ENTRANCE EXAMINATION - PHD

Duration: 360 minutes

During your work:

(1) you MAY use textbooks or course notes;
(2) you SHOULD NOT use external help;
(3) you SHOULD NOT use electronic devices for symbolic computations;
(4) you SHOULD NOT use the internet for search or communication.

BY WRITING THIS TEST,
YOU AGREE TO THE TERMS LISTED ABOVE.

This examination contains 12 pages, including this cover page.

Each page contains a test with problems on one of the following topics:

0. **BASIC MATHEMATICS**
1. **ALGEBRA**
2. **ANALYSIS**
3. **COMBINATORICS**
4. **GEOMETRY**
5. **PROBABILITY THEORY**
6. **SET THEORY**
7. **MEASURE THEORY / FUNCTIONAL ANALYSIS**
8. **OPERATIONS RESEARCH**
9. **TOPOLOGY / DIFFERENTIAL GEOMETRY**

You have to do the test from BASIC MATHEMATICS.

You also have to choose 3 further topics from the subjects 1–9 and attempt all problems from the chosen topics. Each topic is worth 100 marks.

Please, pay attention to the presentation. Send us your work even if you have not solved each problem or have only partial results in a certain problem. Unless otherwise stated, you SHOULD JUSTIFY your answers.

Send the scanned copy of your solutions WITHIN 6 HOURS of receiving the problem sheet to:

`agoston@cs.elte.hu`

A confirmation letter will be sent within the next 24 hours.

GOOD LUCK!

0. BASIC MATHEMATICS: 10 multiple choice questions

Each problem has exactly one correct answer.
Please, write your solutions into the boxes at the bottom of each page.

0/1. How can one *disprove* a statement of the form "If A then B." ?
- a) We prove that if A holds then B is false.
- b) We prove that if B holds then A is false.
- c) We show an example when A holds but B is false.
- d) We show an example when B holds but A is false.

0/2. What is the negation of the following statement? *There exists a town in Dreamland in which every child has a dragon.*
- a) There exists a town in Dreamland, in which no child has a dragon.
- b) There exists a town in Dreamland, in which there exists a child with no dragon.
- c) In every town in Dreamland there exists a child with no dragon.
- d) In every town in Dreamland there exists a child who has a dragon.

0/3. Which set equals to $\{x : x \in A \Longrightarrow x \in B\}$?
- a) $A \cup B$
- b) $A \cap B$
- c) $A \cap \overline{B}$
- d) $\overline{A} \cup B$

0/4. Which set equals to $\{y : |x| < y \Longrightarrow x < 4\}$?
- a) $(-\infty, 4)$
- b) $(-\infty, 4]$
- c) $(0, 4)$
- d) $[0, 4]$

0/5. Which of the following statements is *false*?
- a) Every real number with finite decimal representation is rational.
- b) Every nonzero real number with finite decimal representation has two infinite decimal representations.
- c) Every rational number has two infinite decimal representations.
- d) If a real number has two infinite decimal representations then it is rational.

Solutions to questions 1-5:

1	2	3	4	5

THIS TEST CONTINUES ON THE NEXT PAGE!

0/6. Which of the following statements is true?

a) If a set of real numbers has a finite supremum then it also has a maximum.
b) The sumpremum of a set of real numbers is always an element of the set.
c) If the supremum of a set of real numbers is an element of the set then the set has a maximum.
d) Every bounded set of real numbers has a maximum.

0/7. Let A and B subsets of the reals. What is the connection between the following statements?

P: $(\forall x \in A)\ (\exists y \in B)\ x < y$ Q: $(\forall y \in B)\ (\exists x \in A)\ x < y$

a) $P \Rightarrow Q$ but $Q \not\Rightarrow P$
b) $Q \Rightarrow P$ but $P \not\Rightarrow Q$
c) $P \Leftrightarrow Q$
d) $P \not\Rightarrow Q$ and $Q \not\Rightarrow P$

0/8. Let (a_n) be a sequence of real numbers. Which of the following statements is *false*?

a) If (a_n) is bounded and nondecreasing then it is convergent.
b) If (a_n) tends to infinity then it is nondecreasing.
c) If (a_n) is nondecreasing then it has a (finite or infinite) limit.
d) If (a_n) tends to infinity then it is unbounded.

0/9. Which of the following statements is true?

a) The function f is strictly increasing on A if there exist $x_1, x_2 \in A$ such that $x_1 < x_2$ and $f(x_1) < f(x_2)$.
b) The function f is **not** strictly increasing on A if there exist $x_1, x_2 \in A$ such that $x_1 < x_2$ and $f(x_1) \geq f(x_2)$.
c) The function f is strictly increasing on A if for every $x_1, x_2 \in A$ we have $x_1 < x_2$ and $f(x_1) < f(x_2)$.
d) If the function f is **not** strictly increasing on A then

$$x_1, x_2 \in A, x_1 < x_2 \Longrightarrow f(x_1) \geq f(x_2).$$

0/10. Suppose that $\lim_{x \to 5} f(x) = 7$. What happens if f is increased at 5 by 1 (and remains the same everywhere else)?

a) The new function will tend to 7 at 5.
b) The new function will tend to 8 at 5.
c) The new function will have no limit at 5.
d) We cannot claim for sure any of the above three statements.

Solutions to questions 6-10:

6	7	8	9	10

1. ALGEBRA: 4 problems

1/1. Consider the following matrix:

$$A = \begin{pmatrix} 0 & 2 & -2 \\ 2 & 0 & 2 \\ 2 & -2 & 4 \end{pmatrix}$$

Find the characteristic polynomial, eigenvalues and eigenvectors of A and determine whether the matrix is diagonalizable. If it is, find a matrix T such that $T^{-1}AT$ is a diagonal matrix.

1/2. For a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ the infinite sequence $(r_1, r_2, \ldots, r_k, \ldots)$ will be called be the *rank sequence* of T, if $r_i = \dim \operatorname{Im} T^i$ for every $i = 1, 2, \ldots$. For each of the following sequences determine whether the sequence is the rank sequence of a linear transformation $T : \mathbb{R}^{10} \to \mathbb{R}^{10}$. Justify your answer.

a) $(9, 8, 7, \ldots, 0, 0 \ldots)$;
b) $(10, 9, 8, \ldots, 0, 0 \ldots)$;
c) $(8, 6, 4, 4, \ldots)$;
d) $(8, 6, 4, 4, 2, \ldots)$,

1/3. Let S_n denote the group of all permutations on n letters.

a) Which is the smallest n for which there is an element σ of order 15 in S_n?
b) Determine the number of 2-Sylow subgroups of S_4 and find the intersection of these 2-Sylow subgroups (i.e. list the elements in it).

1/4. Let $\mathbb{Z}_2$ denote the field of residue classes modulo 2 and consider the four factor rings $R_1 = \mathbb{Z}_2[x]/(x^2)$, $R_2 = \mathbb{Z}_2[x]/(x^2 + 1)$, $R_3 = \mathbb{Z}_2[x]/(x^2 + x)$ and $R_4 = \mathbb{Z}_2[x]/(x^2 + x + 1)$. Determine:

a) which of them contain(s) nonzero nilpotent elements;
b) which of them contain(s) zero divisors;
c) which of them form(s) a field;
d) whether any two of these rings are isomorphic to each other.

2. ANALYSIS: 5 problems

2/1. Let $\sum f_n$ be a uniformly convergent sequence of functions on the interval $[a,b]$. Does it imply that the series $\sum \sup\{|f_n(x)|\colon x \in [a,b]\}$ is convergent? Is this true if we also assume that the functions f_n are continuous on $[a,b]$?

2/2. Assume that $\sum a_n$ is convergent. Does it imply that (i) $n \cdot a_n \to 0$ and (ii) $\sum a_n^2$ is convergent? Are (i) and (ii) true if we also assume that the sequence (a_n) is monotone?

2/3. Determine the value of $\sqrt{2}$ with accuracy $1/16$ by using some terms of the Taylor series of $f(x) = \sqrt{1+x}$ and Lagrange's remainder.

2/4. Is the function

$$f(x,y) = \frac{x^2 y}{x^2+y^2}, \quad f(0,0) = 0$$

continuous? Is this function differentiable at $(0,0)$?

2/5. Determine the local maxima and minima for $f(x,y) = x^3 + 3y^3 - 12x - 81y + 8$.

3. COMBINATORICS: 5 problems

3/1. Find the number of eight digit numbers that contain exactly two '1's.

3/2. Let $\chi(G)$ denote the chromatic number of a simple graph G, and let e be the number of edges in G. Show that $e \leq \binom{\chi(G)}{2}$, and characterize graphs attaining equality.

3/3. Consider a 10×10 table filled with positive real numbers. The *neighborhood* of an entry is the set of (at most eight) entries that are tightly around it. An entry a of the table is called *quasi dominant* if there is at most one entry in its neighborhood that is at least as large as a. Find the maximum possible number of quasi dominant entries.

3/4. Determine the number of (consecutive) zero digits at the end of $11^{100} - 1$.

3/5. In a narrow tunnel, n ants are marching, one after the other. The tunnel is so narrow that the ants cannot pass each other. Their way passes a similarly narrow side tunnel, into which one or more ants may enter, wait till some of the other ants pass, and then return to the row and continue marching. Determine the number of different orders the ants may have after passing the side tunnel.

4. GEOMETRY: 4 problems

4/1. Assume that O, A, A', B, and B' are points in the Euclidean plane, $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OA'} = \lambda\mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{OB'} = \mu\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are linearly independent vectors, λ and μ are real numbers such that $\lambda \cdot \mu \neq 1$. Let I be the intersection point of the lines AB' and $A'B$. Express the vector $\overrightarrow{OI}$ as a linear combination of the vectors $\mathbf{a}$ and $\mathbf{b}$.

4/2. Suppose that the vertices of a cube have integer coordinates with respect to a Cartesian coordinate system. Show that both the volume of the cube, the area of its facet are integers. Derive from this that the edge length of the cube is also an integer.

4/3. Consider the triangles contained in a given regular pentagon. Which of these triangles have maximal perimeter?

4/4. Let $K \neq \emptyset$ be a closed bounded convex set in $\mathbb{R}^3$. Show that there is a plane which intersects K at exactly one point.

5. PROBABILITY THEORY: 4 problems

5/1. Let X and Y be independent with standard normal distribution. Calculate $Var(2X - Y)$.

5/2. Let X have a uniform distribution over the interval $[0, 1]$. Give the density function of $(X - 0.5)^2$.

5/3. Let X be an arbitrary random variable and suppose that $E(2^X) = 4$. Prove that $P(X > 3) \leq 1/2$.

5/4. Let X_n have Poisson distribution with parameter n. What is the limit of the sequence $P(X_n < n)$?

6. SET THEORY: 4 problems

6/1. Prove that if $\{x_\xi : \xi < \alpha\}$ is a strictly increasing sequence of reals, α an ordinal, then α is countable.

6/2. $a_0, a_1, \ldots$ are natural numbers, i.e., finite cardinals. If the cardinal $a_0 \cdot a_1 \cdots$ is infinite then what is it?

6/3. Prove that if $(A, <)$ is an ordered set such that $(A, <)$ and $(A, >)$ are both well ordered then A is finite.

6/4. Prove that if A is an infinite set then there are exactly $2^{|A|}$ permutations of A.

7. MEASURE THEORY / FUNCTIONAL ANALYSIS: 4 problems

7/1. Let $A_1, A_2, \ldots$ be Lebesgue-measurable subsets of $[0,1]$ of measure $1/2$, and let A denote the set of points x such that x belongs to infinitely many of the sets A_n. Prove that the Lebesgue measure of A is at least $1/2$.

7/2. For every irrational number $x \in [0,1]$ let $f(x)$ denote the number of digits 7 in the decimal expansion of the number x. Prove that the Lebesgue integral $\int_0^1 f\,dx$ exists, and compute its value.

7/3. Let A be a compact set in a Banach space X, and let $B \subset X$ be closed. Prove that the set $A + B = \{x + y : x \in A,\ y \in B\}$ is closed. Is it true that if $A, B \subset X$ are closed then $A + B$ is also closed?

7/4. Let $f : X \to \mathbb{R}$ be a linear functional on the real Banach space X such that $f^{-1}(0)$ is closed. Prove that f is continuous.

8. OPERATIONS RESEARCH: 4 problems

8/1. Let $A \in \mathbf{R}^{m\times n}$ and $b \in \mathbf{R}^m$. Prove that if the linear inequality system $Ax \leq b$ has no solution, then it has a subsystem of at most $n+1$ inequalities with no solution.

8/2. Write the dual of the following linear programming problem.

$$\begin{gathered} x_1 - x_2 + x_3 \leq 1 \\ -x_1 + x_2 + x_3 = -1 \\ x_1 + x_2 + x_3 \geq 0 \\ x_1, x_2 \geq 0 \\ \min\ 2x_1 + 3x_3. \end{gathered}$$

8/3. Let $D = (V, E)$ be a digraph, and let $\ell : E \to \mathbf{R}_+$ be a nonnegative length function on the arcs. We are given two nodes s and t such that t is reachable from s in D, and the removal of any arc from D does not change the length of the shortest $s-t$ path. Prove that there are 2 arc-disjoint shortest $s-t$ paths in D.

8/4. Let $A \in \mathbf{R}^{m\times n}$ be a real matrix. Prove that we can round each element of A either up or down in such a way that every row sum and every column sum changes by less than 1.

9. TOPOLOGY / DIFFERENTIAL GEOMETRY: 5 problems

9/1. Consider the set

$$X = \{(t, \sin(1/t)) \mid t \in (0,1)\} \cup \{(0,t) \mid t \in [-1,1]\}$$

with the subspace topology inherited from the standard topology of $\mathbb{R}^2$. Is Xconnected? (Recall that a topological space X is said to be **connected** if it cannot be obtained as the union of two disjoint non-empty open subsets of X.)

9/2. Let $X = S^2 \cup [A,B]$ be the union of the sphere

$$\mathbb{S}^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

and the segment $[A,B]$ connecting the points $A = (0,0,1)$ and $B = (0,0,-1)$. X is a topological space with the subspace topology inherited from the standard topology of $\mathbb{R}^3$, which contains the origin $\mathbf{0}$. Compute the fundamental group $\pi_1(X, \mathbf{0})$.

9/3. Let $\gamma\colon (a,b) \to \mathbb{R}^3$ be a regular parameterized curve in $\mathbb{R}^3$. Assume that all normal planes of γ go through a given point P. Show that the set $\{\gamma(t) \mid t \in (a,b)\}$ is contained in a sphere.

9/4. Is the open ball $B = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \sum_{i=1}^n x_i^2 < 1\}$ diffeomorphic to the open cube $C = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \max_{1 \le i \le n} |x_i| < 1\}$?

9/5. Assume that a smooth function $f\colon \mathbb{R}^3 \to \mathbb{R}$ vanishes on a Möbius band $M \subset \mathbb{R}^3$ embedded into $\mathbb{R}^3$ as a smooth submanifold. Show that the gradient vector field $\operatorname{grad} f = (\partial_1 f, \partial_2 f, \partial_3 f)$ of f vanishes at a point of M.