z) \to H^*(\Omega, \partial\Omega)$, similarly for $i'_!$

and $i''_!$, and $p_{A!} : H^*(\Omega, \partial\Omega) \to H^*(A)$. If α' is close enough to α and v is close enough to the zero section, then F and $F'|_\Omega$ are homotopic through maps transversal to $\delta_X(2)$, and sending $\partial\Omega$ to $X \times X \setminus \delta_X(2)$. Thus Z' and Z'' are isotopic in $(\Omega, \partial\Omega)$. This means that $i'_! f'^*_B = i''_!(f_B j)^*$. Finally we remind that Z'' is the zero set of a section transversal to the zero section of E and so by a previous remark $j_!(1) = e(E)$.

Using that $f'_A = p_A i'$, $i'' = ij$, $f_A = p_A i$ and Lemma 2, we finish the proof with the following sequence of equations:

$$\begin{aligned}\alpha^* \beta_!(c) = \alpha'^* \beta_!(c) = f'_A f'^*_B(c) = P_{A!} i'_! f'^*_B(c) = p_{A!} i''_! j^* f^*_B(c) = \\ = p_{A!} i_! j_! j^* f^*_B(c) = (p_A i)_!(j_!(1) \cdot f^*_B(c)) = f_{A!}(e(E) \cdot f^*_B(c)).\end{aligned}$$

□

2.3 The formula

Theorem 1 (Herbert's formula). *Let $f : V^n \to W^m$ be a proper self-transverse immersion. Then the closures of the k-tuple intersection sets, $\bar{N}_k(f)$ and $\bar{M}_k(f)$ carry fundamental classes with $\mathbb{Z}_2$ coefficients. Denoting by n_k and m_k their Poincaré duals in W and V respectively and setting $e = e(\nu_f)$ we have:*

$$m_k = f^*(n_{k-1}) - e \cdot m_{k-1} \qquad (*)$$

If $m - n$ is even and V and W are oriented, $\bar{N}_k(f)$ and $\bar{M}_k(f)$ carry fundamental classes with $\mathbb{Z}$ coefficients, and the above formula is valid in integral cohomology.

Proof. We shall apply Lemma 3 to the following diagram:

$$\begin{tikzcd} \tilde{M}_k(f) \cup \tilde{M}_{k-1}(f) \arrow[r, "p"] \arrow[d, "f_k \cup f_{k-1}"] & \tilde{N}_{k-1}(f) \arrow[d, "g_{k-1}"] \\ V \arrow[r, "f"] & W \end{tikzcd}$$

Here the notations of Section 2.1 are used and

$$p(x_1, [x_2, \ldots, x_k]) = [x_2, \ldots, x_k], p(x_1, [x_2, \ldots, x_{k-1}]) = [x_1, \ldots, x_{k-1}].$$

Since $f^k : V^{(}k) \setminus \Delta_V(k) \to W^{(}k)$ is transversal to $\delta_W(k)$, the above diagram is sub-cartesian. The intersection is transversal at the k-tuple points so the excess bundle is zero on $\tilde{M}_k(f)$ and it is an inclusion at the $k-1$-tuple points so the excess bundle is $f^*_{k-1}(\nu_f)$ on $\tilde{M}_{k-1}(f)$. Let us denote the fundamental class of a manifold by $[\,]$. From Lemma 1 it follows that $f_{k*}([\tilde{M}_k(f)])$ and $g_{k*}([\tilde{N}_k(f)])$ are fundmental classes for $\bar{M}_k(f)$ and $\bar{N}_k(f)$ respectively. Thus $m_k = f_{k!}(1)$ and $n_k = g_{k!}(1)$. Now we can apply Lemma 3 to obtain that

$$\begin{aligned}f^*(n_{k-1}) = f^*(g_{k-1!}(1)) = f_{k!}(e(\emptyset) \cdot p^*(1)) + f_{k-1!}(e(f^*_{k-1}(\nu_f)) \cdot p^*(1)) = \\ = f_{k!}(1) + f_{k-1!}(f^*_{k-1}(e(\nu_f))) = m_k + e(\nu_f) \cdot f_{k-1!}(1) = m_k + e \cdot m_{k-1}\end{aligned}$$

which completes the proof. □

There are two immediate consequences to the (*) formula.

Corollary. $m_k = \sum_{j=0}^{k-1} (-1)^j e^j f^*(n_{k-j-1})$.

Corollary. In the oriented case it is clear that $f_*([\bar{M}_k(f)]) = k \cdot [\bar{N}_k(f)]$ which can be written $f_!(m_k) = k \cdot n_k$ using duality. Let us define for a positive integer h the homomorphism $\varphi_h : H^*(V) \to H^*(V)$ by $\varphi_h(a) = f^* f_!(a) - h(e \cdot a)$. It follows easily from (*) that

$$(k-1)! m_k = \varphi_{k-1} \circ \varphi_{k-2} \circ \cdots \circ \varphi_1(1).$$

3 The universal map method

The aim of this section is to give a new proof of Herbert's formula in the oriented case and even codimension. Our method will only work with $\mathbb{Z}_p$ or $\mathbb{Q}$ coefficients where p is an odd prime, so we shall reprove the formula only modulo 2-torsion. The same method applied to the complex case proves the formula with $\mathbb{Z}$ coefficients. There is however one advantage of this new proof: it can be generalized to maps with certain singularities.

Our main tool will be Szűcs's construction of the so called universal immersion (see [15], [16] in Russian and [17] in English). It is a map $f: Y \to X$ such that whenever $g: M \to N$ is an immersion between smooth manifolds then there exist maps $s: M \to Y$ and $t: N \to X$ for which $t \circ g = f \circ s$. Further the k-tuple points of g are exactly the preimage of the k-tuple points of f by the map s. So if a formula concerning k-tuple points holds for f, then it holds for any immersion g. Thus it is sufficient to show Herbert's formula for the map f. R. Rimányi proves the formula in the complex case modulo torsion by a method based on the construction of the universal immersion. Here we improve this method to obtain a proof in the oriented case modulo 2-torsion. This improved method can also be used to prove the complex case completely.

This section is organized as follows: First we shall outline the construction of X and Y since some basic properties of them are later required. We will also show the (slightly simpler) construction of a map $\bar{f}: \bar{Y} \to \bar{X}$ that is universal (in the same sense as above) for those immersions which are projections of embeddings. These immersions will be called "numbered immersions". Next we will show Herbert's formula for $\bar{f}$ using Rimányi's method described in [11], and then derive the formula for f from this result. Finally we show how the formula can be generalized to maps with certain singularities. Throughout the whole part all cohomologies will be meant either with $\mathbb{Z}_p$ or with $\mathbb{Q}$ coefficients.

3.1 Construction of the universal map

Now we outline the construction of the universal map due to A. Szűcs, concentrating on the properties we need for the proof of the formula. Details can be found in [2]. First we shall describe the construction of $\bar{Y}$ and $\bar{X}$. Let k denote the fixed even codimension. Both spaces are glued together from blocks. We denote the subspaces which consist only of the first s blocks by $\bar{Y}_s$ and $\bar{X}_s$. Let γ^k denote the canonical vector bundle over the base space $BSO(k)$. Further define the bundle

$$\bar{\xi}_s = \varepsilon^0 \times \gamma_2^k \times \cdots \times \gamma_s^k \text{ over the base } \bar{K}_s = B[SO(k) \times SO(k)^{s-1}].$$

Here the indices of the bundles are only a distinctive sign to that they are over different bases. The disc bundle of $\bar{\xi}_s$ will be the s'th block of $\bar{Y}$. Thus $\bar{Y}_s$ is obtained from $\bar{Y}_{s-1}$ by gluing to it the disc bundle of $\bar{\xi}_s$ along its boundary. So $\bar{Y}_s \setminus \bar{Y}_{s-1} = D(\xi_s)$. Similarly define

$$\bar{\eta}_s = \gamma_1^k \times \cdots \times \gamma_s^k \text{ over the base space } \bar{L}_s = B[SO(k)^s].$$

The disc bundle of $\bar{\eta}_s$ will be the s'th block of $\bar{X}$. We can define $\bar{f}|_{\bar{K}_s}$ so that the pullback of $\bar{\eta}_s$ is $\gamma_1^k \times \bar{\xi}_s$. It is easily seen that actually $\bar{f}|_{K_s}$ is the identical map. The idea behind this construction is that given a numbered immersion of codimension k, the normal bundle of its s-tuple point set can be decomposed into the sum of s k-plane bundles.

In the case when the immersion is not numbered, each fiber of the normal bundle of the s-tuple point set breaks up as a sum of s k-planes, but the whole normal bundle does not necessarily split into a sum of s k-plane bundles. Thus the structure group in this situation is $SO(k)^s \rtimes S_s$ instead of $SO(k)^s$ (here S_s is the symmetric group on s elements, and $\rtimes$ is the semidirect product with the obvious S_s action on $SO(k)^s$). Accordingly the spaces Y and X become more complicated.

Just like above, X_s and Y_s will denote those subspaces of X and Y that contain only the first s blocks. The s'th block – $Y_s \setminus Y_{s-1}$ – of the space Y will be the disc bundle of

$$\xi_s = \varepsilon^0 \times (\gamma_2^k \times \cdots \times \gamma_s^k) \rtimes S_{s-1} \text{ over the base space } K_s = B[SO(k) \times (SO(k)^{s-1} \rtimes S_{s-1})].$$

Here the symbol $\rtimes$ in the bundle definition means the following: Given a space T and a group G one can form the product $T \times EG$, where EG is the total space of the universal principal G-bundle. Thus G acts freely on this product with the diagonal action. Define $T \rtimes G$ as $T \times_G EG$. Again we can similarly define the s'th block of X: it will be the disc bundle of

$$\eta_s = (\gamma_1^k \times \cdots \times \gamma_s^k) \rtimes S_s) \text{ over the base space } L_s = B[SO(k)^s \rtimes S_s].$$

The map $f|_{K_s}$ is defined such that the pullback of η_s is $\gamma_1^k \times \xi_s$. It is easily seen that $f|_{K_s}$ is an s-sheeted covering.

3.2 The formula for $\bar{f}$

If $U \subset M$ is a subset whose closure represents a homology class, let $[U]$ denote the cohomology class in $H^*(M)$ that is Poincaré dual to the class represented by the closure of U. Herbert's formula can be written now as

$$[\bar{K}_r] = (\bar{f})^*[\bar{L}_{r-1}] - e(\bar{f})[\bar{K}_{r-1}] \in H^{(r-1)k}(\bar{Y}) \qquad (*)$$

Following Rimányi's method we shall first prove this equality restricted to $\bar{K}_s$, then use a Mayer-Vietoris argument to prove it in $\bar{Y}$.

As $\bar{Y}$ is not a manifold, the notation $[\bar{K}_r]$ needs to be explained. $\bar{K}_r$ is the zero section in the r-th block $\bar{Y}_r \setminus \bar{Y}_{r-1}$ which is a disc bundle. As such it has a dual cohomology class in the Thom space of the bundle, namely the Thom class of the bundle. This class can be viewed as an element of $H^*(\bar{Y}, \bar{Y} \setminus \bar{K}_r)$, and restricting to $\bar{Y}$ we get $[\bar{K}_r]$. Let us compute $[\bar{K}_r]|_{\bar{K}_s}$. First observe that if $s < r$ then the closure of $\bar{K}_{r-1}$ is disjoint from $\bar{K}_s$ so the class restricts to zero. Thus we may assume $r \le s$. The neighbourhood of $\bar{K}_s$ in $\bar{Y}$ is the disc bundle of $Q_1 = \varepsilon^0 \times \gamma_2^k \times \cdots \times \gamma_s^k$. Now the closure of $\bar{K}_r$ intersects this bundle in

$$Q_r = \bigcup span(\gamma_{i_1}^k, \ldots, \gamma_{i_{s-r}}^k)$$

where union is meant for all $s-r$ element subsets $\{i_1, \ldots, i_{s-r}\}$ of $\{2, 3, \ldots, s\}$. This means that

$$[\bar{K}_r]|_{\bar{K}_s} = [Q]|_{\bar{K}_s} = e(\nu(Q_r \subset Q_1)|_{\bar{K}_s}) = e(\nu(Q_r \subset Q_1)|_{\bar{K}_s}) =$$
$$= e\left(\bigoplus(\gamma_{j_1}^k \times \cdots \times \gamma_{j_{r-1}}^k)\right) = \sum e(\gamma_{j_1}^k) \cdots e(\gamma_{j_{r-1}}^k) = \sigma_{r-1}(e(\gamma_2^k), \ldots, e(\gamma_s^k))$$

A similar computation shows that $f^*[\bar{L}_r]|_{\bar{K}_s} = \sigma_r(e(\gamma_1^k), \ldots, e(\gamma_s^k))$. It is obvious that $e(f)|_{\bar{K}_s} = e(\gamma_1^k)$. Thus to prove the formula restricted to $\bar{K}_s$ we only have to show the following equation:

$$\sigma_{r-1}(e(\gamma_2^k), \ldots, e(\gamma_s^k)) = \sigma_{r-1}(e(\gamma_1^k), \ldots, e(\gamma_s^k)) - e(\gamma_1^k)\sigma_{r-2}(e(\gamma_2^k), \ldots, e(\gamma_s^k))$$

which is obviously an identity.

It follows from the previous paragraph that (*) holds in $\bar{Y}_1$, since it simply the case of $s = 1$. Suppose we know it holds in $\bar{Y}_s$ We prove that it holds in $\bar{Y}_{s+1}$ as well. Consider the following portion of the Mayer-Vietoris sequence for the triple $\bar{Y}_{s+1} = \bar{Y}_s \cup D\bar{\xi}_{s+1}$:

$$H^{(r-1)k-1}(\bar{Y}_s) \oplus H^{(r-1)k-1}(D\bar{\xi}_{s+1}) \xrightarrow{1} H^{(r-1)k-1}(\bar{Y}_s \cap D\bar{\xi}_{s+1}) \xrightarrow{2}$$
$$\xrightarrow{2} H^{(r-1)k}(\bar{Y}_s \cup D\bar{\xi}_{s+1}) \xrightarrow{3} H^{(r-1)k}(\bar{Y}_s) \oplus H^{(r-1)k}(D\bar{\xi}_{s+1}).$$

Let us use the following notation:

$$\bar{h}_r = \big([\bar{K}_r] - (\bar{f})^*[\bar{L}_{r-1}] + e(\bar{f})[\bar{K}_{r-1}]\big) \in H^{(r-1)k}(\bar{Y})$$

We have to show that $\bar{h}_r|_{\bar{Y}_{s+1}} = 0$. By our inductional assumption and the previous paragraph, h_r maps to $(0,0)$ at map 3. So it suffices to show that map 2 is the zero map. By exactness it is enough to show that map 1 is surjective. In fact already the map $H^{(r-1)k-1}(D\bar{\xi}_{s+1}) \to H^{(r-1)k-1}(\bar{Y}_s \cap D\bar{\xi}_{s+1})$ is surjective. Observe that this latter cohomology group is equal to $H^{(r-1)k-1}(S\bar{\xi}_{s+1})$ (here S stands for the sphere bundle), and the map is contained in the Gysin sequence (as map 4):

$$H^{(r-1)k-1}(D\bar{\xi}_{s+1}) \xrightarrow{4} H^{(r-1)k-1}(S\bar{\xi}_{s+1}) \xrightarrow{5}$$
$$H^{(r-1)k}(D\bar{\xi}_{s+1}, S\bar{\xi}_{s+1}) \xrightarrow{6} H^{(r-1)k}(D\bar{\xi}_{s+1})$$

So it is sufficient to show that map 4 is surjective. This is equivalent to map 5 being zero, which is further equivalent to map 6 being injective. In view of the Thom isomorphism theorem map 6 is equivalent to multiplication by the Euler class of $\bar{\xi}_{s+1}$ in the base space of the bundle. The cohomology ring of the base ($B[SO(k)^s]$) is a polynomial ring with $\mathbb{Z}_p$ or $\mathbb{Q}$ coefficients and the Euler class is not zero, so this multiplication is clearly injective. This completes the proof of the formula for $\bar{f}$.

3.3 The formula for f

Now we shall use the formula for $\bar{f}$ to prove it for f. Again let us use a notation similar to the one in the previous section:

$$h_r = ([K_r] - (f)^*[L_{r-1}] + e(f)[K_{r-1}]) \in H^{(r-1)k}(Y)$$

We want to show that $h_r = 0$. It is clearly enough to show that $h_r|_{Y_s} = 0$ for every s.

One could try to carry out the proof the same way as in the last section. However to do so one would have to understand the cohomologies of K_s in order to prove that $h_r|_{Y_s/Y_{s-1}} = 0$ which was a key step. These cohomology rings are quite complicated. Rimányi avoids this problem by pulling back to a space with simpler cohomologies. This pullback is an s sheeted covering which can only be handled when one can divide by s in the coefficient ring. So his method only works for rational coefficients.

Here we present an idea of A. Szűcs that allows to get the formula alo with $\mathbb{Z}_p$ coefficients where p is any odd prime. Instead of restricting to the blocks of Y we will use the natural map $g : \bar{Y} \to Y$. This map is obtained by inducing the universal numbered immersion $\bar{f} : \bar{Y} \to \bar{X}$ from the universal unnumbered immersion $f : Y \to X$. (Although $\bar{f}$ is not an immersion, g can be constructed on each block by inducing, and the so obtained g's will fit together after gluing.)

It is obvious that $h_r|_{Y_s} = 0$ if $s < r - 1$ since in this case there are no r- or $(r-1)$-tuple points and thus the relevant homology classes are zero. Let us suppose this holds for $s = r - 1$

as well (we will prove this at the end of the section). On the other hand $g^*(h_r) = \bar{h}_r = 0$. Now we show that $h_r|_{Y_r} = 0$. Consider the following diagram. The rows of the diagram are the exact sequences for the pairs $Y_{r-1} \subset Y_r$ and $\bar{Y}_{r-1} \subset \bar{Y}$. The vertical maps are induced by $g : \bar{Y}_s \to Y_s$, $s = r, r-1$. For shorter notation we shall put $i = (r-1)k$.

$$\begin{array}{ccccccccc}
 & & H^i(Y_r/Y_{r-1}) & \xrightarrow{\alpha} & H^i(Y_r) & \xrightarrow{\beta} & H^i(Y_{r-1}) & \longrightarrow & H^{i+1}(Y_r/Y_{r-1}) \\
 & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle g^*} & & \downarrow & & \downarrow \\
H^{i-1}(\bar{Y}_{r-1}) & \xrightarrow{\bar{\partial}} & H^i(\bar{Y}_r/\bar{Y}_{r-1}) & \xrightarrow{\bar{\alpha}} & H^i(\bar{Y}_r) & \xrightarrow{\bar{\beta}} & H^i(\bar{Y}_{r-1}) & \longrightarrow & H^{i+1}(\bar{Y}_r/\bar{Y}_{r-1})
\end{array}$$

We assumed that $\beta(h_r|_{Y_r}) = h_r|_{Y_{r-1}} = 0 \in H^i(Y_{r-1})$. Thus by exactness there is an element $x \in H^i(Y_r/Y_{r-1})$ for which $\alpha(x) = h_r|_{Y_r}$. By the commutativity of the diagram we have $g^*(\alpha(x)) = \bar{\alpha}(\gamma(x))$. By the formula for $\bar{f}$ we have that $g^*(\alpha(x)) = g^*(h_r|_{Y_r}) = \bar{h}_r|_{\bar{Y}_r} = 0$, so $\bar{\alpha}(\gamma(x)) = 0$. If we show that both $\bar{\alpha}$ and γ are injective, then $x = 0$ and hence $h_r|_{Y_r} = 0$ will follow.

Lemma 4. *$\bar{\alpha}$ is injective.*

Proof. It is sufficient to show that $\bar{\partial}$ is the zero map. We can even say more: the group $H^{i-1}(\bar{Y}_{r-1}) = 0$. Recall that the codimension k is even, so $i - 1 = (r-1)k$ is odd. The spaces $\bar{Y}_s$ have no cohomologies in odd dimensions. This can be proved by induction on s. Consider the exact sequence for an odd number j:

$$H^j(\bar{Y}_s/\bar{Y}_{s-1}) \to H^j(\bar{Y}_s) \to H^j(\bar{Y}_{s-1})$$

Here the last group is 0 by induction. The first group is also 0 since the cohomology ring of $\bar{Y}_s/\bar{Y}_{s-1}$ is the same as the cohomology ring of $\bar{K}_s$ which is a polynomial ring generated by the Pontrjagin classes and possibly the Euler class, which are all even dimensional. (Recall that we have $\mathbb{Z}_p$ or $\mathbb{Q}$ as the coefficient ring, and that k is even.) Thus the middle group must as well be 0 hence the statement indeed follows by induction. $\square$

Lemma 5. *γ is injective.*

Proof. Let us first consider a G finite group that acts freely on the space A. Let us denote A/G by C. As before, EG denotes the total space of the principal universal G-bundle. It is a contractible space on which G acts freely. As we have already seen, we can form the space $A \times_G EG$ which is the factor of $A \times EG$ by the diagonal action of G. As G acts freely on both factors, the action can be shifted to either side, thus obtaining two fibrations: $A \times_G EG \xrightarrow{A} BG$ and $A \times_G EG \xrightarrow{EG} A/G = C$. The latter fibration map is a homotopy equivalence since EG is contracible. Thus we get a fibration of C over BG with A as the fiber: $C \xrightarrow{A} BG$.

Now we use this general construction in the following setting:

$$\begin{array}{c}
A = D(\varepsilon^0 \times \gamma_2^k \times \cdots \times \gamma_r^k) \times ES_{r-1} \\
\downarrow{\scriptstyle /S_{r-1}} \\
A/S_{r-1} = C = D(\varepsilon^0 \times \gamma_2^k \times \cdots \times \gamma_r^k) \rtimes S_{r-1}
\end{array}$$

Here A is homotopy equivalent to the r'th block of $\bar{Y}$ and C is exactly the r'th block of Y. These spaces have boundary, so our general construction yields a fibration pair:

$$\begin{array}{c}
(D\xi_r, S\xi_r) \\
\downarrow{\scriptstyle (D\bar{\xi}_r, S\bar{\xi}_r)} \\
BS_{r-1}
\end{array}$$

By the Thom isomorphism theorem the first non-zero cohomology group of the pair $(D\bar{\xi_r}, S\bar{\xi_r})$ is in dimension $(r-1)k$. So the same stands for homology groups and by Hurewicz's theorem for homotopy groups. Thus $\bar{Y_r}\backslash\bar{Y}_{r-1} = D\bar{\xi_r}\backslash S\bar{\xi_r}$ has a CW-complex structure with no cells in dimensions less than $(r-1)k$. Let us give BS_{r-1} a structure of a CW-complex such that there is only one null-cell. This way we can obtain a CW structure for $(D\xi_r, S\xi_r)$ such that every cell is the product of a cell in the base and a cell in the fiber. Thus the $(r-1)k$-skeleton of $(D\xi_r, S\xi_r)$ is exactly the preimage of the null-cell in the base. So $sk_{(r-1)k}(D\xi_r/S\xi_r) \subset sk_{(r-1)k}(D\bar{\xi_r}/S\bar{\xi_r})$. Thus the map $(D\bar{\xi_r}/S\bar{\xi_r}) = \bar{Y_r}/\bar{Y}_{r-1} \to Y_r/Y_{r-1} = D\xi_r/S\xi_r$ is surjective on the level of $(r-1)k$-cycles, so it is injective on the level of $(r-1)k$-cocycles. Hence the induced map $\gamma : H^{(r-1)k}(Y_r/Y_{r-1}) \to H^{(r-1)k}(\bar{Y_r}/\bar{Y}_{r-1})$ is injective. This finishes the proof. □

To complete the proof of $h_r|_{Y_r} = 0$ we have to show that $h_r|_{Y_{r-1}} = 0$. More generally it is enough to show that if an immersion $g : M \to N$ has no r-tuple points, then $g^*[L_{r-1}] = [K_{r-1}]$. The $r-1$-tuple point set of such an immersion is an embedded manifold both in M and N, and let the homology class represented by them be denoted by $K = K_{r-1}$ and $L = L_{r-1}$ respectively.

Let us consider ν_g the normal bundle of g over M. Denote by $s : M \to \nu_g$ a section of ν_g that is a small transversal perturbation of the zero section. The "push forward" of s defines a small perturbation of g denoted by $\tilde{g}$. This perturbation is transversal to L. If s is sufficiently small then $\tilde{g}(x) \in L$ can only hold if $s(x) = x$. Now by the basic properties of duality one has:

$$g^*[L] = \tilde{g}^*[L] = [\tilde{g}^{-1}(L)] = [K \cap \{s(x) = x\}] = [K] \cup [\{s(x) = x\}] = e(\nu_g)[K]$$

which is exactly what we wanted to prove.

Finally we have to show that $h_r|_{Y_s} = 0$ if $s > r$. This will be done by induction. Consider the cohomological long exact sequence of the pair $Y_{s-1} \subset Y_s$. We use again the simplifying notation $i = (r-1)k$.

$$H^i(Y_s/Y_{s-1}) \to H^i(Y_s) \xrightarrow{(1)} H^i(Y_{s-1}) \to H^{i+1}(Y_s/Y_{s-1})$$

Here the first and last groups are zero, since Y_s/Y_{s-1} is the Thom space of an $(s-1)k$ dimensional vector bundle thus the first $(s-1)k-1$ cohomology groups of Y_s/Y_{s-1} vanish, and $i \leq (s-1)k-1$ as $r < s$. So map (1) is an isomorphism. By induction $h_r|_{Y_{s-1}} = 0$ so by the isomorphism $h_r|_{Y_s} = 0$. So we are finished with the proof of Herbert's formula. This method proves the complex case completely, the real oriented case modulo 2-torsion.

3.4 Maps with singularities

Let us fix r and the even codimension k. We shall consider maps that may have singularities, but we restrict ourselves to such singular maps that take their multisingularities from a given set. We need two properties of this set. We require that only such multisingularities are in the set for which there are no $r-1$ tuple points in the neighbourhood of the singularity. We also require that there is a universal map construction for this set of multisingularities (see [12]). This universal map construction is similar to that described in section 3.1. It even has a similar block structure. The only difference is that there are some blocks which are glued to Y_{r-2} that correspond to the multisingularities that we allow. As we do not allow multisingularities that have $r-1$-tuple points in their neighbourhood, no blocks are glued to $Y_s \setminus Y_{r-2}$ if $s \geq r-1$. For $s \geq r-2$ let us denote by Z_s the space Y_s together with the blocks that were glued to Y_{r-2}. We shall use the notations of the previous sections. Similarly as in the immersion case it is enough to show that $h_r|_{Z_s} = 0$ for $s \geq r$. If $s \geq r$ then in the exact sequence

$$H^i(Z_s/Z_{s-1}) \to H^i(Z_s) \xrightarrow{(1)} H^i(Z_{s-1}) \to H^{i+1}(Z_s/Z_{s-1})$$

the first and last groups are again zero, in fact $Y_s/Y_{s-1} = Z_s/Z_{s-1}$. So by induction it is enough to prove $h_r|_{Z_r} = 0$. Consider the following diagram of long exact sequences:

$$\begin{tikzcd}
H^i(Y_r/Y_{r-1}) \arrow[r, "\alpha"] & H^i(Y_r) \arrow[r, "\beta"] & H^i(Y_{r-1}) \arrow[r] & H^{i+1}(Y_r/Y_{r-1}) \\
H^i(Z_r/Z_{r-1}) \arrow[r, "\tilde{\alpha}"] \arrow[u, "\cong"] & H^i(Z_r) \arrow[r, "\tilde{\beta}"] \arrow[u, "g^*"] & H^i(Z_{r-1}) \arrow[r] \arrow[u] & H^{i+1}(Z_r/Z_{r-1}) \arrow[u, "\cong"]
\end{tikzcd}$$

If $h_r|_{Z_r}$ maps to 0 at g^* we are done, since then there is an $x \in H^i(Z_r/Z_{r-1}) = H^i(Y_r/Y_{r-1})$ such that $\tilde{\alpha}(x) = h_r|_{Z_r}$ and $\alpha(x) = g^*(h_r|_{Z_r}) = 0$. (The last equality is simply the original formula.) But we have shown in the course of the new proof of the formula that α is injective. Thus $x = 0$ and so $h_r|Z_r = 0$.

So we only need to show that $h_r|Z_{r-1} = 0$. This is done by exactly the same argument that was used in the immersion case.

4 Application via cobordism

4.1 The Boardman homomorphism

In this section we shall consider the extraordinary cohomology theory formed by the unoriented cobordism groups. $\mathcal{N}^*(X)$ will denote the unoriented cobordism cohomology groups of X, while $H^*(X)$ will denote the ordinary cohomolgy ring of X with $\mathbb{Z}_2$ coefficients. As before, $e(\xi)$ will denote the Euler class of the vector bundle ξ in cobordism theory and $e_H(\xi)$ will denote the Euler class in ordinary cohomology.

The Boardman homomorphism is a natural transformation from the unoriented cobordism groups to ordinary cohomology. It can be written in the form of a homomorphism:

$$\beta : \mathcal{N}^*(\cdot) \to H^*(\cdot)[[t_1, \dots, t_n, \dots]]$$

where the latter is the field of formal power series of the variables $t_1, t_2, \dots$ with $H^*(X)$ coefficients. It has the following properties:

(i) $\beta(f^*(a)) = f^*(\beta(a))$ for any $f : X \to Y$ and $a \in \mathcal{N}^*(Y)$,

(ii) $\beta(ab) = \beta(a)\beta(b)$,

(iii) for a line bundle l over X we have $\beta(e(l)) = e_H(l) + \sum_{j=1,2,\dots} e_H(l)^j t_j$.

Let ξ be an n-dimensional vector bundle over X. By the splitting principle there exists a space Y and a map $p : Y \to X$ such that $p^*\xi$ splits into a sum of n line bundles $l_1, \dots, l_n$ over Y while $p^* : H^*(X) \to H^*(Y)$ is injective. Thus the image of the characteristic class $w(\xi)$ is $p^*w(\xi) = (1 + e_H(l_1)) \cdots (1 + e_H(l_n))$.

Using the injectivity of p^* we can define a characteristic class in $H^*(X)[[t_1, \dots, t_n, \dots]]$ with the formula

$$p^*w_t(\xi) = \prod_{i=1}^{n} \left(1 + \sum_{k=1}^{\infty} e_H(l_i)^k t_k\right)$$

since the coefficient of any monomial in this expression is a symmetric polynomial in $(e_H(l_i))$ and thus the image of a uniqe element of $H^*(X)$. It is obvious to see that for any $f : X \to Y$ we have $w_t(f^*(\xi)) = f^*(w_t(\xi))$ and $w_t(\xi \oplus \eta) = w_t(\xi)w_t(\eta)$ for ξ, η bundles over X. It also follows from the definition that if l is a line bundle over X, then

$$w_t(l) = 1 + e_H(l)t_1 + e_H(l)^2 t_2 + \cdots .$$

The relation between w_t and β is stated in the following lemma.

Lemma 6.

(1) For a smooth map $f : X \to Y$ between closed manifolds $\beta(f_!(1)) = f_!(w_t(\nu_f))$ where ν_f is the normal bundle of f.

(2) For a vector bundle ξ over X

$$\beta(e(\xi)) = e_H(\xi)w_t(\xi).$$

Proof. The proof of (2) is a straightforward calculation:

$$p^*\beta(e(\xi)) = \beta(p^*(e(\xi)) = \beta(e(l_1)\cdots e(l_n)) =$$
$$= \beta(e(l_1))\cdots\beta(e(l_n)) = \prod_{i=1}^{n}(e_H(l_i) + e_H(l_i)^2 t_1 + \cdots) =$$
$$= e_H(l_1)\cdots e_H(l_n)\cdot\prod_{i=1}^{n}\left(1 + \sum_{k=1}^{\infty} e_H(l_i)^k t_k\right) = p^*(e_H(\xi)w_t(\xi))$$

This proves the statement since p^* is injective in ordinary cohomology. □

4.2 On the Stiefel-Whitney numbers of $\tilde{M}_k(f)$

In this section we will apply Herbert's formula and the results about the Boardman homomorphism to obtain results about the cobordism class of the multiple point manifolds. The methods in this section are due to Kamata (see [8]). We shall use the notations of Section 2.1 and Section 4.1.

Lemma 7. *If a self-transverse immersion $f : V^n \to W^m$ is homotopic to the constant map, then*

$$f_{k!}\left(\frac{1}{w_t(\tau(\tilde{M}_k(f)))}\right) = e_H(\nu_f)^{k-1}\left(\frac{1}{w_t(\tau(V^n))}\right)^k$$

in $H^*(V^n, \mathbb{Z}_2)[[t_1, t_2, \ldots]]$.

Proof. We saw that $m_k = \sum_{j=0}^{k-1}(-1)^j e^j f^*(n_{k-j-1})$ as a corollary of Herbert's formula. In our case f is homotopic to the constant map, so f^* is the zero homomorphism. On the other hand $m_k = f_{k!}(1)$ was the definition of m_k. Combining these two observations we get that $f_{k!}(1) = e(\nu_f)^{k-1}$. Let us apply the Boardman homomorhpism to both sides of this equality: $\beta(f_{k!}(1)) = \beta(e(\nu_f))^{k-1}$. By Lemma 6 we obtain

$$\beta(f_{k!}(1)) = f_{k!}(w_t(\nu_{f_k})) = f_{k!}\left(w_t(\nu_{f_k})\frac{w_t(\tau(\tilde{M}_k(f)))}{w_t(\tau(\tilde{M}_k(f)))}\right) = f_{k!}\left(\frac{f_k^*(w_t(\tau(V^n)))}{w_t(\tau(\tilde{M}_k(f)))}\right) =$$
$$= w_t(\tau(V^n))f_{k!}\left(\frac{1}{w_t(\tau(\tilde{M}_k(f)))}\right)$$

and

$$\beta(e(\nu_f))^{k-1} = (e_H(\nu_f)w_t(\nu_f))^{k-1} = \left(\frac{e_H(\nu_f)w_t(\nu_f)w_t(\tau(V^n))}{w_t(\tau(V^n))}\right)^{k-1} =$$
$$= \left(\frac{e_H(\nu_f)f^*(w_t(\tau(W^m)))}{w_t(\tau(V^n))}\right)^{k-1} = \left(\frac{e_H(\nu_f)}{w_t(\tau(V^n))}\right)^{k-1}$$

Thus the two last terms in these sequences are equal to each other, which is exactly what we wanted to prove. □

Let us take a closer look at the formula of this Lemma. It expresses an equality of two formal power-series. Thus it is equivalent to say that the coefficients of the two series are pairwise equal. Hence using Herbert's single original formula we derived a series of identities concerning Stiefel-Whitney classes of the multiple point manifolds. This approach is used in the next two theorems.

Theorem 2. *If a self-transverse immersion $f : V^n \to W^m$ is homotopic to the constant map, then*

$$\left\langle \frac{1}{w_t(\tau(\tilde{M}_k(f)))}, [\tilde{M}_k(f)] \right\rangle = \left\langle e_H(\nu_f)^{k-1} \left(\frac{1}{w_t(\tau(V^n))} \right)^k, [V^n] \right\rangle$$

Proof. The formula is obtained from Lemma 7 by evaluating both sides on $[V^n]$, and noticing that $\langle f_{k!}(a), [V^n]\rangle = \langle a, [\tilde{M}_k(f)]\rangle$. □

Theorem 3. *If a self-transverse immersion $f : V^n \to W^m$ is homotopic to the constant map and V^n is the boundary of a compact manifold, then $\tilde{M}_k(f)$ is also the boundary of a compact manifold.*

Proof. For a manifold M let us denote by $\nu(M)$ its stable normal bundle. By definition $\tau(M) \oplus \nu(M)$ is trivial, so $w_t(\tau(M))w_t(\nu(M)) = 1$. On the other hand since $f : V^n \to W^m$ is homotopic to the constant map ν_f can be considered as the stable normal bundle of V. Thus in our case we can reformulate Theorem 2 as follows:

$$\langle w_t(\nu(\tilde{M}_k(f))), [\tilde{M}_k(f)]\rangle = \langle e_H(\nu(V^n))^{k-1} w_t(\nu(V^n)), [V^n]\rangle. \qquad (*)$$

Let us now consider an arbitrary n-dimensional vector bundle ξ over X. Using the splitting principle as before, we can interpret $w(\xi)$ as a product $(1+a_1)\cdots(1+a_n)$. Now the coefficient of a monomial in $w_t(\xi)$ is a symmetric polynomial in $a_1, \ldots, a_n$, so it is also a polynomial of the elementary symmetric polynomials of $a_1, \ldots, a_n$, which are exactly the Stiefel-Whitney classes of ξ. So the coefficients in $w_t(\xi)$ are polynomials of the Stiefel-Whitney classes.

On the other hand it can be readily verified that these coefficients form an additive base in the space of symmetric polynomials of $a_1, \ldots, a_n$. This means that every polynomial of the Stiefel-Whitney classes of ξ can be expressed as a linear combination of the coefficients of $w_t(\xi)$.

It is well known that a manifold is null-cobordant if and only if all its normal Stiefel-Whitney numbers are zero. The coefficients of the right hand side of (*) are polynomials of normal Stiefel-Whitney classes evaluated on $[V^n]$. Nonzero coefficients can only be obtained by evaluating n-dimensional cohomology classes. Thus nonzero coefficients are always linear combinations of normal Stiefel-Whitney numbers of V^n. However by assumption V^n is null-cobordant so the right hand side of (*) is indeed zero. Obviously the same holds for the left hand side. But all the normal Stiefel-Whitney numbers of $\tilde{M}_k(f)$ can be expressed as a linear combination of coefficients on the left hand side of (*). As all these coefficients are zero, all the normal Stiefel-Whitney numbers of $\tilde{M}_k(f)$ are zero, thus it is null-cobordant. □

Remark. In the oriented case a similar statement was proved by A. Szűcs in [18]: If a self-transverse immersion $f : V^{2n} \to W^{2m}$ is homotopic to the constant map, and all the Pontrjagin numbers of V^n vanish, then the same is true for every multiple point manifold $\tilde{M}_k(f)$. For an immersion $f : V^{2n+1} \to W^{2m+1}$ this is obvious since every $\tilde{M}_k(f)$ is odd dimensional, so their Pontrjagin numbers must vanish. On the other hand if the codimension of f is odd, then the manifolds $\tilde{M}_k(f)$ can not, or can not canonically be given orientations, and so the problem in this case is not well defined.

Let us denote by s_i the polynomial which expresses $a_1^i + \cdots a_n^i$ in terms of the elementary symmetric polynomials of $a_1, \ldots, a_n$. For an n-dimensional vector bundle ξ over X let $s_i(\xi) = s_i(w_1(\xi), \ldots, w_n(\xi))$. Clearly $s_i(\xi)$ is the coefficient of t_n in $w_t(\xi)$.

Theorem 4. *For a self-transverse immersion $f : \mathbb{RP}^n \to S^{n+d}$ the following formula holds modulo 2:*

$$\langle s_{(n-(k-1)d)}(\tau(\tilde{M}_k(f))), [\tilde{M}_k(f)]\rangle \equiv k(n+1)\binom{n+d}{d}^{k-1}.$$

Proof. Let x denote the Euler class of the tautological line bundle over $\mathbb{RP}^n$. Then $H^*(\mathbb{RP}^n, \mathbb{Z}_2) = \mathbb{Z}[x]/(x^{n+1})$ holds. As $\tau(\mathbb{RP}^n) = (1+x)^{n+1}$ it follows that $w(\nu_f) = 1/(1+x)^{n+1}$. Now we can compute $e_H(\nu_f)$ by observing that

$$\frac{1}{(1+x)^{n+1}} = \left(\frac{1}{1+x}\right)^{n+1} = (1 + x + x^2 + x^3 + \dots)^{n+1}.$$

The coefficient of x^d in the last expression is the number of partitions of d into $n+1$ non-negative summands with respect to their order. It is well known that this number is $\binom{n+d}{d}$. So $e_H(\nu_f) = \binom{n+d}{d} x^d$.

Now we shall use Lemma 7 with our $f : \mathbb{RP}^n \to S^{n+d}$:

$$f_{k!}\left(\frac{1}{w_t(\tau(\tilde{M}_k(f)))}\right) = e_H(\nu_f)^{k-1}\left(\frac{1}{w_t(\tau(\mathbb{RP}^n))}\right)^k \tag{*}$$

Let us compare the coefficient of $t_{n-(k-1)d}$ on the two sides of (*). It is well known that $\tau(\mathbb{RP}^n) \oplus \varepsilon^1 = l_1 \oplus l_2 \oplus \dots \oplus l_{n+1}$ where l_i are all isomorphic to the tautological line bundle over $\mathbb{RP}^n$. Thus $w_t(\tau(\mathbb{RP}^n)) = w_t(l_1)^{n+1} = (1 + xt_1 + x^2t_2 + \cdots)^{n+1}$ and so:

$$\left(\frac{1}{w_t(\tau(\mathbb{RP}^n))}\right)^k = \left(\frac{1}{1 + (xt_1 + x^2t_2 + \cdots)}\right)^{(n+1)k} =$$
$$= \big((1 + (xt_1 + \cdots) + (xt_1 + \cdots)^2 + \cdots)\big)^{(n+1)k}.$$

It is easily seen that the coefficient of t_j in this expression is $k(n+1)x^j$. So the coefficient of $t_{n-(k-1)d}$ on the right hand side of (*) is:

$$\left(\binom{n+d}{d}x^d\right)^{k-1} k(n+1)x^{n-(k-1)d} = k(n+1)\binom{n+d}{d}^{k-1} x^n.$$

To compute the coefficient on the left hand side of (*) let us write for an aribtrary ξ vector bundle $w_t(\xi) = 1 + q(\xi)$. Now we have $1/w_t(\xi) = 1 + q(\xi) + q(\xi)^2 + \cdots$. The coefficient of t_j in this expression is the same as it is in $q(\xi)$ which is in turn the same as in $w_t(\xi)$, since the only power of $q(\xi)$ that contains a t_j term is the first power. Using this with $\xi = \tau(\tilde{M}_k(f))$ and $j = n - (k-1)d$ we get that

$$f_{d!}(s_{n-(k-1)d}(\tau(\tilde{M}_k(f)))) = k(n+1)\binom{n+d}{d}^{k-1} x^n.$$

Finally evaluating this on $[\mathbb{RP}^n]$ finishes the proof. $\square$

Corollary. The multiple point manifold $\tilde{M}_k(f)$ of the immersion $f : \mathbb{RP}^n \to S^{n+d}$ represents an irreducible in the unoriented cobordism ring if and only if $k, n+1$ and $\binom{n+d}{d}$ are odd.

Corollary. Let $\mathbb{RP}^n$ be even dimensional. Take the smallest d for which tere exists an immersion $f : \mathbb{RP}^n \to S^{n+d}$. It is well known that now $\binom{n+d}{d}$ is odd. Thus $\tilde{M}_k(f)$ is irreducible if and only if k is odd.

4.3 Codimension one immersions of even dimensional manifolds

In this section we consider the multiple point manifolds of an $f : V^{2k} \to R^{2k+1}$ immersion. We determine the possible Euler characteristics of these manifolds modulo 2. The first such result is due to Banchoff (see [1]):

Theorem (Banchoff). *If $f : V^2 \to \mathbb{R}^3$ is a generic immersion of a closed surface in $\mathbb{R}^3$, then $\chi(V^2) \equiv t(f)$ mod 2, where $t(f)$ denotes the number of triple points of the immersion.*

In 1998 Akhmetiev, Rimányi and Szűcs proved a generalization of this theorem:

Theorem (Generalized Banchoff). *If $f : V^{2k} \to \mathbb{R}^{2k+1}$ is a generic immersion then*

$$\sum_i \chi(\tilde{N}_i(f)) \equiv 0 \ mod\ 2.$$

The following theorem which is due to Szűcs (see [14]) settles the problem in complete generality. The proof uses deep classical results and theorems proved in the previous sections.

Theorem 5. *Let f be as above. Then one of the following two cases hold:*

(1) the manifold V^{2k} and all the manifolds $\tilde{N}_{2i+1}(f)$ are unoriented boundaries (and so all have even Euler characteristics), or

(2) $k = 0, 1,$ or 3, V^{2k} is cobordant to $\mathbb{RP}^{2k}$ and $\tilde{N}_{2i+1}(f)$ is cobordant to $\mathbb{RP}^{2k-2i}$ for $i = 0, 1, \ldots, k-1$ (and so all of these manifolds have odd Euler characteristics).

Remark. Observe that if the dimensions are as above then $\tilde{N}_{2i}(f)$ is an odd dimensional manifold and thus its Euler characteristic is zero. On the other hand there exists a $(2i+1)$-fold covering from $\tilde{M}_{2i+1}(f)$ to $\tilde{N}_{2i}(f)$. Hence all their Stiefel-Whitney numbers are equal and so they are cobordant to each other.

Proof. First let us apply the corollary of Herbert's formula to f. Since f is obviously null-homotopic we have $f^* = 0$ in positive dimensions. Thus the corollary yields that $m_{2k+1} = e(\nu_f)^{2k} = w_1(\nu_f)^{2k}$ as ν_f is a line-bundle. Evaluating on V^{2k} we get that the number of $(2k+1)$-tuple points of f is equal to $\langle \bar{w}_1^{2k}(V^{2k}), [V^{2k}] \rangle$ modulo 2. By a theorem of Eccles (see [5]) this can be non-zero only if $k = 0, 1$ or 3. (This result relies on Adams' Hopf invariant 1 result.) As ν_f is a line-bundle, all the other normal Stiefel-Whitney classes of V^{2k} are zero, thus V^{2k} is not null-cobordant if and only if f has an odd number of $2k+1$-tuple points.

If V^{2k} is null-cobordant then by Theorem 3 all the multiple point manifolds of f are null-cobordant in which case the proof is finished. Now we may assume that $k = 0, 1$ or 3 and that V^{2k} is not null-cobordant. The case $k = 0$ is trivial. For $k = 1$ the statement is exactly the same as Banchoff's theorem.

The only case left is when $k = 3$. Let us first suppose that $V^6 = \mathbb{RP}^6$. By Theorem 4 we have $\langle s_{6-(i-1)}, [\tilde{M}_i(f)] \rangle \equiv 7i\binom{7}{1}^{i-1} \equiv i$, particularly for $i = 3, 5$ we get that $s_2[\tilde{M}_5(f)] = s_4[\tilde{M}_3(f)] = 1$ (using a comfortable abbreviation). From the covering map between $\tilde{M}_i(f)$ and $\tilde{N}_i(f)$ follows that $s_2[\tilde{N}_5(f)] = s_4[\tilde{N}_3(f)] = 1$ also. As $\tilde{N}_5(f)$ is 2-dimensional, $s_2[\tilde{N}_5(f)] \neq 0$ means that $\tilde{N}_5(f)$ is cobordant to $\mathbb{RP}^2$.

It is well known that the unoriented cobordism group of 4-manifolds is generated by the classes $[\mathbb{RP}^4]$ and $[\mathbb{RP}^2 \times \mathbb{RP}^2]$ where $[M]$ denotes the cobordism class of the manifold M. Hence

$$[\tilde{N}_3(f)] = a[\mathbb{RP}^4] + b[\mathbb{RP}^2 \times \mathbb{RP}^2], \quad a, b \in \mathbb{Z}_2.$$

Since $s_4[\tilde{N}_3(f)] = 1$ the coefficient a must be 1. Applying the Generalized Banchoff Theorem we get that $\chi(\mathbb{RP}^6) + \chi(\tilde{N}_3(f)) + \chi(\tilde{N}_5(f)) + \chi(\tilde{N}_7(f))$ is even. But $\chi(\tilde{N}_3(f)) = 1 + b$ and all the other three terms are odd in the previous sum. Thus $b = 0$.

Now let V^6 be an arbitrary non null-cobordant manifold. As we have seen, the only non-zero normal Stiefel-Whitney number of V^6 is $\langle \bar{w}_1^6(V^6), [V^6] \rangle$. The same stands for $\mathbb{RP}^6$, so V^6

is cobordant to $\mathbb{RP}^6$. Finally choose an immersion $g : \mathbb{RP}^6 \to \mathbb{R}^7$. Such a g exists since $\mathbb{RP}^7$ is parallelizable so by Hirsch theory $\mathbb{RP}^6 \subset \mathbb{RP}^7$ can be immersed into $\mathbb{R}^7$. Consider the disjoint union of f and g, that is $f \cup g : V^6 \cup \mathbb{RP}^6 \to \mathbb{R}^7$. As $V^6 \cup \mathbb{RP}^6$ is null-cobordant, from Theorem 3 we deduce that $\tilde{N}_i(f)$ is cobordant to $\tilde{N}_i(g)$ and this finishes the proof. $\square$

5 Immersions of spheres

5.1 Problem outline

Throughout this section all immersions will be "almost generic". That is, they will be generic except to some closed double curves. So at a double point not belonging to these curves the two tangent planes of the immersion will be in generic position, while at the double points belonging to the exceptional curves the two tangent planes will coincide. Double points of the first type will be denoted as *intersection points* and double points of the second type will be denoted as *touching points*. Our immersions will never intersect themselves at touching points (in the sense as the graph of the function x^3 touches and intersects the x-axis at the same time). We will not allow points of multiplicity greater than two at all. Then the double point set of such an immersion is clearly a compact 1-manifold, that is, a disjoint union of circles. (Throughout the whole section circle will mean an embedded S^1.) Our definition of almost genericity implies that a component of the double point set can consist only of double points of one type. Circles consisting of intersection points will be called *self-crossings* while circles consisting of touching points will be called *self-tangencies*. The preimage of the double point set by the immersion is a subset of S^2 consisting of disjoint circles. A simple argument, also used in the proof of Banchoff's theorem (cf [1]), gives that each circle of the double point set will have two disjoint circles as its preimage.

Given disjoint circles on S^2 one can naturally define a graph associated to these circles: The circles divide S^2 into connected regions. The vertices of the graph will correspond to these regions, while the edges of the graph will corrsepond to the circles. The egde corrseponding to a given circle will connect the two regions separated by this circle. The graph defined this way is always a tree, since it is obviously connected, and has exactly 1 more vertices than edges. When the set of circles on S^2 is the preimage of an immersion without triple points, then there is a pairing of the circles, which gives a pairing of the edges of the tree. We will call what we have just constructed the *paired tree associated to the immersion.* It is easy to see that if two sets of circles have isomorphic trees associated to them, then there is a diffeomorphism of S^2 transforming one set into the other.

This means that the preimage of the double point set is well described by the paired tree associated to the immersion. Now in order to characterize the preimage we only need to identify, which paired trees can be associated to immersions. We shall call a paired tree T *touch-realizable* if there exists an immersion with only self-tangencies and no triple points such that T is the tree associated to this immersion. Similarly a paired tree is called *realizable*, if it is the tree associated to an immersion without triple points or self-tangencies. At first glance one might think that every paired tree is realizable, as "there is plenty of room for twisting the sphere". However this is not true, as there are even trees with only four edges which are not realizable. The characterization of the realizable paired trees given below is purely combinatorial.

We shall perform the charcterization in two steps. First we present a characterization of touch-realizable trees. Then we will show that the realizability of a tree is equivalent to the touch realizability of a more complex tree which can be obtained from the original one by a sequence of surgeries.

5.2 Definitions

From now on, our goal will be to transform the original immersion into one that has only self-tangencies, but no self-crossings. To do so, we shall generalize 2-dimensional concepts introduced by Dehn [4], H. de Fraysseix and P. Ossona de Mendez [6].

5.2.1 Simple switch

Consider one component of the double point set of the immersed sphere in $\mathbb{R}^3$, and its tubular neighborhood. The intersection of a fibre of the neighborhood with the image of the sphere is a cross. Since the sphere is orientable, the crosses in the fibres altogether form two strips, intersecting transversally. Let us replace each cross by two strings touching each other (see figure 1). This results in the two intersecting strips being replaced by two strips which are self-tangent along the circle originally belonging to the double point set.

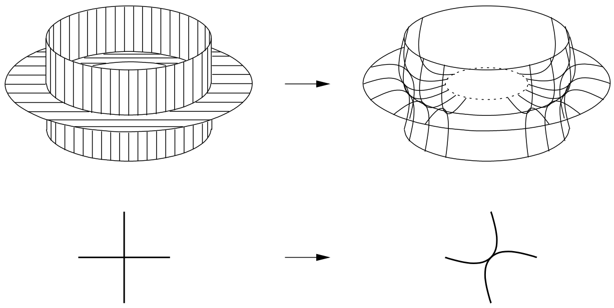

Figure 1:

As a result of this we have constructed an immersion which is similar to the original, except that it has a self-tangency instead of a self-intersection. Actually this surgery can always be done in two ways, since there are two possible pairs of strings we can replace the crosses with. Looking at the source sphere, we can describe the same surgery in a different way: First we cut the sphere along the preimage of a circle in the double point set. At this stage we have two discs and a tube. These altogether have four boundary components. We take two of these, glue them together, and do the same with the other two. There are three possibilities of pairing the four boundary components. In the first we get back the original immersion. The other two correspond to the two surgeries described above. In one of the possibilities the result is an immersion of a sphere and either a torus or a Klein bottle, while in the other we get again an immersion of one sphere (see figure 2). This clearly distinguishes the two ways the surgery can be done. Let us call the surgery where the immersion remains connected a *simple switch*.

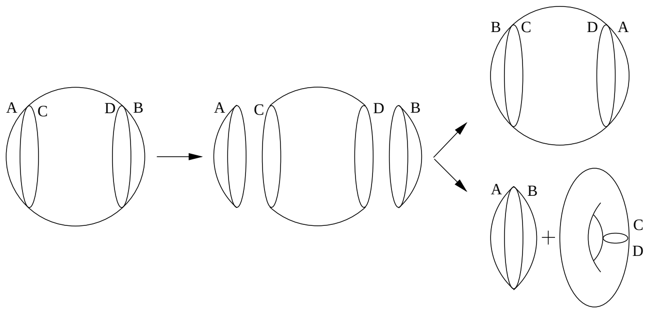

Figure 2:

We can describe a simple switch in terms of the tree associated to the immersion. The switch is done along a circle. In the tree, there are two edges corresponding to this circle. These two edges cut the tree into three parts, which correspond to the three parts of the sphere when cut along the preimages of the circle. The surgery exchanges the two discs attached to the tube, so the tree of the new immersion is constructed from the old one by swapping the outer parts of the tree while leaving the middle part fixed (see figure 3). From now on we shall call this operation *the simple switch of the tree along a pair of edges.*

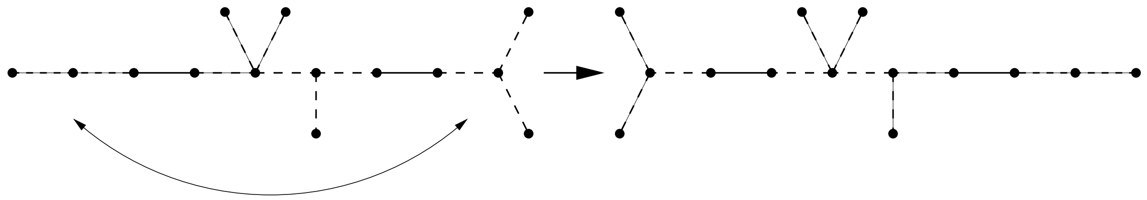

Figure 3:

5.2.2 Double switch

Double switch of an immersion If there is an immersion which is self-tangent along a circle, then we can perform the inverse of the simple switch on it. Here however we have no choice in pairing the four boundary components when replacing the touching strings with crosses. Thus there is nothing that guarantees connectedness of the immersion after the surgery. To resolve this problem, we shall introduce a bit more complicated switch that is called a *double switch.* This is very similar to the simple switch, the difference is only that instead of putting two touching strings in place of the crosses, now we put two strings that touch each other twice. That is, we do a simple switch and then duplicate the resulting circle of self-tangency (see figure 4). Again, we choose the strings such that the new immersion remains connected.

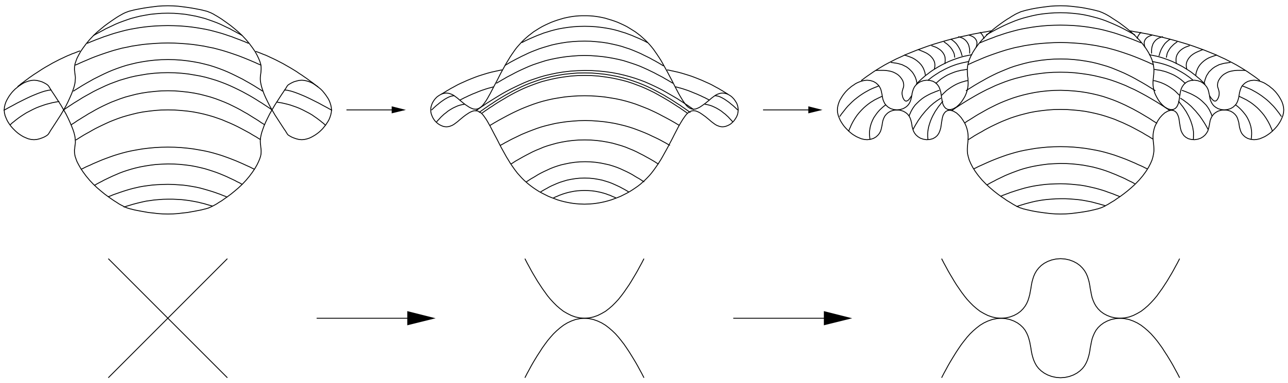

Figure 4:

Double switch of a tree Let us now investigate, what happens to the tree associated to an immersion, when the immersion is double switched along a circle. The tree associated to the new immersion can be constructed from the old one in three steps. First take the two edges corresponding to the circle, and simple switch the tree along these edges. Next split both edges by adding a new vertex to each. As the two original edges have a path connecting them, the same stands for the four edges after the splitting. This path is unique, so we can talk about the order of the four edges. As a special case we can distinguish the two outer edges from the two inner edges. Finally define the pairing on the four new edges such that the two inner edges would be a pair and the two outer edges an other pair. From now on we shall call this operation *the double switch of the tree along a pair of edges.* It is not obvious that the pairing has to be defined as just stated, so we shall prove it. This will be done by studying the inverse of a double switch.

Inverse of a double switch Let us consider an immersion that is self-tangent along two circles (its double point set may contain further components). Suppose that these two circles are very close to each other so that they have a common small tubular neighborhood. Further suppose that each fibre of this neighborhood intersects the image of the sphere in two strings touching each other twice. (We have just described a situation which could occur as a result of a double switch.) Now in each fibre we can replace the two strings with a cross, ie perform the inverse of a double switch. Take a look at the preimage of the two circles before performing the inverse double switch. These four circles are always positioned as shown in figure 5.

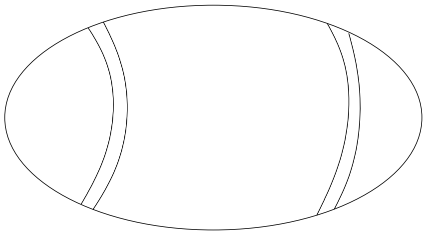

Figure 5:

The order of the four circles (from left to right on figure 5) can either be $a_1a_2b_2b_1$ or $a_1a_2b_1b_2$, where a_i is the pair of $b_i, i = 1, 2$. (The order is determined up to reversing.) What happens to the sphere during the inverse of a double switch is the following: First the sphere is cut along the four circles. Next the two regions between the first two and the second two circles (ie the two thin annular regions) are thrown out. The remaining three have four boundary components labeled a_1, a_2, b_1, b_2. When inserting the crosses in the fibres of the tubular neighborhood, the component labeled a_1 is glued to the component labeled b_2 and similarly a_2 is glued to b_1. So if the original order of the four circles was $a_1a_2b_1b_2$, then after the inverse double switch we obtain an immersion of a sphere and a torus or a Klein bottle, while if the original order was $a_1a_2b_2b_1$, then we obtain an immersion of one sphere.

Now consider an immersion which intersects itself transversally along a circle (its double point set may have further components). Perform a double switch along this circle. The resulting situation will be just as described in the previous paraghraph. So we can perform the inverse double switch, and clearly after this second surgery we obtain the original immersion, which is connected. Thus the order of the four circles after the double switch could have been only $a_1a_2b_2b_1$, otherwise the inverse double switch would have resulted in an unconnected immersion. This completes the proof that only the pairing $a_1a_2b_2b_1$ can occur after a double switch, ie when a transverse double-circle is replaced by two parallel touching circles. Thus we have completed the description of how the tree is transformed by a double switch.

Finally the previous considerations show the advantge of the double switch over the simple switch: a simple combinatorial property (namely the order of four edges in the tree) shows whether an inverse double switch keeps connectedness or not.

5.2.3 Separating pairs

Consider two pairs of circles on the unit sphere. We would like to connect the two circles in one pair by a tube and the two circles in the other pair by an other tube such that these tubes are disjoint and both are inside the unit ball. This is not always possible. When it is not possible, we shall say that the two pairs of circles *separate each other*, or in other words they are *linked.*

In fact it is easy to determine, when two pairs of circles are linked: Let the four circles be a_1, a_2, b_1 and b_2, where a_i is the pair of b_i. Take an embedded tube inside the unit ball that connects a_1 with b_1. Such a tube obviously exists. Strech this tube a bit outside the ball and attach two discs to its ends. This way we obtain a sphere which divides the space into two parts. If a_2 and b_2 lie in different parts, they cannot be connected by a tube disjoint from the other tube. Thus two pairs are linked if the tree associated to them looks like one of the first two trees in figure 6. However if the tree looks like one of the last four in figure 6, then the two disjoint tubes can be easily constructed. As the tree is (up to symmetry) always similar to one of the six below, this gives a full characterization of separating pairs.

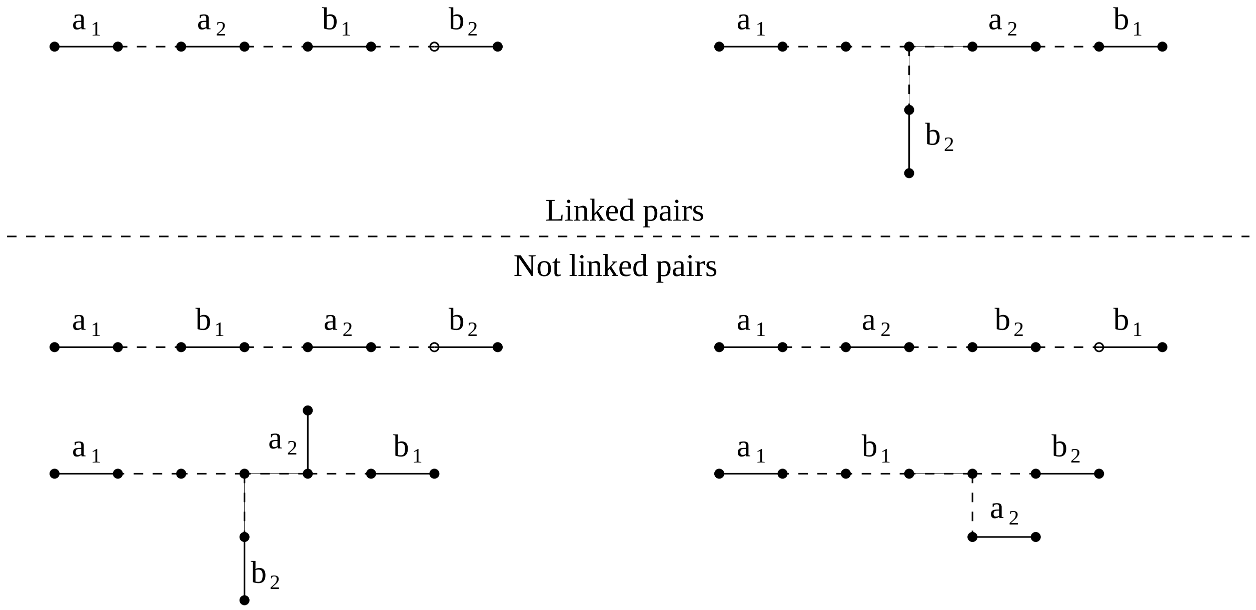

Figure 6:

Given a paired tree, we can construct a graph, called the *linking graph* of this tree. The vertices of the graph will correspond to the pairs of edges in the tree. Two vertices will be connected if the corresponding pairs separate each other.

5.3 Results

Now we are ready to state and prove our main results.

Theorem 6. *A paired tree is touch realizable if and only if its linking graph is bipartite.*

Theorem 7. *Let T be a paired tree. Double switch T along all its original edges (in any order) and denote the resulting tree by T^*. Now T is realizable if and only if the linking graph of T^* is bipartite.*

Proof of Theorem 6. Let T be a paired tree. Suppose that T is the tree associated to an immersion without triple points and only self-tangencies. Now take a circle in the double point set. The immersion touches itself along this circle. Pull the two parts of the immersion a bit away from each other, and glue a small tube to the sphere that connects the two circles along which the immersion touched itself before (see figure 7). This can be done just like we defined the switches above, ie one can do the surgery separately in each fibre (but depending smoothly on the fibre) of a small tubular neighborhood.

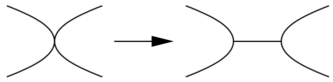

Figure 7:

Repeat this method with all the circles in the double point set. This way we get an embedding of the sphere into $\mathbb{R}^3$. There will be pairs of circles on the sphere, and each pair will be connected by a small tube. These tubes are clearly disjoint. In $\mathbb{R}^3$ any embedding of the 2-sphere is isotopic to the standard embedding. So there exists a diffeomorphism of $\mathbb{R}^3$ which transforms the given embedding into the standard embedding. The tubes are transformed into tubes connecting pairs of circles on the standardly embedded sphere. Some tubes will be inside the sphere and some will be outside, but all the tubes will be disjoint from each other and their interiors will be disjoint from the sphere. Now the bipartiteness of the linking graph is easily seen: As each tube connects a pair of circles, it corresponds to a vertex in the linking graph. Let the vertices corresponding to the tubes inside the sphere form one part, and those corresponding to tubes outside the sphere form the other part of the graph. Consider now two vertices in the first part. The corresponding pairs of circles are connected by disjoint tubes inside the sphere, thus these pairs of circles are not linked. So the first part of the graph contains no edges. The same can be said about the second part of the graph, only a spherical inversion must be applied to the tubes outside the sphere. So the linking graph is indeed bipartite.

Now suppose that the linking graph of T is bipartite. We will try to reverse the steps in the previous paragraph in order to construct an appropriate immersion. To do this we first need a simple lemma.

Lemma 8. *Consider a paired set of disjoint circles on the sphere. There exists a set of disjoint tubes connecting the pairs inside the sphere if and only if the same stands for any two pairs of circles, that is if none of the pairs are linked with each other.*

Proof. This is easily seen by induction on the number of pairs in the set. □

First consider the standard sphere in $\mathbb{R}^3$. Choose a system of circle-pairs on the sphere such that T is the tree associated to this system. This can be done easily. The linking graph of T is bipartite. Take the vertices in the first part. None of the circle-pairs corresponding to these vertices are linked with each other, so they satisfy the condition of the lemma. Thus there is a set of disjoint tubes connecting the pairs inside the sphere. Consider the spherical inversions of these tubes, and glue them to the sphere. These new tubes connect the pairs outside the sphere. Repeat the same with the second part except to the spherical inversion. What we get is a system of disjoint tubes that now connect all the circle-pairs. After this we perform a diffeotopy of $\mathbb{R}^3$ that is identical except in a small neighborhood of the tubes, while in the small neighborhood of a tube it contracts the tube until it degenerates into a circle. (Of course in the last moment this will not be a diffeomorphism.) So the sphere will be stretched along the tubes until the circular ends of the stretched parts will touch each other at the middle circle of the tube (see figure 8).

Thus we have constructed an immersion without triple points and only self-tangencies to which the tree T is associated. So we have shown that T is touch-realizable. □

Proof of Theorem 7. Suppose that the paired tree T is associated to an immersion of the 2-sphere into $\mathbb{R}^3$ without triple points and self-tangencies. Perform a double switch along every

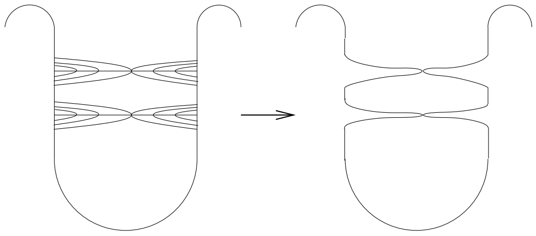

Figure 8:

circle in the double point set of the immersion. The resulting immersion will have only self-tangencies and no triple points. As seen after the definition of the double switch, the tree associated to this new immersion is obtained from T by double switching along all its pairs of edges, in the same order as we did with the pairs of circles. By Theorem 6 this new tree will have a bipartite linking graph. This proves one direction of the statment.

To prove the other direction consider a T_0 paired tree. We would like to construct an f_0 immersion of S^2 into $\mathbb{R}^3$ such that it has no triple points or self-tangencies and T_0 is the tree associated to f_0. This will be done the following way: Double switch along every pair of edges of T_0. This process results in a paired tree we shall later refer to as T_k. Our assumption was that T_k has a bipartite linking graph, hence T_k is touch-realizable by Theorem 6, ie there exists an immersion $f_k : S^2 \to \mathbb{R}^3$ without triple points or self-crossings such that T_k is associated to f_k. The immersion f_0 realizing the paired tree T_0 will then be obtained from f_k by a sequence of inverse double switches.

In order to show the details of the above sketched process, we have to introduce some more notations. Denote the edges of T_0 by $a_1, b_1, \ldots, a_k, b_k$ where a_i is always the pair of b_i. Construct the trees $T_1, T_2, \ldots, T_k$ recursively as follows: T_i is obtained from T_{i-1} by double switching along the pair a_i, b_i. This pair is split into four new edges in T_i. Denote them by a_i, b_i, c_i and d_i. We may assume that a_i in T_{i-1} is replaced by the adjacent edges a_i and c_i in T_i, and b_i in T_{i-1} is replaced by the adjacent edges b_i and d_i in T_i. Let the order of the four edges be a_i, c_i, d_i, b_i. As we have seen after the definition of the double switch, a_i will be the pair of b_i and c_i will be the pair of d_i. As the pair denoted by a_i, b_i is in similar position in T_{i-1} and T_i, this notation will not be confusing. The tree obtained after accomplishing all the double switches is T_k. By our assumption it has a bipartite linking graph. Hence by Theorem 6 it is touch-realizable. Thus there is an immersion $f_k : S^2 \to \mathbb{R}^3$ without triple points or self-crossings such that T_k is the tree associated to f_k.

We shall now construct immersions $f_{k-1}, f_{k-2}, \ldots, f_1, f_0$ such that T_i is associated to f_i for every $i = 0, 1, 2, \ldots, k-1$ and f_0 has no self-tangencies. f_{k-1} is obtained from f_k by performing an inverse double switch on the pairs a_k, b_k and c_k, d_k. By the definition of the double switch, these two pairs of edges are not linked in T_k, so as seen after the definition of the switch, the inverse surgery produces the immersion of a connected surface. So f_{k-1} is indeed an immersion of the 2-sphere, and the tree associated to f_{k-1} is clearly T_{k-1}. Now construct f_{k-2} from f_{k-1} by performing an inverse double switch on the pairs a_{k-1}, b_{k-1} and c_{k-1}, d_{k-1}. As these pairs are not linked in T_{k-1} by definition, the new immersion f_{k-2} will be an immersion of the 2-sphere, and the tree associated to f_{k-2} will be T_{k-2}. Construct the immersions $f_{k-3}, \ldots, f_1, f_0$ similarly. Clearly the tree associated to the self-transverse immersion $f_0 : S^2 \to \mathbb{R}^3$ will be T_0. Furthermore f_0 will have no triple points or self-tangencies.

There is only one problem with this argument. Namely, nothing guarantees that the inverse double switches used above can be accomplished. To perform an inverse double swith, we need the two circles in the double point set to be sufficiently close to each other. This means that for every $i = 1, 2, \dots, k$ the two circles corresponding to a_i and c_i have to be close to each other in the immersion f_i. However f_i is constructed from f_k by a sequence of inverse double switches done along circles different from the two above. It is easy to see that these inverse double switches do not change the position of the two circles. Thus the circles corresponding to a_i and c_i have to be close to each other already in f_k. On the other hand this is clearly a sufficient condition for the inverse double switches to work. All we have to do is construct f_k carefully so that for every $i = 1, 2, \dots, k$ the two circles corresponding to a_i and c_i are close to each other.

We will use the method of Theorem 6 with a slight modification to construct f_k. The main idea there was connecting the pairs of circles with tubes, and then stretch the sphere along the tubes until the two circles touch each other. Now we are going to deal with two tubes at the same time. One connecting a_i with b_i and the other connecting c_i with d_i.

First of all, the edges a_i and c_i always have a common vertex of degree 2. This is so by definition in T_i and clearly never changes during further switches. Thus when we choose the structure of circle-pairs on the sphere, for each a_i, c_i we can choose two circles that are very close to each other so that they form a very thin annulus on the sphere. The same holds for each b_i, d_i.

The next step in the proof of Theorem 6 is to connect a_i with b_i and c_i with d_i using two disjoint tubes. In the lemmas below we shall show that these pairs are not linked in T_k hence their tubes lie on the same side of the sphere. This together with the fact that the ends of the tubes are very close to each other means that we can choose the whole tubes to be close to each other. Now we cut both tubes halfway between a_i and b_i, and modify them as shown in figure 9. (The figure shows a cross-section.)

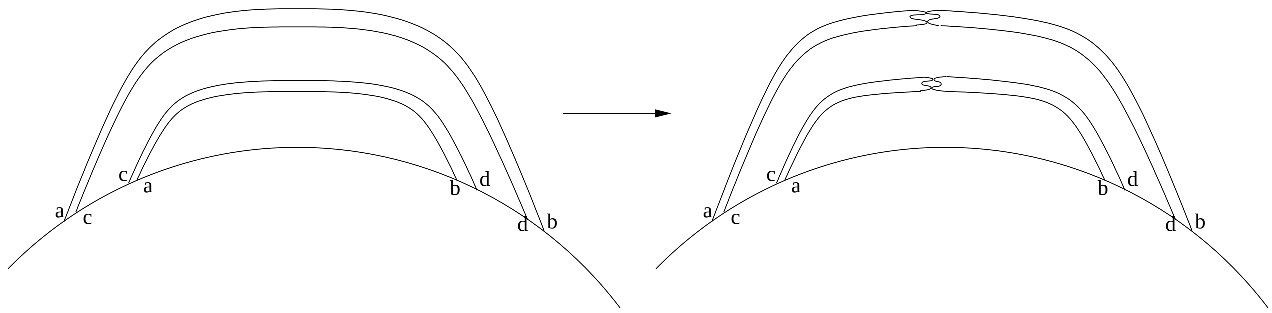

Figure 9:

Finally we dispose both annular regions of the sphere between a_i and c_i, and respectively b_i and d_i. If we perform this process with every quadruple of circles a_i, b_i, c_i, d_i, and smoothen the sharp edges between the sphere and the joining tubes, we get an immersion that can work as f_k. Clearly the tree associated to this immersion is T_k, and all inverse double swiches are accomplishable. This together with the previous arguments finishes the proof of Theorem 7. ☐

The only thing left is to prove that the pairs a_i, b_i and c_i, d_i do not separate each other in T_k. This is done in the following two lemmas.

Lemma 9 (Properties of (double) switching). *(i) If the pairs p and q are linked, then after a switch (or double switch) along q they will still separate each other in the new tree.*

(ii) If p and q do not separate each other, then switching along q does not change the linkedness of p with any other pair.

Proof. The first property can be checked very easily as there are only three ways for q to separate p, as shown in figure 6. The second property comes from the following: An edge can have three different positions relative to the pair p. It can be between the two edges of p, ie. lying on the path connecting them. Or it can "hang" from this path (eg. b_2 has such a position relative to a_1, b_1 in the second tree of figure 6). The remaining case is when it is outside the two edges. Whether the pairs p and r are linked or not depends only on the position of their edges relative to each other. They are linked either if one edge of r is hanging from the path connecting the edges of p and the other does not, or if one edge of r is between the edges of p and the other is outside (see figure 6). When q is not linked with p, then switching along q does not change the position of any edge relative to p. Hanging edges remain hanging, outside edges remain outside, and edges inbetween remain inbetween. So p and r separate each other after the switch if and only if they did before the switch. □

Lemma 10. *If T_k has a bipartite linking graph, then the pairs a_i, b_i and c_i, d_i cannot be linked in T_k.*

Proof. We shall use the following notation: let $l_n(r,s) = 1$ if the pairs r and s are linked in the tree T_n and $l_n(r,s) = 0$ if not. Let p denote the pair a_i, b_i and q denote the pair c_i, d_i. (q exists in T_n only for $n \geq i$.)

Suppose on the contrary that $l_k(p,q) = 1$. As a_i and c_i are neighbours in T_n for every $n \geq i$, and the same holds for b_i and d_i, for every $n \geq i$ and any pair r different from p and q $l_n(r,p) = l_n(r,q)$. As there is no triangle in the linking graph of T_k, this means that $l_k(p,r) = 1$ can hold at most for $q = r$. Consider the largest number n such that there is a pair r other than q in T_n such that $l_n(p,r) = 1$. Now we double switch T_n along a_{n+1}, b_{n+1} to get T_{n+1}. Let us denote the pair a_{n+1}, b_{n+1} by s.

There are two possibilities: $l_n(s,p) = 1$ or $l_n(s,p) = 0$. If $l_n(s,p) = 1$ then the first property of Lemma 9 says $l_{n+1}(s,p) = 1$ which contradicts the choice of n. So $l_n(s,p) = 0$. But then by the second property of Lemma 9 we have $1 = l_n(p,r) = l_{n+1}(p,r)$. This is again impossible. Thus such an n does not exist. By definition $l_i(p,q) = 0$ and the second property in Lemma 9 together with the previous argument says that $l_{n-1}(p,q) = l_n(p,q)$ for every $i < n \leq k$. In particular we get $l_k(p,q) = 0$. This contradicts our original assumption so we have proved that a_i, b_i and c_i, d_i are indeed not linked in T_k. □

References

[1] T. Banchoff, Triple points and surgery of immersed surfaces, *Proc. Amer. Math. Soc.* **46** (1974) 407-413.

[2] G. Barratt, P. J. Eccles, Γ^+-structures I. A free group functor for stable homotopy theory, *Topology* **13** (1974), 25-45.

[3] J. M. Boardman, On manifolds with involution, *Bull. Am. Math. Soc* **73** (1967) 136-138.

[4] M. Dehn: Über kombinatorische Topologie *Acta Math.* **67** (1936), 123–168

[5] P. Eccles, Codimension one immersions and the Kervaire invariant one problem, *Math. Proc. Cambridge Phil. Soc.* **90** (1981) 483-493.

[6] H. de Fraysseix, P. Ossona de Mendez: On a Characterization of Gauss Codes, *Discrete Comput. Geom.* **22** (1999), 287–295

[7] R. J. Herbert, Multiple points of immersed manifolds, *Thesis, University of Minnesota* (1975).

[8] M. Kamata, On multiple points of a self-transverse immersion, *Kyushu J. Math.* **50** (1996) 275-283.

[9] R. Lashof and S. Smale, Self-intersections of immersed manifolds, *Journal of Math. and Mech.* (1959) 143-157.

[10] D. Quillen, Elementary proofs of some results of cobordism theory using Steenrod operations, *Adv. Math.* **7** (1971) 29-56.

[11] R. Rimányi, Thom polinomials, symmetries and incidences of singularities, *Inv. Math* **143** (2001) 499-521.

[12] R. Rimányi, A. Szűcs, Generalized Pontrjagin-Thom construction for maps with singlarities, *Topology*, **37** (1998) 1177-1191.

[13] F. Ronga, On multiple points of smooth immersions, *Comment. Math. Helv.* **55** (1980) 521-527.

[14] A. Szűcs, On multiple points of codimension one immersions of even dimensional manifolds, *Kyushu J. Math.* **57** (2003) 193-196.

[15] A. Szűcs, Gruppy kobordizmov l-pogruzenii I, *Acta Math. Acad. Sci. Hungar.* **27** (1976) 343-358.

[16] A. Szűcs, Gruppy kobordizmov l-pogruzenii II, *Acta Math. Acad. Sci. Hungar.* **28** (1976) 93-102.

[17] A. Szűcs, Cobordism groups of immersions with restricted self-intersection, *Osaka J. Math.* **21** (1984) 71-80.

[18] A. Szűcs, On the multiple points of immersions in Euclidean spaces, *Proc. Amer. Math. Soc* **126** (1998) 1873-1882.