Difference functions of periodic L_p functions

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1 Introduction

1.1 Background

About fifty years ago de Bruijn proved that if a periodic measurable real function f has the property that $f(x+h)-f(x)$ is square integrable for every $h \in \mathbb{R}$, then f itself is also square integrable (see [8]). Thirteen years later Carroll proved the same statement for p integrable functions ($1 \leq p \leq \infty$) and a few years later this result was generalized to every compact Abelian group.

But what happens if we know that the difference functions $f(x+h)-f(x)$ are p integrable only for some restricted values of h? This question has been extensively studied by Tamás Keleti in [1]. In this paper we continue his investigations.

Let us call a set H of the Abelian group G *small* if there is a function f with difference functions in a fixed L_p class for every $h \in H$ but f itself does not belong to L_p. The previous results imply that the whole group is not small. So a small set is a proper subset of G and by continuity arguments one can show that a small set can be covered by a proper F_σ subgroup of G. Tamás Keleti showed that *pseudo-Dirichlet sets* are small. The reason of this is that pseudo-Dirichlet sets are "very periodic" (for the precise definition see below). He also realized that the property of being small is closely related to the non-ejectivity property of sets. It was known that among compact sets the non-ejective sets are exactly the N-sets and the class of N-sets is closed under taking generated subgroup. For these reasons Tamás Keleti conjectured that a set is small if and only if it is an N-set. We prove partially this conjecture.

1.2 Notation and Results

Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ denote the circle group and let L_0 be the class of measurable complex valued functions on $\mathbb{T}$. Let $\mathcal{G} \subset \mathcal{F} \subset L_0$ be two classes of functions on $\mathbb{T}$. Following the notations in [1] we denote by $\mathfrak{H}(\mathcal{F},\mathcal{G})$ those subsets H of $\mathbb{T}$ for which there exists a function $f \in \mathcal{F} \setminus \mathcal{G}$ with difference functions $\Delta_h f(x) = f(x+h) - f(x)$ belonging to $\mathcal{G}$ for every $h \in H$. That is,

$$\mathfrak{H}(\mathcal{F},\mathcal{G}) = \{H \subset \mathbb{T} : (\exists f \in \mathcal{F} \setminus \mathcal{G}) \ \ (\forall h \in H) \ \ \Delta_h f \in \mathcal{G}\}\,.$$

For the general properties of $\mathfrak{H}(\mathcal{F},\mathcal{G})$ see [1].

Let L_p $(0 < p < \infty)$ denote the class of those measurable functions on $\mathbb{T}$ for which $||f||_{L_p} = \left(\int_{\mathbb{T}} |f|^p\right)^{\frac{1}{p}} < \infty$. It is clear that $q \leq p$ implies $L_p \subset L_q$. Let

$$L_{p<} = \bigcap_{0<q<p} L_q,$$

a class containing L_p. For these function classes the classes $\mathfrak{H}(\mathcal{F}, \mathcal{G})$ are related to some classes of thin sets in harmonic analysis. We collect here those classes which arise in our proofs and results.

Definition 1.1. A set $H \subset \mathbb{T}$ is an *N-set* if there exists a trigonometric series on $\mathbb{T}$ which is absolute convergent on H but not absolute convergent everywhere. The class of N-sets on $\mathbb{T}$ will be denoted by $\mathfrak{N}$.

Definition 1.2. A set $H \subset \mathbb{T}$ is a *Dirichlet set* if there is an increasing sequence of positive integers (q_n) such that $\sin(q_n \pi h) \to 0$ uniformly on H. The class of Dirichlet sets on $\mathbb{T}$ will be denoted by $\mathfrak{D}$.

Definition 1.3. A set $H \subset \mathbb{T}$ is a *pseudo-Dirichlet set* if there is an increasing sequence of positive integers (q_n) and a sequence ε_n converging to zero such that for every $x \in H$ there exists a $k(x)$ for which $|\sin(q_n \pi x)| < \varepsilon_n$ if $n > k(x)$. The class of pseudo-Dirichlet sets on $\mathbb{T}$ will be denoted by $\mathfrak{pD}$.

Definition 1.4. A Borel set $H \subset \mathbb{T}$ is a *weak-Dirichlet set* if for every probability measure μ supported by H,

$$\limsup_{|n| \to \infty} |\hat{\mu}(n)| = 1,$$

where

$$\hat{\mu}(n) = \int_{\mathbb{T}} e^{-2\pi i n t} \mathrm{d}\mu(t)$$

stands for the Fourier-transform of μ. The class of weak-Dirichlet sets on $\mathbb{T}$ will be denoted by $\mathfrak{wD}$, while $\mathfrak{cwD}$ will stand for the class of compact weak-Dirichlet sets.

Remark 1.5. An easy computation shows that we get an equivalent definition of weak-Dirichlet sets by requiring

$$\inf_{n \neq 0} \int_{\mathbb{T}} \left|e^{-2\pi i n t} - 1\right|^2 \mathrm{d}\mu(t) = 0$$

for every probability measure μ supported by H.

$\mathfrak{F}_\sigma$ stands for the class of subsets of $\mathbb{T}$ which can be covered by a proper F_σ subgroup of $\mathbb{T}$.

With these notations the results of T. Keleti can be summarized as follows ([1] Theorem 2.5, [2] Theorem 4.10).

Theorem 1.6. *For every $0 < p < q < \infty$,*

$$\mathfrak{p}\mathfrak{D} \subset \mathfrak{H}(L_p, L_q) \subset \mathfrak{F}_\sigma.$$

He also proves the following inclusion ([1] Theorem 3.2).

Theorem 1.7. *If $H \subset \mathbb{T}$ is not an N-set, $f : \mathbb{T} \to \mathbb{C}$ is a measurable function with $\Delta_h f \in L_\infty$ for every $h \in H$, then $f \in L_{\infty^<}$.*

These inclusions motivated the conjecture

$$\mathfrak{H}(L_p, L_q) = \mathfrak{N} \quad (0 < p < q < \infty),$$

which we confirm partially in our main theorem:

Theorem 1.8. *1. For every $0 \le p < q \le 2$,*

$$\mathfrak{H}(L_p, L_q) = \mathfrak{N}.$$

2. For every $0 \le p < q < \infty$,

$$\mathfrak{H}(L_p, L_{q^<}) = \mathfrak{N}.$$

In particular, if $0 \le p < \infty$ and $H \subset \mathbb{T}$ is not an N-set, $f : \mathbb{T} \to \mathbb{C}$ is a measurable function with $\Delta_h f \in L_p$ for every $h \in H$, then $f \in L_{p^<}$.

2 Preliminaries

For proving that the classes in question are contained in $\mathfrak{N}$ we use the fact that on the L_2 class the Fourier transform is an isometric isomorphism. For $0 < p < \infty$ let l_p denote the complex Banach-space of the two-way infinite sequences $c = (c_n)_{n=-\infty}^{\infty}$ of complex numbers for which

$$||c||_{l_p} = \left(\sum |c_n|^p\right)^{\frac{1}{p}} < \infty.$$

For an $f : \mathbb{T} \to \mathbb{C}$ measurable function let

$$\mathfrak{F}f(n) = \int_{\mathbb{T}} f(t) e^{2\pi int} \mathrm{d}t$$

denote the Fourier transform of f. The Theorem of Young and Hausdorff says that $f \in L_2$ if and only if $\mathfrak{F}f \in l_2$, moreover every sequence of l_2 can be obtained as the Fourier transform of a function belonging to the L_2 class. According to the Parseval formula

$$||f||_{L_2} = ||\mathfrak{F}f||_{l_2}.$$

The following two theorems shows the close relation of N-sets and weak-Dirichlet sets (see [7], page 190, Theorem 1.5. and Corollary 1.6. or [6], page 49).

Theorem 2.1. *Any increasing union of compact weak-Dirichlet sets is an N-set.*

Theorem 2.2. *Any N-set can be obtained as an increasing countable union of compact weak-Dirichlet sets.*

Finally we state a result giving an equivalent property of being a weak-Dirichlet set (see [5], page 162. Remark 4.3). We will use this theorem for proving that $\mathfrak{N}$ is contained in the classes we examine. For an $h \in \mathbb{T}$ and $A \subset \mathbb{T}$ we introduce the notation

$$\Delta_h A = ((A+h) \setminus A) \cup (A \setminus (A+h)) .$$

Definition 2.3. *A compact set $H \subset \mathbb{T}$ is non-ejective if for every $0 \leq x \leq 1$,*

$$\inf_{\substack{A \subset \mathbb{T} \\ \lambda(A)=x}} \sup_{h \in H} \lambda(\Delta_h A) = 0.$$

Theorem 2.4. *Let $H \subset \mathbb{T}$ be a compact set. Then H is weak-Dirichlet if and only if H is non-ejective*

Unfortunately the proof of this theorem has not been published yet. In the sequel we will use only the implication weak-Dirichlet $\to$ non-ejective. The other (easy) direction will follow from our results.

In the proofs $\mathbb{N}$ and $\mathbb{Z}$ will stand for the set of positive integers including zero and integers respectively.

The translation on $\mathbb{T}$ by $h \in \mathbb{T}$ will be denoted by γ_h.

For a function $f : \mathbb{T} \to \mathbb{C}$ and $a, b \in \mathbb{R} \cup \{\pm\infty\}$ let

$$[a \leq f < b] = \{x \in \mathbb{T} : a \leq f(x) < b\}.$$

For a set $K \subset \mathbb{T}$, χ_K denotes the characteristic function of K.

3 Basic inclusions

In this section we prove the following two inclusion theorems between the classes $\mathfrak{H}(L_p, L_q)$ and $\mathfrak{H}(L_r, L_{s^<})$.

Theorem 3.1. *For every $0 \leq p < q < \infty$, $0 \leq r < s < \infty$ with $q \leq s$,*

$$\mathfrak{H}(L_p, L_q) \subset \mathfrak{H}(L_r, L_s).$$

Theorem 3.2. *For every $0 \leq p < q < \infty$ the classes*

$$\mathfrak{H}(L_p, L_{q^<})$$

coincide.

They will be necessary to transfer the results obtained for L_1 or L_2 to arbitrary L_p class.

The proof of these theorems uses some properties of powers of functions. These properties are described in Lemma 3.4, Lemma 3.6 and Lemma 3.8.

3.1 Powers less than 1

Definition 3.3. For a measurable function $f : \mathbb{T} \to \mathbb{C}$,

$$\mathcal{L}(f) = \sup \{0 \leq p : f \in L_p\}$$

denotes the *class of f*.

The following lemma describes the behavior of taking power f^α for $\alpha \leq 1$.

Lemma 3.4. *Let $0 \leq p \leq r < \infty$ and $h \in \mathbb{T}$. If $f \in L_p$, $0 \leq f$ and $\Delta_h f \in L_r$ then for every $0 < \alpha \leq 1$ we have $f^\alpha \in L_{\frac{p}{\alpha}}$ and $\Delta_h f^\alpha \in L_{\frac{r}{\alpha}}$ with*

$$||\Delta_h f^\alpha||_{L_{\frac{r}{\alpha}}} \leq ||\Delta_h f||_{L_r}^\alpha.$$

We also have

$$\mathcal{L}(f^\alpha) = \frac{\mathcal{L}(f)}{\alpha}.$$

Proof. The first and third statement being obvious we prove only the second one. It is easy to see that for $0 < \alpha \leq 1$ and $a, b \in \mathbb{C}$,

$$||a|^\alpha - |b|^\alpha| \leq |a-b|^\alpha.$$

From this with

$$a = f(x+h),\ b = f(x)$$

we have

$$|f^\alpha(x+h) - f^\alpha(x)|^{\frac{r}{\alpha}} \leq |f(x+h) - f(x)|^{\alpha\frac{r}{\alpha}} = |f(x+h) - f(x)|^r,$$

so

$$\Delta_h f^\alpha \in L_r$$

and

$$||\Delta_h f^\alpha||_{L_{\frac{r}{\alpha}}} \leq ||\Delta_h f||_{L_r}^\alpha. \blacksquare$$

3.2 Powers greater than 1

Now we examine f^α for $1 < \alpha$. As we shall see, this power does not behave as well as it did for $\alpha \leq 1$, there are more technical difficulties for less result.

Lemma 3.5. *Let $0 < \delta < 1$, $1 \leq b \leq b + b^\delta \leq a$ be reals. Then for every $1 \leq \gamma$ we have*

$$(a^\gamma - b^\gamma) \leq 2^\gamma (a-b)^{\gamma + \frac{(1-\delta)(\gamma-1)}{\delta}}.$$

Proof. We show that the value of the fraction

$$\frac{a^\gamma - b^\gamma}{(a-b)^\gamma}$$

is maximal when $a = b + b^\delta$. This is true since

$$\left(\frac{x^\gamma - b^\gamma}{(x-b)^\gamma}\right)' = (x-b)^{-1-\gamma}\gamma b \left(b^{\gamma-1} - x^{\gamma-1}\right) \leq 0$$

for $b \le x$ and $1 \le \gamma$, so the fraction

$$\frac{x^\gamma - b^\gamma}{(x-b)^\gamma}$$

is a decreasing function of x for $b \le x$.

For every $1 \le \gamma$ and $0 \le t \le 1$ we have

$$(1+t)^\gamma \le 1 + 2^\gamma t, \tag{1}$$

since the left-hand side of the expression is convex while the right-hand side is linear in t, and the inequality holds for $t = 0$ and $t = 1$.

Applying (1) with $t = \frac{1}{y}$ we have

$$(1+y)^\gamma - y^\gamma \le 2^\gamma y^{\gamma-1},$$

and with $y = b^{1-\delta}$ we get

$$\frac{a^\gamma - b^\gamma}{(a-b)^\gamma} \le \frac{\left(b + b^\delta\right)^\gamma - b^\gamma}{(b + b^\delta - b)^\gamma} = \left(1 + b^{1-\delta}\right)^\gamma - b^{(1-\delta)\gamma} \le$$

$$\le 2^\gamma b^{(1-\delta)(\gamma-1)} \le 2^\gamma (a-b)^{\frac{(1-\delta)(\gamma-1)}{\delta}},$$

as stated.■

Lemma 3.6. *Let $0 < q < r < \infty$, $H \subset \mathbb{T}$ and $f : \mathbb{T} \to \mathbb{C}$ such that $\mathcal{L}(f) = q$ and $\Delta_h f \in L_r$ for every $h \in H$. Then there is a continuous function $\rho : (0,q) \to \mathbb{R}$ such that $p < \rho(p)$ and for every $0 < p < q$ there exists a function f_p such that $\mathcal{L}(f_p) = p$ and $\rho(p) \le \mathcal{L}(\Delta_h f_p)$ for every $h \in H$.*

Proof. Let $f_p = |f|^{\frac{q}{p}}$. Again, it is obvious that $\mathcal{L}(f_p) = p$.

Fix an $h \in H$ and let $0 < \delta < 1$, $p < s$ be arbitrary. The value of δ will be chosen later. Let

$$a(x) = \max(|f(x+h)|, |f(x)|),$$

$$b(x) = \min(|f(x+h)|, |f(x)|),$$

$$A = \left\{x \in \mathbb{T} : b(x) + b^\delta(x) \le a(x)\right\},$$

$$A' = \{x \in \mathbb{T} : 1 \le b(x)\}.$$

Then

$$||\Delta_h f_p||_{L_s}^s = \int_{\mathbb{T}} \left| |f(x+h)|^{\frac{q}{p}} - |f(x)|^{\frac{q}{p}} \right|^s \mathrm{d}x =$$

$$= \int_{\mathbb{T}} \left(a^{\frac{q}{p}}(x) - b^{\frac{q}{p}}(x) \right)^s \mathrm{d}x =$$

$$= \int_{A\cap A'} \left(a^{\frac{q}{p}}(x) - b^{\frac{q}{p}}(x) \right)^s \mathrm{d}x +$$

$$+ \int_{(\mathbb{T}\setminus A)\cap A'} \left(a^{\frac{q}{p}}(x) - b^{\frac{q}{p}}(x) \right)^s \mathrm{d}x + \int_{\mathbb{T}\setminus A'} \left(a^{\frac{q}{p}}(x) - b^{\frac{q}{p}}(x) \right)^s \mathrm{d}x.$$

We estimate the integrals separately.

For the first term, by applying Lemma 3.5 with $\gamma = \frac{q}{p}$, we get

$$\int_A \left(a^{\frac{q}{p}}(x) - b^{\frac{q}{p}}(x) \right)^s \mathrm{d}x \le$$

$$\le 2^{\frac{q}{p}} \int_{\mathbb{T}} (a(x) - b(x))^{s\frac{q}{p} + s\frac{(1-\delta)(\frac{q}{p}-1)}{\delta}}. \tag{2}$$

Since $\Delta_h f \in L_r$ means

$$\int_{\mathbb{T}} |f(x+h) - f(x)|^r \mathrm{d}x = \int_{\mathbb{T}} (a(x) - b(x))^r \mathrm{d}x < \infty,$$

(2) is finite if

$$s\frac{q}{p} + s\frac{(1-\delta)(\frac{q}{p}-1)}{\delta} \le r,$$

that is if

$$s \le \frac{rp}{q + \frac{(1-\delta)(q-p)}{\delta}}. \tag{3}$$

For the second term, again from Lemma 3.5 with $\gamma = \frac{q}{p}$, we get

$$\int_{(\mathbb{T}\setminus A)\cap A'} \left(a^{\frac{q}{p}}(x) - b^{\frac{q}{p}}(x) \right)^s \mathrm{d}x \le \int_{\mathbb{T}} \left(\left(b + b^{\delta} \right)^{\frac{q}{p}} - b^{\frac{q}{p}} \right)^s \le$$

$$\le 2^{s\frac{q}{p}} \int_{\mathbb{T}} b^{s\delta\left(\frac{q}{p} + \frac{(1-\delta)(\frac{q}{p}-1)}{\delta} \right)} \le 2^{s\frac{q}{p}} \int_{\mathbb{T}} b^{s\left(\delta + \frac{q}{p} - 1\right)} \le 2^{s\frac{q}{p}} \int_{\mathbb{T}} |f|^{s\left(\delta + \frac{q}{p} - 1\right)},$$

which is finite if

$$s\left(\delta+\frac{q}{p}-1\right)<q,$$

that is if

$$s<\frac{p}{1-(1-\delta)\frac{p}{q}}. \tag{4}$$

To estimate the third term we use that if $0 \leq b \leq 1$, $b \leq a$ then

$$a^{\alpha} \leq 2^{\alpha}(a-b)^{\alpha}+2^{\alpha},$$

for $0 \leq a \leq 2$ by the second term, for $2 \leq a$ by the first term. Applying this with $\alpha = s\frac{q}{p}$, for the third term we have

$$\int_{\mathbb{T}\setminus A'}\left(a^{\frac{q}{p}}(x)-b^{\frac{q}{p}}(x)\right)^{s} \mathrm{d}x \leq \int_{\mathbb{T}\setminus A'} a^{s\frac{q}{p}} \leq \int_{\mathbb{T}\setminus A'} 2^{s\frac{q}{p}}\left((a-b)^{s\frac{q}{p}}+1\right),$$

which is finite whenever

$$s\frac{q}{p} \leq r,$$

that is if

$$s \leq \frac{pr}{q}. \tag{5}$$

With the unique $0<\delta<1$ fulfilling

$$\frac{1-\delta}{\delta}(q-p)=\frac{r-q}{2},$$

the right-hand side of (3) and (4) is bigger than p. Fixing this δ let

$$\rho=\min\left\{\frac{rp}{q+\frac{(1-\delta)(q-p)}{\delta}}, \frac{p}{1-(1-\delta)\frac{q}{p}}, \frac{pr}{q}\right\}.$$

Then $p<\rho$, and (3), (4) and (5) are satisfied for $s<\rho$. This proves the statement.■

3.3 A small power

To put a function f with $\mathcal{L}(f)=0$ in a L_p class for $0<p$ it must be composed by a function which increases more slowly than x^{α} for any $0<\alpha$. This is done in the following lemmas. They will help to deal with the $p=0$ case.

Lemma 3.7. *If $\Phi : [0, \infty) \to [0, \infty)$ is a concave function with*

$$\Phi(1) \leq 1,$$

$f : \mathbb{T} \to \mathbb{R}$ is a function and $\Delta_h f \in L_p$ for some $0 \leq p < \infty$ and $h \in \mathbb{T}$, then

$$\Delta_h \Phi(|f|) \in L_p$$

with $||\Delta_h \Phi(|f|)||_{L_p}$ bounded by a function of $||\Delta_h f||_{L_p}$.

Proof. Since $0 \leq \Phi$, the concavity of Φ and $\Phi(1) \leq 1$ imply $\Phi(x) \leq x + 1$. Using the concavity and $\Phi(x) \leq x + 1$ we get

$$|\Phi(|f(x+h)|) - \Phi(|f(x)|)| \leq \Phi(||f(x+h)| - |f(x)||) - \Phi(0) \leq$$

$$\leq ||f(x+h)| - |f(x)|| + 1 \leq |f(x+h) - f(x)| + 1,$$

which clearly implies the statement.

Lemma 3.8. *Let $(\lambda_i)_{i=0}^{\infty}$ be a given sequence of nonnegative reals with*

$$\sum_{i=0}^{\infty} \lambda_i = 1, \tag{6}$$

$$\sum_{i=1}^{\infty} i\lambda_i = \infty. \tag{7}$$

Then there is a $\Phi : [0, \infty) \to [0, \infty)$ concave function with $\Phi(1) \leq 1$ such that

$$\sum_{i=0}^{\infty} \Phi(i)\lambda_i < \infty, \tag{8}$$

$$\sum_{i=0}^{\infty} \Phi^3(i)\lambda_i = \infty. \tag{9}$$

Proof. From (6) we have an increasing sequence $(i_k)_{k=0}^{\infty}$ of integers with

$$\sum_{i_k \leq i} \lambda_i \leq \frac{1}{2^k}. \tag{10}$$

We define recursively a strictly increasing sequence $(a_k)_{k=0}^{\infty}$ of integers fulfilling

$$\max\{k, i_k\} \leq a_k, \tag{11}$$

$$a_k - a_{k-1} \leq a_{k+1} - a_k \tag{12}$$

and

$$\frac{1}{k^2} \leq \sum_{a_k \leq i < a_{k+1}} \left(k + \frac{i - a_k}{a_k - a_{k-1}} \right) \lambda_i \tag{13}$$

for every $k = 1, 2, \ldots$

Let $a_0 = 0$, $a_1 = i_1$. If a_k is already defined we define a_{k+1} simply by taking its value large enough. Conditions (11) and (12) can easily be satisfied, while (7) shows that (13) also holds for a large a_{k+1}, since the coefficients in (13) grow linearly in i.

If there is a given strictly increasing sequence $(a_k)_{k=0}^{\infty}$ of integers with $a_0 = 0$ and a sequence $(b_k)_{k=0}^{\infty}$ of reals, we denote by $\Phi_{(a_k),(b_k)}$ the function for which $\Phi(a_k) = b_k$ and linear on the intervals $[a_k, a_{k+1}]$. With this notation let $\Phi = \Phi_{(a_k),(k)}$, a concave function by (12) and $\Phi(1) \leq 1$ by (11).

Since

$$\sum_{i=0}^{\infty} \Phi(i)\lambda_i = \sum_{k=0}^{\infty} \sum_{a_k \leq i < a_{k+1}} \Phi(i)\lambda_i \leq$$

$$\leq \sum_{k=0}^{\infty} \sum_{a_k \leq i < a_{k+1}} (k+1)\lambda_i \leq \sum_{k=0}^{\infty} \sum_{a_k \leq i} (k+1)\lambda_i \leq$$

$$\leq \sum_{k=0}^{\infty} \sum_{i_k \leq i} (k+1)\lambda_i \leq \sum_{k=0}^{\infty} \frac{(k+1)}{2^k} < \infty,$$

we have property (8). If we knew

$$\frac{1}{k^2} \leq \sum_{a_k \leq i} \Phi(i)\lambda_i \tag{14}$$

for infinitely many k's, then, since $k \leq \Phi(i)$ for $a_k \leq i$, we would have

$$1 \leq \sum_{a_k \leq i} \Phi^3(i)\lambda_i$$

for infinitely many k's, which would show that the series of (9) cannot converge. So if (14) holds only for a finite set of k's then we modify Φ to have it for infinitely many k's.

If

$$\sum_{a_k \leq i} \Phi(i)\lambda_i < \frac{1}{k^2},$$

then there is a $0 < d$ such that with the sequence $b_l = l$, if $l \leq k$ and $b_l = l+d$, if $k < l$, we have

$$\sum_{a_k \leq i} \Phi_{(a_k),(b_k)}(i)\lambda_i = \frac{1}{k^2} \tag{15}$$

and the function $\Phi_{(a_k),(b_k)}$ is still concave. This follows from (13), which says that if Φ would increase on $[a_k, a_{k+1}]$ as is does on $[a_{k-1}, a_k]$ then

$$\frac{1}{k^2} \leq \sum_{a_k \leq i < a_{k+1}} \Phi(i)\lambda_i.$$

Moreover, if this modification is done for a k bigger than 2, then also $\Phi(1) \leq 1$ remains true.

We repeat this modification step finitely or infinitely many times to get, as a monotone increasing point-wise limit of concave functions, a concave function Φ for which (14) holds for infinitely many k's, so we certainly have property (9). We show that we did not loose property (8).

Either there were only finitely or infinitely many modification steps, in the jth step, say at a_{k_j}, the value of the sum

$$\sum_{i=1}^{\infty} \Phi(i)\lambda_i$$

could increase by at most $\frac{1}{k_j^2}$. Since

$$\sum_{j=1}^{\infty} \frac{1}{k_j^2} \leq \sum_{j=1}^{\infty} \frac{1}{j^2} < \infty,$$

$$\sum_{i=0}^{\infty} \Phi(i)\lambda_i$$

must converge, so the proof is complete.■

As an application, we can prove the last lemma of this section.

Lemma 3.9. *Let $0 < s < \infty$, $H \subset \mathbb{T}$ and $f : \mathbb{T} \to \mathbb{R}$ be such that $\mathcal{L}(f) = 0$ and $\Delta_h f \in L_s$ for every $h \in H$. Then there is a function $0 \leq f_1$ such that $f_1 \in L_1 \setminus L_3$ and $\Delta_h f_1 \in L_4$ for every $h \in H$ with $||\Delta_h f_1||_{L_4}$ bounded by some function of $||\Delta_h f||_{L_s}$.*

Proof. If $s < 4$, then let

$$\tilde{f} = \left[|f|^{\frac{s}{4}}\right],$$

else we set

$$\tilde{f} = [|f|],$$

where [.] denotes the integer part. In the first case, by Lemma 3.4, we have $\Delta_h \tilde{f} \in L_4$ with

$$||\Delta_h \tilde{f}||_{L_4} \leq ||\Delta_h f||_{L_s} + 2 \tag{16}$$

for every $h \in H$. In the second case (16) holds obviously for every $h \in H$. Since $\mathcal{L}(f) = 0$, we have $\mathcal{L}(\tilde{f}) = 0$ in both cases.

By applying Lemma 3.8 with

$$\lambda_i = \lambda\left(\left[\tilde{f} = i\right]\right),$$

we get the concave function Φ for which $f_1 = \Phi \circ \tilde{f} \in L_1 \setminus L_3$. According to Lemma 3.7, $\Delta_h f_1 \in L_4$ with $||\Delta_h f_1||_{L_4}$ bounded by some function of $||\Delta_h f||_{L_s}$ for every $h \in H$. This proves the statement.■

Corollary 3.10. *Let* $0 < s < \infty$, $H \subset \mathbb{T}$ *and* $f : \mathbb{T} \to \mathbb{R}$ *be such that* $\mathcal{L}(f) = 0$ *and* $\Delta_h f \in L_s$ *for every* $h \in H$. *Then*

$$H \in \mathfrak{H}(L_u, L_v) \cap \mathfrak{H}(L_u, L_{v^<})$$

for every $0 \leq u < v < \infty$.

Proof. Let f_1 be as in the previous Lemma.

If $\mathcal{L}(f_1) \leq \frac{u+v}{2}$ or $4 \leq v$, we apply Lemma 3.4 for f_1 with $r = 4$ and

$$\alpha = \min\left\{\frac{4}{v}, \frac{2\mathcal{L}(f_1)}{u+v}\right\}.$$

We get a function $f^\star$ with $u < \mathcal{L}(f^\star) < v$ and $\Delta_h f^\star \in L_v$ for every $h \in H$.

If $\frac{u+v}{2} < \mathcal{L}(f_1)$ and $v < 4$, we apply Lemma 3.6 for f_1 with $q = \mathcal{L}(f_1)$, $r = 4$ to have a $u < p < v$ and again a function $f^\star$ with $\mathcal{L}(f^\star) = p$ and $\Delta_h f^\star \in L_v$ for every $h \in H$. This can be done by the continuous dependence of ρ from p. Thus in both cases we get

$$H \in \mathfrak{H}(L_u, L_v) \cap \mathfrak{H}(L_u, L_{v^<}).■$$

3.4 The inclusion theorems

Now we can prove the two inclusion theorems.

Proof of Theorem 3.1. Suppose that for a set $H \subset \mathbb{T}$ we have $H \in \mathfrak{H}(L_p, L_q)$, that is there is a function $f \in L_p \setminus L_q$ with $\Delta_h f \in L_q$ for every $h \in H$. Then by Lemma 3.4 with $\alpha = \frac{q}{s}$ we have $\tilde{f} = |f|^{\frac{q}{s}} \in L_{\frac{ps}{q}} \setminus L_s$ and $\Delta_h \tilde{f} \in L_s$ for every $h \in H$. So

$$H \in \mathfrak{H}(L_r, L_s)$$

if $r < \mathcal{L}(\tilde{f})$.

Suppose now that $\mathcal{L}(\tilde{f}) \leq r < s$. If $0 < \mathcal{L}(\tilde{f})$, then by applying Lemma 3.4 with $\alpha = \frac{2\mathcal{L}(\tilde{f})}{r+s}$ we have a function $\overline{f} \in L_r \setminus L_s$ with $\Delta_h \overline{f} \in L_{s\frac{r+s}{2\mathcal{L}(\tilde{f})}} \subset L_s$ for every $h \in H$. Thus $H \in \mathfrak{H}(L_r, L_s)$.

If $\mathcal{L}(\tilde{f}) = 0$ then the statement follows from Corollary 3.10.■

Proof of Theorem 3.2. It is enough to prove that

$$\mathfrak{H}(L_p, L_{q^<}) \subset \mathfrak{H}(L_r, L_{s^<})$$

for every $0 \leq p < q < \infty$, $0 \leq r < s < \infty$. Let $H \in \mathfrak{H}(L_p, L_{q^<})$. We distinguish two cases.

Case 1. $(q \leq s)$ We proceed as in the proof of Theorem 3.1. By Lemma 3.4 with $\alpha = \frac{q}{s}$ we have $\tilde{f} = |f|^{\frac{q}{s}} \in L_{\frac{s}{q}} \setminus L_{s^<}$ and $\Delta_h \tilde{f} \in L_{s^<}$ for every $h \in H$, and we are done if $r < \mathcal{L}(\tilde{f})$.

Now if $0 < \mathcal{L}(\tilde{f}) \leq r < s$ then again by Lemma 3.4 with $\alpha = \frac{2\mathcal{L}(\tilde{f})}{r+s}$ we have the function $\overline{f} \in L_r \setminus L_{s^<}$ with $\Delta_h \overline{f} \in L_{\left(s\frac{r+s}{2\mathcal{L}(\tilde{f})}\right)^<} \subset L_s$ for every $h \in H$ which shows $H \in \mathfrak{H}(L_r, L_{s^<})$.

If $\mathcal{L}(\tilde{f}) = 0$ then the statement follows from Corollary 3.10.

Case 2. $(s < q)$ Let us consider first $0 < \mathcal{L}(f)$. By Lemma 3.6 we have a pair u, v satisfying $0 < u < v < s$ and a function $\tilde{f}$ with $\mathcal{L}(\tilde{f}) = u$ and $\Delta_h \tilde{f} \in L_{v^<}$ for every $h \in H$. This $\tilde{f}$ shows that $H \in \mathfrak{H}(L_u, L_{v^<})$. Now we are in the situation of *Case 1.*, so we have $H \in \mathfrak{H}(L_r, L_{s^<})$.

Now if $\mathcal{L}(f) = 0$ then again the statement follows from Corollary 3.10.■

4 Upper bounds

The proof of the upper bounds uses an idea of de Bruijn proving the weak difference property for the L_2 class (see [8]). The trick is to average the Fourier coefficients of the functions $\Delta_h f$ for different h's.

Theorem 4.1. *For every* $1 \leq p < 2$,

$$\mathfrak{H}(L_p, L_2) \subset \mathfrak{N}.$$

Proof. Let $H \subset \mathbb{T}$ be not an N-set and suppose that for a function $f \in L_p$ we have $\Delta_h f \in L_2$ for every $h \in H$. We show that $f \in L_2$.

For $0 < M$ let

$$A_M = \left\{h \in H : ||\Delta_h f||_{L_2}^2 \leq M\right\}.$$

It is easy to see that A_M is compact and for $M < N$ we have $A_M \subset A_N$. Since according to our assumption the increasing union

$$H = \bigcup_{M \in \mathbb{N}} A_M$$

is not an N-set, we have from Theorem 2.1 that for an $M \in \mathbb{N}$ the set A_M is not weak-Dirichlet.

Let $A_M \notin \mathfrak{wD}$. This means that there exists a probability measure μ supported by A_M for which

$$\limsup_{|n| \to \infty} |\hat{\mu}(n)| < 1.$$

Thus for an $\eta > 0$ sufficiently small and an n_0 we have

$$\eta < |\hat{\mu}(n) - 1| \tag{17}$$

for every $|n| > n_0$.

For every $h \in A_M$ we have

$$N(h) = ||\Delta_h f||_{L_2}^2 = ||\mathfrak{F}\Delta_h f||_{l_2}^2 = \sum |\mathfrak{F}(f \circ \gamma_h)(n) - \mathfrak{F}f(n)|^2 =$$

$$= \sum \left|(\mathfrak{F}f)(n)\left(e^{-2\pi i n h} - 1\right)\right|^2.$$

$N(h)$, as a function of h, is measurable and bounded by M on the support of μ. So by the Hölder inequality and the Fubini Theorem,

$$||(\mathfrak{F}f)(\hat{\mu}-1)||_{l_2}^2 = \sum |\mathfrak{F}f|^2(n)\,|\hat{\mu}-1|^2(n) =$$

$$= \sum |\mathfrak{F}f|^2(n) \left| \int_{\mathbb{T}} \left(e^{-2\pi inh}-1\right) \mathrm{d}\mu(h) \right|^2 \leq$$

$$\leq \sum |\mathfrak{F}f|^2(n) \int_{\mathbb{T}} \left|e^{-2\pi inh}-1\right|^2 \mathrm{d}\mu(h) =$$

$$= \int_{\mathbb{T}} \sum \left|(\mathfrak{F}f)(n)\left(e^{-2\pi inh}-1\right)\right|^2 \mathrm{d}\mu(h) = \int_{\mathbb{T}} N(h)\mathrm{d}\mu(h) \leq M.$$

From this and (17) we have

$$||\mathfrak{F}f||_{l_2}^2 \leq \sum_{j=-n_0}^{n_0} |\mathfrak{F}f|^2(j) + \frac{M}{\eta}.$$

Thus $\mathfrak{F}f \in l_2$, which implies $f \in L_2$ by the Theorem of Hausdorff and Young. This proves the statement.■

Corollary 4.2. *1. For every* $0 \leq p < q \leq 2$,

$$\mathfrak{H}(L_p, L_q) \subset \mathfrak{N}.$$

2. For every $0 \leq p < q < \infty$,

$$\mathfrak{H}(L_p, L_{q^<}) \subset \mathfrak{N}.$$

1. Let $0 \leq p < q \leq 2$. By Theorem 3.1 and Theorem 4.1 we have

$$\mathfrak{H}(L_p, L_q) \subset \mathfrak{H}(L_1, L_2) \subset \mathfrak{N},$$

as stated.

2. For $0 \leq p < q < \infty$ we have by Theorem 3.2

$$\mathfrak{H}(L_p, L_{q^<}) \subset \mathfrak{H}(L_1, L_{2^<}).$$

Since

$$\mathfrak{H}(L_1, L_{2^<}) \subset \bigcup_{1<r<2} \mathfrak{H}(L_1, L_r) \subset \mathfrak{N},$$

our statement follows.■

5 Lower bounds

The lower bounds are based on the non-ejectivity property of weak-Dirichlet sets.

5.1 Non-ejective sets

Lemma 5.1. *For every compact non-ejective set $H \subset \mathbb{T}$, there is a function $f : \mathbb{T} \to \mathbb{R}$ such that $\mathcal{L}(f) = 0$ and $\Delta_h f \in L_1$ with*

$$||\Delta_h f||_{L_1} \leq 2$$

for every $h \in H$.

Proof. By the non-ejectivity of H, for every fixed $n \in \mathbb{N}$ and for every $0 < \varepsilon$ there exists a set $K_n(\varepsilon)$ such that

$$\lambda\left(K_n(\varepsilon)\right) = \frac{1}{n^2} \tag{18}$$

and

$$\lambda\left(\Delta_h K_n(\varepsilon)\right) \leq \varepsilon, \ \forall h \in H. \tag{19}$$

Let

$$f_n = 2^n \chi_{K_n(\frac{1}{4^n})}$$

and

$$f = \sum_{n=1}^{\infty} f_n,$$

as a point-wise limit of measurable functions. We show that $f < \infty$ almost everywhere, $\mathcal{L}(f) = 0$ and $\Delta_h f \in L_1$ with $||\Delta_h f||_{L_1} \leq 2$ for every $h \in H$.

By (18) we have

$$\sum \lambda\left(K_n\left(\frac{1}{4^n}\right)\right) = \sum \frac{1}{n^2} < \infty,$$

so the first statement follows from the Borel-Cantelli Lemma.

Since $f_n \leq f$ for every $n \in \mathbb{N}$ and for every $0 < \alpha$ we have

$$||f_n||_{L_\alpha} = \frac{1}{n^{\frac{2}{\alpha}}} 2^n \to \infty$$

if $n \to \infty$, we can conclude $\mathcal{L}(f) = 0$.

From $f < \infty$ almost everywhere we have that the series of the difference functions

$$\sum_{n=1}^{\infty} \Delta_h f_n$$

converges almost everywhere to $\Delta_h f$. Since for every $N, M \in \mathbb{N}$ with $N < M$ we have

$$\left|\left|\sum_{n=N}^{M} \Delta_h f_n\right|\right|_{L_1} \leq \sum_{n=N}^{M} ||\Delta_h f_n||_{L_1} \leq$$

$$\leq \sum_{n=N}^{M} \frac{2}{2^n} \leq \frac{1}{2^{N-2}},$$

the series

$$\sum_{n=1}^{\infty} \Delta_h f_n$$

is Cauchy in L_1, so it is convergent point-wise and in L_1 too, so the point-wise limit $\Delta_h f$ belongs to L_1 with

$$||\Delta_h f||_{L_1} = \left|\left|\sum_{n=1}^{\infty} \Delta_h f_n\right|\right|_{L_1} \leq \sum_{n=1}^{\infty} ||\Delta_h f_n||_{L_1} \leq \sum_{n=1}^{\infty} \frac{2}{2^n} \leq 2,$$

which completes the proof.■

Corollary 5.2. *For every $H \in \mathfrak{cw}\mathfrak{D}$ there is a function $f : \mathbb{T} \to \mathbb{R}$ such that $\mathcal{L}(f) = 0$ and $\Delta_h f \in L_1$ with*

$$||\Delta_h f||_{L_1} \leq 2$$

for every $h \in H$.

Proof. Since by Theorem 2.4 every compact weak-Dirichlet set is non-ejective, the statement follows from the previous lemma.■

5.2 General N-sets

On the account of Theorem 2.2, the remaining task is to examine the behavior of $\mathfrak{H}(L_p, L_q)$ for compact increasing union.

Lemma 5.3. *Let $1 \leq p < q \leq r < \infty$ and suppose we have an increasing sequence of sets (H_i) such that for every $i \in \mathbb{N}$ there is a function $f_i \in L_p \setminus L_q$ such that for every $h \in H_i$ we have $||\Delta_h f_i||_{L_r} \leq K_i$. Then for*

$$H = \bigcup_{i=1}^{\infty} H_i$$

there is a function $f \in L_p \setminus L_q$ such that $||\Delta_h f||_{L_r} < \infty$ for every $h \in H$.

Proof. By taking absolute value we can suppose that $0 \leq f_i$ for every $i \in \mathbb{N}$. For every $M \in \mathbb{R}$ and $i \in \mathbb{N}$, with the notation $f_i \wedge M = \min(f_i, M)$, we have

$$||f_i \wedge M||_{L_p} \leq ||f_i||_{L_p}$$

and

$$||\Delta_h(f_i \wedge M)||_{L_r} \leq ||\Delta_h f_i||_{L_r} \leq K_i$$

for any $h \in H_i$, while $||f_i \wedge M||_{L_q} \to \infty$ for $M \to \infty$. So we can find an $M_i \in \mathbb{R}$ such that

$$2^i \max(K_i, ||f_i||_{L_p}) \leq ||f_i \wedge M_i||_{L_q}.$$

Let

$$f = \sum_{i=1}^{\infty} \frac{f_i \wedge M_i}{2^i \max(K_i, ||f_i||_{L_p})}, \tag{20}$$

as a point-wise limit of measurable functions.

It is obvious that $f \in L_p \setminus L_q$, specially $f < \infty$ almost everywhere. Thus the series (20) is absolute convergent almost everywhere, so for every $h \in H$ the series

$$\sum_{i=1}^{\infty} \Delta_h \frac{f_i \wedge M_i}{2^i \max(K_i, ||f_i||_{L_p})}$$

converges point-wise to $\Delta_h f$ almost everywhere. If $h \in H_j$ for a $j \in \mathbb{N}$, then for every $N, M \in \mathbb{N}$ with $j < N < M$ we have

$$\left|\left|\sum_{i=N}^{M} \Delta_h \left(\frac{f_i \wedge M_i}{2^i \max(K_i, ||f_i||_{L_p})}\right)\right|\right|_{L_r} \leq \sum_{i=N}^{M} \left|\left|\Delta_h \left(\frac{f_i \wedge M_i}{2^i \max(K_i, ||f_i||_{L_p})}\right)\right|\right|_{L_r} =$$

$$= \sum_{i=N}^{M} \frac{||\Delta_h (f_i \wedge M_i)||_{L_r}}{2^i \max(K_i, ||f_i||_{L_p})} \leq \sum_{i=N}^{M} \frac{1}{2^i} \leq \frac{1}{2^{N-1}},$$

so the series

$$\sum_{i=1}^{\infty} \Delta_h \frac{f_i \wedge M_i}{2^i \max(K_i, ||f_i||_{L_p})}$$

is Cauchy, hence convergent in L_r. So we have that $\Delta_h f$ itself is in L_r for every $h \in H$, which completes the proof.■

Finally we can prove the remaining statements of Theorem 1.8.

Theorem 5.4. *For every* $0 \leq p < q < \infty$,

1.

$$\mathfrak{N} \subset \mathfrak{H}(L_p, L_{q^<}).$$

2.

$$\mathfrak{N} \subset \mathfrak{H}(L_p, L_q).$$

Proof.

1. Let $H \in \mathfrak{N}$. By Theorem 2.2 there is an increasing sequence of sets $(H_i) \subset \mathfrak{cw}\mathfrak{D}$ with
$$H = \bigcup_{i=1}^{\infty} H_i.$$
By Corollary 5.2, for any $i = 1, 2, \ldots$ we have a function f_i such that $\mathcal{L}(f_i) = 0$ and $\Delta_h f_i \in L_1$ with
$$||\Delta_h f_i||_{L_1} \leq 2$$
for every $h \in H_i$. We can apply Lemma 3.9 to obtain a sequence of functions $\hat{f}_i$ fulfilling $\hat{f}_i \in L_1 \setminus L_3$ and $\Delta_h \hat{f}_i \in L_4$ with
$$||\Delta_h \hat{f}_i||_{L_1} \leq K$$
for some $K \in \mathbb{R}$, for every $h \in H_i$. Thus by Lemma 5.3 there is a function $f \in L_1 \setminus L_3$ for which $\Delta_h f \in L_4$ for every $h \in H$. This shows $H \in \mathfrak{H}(L_1, L_{4^<})$, and by Theorem 3.2 the proof is complete.

2. Let $H \in \mathfrak{N}$. We know from the preceding that $H \in \mathfrak{H}(L_p, L_{q^<})$ for every $0 \leq p < q < \infty$, so from

$$\mathfrak{H}(L_p, L_{q^<}) \subset \bigcup_{r<q} \mathfrak{H}(L_p, L_r)$$

we have $H \in \mathfrak{H}(L_p, L_r)$ for some $r < q$. By Theorem 3.1 we have

$$\mathfrak{H}(L_p, L_r) \subset \mathfrak{H}(L_p, L_q),$$

thus $H \in \mathfrak{H}(L_p, L_q)$, which completes the proof.■

Corollary 4.2 and Theorem 5.4 together prove our main theorem.

6 Ejectivity

As a corollary of Lemma 5.1 and Theorem 4.2 we get that every non-ejective set is an N-set, in particular every compact non-ejective set is a compact N-set, hence a compact weak-Dirichlet set. This proves Theorem 2.4 in one direction. However, as Imre Ruzsa showed, the averaging technique of Theorem 4.1 can be applied directly to the ejective set, which gives a direct proof [9].

6.1 A minimax theorem

In the main theorem of the preceding we have proved in particular that for a weak-Dirichlet set H there is a function $f \in L_2 \setminus L_3$ such that every $h \in H$ the difference functions $\Delta_h f$ are so small that they may belong to any fixed higher class. The following theorem shows a quantitative analogue of this result in the class L_2. In the proof of Ruzsa Imre this result is the first step for proving that every weak-Dirichlet set is non-ejective. Here we give a different proof.

For a set $H \subset \mathbb{T}$, $\mathcal{M}(H)$ denotes the set of probability measures supported by H. Let

$$\mathcal{F} = \left\{ f : \mathbb{T} \to \mathbb{C} : f \in L_2, ||f||_{L_2} = 1, \int_{\mathbb{T}} f = 0 \right\},$$

while ${}_{\mathbb{R}}l_p \subset l_p$ stands for the *real* Banach-space of two-way infinite p-summable sequences.

Theorem 6.1. *Let $H \subset \mathbb{T}$ be a compact set. Then*

$$\inf_{f\in\mathcal{F}} \sup_{h\in H} ||\Delta_h f||_{L_2}^2 = \sup_{\mu\in\mathcal{M}(H)} \inf_{n\neq 0} \int_{\mathbb{T}} |e^{-2\pi i n h} - 1|^2 \, d\mu(h).$$

Proof. If for an $f \in \mathcal{F}$ we have

$$\sup_{h\in H} ||\Delta_h f||_{L_2}^2 \leq \varepsilon^2,$$

then by the Parseval formula

$$\sum |\mathfrak{F} f(n)|^2 = ||f||_{L_2}^2 = 1$$

and

$$\sum |\mathfrak{F} f(n)|^2 \left|e^{-2\pi i n h} - 1\right|^2 = ||\mathfrak{F}\Delta_h f||_{l_2}^2 = ||\Delta_h f||_{L_2} \leq \varepsilon^2,$$

thus for every $\mu \in \mathcal{M}(H)$ we have by the Fubini Theorem

$$\sum |\mathfrak{F} f(n)|^2 \int_{\mathbb{T}} \left|e^{-2\pi i n h} - 1\right|^2 \mathrm{d}\mu(h) =$$

$$\int_{\mathbb{T}} \sum |\mathfrak{F} f(n)|^2 \left|e^{-2\pi i n h} - 1\right|^2 \mathrm{d}\mu(h) \leq \varepsilon^2.$$

Since $\mathfrak{F} f(0) = 0$ for our $f \in \mathcal{F}$, we get

$$\inf_{n\neq 0} \int_{\mathbb{T}} |e^{-2\pi i n h} - 1|^2 \mathrm{d}\mu(h) \leq \varepsilon^2,$$

which proves

$$\sup_{\mu\in\mathcal{M}(H)} \inf_{n\neq 0} \int_{\mathbb{T}} |e^{-2\pi i n h} - 1|^2 \mathrm{d}\mu(h) \leq \inf_{f\in\mathcal{F}} \sup_{h\in H} ||\Delta_h f||_{L_2}^2.$$

For the other direction let

$$\varepsilon^2 = \inf_{f\in\mathcal{F}} \sup_{h\in H} ||\Delta_h f||_{L_2}^2.$$

We denote by $(c_0, ||.||_\infty)$ the real Banach space of two-way infinite null-convergent sequences, that is $(c_n)_{n=-\infty}^{\infty} \in c_0$ if and only if

$$\lim_{n\to\infty} |c_n| + |c_{-n}| = 0.$$

For every positive $n \in \mathbb{N}$ let

$$e_n = (\dots, 0, \underbrace{1, \dots, 1, 1}_{n}, 0, \underbrace{1, 1, \dots, 1}_{n}, 0, \dots) \in l_\infty.$$

For an $h \in \mathbb{T}$, let

$$a(h) = (a_n(h))_{n=-\infty}^{\infty} = \left(e^{-2\pi inh} - 1\right)_{n=-\infty}^{\infty},$$

and for every $Q \in \mathbb{N}$ consider the following subsets of c_0:

$$W_Q = \left\{P_m\left((|a_n(h)|^2)\right) | m \in \mathbb{N}, Q \leq m, h \in H\right\},$$

where $P_m : l_\infty \to c_0$,

$$P_m\left((c_n)_{n=-\infty}^{\infty}\right) = (\dots, 0, c_{-m}, c_{-m+1}, \dots, c_0, \dots, c_{m-1}, c_m, 0, \dots)$$

is the projection to the central $2m+1$ coordinates. Let K_Q denote the closure of the convex hull of W_Q in the $||.||_\infty$ norm. We show that for every $0 \leq \delta$ and $n, Q \in \mathbb{N}$,

$$(\varepsilon^2 - \delta)e_n \in K_Q.$$

Suppose that this is not true for a Q and n and consider the Minkowsky sum K of K_Q and the segment $[0, \frac{\delta}{2}]e_n$ and let

$$\gamma = \sup\{0 < \tau < 1 : \tau e_n \in K\}.$$

We have $\frac{\delta}{2} \leq \gamma \leq \varepsilon^2 - \frac{\delta}{2}$.

Since K is a convex subset of the locally convex Banach space l_∞ and γe_n is one point on its boundary, by the Theorem of Hahn and Banach there is a functional $\phi \in c_0^\star$ with norm 1 such that

$$\phi(c) \leq \gamma, \quad \forall c \in K \tag{21}$$

and $\phi(e_n) = 1$. Since $c_0^\star = {}_{\mathbb{R}}l_1$, ϕ can be represented as a $(z_n)_{n=-\infty}^{\infty} \in {}_{\mathbb{R}}l_1$ sequence, for which we have

$$\sum_{j=-\infty}^{\infty} |z_j| = ||(z_n)||_{L_1} = 1 = \phi(e_n) = \sum_{j=-n}^{n} z_j,$$

that is $0 \leq z_j$ and $z_m = 0$ for $n < |m|$ and for $n = 0$. Let $(d_n)_{n=-\infty}^{\infty} \in {}_{\mathbb{R}}l_2$ be the sequence for which $0 \leq d_n = \sqrt{z_n}$. Then, from (21), $||\left(a_n(h)d_n\right)||_{l_2}^2 \leq \gamma$ for every $h \in H$.

Consider now $f = \mathfrak{F}\left((d_n)_{n=-\infty}^{\infty}\right)$. By the Parseval formula $||f||_{L_2}^2 = 1$, $\int_{\mathbb{T}} f = 0$, so $f \in \mathcal{F}$, but for every $h \in H$,

$$||\Delta_h f||_{L_2}^2 = ||\mathfrak{F}\left(\Delta_h f\right)||_{l_2}^2 = ||\left(a_n(h)d_n\right)||_{l_2}^2 \leq \gamma \leq \varepsilon^2 - \frac{\delta}{2},$$

which is a contradiction.

This means that for every $0 < \delta$ and $Q \in \mathbb{N}$ there is a convex combination $\left(w_n^Q\right)_{n=-\infty}^{\infty}$ of certain elements of W_Q such that

$$(\varepsilon^2 - \delta) \leq w_j^Q$$

for every $0 \neq |j| \leq Q$. By reformulation, for every $Q \in \mathbb{N}$ there is a μ_Q convex combination of Dirac measures on $\mathbb{T}$ such that for every $|j| \leq Q$, $j \neq 0$,

$$\varepsilon^2 - \delta \leq \left(\int_{\mathbb{T}} P_Q(|a(h)|^2)\mathrm{d}\mu_Q(h)\right)(j) = P_Q\left(\int_{\mathbb{T}} |a(h)|^2\mathrm{d}\mu_Q(h)\right)(j),$$

that is

$$\varepsilon^2 - \delta \leq \int_{\mathbb{T}} |a_j|^2(h)\mathrm{d}\mu_Q(h) = \int_{\mathbb{T}} \left|e^{-2\pi ijh} - 1\right|^2 \mathrm{d}\mu_Q(h)$$

for every $|j| \leq Q$, $j \neq 0$. Moreover μ_Q is a probability measure.

The Banach space of measures on $\mathbb{T}$ being the dual space of the continuous functions on $\mathbb{T}$, we can take a subsequence μ_{Q_j} weakly converging to a μ probability measure, and since $\mathrm{supp}\mu_Q \subset H$ for every $Q \in \mathbb{N}$ and H is compact we have $\mathrm{supp}\mu \subset H$ also.

By the of definition of weak limit we get

$$\varepsilon^2 - \delta \leq \lim_{j\to\infty} \int_{\mathbb{T}} \left|e^{-2\pi inh} - 1\right|^2 \mathrm{d}\mu_{Q_j}(h) = \int_{\mathbb{T}} \left|e^{-2\pi int} - 1\right|^2 \mathrm{d}\mu(h)$$

for every $n \neq 0$. Taking $\delta \to 0$ we obtain

$$\inf_{f\in\mathcal{F}} \sup_{h\in H} ||\Delta_h f||_{L_2}^2 \leq \sup_{\mu\in\mathcal{M}(H)} \inf_{n\neq 0} \int_{\mathbb{T}} |e^{-2\pi inh} - 1|^2\mathrm{d}\mu(h),$$

so the proof is complete.■

Remark 6.2. The argument of Imre Ruzsa for proving that every compact weak-Dirichlet set is non-ejective goes on the following way. If the set $H \subset \mathbb{T}$ is a compact weak-Dirichlet set, then by Remark 1.5,

$$\sup_{\mu \in \mathcal{M}} \inf_{n \neq 0} \int_{\mathbb{T}} |e^{-2\pi i n h} - 1|^2 \mathrm{d}\mu(h) = 0.$$

So by Theorem 6.1 for every $0 < \varepsilon$ there is a function $f^{(\varepsilon)} \in L_2$ with $||f^{(\varepsilon)}||_{L_2} = 1$ and $\int_{\mathbb{T}} f^{(\varepsilon)} = 0$ such that $||\Delta_h f^{(\varepsilon)}||_{L_2} \leq \varepsilon$ for every $h \in H$.

For a sequence $(\varepsilon_n) \in \{-1, +1\}^{\mathbb{Z}}$ let

$$f_{(\varepsilon_n)} = \mathfrak{F}^{-1}((\varepsilon_n (\mathfrak{F} f)(n))),$$

that is the function obtained from f by signing its Fourier coefficients with (ε_n). By averaging on the signing, Imre Ruzsa showed that there is a sequence $(\varepsilon_n) \in \{-1, +1\}^{\mathbb{Z}}$ and a $u \in \mathbb{R}$ such that for the sub-level set $A = [f^{(\varepsilon)}_{(\varepsilon_n)} < u]$ we get

$$\sup_{h \in H} \lambda(\Delta_h A) \leq \varepsilon \lambda(A),$$

from which the non-ejectivity of H follows easily.

6.2 Conclusions

There is no reason to think that the conjecture of Tamás Keleti fails for the class $\mathfrak{H}(L_p, L_q)$ if $2 \leq q$. Our proof brakes down at $q = 2$ only because of that caprice of the Fourier transform that it happens to be isometric isomorphism just on the L_2 class. By using an analogue of the argument of Ruzsa Imre we could avoid the Fourier transform and prove the conjecture for the remaining classes. On the other hand, for $2 < q$ we have no Theorem 6.1 like statement for the L_q class. But from a function $f \notin L_q$ with $\Delta_h \in L_q$ for every $h \in H$ one may can construct a function for which a sub-level set works on the same way. A difficulty can be that not every function has appropriate sub-level sets even in the L_2 case, and for $2 < q$ we cannot average on the Fourier coefficients.

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