Disjoint paths and cuts in planar graphs

by

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Preface

A history of the disjoint paths problem

Finding disjoint paths in graphs is a classical problem, one of the most studied areas in graph theory. Many beautiful results, and many basic techniques, which affected strongly the theory of graphs, have been discovered during the last 50 years. Beside the theoretical results, the area has many real-life applications. One of them is VLSI (Very Large Scale Integration), which is a well developed field in science and technology nowadays.

The first result in this area goes back to 1927, when K. Menger proved that the maximum number of pairwise disjoint paths between a given source and given sink in a graph is equal to the minimum size of a cut separating source and sink. Then in 1956, Ford and Fulkerson gave a combinatorial method finding the maximun flow, which is known now as the max flow min cut algorithm. Ford and Fulkerson was also the first who considered the problem of finding *multicommodity flows* (instead of a pair of source and sink, they considered many pairs of them).

In 1963, T.C. Hu found an algorithm to 2-commodity flow problem, the algorithm gave a half-integer valued flow. He also remarked that it is unlikely that his method could be extended to more general cases and at that time the *linear programming* was the only tool available.

Then in the 1970s, the theorem of complexity originated from J. Edmonds, S.A Cook, and R.M. Karp's works showed that finding integer valued commodity flow is hard. In 1979 L.G. Khachiyan proved that with the ellipsoid method, linear programming can be solved in polynomial time, thus fractional multicommodity flow is polynomially solvable.

In the 1980s two breakthrough results were found by É. Tardos(1985): finding mincost flow in *strongly* polynomial time, and by N. Robertson, P.D. Seymour (1986): finding k vertex disjoint paths in a undirected graph in polynomial time with any fixed k. Since then until the early 1990's increasingly many related results, applications were published. Meanwhile, researches started to considered special cases when there are good characterizations, for the multi-flow problem. Many results of P. Seymour, H. Okamura, A. Schrijver, A. Frank, A. Sebő, A.V. Karzanov, M.V. Lomonosov and others have created a rich, interesting area.

In the 1990s the approximation algorithms became one of the fastest developing area in the theory of algorithms, many results about finding approximate multiflow were published, with quite many interesting techniques.

Recently, a new direction of research on multicommodity flows emerged. The so called *games on networks*, first studied by É. Tardos and T. Roughgarden, has quite many applications in computer science and economics.

Structure of the text

In this work, we are concerned with types of results in the 1980's and the early 1990's, the special cases when there are good characterizations. More concretely, we will consider the cases of the multi-flow problem, when the supply graph is planar, and in almost every case we also assume a parity condition. Many results are due to P. Seymour, H. Okamura and A. Schrijver. In the first chapter we summarize some basic theorems, and give some notation for the remaining parts. In the second chapter, a general technique and its applications will be considered. We show that in some special cases the existence of a fractional solution also implies the existence of integer valued one. In chapter three many results are proved. By proving a polar result - cut packing, we show the existence of fractional solution to the original one. Here we also have a technique dual to the one mentioned in chapter two. Another proof is given to one of the Okamura's theorems in chapter four. The node-capacitated routing problem is solved in chapter six, using some basic knowledge in the fifth chapter and the Okamura's theorem. In the last chapter we mention further results, extension, and some open problems.

New results

The work contains some new results and new proofs. New result in cuts packing realizing some distances (chapter 3), a characterization of the feasibility of the node-capacitated problem in ring network (chapter 6) and a new proof of Okamura theorem (chapter 4).

Thanks

I am grateful to András Frank, my advisor, for giving me this topic and for his help and encouragement. I also would like to express my thanks to Zoltán Király and László Szegő for their useful advice. Many thanks are due to Mihály Bárász for his help with the technical problems of Linux, to my classmates and to the Egerváry Research Group for their patient listening.

Chapter 1

Introduction

In this chapter, we discuss some basic facts and terminology on multiflows and disjoint paths in undirected graphs.

1.1 Disjoint paths, multicommodity flows

Given two undirected graph $G = (V, E); H = (T, F)$, where G is the *supply* graph, H is the *demand* graph, with $T \subset V$ which is called *terminals*. Edges in F are called demand edges, each path in G which connects two end-nodes of a demand edge is called *demand path*, sometimes we say: the path connects the demand edge.

The *disjoint paths* problem is the one of finding a set of disjoint demand paths, which connect all (or as many as possible of) the demand edges.

If every demand and supply edge has an integer valued capacity, the disjoint paths problem of the new pair of demand and supply graphs when every edge is multiplied as many times as its capacity is called the *edge capacitated routing* problem.

When the nodes of the supply graph have capacity, the problem of finding a set of demand paths, for which at every node of G, the number of paths containing it as an inner node does not exceed its capacity is called *node-capacitated routing*

A natural relaxation of the path packing problem is the *multicommodity* flow problem is as follows:

For $s, t \in V$ an $s - t$ flow is a $f : E \to \mathbb{R}_+$ if there exists an orientataion D of G such that f is an st flow in D. A multiflow is a function f on F such that for every $r = (s, t) \in F$ f_r is an $s - t$ flow.

Given a capacity function $g : E \to \mathbb{R}_+$, a demand function $h : F \to \mathbb{R}_+$, we say that an f multi-flow satisfies the demand function if the value of the flow f_r is not smaller than the demand value $h(r)$. The multiflow is *subject to g* if :

$$\sum_{r\in F} f_r(e) \leq g(e)\ \forall e \in E. \tag{1.1}$$

When there is a node capacity function d on V instead of the edge capacity, then instead of (1.1), we have the following:

$$\sum_{r\in F} f_r^v \leq d(v)\ \forall v \in V. \tag{1.2}$$

Here f_r^v denotes the value of the flow f_r "flowing cross" v, i.e using v as an inner node.

A multiflow which is subject to a edge capacity or a node capacity is sometimes called *fractional routing* or *fractional node-capacitated routing.* And it is called *integer valued* if, for every flow $r \in F$ f_r is integer valued, and the definition is similar to the *half integer valued* case.

1.2 Notation

Given a G and an H graphs, $V(G)$ denotes the node set, $E(G)$ de notes the edge set of G. If $V(H) \subset V(G)$, $G + H$ denotes the graph on $V(G)$, and the edges's are $E(G) \cup^* E(V)$ A graph *Eulerian* if every degree is even.

The edges between A and $V(G) - A$ is called a cut and is denoted by $\nabla(A)$. If both A and $V(G) - A$ induce a connected subgraph, then $\nabla(A)$ is called a *bond.*. It is well-known that any cut can be partitioned into bonds.

$d_G(A, B)$ denotes the number of edges of G with one end in $A - B$ and one end in $B - A$. Sometimes when there is a edge capacity function c, $d_c(A, B)$ denotes the capacity sum of edges of G with one end in $A - B$ and one end in $B - A$. We use $d_G(A), d_c(A)$ for $d_G(A, V(G) - A), d_c(A, V(G) - A)$.

If P is a path, a, b are two node on the path, P[a,b] denotes the subpath of P between a and b. $|P|$ refers to the length of P.

If S is a finite set, $X \subset S$ and $h : S \to \mathbb{R}$ is a funtion, we use the notation $h(X) := \sum_{x\in X} h(x)$, and h^+ denotes the fuction $x \to max\{0, h(x)\}$.

1.3 Cut condition, Japanese theorem

A simple necessary condition for the solvability of the edge-disjoint paths problem is the so called edge-cut criterion, which requires: $d_G(X) \geq d_H(X)$

for every subset $X \subset V(G)$. For the multicommodity flow problem, the edge-cut criterion requires

$$d_g(X) \geq d_h(X) \; \forall X \subset V. \tag{1.3}$$

The following useful claim is well-known and easy to proof.

Claim 1.1 *Let the supply graph G be connected. If the edge cut inequality (1.3) holds for every subset X of V for which both X and $V - X$ induce a connected subgraph of G, then it holds for every subset $X \subset V$*

While the cut criterion is sufficient in many cases, such as the 1-commodity flow problem, in general it is not sufficient, even for the existence of the fractional routing. The following important example shows this. Figure 1.1

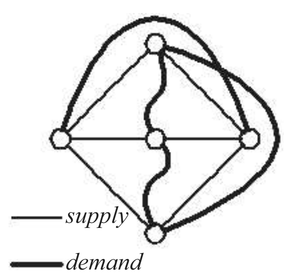

Figure 1.1: The cut criterion holds, but no fractional multiflow exists.

The fractional multiflow problem can easily be described as one of solving a systems of linear inequalities in the variables $f_r(a)$ for $r \in F, a \in E$. The constraint are flow conservation laws and the demand constraint for each flow f_r, together with the capacity constrains and the demand constrains. Therefore the fractional multiflow problem can be solved in polynomial time. Tardos [19] showed that the fractional multiflow problem can be solved in strongly polynomial time by proving that any linear programming problem with $\{0, \pm 1\}$ constraint matrix is solvable in strongly polynomial time. Onaga(1970) [12] gave the following good characterization for the feasibility of fractional routing, which can be derived from Farkas' lemma. This theorem is sometimes called the *Japanese* theorem in literature.

Theorem 1.2 *Given a demand function h on the edge set of a demand graph and a capacity function g on the supply graph, there is a feasible solution to* *the edge capacitated multi-flow problem fulfilling the demand iff the edge-distance criterion holds, that is, for every nonnegative function* $y : E \rightarrow \mathbb{R}_+$,

$$\sum_{e \in E} y(e)g(e) \geq \sum_{r \in F} h(r)dist_y(r), \tag{1.4}$$

where $dist_y(r)$ *denotes the minimum y-cost of a path connecting the two end nodes of demand edge r*

Theorem 1.2 can be formulated so as to the apply to the node-capacitated multicommodity flow problem as well.

Theorem 1.3 *Given a demand function h on the edge set of a demand graph* $H = (V, F)$ *and a capacity fuction c on the node set V of a supply graph* $G = (V, E)$, *there is a feasible solution to the node-capacitated multicommodity flow problem fulfilling the demands iff the node distance holds, that is for every nonnegative function* $y : V \rightarrow \mathbb{R}_+$,

$$\sum_{e \in E} y(v)c(v) \geq \sum_{r \in F} h(r)dist^*_y(r), \tag{1.5}$$

where $dist^*_y(r)$:=*min* $\{\sum_{v \in I(P) y(v)}$: *P a path in G connecting the end-nodes of* $r\}$. *That is,* $dist^*_y(r)$ *is the minimum total cost of internal nodes on path connecting the end nodes of r.*

1.4 The parity condition

In general cases, the existence of fractional routings does not imply the existence of integer valued ones, unlike the case when there is only one demand edge. The simplest example is the following: The demand graph is $C4$, the demand edges are two cross edges. Figure 1.2 Although there is a half-integer valued routing, no integer valued routing exists.

In almost every case, we assume that $G + H$ is Eulerian, without this assumption, we hardly know any good characterizations. Although, the Eulerian condition is helpful in some special cases, Middendorf and Pfeiffer [9] show that the half integer valued multiflow problem with all capacities and demand equal to 1 is NP complete. This indicates that the disjoint paths problem when Eulerian condition holds is still a hard problem.

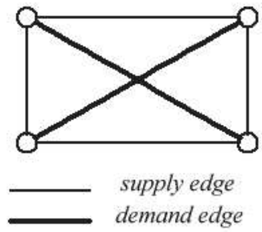

Figure 1.2: There are no integer valued routings

1.5 The polar problem - metric packing

Let Γ be a graph and let V be a finite set. A metric μ on V is called a Γ-*metric* if there is a function $\phi : V \to V(\Gamma)$ with :

$$\mu(u,v) = dist_\Gamma(\phi(u), \phi(v))$$

for all $u, v \in V$. Here $dist_\Gamma(x,y)$ denotes the distance of x and y in Γ. The Γ-*metric condition* for the existence of a feasible fractional multiflow:

$$\sum_{r=st\in F} h(r)\mu(s,t) \leq \sum_{e=uv\in E} g(e)\mu(u,v) \text{ for each } \Gamma\text{-metric } \mu \text{ on } V. \quad (1.6)$$

This is a specialization of condition (1.4).

The cut condition is in fact K_2-metric condition. $K_{2,3}$ is a complete bipatite graph on 5 nodes of which one class consists of 3 and the other has 2 nodes. The distance between two nodes in a class is 2, and between two nodes in different classes is 1. Since $K_{2,3}$ includes the cut condition, the class of supply, demand graphs for which the $K_{2,3}$-metric condition implies the existence of feasible fractional multiflow is larger than the one, when the cut condition is sufficient.

Let $\mathcal{M}$ be a finite class of graphs. $\mathcal{M}$-*metric condition* is the condition which includes all the M-metric condition, for every $M \in \mathcal{M}$.

Let us recall Theorem 1.2 a multiflow problem has a solution if and only if the distance criterion holds. Therefore if we want to show that in a certain case already the cut condition (or $\mathcal{M}$-metric condition) is sufficient, we have to show that cut condition (or $\mathcal{M}$-metric condition) implies the distance criterion. One way to do so is, roughly, to point out that the vector y in (1.4) can be expressed as a non-negative linear combination of cuts (or $\mathcal{M}$-metrics.) Precisely we have the followings:

Definition 1.1 *Let G, H be a pair of supply and demand graphs, w be a nonnegative weight function on E(G). We say w is* in the H-facet of the cone of $\mathcal{M}$-metrics , *if there exist* $\mu_1, \mu_2 \dots \mu_k \in \mathcal{M}$*-metrics, and* $\lambda_1 \dots \lambda_k \in \mathbb{R}_+$ *such that:*

$$w(u,v) \geq \sum_{i=1}^{k} \lambda_i \mu_i(u,v) \text{ for every } u,v \in V$$

and

$$dist_w(u,v) = \sum_{i=1}^{k} \lambda_i \mu_i(u,v) \text{ for every } (uv) \in \text{ demand edges } E(H)$$

Let w *be integer valued, if the* $\lambda_1 \dots \lambda_k$ *values above is integer valued then we say* w *is* $\mathcal{M}$-metrics packable into H.

Lemma 1.4 *Let* G, H *be a pair of supply and demand graphs and every non-negative weight function on* $E(G)$ *is in the* H*-facet of the cone of* $\mathcal{M}$*-metrics, then the* $\mathcal{M}$*-metric condition implies the existence of fractional routings.*

Proof. It is enough to show that if there is a vector y violating (1.4) then there is also a μ $\mathcal{M}$-metric violating (1.6). Let y be such a vector, we have:

$$\sum_{e \in E} y(e)g(e) < \sum_{r \in F} h(r) dist_y(r)$$

But y is in in the H-facet of the cone of $\mathcal{M}$-metrics:

$$\sum_{e \in E} \sum_{i} \lambda_i \mu_i(e) g(e) \leq \sum_{e \in E} y(e)g(e) < \sum_{r \in F} h(r) dist_y(r) = \sum_{r \in F} \sum_{i} h(r) \lambda_i \mu_i(r)$$

$$\Rightarrow \sum_{i=1}^{k} \lambda_i \left(\sum_{e \in E} \mu_i(e) g(e) - \sum_{r \in F} h(r) \mu(r) \right) < 0$$

this shows that there is a μ_j violating (1.6). $\square$

Chapter 2

Split off technique

2.1 Reduction principle

Let us introduce a simple device by which the edges disjoint paths problem can be reduced to a case when every degree in $G + H$ is at most 4. First replace each demand edge by a path of three edges, such that the middle edge is a demand edge, the two other are supply edges. As a result, no demand edge is incident to a node of degree bigger than 2. Next let v be a node with degree at least 5. Replace this node and the incident edges as shown in the Figure 2.1

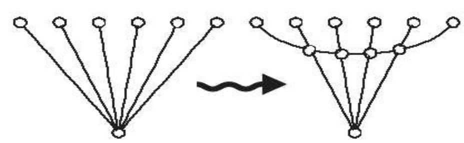

Figure 2.1: Reduction

It is easy to see that the edge-disjoint paths problem is solvable in the original graph if and only if it is solvable in the new graph. Apply this reduction at a node v as long as its degree is larger than 4.

The problem we obtain by eliminating all nodes of degree at least five is not only equivalent to the original problem, but its size is also a polynomial of the original size.

2.2 When fractional solution implies integer valued one

We list here all the known cases, when the supply graph is planar and cut condition and Eulerian condition imply the existence of the integer valued routing.

G is bipartite planar, I, O are two faces of G, p is a node on O. H is one of the following cases:

1. Such that $G + H$ is planar
2. Complete graph on $V(O)$
3. Complete graph on $V(I) \cup$ complete graph on $V(O)$
4. Complete graph on $V(O) \cup \{pv : v \in V(G)\}$
5. Complete graph on $V(O) \cup$ an edge $p_1 p_2$ where $p_1, p_2 \in V(G)$
6. Assume G is drawn on the plane in such a way that O is the outer face, I is a inner face. Let $a_1, a_2 \dots$ on the face O in the cyclist order, and $b_1, b_2 \dots$ on the I face and they are arranged in the reverse cyclist order. The edges of H consists of $a_i b_i$.

The first case is the theorem of Seymour [18]. The second was proved by Okamura and Seymour [10]. The third and fourth cases are due to Okamura [11]. The fifth case was proved by Schrijver and Gerards [5]. The sixth case was a result of Schrijver [14]. One could ask if there is a general theorem to all of these cases. In fact there is a reduction method which can be formulated as follow (This lemma can be found in [2]):

Lemma 2.1 *Let G be a planar supply graph, H be a demand graph such that $G + H$ Eulerian. G, H is one of the above cases. If the cut condition implies the existence of fractional routings, then it also implies the existence of integer valued routings.*

Proof. First, by the reduction principle described above, we assume that in $G + H$ every degree is 4. Let x be the solution to the corresponding multiflow problem (in either of the theorem...) If there is a path P and an inner node v of P such that $x(P) > 0$ and the two edges uv and vz of P are in the same face of G, the splitting off the edges preserves not only the cut criterion, but also the planarity. If no such a path exists, that is for every inner node v of any path P with $x(P) > 0$ goes across v, then for every terminal pair (s,t) there can be only one path P with $x(P) > 0$ connecting s and t. Consequently, x is 0-1 valued, that is x itself is a solution to the corresponding edge-disjoint paths problem.

Remark In fact the lemma is true not only for these six cases, but also true for all the planar cases, when the splitting off as described above preserves the properties of G and H. But the real question here is how to proved the fractional problems. In chapter 3 we will give such a method.

2.3 A thereom of Seymour

Now using the split off technique, we will prove the theorem of Seymour. This proof is in [2]

Theorem 2.2 (Seymour 1981) . *Suppose that $G + H$ is planar and Eulerian. Then the cut criterion is necessary and sufficient for the solvability of the edge-disjoint paths problem.*

Proof. (Z. Zubor 1989) We can assume that every edge $e \in E$ is in a tight cut since otherwise e can be moved from E into F without destroying the cut criterion. By the reduction principle we can assume that in $G + H$ every degree is 2 or 4. Suppose that $G + H$ is a counter-example with a minimum number of nodes of degree 4. Define $w : E \cup F \to \{+1, -1\}$ by:

$$w(e) := \begin{cases} +1 & \text{if } e \in E \\ -1 & \text{if } e \in F \end{cases}$$

The cut criterion is equivalent to: $d_w(X) \geq 0$ for every $X \subset V$. We need the following observation of A.Sebő.

Claim 2.3 *Let $A \subset V$ be a tight, i.e $d_w(A) = 0$, and define*

$$w'(e) := \begin{cases} w(e) & \textit{if } e \notin \nabla(A) \\ -w(e) & \textit{if } e \in \nabla(A) \end{cases}$$

Then $d_{w'}(X) \geq 0$ for every $X \subset V$.

Proof. We have $d_{w'}(X) = d_w(A \oplus X) - d_w(A) = d_w(A \oplus X) \geq 0$. ($A \oplus X$ denotes $(A - X) \cup (X - A)$.) $\square$

By *interchanging along* a cut C we mean an operation that replaces F by $F \oplus C$ and E by $E \oplus C$. By the claim, the theorem holds for $g + H$ iff it holds after interchanging along a tight cut.

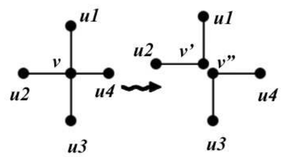

Figure 2.2: "Split off" operation

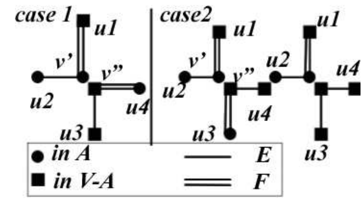

Figure 2.3: Two cases

Let vu_1 be a demand edge. Assume that the four edges $e_i = vu_i, (i = 1,2,3,4)$ incident to v are indexed in cyclic order so that $e_1 \in F, e_2 \in E$. Modify slightly the "splitting off" operation as follows. Replace v by v' and v'' so that v' is connected to u_1, u_2, and v'' is connected to u_3, u_4.

Let $G' = (V', E')$ and $H' = (V', F')$ denote the resulting graphs. If there were a solution to the edge-disjoint paths problem in $G' + H'$, there would be one in $G+H$. There by there is a bond $\nabla'(A)$ for which $d_{G'}(A) < d_{H'}(A)$. We can assume that $v' \in A$. Since the cut criterion holds for $G+H$ we have: $(*) v'' \notin A$ and an edge e_i belong to $\nabla'(A)$ precisely if $e_i \in F$.
There are two cases:

1. Case 1. $e_4 \in F$. By (*) $u_2, u_4 \in A$ and $u_1, u_3 \notin A$. Both A and $V' - A$ induce a connected subgraph of $G' + H'$ contradicting the planarity of $G' + H'$.

2. Case 2.$e_4 \in E$. By $(*) u_2 \in A$ and $u_1, u_4 \notin A$. Now $A - v'$ is tight in $G+H$. By interchanging along $\nabla(A - v')$ (and re indexing the $e_i' s$) we are at Case 1. $\square$

Chapter 3

The dual path method, cut packing

In this chapter we develop the dual path technique, which was introduced by Schrijver in [13] to prove that the cut criterion and the Euler condition are sufficientfor the existence of integer valued routings in cases 2, 3, 4, 5 introduced in Section 2.2. Due to Lemma 1.4, if every nonnegative weight on $E(G)$ is in the cone of cuts, then the cut criterion implies the existence of fractional solution. But these cases satisfy the conditions of Lemma 2.1, so it also implies the integer valued one.

In fact, a stronger result will be showed: Every *bipartite* nonnegative weight function is cut packable. (By bipartite weight function, we refer to an integer valued function, for which every circuit has an even weight). By dividing each edge e of the demand graph G into $w(e)$ edges, we can assume that the weight function is 0-1 valued.

We recall here the definition of "cut-packable" for a pair of supply-demand graph:

Definition 3.1 *We call a pair of supply-demand graphs (G,H)* cut-packable *if there exist pairwise edge-disjoint cuts so that for each demand edge of H h the distance between two end nodes of h in G is equal to the number of cuts separating them.*

Now, as showed above, to prove that the cut criterion and the Euler condition are sufficient for the existence of integer valued routings in cases 2, 3, 4, 5 introduced in Section 2.2, it is enough to prove the following theorem.

Theorem 3.1 *When G is bipartite planar, I, O are two faces of G, p is a node on O. H is one of the following cases:*

1. Complete graph on $V(O)$

2. Complete graph on $V(I) \cup$ complete graph on $V(O)$

3. Complete graph on $V(O) \cup \{pv : v \in V(G)\}$

4. Complete graph on $V(O) \cup$ an edge $p_1 p_2$ where $p_1, p_2 \in V(G)$

then (G, H) is cut-packable.

Proof of the theorem

In the remains of this chapter we will prove this theorem. The original proof of Schrijver was showed for the 2nd case [13]. The first case is a trivial consequence of the 2nd case. It was a result of Hurkens, Schrijver, and Tardos [6]. As we will see the technique used by Schrijver can be extended to the 3rd and 4th cases. The 4th case was announced by Gerards in [5].

Supose that the theorem is not true, and let G be a counterexample with

$$\sum_{F \neq O} 2^{e(F)} \text{ in the 1st, 3rd, 4th cases, and } \sum_{F \neq O, I} 2^{e(F)} \text{ in the 2nd case} \quad (3.1)$$

as small as possible.

Where the sum ranges over almost the faces, and $e(F)$ denotes the number of edges surrounding F. We may assume that O is the unbounded face.

G has no multiple edges: otherwise, either the circuit C formed by them is a face in which we can delete one of the edges, thereby decreasing the sum (3.1), or C contains edges both in its interior and its exterior, in which the case the graph formed by C and its interior or the graph formed by C and its exterior yields a counterexample with smaller sum.

We first show:

Claim 3.2 *Each face $F \neq O$ (in the first three cases), or $F \neq O, I$ (in the 2nd case) is quadrangle (i.e., $e(F) = 4$)*

Proof. Let F be some face forming a k-gon, with $k \neq 4$. Since G is bipartite and has no parallel edges, k is an even and $k \geq 6$. We make a counterexample with a smaller sum than (3.1) as follows. Let $v_1, \ldots, v_k$ be the vertices surrounding F. Add, in the interior of F, new vertices $w_1, \ldots, w_{\frac{k}{2}-2}$ and new edges $(v_1, w_1), (v_{k-1}, w_1), (w_i, W - i + 1)$ where $i = 1, \ldots \frac{1}{2}k - 3$, and $w_{\frac{1}{2}k-2}, v_{\frac{1}{2}k}$.

Note that this modification does not change the distance between any two vertices of the original graph. Therefore after this modification we have again a counterexample to the theorem, with however smaller sum than (3.1) (since $2^k > 2^{k-2} + 2^{k-2} + 2^4$), contradicting our assumption. □

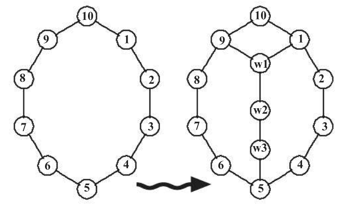

Figure 3.1: When there is a big face

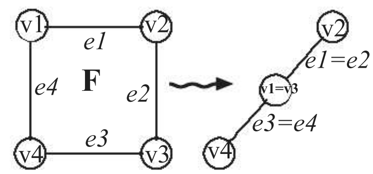

Figure 3.2: Get a smaller graph

Next we show:

Claim 3.3 *Let F be a face with $F \neq O, (\neq O, I$ in the 2nd case), and let $e_1 = (v_1, v_2), e_2 = (v_2, v_3) e_3 = (v_3, v_4) e_4 = (v_4, v_1)$ be the four edges surrounding F. Then there exist vertices v, w, with $(v, w) \in E(H)$, and a shortest path from v to w which uses both e_1, e_2.*

Proof. Suppose no such v, w exist. Identify v_1 and v_3, e_1 and e_2, e_3 and e_4. After this modification, all distances between vertices v, w that connects a demand edge are unchanged. Hence, the new graph is again a counter example. However, the sum (3.1) has decreased, contradicting our assumption. □

Dual path

Now we define *dual paths* $Q_1, \ldots, Q_t$ paths including circuits in the planar dual graph of G. These dual paths are determined by the following properties: each edge of the graph occurs exactly once in $Q_1, \ldots, Q_t$; if F $(\neq O)$ is surrounded by the edges e_1, e_2, e_3, e_4 in this order, then e_1, F, e_3 or e_3, F, e_1 will occurs exactly once of the Q_j; The faces O or (O, I in the 2nd case) only occurs as beginning or end faces in $Q_1, \ldots, Q_t$. More precisely, $Q_1, \ldots, Q_t$ are all possible sequence of the form

$$(F_0, e_1, F_1, e_2, \ldots, F_{k-1}, e_k, F_k) \tag{3.2}$$

satisfying:

– For $i = 1, .., k$: e_i is an edge separating the face F_{i-1} and F_i

– For $i = 1, .., k-1$: K_i is not O, (not O and I in the 2nd case), and e_i, e_{i+1} are opposite edges of F_i.

– Either F_o, F_k is O (O or I in the 2nd case of the theorem) or $F_0 = F_k$ and e_1, e_k are opposite edges of F_0.

Clearly, in the way described the edges of G are partitioned into dual paths and circuits. A path P and a dual paths Q intersect at e_i if $e_i \in P$.

Take a dual path Q:

$$Q = (F_0, e_1, F_1, e_2, \ldots, F_{k-1}, e_k, F_k)$$

Let $a_i, b_i, a_{i+1}, b_{i+1}$ where $i = 0..k$ be the nodes of the faces F_i, while f_i, g_i,e_i, e_{i+1} be the edges of F_i. Figure 3.3. Take a shortest path P, which intersects with Q at e_i, e_j assume that on the path P between these two edges, there are no common edges with Q. Call the area formed by P and sub-path between $b - i$ and b_j: S. (The shaded area in the Figure 3.3.) Now according to Claim 3.3 through g_i, e_{i+1} there exist a shortest demand path U connecting u and w: $(v, a_i, a_{i+1}, b_{i+1}, w)$. Call the subpath of U connecting b_{i+1}, w: α, and the subpath of U connecting a_i, v as β. With this notation, we have the following claims:

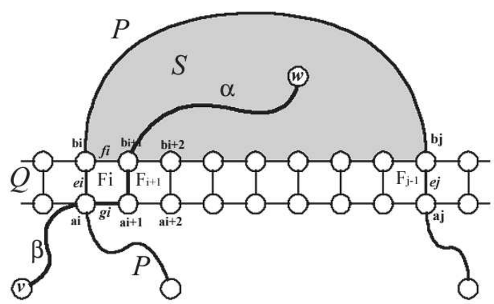

Figure 3.3: P and Q intersect

Claim 3.4 *Take a dual path Q, a shortest demand path P which intersects with Q at e_i, e_j. Let S set node in the area formed by P, Q like defined above. Then exist an end node of a demand edge which is in $S - P$.*

Claim 3.5 *Furthermore, if we choose (P, e_i, e_j) such that P and Q intersect at e_i, e_j and $|i - j|$ is as small as possible. Then $w \in S - P$ and $v \in V - S$*

Proof. (Claim 3.4)

First, we remark that if two shortest paths have two common nodes, then the two subpaths between two node have the same length. We prove the claim by induction with $|S|$ If $w \in S - P$ then we finished the proof. Now, if it is not the situation, that means α and $P[b_i, b_j]$ intersect or there exists $e_l \in \alpha, l \in [i..j]$, where $P[b_i, b_j]$ denotes the sub-path of P between b_i and b_j. In the latter case, we choose U instead of P, by induction, we finish. When exists $p \in \alpha \cap P[b_i, b_j]$ by changing the subpath $P[a_i, p]$ by $(g_i, e_{i+1}, \alpha[b_{i+1}, p])$, we also have a induction step. □

Proof. (Claim 3.5)

It is similar to the above proof to prove $w \in S - P$. Now with v, consider the sub path β, it could not contain an e_l $l \in [i..j]$ edge or a p node of $P[b_i, b_j]$. If $e_l \in \beta$, then the path $(\alpha, e_{i+1}, g_i, \beta)$ and Q intersect at e_{i+1}, e_l, which is not possible due to the choice of P. And when there is a node $p \in P[b_i, b_j] \cup \beta$, then the path P and the shortest path connecting v, w have three common nodes: ai, bi, p. We have:

$$|P[p, bi]| = |U[p, b_i]| = |U[p, a_i]| + 1 = |P[p, a_i]| + 1 = |P[p, b_i]| + 1 + 1.$$

which is a contradiction. □

Since in a planar graph, two nodes of a face cannot be separated by a cycle, we have the following Corollary:

Corollary 3.6 *v, w in Claim 3.5 cannot belong to O, or to a $F \in \{O, I\}$ in the second case of the theorem.* □

We are now ready to prove each case of the theorem separatedly.

3.1 The first case (Hurkens, Schrijver, Tardos)

As shown in Corollary 3.6, if there is a shortest path which intersects with a dual path at least twice, then exist a demand edge (v, w), $v, w \notin O$, contradicting our assumption that the demand edges only connect nodes on $V(O)$. This also implies that the dual paths are not self-intersections. Because, if a Q dual path self intersect at face $F = v_1, v_2, v_3, v_4$, the shortest path through v_1, v_2, v_3 and Q intersect twice.

Now a dual path (a circuit, or a path with the beginning, and ending face is O) makes a $\delta(X)$ cut in the original graph. This cut separates every shortest path at most once. By contracting all edges appearing in this cut, we have a smaller bipartite graph G', with the new distance function $dist'$, by the minimal property of G in G' exist pair wise disjoint cuts $\Delta := \{\delta(X_1), \ldots, \delta(X_t)\}$ satisfying for every demand edge (u, v), there are exactly $dist'(u, v)$ cuts separating them. But then $\Delta \cup \delta(X)$ are pair wise disjoint cuts in G satisfying that exactly $dist(u, v)$ cuts among them separate two nodes u and v on the face O. This is a contradiction to our indirect assumption.

3.2 The second case (Schrijver)

As in the first case, if there is a dual path, which is a circuit, or connects the same face, then we finish the proof.

Now every dual path connects O and I.

Claim 3.7 *No two distinct Q_1, Q_2 have a face $F \neq O, I$ in common.*

Proof. Suppose to the contrary, $Q_1 = (O, \sigma, F, \phi), Q_2 = (O, \tau, F, \psi)$, for strings σ, ϕ, τ, ψ and face $F \neq O, I$. We may assume that σ, τ do not have a face in common, by taking so that σ, τ have minimal length. This gives the following situation: Figure 3.4.

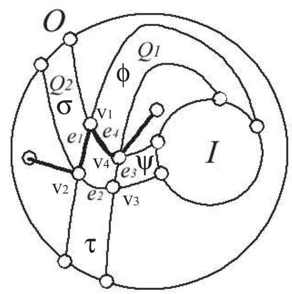

Figure 3.4: Intersecting dual paths

We may assume that e_1 is the last symbol of σ and e_2 is the last symbol of τ. By Claim 3.3, there exist vertices v, w both on O or both on I, and a shortest v, w path P using e_1, e_4. We know that P intersects with Q_1 and with Q_2 only once, at e_1, e_4. Now v and w belong to the same face I or O, But one can easily see that these are impossible.

This claim implies that there are no faces other than O, I(any other face would belong to two different dual path. So G is a simple circuit, for which the theorem trivial holds.

3.3 The third case

It is enough to prove that there exists a dual path, which does not intersect with any shortest demand path more than once. Since if exists such a dual path, by contracting the edges of the dual path, we will have a smaller counterexample, as explained in the first case.

First, we prove no shortest demand paths pv intersect with any dual path twice. Indirect, assume there are such paths. Consider a shortest path pv which intersect with a Q dual path at e_i, e_j such that $|i - j|$ minimal. Let S be the set of nodes in the area formed by P and Q . Now through a_j, a_{j-1}, b_{j-1} there is a shortest demand path U, but one of its end-node is inside S due to Claim 3.5, therefore, this demand path is pw. In this case

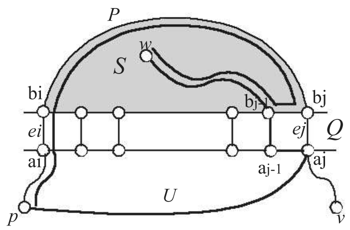

Figure 3.5: pv and Q do not intersect twice

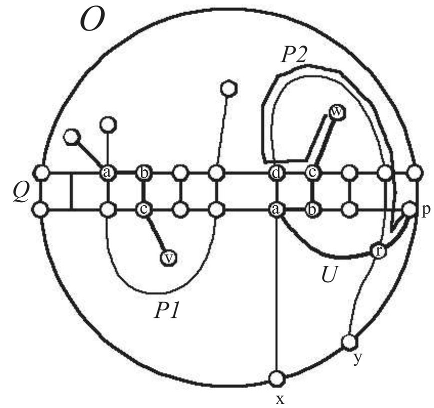

Figure 3.6: xy and Q do not intersect twice

we have: $|P[p, a_j]| = |U[p, a_j]|$, but then the path $(P[p, b_j], b_j b_{j-1}, U[b_{j-1}, w])$ is shorter than U, thus we have a contradiction. Figure 3.5. Now consider the dual path Q beginning with an edge of O incident with p. For the convenience, we draw Q as a simple dual path dividing the graph into the upper and the lower parts, and p is in the lower one. (When Q is not simple, i.e there are faces that appear twice in Q, the proof is the same).

Case 1: If there is a P_1 shortest demand path that intersects with Q twice, and the area S_1 between them is in the lower part of the graph, then consider the shortest demand path crossing a, b, c See Figure 3.6. One of its ends must be in side S_1, therefore, this demand path is pv, but in this case, a path from p to v crossing a, b, c and Q must intersect at least twice, contradicting what we have just proved.

Case 2: Now, if there is a shortest demand path $P2$ connecting x, y intersecting with Q twice, and the area S_2 between them is in the upper part of the graph. x, y must be in the lower part of the graph, otherwise, we return to the case 1. Again the path U through a, b, c ending at w inside S_2, therefore it begins at p. The subpath from p to a and P_2 intersect at r, we have : $|U[p, r]| = |P_2[p, r]|$, and then the path (p, r, d, c, w) is shorter than U, we have a contradiction.

3.4 The fourth case (Gerards, Schrijver)

Again we will show that there exists a dual path that does not intersect with any shortest demand paths more than once.

If a shortest path P connecting p_1p_2 and a Q dual path intersect more than once. Consider two intersecting edges e_i, e_j which are as close to each other as possible. ($|i-j|$ is as small as possible like in the Claim 3.4). One can see just like in that claim (using the same figure and notation), that there exist a demand edge (v, w) for which there is a shortest path connecting them using g_i, e_{i+1}. But then, v, w do not belong to O due to Corollary 3.6 , hence $(v, w) = (p_1, p_2)$, but that is also impossible since both $p_1, p_2 \notin S$

Assume that for every Q dual path, there is a demand shortest path P connecting $x, y \in O$ intersecting with Q at least twice.

If there is a dual path Q, which is a circuit. For the simplicity, let assume that Q is simple circuit, let a demand path connecting x, y intersect with Q twice. It divides the interior of Q in to S_1, S_2. Now according to Claim 3.4, there are nodes in S_1, and a node in S_2, which are terminals of demand edges. This only happens if these two node are p_1, p_2. So $p_1 \in S_1, p_2 \in S_2$. Take a shortest demand path P that intersects with Q at least twice, and the area formed by P and Q containing p_1 is minimal sized, similar to the previous proof, the shortest demand path through a, b, c. See Figure 3.7 connects p_1, p_2, but in this case, this path intersects with Q at least twice, contradicting the above: every shortest path connecting p_1p_2 intersect with a dual path at most once.

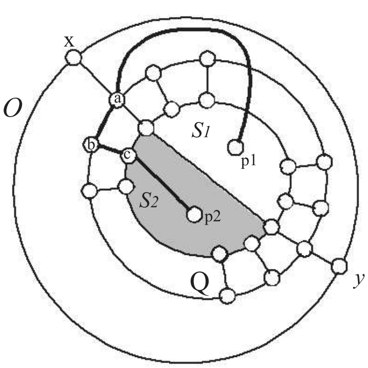

Figure 3.7: A dual circuit

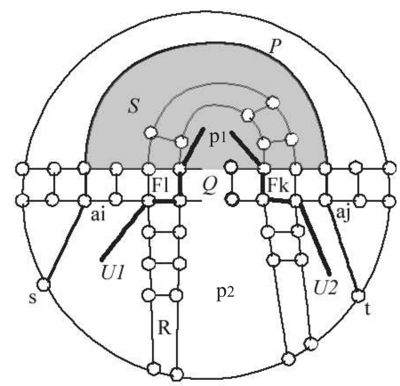

Figure 3.8: When a demand path and Q intersect twice

Now if there is a shortest path P connecting st intersecting a dual path Q more than once, where $s, t \in O$. Among them choose the $< P, Q >$ pair such that P and Q intersect at e_i, e_j and the area S formed by them is smallest. See Figure 3.8. Now the length of the sub path of P between a_i, a_j is shorter than the sub path between them along the dual path Q. Hence there exists a dual path R, which has common faces F_l, F_k with Q, where $l, k \in [i..j-1]$

and the subpath $P[a_i, a_j]$ and the subdual path of R between Fl, F_k do not intersect. Consider the shortest demand paths going through $F_l, F_k : U_1, U_2$. (See Figure 3.8). These could not connects nodes on V(O), since otherwise for example, (U_1, Q) if U_1 cross out the area bounded by Q and R through Q or (U1,R) if U_1 cross out through R will be a better choice, since the area formed by them is smaller than S. Now U_1, U_2 connect p_1 and p_2, in every situations of p_1, p_2 one of these paths intersect with Q or R at least twice, contradicting what we have proved before.

Chapter 4

Another proof of Okamura's theorem

In this chapter we will give another proof to the Okamura's theorem: The cut condition and the Eulerian condition are sufficient for the path packing problem of a planar graph and the demand edges are on one face, or have a fixed end-node on that face. This proof uses an equality of G. Tardos which can be found in [2]. G. Tardos used this to prove another theorem of Okamura. (The 3rd case introduced in Section 2.2)

4.1 Introduction

In this section we recall some notation, and basic background of the edge disjoint paths problem mentioned in the first Chapter. So given a supply graph G, demand graph H, we use $d_G(X)$ or $d_H(X)$ for the number of edges in G or in H , which has exactly one end-node in X. $d(A,B)$ $(d_G(A,B), d_H(A,B))$ denote the number of edges connecting A and B.

Cut condition: A trivial condition for the existence of the disjoint paths problem is the cut condition. $o(X) := d_G(X) - d_H(X) \geq 0 \; \forall X \subset V(G)$. When the equality occurs, we call X a *tight* cut. Although, in some cases this condition is enough, but more general condition is required, even for the existence of the fractional relaxing of the problem: the *distance condition.*

Non-separating condition: This condition is a special case of the distance condition. We call a node pair st is separated by tight cuts $X_1, \dots X_k$, if we delete the set $X := \bigcup_i X_i$ from the supply graph, s and t fall into two different components. Clearly, if there exist solutions to the disjoint paths problem, then no demand edge is separated by a tight cuts family.

Eulerian condition: In almost every known case where the cut condition is sufficient to the solution of the disjoint paths, the condition that $d_G(v) + d_H(v)$ is an even number for every node v is required.

Let A, B be sets of nodes in a graph, we have the following equality:

$$d(A) + d(B) = d(A \cup B) + d(A \cap B) + 2d(A, B) \qquad (4.1)$$

As a consequence we have the claims:

Claim 4.1 *Assume that the cut condition holds. If A, B are tight, $d_H(A, B)$ $= 0$ then $A \cap B$, $A \cup B$ are also tight. If $(A \cup B)$, $A \cap B$ are tight, $d_G(A, B) = 0$, then A, B tight and $d_H(A, B) = 0$.*

Proof. Using (4.1), we have:

$$o(A) + o(B) = o(A \cup B) + o(A \cap B) + 2d_G(A, B) - 2d_H(A, B).$$

From this is $d_H(A, B) = 0$ A, B are tight, then the left-hand side is 0 when the right-hand side is not negative. The equality occurs if $A \cap B$, $A \cup B$ are also tight and $d_G(A, B) = 0$. If If $(A \cup B)$, $A \cap B$ is tight, $d_G(A, B) = 0$, the right-hand side is not positive, while the left hand side is not negative, this can only happen when A, B are tight and $d_H(A, B) = 0$.□

Claim 4.2 *If (s, t) is a demand edge, the cut condition holds, then s and t cannot be separated by an X tight cut.*

Proof. Assume there is an tight cut X separating s, t. Let S, T be two disjoint sets of nodes of $G - X$ such that $S \cup T = G - X$, $s \in S, t \in T$. Let $A := S \cup X, B := T \cup X$, A, B satisfy the second part of Claim 4.1, then $d_H(A, B) = 0$, but st is a demand edge connecting A, B, we have the contradiction.□

The following equality is originated from G. Tardos. Suppose that the node set of a graph is partitioned into five sets: A, B, C, S, T then:

$$d(A \cup C) + d(B \cup C) + 2d(S, T) = d(A \cup T) + d(B \cup T) + 2d(S, C). \qquad (4.2)$$

From this we have the following claim:

Claim 4.3 (G. Tardos) *Assume that the cut condition holds, s, t are separated by X, Y tight cuts. Let S, T are two disjoint sets of nodes of $G - (X \cup Y)$ such that $S \cup T = G - (X \cup Y)$, $s \in S, t \in T$, $d_G(S, T) = 0$. $d_H(S, X \cap Y) = 0$, then st is not a demand edge.*

Proof. Let $C := X \cap Y, A := X - C, B := Y - C, S := S, T := T$, using (4.2), we have:

$$o(X) + o(Y) + 2d_G(S,T) - 2d_H(S,T) =$$
$$o(A \cup T) + o(B \cup T) + 2d_G(S,C) - 2d_H(S,C)$$

But $d_G(S,T) = 0$. $d_H(S,C) = 0$:

$$0 = o(X) + o(Y) = o(A \cup T) + o(B \cup T) + 2d_G(S,C) + 2d_H(S,T) \geq 0$$

From this $d_H(S,T) = 0$, and st can not be a demand edge. □

4.2 The theorem

Theorem 4.4 *Let $G = (V,E)$ be a planar supply graph, H be the set of demand edges such that there is a vertex q on the boundary of G with the property that each edge of H is spanned by the outer boundary of G or it contains q. Furthermore, G, H satisfy the Eulerian condition. Then the edge-disjoint paths problem has a solution if and only if the cut condition holds.*

Proof. The necessity being trivial. We show the sufficiency by induction with $|E|$. If there is a demand edge and a supply edge that span a same pair of nodes, then we can delete these edges without violating either the cut condition or the Eulerian condition. Thus by induction, we finish the proof. Now, take a demand edge (qr), r's neighbor nodes are $r_1, r_2, \ldots r_k$, we can assume that they are drawn on the plane in this order as we go around the node r in a clockwise cyclist order. We try to "go" from r to q through r_i by deleting rr_i supply edge, qr demand edge and adding qr_i as a new demand edge. It is easy to see that the new supply, demand graphs G', H' satisfy the Eulerian condition, $|E(G')| < |E(G)|$. If we can show that there is a r_i node such that after this operation the cut condition also holds, then by induction the proof is finished. Assume that is not the situation: Going from r to q through each r_i node violates the cut condition, we know that G, H satisfy the condition. So this could only happen when for every r_i there exists a *tight* cut X_i such that $r_i \in X_i$, $r, q \notin X_i$. (X is tight cut if $X \subset V(G)$, $d_G(X) = d_H(X)$.) Call the outer boudary of G: O. It is known that we can assume that $G[X_i]$ and $G - X_i$ are connected, thus, $G[X_i] \cap O$, if not empty, is an arc of O.

If all the X_i cuts do not intersect with O, then there are no demand edges between X_i cuts. By Claim 4.1 $X := \bigcup_i X_i$ is also tight. But then s, t are separated by X. Contradicting Claim 4.2. Now it is not the case. Imagine the graph is drawn on a disk such that the O is still the outer cycle, r is in the center of the disk, $r_1, r_2, \ldots r_k$ are on a smaller cycle with center at r: I . See Figure 4.1. Here we assumed that $r, r_i \notin O$, but the cases when r or $r_i \in O$, the two cycles intersect are proved exactly in the same way. Now each X_i cut is drawn as an connected area on the plane since its graph is connected. From a node r_i, we try to draw a curve to the outer boundary. If $X_i \cap O \neq \emptyset$ then we draw this line in the area of X_i. If $X_i \cap O = \emptyset$, then we draw a curve along closely with I from r_i to a point near r_{i+1} and check if $X_{i+1} \cap O \neq \emptyset$ then continue the curve to O inside X_{i+1}, else continue drawing the curve along I to a point near r_{i+2} and so on... We can also assume every two curves have no common sections. There is a tight cut belonging to each curve, for example, a curve c from x_i passes $x_{i+1}, x_{i+2}, \ldots x_{i+l}$ and goes to O, then the cut belonging is:

$$X_c := \bigcup_{1 \leq j \leq l} X_{i+j}.$$

Due to the description of the curve's drawing and Claim 4.1, X_c is tight.

Now imagine that we use scissors to cut the area between I and O along the curves. Every two cuts from r_i and r_j divide this area into at least two pieces, one of them contains q, call it Q. Q either contains no neighbors of r (in Figure 4.1 curves 1 and 2 are in this case) or r_i, r_j arc (curves 2 and 3 are in this case). It is easy to see if Q does not contain any neighbors of r, or when $i = j + 1$ and Q contains no neighbor of r except r_i, r_{i+1}, then the cuts belonging to these curves separate q and r. We will show now that there always exist two curves satisfying one of these cases. If the cuts belonging to theses curves are the same, then again, there is a tight cut that separates q and r, which is impossible due to Claim 4.2. If these cuts are different, we will show that they satisfy the conditions of Claim 4.3 and thus, we will have the contradiction, since qr is a demand edge.

Indirect, assume there are no such curves, that means after cutting along every two curves from r_i, r_j Q_{ij} contains r_i, r_j arc. Let r_i, r_j be the two nodes that on this arc there are as few neighbors of r as possible. Because of the indirect assumption, exists $r_l, l \neq i, j$ on this arc, thus we can draw a path from q to r_l inside Q_{ij}, of course this path is disjoint with the curves from r_i, r_j. Now consider the curve from r_l, if cutting along it, and along r_i, Q_{li} must contain $r_i r_j r_l$ arc due to the r_i, r_j choice. We now can draw a path from q to j which is disjoint with the curves from r_i, r_l. Similarly, we can also draw a path from q to i which is disjoint with the curves from r_j, r_l. But these are impossible, since two of these paths, for example qr_i,qr_j will "close"

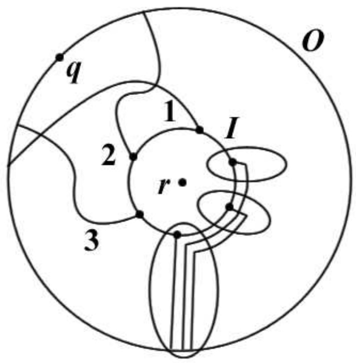

Figure 4.1: Curves and cuts.

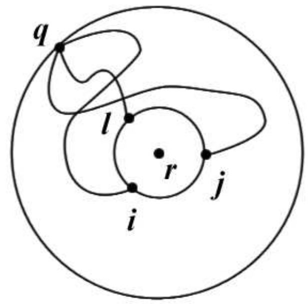

Figure 4.2: Paths from q

the third node r_l, and no curves connecting r_l and O exist which is disjoint with qr_i and qr_j.

Now it remains to show that the tight cuts belonging to the curves satisfy the conditions in Claim 4.3. Let these cut be X and Y. X, Y separate q and r, let R be the component containing r in $G - (X \cup Y)$. Because X is the union of a cut X_i where $X_i \cap O$ is an arc, and other cuts which are disjoint from O, then $X \cap O$ is also an a_X arc. Similarly $Y \cap O$ is an a_Y arc. If $a_x \cap a_Y = \emptyset$ then $(X \cap Y) \cap O = \emptyset$ thus there are no demand edges between $(X \cap Y)$ and R. If $a_x \cap a_Y \neq \emptyset$, then $a_x \cup a_Y$ is an arc not containing q, therefore $R \cap O = \emptyset$, otherwise there a path from r to q through $R \cap O$, contradicting the separation of X and Y. And also in this case, there are no demand edges between $(X \cap Y)$ and R. By this we finished the proof. □

Chapter 5

Linear inequality systems and bidirected graphs

5.1 Some definitions

In this chapter we will give necessary and sufficient conditions and also strongly polynomial algorithms for the feasibility of a special class of linear inequality systems.

Definition 5.1 *Let* $A = (a_{ij})$ *be an* $m \times n$ *matrix of* $\{1, -1, 0, 2, -2\}$. *We call* A simple *if* $\sum_i |a_{ij}| = 2 \ \forall \ 1 \leq i \leq m$.

It is known that the Fourier Motzkin elimination of the variables is a polynomial algorithm for finding solutions or integer valued solutions of $Ax \leq b$. Besides the algorithm, necessary and sufficient conditions are also known in Schrijver [15]. Here we will use the different terminology from the one in their papers. The algorithm and characterization is done with the help of a *bidirected representation graph* produced from the inequality system: Let G be a graph with n nodes, m edges. Each node corresponds to a column of A, while each edge corresponds to a row of A. Parallel edges and curves are allowed in G. Every edge has a *length or weight*, which is in fact the value of the proper coordinate of the vector b. For $(v_i, v_j) i \neq j$ node pair, a row of A where the values at i, j are not zero could be one of four cases: $(1, 1); (1, -1); (-1, 1); (-1, -1)$, and when in a row there is only one non-zero value it could be 2 or -2. Correspondingly, the edges in G can be: $\oplus$ signed; directed from j to i; directed from i to j; $\ominus$ signed; or a $\oplus$ signed curve; $\ominus$ signed curve.

An x satisfying $Ax \leq$ is also a *potential* function on $V(G)$. By potential function we mean a function on $V(G)$, satisfying $\pm x(v_i) \pm x(v_j) \leq b(e)$.

Here, $\pm$ is $+$ or $-$ depending on the kind of the edge e on (v_i, v_j) nodes. The following operation will make the directed edges and curves become $\oplus$ and $\ominus$ signed edges. For each (+1,-1) edge on v_i, v_j, then delete it, add an u node, and let (v_i, u) be $\oplus$ signed edge with a length of 0, (u, v_j) $\ominus$ signed edge with a length of the original edge. Do the same with (-1,+1) typed edge. For each (v_i, v_i) is a $+$ signed curve, delete it and add (v_i, a) $\oplus$ signed edge, (a, b) $\ominus$ signed edge with a length of 0 and (b, v_j) $\oplus$ signed edge with a length of the original curve. Do the same with $\ominus$ signed curves. It is clear that a potential on the new graph is also a potential on the original graph.

Let $Ax \leq b$ be an inequality system, where A is a simple matrix with every row $(+1, +1)$ and $(-1, -1)$ typed, and G be the representation bidirected graph without curves, and directed edges, b is the weight (length) function on $E(G)$. The definitions, lemmas, corollaries in this chapter are about them.

Definition 5.2 *In the bidirected graph a sequence* $(v_1, e_1, v_2, e_2, \ldots, e_k, v_{k+1})$ *is a* chain of edges *if:* e_i *is an edge on* v_i *and* v_{i+1}, e_i, e_{i+1} *are differently signed for every* $1 \leq i \leq k$. *(*v_i, v_j *or* e_i, e_j *are not necessarily different nodes). When* $v_i, v_j, i < j, i \neq 1, j \neq k+1$ *are different nodes we call the chain of edge* simple. *We call* v_1, v_{k+1} end-nodes, *while* e_1, e_k end-edges *of the chain.*

Definition 5.3 Closed walk *in the representation graph is a chain of edges* W, *of which the begining and the ending nodes* v_1, v_{k+1} *are the same, and* e_1, e_k *are differently signed. The close walk is called* negative *when* $b(W) < 0$, *or* tight *if* $b(W) = 0$. The graph *of the closed walk is the graph with nodes and edges appearing in the closed walk. Different closed walks may have the same graph.*

Definition 5.4 Double walk *is a closed walk in the representation graph, on which every node appears at most twice.*

Definition 5.5 Handcuff walk *is a closed walk:* $H = (P, K_1, P^-, K_2)$ *where* P, K_1, K_2 *are chains of edges, and* P^- *is the reverse chain of* P. *See Figure 5.1. The node connecting* P *and* K_1, P *and* K_2, *are called* roots *of* K_1 *or* K_2. *(When* P *is a node the two roots are the same).* Size *of the handcuff is defined by* $|K_1| + |K_2| + 2|P|$. *When* b *is integral function,* H *is called* integrally negative *if:*

$$\left\lfloor \frac{b(K_1)}{2} \right\rfloor + b(P) + \left\lfloor \frac{b(K_2)}{2} \right\rfloor < 0.$$

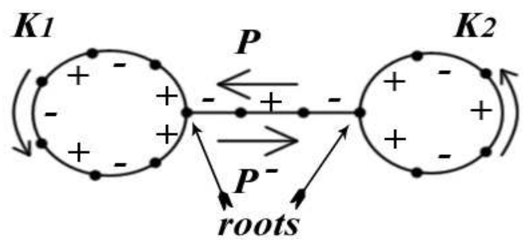

Figure 5.1: A handcuff.

Definition 5.6 Handcuff *is a handcuff walk* (P, K_1, P^-, K_2)*, where* K_1*,* K_2*,* P *are simple. Furthermore* K_i $(i = 1, 2)$ *and* P *have only one intersection node, which is the root of* K_i*.*

Remark The negative-ness and tightness are also valid for double walks, handcuff walks and handcuffs as just in Definition 5.3.

Definition 5.7 *Let* x *be a potential fuction of* G*. An* uv *edge is* tight *respected to* x *if* $x(u) + x(v) = b(uv)$ *if* uv *is* $\oplus$ *signed, and* $-x(u) - x(v) = b(uv)$ *if* uv *is* $\ominus$ *signed.*

Definition 5.8 *An edge of* G *is* strongly tight *if it is on a tight closed walk.*

Claim 5.1 *Let* x *be a potential of* G*,* W *be a closed walk and every edges in* W *are tight with respect to* x*, then* W *is tight. On the other hand, strongly tight edges are tight with respect to every potential function.*

Proof. Adding the x values along the edges of W, on one hand we have 0 because W is closed walk, on the other hand we get $b(W)$ because the edges are tight. From this W is tight.

It is clear that when x is a potential function, adding the x values along the edges of a tight W, the value on the left side is greater or equal than on the right side, but both are 0, The equality happens when for every W's edge the equalities occur. That mean W's edges are tight with respect to x. □

Claim 5.2 *Let* $H = (P, K_1, P^-, K_2)$ *be a tight handcuff, then for every potential functions* x, x' $x(v) = x'(v)$ *for every* $v \in H$*.*

Proof. H is tight, then every edge of H is tight with respect to x and x'. Adding the values of x, or x' along K_1 we get $\pm 2x(r_1) = b(K_1) = \pm 2x'(r_1)$. We have $x(r_1) = x'(r_1)$. Here r_1 is the root of K_1. Let v be a node on H, then exists a chain of tight edges L from r_1 to v. Adding x, x' values along L, depending on the signs of the two end edges of L we get one of the four cases:

$$\begin{aligned} x(r_1) + x(v) &= b(L) = x'(r_1) + x'(v) \\ x(r_1) - x(v) &= b(L) = x'(r_1) - x'(v) \\ -x(r_1) + x(v) &= b(L) = -x'(r_1) + x'(v) \\ -x(r_1) - x(v) &= b(L) = -x'(r_1) - x'(v). \end{aligned}$$

From this $x(v) = x'(v)$ for every $v \in H$. □

Claim 5.3 *If the bidirected graph G has no negative closed walks, b is integral, then a handcuff $H = (P, K_1, P^-, K_2)$ is only integrally negative if it is tight and $b(K_1)$ is odd.*

Proof. Indirect, assume that the claim is not true, we have one of three following cases which all contradict the assumption that H is integrally negative.

$b(H) \geq 2$ then :

$$\lfloor \frac{b(K_1)}{2} \rfloor + \lfloor \frac{b(K_2)}{2} \rfloor + b(P) \geq \frac{b(K_1) - 1}{2} + \frac{b(K_2) - 1}{2} + b(P) = \frac{b(H)}{2} - 1 \geq 0$$

$b(H) = 1$ then $b(K_1) \not\equiv b(K_2) \pmod 2$, let say $b(K_1)$ is odd, we have :

$$\lfloor \frac{b(K_1)}{2} \rfloor + \lfloor \frac{b(K_2)}{2} \rfloor + b(P) = \frac{b(K_1) - 1}{2} + \frac{b(K_2)}{2} + b(P) = 0$$

$b(H) = 0$ and $b(K_1) \equiv b(K_2) \equiv 0 \pmod 2$ then :

$$\lfloor \frac{b(K_1)}{2} \rfloor + \lfloor \frac{b(K_2)}{2} \rfloor + b(P) = \frac{b(H)}{2} = 0. \ \square$$

5.2 Characterizations

Lemma 5.4 *The followings are equivalent:*

1. *$Ax \leq b$ has solutions.*
2. *There are no negative closed walks in the representation graph.*

3. *There are no negative double walks in the representation graph.*

And if b is integral, then $Ax \leq b$ has solutions also implies that it has half-integral solutions.

Proof.

$1 \Rightarrow 2$ If we have a solution x, and a negative closed walk W, adding the value of x along W. On the left side, we get 0 and on the right side, we get the weight of the closed walk, which is negative. But this is a contradiction because the left side is smaller then the right one.

$2 \Rightarrow 3$ is trivial.

$3 \Rightarrow 1$ To proof the sufficient condition, we define an auxiliary directed graph G^* as follows. Double every v node of the representation graph G into v_1 and v_2. If an edge uv of G is $\oplus$ signed then in G^* let $(u_1, v_2), (v_1, u_2)$ be directed edges. $(v-2, u_1), (u_2, v_1)$ be directed edge while uv is $\ominus$ signed. Their weight are the weight of (uv).

One can easily check that the non-negativeness of double walks in G is equivalent with the non-negativeness of directed cycle in G^*, which guarantees that there is a potential function π on $V(G^*)$ satisfying $\pi(s) - \pi(t) \geq b(ts)$ if st is an directed edge. Now let :

$$x(v) \stackrel{def}{=} \frac{\pi(v_2) - \pi(v_1)}{2}.$$

It is not difficult to see that x is a potential function of the bidirected graph, and is a solution of $Ax \leq b$.

Besides, it is clear that if b is integral, then π can be chosen as integer, and then x is half-integral. □

Lemma 5.5 *b is integral, $Ax \leq b$ has solutions. The followings are equivalent:*

1. *$Ax \leq b$ has integral solutions.*

2. *There are no integrally negative handcuff walks in G.*

3. *There are no integrally negative handcuffs in G.*

4. *There are no tight handcuffs in G (P, K_1, P^-, K_2) with $b(K_1)$ odd.*

Proof. :

1 ⇒ 2: Indirect, assume that there is an integral solution x, and an $H = (P, K_1, P^-, K_2)$ integrally negative handcuff walk. Adding the values of x along K_1 we get $\pm 2.x(r_1) \leq b(K_1)$. (Where r_1 is the root of K_1). Thus, $\pm x(r_1) \leq \lfloor \frac{b(K_1)}{2} \rfloor$. Here $\pm x(r_1)$ refers to $x(r_1)$ or $-x(r_1)$ depending on the sign of K_1's edges at r_1. Similarly if r_2 is root of K_2 then $\pm x(r_2) \leq \lfloor \frac{b(K_2)}{2} \rfloor$. Adding these inequalities with the values of x along P, we will have:

$$0 \leq \left\lfloor \frac{b(K_1)}{2} \right\rfloor + \left\lfloor \frac{b(K_2)}{2} \right\rfloor + b(P).$$

This contradicts the fact that H is integrally negative.

2 ⇒ 1: b is integral, $Ax \leq b$ has solutions. According to the Lemma 5.4, there is a half-integral solution x. Let G_1 be the nodes of G where the x is integral, and G_2 be the nodes of G where x is half-integral. For $v_1 \in G_1, v_2 \in G_2 : x(v_1) + x(v_2)$ is not integral, so the edges connecting v_1, v_2 are not tight, we can add $\pm\frac{1}{2}$ to each $x(v), v \in G_2$ without violating the inequalities on edges connecting G_1, G_2.

Let v be a node of G_2, add $\frac{1}{2}$ to $x(v)$. To maintain the inequalities, we have to add $-\frac{1}{2}$ to $x(u)$ for all $\oplus$ signed tight edges with two end nodes are u and v, then we have to add $\frac{1}{2}$ to $x(w)$ for all $\ominus$ signed tight edges ending at u and w, and so on. This repairing operation works if there is not a node r_1, where we have to add both $\frac{1}{2}$ and $-\frac{1}{2}$ to. That means there are P_1, K_1 chains of tight edges, such that v and r_1 are end-nodes of P_1, K_1 ends at r_1 and its two end-edges have the same sign, which is different from the sign of the end-edge of P_1 at r_1. If this r_1 exists, then we try to add $-\frac{1}{2}$ to $x(v)$, if we cannot then there is a "bad" P_2, K_2 chain of tight edges similar to the first case. But now $P_2 \cup P_1, K_1, P_1^- \cup P_2^-, K_2$ is an integrally negative handcuff walk. (Figure 5.2)

3 ⇒ 2: is trivial

2 ⇒ 3: It is enough to show that if there is an integrally negative handcuff walk, then there is also an integrally negative handcuff. Let $H = (P, K_1, P^-, K_2)$ be an integrally negative handcuff walk. If there is a node v appearing more than twice, let v_1, v_2 be two places that v appears on K_1. Going around K_1 in clockwise order, the root of K_1 and v_1, v_2 divide K_1 into l_1, l_2, l_3. See Figure 5.3. If the *changing order*

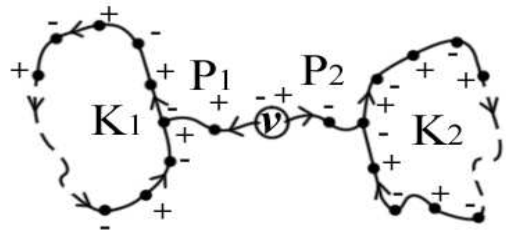

Figure 5.2: When the repairing does not work.

of edges' signs (by this we mean the changing of signs of two edges from ⊖ to ⊕ or from ⊕ to ⊖) at v_1, v_2 are the same then l_2 is a closed walk, $b(l_2) \geq 0$. By deleting l_2 from H, we get a smaller integrally negative handcuff walk. Now if at two places of v the changing oder of the edges' signs are different. Then, if $b(l_1) \geq b(l_3)$ delete l_1 from H, let $K_1^{'} := l_2, P^{'} := P \cup l_3, K_2^{'} := K_2$, by this we get a smaller integrally negative handcuff walk.

We also know that the H handcuff walk can be also observed as $(P \cup K_1 \cup P^-, K_2)$, (instead of K_1 we consider $P \cup K_1 \cup P^-$). That is why when a node appears more than once in $K_1 \cup P$, the same deleting operation can be applied. Continue doing this, until we get an H handcuff.

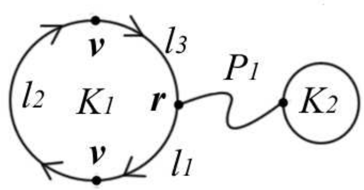

Figure 5.3: A handcuff from a handcuff walk

3 ⇔ 4: It is clear due to the Claim 5.3. □

Remark The proofs of Lemma 5.4 and Lemma 5.5 also yield an trongly polinomial algorithms for finding solution or integral solution of $Ax \leq b$.

Corollary 5.6 *If b is even, then the existence of solution to $Ax \leq b$ also implies the existence of integral solutions.* □

Definition 5.9 *Assume that b is even, $Ax \leq b$ has solutions. The parity of a tight handcuff $H = (P, K_1, P^-, K_2)$ is defined by:*

$$parity(H) := \frac{b(K_1)}{2} \pmod 2.$$

Claim 5.7 *Assume that b is even, $Ax \leq b$ has solutions. Every two tight handcuffs which have common nodes have the same parity, and as a consegence the parity of H only depends on its graph.*

Proof. According to the Corollary 5.6, then exists an x_0 integral solution to $Ax \leq b$. The parity of H is in fact the parity of $x_0(r_1)$, (r_1 is the root of K_1). Besides, every edge of H is tight with respect to x_0, and has even-length, then the $parity(H) \equiv x_0(v) \pmod 2$ for every $v \in V(H)$. Tight handcuffs H_1, H_2 have a u common node, then $parity(H_1) \equiv x_0(v) \equiv parity(H_2) \pmod 2$.

It is clear that the parity of H only depends on its description because if there is another tight handcuffs $H^{'}$ on the same graph, then $H^{'}$ has common nodes with H, and therefore their parity is the same. □

Remark From the definition, the parity does not depend on x_0. But one can also see this by Claim 5.2.

Theorem 5.8 *$Ax \leq b$ has solution, b is even. $Ax \leq b$ has a solution such that $x_i \equiv x_j \pmod 2$ if and only if there are no two disjoint tight handcuffs $H, H^{'}$ where $parity(H) \neq parity(H^{'})$. Furthermore, if x be a solution, no tight edges with respect to x connect H and $H^{'}$.*

Proof. The proof of the necessary condition is trivial. Let $v, v^{'}$ be the nodes of H and $H^{'}$. $parity(H) \equiv x(v) \pmod 2$, $parity(H^{'}) \equiv x(v^{'}) \pmod 2$. But $x(v) \equiv x(v^{'}) \pmod 2$, we have $parity(H) \equiv parity(H^{'}) \pmod 2$.

$Ax \leq b$ has an x solution such that $x_i \equiv x_j \pmod 2$ if one of the following two cases is satisfied:

Case 1 : $Ax \leq b$ has an even solution. This means $Az \leq \frac{b}{2}$ has integral solution. Applying Lemma 5.5: for every (P, K_1, P^-, K_2) tight handcuff (with respect to $\frac{b}{2}$), $\frac{b}{2}(K_1)$ is even.

Case 2 : $Ax \leq b$ has an odd solution. This means $Az \leq \frac{b}{2} - 1$ has integral solution. Applying Lemma 5.5: for every $H = (P, K_1, P^-, K_2)$ tight handcuff (with respect to $\frac{b}{2} - 1$), $(\frac{b}{2} - 1)(K_1)$ is even. But $|K_1|$ is odd, we have $\frac{b}{2}(K_1)$ is odd.

Furthermore, if a closed walk is tight with respect to a function f, then it is also tight with respect to $\alpha f + \beta \ \forall \alpha, \beta \in \mathbb{R}$. A handcuff is also tight with respect to b if it is tight with respect to $\frac{b}{2} - 1$ or $\frac{b}{2}$. So we have, in the first case $parity(H) = 0$ and in the second case $parity(H) = 1$ for every H tight handcuff.

It remains to prove that if we have two tight handcuffs H, H' where $parity(H) \neq parity(H')$, then they are disjoint, and no tight edges with respect to x connect them. The disjointness is clear due to Claim 5.7. Now let x_0 be a integer valued solution (x_0 exists because of Corollary 5.6). If $v \in V(H), v' \in V(H')$, due to Claim 5.2, $x_0(v) = x(v), x_0(v') = x(v')$, on the other hand $x_0(v) = parity(H) \neq parity(H') = x_0(v')$, that means $x(v) \neq x(v') \pmod 2$. Besides, every edge has even length, that is why there are no tight edge with respect to x connecting H, H'. □

The following observation is important for our main theorem in the next chapter.

Theorem 5.9 *In a bidirected graph G, let H is a minimal handcuff. By minimal handcuff, we understand an H handcuff such that there are no smaller sized handcuff on $V(H)$. M is a set of $\ominus$ signed edges on the H's nodes, which don't have the roots of H as end nodes. Then for every $f \in M$ one of the three positions in the Figure 5.4 is permitted. (In the proof we will explain in details the figure.)*

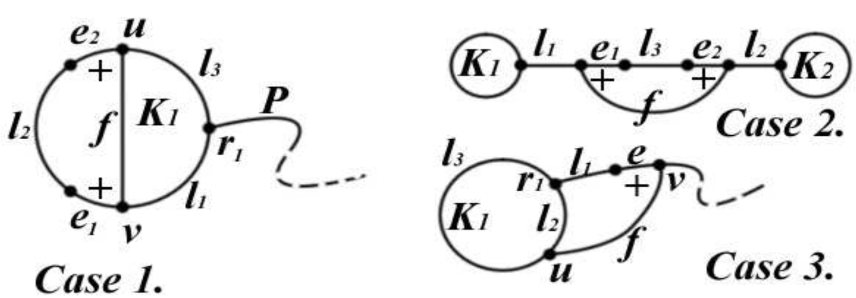

Figure 5.4: Cases of a $\ominus$ signed edge on minimal handcuff.

Proof.

Case 1: When f connects two nodes of K_1 (or K_2). f divides the K_1 in to two cycles. One contains the root and the other does not. Let e_1, e_2 be the neighbor edges of f on the cycle which does not contain the root r_1. Together with r_1, e_1, e_2 divide K_1 in to l_1, l_2, l_3. If e_1, e_2 are

$\ominus$ signed, then delete e_1, e_2, l_2 from $H \cup f$ we could obtain a smaller handcuff. And when e_1 is $\ominus$ signed, e_2 is $\oplus$ signed, then delete l_3 we also get a smaller handcuff, which both contradict the fact that H is minimal. That means e_1, e_2 are $+$ signed.

Case 2: When f connects two nodes of P. Adding f in to H, we get a cycle made by f and a sub path of P. Let e_1, l_3, e_2 be this sub path, such that e_1, e_2 are neighbor edges of f. l_1, l_2 are the remaining sub paths of P. Here we can also have that both e_1, e_2 are $\oplus$ signed. It is true because: If e_1, e_2 are $\ominus$ edge, then delete $e_1 \cup\ l_3 \cup e_2$ from $H \cup f$, or if e_1, e_2 are different signed, for example e_1 is $\oplus$ signed, e_2 is $\ominus$ signed, then delete $K_1 \cup l_1$ from $H \cup f$, we will have a smaller handcuff on $V(H)$.

Case 3: When f connects one node v of P and one node u of K_1. (The situation is the same as K_2). Let r_1 is the root of K_1. u, r_1 divide K_1 into l_2, l_3. The sub path of P connecting r_1 and v is $l_1 \cup e$ where e is the neighbor of f. If e is $\ominus$ signed, then we can delete either l_2 or l_3 to get a smaller handcuff. That means e is $\oplus$ signed. $\square$

Chapter 6

Node capacitated routing in ring networks

6.1 Introduction

In the previous chapters we discussed about the edge-disjoint paths problem. Many results were mentioned. One could ask what about the node disjoint version? As a mater of fact, few results are known about vertex-disjoint paths in undirected graphs. We think that it is a hard problem, since it generalizes the edge disjoint one when one divides every edge of the supply graph by a node and consider the node constraints on these instead of the edge constraints.

In this chapter we will give necessary and sufficient conditions and strongly polynomial algorithm for the node-capacitated multiflows problem on ring networks. As we will see, the condition includes a criterion for existence of fractional solutions based on Farkas's lemma and a criterion of parity. But first, let us formulate the exact problem.

The problem

Given a ring supply graph, by a ring we mean an undirected graph $C = (V, E)$, where $V = v_1, v_2..v_n$ and $E = e_1, e_2, .., e_n$ and $e_i = v_i v_{i+1}, i = 1, .., n$ (Notation $v_{n+1} = v_n$). We will intuitively think that nodes of G are drawn in the plane in a clockwise cyclic order. We are also given a demand graph $H(V, F)$ with the demands $h : F \to \mathbb{R}_+$ and through-node capacities $c : V \to \mathbb{R}_+$. For a demand edge f, $\mathcal{P}_f$ denotes the set of paths in the ring graph connecting the end-node of f. and $\mathcal{P}$ denotes the set of paths in the ring.

As in the introduction, we define an $x : \mathcal{P} \to \mathbb{R}_+$ a solution to the commondity flow problem with node-capacity constraints if satisfying the demand condition:

$$\sum \{x(P)|P \in \mathcal{P}_f\} \geq h(f) \; \forall f \in F \tag{6.1}$$

and the node-capacity constraint with respect to c:

$$o_x(v) := \sum \{x(P)|v \text{ is an inner vertex of } P\} \leq c(v) \tag{6.2}$$

When c, h are integer valued, x integer valued (half-integer valued) solution of the multicommodity flow problem is called *integral routing half-integral routing*

Some approaches

The edge-capacity version of the problem was well studied. So a function $u : E \to \mathbb{N}$ is given and the node-capacity constraint (6.2) should be replaced by the edge-capacity constraint with respect to u :

$$o_x(e) := \sum \{x(P)|e \text{ is an edge of } P\} \leq u(v) \; \forall e \in E. \tag{6.3}$$

In [10], H. Okamura and P.D. Seymour proved that the cut-condition:

$$u(e_i) + u(e_j) \geq L_h(e_i, e_j) \;\; \forall \;\; 0 \leq i < j < n. \tag{6.4}$$

is sufficient for the existence of half integral routings. Where $L_h(e_i, e_j)$ is the load of the cut e_i, e_j, that is the sum of demands between two components of G after deleting e_i, e_j. Furthermore, if $d_{u+h}(v)$ is even for every $v \in V$, then the cut condition (6.4) also indicates the existence of integral routings.

A. Frank in [1] gave a necessary and sufficient for the edge-capacitated integral routing: There exists integer valued routing of the edge-capacitated ring network iff the cut condition holds and and for any two crossing saturated edge-cuts e_i, e_j and e_k, e_l where $i < k < j < l$

$$2|L_h(e_i, e_k) + u(e_i) + u(e_k). \tag{6.5}$$

The problem becomes more complicated when one considers the node capacities instead of the edge capacities. As mentioned in the introduction, by Farkas' lemma, one could extract conditions for the existence of the feasible path packing, the *node-distance condition*:

$$r(y) := \sum_{v \in V} y(v)c(v) - \sum_{f \in F} h(f)s(f, y) \geq 0, \quad \text{for every } y : V \to \mathbb{R} \tag{6.6}$$

In [4], necessary and sufficient conditions for the existence of half-integral routings were proved. The condition is (6.6) restricted to $y : V \to \{0, 1, 2\}$.

Some definitions and notation

In fact, we will show later that it is enough to check the condition for vectors of $\{0,1,2\}$ that every two consercutive coordinates' sum is not greater than 2. We use the same terminology as in [4] , so an vector y of $\{0,1,2\}$ is a *double cut* if $y(v_i)+y(v_{i+1}) \leq 2$ for every $1 \leq i \leq n$. A double cut y is *tight* if in (6.6) equality occurs.

We call function value $r(y)$ defined by (6.6) the *redundancy* of cut y.

Let v_i be a node, e_j be an edge of the ring network. Assume that for every solution $x: \mathcal{P} \to \mathbb{R}_+$ of the multiflow problem,

$$\sum(x(P)|v_i \text{ is an inner node of } P) = c(v_i)$$

and there is no positive valued paths using both e_j and v_i, here v_i is not necessarily an inner node. We call node-edge pair (v_i, e_j) *good pair*.

For a good pair (v_i, e_j), we call the value $L_h(v_i, e_j) - c(v_i) \pmod 2$ as the *parity* of (v_i, e_j)

6.2 A parity condition

Let (v_i, e_j) is a good pair, then for every solution x of the multiflow problem, no positive valued paths contain both v_i and e_j, thus we have:

$$\sum(x(P)|e_j \text{ is an edge of } P) = L_h(v_i, e_j) - c(v_i) \tag{6.7}$$

where $L_h(v_i, e_j)$ is the load of the cut (v_i, e_j), that is $L(v_i, e_j) = d_h(R, L)$, here R, L are the two components of G after deleting e_j and v_i from the graph.

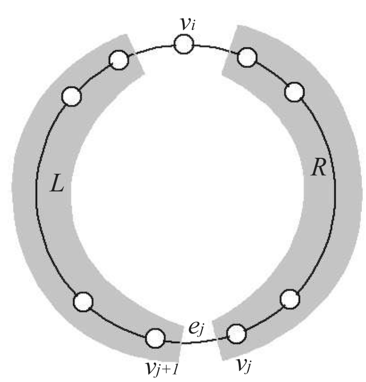

Figure 6.1: A good pair

Observe that, for an x integral routing,

$$\sum(x(P)|e_j \text{ is an edge of } P) + \sum(x(P)|e_l \text{ is an edge of } P) + L(e_j, e_l)$$

is always an even number, so if we have two good pairs (v_i, e_j) and (v_k, e_l) and assume that there exists integer valued solution of the multiflow problem (routing), then:

$$2|(L_h(v_i, e_j) - c(v_i)) + (L_h(v_k, e_l) - c(v_k)) + L_h(e_j, e_l). \qquad (6.8)$$

One could ask: how can good pairs be found? Here we give some examples of good pairs.

Claim 6.1 *Let y be a tight cut, for which $y(v_i) = y(v_j) = 1, y(v_k) = 0 \forall k \neq i, j$. Also assume that no demands connect (i, j). Then all the (v_i, e_j), (v_i, e_{j-1}),(v_{i-1}, e_j), (v_{i-1}, e_{j-1}) are good pairs.*

Claim 6.2 *Let y be a tight (double) cut. v_i, e_j pair satisfies: i) $y(v_i) > 0$, ii) $y(L) > y(R) - y(v_j)$, iii) $y(R) > y(L) - y(v_{j+1})$, then (v_i, e_j) is a good pair. Here R, L are defined as above. (Figure 6.1).*

Proof. y is a tight (double) cut then for every x solution of the multiflow problem, the x sum of the path using a node v_i, where $y(v_i) > 0$ is exactly $c(v_i)$. Furthermore, for every P, where $x(P) > 0$, $y(P) \leq y(C - P)$. This is true because at every solution the positive valued paths are the shortest with respect to y, since y is a tight double cut.

It is easy to check that in the above cases if there is a positive valued demand path P using both e_j and v_i, then $y(P) > y(C - P)$, contradicting what we explained above. □

We say that the *parity condition* holds if for every two good pairs described in Claim 6.1 and Claim 6.2, condition (6.8) holds.

6.3 The main theorems

In the remain of the chapter, we will reprove the theorem of A. Frank, V. Zoltán, B. Shepherd, V. Tandon in [4] and show that the double cut condition and the parity condition discribed above are sufficient for the existence of integral routings. That is we will finish the following theorems:

Theorem 6.3 (A. Frank, V. Zoltán, B. Shepherd, V. Tandon) *Let C be the ring network with node capacity function c, H be the demand graph with demand function h. There is a solution to the node capacitated multi-flow problem if and only if the double cut criterion holds. If in addition, both h and c are integer valued, then the double cut condition is necessary and sufficient for the existence of a half-integral routing.*

Theorem 6.4 *Let C be the ring network with node capacity function c, H be the demand graph with demand function h, both c and h are integer valued. If double cut criterion and parity condition hold, then there exists an integer valued solution.*

Before prove these, let us give some lemmas and theorems.

A derivation to the edge constraints

In [4] the authors derived the node-capacitated routing problem from the the edge-capacitated one. For the completeness we provide this proof.
Claim 6.5 *Given c and h, there is an Eulerian demand function h' so that the routing problem has solutions (half integer valued or integer valued solutions) with respect to and h if and only if it does with respect to c and h'. Furthermore, $r_{ch}(y) = r_{ch'}(y)$ for every y.*

Proof. Let v_i and v_{i+1} be two subsequent nodes in the ring. As a one-edge path connecting v_i and v_{i+1} has no inner node, increasing the demand between them does not affect the solvability of the routing problem.

If $d_h(v)$ is not even everywhere, then there are two nodes v_i and v_j so that both $d_h(v_i)$ and $d_h(v_j)$ are odd and so that $d_h(v_l)$ for each l with $i < l < j$. We can then increase by one the h-values on all demand edges $(v_iv_{i+1}), (v_{i+1}v_{i+2}), \ldots, (v_{j-1}v_j)$ and this way both $d_h(v_i)$ and $d_h(v_j)$ is even while all other $d_(v_l)$ values remain even for $i < l < j$. Repeating this procedure, we obtain the desired Eulerian function h'. It's clear that $s(v_lv_{l+1}, y) = 0 \ \forall y$, for which the $r(y)$ function does not change. □

Lemma 6.6 *Let $g : E \to \mathbb{R}$ be a function satisfying the cut condition and*

$$g(e_{i-1}) + g(e_i) \leq d_h(v_i) + 2c(v_i) \text{ for every } v_i \in V \qquad (6.9)$$

Suppose that a path packing $x : \mathcal{P} \to \mathbb{R}_+$ fufills the demand h (6.1), and satisfy the edge-capactity constraint with respect to g^+ (6.3). Then x satisfies node-capacity constraint (6.2). Moreover, if g is integral valued and there is an edge $e \in E$ with $g(e) \leq 0$, then x is integer valued.

Proof. Let us consider an arbitrary node v_i. If $g(e_i) \leq 0$ then $g(e_i) = 0$ and, hence, $x(P) = 0$ for every path P using e_i. Therefore, $x(P) = 0$ for every path P using v_i as an inner node, that is, in this case (6.2) is satisfied for v_i. The same argument works when $g(e_{i-1}) \leq 0$.

In the remaining case, $g(e_i) > 0$ and $g(e_{i-1}) > 0$. For each i, let $P(v_i)$ be the set of paths that have v_i as an end point. Let

$$\alpha := \sum_{\substack{P \in \mathcal{P}(v_i) \\ e_i \in E(P)}} x(P) \quad \text{and} \quad \beta := \sum_{\substack{P \in \mathcal{P}(v_i) \\ e_{i-1} \in E(P)}} x(P).$$

Then $\alpha + \beta \geq d_h(v_i)$ we have $o_x(v_i) + \alpha \leq g^+(e_i) = g(e_i)$ and $o_x(v_i) + \beta \leq g^+(e_{i-1}) = g(e_{i-1})$. By combining these with (6.9) we obtain $\alpha + \beta + 2o_x(v_i) \leq g(e_{i-1}) + g(e_i) \leq d_h(v_i) + 2c(v_i) \leq \alpha + \beta + 2c(v_i)$, from which $o_x(v_i) \leq c(v_i)$ follows, that is, x satisfies (6.2).

Let $g(e) \leq 0$ for some $e \in E$ that is $g^+(e) = 0$. Let f be any demand edge and P_1, P_2 the two paths connecting its end nodes. If P_1 contains e then $x(P_1)$ must be zero, and hence, $x(P_2) = h(f)$, an integer. □

Theorem 6.7 *Given a ring C, demand graph H with arbitrary demand function h and node-capacty function c.*

a) *The feasibility of the multi-flow problem in node-capacitated ring networks, defined by (6.1) and (6.2) is equivalent with the existence of z in the inequality system (6.10).*

b) *If h, c are both integer valued, h is Eulerian, then existence of z in the system (6.10) indicates the existence of a half-integral routing.*

c) *If h, c are both integer valued, h is Eulerian, then the existence of integral routing is equivalent with the existence of integral z in the system (6.10) and (6.11).*

$$\begin{array}{rcll} z(e_{i-1}) + z(e_i) & \leq & d_h(v_i) + 2c(v_i) & \forall 1 \leq i \leq n \\ -z(e_i) - z(e_j) & \leq & -L_h(e_i, e_j) & \forall 1 \leq i < j \leq n \end{array} \tag{6.10}$$

$$z(e_i) \equiv z(e_j) \pmod 2 \text{ for every } 0 \leq i < j < n. \tag{6.11}$$

Proof. :

a) Let x be a solution to the multi-flow problem. For $e \in E$ define $z(e_i) = \sum(x(P) : e \in P \in \mathcal{P})$, that is $z(e_i)$ denotes the sum of paths using e. Obviously z satisfies the edge-cut condition, as x is an edge feasible routing with respect to z. Let v_i be a node of C, and $\alpha := \sum(x(P) : P$ using v_i as an inner node). Because the paths determined by x that end at v_i uses exactly one of e_i and e_{i-1}, their sum is $d_h(v_i)$. The number of paths using e_{i-1} and e_i is $z_{i-1} + z_i = d_h(v_i) + 2\alpha$, The node condition($\alpha \leq c$) together with the edge-cut condition follows (6.10).

For the reverse direction, let g be a solution of the system (6.10). The second inequality of (6.10) shows that g satisfies the edge-cut condition, $g^+ \geq g$ also satisfies it. According to the theorem of H. Okamura and P.D. Seymour, there exists an x routing with respect to the edges constraint of g^+. But due to Lemma (6.6), x then, is also a routing with respect to the node constraint c.

b) When h is Eulerian, $-L_h(e_i, e_j)$ and $d_h(v_i) + 2c(v_i)$ are even for every $i \neq j$. Due to Corollary 5.6 the feasibility of (6.10) also indicates that it has integral solution. Let g be an integer valued solution of (6.10), then g^+ is also integer valued. Similarly to **a)**, there exist an x routing, with respect to g^+, and x could be chosen as a half-integer value. Again, due to Lemma (6.6), x is also a routing with respect to the node constraint c.

c) Similarly to **a)**:

If x is integer valued then z (defined as just like in **a)**) is also integer valued. And $z(e_i) \equiv z(e_j) \pmod 2$ for every $0 \leq i < j < n$ because $z_{i-1} + z_i = d_h(v_i) + 2\alpha$ and h is Eulerian.

Now let g be an integral solution of (6.10) and (6.11). If $g(e_i)$ is positive for every e_i, $g^+ = g$ satisfies the edge-cut condition and from (6.11) condition, $d_{g+h}(v_i) = g(e_{i-1}) + g(e_i) + d_h(v_i)$ is Eulerian. According to Okamura Seymour theorem, exists an x integral routing satisfying the edge capacity constrant g^+, and also the node capacity constrant c. When $g(e) \leq 0$ for an $e \in E$, using the Lemma (6.6), x is then, also an integral routing. □

Connection between double cuts and double walks

Let G be the representation graph of (6.10), b is the weight(length) function on the edges of G, where b is the vector on the most right side of (6.10). Let p_i be the node of G corresponding to the i-th column, γ_i be the $\oplus$ signed edge connecting (p_{i-1}, p_i), δ_{ij} be the $\ominus$ signed edge connecting (p_i, p_j). Imagine that G is drawn together with C on the plane in such a way that p_i node lies on the ring's arc between v_i and v_{i+1}. (See Figure 6.2.)

We can say γ_i corresponds to v_i, while δ_{kl} corresponds to the edge-cut (e_k, e_l).

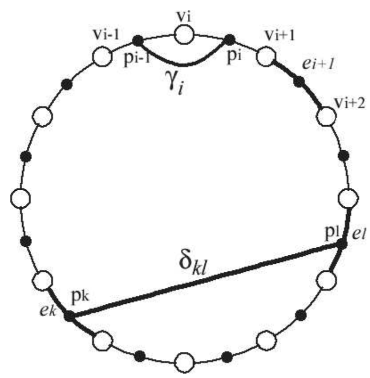

Figure 6.2: The ring and the representation graph.

Let Y be the set of double cuts of the ring network C, $\mathcal{D}$ be the set of double walks in the bidirected graph G. The following lemma is about the relation between $\mathcal{D}$ and Y.

Lemma 6.8 *There exists a function $\Phi : \mathcal{D} \to Y$, such that for every $D \in \mathcal{D}, b(D) \geq 2r(\Phi(D))$. That is the weight of the double cut y is not smaller than two times the redundancy of the double cut $\Phi(y)$.*

Proof. Let $D \in \mathcal{D}$. We define the Φ function as follows: The $y := \Phi(D)$ vector's i-th co-ordinate be the times γ_i appears in D as a $\oplus$ signed edge. Clearly, y is double cut because in D no p_i nodes appear more than twice, which is equivalent with $y(v_i) + y(v_{i+1}) \leq 2$.

It remains to prove $b(D) \geq 2r(\Phi(D))$. Let:

$$D^+ = \{\gamma_{i_1}, \gamma_{i_2}, \ldots, \gamma_{i_k}\}$$

be the $\oplus$ signed edges in D, then let:

$$D^- = \{\delta_{i_1 i_2}, \delta_{i_2 i_3}, \ldots, \delta_{i_{k-1} i_k}, \delta_{i_k i_1}\}$$

be the $\ominus$ signed edges of D.

Let $f = v_i v_j \in F$ be a demand edge. For convenience, we assume that the ring is drawn in the plane in such a way that f is vertical and v_i is above

v_j. (See Figure 6.2.) We say that a $\ominus$ signed edge of G *crosses* f if one of its end nodes in on the left-hand side and the other is on the right-hand side.

Let $\alpha(f)$ denotes the number of edges in D^- crossing f. By counting in two ways the number of (edge-cuts$(e_i, e_j), f)$ pairs where $\delta_{ij} \in D^-, f \in F$ and f crosses δ_{ij}, we have:

$$\sum_{\delta_{ij} \in D^-} L_h(e_i, e_j) = \sum_{f \in F} \alpha(f)h(f).$$

We claim first that :

$$\alpha(f) \leq 2s(f, s) + y(v_i) + y(v_j). \tag{6.12}$$

To see this , by symmetry, we may suppose that: $y(I(P_{v_i v_j} \leq y(I(P_{v_j v_i})$. That means the distance on the right side is not greater than on the right side. It is clear that:

- $\alpha(f) \leq$ *numbers of edges of* D^- *with at least one end on the right-hand side.*

Because the double walk D is a chain of edges, one $\oplus$ signed edge has two $\ominus$ signed neighbor edges, we have:

- *Numbers of edges of* D^- *with one end on the right-hand side* $= 2\times$ *number of edges of* D^+ *with both ends on the right-hand side* + *number of* D^+ *edges with one end on the right side.*

But:

- *Number of edges of* D^+ *with both ends on the right-hand side* $= y(I(P_{v_i v_j})) = s(f, y)$.

- *Number of edges of* D^+ *with one end on the right-hand side* $= y(v_i) + y(v_j)$.

From these, (6.12) is proved.

By (6.12):

$$\begin{aligned}\sum_{f \in F} \alpha(f)h(f) \leq \sum_{f = v_i v_j \in F} h(f).[2s(f, y) + y(v_i) + y(v_j)] = \\ = 2\sum_{f \in F} h(f)s(f, y) + \sum_{v \in V} y(v)d_h(v).\end{aligned} \tag{6.13}$$

Now:

$$\begin{aligned}
b(D) &= \sum_{\gamma_i \in D^+} (d_h(v_i) + 2c(v_i)) - \sum_{\delta_{ij} \in D^-} L_h(e_i, e_j) = \\
&= \sum_{v \in V} y(v)(d_h(v) + 2c(v)) - \sum_{f \in F} \alpha(f) h(f) \geq (\text{ using } (6.13)) \\
&\geq \sum_{v \in V} y(v)(d_h(v) + 2c(v)) - 2\sum_{f \in F} h(f) s(f, y) - \sum_{v \in V} y(v) d_h(v) = \\
&= 2\left(\sum_{v \in V} y(v) c(v) - \sum_{f \in F} h(f) s(f, y)\right) = 2r(y). \ \square
\end{aligned}$$

Corollary 6.9 *Assume that there exist solutions to the multiflow problem. Let W be tight double walk, then the corresponding double cut y is tight.*

Proof. Due to Lemma 6.8 we have: $0 = b(W) \geq 2r(y) \geq 0$. From this y is tight. $\square$

Proof of Theorem 6.3

The necessity of Theorem 6.3 is trivial. It remains to prove the sufficency: According to Theorem 6.7-a the existence of the multiflow problem in node-capacitated ring networks is equivalent to (6.10), so if it has no solutions, then (6.10) has no solutions, either. That means, from Lemma 5.4, there is a negative D double walk in G, but then $y := \Phi(D)$ is a double cut that violated the double cut condition, since due to Lemma 6.8: $r(y) \leq \frac{b(D)}{2} < 0$.

When h, c is integer valued, let h' be the Eulerian function as defined in Claim 6.5. If the double cut condition is satisfied with respect to c, h, it is also satisfied with respect to c, h'. Then there are no negative double walk in G. That means (6.10) has solutions. And due to Theorem 6.7-b, there is a half-integer valued path packing. $\square$

Basic handcuffs

We give now a definition of a special class of handcuffs on the representation graph G, which are described in Figure 6.3.

Definition 6.1 *Using the drawing of the representation graph G in Figure 6.2 We call a $H = (P, K_1, P^-, K_2)$ handcuff in G graph* basic *if its $\ominus$ signed edges connect nodes in two disjoint arcs (p_{i-1}, p_j) and (p_{k-1}, p_l). K_1, K_2 are triangles with two $\ominus$ signed edges. Every $\ominus$ signed edge of P intersects with each other, and intersects with every $\ominus$ signed edge of K_1 and K_2.*

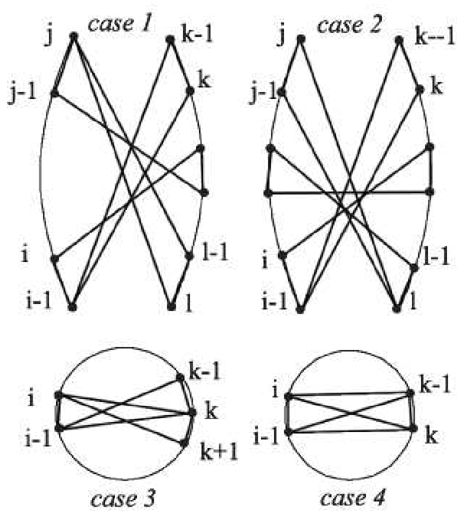

Figure 6.3: Basic handcuffs

Furthermore, when $|P| > 1$*, depending on the roots of* H *are on the same arc or not* H *is like in* case 1*, and* case 2 *in Figure 6.3. When* $|P| = 1$*,* H *is a graph on 5 nodes like in* case 3*:* $P = (p_i, p_{i-1}), K_1 = (p_i p_{k+1} p_k), K_2 = (p_{i-1} p_{k-1} p_k)$*. or a complete graph on 4 nodes like in* case 4*:* $P = (p_i, p_{i-1}), K_1 = (p_i p_{k-1} p_k), K_2 = (p_{i-1} p_{k-1} p_k)$*.*

We will need the following observation: Let x_0 be an solution to the multi-flow node-capacitated problem of the ring network, then for $e \in E$ let $z_0(e_i) = \sum(x_0(P) : e \in P \in \mathcal{P})$. Consider the representation graph G, and this z_0 potential function. We claim:

Claim 6.10 *For* i, j, k, l *lying in clock-wise order of the ring. If two* $\ominus$ *signed edges* δ_{ij}, δ_{kl} *are tight with respect to* z_0 *then* δ_{ab} *is also tight with respect to* z_0 *for every a,b lying on the ring in such a way that* i, j, a, k, l, b *are in clock-wise order. As special cases* δ_{ik}, δ_{jl} *are also tight. Let* s, t *be indices such that* i, j, s, k, l, t *are in clock wise order of the ring, then no demands connect* v_s, v_t*.*

(So if there are two parallel tight $\ominus$ *signed edges respect to* z_0*, then every* $\ominus$ *signed edge lies between them is also tight and there are no demands lying between them).*

Proof. A δ_{ij} is tight with respect to z_0 if and only if there is no P path for which $x_0(P) > 0$ that contains both $e_i = (v_i v_{i+1})$ and $e_j = (v_j v_{j+1})$. So δ_{ij}, δ_{kl} are tight with respect to z_0 means there is not a path P, $x_0(P) > 0$

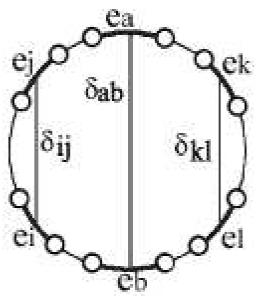

Figure 6.4: Parallel tight edges

that P contains both e_i and e_j or both e_k and e_l. But in this case P can not contain both e_a and e_b from which δ_{ab} is also tight with respect to z_0. Furthermore, in this case, no demands connect v_s, v_t, when i, j, s, k, l, t are in clock wise order of the ring. See Figure 6.4. □

Lemma 6.11 *Assume that h is Eulerian, and the multiflow problem has solutions. Let H, H' be the two tight handcuffs with different parity in the representation graph G, such that they are* minimal *sized. (H is a minimal handcuff, if there are no smaller sized handcuff on $V(H)$), then H, H' are basic.*

Proof. Let δ_{ab} be an $\ominus$ signed edge of H'. δ_{ab} divides C into two arcs, A_1, A_2. For the convenience let suppose δ_{ab} be drawn vertically, and A_1 be on the left-hand side, A_2 be on the right side. Let $H = (P, K_1, P^-, K_2)$. If a δ_{ij} $\ominus$ signed edge of H is parallel with δ_{ab} then due to Claim 6.10, either δ_{ia} or δ_{ib} is tight edge, which connects H and H', contradicting Theorem 5.8. (Two tight handcuffs with diffirent parity are not connected by a tight edge). This means every $\ominus$ signed edge of H connects nodes in two different arcs.

Suppose that there are two $\ominus$ signed parallel edges δ_{ij}, δ_{kl} of H. If they are both in P, K_1 or K_2, or one is in P, and the other one in K_1 or K_2, then we can connect nodes of H with further $\ominus$ signed tight edges. It is an easy practice to show that at least one of those is not situated as discribed in Theorem 5.9(Figure 5.4). Then we can get a smaller handcuff on $V(H)$. And due to Claim 5.7, this handcuff has the same parity with H. Therefore, we get a contradiction to the assumption that (H, H') are minimal handcuffs with different parity. So we have showed that there are no such two parallel $\ominus$ signed edges on H and H'.

Now consider an L simple chain of edges, of which every $\ominus$ signed connects A_1 and A_2 and intersects each other, the $\oplus$ signed edges are on the arcs. See Figure 6.5. Suppose L starts with (p_{i_1}, p_{j_1}). (For convenience we just write the indices (i_1, j_1)). From j_1 there is an $\oplus$ signed edge, by symmetry assume

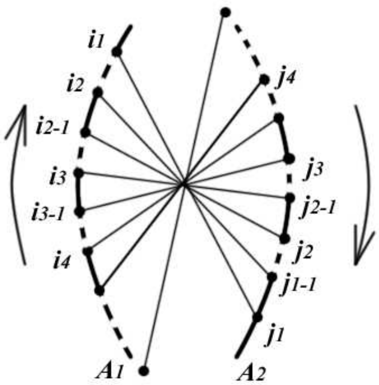

Figure 6.5: When the $\ominus$ signed edges intersect.

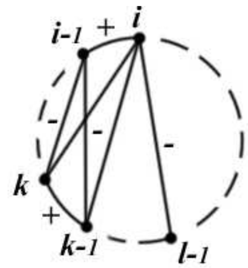

Figure 6.6: When there are parallel edges.

that it is (j_1, j_1-1), the next edge is (j_1-1, i_2), which crosses $(i_1 j_1)$. Consider then the $\oplus$ signed edge from i_2, it should be (i_2, i_2-1) since if it is (i_2, i_2+1) then the next $\ominus$ signed edge from i_2+1 will parallel with either (i_1, j_1) or (i_2, j_1-1). The $\ominus$ signed edge from i_2-1 now is (i_2-1, j_2), again from j_2 the only chance for $\oplus$ signed edge to be set is $(j_2 j_2-1)$ in order to maitain the intersection condition for the next $\ominus$ signed edge. As we see the i_t are more and more far from j_t and i_1. So if L is a cycle then it is a triangles with two $\ominus$ signed edges.

When $|P| > 1$ using the above explanation first for P, then for K_1, K_2, which are triangles with the $\ominus$ signed edges intersect with $\ominus$ signed edges of P. If the two end nodes of P are on the same arc, then we have *case 1* of Figure 6.3, when they are in different arcs we have *case 2*.

When $|P| = 1$, that is P is a $\oplus$ signed edge: (p_{i-1}, p_i), K_1, K_2 are triangles.

- If there aren't two $\ominus$ signed edges of $K_1 \cup K_2$ which are parallel then we have the figure as in the case 1 (with $(i = j)$)) when K_1, K_2 are disjoint, and *case 3* where they have a common node $p_k = p_{l-1}$.

- If there are two parallel $\ominus$ signed edges. Let the two other $\oplus$ signed edges of H is (p_{k-1}, p_k) and (p_{l-1}, p_l), assume $l-1, k-1, k, i-1, i$ are on clock-wise order. See Figure 6.6. If $k = l$, H is the K_4- *case 4* of Figure 6.3. If $k \neq l$, because there are two parallel $\ominus$ edges, K_1, K_2 could not be $(p_{i-1} p_l p_{l-1})$ and $(p_i p_{k-1} p_k)$, then $(p_{i-1} p_k)$ and $(p_i p_{l-1})$ are $\ominus$ signed the edges of H. Because they are edges of a tight handcuff, they are tight. therefore $(p_i, p_{k-1}), (p_{i-1}, p_{k-1}), (p_i, p_k)$ are also tight, due to Claim 6.10. But then complete graph K_4 on $p_{i-1}, p_i, p_{k-1}, p_k$, has all the edges tight, thus it is a tight handcuff, which is smaller than H. We have a contradiction. □

Lemma 6.12 *Assume that h is Eulerian and the multiflow problem has solutions. Let y be the corresponding double cuts of tight basic handcuff H. Then pair (v_l, e_j), if H is in case 1; pair (v_j, e_l), if H is in case 2; pair (v_k, e_{i-1}), when H is case 3 and pair (v_i, e_k), if H is case 4 are good pairs. And their parity is exactly the parity of H.(Figure 6.7).*

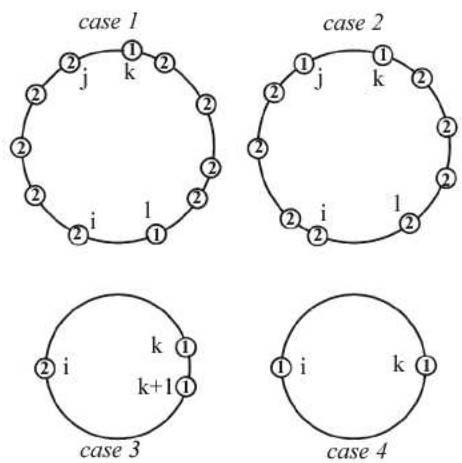

Figure 6.7: Corresponding cuts of basic handcuffs

Proof. It is easy to check that these are good pairs. Considering the tight double cut y, the 1st, 2nd, 3rd cases are in fact the case of Claim 6.2.

For the 4th case, if we show that there are no demand between v_i and v_j, then this good pair is exaclty the case of Claim 6.1. It is true since consider a tight hancuff H of the 4th case in Figure 6.3, then in H $\delta_{i,k-1}$ and $\delta_{i-1,k}$ are strongly tight, these two edges are parallel, thus there are no demands connecting v_i and v_j.

Let $H = (P, K_1, P^-, K_2)$ be a tight basic handcuff. The parity of H is defined as $\frac{b(K_1)}{2} \equiv \frac{b(K_2)}{2} \pmod 2$ when the weight b is even, (Definition 5.9). Here K_1, K_2 are triangules, thus for example in the first case of Figure 6.3, let consider $K_1 = (p_j p_l, p_{l-1})$:

$$\begin{aligned} \text{parity}(H) &= \frac{b(\delta_{j,l}) + b(\delta_{j,l-1}) + b(\gamma_l)}{2} && \pmod 2 \\ &= \frac{-L_h(e_j, e_l) - L_h(e_j, e_{l-1}k) + d_h(v_l) + 2c(v_l)}{2} && \pmod 2 \\ &= -d_h(R, L) + c(v_l) && \pmod 2 \\ &= \text{parity}(v_l, e_j) \end{aligned}$$

here R, L are the two components of the ring C after deleting v_l, e_j from C.

For the remaining cases, the proof is the same. □

Proof of Theorem 6.4

The necessity is true due to the explanation in Section 6.2. It remains to show the sufficiency.

Due to Claim 6.5, it is enough to prove the sufficiency when h is Eulerian. Now indirect, we assume that the problem has solutions but has NO integral solutions. Because of Theorem 6.7, (6.10) does not have such a z solution that $z(v_i) \equiv z(v_j) \pmod 2$. That means, due to Theorem 5.8, in the represenation graph G, there exists H, H' tight handcuffs with different parity. Among them take the minimal H, H' pair.

Due to Lemma 6.11 H, H' are basic. Consider the good pairs obtained from the corresponding tight double cuts as in Lemma 6.12, they have different parity. That means $(L_h(v_i, e_j) - c(v_i)) + (L_h(v_k, e_l) - c(v_k))$ is an odd number, but h is Eulerian thus,

$$(L_h(v_i, e_j) - c(v_i)) + (L_h(v_k, e_l) - c(v_k)) + L_h e_j, e_l \equiv 1 \pmod 2$$

This violating the parity condition (6.8). By this we finished the theorem. □

Chapter 7

Extensions and open problems

In this chapter we list some more results and questions related to the area.

In [14] Schrijver proved the 6th case mentioned in Section 2.2:

Theorem 7.1 *Assume that G is planar and is drawn on the plane in such a way that O is the outer face, I is a inner face. Let $a_1, a_2 \ldots, a_k$ on the face O in the cyclist order, and $b_1, b_2 \ldots b_k$ on the I face and they are arranged in the reverse cyclist order. The edges of H consists of $(a_i b_i), i = 1..k$. Then The Euler condition and the cut condition are sufficient for the existence of integral edge-capacitated routings.*

But his proof was not algorithmic. Finding such a proof is still an interesting open question.

Karzanov [7] and [8] showed that Okamura's theorem and the dual cut packing result can be extended in a certain way to planar graphs where the demands are on three or more faces. In [7] he proved:

Theorem 7.2 *Let $G = (V, E)$ be a bipartite planar graph and let $\mathcal{F}$ be a set of three of its faces. Then there exist $K_{2,3}$-metrics $\mu_1, \ldots \mu_k$ such that $dist_G(u, v) \geq \mu_1(u, v) + \ldots + \mu_k(u, v)$ for all $u, v \in V$, with equality if there is an $F \in \mathcal{F}$ with both u and v incident with F.*

That is, if consider the edges connecting vertices on the same face F, with $F \in \mathcal{F}$ as demand edges E(H). (H is the demand graph). Then the metric on G is in the H facet of the cone of $K_{2,3}$-metrics. This proof was based on the dual path technique discussed in Chapter 3. And using this Theorem Karzanov showed in [8] that:

Theorem 7.3 *Let $G = (V, E)$ be a bipartite planar graph and let F_1, F_2, F_3 be three of its faces, and let $H = (V, F)$ be a graph such that for each $r = st \in F$, there is an $i = 1, 2, 3$ with s and t on the boundary of F_i. Let $G + H$ be Eulerian. Then there exist solutions to the edge-disjoint paths problem with respect to supply-demand graphs G, H if and only if the $K_{2,3}$-metric condition holds.*

It would be interesting to know if there are more cases, when the $K_{2,3}$-metric and parity conditions imply the existence of integral solution to the edge-disjoint paths problem.

Frank [1] showed an interesting extension of the Okamura-Seymour theorem to the case where the parity condition is only required for the vertices not on the outer boundary.

Theorem 7.4 *Let $G = (V, E)$ be a planar graph such that each vertex not o the boundary has even degree. Let F be a set of demands connecting vertices on the outer boundary of G. Then there exist edge-disjoint demand paths if and only if*

$$\sum_{j=1}^{l} (d_E(X_j) - d_F(X_j)) \geq \frac{q}{2}$$

for each collection of subsets $X_1, \ldots X_l$. Here q denotes the number of components K of $G' := G - \nabla(X_1) - \ldots - \nabla(X_l)$ with $d_E(K) + d_F(K)$ odd.

It is still an open problem to decide if the edge-disjoint paths problem is polynomial-time solvable for planar graphs with all demand on the outer boundary or it is NP-complete. A result for a similar question is due to Middendorf and Pfeiffer. In [9] they showed that if $G + H$ is planar, but not necessarily Eulerian, then the edge-disjoint paths problem is NP-complete.

Frank in [3] observed the following, which he called *intersection criterion*:

$$d_{G+H}(X \cap Y) \text{ is even for any two tight sets } X, Y \subset V$$

This is a necessary condition for the existence of edge-disjoint paths: If paths as required exist, then for each tight X all the edge in $\nabla(X)$ are used by these paths, hence if X and Y are tight, all edges in $\nabla(X \cap Y)$ are used, thus $d_G(X \cap Y) \equiv d_H(X \cap Y) \pmod 2$, that is $d_{G+H}(X \cap Y)$ is even. And in [3] he showed:

Theorem 7.5 *Let $G = (V, E)$, and $H = (V, F)$ be supply and demand graphs such that $G + H$ is planar and such that the edges of H are on at most two of the faces of G. Then there exist edge-disjoint demand paths if and only if the cut condition and the intersection criterion hold.*

Our theorem in Chapter 6 also has a parity condition like the intersection criterion mentioned above.

In the survey of Frank [2] many other cases could be found when good characterization exists for the disjoint paths problem. In those cases the supply graph is not planar, but then there are strong restrictions to the demand graph.

The edge-disjoint paths problem is also extended to matroids. One can find many interesting results in the work of Schwärzler, Sebő [16] and Seymour [17].

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