Hyperbolic Semigroups and Fourier Multipliers

Master's Thesis

András Serény

May 2004

Supervisor: András Bátkai

Introduction

Various differential equations can be written in the form

$$\dot{u}(t) = Au(t), \tag{$*$}$$

where the function $u(t)$ takes its values in a Banach space X, and A is a linear operator on X. Under certain conditions the equation gives rise to an operator semigroup $T(t)$, such that the solution $u(t)$ belonging to the initial data $u(0) = x \in X$ is given by $u(t) = T(t)x$. The main motivation for the development of the asymptotic theory of operator semigroups is to facilitate the deduction of asymptotic properties of the solutions from information about A.

A particular type of asymptotic behaviour is the exponential dichotomy of $(*)$; in this case the generated semigroup is said to be hyperbolic. It means that the space X can be split into a direct sum of two subspaces in a way that solutions originating in one subspace are exponentially stable forward in time, and solutions originating in the other subspace are exponentially stable backward in time. Exponential dichotomy is one of the main tools in the study of linear differential equations in Banach spaces.

It has long been known that $(*)$ admits exponential dichotomy on $\mathbb{R}$ if and only if the inhomogenous equation

$$\dot{u}(t) = Au(t) + f(t) \tag{$**$}$$

has a unique mild solution u for each f taken from some class of X-valued functions. Applying the Fourier transform $\mathcal{F}$ to $(**)$ we see that the solution is given by $u = \mathcal{F}^{-1}((i\cdot - A)^{-1}\mathcal{F}f)$. Thus we are led to the examination, how multiplication by $s \mapsto (is - A)^{-1}$ acts on the space of Fourier transforms, that is, to the Fourier multiplier properties of the resolvent of A. Our main aim is to study hyperbolicity and its relation to Fourier multipliers. Our primary source is [LS]; see the notes at the end of the text.

The thesis is organised as follows. After introducing notation, we briefly review the relevant properties of semigroups and Fourier transforms in Sections 2 and 3, respectively. We concisely discuss uniform exponential stability, then we deal with the basic features of hyperbolic semigroups and with Green's function joined to each hyperbolic semigroup in Section 5. Among the spectral mapping theorems we formulate the theorem of Greiner, which leads to a description of hyperbolicity in Section 7, where we state several equivalent conditions of hyperbolicity in terms of Fourier multipliers. The arising notions and assertions are adapted to spaces of smooth solutions in

Section 8 and to the discrete case in Section 9. Finally, we study the connection between the Fourier multiplier operator induced by the resolvent $s \mapsto (is - A)^{-1}$ and mild solutions of $(**)$.

It is a pleasure to express my warmest thanks to András Bátkai, my thesis supervisor, for his invaluable help, for his cheerful attitude and for the idea of the proof of Lemma 8.1.

1 Conventions and Notation

Γ denotes the unit circle. We freely use the identification $\Gamma = \mathbb{R}/(2\pi\mathbb{Z}) = [0, 2\pi]/_{\{0=2\pi\}}$, so, for example, we identify the space $C(\Gamma)$ of continous functions on the unit circle with the space of 2π-periodic continous functions on the real line. If A is a linear operator on a Banach space X, we label different subsets of its spectrum $\sigma(A)$ as $\mathrm{P}\sigma(A) = \{\lambda \in \mathbb{C} \mid \lambda I - A \text{ is not injective}\}$, $\mathrm{A}\sigma(A) = \{\lambda \in \mathbb{C} \mid \lambda I - A \text{ is not injective or } \mathrm{Im}\,(\lambda I - A) \text{ is not closed}\}$, $\mathrm{R}\sigma(A) = \{\lambda \in \mathbb{C} \mid \mathrm{Im}\,(\lambda I - A) \text{ is not dense in } X\}$, known as the point, approximate and residual spectrum, respectively. The resolvent $R(\lambda, A)$ of A is an operator-valued analytic function on the resolvent set $\varrho(A) = \mathbb{C}\backslash\sigma(A)$. The so-called resolvent identity states that if $\lambda, \mu \in \varrho(A)$, then

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\lambda, A)R(\mu, A). \tag{1.1}$$

Vector-valued integrals and $L_p(\mathbb{R}, X)$ spaces are understood in the sense of Bochner; [HP, Section III. 3] and [DU, Section II.2] provide all-embracing expositions, or consult [EN, Appendix C] for a summary on the subject. For an operator valued function $B(s) : \mathbb{R} \mapsto \mathcal{B}(X)$ we write $B \in L_p(\mathbb{R}, \mathcal{B}(X))$ $(1 \le p \le \infty)$, if there is a constant C such that $\|B(s)x\|_{L_p(\mathbb{R},X)} \le C\|x\|$ for all $x \in X$. If f is an $\mathbb{R} \mapsto X$ function, or a_k $(k \in \mathbb{Z})$ is a sequence in X, we use the notation

$$(C)\int_{\mathbb{R}} f(s)\,ds = \lim_{N\to\infty} \frac{1}{N}\int_0^N \int_{-l}^{l} f(s)\,dsdl,$$

$$(C)\sum_{k\in\mathbb{Z}} a_k = \lim_{N\to\infty} \frac{1}{N+1}\sum_{l=0}^{N}\sum_{k=-l}^{l} a_k$$

for Cesàro integrals and sums.

The symbols $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}(\mathbb{R}, X)$ stand for the classes of complex and X-valued Schwartz functions, respectively. For a distribution r on $\mathbb{R}$ and $\Phi \in \mathcal{S}(\mathbb{R})$, the value of r at Φ is denoted by (r, Φ).

1.1 Lemma *Let X be a Banach space. Then functions $\Phi(s) : \mathbb{R} \mapsto X$ of the form $\Phi(s) = \sum_{j=1}^{n} \Phi_j(s)x_j$, where $x_j \in X$, $\Phi_j(s) \in \mathcal{S}(\mathbb{R})$, and the supports of the functions Φ_j are disjoint and compact, comprise a dense subset in $L_p(\mathbb{R}, X)$ $(1 \leq p < \infty)$.*

Proof. By the definition of Bochner integrability, countably valued measurable functions are dense in $L_p(\mathbb{R}, X)$. As in the scalar-valued case, countably valued measurable functions can be approximated by continous functions with compact support, which, in turn, can be approximated by functions of the desired type. $\square$

2 Strongly Continous Semigroups

2.1 Definition Let X denote a Banach space. The operator-valued mapping $T : [0, \infty) \mapsto \mathcal{B}(X)$ is called a strongly continous operator semigroup on X, if $T(0) = I$, the identity on X, $T(t+s) = T(t)T(s)$ for all $t, s \geq 0$, and the mapping $t \mapsto T(t)x$ $(t \geq 0)$ is continous for all $x \in X$.

Throughout the text, a strongly continous operator semigroup will be referred to as a semigroup, and, in an attempt to abide by the tradition, will invariably be denoted by $T(t)$. In this section we gather the basic facts that we shall need; all proofs along with a great deal of additional information on semigroup theory can be found in [EN].

To a semigroup $T(t)$ on a Banach space X we attach a linear operator A on X called the generator of $T(t)$ defined by

$$Ax = \lim_{t \downarrow 0} \frac{T(t)x - x}{t} \quad (x \in D(A)),$$

where $D(A)$ is the domain of A given naturally by

$$D(A) = \{x \in X \mid \lim_{t \downarrow 0} \frac{T(t)x - x}{t} \text{ exists}\}.$$

The generator of a semigroup is densely defined, closed and determines the semigroup uniquely. Loosely speaking, the resolvent of the generator is sometimes called the resolvent of the semigroup and is denoted by $R(\lambda, A) = R(\lambda)$.

The formulae below are frequently applied.

2.2 Proposition *If A is the generator of the semigroup $T(t)$, and $\lambda \in \mathbb{C}$, then $\int_0^t e^{\lambda(t-s)}T(s)x\,ds \in D(A)$ for all $x \in X, t \geq 0$ and*

$$e^{\lambda t}x - T(t)x = (\lambda - A)\int_0^t e^{\lambda(t-s)}T(s)x\,ds \quad (x \in X) \tag{2.1}$$

$$= \int_0^t e^{\lambda(t-s)}T(s)(\lambda - A)x\,ds \quad (x \in D(A)). \tag{2.2}$$

Moreover, if $R(\lambda)$ exists, then

$$T(t)R(\lambda)x = R(\lambda)e^{\lambda t}x - \int_0^t e^{\lambda(t-s)}T(s)x\,ds. \tag{2.3}$$

2.3 Definition We define the growth bound $\omega_0(T)$ for the semigroup $T(t)$ by $\omega_0(T) = \inf\{\,\omega \in \mathbb{R}\,|\,\|T(t)\| \leq Me^{\omega t}$ for some M and all $t > 0\,\}$, and the abscissa of uniform boundedness of the resolvent by $s_0(A) = \inf\{\,\omega \in \mathbb{R}\,|\,\{\operatorname{Re}\lambda > \omega\} \subset \varrho(A)$ and $\sup_{\operatorname{Re}\lambda>\omega}\|R(\lambda, A)\| < \infty\}$.

It is easily seen that $\omega_0(T) < \infty$, and by the spectral radius formula $r(T(t)) = \lim_{n\to\infty}(T(t)^n)^{\frac{1}{n}}$ we in fact have

$$\omega_0(T) = \inf_{t>0}\frac{1}{t}\log\|T(t)\| = \lim_{t\to\infty}\frac{1}{t}\log\|T(t)\| = \frac{1}{t_0}\log r(T(t_0)) \tag{2.4}$$

for all $t_0 > 0$. Whenever $\lambda \in \mathbb{C}$ and $\int_0^\infty e^{-\lambda t}T(t)x\,dt$ exists for all $x \in X$, we have $\lambda \in \varrho(A)$ and

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t}T(t)x\,dt. \tag{2.5}$$

In particular, (2.5) holds for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega_0(T)$, consequently, $s_0(A) \leq \omega_0(T)$.

A closed subspace $Y \subset X$ is called a $T(t)$-invariant subspace, if Y is invariant to all the operators $T(t)$, i.e. $T(t)Y \subset Y$ for all $t \geq 0$. A $T(t)$-invariant subspace is an invariant subspace of the generator A and of the resolvent operators $R(\lambda)$ as well.

We shall use the following rescaling argument. If A is the generator of a semigroup $T(t)$, and $\lambda \in \mathbb{C}, \alpha > 0$, then the operator $\alpha A - \lambda$ is the generator of the semigroup $t \mapsto e^{-\lambda t}T(\alpha t)$; further, $\sigma(\alpha A - \lambda) = \alpha\sigma(A) - \lambda$, $R(\mu, \alpha A - \lambda) = \frac{1}{\alpha}R(\frac{1}{\alpha}(\mu + \lambda), A)$.

Given a semigroup $T(t)$ with generator A on the Banach space X, one can consider the adjoint operators $T^*(t)$ on the dual space X^*. However $T^*(t)$ is not necessarily strongly continous on X^*, so we introduce the so-called sun dual of $T(t)$ on the subspace $X^\odot = \{x^* \in X^* \mid \lim_{t\downarrow 0} \|T^*(t)x^* - x^*\| = 0\}$ of X^* defined by $T^\odot(t) = T^*(t)|_{X^\odot}$. The definition is correct, and yields a (strongly continous) semigroup on $X^\odot$, whose generator $A^\odot$ is the part of A^* in $X^\odot$.

With a slight abuse of notation, we write

$$\langle x^*, R(\varrho + is)x\rangle = r_\varrho(s, x, x^*) = r_\varrho(s) \quad (x \in X, x^* \in X^*)$$

for the matrix elements of the resolvent on vertical lines and $r(s) = r_0(s)$; the reader should keep in mind that there are x and x^* hidden in r_ϱ. In this context, ϱ always denotes a real number.

Let A be the generator of the semigroup $T(t)$, and consider the inhomogeneous abstract Cauchy problem

$$\begin{aligned} \dot{u}(t) &= Au(t) + f(t) \quad (t \geq 0) \\ u(0) &= x, \end{aligned}$$

where $x \in X, f \in L_p(\mathbb{R}_+, X)$. This problem has a unique mild solution given by

$$u(t) = T(t)x + (T * f)(t),$$

where the convolution is defined by

$$(T * f)(t) = \int_0^t T(s)f(t-s)\,ds \quad (t \geq 0).$$

This motivates us to investigate how convolution with the semigroup acts on the space $L_p(\mathbb{R}_+, X)$, see Theorem 4.3.

One can also take the real line instead of the positive halfline and examine the mild solutions of the problem

$$\dot{u}(t) = Au(t) + f(t) \quad (t \in \mathbb{R}),$$

where $f \in L_p(\mathbb{R}, X)$; this study is carried out in Section 10.

3 Fourier Transform and Multipliers

If $f \in L_1(\mathbb{R}, X)$, we write

$$(\mathcal{F}f)(t) = \hat{f}(t) = \int_{\mathbb{R}} f(s)e^{-ist}\,ds$$

$$(\mathcal{F}^{-1}f)(t) = \check{f}(t) = \frac{1}{2\pi}\int_{\mathbb{R}} f(s)e^{ist}\,ds$$

for the Fourier transform and its inverse. Fourier coefficients of any $f \in L_1(\Gamma, X)$ are also denoted by $(\mathcal{F}f)(k) = \hat{f}(k)$ $(k \in \mathbb{Z})$. All these symbols are extended to stand for the Fourier transform of tempered distributions as well; the Fourier transform of a distribution is defined by the usual duality relation $(\mathcal{F}r, \Phi) = (r, \mathcal{F}\Phi)$ $(\Phi \in \mathcal{S}(\mathbb{R}))$.

We need a few lemmas on the relation between functions and their Fourier transforms. The first one is widely known as Fejér's theorem.

3.1 Lemma *Let X be a Banach space.*

(a) *If $f \in L_1(\mathbb{R}, X)$, then $\frac{1}{2\pi}(C)\int_{\mathbb{R}} e^{ist}\hat{f}(s)\,ds = f(t)$ for almost all t in $\mathbb{R}$. Furthermore, $\frac{1}{2\pi}(C)\int_{\mathbb{R}} e^{ist}\hat{f}(s)\,ds = \frac{1}{2}(\lim_{\tau\downarrow t} f(\tau) + \lim_{\tau\uparrow t} f(\tau))$, whenever the right-hand side exists.*

(b) *If $f \in L_1(\Gamma, X)$, then $\frac{1}{2\pi}(C)\sum_{k\in\mathbb{Z}} \hat{f}(k)e^{ikt} = f(t)$ for almost all t in Γ. Furthermore, $\frac{1}{2\pi}(C)\sum_{k\in\mathbb{Z}} \hat{f}(k)e^{ikt} = \frac{1}{2}(\lim_{\tau\downarrow t} f(\tau) + \lim_{\tau\uparrow t} f(\tau))$, whenever the right-hand side exists.*

The next lemma, as the very first application of Lemma 3.1, shows how the semigroup is regained from its resolvent, and explains why Cesàro convergent integrals pervade the whole theory.

3.2 Lemma *Let $T(t)$ be a semigroup with generator A and resolvent $R(\lambda)$. Then for all $\varrho > s_0(A)$, $x \in X$ we have*

$$\frac{1}{2\pi}(C)\int_{\mathbb{R}} e^{its}R(\varrho + is)x\,ds = \begin{cases} e^{-\varrho t}T(t)x & \text{for } t > 0 \\ \frac{1}{2}x & \text{for } t = 0 \\ 0 & \text{for } t < 0 \end{cases}.$$

See [LS, Corollary 3.5] for a somewhat more general statement.

Proof. For $\varrho > \omega_0(T)$, the claim follows from (2.5) and Lemma 3.1 (applied to the function $t \mapsto e^{-\varrho t}T(t)\chi_{\mathbb{R}_+}(t)$). If $s_0(A) < \omega_0(T)$, fix $a, b \in \mathbb{R}$ such that $s_0(A) < a < \omega_0(T) < b$. The Riemann-Lebesgue lemma yields $\lim_{s\to\infty} \|R(b + is)x\| = 0$ and from the resolvent identity (1.1) we infer

$$\|R(w + is)x\| \leq (1 + K(b - a))\|R(b + is)x\| \quad (x \in X),$$

for a constant K and all $w \in [a, b]$, thus we obtain $\lim_{s\to\infty} \|R(w + is)x\| = 0$ uniformly in $w \in [a, b]$. Cauchy's theorem now allows to shift the path of integration $b + i\mathbb{R}$ to $a + i\mathbb{R}$. $\square$

3.3 Lemma *Suppose $f \in L_1(\mathbb{R}, X)$ and let us denote*

$$f_N(t) = \frac{1}{2\pi N} \int_0^N \int_{-l}^{l} \hat{f}(s) e^{ist}\, ds dl.$$

Then $f_N \to f$ in $L_1(\mathbb{R}, X)$ as $N \to \infty$.

Proof. We recall that

$$\frac{1}{2\pi N} \int_0^N \int_{-l}^{l} \hat{f}(s) e^{ist}\, ds dl = \frac{1}{2\pi} \int_{-N}^{N} \hat{f}(s) \left(1 - \frac{|s|}{N}\right) e^{ist}\, ds$$
$$= \int_{\mathbb{R}} f(s) K_N(t-s)\, ds,$$

where

$$K_N(t) = \frac{1}{2\pi} \int_{-N}^{N} \left(1 - \frac{|s|}{N}\right) e^{ist}\, ds = \frac{N}{2\pi} \left(\frac{\sin \frac{Nt}{2}}{\frac{Nt}{2}} \right)^2.$$

We have $K_N \geq 0, \int_{\mathbb{R}} K_n = 1$ and $\int_{|s| \geq \delta} K_N(s)\, ds \to 0$ as $N \to \infty$ for all $\delta > 0$ (see, for example [K, Lemma 49.2]), that is, K_N is an approximative unity in $L_1(\mathbb{R})$. Hence $\int_{\mathbb{R}} f(s) K_N(t-s)\, ds \to f(s)$ in $L_1(\mathbb{R}, X)$ for all $f \in L_1(\mathbb{R}, X)$, as stated. □

3.4 Lemma *Let $f \in L_\infty(\Gamma)$, and let $a_k = \frac{1}{2\pi} \int_\Gamma e^{ikt} f(t)\, dt$ denote its Fourier coefficients. Then the partial Cesàro sums $\sigma_N = \frac{1}{N+1} \sum_{l=0}^{N} \sum_{k=-l}^{l} a_l$ satisfy $|\sigma_N| \leq \|f\|_\infty$.*

Proof. As is well known, $\sigma_N = \int_\Gamma f(t) F_N(t)\, dt$, where

$$F_N(t) = \frac{1}{2\pi(N+1)} \frac{1 - \cos(N+1)t}{1 - \cos t}$$

is the Fejér kernel. Since $F_N \geq 0$ and $\int_\Gamma F_N = 1$ we obtain $|\sigma_N| \leq \int_\Gamma |f(t)| F_N(t)\, dt \leq \|f\|_\infty$; for details see [K, Lemma 2.2], or any textbook on Fourier analysis. □

We introduce a highly important concept.

3.5 Definition Let X, Y be Banach spaces, $1 \leq p < \infty$, $1 \leq s \leq \infty$. The operator-valued function $M(s) \in L_\infty(\mathbb{R}, \mathcal{B}(X, Y))$ is called a Fourier multiplier from $L_p(\mathbb{R}, X)$ to $L_s(\mathbb{R}, Y)$, if there exists a constant K such that

$$\left\| \mathcal{F}^{-1}(M(s)(\mathcal{F}\Phi)(s)) \right\|_{L_s(\mathbb{R}, Y)} \leq K \|\Phi\|_{L_p(\mathbb{R}, X)} \quad \text{for all } \Phi \in \mathcal{S}(\mathbb{R}, X).$$

The infimum of the set of appropriate K-s is called the multiplier norm of $M(s)$.

In this case, the operator $\Phi \mapsto \mathcal{F}^{-1}(M\mathcal{F}\Phi)$, originally defined on the Schwartz functions, extends to a bounded operator from $L_p(\mathbb{R}, X)$ to $L_s(\mathbb{R}, Y)$. We shall use the notation $L_p(\mathbb{R}, X) \overset{m}{\mapsto} L_s(\mathbb{R}, Y)$ to indicate the Fourier multiplier property. Fourier multipliers on the spaces $L_p(\Gamma, X)$ (so called discrete multipliers) are defined analogously.

3.6 Example If $E(t) \in L_1(\mathbb{R}, \mathcal{B}(X))$ and $M(s)x = \int_{\mathbb{R}} e^{-ist} E(t)x\, dt$ ($x \in X$), then $M(s)$ is an $L_1(\mathbb{R}, X) \overset{m}{\mapsto} L_1(\mathbb{R}, X)$ Fourier multiplier.

Proof. Young's inequality ([ABHN, Proposition 1.3.5]) states that the convolution operator $f \mapsto E * f$ is bounded on $L_1(\mathbb{R}, X)$, where the convolution is defined as

$$(E * f)(s) = \int_{\mathbb{R}} E(t) f(s-t)\, dt.$$

The usual argument (i.e. Fubini's theorem) works to verify that the Fourier transform of the convolution is the product of the Fourier transforms, that is $(\mathcal{F}(E * f))(s) = M(s)(\mathcal{F}f)(s)$, so at least for any $\Phi \in \mathcal{S}(\mathbb{R}, X)$ we have

$$\|\mathcal{F}^{-1}(M\mathcal{F}\Phi)\|_{L_1(\mathbb{R},X)} = \|E * \Phi\|_{L_1(\mathbb{R},X)} \leq C\|\Phi\|_{L_1(\mathbb{R},X)}.$$

□

3.7 Example Let H denote a Hilbert space and let $M(s) \in L_\infty(\mathbb{R}, \mathcal{B}(H))$. Then $M(s)$ is an $L_2(\mathbb{R}, H) \overset{m}{\mapsto} L_2(\mathbb{R}, H)$ multiplier. Indeed, if $\|M(s)\| \leq C$ for all $s \in \mathbb{R}$, then Plancherel's theorem (see [ABHN, Theorem 1.8.2]) yields

$$\begin{aligned}\|\mathcal{F}^{-1}\left(M(s)(F\Phi)(s)\right)\|_{L_2(\mathbb{R},H)} &= \frac{1}{\sqrt{2\pi}}\|M(s)(\mathcal{F}\Phi)(s)\|_{L_2(\mathbb{R},H)} \\ &\leq C\frac{1}{\sqrt{2\pi}}\|\mathcal{F}\Phi\|_{L_2(\mathbb{R},H)} = C\|\Phi\|_{L_2(\mathbb{R},H)}\end{aligned}$$

for any $\Phi \in \mathcal{S}(\mathbb{R}, H)$.

There are a couple of less obvious conditions to make an operator-valued function a multiplier, see [W].

4 Uniform Exponential Stability

One of the most basic types of asymptotic behaviour a semigroup can exhibit is the uniform exponential decay of the orbits, that is, $\|T(t)x\| \leq Me^{-\omega t}\|x\|$ holds for some M and $\omega > 0$.

4.1 Definition A semigroup $T(t)$ is called uniformly exponentially stable, if $\omega_0(T) < 0$.

The equalities (2.4) easily yield the following characterizations of uniform exponential stability.

4.2 Proposition *For a semigroup $T(t)$ the following assertions are equivalent.*

(a) *$T(t)$ is uniformly exponentially stable.*

(b) $\lim_{t\to\infty} \|T(t)\| = 0$.

(c) *$\|T(t_0)\| < 1$ for some $t_0 > 0$.*

(d) *$r(T(t_1)) < 1$ for some $t_1 > 0$.*

An equivalent description is obtained by considering the convolution with the semigroup. Since the resolvent is the Fourier transform of the semigroup, a condition involving the resolvent as a multiplier emerges.

4.3 Theorem *Suppose $T(t)$ is a semigroup on the Banach space X, and let $R(\lambda)$ denote the resolvent of its generator. Then the following are equivalent.*

(a) *$T(t)$ is uniformly exponentially stable.*

(b) *For some/all $1 \le p < \infty$, $\|T * f\|_{L_p(\mathbb{R}_+,X)} \le C\|f\|_{L_p(\mathbb{R}_+,X)}$ for a constant C and all $f \in L_p(\mathbb{R}_+, X)$.*

(c) *$s_0(A) < 0$ and the function $s \mapsto R(is)$ is a Fourier multiplier on $L_p(\mathbb{R}, X)$ for some/all $1 \le p < \infty$.*

Proof. The equivalence (a) $\Leftrightarrow$ (b) is the Datko-van Neerven theorem ([vN, Theorem 3.3.1]), a short proof can be found in [CL, Theorem 2.44]. The equivalence (a) $\Leftrightarrow$ (c) will be contained in Corollary 7.5. □

5 Hyperbolicity

Even when the semigroup $T(t)$ is not uniformly exponentially stable, we may be able to decompose the underlying space X into a direct sum $X_+ \oplus X_-$ such that $T(t)$ is "forward" exponentially stable on X_+ and "backward" exponentially stable on X_-. We shall call these semigroups hyperbolic.

5.1 Definition A semigroup $T(t)$ on a Banach space X is said to be hyperbolic, if the space X decomposes into $X = X_+ \oplus X_-$, where X_+ and X_- are $T(t)$-invariant closed subspaces; there are real numbers $M > 0$ and $\omega < 0$ such that the restricted semigroup $T_+(t)$ on X_+ satisfies $\|T_+(t)\| \leq Me^{\omega t}$ $(t > 0)$, furthermore, the restricted semigroup $T_-(t)$ on X_- extends to a group of operators on X_- defined by the rule $T_-(-t) = T_-(t)^{-1}$, and $\|T_-(-t)\| \leq Me^{\omega t}$ $(t > 0)$.

Such a decomposition clearly induces a decomposition of the semigroup's generator A.

5.2 Definition A linear operator A on a Banach space X is called bisectional, if there are closed subspaces X_+, X_- invariant to A such that $X = X_+ \oplus X_-$ and $\sigma(A_+) \subset \{\lambda \in \mathbb{C} | \operatorname{Re} \lambda < 0\}$, $\sigma(A_-) \subset \{\lambda \in \mathbb{C} | \operatorname{Re} \lambda > 0\}$, where A_+ and A_- denote the restriction of A to X_+ and X_-, respectively, with $D(A_+) = X_+ \cap D(A),\ D(A_-) = X_- \cap D(A)$. A projection $P \in \mathcal{B}(X)$ with $\operatorname{Ker} P = X_+$, $\operatorname{Im} P = X_-$ is called a splitting projection for the operator A.

5.3 Definition A bisectional linear operator A is exponentially dichotomous, if A_+ and $-A_-$ are generators of uniformly exponentially stable semigroups on X_+ and X_-.

Observe that, in accordance with the invertibility and "backward" stability of $T(t)_-$, we required $-A_-$ to be a generator.

5.4 Proposition *Let $T(t)$ be a semigroup with generator A. Then $T(t)$ is hyperbolic if and only if A is exponentially dichotomous.*

For an exponentially dichotomous operator A, the splitting projection will turn out to be unique.

5.5 Definition To a hyperbolic semigroup $T(t)$, whose generator has splitting projection P, we associate the operator-valued Green's function defined by

$$G_T(t) = \begin{cases} T(t)(I-P) & \text{for } t > 0 \\ \frac{1}{2}I - P & \text{for } t = 0 \\ -T(t)P & \text{for } t < 0 \end{cases} .$$

The Green's function of a hyperbolic semigroup is exponentially decaying and strongly continous except at the origin, where it has a jump discontinuity.

Linearity transmits formula (2.5) to hyperbolic semigroups in the subsequent

5.6 Proposition *Assume that the semigroup $T(t)$ is hyperbolic, and let $R(\lambda), G_T(t)$ denote its resolvent and Green's function, respectively. Then there exists a $\varrho_0 > 0$ such that*

$$R(\varrho + is)x = \int_{\mathbb{R}} e^{-(\varrho+is)t} G_T(t)x\,dt \quad (x \in X) \tag{5.1}$$

for all $\varrho \in \mathbb{R}, |\varrho| < \varrho_0$. In particular, the resolvent on the imaginary axis is the Fourier transform of Green's function.

Proof. Consider the decomposition $X = X_+ \oplus X_-$ of the underlying Banach space described in Definiton 5.1, and for any $x \in X$ we write $x = x_+ + x_-$, where $x_+ \in X_+$, $x_- \in X_-$ are uniquely determined. The subspaces X_+ and X_- are invariant to the generator A, which can accordingly be split as $A = A_+ \oplus A_-$. Further, $x_- = Px, x_+ = (I-P)x$ with the splitting projection belonging to A and by introducing the notation $R_+(\lambda) = R(\lambda, A_+), R_-(\lambda) = R(\lambda, A_-)$ we have $R(\lambda) = R_+(\lambda) \oplus R_-(\lambda)$. The restriction of $T(t)$ to X_+, that is, the semigroup $T_+(t)$ is generated by A_+, so equality (2.5) gives

$$R_+(\varrho + is)x_+ = \int_0^\infty e^{-(\varrho+is)t} T_+(t)x_+\,dt = \int_0^\infty e^{-(\varrho+is)t} T(t)(I-P)x\,dt$$

for all $\varrho > \omega_0(T_+)$. Similarly, the operator $-A_-$ generates $T_-(-t)$ on X_- and because of $R(\varrho + is, -A_-) = -R_-(-(\varrho + is))$, we obtain

$$R_-(\varrho + is)x_- = \int_0^\infty e^{(\varrho+is)t} T_-(-t)x_-\,dt = -\int_{-\infty}^0 e^{-(\varrho+is)t} T(t)Px\,dt$$

for all $\varrho < -\omega_0(T_-(-t))$. The definition of hyperbolicity implies that $\omega_0(T_+(t)) < 0$, $\omega_0(T_-(-t)) < 0$, thus if we choose ϱ_0 to be the minimum of $-\omega_0(T_+(t))$ and $-\omega_0(T_-(-t))$), then $\varrho_0 > 0$ and both equalities above hold for any $|\varrho| < \varrho_0$; hence addition gives the desired result. □

5.7 Corollary *If $T(t)$ is hyperbolic, then*

$$G_T(t)x = \frac{1}{2\pi}(C)\int_{\mathbb{R}} R(is)e^{ist}x\,ds \quad (x \in X),$$

for all $t \in \mathbb{R}$; in particular, $P = \frac{1}{2}I - G_T(0)$ and the splitting projection P is unique.

Proof. Apply Lemma 3.1 to equality (5.1) above. □

It is already clear that the operator $x \mapsto (C) \int_{\mathbb{R}} R(is)x\, ds$ is closely related to the splitting projection. As a matter of fact, it is sufficient to assume the boundedness of $R(is)$ and the convergence of $(C) \int_{\mathbb{R}} R(is)x\, ds$ $(x \in X)$ to obtain bisectionality (combine [KVL, Lemma 2.2] with [BGK, Theorem 3.1]).

Analogously to uniformly exponentially stable semigroups, which are characterized by the semigroup operators' spectra being contained in the unit disk, hyperbolic semigroups also have spectral description.

5.8 Proposition *For a semigroup $T(t)$ the following assertions are equivalent.*

(a) *$T(t)$ is hyperbolic.*

(b) *$\sigma(T(t)) \cap \Gamma = \emptyset$ for all $t > 0$.*

(c) *$\sigma(T(t_0)) \cap \Gamma = \emptyset$ for some $t_0 > 0$.*

Proof. First we prove (a) $\Rightarrow$ (b). Since $X = X_+ \oplus X_-$ is a decomposition into $T(t)$-invariant subspaces, $\sigma(T(t)) = \sigma(T_+(t)) \cup \sigma(T_-(t))$. Here $T_+(t)$ is uniformly exponentially stable, hence Proposition 4.2 gives $r(T_+(t)) < 1$, thus $\sigma(T_+(t)) \cap \Gamma = \emptyset$. Similarly, $r(T_-(t)^{-1}) < 1$ and from

$$\sigma(T_-(t)) = \left\{ \frac{1}{\lambda} \mid \lambda \in \sigma\left(T_-(t)^{-1}\right) \right\}$$

we conclude that $|\lambda| > 1$ for all $\lambda \in \sigma(T_-(t))$, hence $\sigma(T_-(t)) \cap \Gamma = \emptyset$.

It is clear that (b) implies (c). To prove (c) $\Rightarrow$ (a), fix $t_0 > 0$ such that $\sigma(T(t_0)) \cap \Gamma = \emptyset$, and let

$$Sx = \frac{1}{2\pi i} \int_\Gamma R(\lambda, T(t_0))x\, d\lambda \quad (x \in X)$$

be the spectral projection belonging to $\sigma(T(t_0)) \cap \{\, |\lambda| < 1 \,\}$. Since $R(\lambda, T(t_0))T(t) = T(t)R(\lambda, T(t_0))$ for all $t \geq 0$, we have $ST(t) = T(t)S$ by the definition of S. Further, let $P = I - S$. Then P is a projection, which commutes with all the semigroup operators $T(t)$, so $X = X_+ \oplus X_-$, the direct sum of $T(t)$-invariant subspaces, where $X_+ = \operatorname{Ker} P$, $X_- = \operatorname{Im} P$. The restriction $T_+(t_0)$ of $T(t_0)$ to X_+ has spectrum $\sigma(T(t_0)) \cap \{\, |\lambda| < 1 \,\}$, so $r(T_+(t_0)) < 1$ and $T_+(t)$ is uniformly exponentially stable (Proposition 4.2). Similarly, the restriction $T_-(t_0)$ of $T(t_0)$ to X_- has spectrum $\sigma(T(t_0)) \cap \{\, |\lambda| > 1 \,\}$, thus $T_-(t_0)$ is invertible on X_-, hence all the operators $T_-(t)$ are invertible. Moreover, we have $r(T_-(t_0)^{-1}) < 1$, which once again implies the uniform exponential stability for the semigroup $T_-(t)^{-1}$, and the proof is complete. □

No matter how pleasant the previous result may seem, in numerous applications one is only given the generator, while the semigroup itself remains unknown. Therefore, we would prefer a characterization of hyperbolicity in terms of the generator and its spectrum. Sometimes this is quite straightforward (see Examples 6.5, 6.6), but in the case of a typical semigroup the situation is far less satisfactory. We may, however, presume the following

5.9 Proposition *Let A generate the semigroup $T(t)$ and suppose that*

$$\sigma(T(t)) \subset \Gamma \cdot \exp(t\sigma(A)).$$

Then $T(t)$ is hyperbolic if and only if $i\mathbb{R} \subset \varrho(A)$.

Hence the question emerges: under what circumstances is the validity of the condition above guaranteed? Theorems relating the spectra of the generator and the semigroup are abounding sources of information.

6 Spectral Mapping Theorems

The unmarred formula we lust for, that is,

$$\sigma(T(t)) = \exp(t\sigma(A)),$$

may, either plainly or subtly, fail in the general context. However, surprisingly much still remains accessible.

6.1 Theorem *If A generates the semigroup $T(t)$, then*

$$\mathrm{P}\sigma(T(t))\backslash\{0\} = \exp(t\mathrm{P}\sigma(A)).$$

Proof. By $0 \notin \exp(t\mathrm{P}\sigma(A))$ and the rescaling argument it is sufficient to prove that $1 \in \mathrm{P}\sigma(T(2\pi))$ if and only if there exists $k \in \mathbb{Z}$ such that $ik \in \mathrm{P}\sigma(A)$. Taking $t = 2\pi$, $\lambda = ik$ in (2.2) yields

$$x - T(2\pi)x = \int_0^{2\pi} e^{-ikt}T(t)(ik - A)x\,dt$$

for all $x \in D(A)$, hence $T(2\pi)x = x$ provided that $Ax = ikx$. Conversely, if there is a $0 \neq x_0 \in X$ such that $T(2\pi)x_0 = x_0$, then the orbit $t \mapsto T(t)x_0$ is 2π -periodic, and now (2.1) gives

$$0 = (ik - A)\int_0^{2\pi} e^{-iks}T(s)x_0\,ds,$$

where the Fourier coefficients $\int_0^{2\pi} e^{-iks}T(s)x_0\,ds$ are in $D(A)$ (see Proposition 2.2) and (by Lemma 3.1) not all of them vanish. □

6.2 Corollary *If A generates the semigroup $T(t)$, then*

$$\mathrm{R}\sigma(T(t))\backslash\{0\} = \exp(t\mathrm{R}\sigma(A)).$$

Proof. One can prove that $\mathrm{R}\sigma(T(t)) = \mathrm{P}\sigma(T^*(t)) = \mathrm{P}\sigma(T^{\odot}(t)), \mathrm{R}\sigma(A) = \mathrm{P}\sigma(A^*) = \mathrm{P}\sigma(A^{\odot})$ and apply Theorem 6.1 to $T^{\odot}(t)$; the details are in [vN, Theorem 2.1.3]. □

Consequently, the point and residual spectra of the generator exhibit fair behaviour. However, the approximate spectrum does not do, and this is the reason for $i\mathbb{R} \subset \varrho(A)$ not being sufficient for hyperbolicity, cf. Example 6.6 (althought it is necessary, see Corollary 6.4). We need additional hypotheses to control $A\sigma(T(t))$; a possible condition is given in the next proposition.

6.3 Proposition *If $T(t)$ is the semigroup generated by A on the Banach space X and $t_0 > 0$, then $\lambda \in \exp(t_0\mathrm{A}\sigma(A))$ if and only if there is a sequence $x_n \in X$, $\|x_n\| = 1$ satisfying $\|T(t_0)x_n - \lambda x_n\| \to 0$ as well as $\lim_{t\to 0}\sup_n \|T(t)x_n - x_n\| = 0$.*

We refer the reader to [EN, Lemma IV.3.9] for a proof. One can apply Proposition 6.3 to gain a spectral mapping theorem for the large class of eventually norm continous semigroups (see [EN, Theorem IV.3.10]), so for these semigroups hyperbolicity and $i\mathbb{R} \subset \varrho(A)$ are equivalent. However, we shall take a different route through Theorem 6.7, and try to characterize hyperbolicity without any further assumption on the semigroup itself.

6.4 Corollary *The spectrum $\sigma(A)$ of the generator and the spectrum $\sigma(T(t))$ of the semigroup operator $T(t)$ satisfy $\exp(t\sigma(A)) \subset \sigma(T(t))$ for $t \geq 0$.*

Proof. Besides being a corollary of the previous statements, this can directly be derived from (2.1) and (2.2). □

6.5 Example Suppose A is a bounded operator on a Banach space. Then A has holomorphic functional calculus (see [R, Definition 10.26]), in particular, $\exp(tA)$ makes sense and by differentiation one can check that A is the generator of the semigroup $T(t) = \exp(tA)$. So $\sigma(T(t)) = \exp(t\sigma(A))$; consequently, A is exponentially dichotomous if and only if $i\mathbb{R} \subset \varrho(A)$.

6.6 Example Let M_q be the generator of a multiplication semigroup on $L_p(\Omega)$. A straightforward computation shows that M_q is bisectional if and only if $i\mathbb{R} \subset \varrho(M_q)$ and M_q is exponentially dichotomous if and only if there is an $\omega > 0$ such that the strip $\{\lambda \in \mathbb{C} \mid |\mathrm{Re}\,\lambda| < \omega\}$ is contained in $\varrho(M_q)$.

The following theorem, due to Greiner ([G]), may come as a surprise, but it does have the character of a spectral mapping theorem. Motivation is postponed until the next section, and a comprehensive treatment until Section 9 (Theorem 9.1); see also [vN, Theorem 2.2.1].

6.7 Theorem *Let $T(t)$ be a semigroup with generator A. Then $1 \in \varrho(T(2\pi))$ if and only if $i\mathbb{Z} \subset \varrho(A)$ and $(C)\sum_{k\in\mathbb{Z}} R(ik)x$ is convergent for all $x \in X$.*

Proof. If we take $t = 2\pi$, $\lambda = ik$ in (2.3), we obtain

$$\frac{1}{2\pi}R(ik)(I - T(2\pi))x = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikt}T(t)x\,dt,$$

hence Lemma 3.1 yields

$$\frac{1}{2\pi}(C)\sum_{k\in\mathbb{Z}} R(ik)(I - T(2\pi))x = \frac{1}{2}(I + T(2\pi))x. \tag{6.1}$$

If $1 \in \varrho(T(2\pi))$, then by the spectral inclusion (Corollary 6.4) $i\mathbb{Z} \subset \varrho(A)$. Furthermore, $I - T(2\pi)$ is surjective, thus for any $y \in X$ we can take $x \in X$ such that $y = (I - T(2\pi))x$ and from (6.1) we deduce

$$\frac{1}{2\pi}(C)\sum_{k\in\mathbb{Z}} R(ik)y = \frac{1}{2}(I + T(2\pi))x,$$

in particular, $(C)\sum_{k\in\mathbb{Z}} R(ik)y$ is convergent.

Conversely, if we write $Sx = \frac{1}{2\pi}(C)\sum_{k\in Z} R(ik)x$, then (6.1) gives

$$\left(\frac{1}{2}I + S\right)(I - T(2\pi))\,x = (I - T(2\pi))\left(\frac{1}{2}I + S\right)x = x,$$

so $I - T(2\pi)$ is injective and surjective, hence (continously) invertible. □

An application of the rescaling argument yields

6.8 Corollary *The semigroup $T(t)$ is hyperbolic if and only if $i\mathbb{R} \subset \varrho(A)$ and $(C)\sum_{k\in\mathbb{Z}} R(i(k+\xi))x$ is convergent for all $\xi \in [0,1)$ and $x \in X$.*

7 A Characterization of Hyperbolicity

By virtue of the spectral mapping theorem 6.7 we are able to formulate an equivalent condition of hyperbolicity with reference to the resolvent only; the classical result of Kaashoek and Verduyn Lunel (see [KVL]) has been strengthened by Latushkin and Shvydkoy in [LS].

7.1 Proposition *If $T(t)$ is a hyperbolic semigroup whose resolvent has matrix elements r_ϱ, then there exists a $\varrho_0 > 0$ such that $\|\check{r}_\varrho\|_\infty \le K_\varrho \|x\| \|x^*\|$ for all $|\varrho| < \varrho_0$.*

Proof. Let us write $g_\varrho(t) = \langle x^*, e^{-\varrho t} G_T(t) x\rangle$, where $x \in X, x^* \in X^*$ and $G_T(t)$ is Green's function. $G_T(t)$ is exponentially decaying, so $\|e^{-\varrho t} G_T(t)\| \le K_\varrho$ for some K_ϱ and all $t \in \mathbb{R}$, provided that $|\varrho|$ is small enough. We obtain from Proposition 5.6 that $\hat{g}_\varrho = r_\varrho$, therefore $\|\check{r}_\varrho\|_\infty = \|g_\varrho\|_\infty \le K_\varrho \|x\| \|x^*\|$, as asserted. $\square$

Remarkably, the easily obtained consequence above is also sufficient for hyperbolicity. Before we would embark upon the proof, we show that there is a firm connection between the functions $\check{r}_\varrho$; this will immediately yield that the boundedness of $\check{r}_\varrho$ implies the exponential decay of $\check{r}$.

7.2 Lemma *Suppose the resolvent is uniformly bounded on a vertical strip containing the imaginary axis, i.e.* $\sup\{\|R(\varrho + is)\| \mid s \in \mathbb{R}, |\varrho| < \varrho_0\} < \infty$. *Then $\check{r} = e^{\varrho t} \check{r}_\varrho$ for all ϱ with $|\varrho| < \varrho_0$.*

Here $\check{r}$ and $\check{r}_\varrho$ are distributions and $e^{\varrho t}$ is a function on $\mathbb{R}$.
Proof. If $\Phi \in \mathcal{S}$ has compact support, then $\check{\Phi}$ is an entire function and $\lim_{\alpha\to\infty} \check{\Phi}(\alpha + i\beta) = 0$ uniformly in β on bounded sets, since

$$2\pi\check{\Phi}(\alpha + \beta i) = \int_{\mathbb{R}} e^{\beta s}\Phi(s) e^{i\alpha s}\, ds = -\frac{1}{\alpha i}\int_{\mathbb{R}} (\beta e^{\beta s}\Phi(s) + e^{\beta s}\Phi'(s)) e^{i\alpha s}\, ds\,.$$

Cauchy's theorem now gives

$$\begin{aligned}
(\check{r}_\varrho, \Phi) = (r_\varrho, \check{\Phi}) &= \int_{\mathbb{R}} \langle x^*, R(\varrho + is)x\rangle \check{\Phi}(s)\, ds = \int_{\mathbb{R}} \langle x^*, R(i(s - i\varrho))x\rangle \check{\Phi}(s)\, ds \\
&= \int_{\mathbb{R}-i\varrho} \langle x^*, R(is)x\rangle \check{\Phi}(s + i\varrho)\, ds = \int_{\mathbb{R}} \langle x^*, R(is)x\rangle \check{\Phi}(s + i\varrho)\, ds \\
&\quad + 2i \lim_{\alpha\to\pm\infty} \int_0^\varrho \langle x^*, R(i(\alpha + i\beta))x\rangle \check{\Phi}(\alpha + i(\beta + \varrho))\, d\beta \\
&= \int_{\mathbb{R}} \langle x^*, R(is)x\rangle \check{\Phi}(s + i\varrho)\, ds = (r, \check{\Phi}(\cdot + i\varrho)) = (\check{r}, e^{-\varrho t}\Phi),
\end{aligned}$$

as stated. $\square$

The central theorem of this section is at hand.

7.3 Theorem *The semigroup $T(t)$ is hyperbolic if and only if there is a $\varrho_0 > 0$ such that $\sup_{|\mathrm{Re}\,\lambda|<\varrho_0} \|R(\lambda)\| < \infty$ and the matrix elements r_ϱ satisfy $\|\check{r}_\varrho\|_\infty \leq K_\varrho \|x\| \|x^*\|$ for some K_ϱ for all $|\varrho| < \varrho_0$ and $x \in X, x^* \in X^*$.*

Proof. Necessity has been stated in Proposition 7.1; it remains to prove sufficiency. By Lemma 7.2, $\check{r}(s) = e^{-\varrho s}\check{r}_{-\varrho}(s)$ a.e. and $\check{r}(s) = e^{\varrho s}\check{r}_\varrho(s)$ a.e. for some $\varrho > 0$ small enough. By the assumption $|\check{r}(s)| \leq e^{-\varrho|s|}K\|x\|\|x^*\|$ a.e., where $K = \max\{K_\varrho, K_{-\varrho}\}$. Observe that $\Gamma = \mathbb{R}/(2\pi\mathbb{Z})$ and for any $f \in L_1(\Gamma)$ let f denote its lifted version as well. (In other words, identify Γ with $[0, 2\pi]$ and extend f periodically.) Then for all $\xi \in [0, 1)$, $f \mapsto \int_{\mathbb{R}} f(s)e^{-i\xi s}\check{r}(s)\,ds$ is a continous linear functional on $L_1(\Gamma)$ and by duality there are functions $m_\xi \in L_\infty(\Gamma)$, $\|m_\xi\|_\infty \leq C\|x\|\|x^*\|$ for some constant C such that

$$\int_{\mathbb{R}} f(s)e^{-i\xi s}\check{r}(s)\,ds = \int_\Gamma f(s)m_\xi(s)\,ds,$$

in particular, for any $k \in \mathbb{Z}$ and $f(s) = e^{-isk}$

$$r(k+\xi) = \int_{\mathbb{R}} e^{-is(k+\xi)}\check{r}(s)\,ds = \int_\Gamma e^{-isk}m_\xi(s)ds.$$

Let us introduce the partial Cesàro sum operators $S_N \in \mathcal{B}(X)$, $S_N x = \frac{1}{N+1}\sum_{l=0}^N \sum_{k=-l}^l R(i(k+\xi))x$. By the argument above and Lemma 3.4, we have

$$|\langle x^*, S_N x\rangle| = \left| \frac{1}{N+1}\sum_{l=0}^N \sum_{k=-l}^l r(k+\xi) \right| \leq \|m_\xi\|_\infty \leq C\|x\|\|x^*\|,$$

therefore, by duality, $\|S_N\| \leq C$. Since $R(\lambda)x = \frac{1}{\lambda^2}R(\lambda)A^2x + \frac{1}{\lambda}x + \frac{1}{\lambda^2}Ax$ for all $x \in D(A^2)$, it is clear that $\lim_{n\to\infty} S_N x$ exists for all $x \in D(A^2)$ (see also Lemma 8.3), and then the density of $D(A^2)$ and the uniform boundedness of S_N yield the existence of the limit for all $x \in X$. Therefore $(C)\sum_{k\in\mathbb{Z}} R(i(k+\xi))x$ converges, and an application of Corollary 6.8 completes the proof. □

The next theorem reveals that the hypothesis above is actually a condition on the multiplier properties of the resolvent. Relevance of Fourier multipliers are indicated by Theorem 4.3 as well.

7.4 Theorem *Suppose there exists a $\varrho_0 > 0$ such that resolvent set $\varrho(A)$ of the semigroup generator A satisfies $\{\lambda \in \mathbb{C} \mid |\mathrm{Re}\,\lambda| < 2\varrho_0\} \subset \varrho(A)$. For the resolvent $R(z)$ the following assertions are equivalent:*

(a) *For some/all $1 \leq p < \infty$ the resolvent on the imaginary axis $R(i\cdot)$ is an $L_p(\mathbb{R}, X) \overset{m}{\mapsto} L_p(\mathbb{R}, X)$ multiplier.*

(b) *There exist* $1 \le p < \infty$ *and* $1 \le s \le \infty$ *such that* $R(i\cdot+\varrho)$ *is an* $L_p(\mathbb{R},X) \overset{m}{\mapsto} L_s(\mathbb{R},X)$ *multiplier for all* $|\varrho| < \varrho_0$.

(c) *There exists a* $1 \le p < \infty$ *for which* $R(i\cdot+\varrho)$ *is an* $L_p(\mathbb{R},X) \overset{m}{\mapsto} L_\infty(\mathbb{R},X)$ *multiplier for all* $|\varrho| < \varrho_0$.

(d) *For all* $|\varrho| < \varrho_0$, $R(i\cdot+\varrho)$ *is an* $L_1(\mathbb{R},X) \overset{m}{\mapsto} L_\infty(\mathbb{R},X)$ *multiplier.*

(e) $\sup_{|\mathrm{Re}\,\lambda|<\varrho_0} \|R(\lambda)\| < \infty$ *and* $|(r_\varrho,\Phi)| \le K_\varrho\|x\|\|x^*\|\|\check{\Phi}\|_1$ *holds for all* $|\varrho| < \varrho_0$ *and* $\Phi \in \mathcal{S}(\mathbb{R})$.

(f) $\sup_{|\mathrm{Re}\,\lambda|<\varrho_0} \|R(\lambda)\| < \infty$ *and* $\|\check{r}_\varrho\|_\infty \le K_\varrho\|x\|\|x^*\|$ *for all* $|\varrho| < \varrho_0$.

Proof. It is immediately seen that (d) $\Rightarrow$ (e) $\Rightarrow$ (f); we now prove (f) $\Rightarrow$ (a). As we have seen in the proof of Theorem 7.3 by applying Lemma 7.2, $\check{r}(s) = e^{\varrho s}\check{r}_\varrho(s)$ a.e. for some $\varrho > 0$ small enough, therefore $|\hat{r}(s)| \le e^{-\varrho|s|}K\|x\|\|x^*\|$. Consider a function $\Phi(s) = \sum_{k=1}^n \Phi_k(s)x_k \in \mathcal{S}(\mathbb{R},X)$ where $\Phi_k \in \mathcal{S}(\mathbb{R})$ and have disjoint supports. We have $\|\Phi\|^p_{L_p(\mathbb{R},X)} = \sum_{k=1}^n \|x_k\|^p\|\Phi_k\|_p^p$ and by Lemma 1.1, functions of this type form a dense subset of $L_p(\mathbb{R},X)$. The following computation verifies our assertion.

$$\begin{aligned}
\int_{\mathbb{R}} \left\| \int_{\mathbb{R}} R(is)\hat{\Phi}(s)e^{ist}\,ds \right\|^p dt &= \int_{\mathbb{R}} \sup_{\|x^*\|\le 1} \left| \sum_{k=1}^n \int_{\mathbb{R}} r(s,x_k,x^*)\hat{\Phi}_k(s)e^{ist}\,ds \right|^p dt \\
&= \int_{\mathbb{R}} \sup_{\|x^*\|\le 1} \left| \sum_{k=1}^n \int_{\mathbb{R}} \hat{r}(s,x_k,x^*)\Phi_k(t+s)\,ds \right|^p dt \\
&\le K^p \int_{\mathbb{R}} \left(\sum_{k=1}^n \int_{\mathbb{R}} e^{-\varrho|s|}\|x_k\||\Phi_k(t+s)|\,ds \right)^p dt \\
&= K^p \int_{\mathbb{R}} \left(\int_{\mathbb{R}} e^{-\varrho|s|} \left(\sum_{k=1}^n \|x_k\||\Phi_k(t+s)| \right) ds \right)^p dt \\
&\le C_\varrho K^p \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-\varrho|s|} \left(\sum_{k=1}^n \|x_k\||\Phi_k(t+s)| \right)^p dsdt \\
&= C_\varrho K^p \int_{\mathbb{R}} e^{-\varrho|s|} \int_{\mathbb{R}} \left(\sum_{k=1}^n \|x_k\||\Phi(t+s)| \right)^p dtds \\
&= C_\varrho K^p \int_{\mathbb{R}} e^{-\varrho|s|} \sum_{k=1}^n \|x_k\|^p\|\Phi_k\|_p^p\,ds = C'_\varrho K^p\|\Phi\|^p_{L_p(\mathbb{R},X)}
\end{aligned}$$

We prove the implication (a) $\Rightarrow$ (b). Take an arbitrary $1 \le p < \infty$. The function $s \mapsto R(is)$ is an $L_p(\mathbb{R}, X) \overset{m}{\mapsto} L_p(\mathbb{R}, X)$ multiplier, therefore $\|R(is)\|$ is bounded. Then there is a $\varrho_1 > 0$ such that $(I + \varrho R(is))^{-1}$ exists for $|\varrho| \le \varrho_1$, and the resolvent identity yields

$$R(is + \varrho) = R(is)(I + \varrho R(is))^{-1},$$

that is to say, $R(is + \varrho)$ exists, and satisfies the equation ($|\varrho| < \varrho_1$). Generally speaking, let $M_1(s), M_2(s) : \mathbb{R} \mapsto \mathcal{B}(X)$ be smooth $L_p(\mathbb{R}, X) \overset{m}{\mapsto} L_p(\mathbb{R}, X)$ Fourier multipliers with multiplier norm m_1, m_2, respectively. For any $\Phi(s) \in \mathcal{S}(\mathbb{R}, X)$ we have $\Psi(s) = M_2(s)\mathcal{F}(\Phi)(s) \in \mathcal{S}(\mathbb{R}, X)$, $\|\mathcal{F}^{-1}\Psi\|_{L_p(\mathbb{R},X)} \le m_2 \|\Phi\|_{L_p(\mathbb{R},X)}$, so

$$\begin{aligned}\mathcal{F}^{-1}(M_1(s)M_2(s)\mathcal{F}(\Phi)(s)) &= \mathcal{F}^{-1}\left(M_1(s)\mathcal{F}\mathcal{F}^{-1}\left(M_2(s)\mathcal{F}(\Phi)(s)\right)\right) \\ &= \mathcal{F}^{-1}\left(M_1(s)\mathcal{F}(\Psi)(s)\right).\end{aligned}$$

Thus

$$\|\mathcal{F}^{-1}(M_1(s)M_2(s)\mathcal{F}(\Phi)(s))\|_{L_p(\mathbb{R},X)} \le m_1 m_2 \|\Phi\|_{L_p(\mathbb{R},X)}.$$

Choose ϱ so small that $s \mapsto \varrho R(is)$ have multiplier norm $m < 1$. By the arguments above, $(\varrho R(is))^n$ has multiplier norm less than m^n, therefore

$$\begin{aligned}&\left\|\mathcal{F}^{-1}\left((I + \varrho R(is))^{-1}(\mathcal{F}\Phi)(s)\right)\right\|_{L_p(\mathbb{R},X)} \\ &\qquad \le \left\|\mathcal{F}^{-1}\left(\sum_{n=0}^{\infty}(-\varrho R(is))^n (F\Phi)(s)\right)\right\|_{L_p(\mathbb{R},X)} \\ &\le \sum_{n=0}^{\infty}\left\|\mathcal{F}^{-1}\left((-\varrho R(is))^n(\mathcal{F}\Phi)(s)\right)\right\|_{L_p} \le \sum_{n=0}^{\infty} m^n \|\Phi\|_{L_p(\mathbb{R},X)} = \frac{1}{1-m}\|\Phi\|_{L_p}.\end{aligned}$$

Then $(I+\varrho R(is))^{-1}$ is an $L_p(\mathbb{R}, X) \overset{m}{\mapsto} L_p(\mathbb{R}, X)$ multiplier and so is $R(is+\varrho)$, whenever $|\varrho|$ is small enough, which confirms (b) with $p = s$.

We go on to show that (b) $\Rightarrow$ (c). Having chosen any ϱ with $|\varrho| < \varrho_0$, from (b) we conclude that there is a constant C satisfying

$$\left\|\int_{\mathbb{R}} R(iu + \varrho)\hat{\Phi}(u)e^{iu\cdot}\,du\right\|_{L_s(\mathbb{R},X)} \le C\|\Phi\|_{L_p(\mathbb{R},X)},$$

for all $\Phi \in \mathcal{S}(\mathbb{R}, X)$. It implies that for all $n \in \mathbb{Z}$ there is a $t \in [n, n+1]$ such that

$$\left\|\int_{\mathbb{R}} R(iu + \varrho)\hat{\Phi}(u)e^{iut}\,du\right\|_X \le C\|\Phi\|_{L_p(\mathbb{R},X)}.$$

Fix any $\tau \in [0,2]$ and M with $\|T(u)\| \le M$ $(u \in [0,2])$ then apply the operator $T(\tau)$ to the vector on the left-hand side to obtain

$$\left\| \int_{\mathbb{R}} T(\tau) R(iu+\varrho) \hat{\Phi}(u) e^{iut} \, du \right\|_X \le MC \|\Phi\|_{L_p(\mathbb{R},X)}.$$

From formula (2.3) we infer

$$\begin{aligned} &\int_{\mathbb{R}} R(iu+\varrho) \hat{\Phi}(u) e^{iu(t+\tau)} \, du \\ &\qquad = \int_{\mathbb{R}} T(\tau) R(iu+\varrho) \hat{\Phi}(u) e^{iut} \, du + \int_{\mathbb{R}} \int_0^\tau T(v) \hat{\Phi}(u) e^{iu(t+\tau-v)} \, dv \, du. \end{aligned}$$

Here

$$\begin{aligned} \left\| \int_{\mathbb{R}} \int_0^\tau T(v) \hat{\Phi}(u) e^{iu(t+\tau-v)} \, dv du \right\|_X &= 2\pi \left\| \int_0^\tau T(v) \Phi(t+\tau-v) \, dv \right\|_X \\ &\le 2\pi M \int_t^{t+\tau} \|\Phi(v)\|_X \, dv \le 4\pi M \|\Phi\|_{L_p(\mathbb{R},X)}; \end{aligned}$$

therefore, by the triangle inequality,

$$\left\| \int_{\mathbb{R}} R(iu+\varrho) \hat{\Phi}(u) e^{iu(t+\tau)} \, du \right\|_X \le (4\pi + C) M \|\Phi\|_{L_p(\mathbb{R},X)},$$

and the observation that any real number is the sum of an appropriate t and τ allows us to arrive at the conclusion (c).

Finally, we prove (c) $\Rightarrow$ (d) by duality. For the sake of simplicity in notation, take $\varrho = 0$. Recall that $X^\odot$ denotes the sun dual of X and $R^\odot$ denotes the resolvent of the sun dual generator $A^\odot$. Let q be the conjugate exponent of p, that is, $\frac{1}{p} + \frac{1}{q} = 1$. Then $L_q(\mathbb{R}, X^\odot) \subset L_q(\mathbb{R}, X^*) \subset L_p(\mathbb{R}, X)^*$ with the natural duality pairing, see [DU, Section IV.1] for an elementary explanation of the second inclusion. Let $\Phi^\odot(s) = \sum_{j=1}^n \Phi_j(s) x_j^\odot$, where $\Phi_j \in \mathcal{S}(\mathbb{R}), x_j^\odot \in X^\odot$, the functions Φ_j have disjoint supports, and let $\Psi \in \mathcal{S}(\mathbb{R}, X)$. Then

$$\begin{aligned} &\|\mathcal{F}^{-1} \left(R^\odot(is)(\mathcal{F}\Phi^\odot)(s) \right) \|_{L_q(\mathbb{R},X^\odot)} \\ &\qquad = \sup_{\|\Psi\|_{L_p(\mathbb{R},X)} \le 1} \left| \left\langle \mathcal{F}^{-1} \left(R^\odot(is)(\mathcal{F}\Phi^\odot)(s) \right), \Psi \right\rangle \right| \\ &\qquad = \frac{1}{2\pi} \sup \left| \int_{\mathbb{R}} \left\langle \int_{\mathbb{R}} R^\odot(is) \hat{\Phi}^\odot(s) e^{ist} \, ds, \Psi(-t) \right\rangle dt \right| \end{aligned}$$

$$= \frac{1}{2\pi} \sup \left| \int_{\mathbb{R}} \int_{\mathbb{R}} \left\langle R^{\odot}(is)\hat{\Phi}^{\odot}(s), \Psi(-t)e^{ist} \right\rangle dtds \right|$$
$$= \frac{1}{2\pi} \sup \left| \int_{\mathbb{R}} \left\langle R^{\odot}(is)\hat{\Phi}^{\odot}(s), \int_{\mathbb{R}} \Psi(-t)e^{ist}\, dt \right\rangle ds \right|$$
$$= \frac{1}{2\pi} \sup \left| \sum_{j=1}^{n} \int_{\mathbb{R}} \left\langle R^{\odot}(is)x_j^{\odot}, \hat{\Phi}_j(s)\hat{\Psi}(s) \right\rangle ds \right|$$
$$= \frac{1}{2\pi} \sup \left| \sum_{j=1}^{n} \int_{\mathbb{R}} \left\langle x_j^{\odot}, R(is) \left(\hat{\Phi}_j(s)\hat{\Psi}(s) \right) \right\rangle ds \right|$$
$$= \frac{1}{2\pi} \sup \left| \sum_{j=1}^{n} \left\langle x_j^{\odot}, \int_{\mathbb{R}} R(is)\mathcal{F}(\Phi_j * \Psi)(s)\, ds \right\rangle \right|.$$

By Young's inequality, $\|\Phi_j * \Psi\|_{L_p(\mathbb{R},X)} \leq \|\Psi\|_{L_p(\mathbb{R},X)}\|\Phi_j\|_{L_1(\mathbb{R})}$. Condition (c) asserts that $R(is)$ is an $L_p(\mathbb{R},X) \overset{m}{\mapsto} L_\infty(\mathbb{R},X)$ multiplier, therefore

$$\left\| \int_{\mathbb{R}} R(is)\mathcal{F}(\Phi_j * \Psi)(s)\, ds \right\|_X \leq C\|\Phi_j * \Psi\|_{L_p(\mathbb{R},X)} \leq C\|\Psi\|_{L_p(\mathbb{R},X)}\|\Phi_j\|_{L_1(\mathbb{R})},$$

and we may continue the line of inequalities to obtain

$$\begin{aligned} &\|\mathcal{F}^{-1}\left(R^{\odot}(is)(\mathcal{F}\Phi^{\odot})(s)\right)\|_{L_q(\mathbb{R},X^{\odot})} \\ &\qquad \leq \frac{C}{2\pi} \sup \sum_{j=1}^{n} \|x_j^{\odot}\| \|\Phi_j\|_{L_1(\mathbb{R})} \|\Psi\|_{L_p(\mathbb{R},X)} \leq \frac{C}{2\pi}\|\Phi^{\odot}\|_{L_1(\mathbb{R},X^{\odot})}. \end{aligned}$$

Consequently, $R^{\odot}(is)$ is an $L_1(\mathbb{R},X^{\odot}) \overset{m}{\mapsto} L_q(\mathbb{R},X^{\odot})$ multiplier. Implication (b) $\Rightarrow$ (c) above shows that in this case $R^{\odot}(is)$ is an $L_1(\mathbb{R},X^{\odot}) \overset{m}{\mapsto} L_\infty(\mathbb{R},X^{\odot})$ multiplier as well. There exists a constant C dependent only on the semigroup such that $\|x\| \leq \sup|\langle x, x^{\odot}\rangle|$, where the supremum is taken over all $x^{\odot} \in X^{\odot}, \|x^{\odot}\| \leq C$, see [EN, Section II.2.6]. Thus, if $\Phi(s) = \sum_{j=1}^{n} \Phi_j(s)x_j$, where $x_j \in X$ and $\Phi_j \in \mathcal{S}(\mathbb{R})$ with disjoint supports, then

$$\left\| \int_{\mathbb{R}} R(is)\hat{\Phi}(s)e^{ist}\, ds \right\|_X \leq \sup_{x^{\odot}} \left| \left\langle x^{\odot}, \int_{\mathbb{R}} R(is)\hat{\Phi}(s)e^{ist}\, ds \right\rangle \right|$$
$$= \sup_{x^{\odot}} \left| \int_{\mathbb{R}} \sum_{j=1}^{n} \left\langle x^{\odot}, \hat{\Phi}_j(s)R(is)x_j e^{ist} \right\rangle ds \right|$$

$$= \sup_{x^\odot} \left| \sum_{j=1}^{n} \left\langle \int_{\mathbb{R}} R^\odot(is)\hat{\Phi}_j(s) x^\odot e^{ist}\, ds, x_j \right\rangle \right|$$

$$\leq \sup_{x^\odot} K \sum_{j=1}^{n} \|\Phi_j\|_{L_1(\mathbb{R})} \|x_j\| \|x^\odot\|_{X^\odot} \leq CK\|\Phi\|_{L_1(\mathbb{R},X)},$$

for all $t \in \mathbb{R}$, because $R^\odot$ is a multiplier. Hence $R(is)$ is an $L_1(\mathbb{R}, X) \overset{m}{\mapsto} L_\infty(\mathbb{R}, X)$ multiplier, which is the assertion in (d). The proof is complete. □

Clearly, any of the conditions above is equivalent to the hyperbolicity of $T(t)$.

7.5 Corollary If $s_0(A) < 0$ and the semigroup is hyperbolic, then by Lemma 3.2 and Corollary 5.7 the splitting projection coincides with the zero operator, that is, the semigroup is uniformly exponentially stable. In particular, we have proved that $s_0(A) < 0$ and $R(is)$ is a multiplier on $L_p(\mathbb{R}, X)$ for some $1 \leq p < \infty$ if and only if the semigroup is uniformly exponentially stable (cf. Theorem 4.3).

7.6 Corollary Let $T(t)$ be a semigroup on a Hilbert space H. Example 3.7 shows that that the resolvent $R(is)$ on the imaginary axis is an $L_2(\mathbb{R}, H) \overset{m}{\mapsto} L_2(\mathbb{R}, H)$ multiplier if and only if there exists a constant C such that $\|R(is)\| \leq C$. Consequently, $T(t)$ is hyperbolic if and only if the resolvent is uniformly bounded on the imaginary axis.

As a further corollary, we gain a result on the representation of the resolvent as a Fourier transform, cf. [BGK, Theorem 4.1].

7.7 Theorem *Let A be a semigroup generator on the Banach space X. Then A is exponentially dichotomous if and only if $i\mathbb{R} \subset \varrho(A)$ and*

$$R(is, A)x = \int_{\mathbb{R}} e^{-ist} E(t)x\, dt \quad (x \in X),$$

where $E(t) \in L_1(\mathbb{R}, \mathcal{B}(X))$. In that case $E(t)$ is the Green's function of the hyperbolic semigroup generated by A.

Proof. If $R(is, A)$ is represented by the integral above, then it is a multiplier by Example 3.6, so Theorem 7.4 and Theorem 7.3 give that the semigroup generated by A is hyperbolic, hence A is exponentially dichotomous. Conversely, if A is exponentially dichotomous, then it generates a hyperbolic semigroup, so Proposition 5.6 yields that the resolvent on the imaginary axis is the Fourier transform of Green's function.□

8 Generalization of Hyperbolicity

Arguments from the previous section on the multiplier properties of the resolvent are valid in a more general setting. For any $\alpha > 0$ we introduce the space X_α of "α-times" differentiable solutions and restrict the resolvent multiplier to the space X_α. The main result of this section is an extension of Theorem 7.4 to the case, when the resolvent is an $L_p(\mathbb{R}, X_\alpha) \overset{m}{\mapsto} L_p(\mathbb{R}, X)$ multiplier. The convergence of $(C) \int_{\mathbb{R}} R(is)e^{ist}x\,ds$ is also shown for $x \in X_\alpha$.

We now recall the notion of the fractional power of an operator A; for the sake of simplicity we keep on assuming that A is the generator of some semigroup $T(t)$ on X; though, this is in fact not necessary for the following summary. Fix any $w > \max\{\omega_0(T) + 2, 2\}$ and let us denote $A_w = A - wI$. Then $(0, \infty) \subset \varrho(A_w)$ and $\|R(\lambda, A_w)\| \leq \frac{M}{1+\lambda}$ $(\lambda > 0)$ for some constant M. By [EN, Proposition II.5.24] there exists an open sector Σ such that $(0, \infty) \subset \Sigma \subset \varrho(A)$ and $\|R(\lambda, A_w)\| \leq \frac{2M}{1+|\lambda|}$ $(\lambda \in \Sigma)$; this allows us to define

$$A_w^{-\alpha}x = \frac{1}{2\pi i}\int_\gamma \lambda^{-\alpha} R(\lambda, A_w)x\,d\lambda \quad (x \in X),$$

where γ is a path in $\Sigma\backslash(0, \infty)$ going from $\infty e^{-i\delta}$ to $\infty e^{i\delta}$ $(0 < \delta < \frac{\pi}{2})$; the integral does not depend on the choice of γ. Since $A_w^{-\alpha}$ is injective ([EN, Propositon II.5.30]), we define A_w^α as the inverse of $A_w^{-\alpha}$ on the domain $D(A_w^\alpha) = \operatorname{Im} A_w^{-\alpha} = X_\alpha$. Endowed with the norm $\|x\|_\alpha = \|A_w^\alpha x\|$, X_α is a Banach space and up to equivalent norms it does not depend on the choice of w (see [Ko, Theorem 6.4]).

Prior to treating the analogue of Theorem 7.4 we establish a lemma showing that the condition in the previous section on the boundedness of the resolvent should now be replaced by a condition on its polynomial growth.

8.1 Lemma *Let $\alpha > 0$ and let S be any subset of $\varrho(A)$. Then conditions*

$$\sup_{\lambda \in S} \frac{\|R(\lambda)\|}{1 + |\lambda|^\alpha} < \infty \quad \textit{and} \quad \sup_{\lambda \in S} \|R(\lambda)A_w^{-\alpha}\| < \infty$$

are equivalent.

Proof. There exists an $\varepsilon > 0$ such that $R(\lambda)$ is bounded on $D = \{\lambda \in \mathbb{C} \mid |\lambda - w| < \varepsilon\}$; from now on we suppose that $D \cap S = \emptyset$. By induction we have

$$R(\lambda)A_w^{-n} = \frac{1}{(w-\lambda)^n}R(\lambda) - \sum_{k=0}^{n-1} \frac{1}{(w-\lambda)^k} A_w^{-(n-k)}$$

for all $n \in \mathbb{N}$. Here the operators $\sum_{k=0}^{n-1} \frac{1}{(w-\lambda)^k} A_w^{-(n-k)}$ are uniformly bounded in $\lambda \in S$. Further, for all $\alpha > 0$ and $\lambda \in S$, $\frac{R(\lambda)}{(w-\lambda)^\alpha} = \frac{1+|\lambda|^\alpha}{(w-\lambda)^\alpha} \frac{R(\lambda)}{1+|\lambda|^\alpha}$, and

$$c_\alpha \leq \frac{1+|\lambda|^\alpha}{(|\lambda|+w)^\alpha} \leq \frac{1+|\lambda|^\alpha}{|w-\lambda|^\alpha} \leq d_\alpha$$

for constants $c_\alpha, d_\alpha > 0$. Hence we have proved that

$$\|R(\lambda)A_w^{-\alpha}\| \leq C_\alpha \frac{\|R(\lambda)\|}{1+|\lambda|^\alpha} + D_\alpha, \frac{\|R(\lambda)\|}{1+|\lambda|^\alpha} \leq C_\alpha \|R(\lambda)A_w^{-\alpha}\| + D_\alpha \tag{8.1}$$

for $\alpha = n \in \mathbb{N}$ and constants C_α, D_α. If α is any non-integer positive number, take $n \in \mathbb{N}$ such that $n < \alpha < n+1$. Since $R(\lambda)A_w^{-\alpha} = R(\lambda)A_w^{-n}A_w^{-(\alpha-n)} = \frac{1}{(w-\lambda)^n} R(\lambda) A_w^{-(\alpha-n)} - \sum_{k=0}^{n-1} \frac{1}{(w-\lambda)^k} A_w^{k-\alpha}$, we see that it is sufficient to prove the relations (8.1) for $0 < \alpha < 1$. To this end, we apply the moment inequality (see [EN, Theorem 5.34]) with exponents $0 < \alpha < 1$ and we get

$$\begin{aligned}\|R(\lambda)x\| = \|A_w^\alpha R(\lambda)A_w^{-\alpha}x\| &\leq L\|R(\lambda)A_w^{-\alpha}x\|^{1-\alpha}\|A_w R(\lambda)A_w^{-\alpha}x\|^\alpha \\ &\leq L\|R(\lambda)A_w^{-\alpha}\|^{1-\alpha}\|(-I+(\lambda-w)R(\lambda))A_w^{-\alpha}\|^\alpha\|x\| \\ &\leq (1+|\lambda|^\alpha)(C_\alpha\|R(\lambda)A_w^{-\alpha}\| + D_\alpha)\|x\|.\end{aligned}$$

In order to obtain the converse inequality, we use the moment inequality with exponents $-1 < -\alpha < 0$, so

$$\begin{aligned}\|R(\lambda)A_w^{-\alpha}x\| &\leq L\|A_w^{-1}R(\lambda)x\|^\alpha\|R(\lambda)x\|^{-\alpha+1} \\ &\leq L\|\frac{1}{w-\lambda}(R(w)-R(\lambda))\|^\alpha\|R(\lambda)\|^{1-\alpha}\|x\| \leq C_\alpha\frac{\|R(\lambda)\|}{1+|\lambda|^\alpha} + D_\alpha.\end{aligned}$$

Hence the statement of the lemma follows. □.

8.2 Theorem *Suppose there exists a $\varrho_0 > 0$ such that resolvent set $\varrho(A)$ of the semigroup generator A satisfies $\{\lambda \in \mathbb{C} \mid |\mathrm{Re}\,\lambda| < 2\varrho_0\} \subset \varrho(A)$. The following assertions are equivalent:*

(a) *For all $1 \leq p < \infty$ and $|\varrho| < \varrho_0$, $R(i\cdot+\varrho)$ is an $L_p(\mathbb{R}, X_\alpha) \overset{m}{\mapsto} L_p(\mathbb{R}, X)$ multiplier.*

(b) *There exist $1 \leq p < \infty$ and $1 \leq s \leq \infty$ such that $R(i\cdot+\varrho)$ is an $L_p(\mathbb{R}, X_\alpha) \overset{m}{\mapsto} L_s(\mathbb{R}, X)$ multiplier for all $|\varrho| < \varrho_0$.*

(c) *For all $|\varrho| < \varrho_0$, $R(i\cdot+\varrho)$ is an $L_1(\mathbb{R}, X_\alpha) \overset{m}{\mapsto} L_\infty(\mathbb{R}, X)$ multiplier.*

(d) $\sup_{|\operatorname{Re}\lambda|<\varrho_0} \frac{\|R(\lambda)\|}{1+|\lambda|^\alpha} < \infty$ *and* $|(r_\varrho, \Phi)| \le K_\varrho \|x\|_\alpha \|x^*\| \|\check{\Phi}\|_1$ *holds for all* $|\varrho| < \varrho_0$, $\Phi \in \mathcal{S}(\mathbb{R}), x \in X_\alpha, x^* \in X^*$ *and* $|\varrho| < \varrho_0$.

Note that in the general case we do not gain an equivalent condition by assuming the multiplier property on the imaginary axis only (cf. [LR, Example 3.1] for the case $\alpha = 1$).

Proof. Since Lemma 8.1 makes Lemma 7.2 subsist on vectors from X_α, the proof of Theorem 7.4 carries over. □

In the hyperbolic case, $R(is)$ has been shown to be the Fourier transform of Green's function G_T, which, in turn, has implied the convergence of the Cesàro integral $(C) \int_{\mathbb{R}} R(is)e^{ist}x\, ds$. Even though there is no splitting projection available to define Green's function in the general case, the convergence of the integral can still be proved for $x \in X_\alpha$, despite the proof being a bit more laborious.

To begin with, we show convergence for $x \in X_{\alpha+2}$. The lemma requires merely the polynomial boundedness of the resolvent on the imaginary axis.

8.3 Lemma *Suppose* $\|R(is)A_w^{-\alpha}\| < C$ *for* $s \in \mathbb{R}$. *Then* $(C) \int_{\mathbb{R}} R(is)e^{ist}x\, ds$ *exists for all* $x \in X_{\alpha+2}$.

Proof. The identity $R(\lambda) = \frac{1}{\lambda^2}R(\lambda)A^2 + \frac{1}{\lambda}I + \frac{1}{\lambda^2}A$ holds whenever it has a meaning, so we have

$$
\begin{aligned}
\frac{1}{2\pi}\frac{1}{N}\int_0^N \int_{-l}^{l} R(is)xe^{ist}\, dsdl &= \frac{1}{2\pi}\frac{1}{N}\int_0^1 \int_{-l}^{l} R(is)xe^{ist}\, dsdl \\
&+ \frac{1}{2\pi}\frac{1}{N}\int_1^N \int_{-1}^{1} R(is)xe^{ist}\, dsdl + \frac{1}{2\pi}\frac{1}{N}\int_1^N \int_{1\le|s|\le l} R(is)xe^{ist}\, dsdl \\
&= \frac{1}{2\pi}\frac{1}{N}\int_0^1 \int_{-l}^{l} R(is)xe^{ist}\, dsdl + \frac{1}{2\pi}\frac{N-1}{N}\int_{-1}^{1} R(is)xe^{ist}\, dsdl \\
&\qquad + \frac{1}{2\pi}\frac{1}{N}\int_0^N \int_{1\le|s|\le l} \left(-\frac{R(is)A^2x}{s^2} + \frac{x}{is} - \frac{Ax}{s^2}\right) e^{ist}\, dsdl.
\end{aligned}
$$

Then

$$
\begin{aligned}
\lim_{N\to\infty} \frac{1}{2\pi}\frac{1}{N}\int_0^N \int_{-l}^{l} R(is)xe^{ist}\, dsdl &= \frac{1}{2\pi}\int_{-1}^{1} R(is)xe^{ist}\, ds \\
- \frac{1}{2\pi}\int_{|s|\ge 1} \left(\frac{Ax}{s^2} + \frac{R(is)A^2x}{s^2}\right) e^{ist}\, ds &+ \frac{x}{\pi}\chi_{\mathbb{R}\setminus\{0\}}(t)\left(\int_{|t|}^{\infty} \frac{\sin s}{s}\, ds\right),
\end{aligned}
$$

and $R(is)A^2x$ is bounded for any $x \in X_{\alpha+2}$, thus the integrals are convergent. □

8.4 Proposition *If $R(i\cdot)$ is an $L_1(\mathbb{R}, X_\alpha) \overset{m}{\mapsto} L_1(\mathbb{R}, X)$ multiplier, then the integral*

$$G(t)x = \frac{1}{2\pi}(C)\int_{\mathbb{R}} R(is)xe^{ist}\,ds$$

converges for all $x \in X_\alpha$.

Proof. We show that the operators $I_N^t(x) : X_\alpha \mapsto X$,

$$I_N^t(x) = \frac{1}{2\pi}\frac{1}{N}\int_0^N \int_{-l}^{l} R(is)xe^{ist}\,ds\,dl$$

are uniformly bounded in $\mathcal{B}(X_\alpha, X)$. Then the first assertion will follow by Lemma 8.3 from the existence of $\lim_{N\to\infty} I_N^t(x)$, x being taken from a dense subset of X_α.

Take $f \in L_1(\mathbb{R})$ and $x \in X_\alpha$. By the assumption $\mathcal{F}^{-1}(R\hat{f}x) \in L_1(\mathbb{R}, X)$, hence by Lemma 3.1 there is a $\tau \in [-1, 0]$ such that

$$\mathcal{F}^{-1}(R\hat{f}x)(\tau) = \frac{1}{2\pi}(C)\int_{\mathbb{R}} R(is)x\hat{f}(s)e^{is\tau}\,ds.$$

If we apply the operator $T(-\tau)$ and use (2.3), then we obtain

$$\begin{aligned} T(-\tau)(\mathcal{F}^{-1}(R\hat{f}x)(\tau)) &= \frac{1}{2\pi}(C)\int_{\mathbb{R}} R(is)T(-\tau)xe^{ist}\hat{f}(s)\,ds \\ &= \frac{1}{2\pi}(C)\int_{\mathbb{R}} \left(R(is)x\hat{f}(s) - \int_0^{-\tau} T(r)xe^{-isr}\,dr\cdot\hat{f}(s)\right)\,ds. \end{aligned}$$

Here

$$(C)\int_{\mathbb{R}}\int_0^{-\tau} T(r)xe^{-isr}\hat{f}(s)\,dr\,ds = \int_0^{-\tau} T(r)x\cdot\left((C)\int_{\mathbb{R}}\hat{f}(s)e^{-isr}\,ds\right)\,dr,$$

because, by Lemma 3.3, $(C)\int_{\mathbb{R}}\hat{f}(s)e^{-is\cdot}\,ds$ is the L_1-limit of the averages $\frac{1}{N}\int_0^N\int_{-l}^{l}\hat{f}(s)e^{-is\cdot}\,dsdl$ and $g \mapsto \int_0^{-\tau} T(r)x\cdot g(r)\,dr$ is a continous $L_1(\mathbb{R}) \mapsto X$ operator. Consequently, G(0,f) $= \frac{1}{2\pi}(C)\int_{\mathbb{R}} R(is)x\hat{f}(s)\,ds$ converges, hence for all $t \in \mathbb{R}$

$$G(t, f) = G(0, f(\cdot - t)) = \frac{1}{2\pi}(C)\int_{\mathbb{R}} R(is)x\hat{f}(s)e^{ist}\,ds$$

also do. Let us denote

$$S_N^t(f, x) = \frac{1}{2\pi}\frac{1}{N}\int_0^N\int_{-l}^{l} R(is)x\hat{f}(s)e^{ist}\,dsdl;$$

this is an $L_1(\mathbb{R})\times X_\alpha \mapsto X$ operator and it is easy to see that $\|S_N^t(f,x)\|_X \le C_N\|f\|_{L_1}\|x\|_\alpha$. We have proved that $G(t,f)x = \lim_{N\to\infty} S_N^t(f,x)$ exists for all $f \in L_1(\mathbb{R}), x \in X_\alpha$. By the uniform boundedness principle we obtain $\|S_N^t\| \le C$, where C depends neither on N nor on t. Take an approximative unit in $L_1(\mathbb{R})$, that is, a function $f_\varepsilon \in L_1(\mathbb{R})$ for all $\varepsilon > 0$, such that $f_\varepsilon * g \to g$ $(\varepsilon \to 0)$ in $L_1(\mathbb{R})$ for all $g \in L_1(\mathbb{R})$. Then $I_N^t(x) = \lim_{\varepsilon\to 0} S_N^t(f_\varepsilon, x)$, hence $\|I_N^t\| \le C$. This completes the proof. $\square$

9 Discrete Multipliers

We present a result which connects a general version of Theorem 6.7 to the adaptation of the previous section for the discrete case.

9.1 Theorem *Suppose $i\mathbb{Z} \subset \varrho(A)$. Then the following conditions are equivalent.*

(a) $X_\alpha \subset \mathrm{Im}\,(I - T(2\pi))$.

(b) *The sum $(C)\sum_{k\in\mathbb{Z}} R(ik)x$ exists (in X-norm) for all $x \in X_\alpha$.*

(c) *The function $\mathbb{Z} \mapsto \mathcal{B}(X)$, $k \mapsto R(ik)$ is an $L_p(\Gamma, X_\alpha) \overset{m}{\mapsto} L_p(\Gamma, X)$ Fourier multiplier for all $1 \le p < \infty$.*

(d) *There exist $1 \le p < \infty$, $1 \le s \le \infty$ such that $k \mapsto R(ik)$ is an $L_p(\Gamma, X_\alpha) \overset{m}{\mapsto} L_s(\Gamma, X)$ multiplier.*

(e) *The function $k \mapsto R(ik)$ is an $L_1(\Gamma, X_\alpha) \overset{m}{\mapsto} L_\infty(\Gamma, X)$ multiplier.*

(f) *There exists a constant $K > 0$ such that $\left|\sum_{k\in\mathbb{Z}} r(k,x,x^*)\Phi(k)\right| \le K\|x\|_\alpha\|x^*\|\|\check{\Phi}\|_{L_1(\Gamma)}$ for all $\Phi \in \mathcal{S}(\mathbb{Z})$.*

The analogy with the continous case (i.e. Theorem 8.2) leaps to the eye. The convergence of the sum $(C)\sum_{k\in\mathbb{Z}} R(ik)x$ corresponds to the convergence of the integral $(C)\int_{\mathbb{R}} R(is)x\,ds$. However, in the continous case the convergence of the integral is not sufficient to make $R(is)$ a multiplier.

Proof. The equivalence of statements (a) and (b) has already been proved in Theorem 6.7 for $\alpha = 0$; the same proof works for any $\alpha > 0$.

We now prove (a) $\Rightarrow$ (c). Let $K : L_p(\Gamma, X) \mapsto L_p(\Gamma, X)$ denote the convolution operator with the semigroup, that is,

$$Kf(t) = \int_\Gamma T(s)f(t-s)\,ds \quad (f \in L_p(\Gamma, X)).$$

Clearly, K is bounded on $L_p(\Gamma, X)$ for all $1 \le p < \infty$. Now for any trigonometric polynomial $f(t) = \sum_{j=-n}^{n} e^{ijt} x_j$ $(x_j \in X_\alpha)$ define the discrete multiplier operator

$$(Lf)(t) = \sum_{k\in\mathbb{Z}} R(ik)\hat{f}(k)e^{ikt} \quad (t \in \Gamma).$$

Then

$$\begin{aligned}(L(I-T(2\pi))f)(t) &= \left(L(I-T(2\pi))\left(\textstyle\sum_{j=-n}^{n} e^{ij\cdot}x_j\right)\right)(t)\\ &= \sum_{k=-n}^{n} R(ik)(I-T(2\pi))x_k e^{ikt} = \sum_{k=-n}^{n}\int_0^{2\pi} e^{-iks}T(s)x_k\,ds\,e^{ikt}\\ &= \int_0^{2\pi} T(s)\sum_{k=-n}^{n} e^{ik(t-s)}x_k\,ds = (Kf)(t). \quad (9.1)\end{aligned}$$

As a consequence of the assumption $i\mathbb{Z} \subset \varrho(A)$ and the point spectral mapping theorem 6.1, the operator $I - T(2\pi)$ is injective. By condition (a), there is an operator $U : X_\alpha \mapsto X$ such that $(I - T(2\pi))U = I$ on X_α. Since convergence in α-norm implies convergence in X-norm, it is easily verified that U is a closed operator, therefore U is bounded. Then (9.1) gives $KU = L$ on the X_α-valued trigonometric polynomials, so L extends to a bounded mapping from $L_p(\Gamma, X_\alpha)$ into $L_p(\Gamma, X)$ $(1 \le p < \infty)$, as is stated in (c).

The implication (c) $\Rightarrow$ (d) is straightforward. To prove (d) $\Rightarrow$ (e), we repeat the line of arguments in the proof of Theorem 7.4. If $R(ik)$ is an $L_p(\Gamma, X_\alpha) \overset{m}{\mapsto} L_s(\Gamma, X)$ multiplier, then $R(ik)A_w^{-\alpha}$ is an $L_p(\Gamma, X) \overset{m}{\mapsto} L_s(\Gamma, X)$ multiplier, and the same way we have proved the implication (b) $\Rightarrow$ (c) at Theorem 7.4 we derive that $R(ik)$ is an $L_p(\Gamma, X_\alpha) \overset{m}{\mapsto} L_\infty(\Gamma, X)$ multiplier. (The proof is in fact somewhat simpler now.) By duality, $(R(ik)A_w^{-\alpha})^*$ is an $L_1(\Gamma, X^\odot) \overset{m}{\mapsto} L_q(\Gamma, X^\odot)$, thus also an $L_1(\Gamma, X^\odot) \overset{m}{\mapsto} L_\infty(\Gamma, X^\odot)$ multiplier. Once again by duality $R(ik)A_w^{-\alpha}$ is an $L_1(\Gamma, X) \overset{m}{\mapsto} L_\infty(\Gamma, X)$ multiplier, and this proves (e).

If (e) holds, then for any $\Phi(s) \in \mathcal{S}(\Gamma), x \in X_\alpha$ we have

$$\left\|\sum_{k\in\mathbb{Z}} R(ik)\hat{\Phi}(k)xe^{ikt}\right\|_X \le K\|x\|_\alpha \|\Phi\|_{L_1(\Gamma)}$$

for some constant K and almost all $t \in \Gamma$; in fact, by continuity, for all $t \in \Gamma$. Let us take $t = 0$ and we obtain (f).

Finally, we show that (f) $\Rightarrow$ (b). Given any $k \in \mathbb{Z}$, let us take the function $\Phi \in \mathcal{S}(\mathbb{Z}), \Phi(k) = 1, \Phi(j) = 0$ $(j \ne k)$. The assumption (e) implies $|r(k)| \leq K\|x\|_\alpha\|x^*\|$, hence the operators $R(ik)$ from X_α to X are uniformly bounded. From this and from the equality $R(ik)x = \frac{1}{(ik)^2}R(ik)A^2x + \frac{1}{ik}x + \frac{1}{(ik)^2}Ax$ $(x \in X_{\alpha+2})$ it follows that $(C)\sum_{k\in\mathbb{Z}} R(ik)x$ converges for $x \in X_{\alpha+2}$. Furthermore, the condition $|(r,\Phi)| = \left|\sum_{k\in\mathbb{Z}} r(k,x,x^*)\Phi(k)\right| \leq K\|x\|_\alpha\|x^*\|\|\check{\Phi}\|_{L_1(\Gamma)}$ and the duality relation $(\mathcal{F}r,\Phi) = (r,\mathcal{F}\Phi)$ imply $|(\mathcal{F}r,\Phi)| \leq K\|x\|_\alpha\|x^*\|\|\Phi\|_{L_1(\Gamma)}$, therefore $\mathcal{F}(r(k,x,x^*)) \in L_\infty(\Gamma)$, $\|\mathcal{F}(r(k,x,x^*))\|_\infty \leq K\|x\|_\alpha\|x\|^*$, when $x \in X_\alpha$. As in the proof of Theorem 7.3, the operator family $S_N : X_\alpha \mapsto X$, $S_N x = \frac{1}{N+1}\sum_{l=0}^{N}\sum_{k=-l}^{l} R(ik)x$ satisfies

$$|\langle x^*, S_N x\rangle| = \left|\frac{1}{N+1}\sum_{l=0}^{N}\sum_{k=-l}^{l} r(k)\right| \leq K\|x\|_\alpha\|x\|^*,$$

because Lemma 3.4 applies. Then $\|S_N\| \leq K$ and the proof is complete. □

10 Mild Solutions

Hyperbolicity and its generalizations are connected to the existence and uniqueness of mild solutions of the inhomogenous Cauchy problem

$$\dot{u}(t) = Au(t) + f(t). \tag{10.1}$$

We give a discrete and a continous version of this result in the general setting in terms of multiplier properties of the resolvent.

The discrete theory of multipliers leads to periodic solutions; that is, let us consider (10.1) for $0 \leq t \leq 2\pi$. If $f \in L_p(\Gamma, X)$, the function $u \in C(\Gamma, X)$ is a mild solution of (10.1) if

$$u(t) = T(t)u(0) + \int_0^t T(t-s)f(s)\,ds \quad (0 \leq t \leq 2\pi). \tag{10.2}$$

10.1 Theorem *Suppose $i\mathbb{Z} \subset \varrho(A)$. Then for each $f \in L_p(\Gamma, X_\alpha)$ $(1 \leq p < \infty)$ there is a unique $u \in C(\Gamma, X)$ satisfying (10.2) if and only if $X_\alpha \subset \operatorname{Im}(T(2\pi) - I)$.*

This is indeed a multiplier-type result by Theorem 9.1.
Proof. To prove that $X_\alpha \subset \operatorname{Im}(T(2\pi) - I)$, take $x \in X_\alpha$. For the function $f(s) = \frac{1}{2\pi}T(s)x$ there exists $u \in C(\Gamma, X)$ that satisfies (10.2), thus for $t = 2\pi$ we obtain

$$(T(2\pi) - I)u(0) = \int_0^{2\pi} T(2\pi - s)f(s)\,ds$$

$$= \frac{1}{2\pi}\int_0^{2\pi} T(2\pi - s)T(s)x\,ds = T(2\pi)x = (T(2\pi) - I)x + x,$$

therefore $x \in \mathrm{Im}\,(T(2\pi) - I)$.

Conversely, suppose $i\mathbb{Z} \subset \varrho(A)$ and $X_\alpha \subset \mathrm{Im}\,(T(2\pi) - I)$. The point spectral mapping theorem 6.1 yields that $T(2\pi) - I$ is injective; let us denote $S = (T(2\pi) - I)^{-1}$ with $D(S) = \mathrm{Im}\,(T(2\pi) - I) \supset X_\alpha$. For any $f \in L_p(\Gamma, X_\alpha)$, $\int_0^{2\pi} T(s)f(2\pi - s)\,ds \in X_\alpha$ and let us define $u(0) = -S\left(\int_0^{2\pi} T(s)f(2\pi - s)\,ds\right)$, $u(t) = T(t)u(0) + \int_0^t T(s)f(t-s)\,ds$. Observe that, by definition, u satisfies (10.2), that u is continous and $u(2\pi) = T(2\pi)u(0) - (T(2\pi) - I)u(0) = u(0)$, thus $u \in C(\Gamma, X)$. We have therefore proved the existence of the solution. To obtain uniqueness, assume that (10.2) holds for some function $u \in C(\Gamma, X)$ and a given f; take $t = 2\pi$ and by $u(0) = u(2\pi)$ we get $(I - T(2\pi))u(0) = \int_0^{2\pi} T(t-s)f(s)\,ds$. By the injectivity of $T(2\pi) - I$, $u(0)$ is uniquely determined, and then (10.2) prescribes $u(t)$ for all $0 \le t \le 2\pi$. □

Now we consider (10.1) for $t \in \mathbb{R}$. For $f \in L_p(\mathbb{R}, X)$, the function $u : \mathbb{R} \mapsto X$ is a mild solution of (10.1) on $\mathbb{R}$, if

$$u(t) = T(t-s)u(s) + \int_s^t T(t-r)f(r)\,dr \quad (t, s \in \mathbb{R},\ t \ge s). \tag{10.3}$$

Note that mild solutions are continous.

10.2 Theorem *Suppose $i\mathbb{R} \subset \varrho(A)$. Then for every $f \in L_p(\mathbb{R}, X_\alpha)$, (10.3) has a unique solution $u \in L_p(\mathbb{R}, X)$ if and only if $R(i\cdot)$ is an $L_p(\mathbb{R}, X_\alpha) \overset{m}{\mapsto} L_p(\mathbb{R}, X)$ multiplier.*

Proof. First, suppose that (10.3) has a unique solution for every $f \in L_p(\mathbb{R}, X_\alpha)$. Let us define the linear operator $Lf = u$, which assigns the solution u to each f. It is easy to check using the closed graph theorem that $L : L_p(\mathbb{R}, X_\alpha) \mapsto L_p(\mathbb{R}, X)$ is bounded. For $\Phi \in \mathcal{S}(\mathbb{R}, X_\alpha)$ let us denote $(M\Phi)(t) = \frac{1}{2\pi}\int_{\mathbb{R}} R(is)\hat{\Phi}(s)e^{ist}\,ds$. Then for all $\Phi \in \mathcal{S}(\mathbb{R}, X_\alpha)$,

$$(M\Phi)(t) = T(t-s)(M\Phi)(s) + \int_s^t T(t-r)f(r)\,dr,$$

because, by (2.3),

$$2\pi\left((M\Phi)(t) - T(t-s)(M\Phi)(s)\right) = \int_{\mathbb{R}} R(iu)\left(e^{iut} - T(t-s)e^{ius}\right)\hat{\Phi}(u)\,du$$

$$= \int_{\mathbb{R}} e^{ius} \left(e^{iu(t-s)} R(iu) - T(t-s) R(iu) \right) \hat{\Phi}(u)\, du$$

$$= \int_{\mathbb{R}} \int_0^{t-s} e^{iu(t-r)} T(r) \hat{\Phi}(u) dr du = \int_0^{t-s} T(r) \int_{\mathbb{R}} \hat{\Phi}(u) e^{iu(t-r)} du dr$$

$$= 2\pi \int_0^{t-s} T(r) \Phi(t-r) du dr = 2\pi \int_s^t T(t-r) \Phi(r) dr. \quad (10.4)$$

Therefore the operators L and M coincide on the set $\mathcal{S}(\mathbb{R}, X_\alpha)$ which is dense in $L_p(\mathbb{R}, X_\alpha)$. L is a bounded operator on the whole $L_p(\mathbb{R}, X_\alpha)$, so M also extends to $L_p(\mathbb{R}, X_\alpha)$, as asserted.

Conversely, suppose that M is bounded from $L_p(\mathbb{R}, X_\alpha)$ to $L_p(\mathbb{R}, X)$. The identity (10.4) shows that $M\Phi$ is a solution for any $\Phi \in \mathcal{S}(\mathbb{R}, X_\alpha)$. Let $g \in L_p(\mathbb{R}, X_\alpha)$ be arbitrary and take a sequence $g_n \in \mathcal{S}(\mathbb{R}, X_\alpha)$, $g_n \to g$ in $L_p(\mathbb{R}, X_\alpha)$. Then $u_n = Mg_n \to Mg = u$ in $L_p(\mathbb{R}, X)$ and, without loss of generality, pointwise on a set $E \subset \mathbb{R}$, where the complement of E has Lebesgue measure zero. Choose a decreasing sequence $s_n \in E, s_n \to -\infty$, and put

$$v_n(t) = T(t - s_n) u(s_n) + \int_{s_n}^t T(t-r) f(r)\, dr \quad (t \geq s_n).$$

For any integers $m > n$ we have $u(t) = v_n(t) = v_m(t)$ $(t \in E, t \geq s_n)$, and by continuity of v_n and v_m, $v_n(t) = v_m(t)$ for all $t \geq s_n$. Thus the definition $v(t) = v_n(t)$ $(t \geq s_n)$ is correct, and gives a solution v of (10.3) for all $t, s \in \mathbb{R}, t \geq s$. To see uniqueness, we may take $f = 0$ in (10.3), and then we have, in particular, that $u(t+1) = T(1)u(t)$ for all $t \in \mathbb{R}$. Thus u is periodic and $u \in L_p(\mathbb{R}, X)$, so $u = 0$. □

10.3 Corollary *The semigroup $T(t)$ is hyperbolic if and only if $i\mathbb{R} \subset \varrho(A)$ and (10.3) has a unique solution $u \in L_p(\mathbb{R}, X)$ for any $f \in L_p(\mathbb{R}, X)$.*

Notes

The material contained in Sections 2 and 3 is standard; definitions and propositions in Section 2 come from [EN], Lemmas 3.1 and 3.2 are in [vN], and Lemma 3.3 is taken from [LS]; Lemma 3.4 must also be well known.

The equivalence of (a) and (b) in Theorem 4.3 comes from [vN]; the equivalence of (a) and (c) is formulated in [LR]. Most of Section 5 can be found in [KVL]; Theorem 5.8 is proved in [EN].

Theorem 6.1 is the famous point spectral mapping theorem, stated for example in [vN]; our proof given here is a simplification of the former ones.

Corollaries 6.2 and 6.4 can be found in [vN] as well as in [EN]. Theorem 6.7 was first proved in [G]; the proof here is slightly different from the original and is based on [LS].

Propositon 7.1 first appeared in [KVL]; the crucial Lemma 7.2 was proved in [LS]; Theorem 7.3 is a stronger version of the main result in [KVL] and was formulated in [LS]. The proof of Theorem 7.3 as given here is due to the author; though, it is a straightforward combination of the previous ideas and may have very well appeared in the literature before. Theorem 7.4 together with the ensuing corollaries is taken from [LS]; minor modifications have been made for the sake of a more concise exposition.

Sections 8 and 9 are based on [LS]; here we give Lemma 8.1 in a general form and with a new proof. Theorem 10.1 is taken from [LR] and Theorem 10.2 from [LS]; the proof of the latter has been changed a bit to avoid reference to evolution semigroups.

References

[ABHN] Wolfgang Arendt, Charles Batty, Matthias Hieber, Frank Neubrander: Vector-Valued Laplace Transforms and Cauchy Problems, 2000.

[BGK] H. Bart, I. Gohberg, M.A. Kaashoek: Wiener-Hopf Factorization, Inverse Fourier Transforms and Exponentially Dichotomous Operators, Journal of Functional Analysis 68, (1986) pp.1-42.

[CL] Carmen Chicone, Yuri Latushkin: Evolution Semigroups in Dynamical Systems and Differential Equations, Mathematical Surveys and Monographs 70, AMS, 1999.

[DU] J. Diestel, J.J. Uhl: Vector Measures, Mathematical Surveys, AMS, Providence, RI, 1977.

[HP] E. Hille, R.S. Phillips: Functional Analysis and Semi-Groups, Amer. Math. Soc. Coll. Publ., Volume XXXI, AMS, 1957.

[EN] Klaus-Jochen Engel, Rainer Nagel: One-Parameter Semigroups for Linear Equations, Springer, New York, 2000.

[G] Günther Greiner, Manfred Schwarz: Weak Spectral Mapping Theorems for Functional Differential Equations, Journal of Differential Equations 94 (1991), pp. 205-216.

[K] T.W. Körner: Fourier Analysis, Cambridge University Press, Cambridge, 1988.

[KVL] M. A. Kaashoek, S. M. Verduyn Lunel: An Integrability Condition on the Resolvent for Hyperbolicity of the Semigroup, Journal of Differential Equations 112 (1994), pp. 374-406.

[Ko] H. Komatsu: Fractional Powers of Operators, Pacific Journal of Mathematics 19 (1966), pp. 285-346.

[LR] Y. Latushkin, F. Räbiger: Fourier Multipliers and Stability, preprint.

[LS] Yuri Latushkin, Roman Shvydkoy: Hyperbolicity of semigroups and Fourier multipliers, in: Systems, Approximation, Singular Integral Operators and Related Topics, IWOTA 2000, Oper. Theory Adv. Appl. 129 (2001) pp.341-364.

[vN] Jan van Neerven: The Asymptotic Behaviour of Semigroups of Linear Operators, Birkhäuser, Basel, 1996.

[R] Walter Rudin: Functional Analysis, McGraw-Hill, New York, 1973.

[W] Lutz Weis: Operator-valued Fourier multiplier theorems and maximal L_p-regularity, Mathematische Annalen 319 (2001), pp.735-758.