Level sets of Lipschitz quotient mappings from super-reflexive Banach spaces to $\mathbb{R}$

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Chapter 1

Introduction

A mapping $f\colon X \to Y$ between the metric spaces X and Y is called *uniform quotient*, if it satisfies

$$B(f(x), \omega(r)) \subseteq f(B(x, r)) \subseteq B(f(x), \Omega(r))$$

for all $x \in X$ and $r > 0$ with some functions $\omega, \Omega\colon (0, \infty) \to (0, \infty)$, $\Omega(r) \xrightarrow{r \to 0} 0$. If f satisfies the left-hand side containment, we say that f is *co-uniformly continuous*. Analogously, f is called *Lipschitz quotient*, if it satisfies

$$B(f(x), cr) \subseteq f(B(x, r)) \subseteq B(f(x), Lr)$$

for all $x \in X$ and $r > 0$ with some constants c, $L > 0$, and f is called *co-Lipschitz*, if it satisfies the left-hand side containment. The concepts of co-uniformly continuous and co-Lipschitz mappings are in a certain sense dual to the classical concepts of uniformly continuous and Lipschitz mapping as they describe the extent to which a mapping is "uniformly" open. In functional analysis, uniform and Lipschitz quotient mappings can sometimes be used instead of linear quotient mappings. However, the existence of a Lipschitz quotient mapping between two Banach spaces does not, in general, imply the existence of a linear quotient mapping between these spaces, even in the separable setting (see [3] for counterexamples). Though the uniform and Lipschitz quotient mappings hold some of the structure-preserving properties of the bi-uniform and bi-Lipschitz homeomorphisms and linear quotient mappings, they often behave unlike them: for example, the *Gorelik principle* turns out to be false for Lipschitz quotient mappings. Even the finite dimensional version, which would say that a Lipschitz quotient mapping between finite dimensional Banach spaces cannot carry a subspace into a subspace of higher codimension, is not true. In [2], a Lipschitz quotient mapping from $\mathbb{R}^3$ to $\mathbb{R}^2$ is constructed that sends a 2-dimensional plane to the origin. This example shows that even in the finite dimensional context, uniform and Lipschitz quotient mappings lead to interesting questions.

In the case of Lipschitz quotient mappings from $\mathbb{R}^n$ to $\mathbb{R}^n$, there is a connection with the quasiregular mappings. A Lipschitz mapping is called quasiregular if the n by n matrix $D(x)$ of partial derivatives of f satisfies $\|D(x)\|^n \leq K \det D(x)$ with a constant K. If $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a Lipschitz quotient mapping, then we have a weaker statement: for every x at which f is differentiable $\|D(x)\|^n \leq K|\det D(x)|$ with $K = (L/c)^{n-1}$, where L (respectively, c) is the Lipschitz (respectively, co-Lipschitz) constant of f.

In [1], it is proved that the level sets of a Lipschitz quotient mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$ are discrete. This result implies that f is a quasiregular mapping, up to orientation. In connection with this, there is a conjecture, that this is also true in higher dimensions: a Lipschitz quotient mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$ has discrete level sets.

In [4] it is furthermore shown that every Lipschitz quotient mapping $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a reparametrisation of a complex polynomial P. That is, $f = P \circ h$ where h is a homeomorphism of the plane. Hence there is a number n such that each level set consists of at most n points and there are at most $n-1$ level sets containing fewer than n points. An analogous result is proved in [7] for mappings from $\mathbb{R}^2$ to $\mathbb{R}$. All level sets of a co-Lipschitz uniformly continuous mapping $f\colon \mathbb{R}^2 \to \mathbb{R}$ are hereditarily locally connected, closed, locally compact, have no end points and contain no simple curve. This implies that every level set of f has a finite graph structure (i.e. is a union of homeomorphic images of $[0,1]$ and $[0,\infty)$, which only intersect at the images of their end points, and the distance of the images of $[N,\infty)$ from the origin tends to infinity as N tends to infinity). It is also shown that there exists a number n such that $\mathbb{R}^2 \setminus f^{-1}(t)$ has exactly $n+1$ components for every $t \in \mathbb{R}$ (for precise bounds on n see [6]). Moreover, there exists a finite set $T \subset \mathbb{R}$ with $|T| \leq 2n-1$, such that $f^{-1}(t)$ has n components for all $t \notin T$, all these components are homeomorphic to the real line and separate the plane into two components. The paper [7] also contains an example of a uniform quotient mapping from $\mathbb{R}^2$ to $\mathbb{R}$ that has non-locally connected level sets.

In higher dimension we cannot expect similar results to hold: the Lipschitz quotient mapping $f\colon \mathbb{R}^3 \to \mathbb{R}, f(x,y,z) = \|(x,y)\| - |z|$ has level sets $f^{-1}(t)$ connected for $t \geq 0$ and disconnected (having two connected components) for $t < 0$. In this paper we prove that for Lipschitz quotient mappings $f\colon \mathbb{R}^n \to \mathbb{R}$, level sets are necessarily locally connected, but need not be simply connected or locally simply connected. This answers a question asked in [7]. We also obtain a characterisation of the subsets of any super-reflexive Banach space X that can occur as level sets of Lipschitz quotient mappings to $\mathbb{R}$. As a consequence, we get that in case of infinite dimensional spaces X the level sets of Lipschitz quotient mappings from X to $\mathbb{R}$ need not be locally connected, and may have connected components consisting of a single point (hence not separating X).

Chapter 2

Preliminaries

In this paper, we use the following notations:

Definition 2.1. *In a metric space X, the (open)* ***ball*** *with centre $x \in X$ and radius $r > 0$ is the set*

$$B(x, r) = \{y \in X \colon d(x, y) < r\},$$

the ***closed ball*** *with centre $x \in X$ and radius $r > 0$ is the set*

$$\overline{B}(x, r) = \{y \in X \colon d(x, y) \leq r\},$$

and the ***sphere*** *with centre $x \in X$ and radius $r > 0$ is the set*

$$S(x, r) = \{y \in X \colon d(x, y) = r\}.$$

Definition 2.2. *A map $f \colon X \to Y$, where X and Y are metric spaces, is called:*

- ***uniformly continuous****, if there exists a function $\Omega \colon (0, \infty) \to (0, \infty)$ with $\lim\limits_{t \to 0} \Omega(t) = 0$ such that for all $x \in X$ and $r > 0$*

  $$f(B(x, r)) \subseteq B(f(x), \Omega(r));$$

  a function Ω satisfying this condition is called a modulus of uniform continuity of f.

- ***Lipschitz***, *if the function* Ω *mentioned above can be chosen to be linear, i.e. if there exists a constant* $L > 0$ *such that for all* $x \in X$ *and* $r > 0$
$$f(B(x,r)) \subseteq B(f(x), Lr);$$
the infimum of such constants is called the Lipschitz constant of f.

- ***co-uniformly continuous***, *if there exists a function* $\omega\colon (0,\infty) \to (0,\infty)$ *such that for all* $x \in X$ *and* $r > 0$
$$f(B(x,r)) \supseteq B(f(x), \omega(r));$$
a function ω *satisfying this condition is called a modulus of co-uniform continuity of* f.

- ***co-Lipschitz***, *if the function* ω *mentioned above can be chosen to be linear, i.e. if there exists a constant* $c > 0$ *such that for all* $x \in X$ *and* $r > 0$
$$f(B(x,r)) \supseteq B(f(x), cr);$$
the supremum of such constants is called the co-Lipschitz constant of f.

- ***uniform quotient***, *if* f *is both uniformly continuous and co-uniformly continuous, and*

- ***Lipschitz quotient***, *if* f *is both Lipschitz and co-Lipschitz.*

Remark: If f is uniformly continuous, then we can always assume without loss of generality that Ω is continuous and strictly monotone increasing, with $\lim_{t\to\infty} \Omega(t) = \infty$.

Remark: If all closed balls are compact in X, and $f\colon X \to Y$ is co-Lipschitz with co-Lipschitz constant c, then the definition of the co-Lipschitz property also holds with closed balls instead of open ones:
$$f(\overline{B}(x,r)) \supseteq \overline{B}(f(x), cr) \text{ for all } x \in X,\ r > 0.$$
Indeed, for every $n \in \mathbb{N}$ we have $f\left(B\left(x, r+\frac{1}{n}\right)\right) \supseteq B\left(f(x), cr + \frac{c}{n}\right) \supseteq \overline{B}(f(x), cr)$ by the co-Lipschitz property, and hence for every point $z \in \overline{B}(f(x), cr)$ we have a point $w_n \in B\left(x, r + \frac{1}{n}\right)$ such that $f(w_n) = z$. Since $\overline{B}(x, r+1)$ is compact by assumption and contains w_n for all $n \in \mathbb{N}$, there

exists a convergent subsequence $(w'_n)_{n\in\mathbb{N}} \subseteq (w_n)_{n\in\mathbb{N}}$, denote $w = \lim\limits_{n\to\infty} w'_n$. Then we have

$$d(x,w) = \lim_{n\to\infty} d(x,w'_n) \leq \limsup_{n\to\infty} d(x,w_n) \leq r$$

and

$$f(w) = \lim_{n\to\infty} f(w'_n) = z,$$

so

$$z = f(w) \in f(\overline{B}(x,r))$$

holds. Since $z \in \overline{B}(f(x),cr)$ was arbitrary, $f(\overline{B}(x,r)) \supseteq \overline{B}(f(x),cr)$ as claimed.

Remark: To prove that a continuous mapping $f\colon X \to Y$ is Lipschitz quotient it is enough to check the Lipschitz quotient property on a dense subset of X. Indeed, assume f satisfies the Lipschitz quotient property with the Lipschitz and co-Lipschitz constants L and c, respectively, at the points of $D \subseteq X$ where D is dense in X. Let $x \in X$ be an arbitrary point; since D is dense in X, we can take a sequence $(x_n)_{n\in\mathbb{N}} \subset D$ that converges to x. Then for any point $z \in X$ we have

$$d(f(x),f(z)) = \lim_{n\to\infty} d(f(x_n),f(z)) \leq \lim_{n\to\infty} Ld(x_n,z) = Ld(x,z),$$

proving the Lipschitz property at x. On the other hand, for all $0 < \tilde{c} < c$ and $r > 0$ we have

$$f(B(x,r)) \supseteq f\left(B\left(x_n, \frac{\tilde{c}}{c}r\right)\right) \supseteq B(f(x_n),\tilde{c}r)$$

for all n that are large enough to satisfy

$$d(x_n,x) \leq \frac{c-\tilde{c}}{c}r$$

by the co-Lipschitz property at the point $x_n \in D$, hence

$$f(B(x,r)) \supseteq \bigcup_{n=N}^{\infty} B(f(x_n),\tilde{c}r) \supseteq B(f(x),\tilde{c}r)$$

holds. But this implies that

$$f(B(x,r)) \supseteq \bigcup_{n=2}^{\infty} B\left(f(x), \frac{n-1}{n}cr\right) = B(f(x),cr)$$

holds for all $r > 0$, finishing the proof of the co-Lipschitz property of f at x.

It is easy to see that if X and Y are normed spaces and we change their norms to equivalent norms, then the classes of functions defined above are the same (only with changed moduli or constants). In $\mathbb{R}^n$ we will always use the Euclidean norm, as all norms on a finite dimensional space are equivalent.

Definition 2.3. *A normed space X is called* ***uniformly convex****, if there exists a function $\delta\colon (0,\infty) \to (0,\infty)$ such that for all x, $y \in X$ with $\|x\| = \|y\| = 1$ we have $\|x+y\| \le 2 - \delta(\|x-y\|)$; we call such a function δ a* ***modulus of uniform convexity*** *of X. If X has an equivalent uniformly convex norm, we call X* ***super-reflexive****.*

Remark: In the following, we will always assume without loss of generality that $\delta(r) < 1$ for all $r \in (0,\infty)$ and δ is decreasing.

The notion of Lipschitz quotient mappings is suitably stable under taking pointwise limits, as the following lemma shows us.

Lemma 2.4. *Let $f_n\colon \mathbb{R}^k \to Y$ be Lipschitz quotient mappings with Lipschitz constant L and co-Lipschitz constant c, and assume that f is the pointwise limit of the sequence $(f_n)_{n\in\mathbb{N}}$. Then f is Lipschitz quotient, has Lipschitz constant L and co-Lipschitz constant c.*

Proof. Observe that since $f_n \to f$ and $f_n(B(x,r)) \subseteq B(f_n(x), Lr)$, for all $y \in B(x,r)$ we have $d(f_n(y), f_n(x)) < Ld(x,y)$ and hence

$$d(f(y), f(x)) = \lim_{n\to\infty} d(f_n(y), f_n(x)) \le Ld(x,y).$$

This proves the Lipschitz part of the Lipschitz quotient definition. Also, for all $\varepsilon > 0$ we have

$$f_n(B(x,r)) \supseteq B(f_n(x), cr) \supseteq B(f(x), cr - \varepsilon)$$

for all n that are large enough to satisfy $d(f_n(x), f(x)) < \varepsilon$. Hence for every $z \in B(f(x), cr-\varepsilon)$ there are points $w_n \in B(x,r)$ with the property $f_n(w_n) = z$. Select a convergent (in $\overline{B(x,r)}$) subsequence, $(w_{n_m})_{m\in\mathbb{N}}$, converging to $w \in \overline{B(x,r)}$. We obtain

$$|f(w) - z| = \lim_{m\to\infty} |f_{n_m}(w) - z| = \lim_{m\to\infty} |f_{n_m}(w) - f_{n_m}(w_{n_m})|$$
$$\le \lim_{m\to\infty} L|w_{n_m} - w| = 0.$$

So $f(w) = z$. This means $f(\overline{B(x,r)}) \supseteq B(f(x), cr - \varepsilon)$ for all $\varepsilon > 0$ and $r > \varepsilon/c$. Taking $\varepsilon_n = cr/n$ we have

$$\begin{aligned} f(B(x,r)) &= \bigcup_{n=3}^{\infty} f\left(\overline{B\left(x, r - \frac{r}{n}\right)}\right) \\ &\supseteq \bigcup_{n=3}^{\infty} B\left(f(x), cr - \frac{cr}{n} - \frac{cr}{n}\right) = B(f(x), cr), \end{aligned}$$

hence f is co-Lipschitz with co-Lipschitz constant c. □

Lemma 2.5. *(Proposition 4.4. in [1]) Let X be an arbitrary metric space. Suppose that a mapping $f\colon \mathbb{R}^n \to X$ is continuous and co-Lipschitz with co-Lipschitz constant one, $f(x) = y$, $\xi\colon [0,\infty) \to X$ is a curve with Lipschitz constant one, and $\xi(0) = y$. Then there is a curve $\varphi\colon [0,\infty) \to \mathbb{R}^n$ with Lipschitz constant at most one such that $\varphi(0) = x$ and $f(\varphi(t)) = \xi(t)$ for $t \geq 0$.*

Proof. For $m = 1, 2, \ldots$, we define curves $\varphi_m\colon [0,\infty) \to \mathbb{R}^n$ with Lipschitz constant at most one, starting from x and with the property

$$f\left(\varphi_m\left(\frac{k}{m}\right)\right) = \xi\left(\frac{k}{m}\right) \text{ for all } k = 0, 1, \ldots$$

Let $\varphi_m(0) = x$ for every m. Using the co-Lipschitz property of f for closed balls $\overline{B}(\varphi_m(k/m), 1/m)$ we can inductively choose $\varphi_m(1/m)$, $\varphi_m(2/m)$, ... such that

$$\left\|\varphi_m\left(\frac{k+1}{m}\right) - \varphi_m\left(\frac{k}{m}\right)\right\| \leq \frac{1}{m}$$

and

$$f\left(\varphi_m\left(\frac{k+1}{m}\right)\right) = \xi\left(\frac{k+1}{m}\right)$$

hold for all $k \geq 0$. Then we extend φ_m linearly onto the intervals $\left(\frac{k}{m}, \frac{k+1}{m}\right)$. The resulting curve has Lipschitz constant at most one since on the intervals $\left(\frac{k}{m}, \frac{k+1}{m}\right)$ it runs with speed

$$m\left\|\varphi_m\left(\frac{k+1}{m}\right) - \varphi_m\left(\frac{k}{m}\right)\right\| \leq 1.$$

Select a pointwise convergent subsequence $(\varphi_{m_k})_{k\in\mathbb{N}}$ of the sequence $(\varphi_m)_{m\in\mathbb{N}}$, and let $\varphi = \lim_{k\to\infty} \varphi_{m_k}$. Then φ satisfies our conditions: it has Lipschitz constant at most one, $\varphi(0) = x$, and since for every $t \geq 0$

$$\begin{aligned} d\big(f(\varphi(t)), \xi(t)\big) &= \lim_{k\to\infty} d\big(f(\varphi_{m_k}(t)), \xi(t)\big) \\ &\leq \lim_{k\to\infty} d\left(f(\varphi_{m_k}(t)), f\left(\varphi_{m_k}\left(\frac{\lfloor tm_k \rfloor}{m_k}\right)\right)\right) \\ &\qquad + \lim_{k\to\infty} d\left(f\left(\varphi_{m_k}\left(\frac{\lfloor tm_k \rfloor}{m_k}\right)\right), \xi(t)\right) \\ &\leq d\big(f(\varphi(t)), f(\varphi(t))\big) + \lim_{k\to\infty} \frac{1}{m_k} = 0, \end{aligned}$$

we have $f \circ \varphi = \xi$, as required. □

Definition 2.6. *We say that a curve φ satisfying the conditions of Lemma 2.5 is a **lift** of ξ from the point x.*

Chapter 3

Local connectedness

We need two topological lemmas to prove Theorem 3.5.

Lemma 3.1. *Let F_1 and F_2 be disjoint closed subsets of $\mathbb{R}^n$, and $F = F_1 \cup F_2$. If F separates between points P and Q, then at least one of the sets F_1, F_2 separates P from Q.*

The proof of Lemma 3.1 combines the following two theorems from [5]:

Theorem 3.2. *(Theorem 59.II.11 in [5]) If none of the closed sets F_0 and F_1 cuts $\mathbb{S}^n$ between the points p and q and if $\dim(F_0 \cap F_1) \leq n - 3$, their union $F_0 \cup F_1$ does it neither.*

Theorem 3.3. *(Theorem 61.I.7 in [5]) Let B_0 and B_1 be two closed or two open sets [in $\mathbb{S}^2$]. If none of these sets is a cut between p_0 and p_1 and if $B_0 \cap B_1$ is connected, then $B_0 \cup B_1$ is not a cut between p_0 and p_1 either.*

Proof (of Lemma 3.1). We consider the space $\mathbb{R}^n$ embedded into its one-point compactification $\mathbb{R}^n \cup \{*\} \simeq \mathbb{S}^n$. Define sets $\tilde{F}_1 = F_1 \cup \{*\}$ and $\tilde{F}_2 = F_2 \cup \{*\}$. With these sets, $\tilde{F}_1 \cup \tilde{F}_2$ separates P from Q in $\mathbb{S}^n$, as $\mathbb{S}^n \setminus (\tilde{F}_1 \cup \tilde{F}_2) = \mathbb{R}^n \setminus (F_1 \cup F_2)$ has P and Q in different connected components. $\tilde{F}_1 \cap \tilde{F}_2 = (F_1 \cap F_2) \cup \{*\} = \{*\}$ is connected, and in case of $n \geq 3$, also has dimension at most $n-3$, so we can apply Theorem 3.2 if $n \geq 3$ and Theorem 3.3 if $n = 2$ to points P and Q and closed sets $\tilde{F}_1$ and $\tilde{F}_2$. Hence at least one of the sets $\tilde{F}_1$ or $\tilde{F}_2$ separates P from Q in $\mathbb{S}^n$, so either $\mathbb{S}^n \setminus \tilde{F}_1 = \mathbb{R}^n \setminus F_1$ or $\mathbb{S}^n \setminus \tilde{F}_2 = \mathbb{R}^n \setminus F_2$ has P and Q in different connected components. But that means exactly that at least one of the sets F_1, F_2 separates P from Q. □

We quote the following lemma without proof:

Lemma 3.4. *(Theorem 61.I.5 in [5]) Let A_0 and A_1 be two closed or two open sets [in $\mathbb{S}^2$]. If these sets are connected, whereas their intersection $A_0 \cap A_1$ is not, then their union $A_0 \cup A_1$ is a cut of $\mathbb{S}^2$.*

Remark: The statement of this lemma remains true if $\mathbb{S}^2$ is changed to $\mathbb{R}^2$ and the sets A_0 and A_1 are compact. To prove this, embed $\mathbb{R}^2$ in its one-point compactification, $\mathbb{S}^2 \simeq \mathbb{R}^2 \cup \{*\}$. The images of A_0 and A_1 are compact and connected by the continuity of the embedding, and the disconnectedness of the bounded set $A_0 \cap A_1$ is also preserved. By Lemma 3.4 the image of $A_0 \cup A_1$ separates $\mathbb{S}^2$. But a neighbourhood of $\{*\}$ is disjoint from $A_0 \cup A_1$ and remains connected after removing $\{*\}$, so $A_0 \cup A_1$ separates $\mathbb{S}^2 \setminus \{*\} \simeq \mathbb{R}^2$, as claimed.

We are now ready to prove the main result of this section:

Theorem 3.5. *If $f\colon \mathbb{R}^n \to \mathbb{R}$ is a co-Lipschitz uniformly continuous mapping, then for each $t \in \mathbb{R}$ the set $f^{-1}(t)$ is locally connected.*

Proof. We assume without loss of generality that $t = 0$ and $f(0) = 0$. Let

$$H = f^{-1}(\mathbb{R} \setminus \{0\}) \text{ and } F = f^{-1}(0).$$

We first show that for $r > 0$ and $0 < \varepsilon < 1$, the number of connected components of $H \cap B(0, r)$ that intersect $B(0, r(1-\varepsilon))$ is bounded by a constant depending only upon n, ε, the co-Lipschitz constant c and the modulus of uniform continuity Ω of f. We also assume without loss of generality that Ω is continuous and strictly monotone increasing with $\lim_{t\to\infty} \Omega(t) = \infty$, so that Ω^{-1} is a well-defined function.

Suppose that $Q_1, Q_2, \ldots, Q_m \in H \cap B(0, r(1-\varepsilon))$ belong to distinct connected components of $H \cap B(0, r)$. By Lemma 2.5, applied to f/c, there are curves $\gamma_1, \gamma_2, \ldots, \gamma_m\colon [0, \infty) \to \mathbb{R}^n$ that are Lipschitz with Lipschitz constant at most 1, such that

$$\gamma_1(0) = Q_1,\ \gamma_2(0) = Q_2,\ \ldots,\ \gamma_m(0) = Q_m$$

and

$$|f(\gamma_j(t))| = |f(Q_j)| + ct \text{ for each } j \in \{1, 2, \ldots, m\}.$$

Indeed, we can choose γ_j to be a lift of the curve

$$\xi_j(t) = f(Q_j)\left(1 + \frac{t}{|f(Q_j)|}\right).$$

Since $f(B(0,r)) \subseteq B(0,\Omega(r))$, the curves γ_j are not contained in $B(0,r)$. So there are $t_1, t_2, \ldots, t_m > 0$ such that

$$\gamma_j(t_j) \in S(0,r) \text{ and } \gamma_j([0,t_j)) \subset B(0,r) \text{ for each } j \in \{1, 2, \ldots, m\}.$$

As each γ_j has Lipschitz constant at most 1, it follows that $t_j \geq r\varepsilon$ for each j and so

$$|f(\gamma_j(t_j))| = |f(Q_j)| + ct_j > ct_j \geq cr\varepsilon.$$

Hence the balls defined by $B_j = B(\gamma_j(t_j), \Omega^{-1}(cr\varepsilon))$ do not intersect $f^{-1}(0)$; since if $d(x, \gamma_j(t_j)) < \Omega^{-1}(cr\varepsilon)$, then

$$|f(x)| \geq |f(\gamma_j(t_j))| - \Omega(d(x, \gamma_j(t_j))) > cr\varepsilon - \Omega(\Omega^{-1}(cr\varepsilon)) = 0.$$

Moreover, the balls B_j are disjoint. Indeed, if $B_i \cap B_j \neq \emptyset$ for some $i \neq j$, then $(B_i \cup B_j) \cap B(0,r)$ would be path-connected and intersect both $\gamma_i([0,t_i))$ and $\gamma_j([0,t_j))$. But this would imply that Q_i and Q_j are in the same connected component of $H \cap B(0,r)$, a contradiction.

Since each ball B_i is of the same radius $\Omega^{-1}(cr\varepsilon)$, it covers a fixed proportion of the area of $S(0,r)$, and so it follows that m is bounded by a constant depending only on n, ε, c and Ω.

Let $\{F_\alpha \colon \alpha < \kappa\}$ be the collection of connected components of $F \cap B(0,r)$ with $0 \in F_0$. We claim that for each $\varepsilon \in (0,1)$,

$$I_\varepsilon = \{\alpha \colon \alpha < \kappa \text{ and } F_\alpha \cap B(0, r(1-\varepsilon)) \neq \emptyset\}$$

is a finite set.

Fix $\varepsilon \in (0,1)$ and suppose indirectly that I_ε is not finite. For each $\alpha \in I_\varepsilon$ choose $P_\alpha \in F_\alpha \cap B(0, r(1-\varepsilon))$. When the curves $\gamma^+(t) = t$ and $\gamma^-(t) = -t$ are lifted from P_α, their lifts meet two distinct connected components of $H \cap B(0,r)$, H_α^+ and H_α^-, say. Since there are only finitely many components of H intersecting $B(0, r(1-\varepsilon))$, it follows that there are only finitely many pairs (H_α^+, H_α^-). Hence as I_ε is not finite, there are distinct $\alpha, \beta \in I_\varepsilon$ for which

$$H_\alpha^+ = H_\beta^+ \text{ and } H_\alpha^- = H_\beta^-.$$

From now on, we restrict ourselves to the ball $B(0,r)$ and work with the topology relative to that ball. Let G_1 be the connected component of $B(0,r) \setminus \overline{H_\alpha^+}$ that covers H_α^- and let $G_2 = B(0,r) \setminus \overline{G_1}$. Since G_1 is constructed as a component of the complement of the closure of an open set, and G_2 is the interior of the complement of the closure of an open set, we have $\mathrm{int}(\overline{G_1}) = G_1$ and $\mathrm{int}(\overline{G_2}) = G_2$. Hence $\partial\overline{G_1} = \partial\overline{G_2} = \partial G_1 = \partial G_2$. Both G_1 and G_2 are open, G_1 is connected by definition, and we claim that $\overline{G_2}$ is also connected. Indeed, if $\overline{G_2} = V_1 \cup V_2$ where V_1 and V_2 are nonempty clopen sets in $\overline{G_2}$, then only one of V_1 and V_2, V_1 say, can intersect the connected $\overline{H_\alpha^+}$. Thus, if $v \in \partial V_2$, then, as V_1 is closed (in $\overline{G_2}$, so also in the whole $B(0,r)$), there is an open ball centered at v, B_v, with $B_v \subseteq B(0,r) \setminus V_1 = V_2 \cup G_1$. But

$$v \in \partial V_2 \subset \partial\overline{G_2} = \partial G_1$$

and so $G_1 \cup B_v$ is an open, connected set (in $B(0,r)$) disjoint from H_α^+ contradicting the choice of G_1.

We claim that $\partial G_1 = \partial G_2$ is connected. Suppose indirectly that ∂G_1 is disconnected. Then $\partial G_1 = W_1 \cup W_2$ with some nonempty disjoint subsets W_1 and W_2 of ∂G_1, that are clopen in ∂G_1. Observe that since both W_1 and W_2 are closed in ∂G_1, which is a closed subset of $B(0,r)$, they are themselves closed. Hence they can be separated by open sets, i.e. there are disjoint open sets V and $\tilde{V}$ in $B(0,r)$ such that

$$W_1 \subset V \text{ and } W_2 \subset \tilde{V}.$$

But then $\overline{V} \cap \tilde{V} = \emptyset$ as $\tilde{V}$ is open, so we obtain that

$$\partial V \cap \partial G_1 \subseteq (B(0,r) \setminus (V \cup \tilde{V})) \cap (W_1 \cup W_2) = (W_1 \cup W_2) \setminus (V \cup \tilde{V}) = \emptyset$$

and hence

$$\partial V = (\partial V \cap G_1) \cup (\partial V \cap G_2).$$

Observe that $\partial V \cap G_1$ and $\partial V \cap G_2$ are both closed sets, since they are, by definition, open in the closed set ∂V and ∂V is their disjoint union. Now $\partial G_1 \cap V$ and $\partial G_1 \setminus V$ are contained in both of the connected sets $\overline{G_2}$ and $\overline{G_1}$, so no set entirely within G_1 or G_2 respectively can separate them. In particular, neither $\partial V \cap G_1$ nor $\partial V \cap G_2$ separates $\partial G_1 \cap V$ from $\partial G_1 \setminus V$, so by Lemma 3.1 their union, ∂V does not separate $\partial G_1 \cap V$ from $\partial G_1 \setminus V$ either. But it is clear that ∂V separates $\partial G_1 \cap V = W_1 \neq \emptyset$ from $\partial G_1 \setminus V = W_2 \neq \emptyset$, and we have a contradiction. This proves that $\partial G_1 = \partial G_2$ is connected.

F_α and F_β both contain points from $\partial H_\alpha^+ \cap \partial H_\alpha^- \subseteq \partial G_1$, since $H_\alpha^- \subseteq G_1$ and $H_\alpha^+ \subseteq G_2$. Thus the connectedness of $\partial G_1 (= \partial G_2)$ implies that both F_α and F_β contain ∂G_1. This contradicts $F_\alpha \cap F_\beta = \emptyset$ and so I_ε is finite as claimed.

Let $U = B(0,r) \setminus (F \setminus F_0)$. Since

$$\begin{aligned} U &= \bigcup_{j=2}^{\infty} \big(U \cap B(0, r(1-1/j))\big) \\ &= \bigcup_{j=2}^{\infty} \left(B(0, r(1-1/j)) \setminus \bigcup_{\alpha<\kappa} \big(F_\alpha \cap B(0, r(1-1/j))\big) \right) \\ &= \bigcup_{j=2}^{\infty} \left(B(0, r(1-1/j)) \setminus \bigcup_{\alpha \in I_{1/j}} \big(F_\alpha \cap B(0, r(1-1/j))\big) \right) \end{aligned}$$

is a union of open sets, U is open, and contains F_0. Thus it is a neighbourhood of the origin and $F \cap U = F_0$ is connected. Hence for all $r > 0$, we can find a neighbourhood $U \subseteq B(0,r)$ of the origin such that $f^{-1}(0)$ is connected in U. That is, $f^{-1}(0)$ is locally connected. □

Remark: Theorem 3.5 remains true if f is continuous and co-Lipschitz. Indeed, in the proof we only used the properties of the function f on a bounded set and f is uniformly continuous on every bounded set. So the conclusion, which does not depend on Ω, also holds in this case.

We now show that in general the level sets of Lipschitz quotient mappings from $\mathbb{R}^n$ to $\mathbb{R}$ do not have much more structure (e.g. manifold structure, local contractibility, etc). We construct a Lipschitz quotient mapping from $\mathbb{R}^3$ to $\mathbb{R}$ with a level set that is not locally simply connected.

Let $g_0 \colon \mathbb{R}^3 \to \mathbb{R}$ be the last coordinate function, $g_0((x_1, x_2, x_3)) = x_3$. We modify it within a spherical shell $\{x \colon 0 < r < \|x\| < R\}$ while preserving the Lipschitz quotient property, to have the level set $g^{-1}(0)$ not simply connected. To achieve this, let the surface F be defined as the union of the following two sets:

$$\big\{(x,y,z) \in \mathbb{R}^3 : z = 0 \text{ and } d\big((x,y),(2,0)\big), d\big((x,y),(-2,0)\big) \geq 1\big\}$$

and

$$\big\{(x,y,z) \in \mathbb{R}^3 : z > 0 \text{ and } d\big((x,y,z), \{(2\cos\alpha, 0, 2\sin\alpha) \colon \alpha \in (0,\pi)\}\big) = 1\big\},$$

i.e. the xy coordinate plane with two holes and a "handle" attached to these holes. It separates the entire space into two components, one covering the halfspace $\mathbb{R}^2 \times (-\infty, 0)$, we call it U^-, and the other one, U^+. Let C_1 denote the cone with the vertex $(-2,0,-3)$ and with base circle in the xy coordinate plane with radius 1 centered at $(-2,0,0)$, and let C_2 be the cone with the vertex $(2,0,-3)$ and with base circle in the xy coordinate plane with radius 1 centered at $(2,0,0)$, respectively. That is,

$$C_1 = \{(x,y,z) \in \mathbb{R}^2 \times (-\infty,0) \colon 3d((x,y),(-2,0)) - z \leq 3\}$$

and

$$C_2 = \{(x,y,z) \in \mathbb{R}^2 \times (-\infty,0) \colon 3d((x,y),(2,0)) - z \leq 3\}.$$

We write

$$U^- = H \cup G,$$

where

$$H = \big(U^- \cap (\mathbb{R}^2 \times [0,\infty))\big) \cup C_1 \cup C_2$$

(the solid "handle" with the cones C_1, C_2), and

$$G = U^- \setminus H.$$

We first define g on $\overline{U^-}$. Let $g = g_0$ on $\overline{G}$ and constant 0 on ∂U^-. On H, we define g to be linear on the line segments connecting the boundary of H with the closest point of

$$L = \big\{(2\cos\alpha, 0, 2\sin\alpha) \colon \alpha \in (0,\pi)\big\} \cup \big\{(\pm 2, 0, t) \colon t \in [-3,0]\big\}.$$

Since g is already determined on ∂H, we need to define g on L only. Let

$$g((\pm 2, 0, t)) = -2 + \frac{t}{3} \text{ for } t \in [-3,0],$$

and

$$g((2\cos\alpha, 0, 2\sin\alpha)) = -1 - \frac{2}{\pi}\left|\alpha - \frac{\pi}{2}\right| \text{ for } \alpha \in (0,\pi).$$

It is easy to see that g is Lipschitz ("sewn together" from finitely many Lipschitz bits). At the points of the complement of $L \cup \partial H$ there are directions in which g is locally linear. It has gradient with length 1 at the points of G, gradient with length 3 at the points of $(\mathrm{int}(C_1) \cup \mathrm{int}(C_2)) \setminus L$, and gradient with length somewhere between 1 and 2 at the points of $(U^- \cap \mathbb{R}^2 \times (0,\infty)) \setminus L$, respectively. At the points of the lateral surface of the cones C_1 and C_2 we have g decreasing with speed 1 in the direction $(0,0,-1)$ and increasing with speed $3/\sqrt{10}$ in the direction pointing "away" from the vertex of the appropriate cone. On the line segments contained in L, g is linear with gradient having length $1/3$ (in the vertex the previous estimate applies). On the arc contained in L, g decreases with speed $2/\pi$ in the direction of the arc and increases with speed at least 1 in any direction perpendicular to that direction. This proves that g is also co-Lipschitz as a function to $(-\infty, 0])$ with a co-Lipschitz constant $1/3$.

It is easy to see that the sets $\overline{U^-}$ and $\overline{U^+}$ are homeomorphic; moreover, there is a bi-Lipschitz homeomorphism $\Theta\colon \overline{U^-} \to \overline{U^+}$ such that it coincides with $(x, y, z) \to (x, y, -z)$ outside a ball and maps $(0, 0, 3)$ to the origin (the image of $\overline{U^-}$ under the mapping $(x, y, z) \to (x, y, -z)$ differs from $\overline{U^+}$ only in the position and the shape of the "handle"). By an infinitesimal modification of this mapping Θ within a neighbourhood of the point $(0, 0, 3)$ we can achieve $g_0(\Theta(x)) = -g(x)$. Finally let

$$g(x) = -g(\Theta^{-1}(x)) \text{ for all } x \in U^+.$$

By the properties derived above, g is a Lipschitz quotient mapping on the entire space $\mathbb{R}^3$, g coincides with g_0 outside a spherical shell $\{x\colon 0 < r < \|x\| < R\}$ and the pre-image of 0 is F, which is not simply connected.

With this g constructed, let f be defined as follows:

$$f(x) = g_0(x) + \sum_{j=1}^{\infty} \left(\frac{r}{2R}\right)^j \left(g\left(\left(\frac{2R}{r}\right)^j x\right) - g_0\left(\left(\frac{2R}{r}\right)^j x\right)\right).$$

Observe that the terms of the infinite sum part of this expression have disjoint support and are Lipschitz with the same Lipschitz constant (they are just scaled copies of the same Lipschitz function). The partial sums are also co-Lipschitz with the same co-Lipschitz constant as at every point there are directions (the same as in the construction of g) in which the partial sums decrease with speed c or increase with speed c. By Lemma 2.4 f is a Lipschitz quotient mapping as it is a pointwise limit of Lipschitz quotient functions with the same Lipschitz and co-Lipschitz constants. With this f, $f^{-1}(0)$ is the xy coordinate plane with infinitely many handles (decreasing in size) attached "around" the origin. Hence in any neighbourhood of the origin $f^{-1}(0)$ contains a complete handle, which destroys simple connectedness. So $f^{-1}(0)$ is not locally simply connected at the origin.

Using this construction, one can produce Lipschitz quotient maps from $\mathbb{R}^3$ to $\mathbb{R}$ with countably many level sets not locally simply connected. One just has to pick countably many disjoint balls with radius $2R$ and with centres having different third coordinate, and add the shifts of the function $f - g_0$ by the position vectors of the centres of the balls to g_0. Exactly as above, the resulting function retains the Lipschitz and co-Lipschitz properties of its "parts", and hence is Lipschitz quotient, and the selected level sets will not be locally simply connected.

Chapter 4

Characterisation of level sets of Lipschitz quotient mappings to $\mathbb{R}$

In the proof of Theorem 3.5 we did not "really" use the function itself; we only used a property of the "positive" and "negative" components of the complements of the level sets. This is the following: for every point x in their closure, they contain a whole "thinned cone" with the vertex x, a set of form $\bigcup_{t>0} B(\xi(t), \Omega(t))$ with $\xi\colon [0,\infty) \to \mathbb{R}^n$, $\xi(0) = x$ and a fixed $\Omega\colon (0,\infty) \to (0,\infty)$ (the modulus on uniform continuity of the underlying function). In the Lipschitz case, these sets are "really cones", sets of form $\cup_{t>0} B(\xi(t), \lambda t)$ with $\xi\colon [0,\infty) \to \mathbb{R}^n$, $\xi(0) = x$ and a fixed $\lambda > 0$. Moreover, if $(X, \|\cdot\|)$ is an arbitrary normed space and we consider a Lipschitz quotient mapping $f\colon X \to \mathbb{R}$ with Lipschitz constant L and co-Lipschitz constant c, and $x \in X \setminus f^{-1}(0)$, then an analogous implication holds, even though Lemma 2.5 is no longer applicable (in fact, the statement of Lemma 2.5 is false for example in l_2). Indeed, we may assume that $f(x) > 0$ and $L > 1$. Now choose $0 < \lambda < c/L$ and define recursively the points $x_0, x_1, \ldots$ as follows: first let $x_0 = x$. If we have already defined x_{n-1} for some $n \geq 1$, then pick x_n with the properties $\|x_{n-1} - x_n\| \leq f(x)/L(1+\lambda)$ and $|f(x_n)| = |f(x_{n-1})| + f(x)\lambda/(1+\lambda)$ (using the co-Lipschitz property of f for the ball $B(x_{n-1}, f(x)/L(1+\lambda)))$. Consider the curve $\gamma\colon [0,\infty) \to X$ which connects these points:

$$\gamma\left(\frac{f(x)}{L(1+\lambda)}t\right) = (n-t)x_{n-1} + (t-(n-1))x_n \text{ for } t \in [n-1, n],\ n \geq 1.$$

By the construction of the sequence (x_n) the curve γ has Lipschitz constant at most one, and it also has the property $\gamma(0) = x$. For all $t > 0$, let $n = \lfloor tL(1+\lambda)/f(x)\rfloor$ be the largest integer such that $nf(x)/L(1+\lambda) \leq t$; then we have

$$0 \leq t - n\frac{f(x)}{L(1+\lambda)} < \frac{f(x)}{L(1+\lambda)}$$

and hence

$$\begin{aligned} B(\gamma(t),\, \lambda t) &\subseteq B\left(\gamma\left(n\frac{f(x)}{L(1+\lambda)}\right),\, \lambda t + \left\|\gamma(t) - \gamma\left(n\frac{f(x)}{L(1+\lambda)}\right)\right\|\right) \\ &\subseteq B\left(x_n,\, \left(\lambda n\frac{f(x)}{L(1+\lambda)} + \lambda\frac{f(x)}{L(1+\lambda)}\right) + \frac{f(x)}{L(1+\lambda)}\right) \\ &= B\left(x_n,\, \frac{\lambda n + \lambda + 1}{L(1+\lambda)}f(x)\right). \end{aligned}$$

Due to the Lipschitz property of f this implies that

$$f(B(\gamma(t),\, \lambda t)) \subseteq B\left(f(x_n),\, f(x)\frac{\lambda n + \lambda + 1}{1+\lambda}\right)$$

$$= B\left(f(x) + n\frac{\lambda}{1+\lambda}f(x),\, f(x) + f(x)\frac{n\lambda}{1+\lambda}\right) \subseteq (0, \infty),$$

as required.

In Theorem 4.2 we prove that this property is indeed sufficient to describe the "positive" and "negative" components of the complements of the level sets of Lipschitz quotient mappings from X to $\mathbb{R}$, if X is uniformly convex. One of the reasons that X is required to be uniformly convex is captured in the following lemma:

Lemma 4.1. *Let e and f be distinct unit vectors in a uniformly convex Banach space X with modulus of uniform convexity δ. Then for all $\mu \geq 0$ the following inequality holds:*

$$\|f - \mu e\| \geq 1 - \mu(1 - \delta(\|e - f\|)).$$

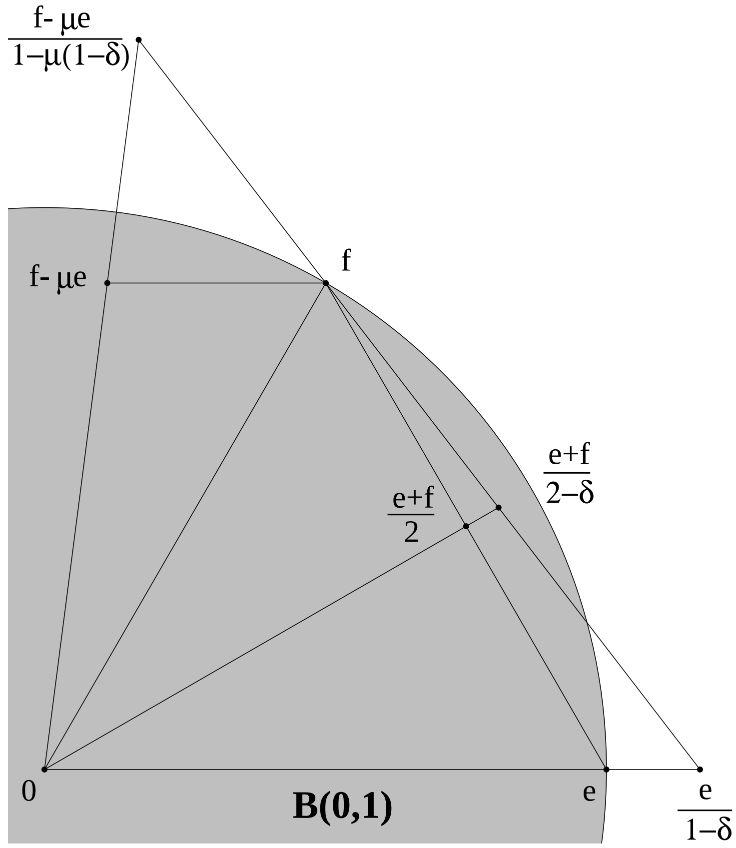

Figure 4.1: The configuration in the proof of Lemma 4.1

Proof. Denote $\delta_0 = \delta(\|e - f\|)$. If $\mu(1-\delta_0) \geq 1$, then $\|f - \mu e\| \geq 0 \geq 1 - \mu(1-\delta_0)$ holds trivially, so we assume that $\mu(1-\delta_0) < 1$.

Using the uniform convexity of X on e and f, we obtain $\|e+f\| \leq 2-\delta_0$. So, if we consider the function

$$l(\alpha) = \left\| \alpha f + (1-\alpha)\frac{e+f}{2-\delta_0} \right\|,$$

we have $l(0) = \|e+f\|/(2-\delta_0) \leq 1$ and $l(1) = \|f\| = 1$. But l is convex, as

$$l(\lambda\alpha + (1-\lambda)\beta) \;=\; \left\| (\lambda\alpha + (1-\lambda)\beta)f + (1 - \lambda\alpha - (1-\lambda)\beta)\frac{e+f}{2-\delta_0} \right\|$$

$$\begin{aligned} &= \left\| \lambda \left(\alpha f + (1-\alpha)\frac{e+f}{2-\delta_0} \right) \right. \\ &\qquad \left. +(1-\lambda)\left(\beta f + (1-\beta)\frac{e+f}{2-\delta_0} \right) \right\| \\ &\le |\lambda| l(\alpha) + |1-\lambda| l(\beta) = \lambda l(\alpha) + (1-\lambda) l(\beta) \end{aligned}$$

holds for all $\lambda \in [0,1]$ and $\alpha, \beta \in \mathbb{R}$. Hence if $\alpha \ge 1$, then $l(\alpha) \ge l(1) + (\alpha - 1)(l(1) - l(0)) \ge l(1) = 1$, and in particular for $\alpha = \frac{1+\mu}{1-\mu(1-\delta_0)} \ge 1 + \mu \ge 1$ we get

$$\begin{aligned} 1 \le l(\alpha) &= \left\| \frac{1+\mu}{1-\mu(1-\delta_0)} f + \left(1 - \frac{1+\mu}{1-\mu(1-\delta_0)}\right) \frac{e+f}{2-\delta_0} \right\| \\ &= \left\| \frac{1+\mu+\frac{1-\mu(1-\delta_0)-(1+\mu)}{2-\delta_0}}{1-\mu(1-\delta_0)} f + \frac{1-\mu(1-\delta_0)-(1+\mu)}{(1-\mu(1-\delta_0))(2-\delta_0)} e \right\| \\ &= \left\| \frac{1+\mu-\mu}{1-\mu(1-\delta_0)} f + \frac{\delta_0\mu - 2\mu}{(1-\mu(1-\delta_0))(2-\delta_0)} e \right\| = \left\| \frac{f-\mu e}{1-\mu(1-\delta_0)} \right\|, \end{aligned}$$

and this implies the statement of the lemma. □

Theorem 4.2. *Let X be a uniformly convex Banach space with modulus of uniform convexity δ, let $U \subsetneq X$ be an open set and $c \in (0,1)$. Assume that for every $x \in U$ there exists a Lipschitz curve $\xi_x \colon [0,\infty) \to U$ with Lipschitz constant at most 1 such that $\xi_x(0) = x$ and $B(\xi_x(t), ct) \subseteq U$ for all $t > 0$. Then there exists a constant $\tilde{c} = \tilde{c}(c)$ and a Lipschitz quotient mapping $g \colon \overline{U} \to [0,\infty)$ with Lipschitz constant at most 1, co-Lipschitz constant $\tilde{c}$ and $g^{-1}(0) = \partial U$.*

Proof. If U is empty, then the statement of the theorem holds trivially. From now on we suppose that this is not the case, $U \neq \emptyset$ and so $\partial U \neq \emptyset$.

In the following, we assume without loss of generality that $c < \delta(1)$.

Let $f \colon U \to \mathbb{R}$ be the distance from the boundary, $f(x) = d(x, \partial U)$. We define the function $\overline{f} \colon U \to \mathbb{R}$ in the following way:

$$\overline{f}(x) = \sup \left\{ W(\gamma) \middle| \gamma \colon [0,\infty) \to U \text{ 1-Lipschitz}, \gamma(0) = x \right\}, \tag{4.1}$$

$$\text{where } W(\gamma) = \inf_{t \ge 0} \left(f(\gamma(t)) - \frac{c}{2} t \right),$$

i.e. $\overline{f}(x)$ is the supremum of the sizes of such open balls centered at x and growing with speed $c/2$, that can be "towed out to infinity" with speed 1 without touching ∂U (this makes sense if $\overline{f}(x) > 0$). The function $\overline{f}$ satisfies

$$\frac{c}{2(1+c)} f(x) \le \overline{f}(x) \le f(x)$$

for every $x \in U$. Indeed, the upper bound is obtained by taking $t = 0$ in (4.1), and to obtain the lower bound, take $\gamma = \xi_x$:

$$\begin{aligned} W(\xi_x) &= \inf_{t\ge 0}\left(f(\xi_x(t)) - \frac{c}{2}t\right) \\ &= \min\left\{\inf_{0\le t\le \frac{f(x)}{1+c}}\left(f(\xi_x(t)) - \frac{c}{2}t\right), \inf_{t\ge \frac{f(x)}{1+c}}\left(f(\xi_x(t)) - \frac{c}{2}t\right)\right\} \\ &\ge \min\left\{\inf_{0\le t\le \frac{f(x)}{1+c}}\left(f(\xi_x(0)) - t - \frac{c}{2}t\right), \inf_{t\ge \frac{f(x)}{1+c}}\left(ct - \frac{c}{2}t\right)\right\} \\ &= \min\left\{f(x) - \frac{(2+c)f(x)}{2(1+c)}, \frac{cf(x)}{2(1+c)}\right\} = \frac{c}{2(1+c)}f(x). \end{aligned}$$

Observe furthermore, that for any $\gamma\colon [0,\infty) \to U$ and $\theta > 0$ we have

$$\begin{aligned} W(\gamma(\cdot+\theta)) &= \inf_{t\ge 0}\left(f(\gamma(t+\theta)) - \frac{c}{2}t\right) = \inf_{t'\ge\theta}\left(f(\gamma(t')) - \frac{c}{2}t' + \frac{c}{2}\theta\right) \\ &\ge W(\gamma) + \frac{c}{2}\theta. \end{aligned} \tag{4.2}$$

In the following, for all $x \in U$ and $\varepsilon > 0$ we fix a curve $\psi_{x,\varepsilon}\colon [0,\infty) \to U$ such that it has Lipschitz constant at most one, $\psi_{x,\varepsilon}(0) = x$ and $W(\psi_{x,\varepsilon}) \ge \overline{f}(x) - \varepsilon$.

We claim that $\overline{f}$ is Lipschitz with Lipschitz constant at most one. Indeed, if $\overline{f}(x) = r$, then for all $\varepsilon > 0$ we can "tow away" a ball with initial radius $r - \varepsilon$ from the point x, using the curve $\psi_{x,\varepsilon}$. Hence, considering the curve

$$\gamma(t) = \begin{cases} z + \frac{t}{\|x-z\|}(x-z) & \text{if } 0 \le t < \|x-z\| \\ \psi_{x,\varepsilon}(t - \|x-z\|) & \text{if } t \ge \|x-z\| \end{cases}$$

we obtain

$$\overline{f}(z) \ge W(\gamma) = \min\left\{W(\psi_{x,\varepsilon}) - \frac{c}{2}\|x-z\|, \inf_{0\le t<\|x-z\|}\left(d(\gamma(t), \partial U) - \frac{c}{2}t\right)\right\}$$

$$= \min\left\{r-\varepsilon-\frac{c}{2}\|x-z\|,\, r-\|x-z\|\right\} \geq r-\varepsilon-\|x-z\|,$$

that is, for all points $z \in B(x,r)$ we can first "tow" a ball with initial radius $r-\varepsilon-\|x-z\|$ from z to x within the ball $B(x,r)$ (which is disjoint from ∂U), then continue "towing" the growing ball from x. Since $\varepsilon > 0$ was arbitrary, we get $\overline{f}(z) \geq r - \|x-z\|$. On the other hand, if we take $z \notin B(x,r)$, then $\overline{f}(z) \geq 0 \geq r-\|x-z\|$ still holds. So for all $z \in U$ we have $\|x-z\| \geq \overline{f}(x)-\overline{f}(z)$. But x was arbitrary, so by interchanging x and z we have $\|z-x\| \geq \overline{f}(z)-\overline{f}(x)$ as well. This implies $|\overline{f}(x)-\overline{f}(z)| \leq \|x-z\|$ for all x, $z \in U$, which is exactly the Lipschitz property with constant 1.

Now fix an arbitrary $\varepsilon \in (0, cf(x)/8(1+c))$, let $y \in \partial U$ be such that $\|x-y\| \leq d(x,\partial U)+\varepsilon$ and let $z \in [x,y]$ be a point satisfying $\|x-z\| \leq f(x)$. We prove that in this case

$$\overline{f}(x) \geq \overline{f}(z) - \frac{c}{2}\left(1-\delta(\lambda)\right)\|x-z\| - 2\varepsilon \tag{4.3}$$

holds, where $\lambda = \frac{c\Delta}{c\Delta+3c+4}$ with $\Delta = \delta(1)$.

If $\overline{f}(z) \leq \overline{f}(x)$, then we have the desired inequality; so we assume that $\overline{f}(z) > \overline{f}(x)$. Using the notation $R_0 = \frac{c}{2(1+c)}f(x)$, since we already know that $\overline{f}(x) \geq R_0$, we get that $\overline{f}(z) > \overline{f}(x) \geq R_0$ holds as well.

Denote $\psi = \psi_{z,\varepsilon}$. If for some $\tau \geq 0$ we have $x = \psi(\tau)$, then we immediately obtain $\overline{f}(x) \geq W(\psi(\cdot+\tau)) \geq W(\psi) \geq \overline{f}(z) - \varepsilon$ and so the inequality (4.3) holds, so in the following we assume that $x \neq \psi(\tau)$ for all $\tau \geq 0$.

Consider the function

$$A(t) = \left\| \frac{\psi(t)-x}{\|\psi(t)-x\|} - \frac{y-x}{\|y-x\|} \right\|$$

for $t \geq 0$. Since $x \neq \psi(t)$ for all $t \geq 0$, the function A is continuous. For $t=0$ we have

$$A(0) = \left\| \frac{z-x}{\|z-x\|} - \frac{y-x}{\|y-x\|} \right\| = 0.$$

Let now t be such that $\|\psi(t)-x\| = f(x)$; such a t exists, since $f \circ \psi$ is unbounded and hence the range of ψ cannot be contained in a bounded region of U. For any such t we have

$$A(t) = \left\| \frac{\psi(t)-x}{\|\psi(t)-x\|} - \frac{y-x}{\|y-x\|} \right\| = \frac{1}{f(x)} \left\| \psi(t)-x-\frac{f(x)}{\|y-x\|}(y-x) \right\|$$

$$
\begin{aligned}
&\geq \frac{1}{f(x)}\left(\|\psi(t)-x-(y-x)\| - \left\|(y-x) - \frac{f(x)}{\|y-x\|}(y-x)\right\|\right)\\
&= \frac{1}{f(x)}\big(\|\psi(t)-y\| - \big|\|x-y\| - f(x)\big|\big) \geq \frac{1}{f(x)}\big(d(\psi(t),\partial U)-\varepsilon\big)\\
&\geq \frac{1}{f(x)}\left(\overline{f}(z)+\frac{c}{2}t-2\varepsilon\right) \geq \frac{\overline{f}(x)-2\varepsilon}{f(x)} \geq \frac{c}{2(1+c)} - \frac{c}{4(1+c)}\\
&= \frac{c\Delta}{4(1+c)\Delta} \geq \frac{c\Delta}{c\Delta+3c+4} = \lambda,
\end{aligned}
$$

and hence there exists a $\tau \geq 0$ with the property $\|\psi(\tau)-x\| \leq f(x)$ such that $A(\tau)=\lambda$ (for example, take τ in $[0,t]$ with the smallest t satisfying the above condition). Denote $w=\psi(\tau)$, $R=\overline{f}(z)+\frac{c}{2}\tau-\varepsilon$ and $r=\|w-x\|$.

By definition of $\psi=\psi_{z,\varepsilon}$ we have

$$
\begin{aligned}
R &\leq d(w,\partial U) \leq \|w-y\|\\
&\leq \left\|w-\left(x+\frac{\|w-x\|}{\|y-x\|}y-x\right)\right\| + \left\|\left(x+\frac{\|w-x\|}{\|y-x\|}(y-x)\right)-y\right\|\\
&= r\left\|\frac{w-x}{\|w-x\|}-\frac{y-x}{\|y-x\|}\right\| + \|x-y\|\left|1-\frac{\|w-x\|}{\|y-x\|}\right|\\
&= rA(\tau)+\big|\|x-y\|-\|w-x\|\big| = r\lambda+\|x-y\|-r\\
&\leq f(x)+\varepsilon-r(1-\lambda) \qquad (4.4)
\end{aligned}
$$

and hence, using the fact that $R \geq \overline{f}(z)-\varepsilon \geq R_0-(R_0/4)=3R_0/4$, we get that $r(1-\lambda) \leq f(x)+\varepsilon-R \leq f(x)+(R_0/4)-(3R_0/4) \leq f(x)-(R_0/2)$. So we have

$$
\begin{aligned}
r+(1-\Delta)(R-\varepsilon) &\leq r+(1-\Delta)\big(f(x)-r(1-\lambda)\big)\\
&= (1-\Delta)f(x)+r\big(1-(1-\Delta)(1-\lambda)\big)\\
&\leq f(x)(1-\Delta)+\frac{f(x)-\frac{R_0}{2}}{1-\lambda}\big(\Delta(1-\lambda)+\lambda\big)\\
&= f(x)-\frac{R_0}{2}\Delta+\frac{\lambda}{1-\lambda}\left(f(x)-\frac{R_0}{2}\right)\\
&= f(x)-\frac{cf(x)\Delta}{4(1+c)}+\frac{c\Delta}{3c+4}\left(f(x)-\frac{c}{4(1+c)}f(x)\right)\\
&= f(x), \qquad (4.5)
\end{aligned}
$$

and (4.4) also implies

$$
(R-\varepsilon)+r(1-\Delta) \leq (R-\varepsilon)+r(1-\lambda) \leq f(x), \qquad (4.6)
$$

since $\lambda = \frac{c}{4+(3+\Delta)c}\Delta < \Delta$.

We claim that

$$\operatorname{conv}\big(B(w,R-\varepsilon)\cup B(x,R-\varepsilon)\big) \subseteq B(w,R-\varepsilon)\cup B(x,f(x)). \tag{4.7}$$

Indeed, let $\overline{v} \in \operatorname{conv}(B(w,R-\varepsilon)\cup B(x,R-\varepsilon))$, and denote v the closest point of the line segment $[x,w]$ to $\overline{v}$ (this point exists since $[x,w]$ is compact and is unique since X is uniformly and hence strictly convex). Since $\overline{v} \in \operatorname{conv}(B(w,R-\varepsilon)\cup B(x,R-\varepsilon)) = B(0,R-\varepsilon)\oplus[x,w]$, we have $\|\overline{v}-v\| \le R-\varepsilon$. If now $v \in \{x,w\}$, then $\overline{v} \in B(x,R-\varepsilon)\cup B(w,R-\varepsilon)$ and we are done. So we assume that $v \notin \{x,w\}$; in this case v is the closest point of the line $\{tx+(1-t)w\colon t\in\mathbb{R}\}$ to $\overline{v}$ (the function $t \to \|\overline{v}-(tx+(1-t)w\|$ is convex, so any local minimum is the global minimum). Considering the points

$$\overline{x} = x + \frac{f(x)}{\|\overline{v}-v\|}(\overline{v}-v)$$

and

$$\overline{w} = w + \frac{R-\varepsilon}{\|\overline{v}-v\|}(\overline{v}-v),$$

we get that the closest point of the line $\{tx+(1-t)w\colon t\in\mathbb{R}\}$ to $\overline{x}$ is x. Hence in particular

$$\left\|\overline{x} - \left(x + f(x)\frac{w-x}{\|w-x\|}\right)\right\| \ge \|\overline{x}-x\| = f(x),$$

so applying the uniform convexity property of X to the unit vectors $\frac{w-x}{r}$ and $\frac{\overline{x}-x}{f(x)}$ we get

$$\left\|\frac{\frac{w-x}{r}+\frac{\overline{x}-x}{f(x)}}{2-\Delta}\right\| \le 1$$

and hence for any $\alpha,\ \beta \ge 0$ satisfying both $\alpha+(1-\Delta)\beta \le 1$ and $(1-\Delta)\alpha+\beta \le 1$ we have (see Figure 4.2)

$$\alpha\frac{w-x}{r} + \beta\frac{\overline{x}-x}{f(x)} \in \operatorname{conv}\left\{0,\ \frac{w-x}{r},\ \frac{\overline{x}-x}{f(x)},\ \frac{\frac{w-x}{r}+\frac{\overline{x}-x}{f(x)}}{2-\Delta}\right\}$$

and consequently

$$\left\|\alpha\frac{w-x}{r} + \beta\frac{\overline{x}-x}{f(x)}\right\| \le 1.$$

Taking $\alpha = \frac{r}{f(x)}$, $\beta = \frac{R-\varepsilon}{f(x)}$ (inequalities (4.5) and (4.6) show that the conditions on α and β are satisfied) we get

$$f(x) \geq \left\| r\frac{w-x}{r} + (R-\varepsilon)\frac{\overline{x}-x}{f(x)} \right\| = \|(w-x) + (\overline{w}-w)\| = \|\overline{w}-x\|,$$

so $\overline{w} \in B(x, f(x))$ and hence we get

$$\overline{v} \in \left[v, v + (R-\varepsilon)\frac{\overline{v}-v}{\|\overline{v}-v\|}\right] \subset \operatorname{conv}\left(\left[x, x + \frac{R-\varepsilon}{f(x)}(\overline{x}-x)\right] \cup [w, \overline{w}]\right)$$

$$\subset \operatorname{conv}\big(B(x, R-\varepsilon) \cup B(x, f(x))\big) = B(x, f(x))$$

(the last equality holds since $R - \varepsilon \leq f(x) - r(1-\lambda) < f(x)$), proving our claim.

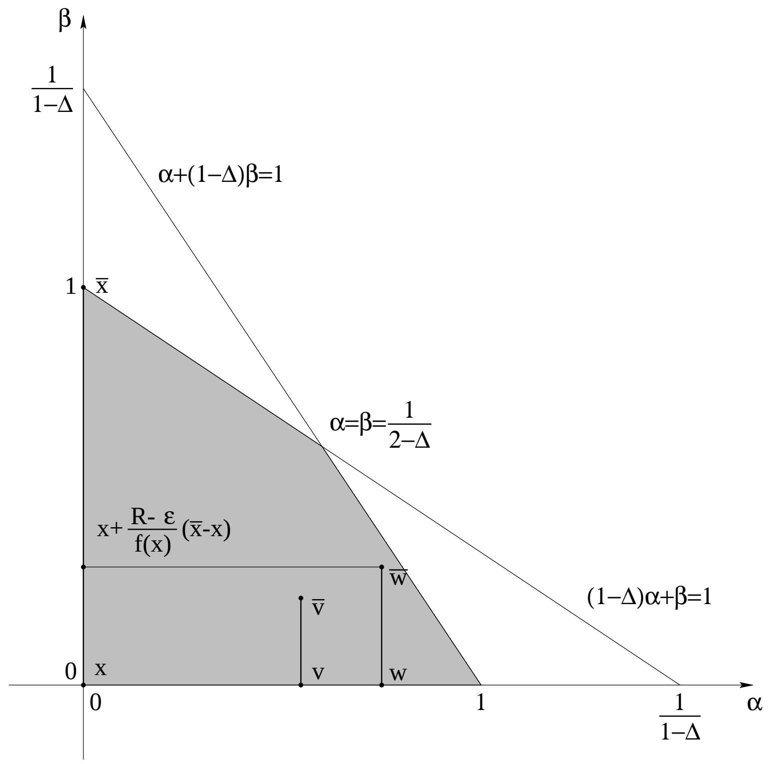

Figure 4.2: The configuration in the proof of (4.7)

Consider now the curve

$$\gamma(t) = \begin{cases} x + \frac{t}{r}(w - x) & \text{if } 0 \le t < r \\ \psi(t - r + \tau) & \text{if } t \ge r. \end{cases}$$

The containment (4.7) implies that the set

$$\bigcup_{0 \le t \le r} B\left(\gamma(t), R - \varepsilon - \frac{c}{2}(t - r)\right) \subset \operatorname{conv}\left(B(x, R - \varepsilon) \cup B(w, R - \varepsilon)\right)$$

is disjoint from ∂U and hence we get that

$$\inf_{0 \le t \le r} \left(f(\gamma(t)) - \frac{c}{2}t\right) \ge R - \varepsilon - \frac{c}{2}r = \overline{f}(z) + \frac{c}{2}(\tau - r) - 2\varepsilon.$$

On the other hand, using the property (4.2) we also have

$$\begin{aligned} \inf_{t \ge r} \left(f\left(\gamma(t)\right) - \frac{c}{2}t\right) &= W(\psi(\cdot + \tau)) - \frac{c}{2}r \ge W(\psi) + \frac{c}{2}(\tau - r) \\ &\ge \overline{f}(z) + \frac{c}{2}(\tau - r) - \varepsilon. \end{aligned}$$

Combining these two estimates, we get that

$$\overline{f}(x) \ge W(\gamma) \ge \overline{f}(z) + \frac{c(\tau - r)}{2} - 2\varepsilon \ge \overline{f}(z) + \frac{c}{2}(\|z - w\| - \|x - w\|) - 2\varepsilon.$$

But applying Lemma 4.1 to $\mu = \frac{\|z-x\|}{\|w-x\|}$ and the vectors $e = \frac{y-x}{\|y-x\|} = \frac{z-x}{\|z-x\|}$, $f = \frac{w-x}{\|w-x\|}$, we get that

$$\begin{aligned} \|z - w\| &= \big\| \|z - x\|e - \|w - x\|f \big\| = \|w - x\| \|f - \mu e\| \\ &\ge \|w - x\| \left(1 - \mu\left(1 - \delta\left(\left\|\frac{w - x}{\|w - x\|} - \frac{y - x}{\|y - x\|}\right\|\right)\right)\right) \\ &= \|w - x\|\big(1 - \mu(1 - \delta(\lambda))\big) = \|w - x\| - \|z - x\|(1 - \delta(\lambda)) \end{aligned}$$

and consequently

$$\overline{f}(x) \ge \overline{f}(z) + \frac{c}{2}(\|z - w\| - \|x - w\|) - 2\varepsilon \ge \overline{f}(z) - \frac{c}{2}\|z - x\|(1 - \delta(\lambda)) - 2\varepsilon,$$

establishing (4.3).

Now we define the function g as

$$g(x) = \frac{1}{1+\frac{c}{2}\left(1-\frac{\delta(\lambda)}{2}\right)}\left(\overline{f}(x)+\frac{c}{2}\left(1-\frac{\delta(\lambda)}{2}\right)f(x)\right)$$

for all $x \in U$ and $g(x) = 0$ for all $x \in \partial U$. This function g is obviously non-negative and continuous (both f and $\overline{f}$ extend continously to have value 0 on ∂U), has Lipschitz constant at most

$$\frac{1}{1+\frac{c}{2}\left(1-\frac{\delta(\lambda)}{2}\right)}\left(\|\overline{f}\|_{Lip}+\frac{c}{2}\left(1-\frac{\delta(\lambda)}{2}\right)\|f\|_{Lip}\right) \leq 1$$

in U. The boundary condition is satisfied, so we only need to show that the co-Lipschitz property also holds at the points of U (since U is dense in $\overline{U}$ and g is continuous on $\overline{U}$; see Remark after Definition 2.2). As for the "upper" part, $g(B(x,r)) \supseteq [g(x), g(x)+\tilde{c}r)$ for some $\tilde{c} > 0$, we have the following estimate for all points $x \in U$ and $\varepsilon > 0$, denoting $\psi = \psi_{x,\varepsilon}$:

$$\begin{aligned}
g(\psi(r)) - g(x) &= \frac{1}{1+\frac{c}{2}\left(1-\frac{\delta(\lambda)}{2}\right)}\left(\overline{f}(\psi(r))+\frac{c}{2}\left(1-\frac{\delta(\lambda)}{2}\right)f(\psi(r))\right. \\
&\qquad \left. -\overline{f}(x)-\frac{c}{2}\left(1-\frac{\delta(\lambda)}{2}\right)f(x)\right) \\
&\geq \frac{1}{1+\frac{c}{2}\left(1-\frac{\delta(\lambda)}{2}\right)}\left(W(\psi(\cdot+r))-\overline{f}(x)\right. \\
&\qquad \left. -\frac{c}{2}\left(1-\frac{\delta(\lambda)}{2}\right)\|\psi(r)-x\|\right) \\
&\geq \frac{1}{1+\frac{c}{2}\left(1-\frac{\delta(\lambda)}{2}\right)}\left(\frac{c}{2}r-\varepsilon-\frac{c}{2}\left(1-\frac{\delta(\lambda)}{2}\right)r\right) \\
&\geq \frac{\frac{c\delta(\lambda)}{4}-\varepsilon}{1+\frac{c}{2}\left(1-\frac{\delta(\lambda)}{2}\right)}r \xrightarrow{\varepsilon\to 0} \frac{c\delta(\lambda)}{4+c(2-\delta(\lambda))}r.
\end{aligned}$$

This tells us the "upper" part of the co-Lipschitz property is satisfied with $\tilde{c} = \frac{c\delta(\lambda)}{4+c(2-\delta(\lambda))}$:

$$g(B(x,r)) \supseteq \bigcup_{n=1}^{\infty}[g(x), g(\psi_{x,1/n}(r))) \supseteq [g(x), g(x)+\tilde{c}r).$$

To get the "lower" estimate, $g(B(x,r)) \supseteq [0,\infty) \cap (g(x) - \tilde{c}r, g(x)]$, observe that for $r > d(x, \partial U)$ there exists a point $y \in \partial U$ with $\|x - y\| < r$ and $[x,y] \in \overline{U}$, and hence we have $g(B(x,r)) \supseteq [g(y), g(x)] = [0, g(x)] \supseteq [0,\infty) \cap (g(x) - \tilde{c}r, g(x)]$ independently of $\tilde{c}$, so we only have to care about the case $r \leq d(x, \partial U)$. But then for all $0 < \varepsilon < cf(x)/(8(1+c))$ we can take a point $y_\varepsilon \in \partial U$ such that $\|x - y_\varepsilon\| \leq d(x, \partial U) + \varepsilon$ and $[x, y_\varepsilon] \in \overline{U}$. Setting $z_\varepsilon \in [x, y_\varepsilon]$ to be the point satisfying $\|x - z_\varepsilon\| = r$, the inequality (4.3) gives us

$$\begin{aligned}
g(x) - g(z_\varepsilon) &= \frac{1}{1 + \frac{c}{2}\left(1 - \frac{\delta(\lambda)}{2}\right)} \left(\overline{f}(x) + \frac{c}{2}\left(1 - \frac{\delta(\lambda)}{2}\right) d(x, \partial U) \right. \\
&\qquad \left. - \overline{f}(z_\varepsilon) - \frac{c}{2}\left(1 - \frac{\delta(\lambda)}{2}\right) d(z_\varepsilon, \partial U) \right) \\
&\geq \frac{1}{1 + \frac{c}{2}\left(1 - \frac{\delta(\lambda)}{2}\right)} \left(-\frac{c}{2}\|x - z_\varepsilon\|(1 - \delta(\lambda)) - 2\varepsilon \right. \\
&\qquad \left. + \frac{c}{2}\left(1 - \frac{\delta(\lambda)}{2}\right) (\|x - y_\varepsilon\| - \varepsilon - \|z_\varepsilon - y_\varepsilon\|) \right) \\
&= \frac{1}{1 + \frac{c}{2}\left(1 - \frac{\delta(\lambda)}{2}\right)} \left(-\frac{c}{2}(1 - \delta(\lambda))r + \frac{c}{2}\left(1 - \frac{\delta(\lambda)}{2}\right)(r - \varepsilon) \right) \\
&\geq \frac{\frac{c\delta(\lambda)}{4}}{1 + \frac{c}{2}\left(1 - \frac{\delta(\lambda)}{2}\right)} r - \varepsilon = \tilde{c}r - \varepsilon
\end{aligned}$$

with the previously defined $\tilde{c}$. This proves that

$$g(B(x,r)) \supseteq \bigcup_{n=1}^{\infty} (g(z_{1/n}), g(x)] \supseteq (g(x) - \tilde{c}r, g(x)],$$

finishing the proof of the co-Lipschitz property of g (we obtained $g(B(x,r)) \supseteq [g(x), g(x) + \tilde{c}r) \cup ([0,\infty) \cap (g(x) - \tilde{c}r, g(x)]) = (g(x) - \tilde{c}r, g(x) + \tilde{c}r) \cap [0,\infty)$ as required) at the points of U, and hence at the points of $\overline{U}$. □

From Theorem 4.2 we obtain the following corollary:

Corollary 4.3. *A set $F \subset X$, where X is uniformly convex, is a level set of some Lipschitz quotient mapping from X to $\mathbb{R}$ if and only if there is a partition of the complement of the set F into two disjoint open sets, $X \setminus F = U^+ \cup U^-$, such that $F = \partial U^- = \partial U^+$ and both of the sets U^- and U^+ satisfy the requirement of Theorem 4.2.*

Proof. If $F = f^{-1}(t)$ for some Lipschitz quotient mapping f and $t \in \mathbb{R}$, then $X \setminus F = f^{-1}((-\infty, t)) \cup f^{-1}((t, \infty)) = U^- \cup U^+$ satisfies the requirements. On the other hand, if the sets U^- and U^+ satisfy our requirements, then by Theorem 4.2 there exist Lipschitz functions $f^-\colon \overline{U^-} \to [0, \infty)$ and $f^+\colon \overline{U^+} \to [0, \infty)$ with Lipschitz constant at most one, the co-Lipschitz property and $(f^-)^{-1}(0) = \partial U^-$, $(f^+)^{-1}(0) = \partial U^+$. Then $f = f^+ \cup (-f^-)$ is Lipschitz with Lipschitz constant at most one, $f^{-1}(0) = F$, and the co-Lipschitz property is satisfied in points of F; combining this with the co-Lipschitz property of $-f^-$ and f^+, we get that the co-Lipschitz property is satisfied everywhere. $\square$

Remark: Since the class of Lipschitz quotient mappings does not change if we change the norms of the involved spaces to equivalent ones, the results of Theorem 4.2 and consequently Corollary 4.3 hold in the case when X is a super-reflexive Banach space.

With the help of Corollary 4.3, we can now construct a connected subset of, say, l_2, that is a level set of a Lipschitz quotient mapping from l_2 to $\mathbb{R}$ and is not locally connected. Take $U_0^+ \subset l_2$ to be the union of open cones with the n^{th} cone having the vertex e_n/n, direction e_n and angle $\arctan(1/2)$, where $\{e_n\}_{n\in\mathbb{N}}$ is the standard basis of l_2, i.e. the following set:

$$U_0^+ = \bigcup_{n=1}^{\infty} \left\{ (x_1, x_2, \dots) \in l_2\colon \|(x_1, \dots, x_{n-1}, x_{n+1}, \dots)\|_2 < \frac{x_n}{2} - \frac{1}{2n} \right\}.$$

If we define $F_0 = \partial U_0^+$ and $U_0^- = l_2 \setminus \overline{U_0^+}$, it is fairly easy to see that the requirements of Corollary 4.3 will be satisfied, and F_0 is not locally connected at $0 \in F_0$; however, F_0 is not connected. So we "connect" the components of U_0^+, defining

$$U^+ = U_0^+ \cup \bigcup_{n=1}^{\infty} \operatorname{conv}\left(B(4e_n, 1) \cup B(4e_{n+1}, 1)\right).$$

Define $U^- = \operatorname{int}(l_2 \setminus U^+)$ and $F = \partial U^+$; it is straightforward to see that these sets still satisfy the requirements of Corollary 4.3 and now F is a connected, not locally connected subset of l_2 that is a level set of a Lipschitz quotient mapping from l_2 to $\mathbb{R}$. Hence Theorem 3.5 does not hold if we release the assumption that the domain space is finite dimensional.

Observe that the set F_0 has a connected component consisting of a single point, namely $\{0\}$. Indeed, the sets

$$F_0^n = \left\{ (x_1, x_2, \dots) \in l_2\colon \|(x_1, \dots, x_{n-1}, x_{n+1}, \dots)\|_2 = \frac{x_n}{2} - \frac{1}{2n} \right\}$$

are connected and $d(F_0^n, F_0 \setminus F_0^n) = 1/n > 0$, so F_0^n is a connected component of F_0 for all n. But then $\{0\} = F \setminus \bigcup_{n=1}^{\infty} F_0^n$ is a union of connected components of F_0, hence $\{0\}$ is a connected component of F_0 itself, as claimed. This example shows that if we do not restrict ourselves to finite dimensional spaces, then in general the connected components of the level sets of a Lipschitz quotient mapping from X to $\mathbb{R}$ need not separate X.

Bibliography

[1] S. Bates, W.B. Johnson, J. Lindenstrauss, D. Preiss and G. Schechtman, *Affine approximation of Lipschitz functions and nonlinear quotients*, Geom. Funct. Anal., 9 (1999), pp. 1092-1127

[2] M. Csörnyei, *Can one squash the space into the plane without squashing?*, Geom. Funct. Anal., 11 (2001), pp. 933-952

[3] W.B. Johnson, J. Lindenstrauss, D. Preiss and G. Schechtman, *Lipschitz quotients from metric trees and from Banach spaces containing l_1*, Journal of Functional Analysis, 194 (2002), pp. 332-346

[4] W.B. Johnson, J. Lindenstrauss, D. Preiss and G. Schechtman, *Uniform quotient mappings of the plane*, Michigan Math. J., 47 (2000), pp. 15-31

[5] K. Kuratowski, *Topology, Vol. II*, Academic Press, New York, 1968.

[6] O. Maleva, *Additional remarks on the structure of level sets*, preliminary version

[7] B. Randrianantoanina, *On the structure of level sets of uniform and Lipschitz quotient mappings from $\mathbb{R}^n$ to $\mathbb{R}$*, submitted to Geom. Funct. Anal.